CENG 384 - Signals and Systems for Computer Engineers
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Homework 3

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1.

$$\int_{-\infty}^{t} x(s)ds = \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 s} ds$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k \cdot \frac{e^{jkw_0 t}}{jkw_0} \Big|_{-\infty}^{t} \right)$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k \cdot \frac{e^{jkw_0 t}}{jkw_0} - a_k \cdot \frac{e^{jkw_0(-\infty)}}{jkw_0} \right)$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k \cdot \frac{e^{jkw_0 t}}{jkw_0} - a_k \cdot \frac{0}{jkw_0} \right)$$

$$= \sum_{k=-\infty}^{\infty} \left(a_k \cdot \frac{e^{jkw_0 t}}{jkw_0} \right)$$

This equation is in the synthesis equation form where $a_k \frac{1}{jkw_0}$ is the Fourier series coefficients of the integrated signal.

Since w_0 is the frequency of the signal, $w_0 = \frac{2\pi}{T}$ where T is the period of the signal.

Substituting w_0 in the equation above, we prove the integration property of the Fourier series.

2. (a) $x(t)x(t) \leftrightarrow a_k * a_k$ (Multiplication Property)

   (b) $\mathcal{E}v\{x(t)\} \leftrightarrow b_k$ (Even Property)

$$b_k = \begin{cases} a_k & k \geq 0 \\ a_{-k} & k < 0 \end{cases}$$

   (c) $x(t + t_0) + x(t - t_0) \leftrightarrow a_k e^{jkw_0 t_0} + a_{-k} e^{-jkw_0 t_0}$ (Shifting and Linearity Properties)

3.

$$x(t) = \begin{cases} 2 & x \in (0, 1) \\ 0 & x \in (1, 2) \\ -2 & x \in (2, 3) \\ 0 & x \in (3, 4) \\ \text{Periodic} & x \notin (0, 4) \end{cases}$$

$$a_k = \frac{1}{T} \int_0^T x(t)e^{-jkw_0t}dt$$

$$= \frac{1}{4}\left(\int_0^1 2e^{-jkw_0t}dt + \int_1^2 0dt + \int_2^3 -2e^{-jkw_0t}dt + \int_3^4 0dt\right)$$

$$= \frac{1}{4}\left(2\frac{e^{-jkw_0t}}{-jkw_0}\Big|_0^1 - 2\frac{e^{-jkw_0t}}{-jkw_0}\Big|_2^3\right)$$

$$= \frac{1}{4}\left(2\frac{e^{-jkw_0}}{-jkw_0} - \frac{2}{-jkw_0} - 2\frac{e^{-3jkw_0}}{-jkw_0} + 2\frac{e^{-2jkw_0}}{-jkw_0}\right)$$

$$= \frac{1}{-2jkw_0}(e^{-jkw_0} - 1 - e^{-3jkw_0} + e^{-2jkw_0})$$

Substitute $w_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$a_k = \frac{1}{-2jk\frac{\pi}{2}}(e^{-jk\frac{\pi}{2}} - 1 - e^{-3jk\frac{\pi}{2}} + e^{-2jk\frac{\pi}{2}})$$

$$= \frac{1}{-jk\pi}(e^{-jk\frac{\pi}{2}} - 1 - e^{-3jk\frac{\pi}{2}} + e^{-jk\pi})$$

$$= \frac{1}{-jk\pi}(cos(-k\frac{\pi}{2}) + jsin(-k\frac{\pi}{2}) - 1 - cos(-3k\frac{\pi}{2}) - jsin(-3k\frac{\pi}{2}) + cos(-k\pi) + jsin(-k\pi))$$

$$= \frac{1}{-jk\pi}(-2jsin(k\frac{\pi}{2}) - 1 + cos(-k\pi))$$

4. (a)

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4})$$

$$= 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t)\cos(\frac{\pi}{4}) - \sin(2\omega_0 t)\sin(\frac{\pi}{4})$$

$$= 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \frac{\sqrt{2}}{2}\cos(2\omega_0 t) - \frac{\sqrt{2}}{2}\sin(2\omega_0 t)$$

$$= 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + 2\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + \frac{\sqrt{2}}{2}\frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} - \frac{\sqrt{2}}{2}\frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j}$$

$$= 1 + \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{\sqrt{2}}{4}e^{j2\omega_0 t} + \frac{\sqrt{2}}{4}e^{-j2\omega_0 t} - \frac{\sqrt{2}}{4j}e^{j2\omega_0 t} + \frac{\sqrt{2}}{4j}e^{-j2\omega_0 t}$$
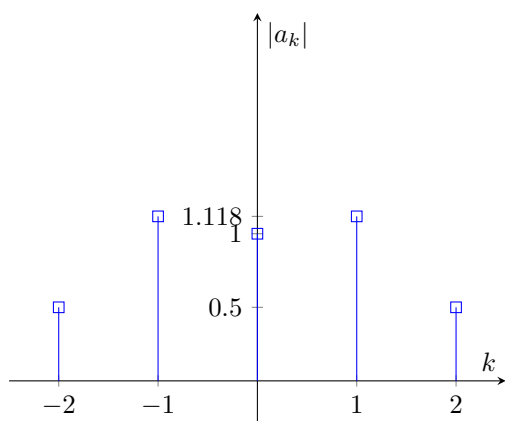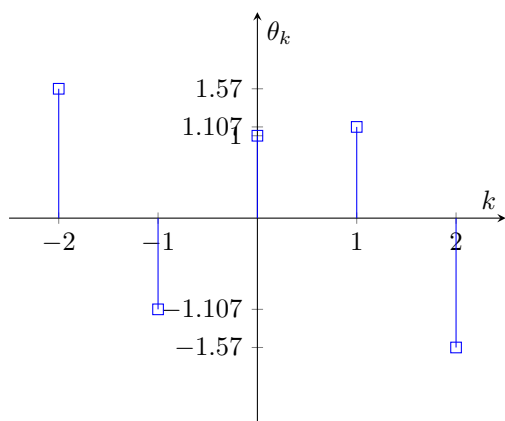
$$\alpha_0 = 1$$

$$\alpha_1 = 1 + \frac{1}{2j}$$

$$\alpha_{-1} = 1 - \frac{1}{2j}$$

$$\alpha_2 = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4j}$$

$$\alpha_{-2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4j}$$

2

θ_k

1.57

1.107
1

k

−2   −1   1   2

−1.107

−1.57

$|a_k|$

1.118
1

0.5

k

−2   −1   1   2

(b)
(c)
(d)

5. (a)

$$x[n] = sin(\frac{\pi}{2}n)$$
$$= \frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j}$$
$$= \frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\frac{\pi}{2}n}$$
$$\alpha_1 = \frac{1}{2j}$$
$$\alpha_{-1} = -\frac{1}{2j}$$

(b)

$$y[n] = 1 + cos(\frac{\pi}{2}n)$$
$$= 1 + \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2}$$
$$= 1 + \frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n}$$
$$\alpha_0 = 1$$
$$\alpha_1 = \frac{1}{2}$$
$$\alpha_{-1} = \frac{1}{2}$$

(c)

$$x[n]y[n] \leftrightarrow \alpha_k * \beta_k$$

$$= \sum_{k=0}^{N-1} \alpha_l \beta_{k-l}$$

$$= \sum_{k=0}^{3} \alpha_l \beta_{k-l}$$

$$= \alpha_0 \beta_{k-0} + \alpha_1 \beta_{k-1} + \alpha_2 \beta_{k-2} + \alpha_3 \beta_{k-3}$$

$$c_k = \frac{1}{2}\beta_{k-1} + \frac{1}{2}\beta_{k-3}$$

$$c_1 = 0$$

$$c_2 = \frac{-1}{2j}$$

$$c_3 = 0$$

$$c_4 = \frac{1}{2j}$$
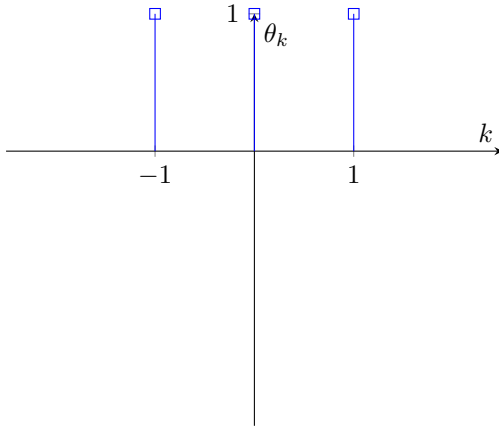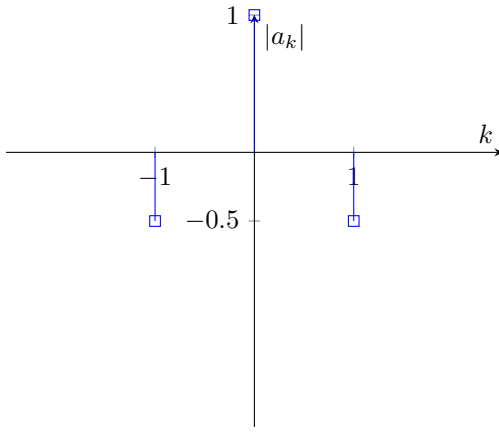
(d)

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]y[n]e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{4}\sum_{n=0}^{3} x[n]y[n]e^{-j\frac{2\pi}{4}kn}$$

$$= \frac{1}{4}\left(x[0]y[0]e^{-j\frac{2\pi}{4}k0} + x[1]y[1]e^{-j\frac{2\pi}{4}k1} + x[2]y[2]e^{-j\frac{2\pi}{4}k2} + x[3]y[3]e^{-j\frac{2\pi}{4}k3}\right)$$

$$= \frac{1}{4}\left(0 \cdot 2 \cdot e^{-j\frac{2\pi}{4}k0} + 1 \cdot 1 \cdot e^{-j\frac{2\pi}{4}k1} + 0 \cdot 0 \cdot e^{-j\frac{2\pi}{4}k2} + (-1) \cdot 1 \cdot e^{-j\frac{2\pi}{4}k3}\right)$$

$$= \frac{1}{4}\left(e^{-j\frac{2\pi}{4}k} - e^{-j\frac{2\pi}{4}k3}\right)$$

$$c_1 = 0$$

$$c_2 = \frac{-1}{2j}$$

$$c_3 = 0$$

$$c_4 = \frac{1}{2j}$$

The results are the same.

6. (a)

$$x[n] = 1 - \cos(\frac{n\pi}{2})$$

$$= 1 - \frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2}$$

$$= 1 - \frac{1}{2}e^{j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}}$$

$$\alpha_0 = 1$$

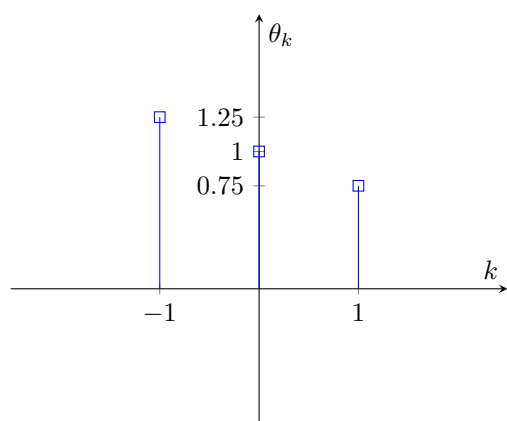$$\alpha_1 = -\frac{1}{2}$$

$$\alpha_{-1} = -\frac{1}{2}$$

1 θ_k

-1

1

k

(b)   i.

ii.

$$y[n] = 1 + \sin(\frac{n\pi}{2}) - \cos(\frac{n\pi}{2})$$

$$= 1 + \frac{e^{j\frac{n\pi}{2}} - e^{-j\frac{n\pi}{2}}}{2j} - \frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2}$$

$$= 1 + \frac{1}{2j}e^{j\frac{n\pi}{2}} - \frac{1}{2j}e^{-j\frac{n\pi}{2}} - \frac{1}{2}e^{j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}}$$

$$= 1 - \frac{1}{2}e^{j\frac{n\pi}{2}} + \frac{1}{2j}e^{j\frac{n\pi}{2}} - \frac{1}{2j}e^{-j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}}$$

$$\alpha_0 = 1$$

$$\alpha_1 = -\frac{1}{2} + \frac{1}{2j}$$

$$\alpha_{-1} = -\frac{1}{2} - \frac{1}{2j}$$

$|a_k|$

1.414

1

-1

1

k

7. (a)
   (b)

8.