CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 3

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1.

$$\begin{aligned}
\int_{-\infty}^{t} x(s)ds &= \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 s} ds \\
&= \sum_{k=-\infty}^{\infty} (a_k \cdot \frac{e^{jkw_0t}}{jkw_0}\Big|_{-\infty}^{t}) \\
&= \sum_{k=-\infty}^{\infty} (a_k \cdot \frac{e^{jkw_0t}}{jkw_0} - a_k \cdot \frac{e^{jkw_0(-\infty)}}{jkw_0}) \\
&= \sum_{k=-\infty}^{\infty} (a_k \cdot \frac{e^{jkw_0t}}{jkw_0} - a_k \cdot \frac{0}{jkw_0}) \\
&= \sum_{k=-\infty}^{\infty} (a_k \cdot \frac{e^{jkw_0t}}{jkw_0})
\end{aligned}$$

This equation is in the synthesis equation form where $a_k \frac{1}{jkw_0}$ is the Fourier series coefficients of the integrated signal.

Since w_0 is the frequency of the signal, $w_0 = \frac{2\pi}{T}$ where T is the period of the signal.

Substituting w_0 in the equation above, we prove the integration property of the Fourier series.

2. (a) $x(t)x(t) \leftrightarrow a_k * a_k$ (Multiplication Property)
   (b) $\mathcal{E}v\{x(t)\} \leftrightarrow b_k$ (Even Property)

$$b_k = \begin{cases} a_k & k \geq 0 \\ a_{-k} & k < 0 \end{cases}$$

   (c) $x(t+t_0) + x(t-t_0) \leftrightarrow a_k e^{jkw_0t_0} + a_{-k}e^{-jkw_0t_0}$ (Shifting and Linearity Properties)

3.

$$x(t) = \begin{cases} 2 & x \in (0,1) \\ 0 & x \in (1,2) \\ -2 & x \in (2,3) \\ 0 & x \in (3,4) \\ \text{Periodic} & x \notin (0,4) \end{cases}$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_0^T x(t)e^{-jkw_0t}dt \\
&= \frac{1}{4}(\int_0^1 2e^{-jkw_0t}dt + \int_1^2 0dt + \int_2^3 -2e^{-jkw_0t}dt + \int_3^4 0dt) \\
&= \frac{1}{4}(2\frac{e^{-jkw_0t}}{-jkw_0}\Big|_0^1 - 2\frac{e^{-jkw_0t}}{-jkw_0}\Big|_2^3) \\
&= \frac{1}{4}(2\frac{e^{-jkw_0}}{-jkw_0} - \frac{2}{-jkw_0} - 2\frac{e^{-3jkw_0}}{-jkw_0} + 2\frac{e^{-2jkw_0}}{-jkw_0}) \\
&= \frac{1}{-2jkw_0}(e^{-jkw_0} - 1 - e^{-3jkw_0} + e^{-2jkw_0})
\end{aligned}
$$

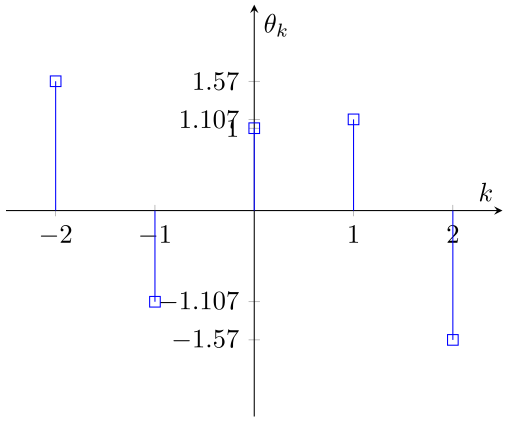
Substitute $w_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

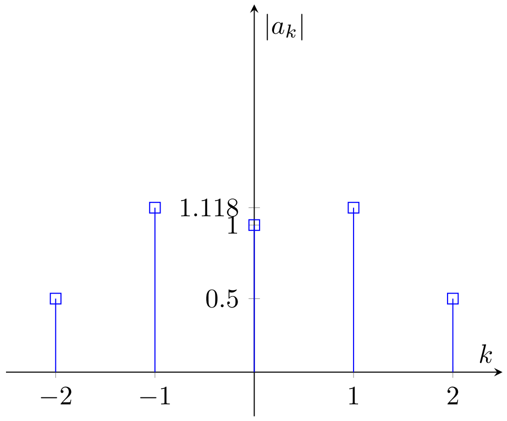
$$
\begin{aligned}
a_k &= \frac{1}{-2jk\frac{\pi}{2}}(e^{-jk\frac{\pi}{2}} - 1 - e^{-3jk\frac{\pi}{2}} + e^{-2jk\frac{\pi}{2}}) \\
&= \frac{1}{-jk\pi}(e^{-jk\frac{\pi}{2}} - 1 - e^{-3jk\frac{\pi}{2}} + e^{-jk\pi}) \\
&= \frac{1}{-jk\pi}(cos(-k\frac{\pi}{2}) + jsin(-k\frac{\pi}{2}) - 1 - cos(-3k\frac{\pi}{2}) - jsin(-3k\frac{\pi}{2}) + cos(-k\pi) + jsin(-k\pi)) \\
&= \frac{1}{-jk\pi}(-2jsin(k\frac{\pi}{2}) - 1 + cos(-k\pi))
\end{aligned}
$$

4. (a)

$$
\begin{aligned}
x(t) &= 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4}) \\
&= 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t)\cos(\frac{\pi}{4}) - \sin(2\omega_0 t)\sin(\frac{\pi}{4}) \\
&= 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \frac{\sqrt{2}}{2}\cos(2\omega_0 t) - \frac{\sqrt{2}}{2}\sin(2\omega_0 t) \\
&= 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + 2\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + \frac{\sqrt{2}}{2}\frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} - \frac{\sqrt{2}}{2}\frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} \\
&= 1 + \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{\sqrt{2}}{4}e^{j2\omega_0 t} + \frac{\sqrt{2}}{4}e^{-j2\omega_0 t} - \frac{\sqrt{2}}{4j}e^{j2\omega_0 t} + \frac{\sqrt{2}}{4j}e^{-j2\omega_0 t}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_0 &= 1 \\
\alpha_1 &= 1 + \frac{1}{2j} \\
\alpha_{-1} &= 1 - \frac{1}{2j} \\
\alpha_2 &= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4j} \\
\alpha_{-2} &= \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4j}
\end{aligned}
$$





(b)

(c)

(d)

5. (a)

$$\begin{aligned}
x[n] &= sin(\frac{\pi}{2}n) \\
&= \frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j} \\
&= \frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\frac{\pi}{2}n} \\
\alpha_1 &= \frac{1}{2j} \\
\alpha_{-1} &= -\frac{1}{2j}
\end{aligned}$$

(b)

$$\begin{aligned}
y[n] &= 1 + cos(\frac{\pi}{2}n) \\
&= 1 + \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2} \\
&= 1 + \frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n} \\
\alpha_0 &= 1 \\
\alpha_1 &= \frac{1}{2} \\
\alpha_{-1} &= \frac{1}{2}
\end{aligned}$$

(c)

$$
\begin{aligned}
x[n]y[n] &\leftrightarrow \alpha_k * \beta_k \\
&= \sum_{k=0}^{N-1} \alpha_l \beta_{k-l} \\
&= \sum_{k=0}^{3} \alpha_l \beta_{k-l} \\
&= \alpha_0\beta_{k-0} + \alpha_1\beta_{k-1} + \alpha_2\beta_{k-2} + \alpha_3\beta_{k-3} \\
c_k &= \frac{1}{2}\beta_{k-1} + \frac{1}{2}\beta_{k-3}
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= \frac{-1}{2j} \\
c_3 &= 0 \\
c_4 &= \frac{1}{2j}
\end{aligned}
$$

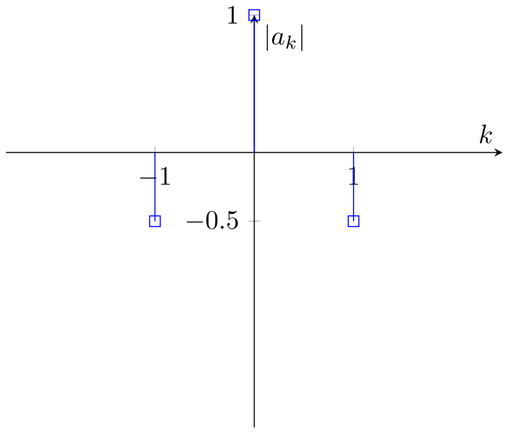
(d)

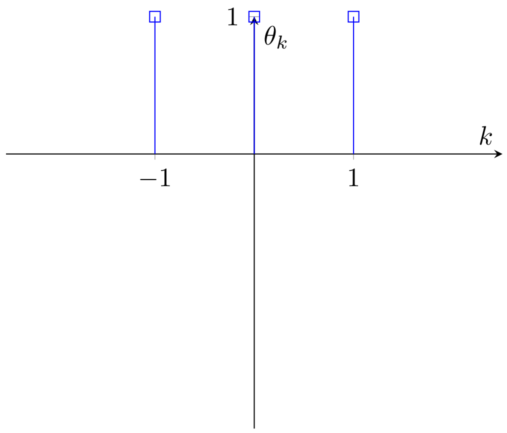
$$
\begin{aligned}
c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]y[n]e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{4}\sum_{n=0}^{3} x[n]y[n]e^{-j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}(x[0]y[0]e^{-j\frac{2\pi}{4}k0} + x[1]y[1]e^{-j\frac{2\pi}{4}k1} + x[2]y[2]e^{-j\frac{2\pi}{4}k2} + x[3]y[3]e^{-j\frac{2\pi}{4}k3}) \\
&= \frac{1}{4}(0\cdot 2\cdot e^{-j\frac{2\pi}{4}k0} + 1\cdot 1\cdot e^{-j\frac{2\pi}{4}k1} + 0\cdot 0\cdot e^{-j\frac{2\pi}{4}k2} + (-1)\cdot 1\cdot e^{-j\frac{2\pi}{4}k3}) \\
&= \frac{1}{4}(e^{-j\frac{2\pi}{4}k} - e^{-j\frac{2\pi}{4}k3}) \\
c_1 &= 0 \\
c_2 &= \frac{-1}{2j} \\
c_3 &= 0 \\
c_4 &= \frac{1}{2j}
\end{aligned}
$$

The results are the same.

6. (a)

$$
\begin{aligned}
x[n] &= 1 - \cos(\frac{n\pi}{2}) \\
&= 1 - \frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2} \\
&= 1 - \frac{1}{2}e^{j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
\alpha_0 &= 1 \\
\alpha_1 &= -\frac{1}{2} \\
\alpha_{-1} &= -\frac{1}{2}
\end{aligned}
$$

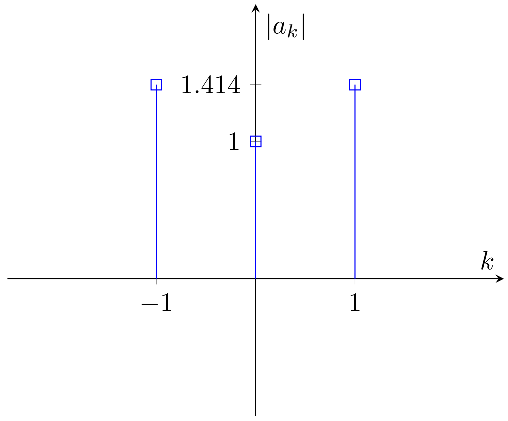


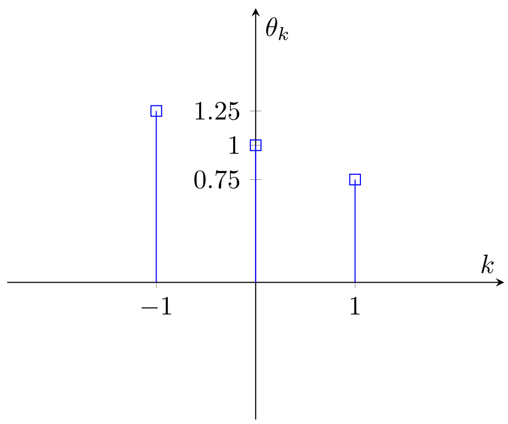


(b) i.

ii.

$$
\begin{aligned}
y[n] &= 1 + \sin(\frac{n\pi}{2}) - \cos(\frac{n\pi}{2}) \\
&= 1 + \frac{e^{j\frac{n\pi}{2}} - e^{-j\frac{n\pi}{2}}}{2j} - \frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2} \\
&= 1 + \frac{1}{2j}e^{j\frac{n\pi}{2}} - \frac{1}{2j}e^{-j\frac{n\pi}{2}} - \frac{1}{2}e^{j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
&= 1 - \frac{1}{2}e^{j\frac{n\pi}{2}} + \frac{1}{2j}e^{j\frac{n\pi}{2}} - \frac{1}{2j}e^{-j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
\alpha_0 &= 1 \\
\alpha_1 &= -\frac{1}{2} + \frac{1}{2j} \\
\alpha_{-1} &= -\frac{1}{2} - \frac{1}{2j}
\end{aligned}
$$





7. (a)

   (b)

Figure 1: Function for Spectral Coefficients

(a)
```
from numpy import exp, pi

def spectral_coefficients(signal, period, num_coefficients):
    coefficients = []
    for k in range(num_coefficients + 1):
        S = 0
        for n in range(period):
            S += signal[n] * exp(-1j * 2 * pi * n * k / period)
        coefficients.append(S / period)
    return coefficients
```

Figure 2: Class for Approximating from Spectral Coefficients

(b)
```
from matplotlib import pyplot
from numpy import exp, pi

SAVE_FOLDER = "figures"

class SignalFromSpectralCoefficients:
    def __init__(self, coefficients, period):
        self.coefficients = coefficients
        self.period = period

    def __getitem__(self, n):
        S = 0
        for k, coefficient in enumerate(self.coefficients):
            S += coefficient * exp(1j * 2 * pi * n * k / self.period)
        return S

    def __iter__(self):
        for n in range(self.period):
            yield self[n]

    def __len__(self):
        return self.period

    def plot(self, title, save_path):
        pyplot.title(title)
        pyplot.plot(range(self.period), [abs(item) for item in self])
        pyplot.savefig(SAVE_FOLDER + "/" + save_path)
        pyplot.clf()
```

Figure 3: Code for Reconstructing the Square Wave

(c)
```
import numpy
from matplotlib import pyplot
from scipy.signal import sawtooth
from q8a import spectral_coefficients
from q8b import SignalFromSpectralCoefficients

t = numpy.linspace(-0.5, 0.5, 1000)

square_wave = [-10] * 500 + [10] * 500
for n in (1, 5, 10, 50, 100):
    pyplot.plot(t, square_wave, label="square wave")
    coefficients = spectral_coefficients(square_wave, len(square_wave), n)
    reconstructed = SignalFromSpectralCoefficients(coefficients, 1000)
    reconstructed.plot("Reconstructed Square Wave", f"square_wave_{n}.png")
```

Figure 4: Code for Reconstructing the Sawtooth Wave

(d)
```
import numpy
from matplotlib import pyplot
from scipy.signal import sawtooth
from q8a import spectral_coefficients
from q8b import SignalFromSpectralCoefficients

t = numpy.linspace(-0.5, 0.5, 1000)

sawtooth_wave = sawtooth(2 * numpy.pi * t)
for n in (1, 5, 10, 50, 100):
    pyplot.plot(t, sawtooth_wave, label="sawtooth wave")
    coefficients = spectral_coefficients(sawtooth_wave, len(sawtooth_wave), n)
    reconstructed = SignalFromSpectralCoefficients(coefficients, 1000)
    reconstructed.plot("Reconstructed Sawtooth Wave", SAVE_PATH + f"sawtooth_wave_{n}.png")
```