

CENG 384 - Signals and Systems for Computer Engineers  
Spring 2023  
Homework 3

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1.

$$\begin{aligned}\int_{-\infty}^t x(s)ds &= \int_{-\infty}^t \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 s} ds \\ &= \sum_{k=-\infty}^{\infty} \left( a_k \cdot \frac{e^{jk\omega_0 t}}{jk\omega_0} \Big|_{-\infty}^t \right) \\ &= \sum_{k=-\infty}^{\infty} \left( a_k \cdot \frac{e^{jk\omega_0 t}}{jk\omega_0} - a_k \cdot \frac{e^{jk\omega_0(-\infty)}}{jk\omega_0} \right) \\ &= \sum_{k=-\infty}^{\infty} \left( a_k \cdot \frac{e^{jk\omega_0 t}}{jk\omega_0} - a_k \cdot \frac{0}{jk\omega_0} \right) \\ &= \sum_{k=-\infty}^{\infty} \left( a_k \cdot \frac{e^{jk\omega_0 t}}{jk\omega_0} \right)\end{aligned}$$

This equation is in the synthesis equation form where  $a_k \frac{1}{jk\omega_0}$  is the Fourier series coefficients of the integrated signal.

Since  $\omega_0$  is the frequency of the signal,  $\omega_0 = \frac{2\pi}{T}$  where  $T$  is the period of the signal.

Substituting  $\omega_0$  in the equation above, we prove the integration property of the Fourier series.

2. (a)  $x(t)x(t) \leftrightarrow a_k * a_k$  (Multiplication Property)

(b)  $\mathcal{E}v\{x(t)\} \leftrightarrow b_k$  (Even Property)

$$b_k = \begin{cases} a_k & k \geq 0 \\ a_{-k} & k < 0 \end{cases}$$

(c)  $x(t+t_0) + x(t-t_0) \leftrightarrow a_k e^{jk\omega_0 t_0} + a_{-k} e^{-jk\omega_0 t_0}$  (Shifting and Linearity Properties)

3.

$$x(t) = \begin{cases} 2 & x \in (0, 1) \\ 0 & x \in (1, 2) \\ -2 & x \in (2, 3) \\ 0 & x \in (3, 4) \\ \text{Periodic} & x \notin (0, 4) \end{cases}$$

$$\begin{aligned}
a_k &= \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \\
&= \frac{1}{4} \left( \int_0^1 2e^{-jk\omega_0 t} dt + \int_1^2 0 dt + \int_2^3 -2e^{-jk\omega_0 t} dt + \int_3^4 0 dt \right) \\
&= \frac{1}{4} \left( 2 \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \Big|_0^1 - 2 \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \Big|_2^3 \right) \\
&= \frac{1}{4} \left( 2 \frac{e^{-jk\omega_0}}{-jk\omega_0} - \frac{2}{-jk\omega_0} - 2 \frac{e^{-3jk\omega_0}}{-jk\omega_0} + 2 \frac{e^{-2jk\omega_0}}{-jk\omega_0} \right) \\
&= \frac{1}{-2jk\omega_0} (e^{-jk\omega_0} - 1 - e^{-3jk\omega_0} + e^{-2jk\omega_0})
\end{aligned}$$

Substitute  $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\begin{aligned}
a_k &= \frac{1}{-2jk\frac{\pi}{2}} (e^{-jk\frac{\pi}{2}} - 1 - e^{-3jk\frac{\pi}{2}} + e^{-2jk\frac{\pi}{2}}) \\
&= \frac{1}{-jk\pi} (e^{-jk\frac{\pi}{2}} - 1 - e^{-3jk\frac{\pi}{2}} + e^{-jk\pi}) \\
&= \frac{1}{-jk\pi} (\cos(-k\frac{\pi}{2}) + j\sin(-k\frac{\pi}{2}) - 1 - \cos(-3k\frac{\pi}{2}) - j\sin(-3k\frac{\pi}{2}) + \cos(-k\pi) + j\sin(-k\pi)) \\
&= \frac{1}{-jk\pi} (-2j\sin(k\frac{\pi}{2}) - 1 + \cos(-k\pi))
\end{aligned}$$

4. (a)

$$\begin{aligned}
x(t) &= 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4}) \\
&= 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos(2\omega_0 t) \cos(\frac{\pi}{4}) - \sin(2\omega_0 t) \sin(\frac{\pi}{4}) \\
&= 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \frac{\sqrt{2}}{2} \cos(2\omega_0 t) - \frac{\sqrt{2}}{2} \sin(2\omega_0 t) \\
&= 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + 2 \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + \frac{\sqrt{2}}{2} \frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} - \frac{\sqrt{2}}{2} \frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} \\
&= 1 + \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{\sqrt{2}}{4} e^{j2\omega_0 t} + \frac{\sqrt{2}}{4} e^{-j2\omega_0 t} - \frac{\sqrt{2}}{4j} e^{j2\omega_0 t} + \frac{\sqrt{2}}{4j} e^{-j2\omega_0 t}
\end{aligned}$$

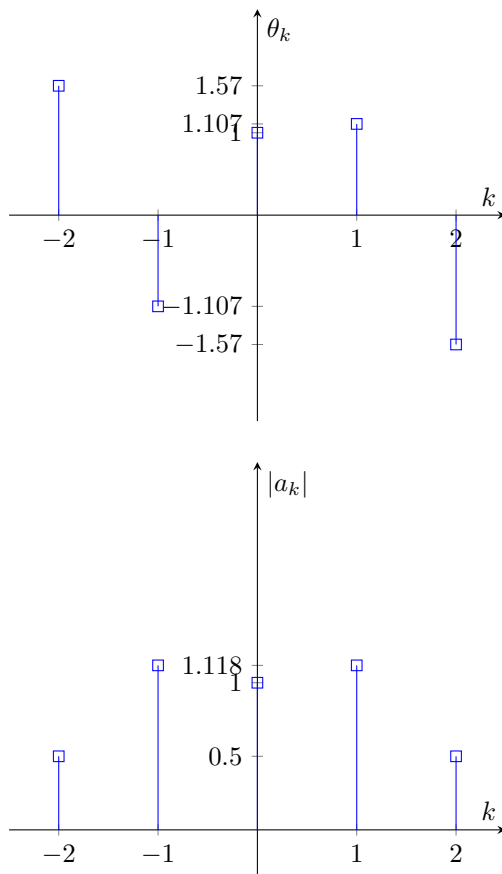
$$\alpha_0 = 1$$

$$\alpha_1 = 1 + \frac{1}{2j}$$

$$\alpha_{-1} = 1 - \frac{1}{2j}$$

$$\alpha_2 = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4j}$$

$$\alpha_{-2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4j}$$



(b)

$$\begin{aligned}
 x(t) &= e^{j\omega_0 t} \\
 y(t) &= H(jk\omega_0)e^{jk\omega_0 t} \\
 y'(t) + y(t) &= x(t) \\
 jk\omega_0 H(jk\omega_0)e^{jk\omega_0 t} + H(jk\omega_0)e^{jk\omega_0 t} &= e^{jk\omega_0 t} \\
 H(jk\omega_0) &= \frac{1}{1 + jk\omega_0}
 \end{aligned}$$

(c)

$$\begin{aligned}
 y(t) &= H(jk\omega_0)e^{jk\omega_0 t} = \frac{1}{1 + jk\omega_0}e^{jk\omega_0 t} \\
 b_k &= a_k \frac{1}{1 + jk\omega_0} \\
 b_1 &= \frac{1 + \frac{1}{2j}}{1 + j\omega_0} \\
 b_{-1} &= \frac{1 - \frac{1}{2j}}{1 - j\omega_0} \\
 b_2 &= \frac{\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4j}}{1 + 2j\omega_0} \\
 b_{-2} &= \frac{\frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4j}}{1 - 2j\omega_0}
 \end{aligned}$$

(d)

$$\begin{aligned}
 y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} \\
 &= b_1 e^{j\omega_0 t} + b_{-1} e^{-j\omega_0 t} + b_2 e^{j2\omega_0 t} + b_{-2} e^{-j2\omega_0 t}
 \end{aligned}$$

5. (a)

$$\begin{aligned}
 x[n] &= \sin\left(\frac{\pi}{2}n\right) \\
 &= \frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j} \\
 &= \frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\frac{\pi}{2}n} \\
 \alpha_1 &= \frac{1}{2j} \\
 \alpha_{-1} &= -\frac{1}{2j}
 \end{aligned}$$

(b)

$$\begin{aligned}
 y[n] &= 1 + \cos\left(\frac{\pi}{2}n\right) \\
 &= 1 + \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2} \\
 &= 1 + \frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n} \\
 \alpha_0 &= 1 \\
 \alpha_1 &= \frac{1}{2} \\
 \alpha_{-1} &= \frac{1}{2}
 \end{aligned}$$

(c)

$$\begin{aligned}
 x[n]y[n] &\leftrightarrow \alpha_k * \beta_k \\
 &= \sum_{k=0}^{N-1} \alpha_l \beta_{k-l} \\
 &= \sum_{k=0}^3 \alpha_l \beta_{k-l} \\
 &= \alpha_0 \beta_{k-0} + \alpha_1 \beta_{k-1} + \alpha_2 \beta_{k-2} + \alpha_3 \beta_{k-3} \\
 c_k &= \frac{1}{2} \beta_{k-1} + \frac{1}{2} \beta_{k-3} \\
 c_1 &= 0 \\
 c_2 &= \frac{-1}{2j} \\
 c_3 &= 0 \\
 c_4 &= \frac{1}{2j}
 \end{aligned}$$

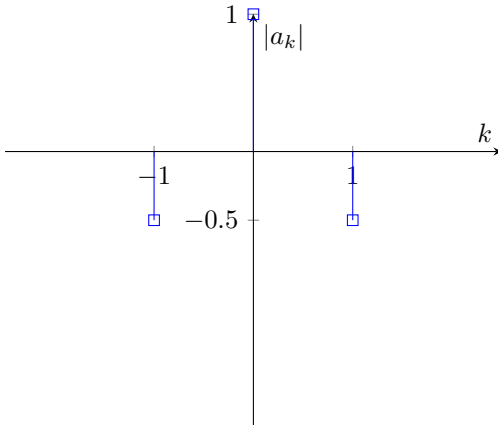
(d)

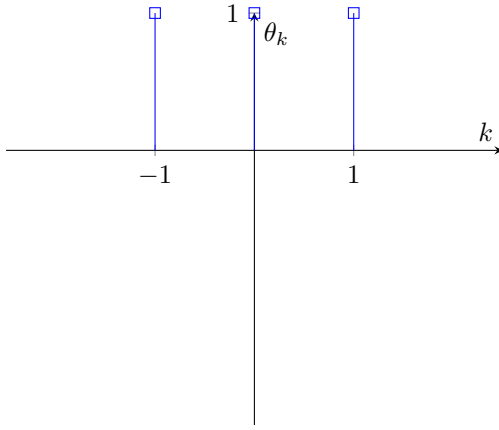
$$\begin{aligned}
c_k &= \frac{1}{N} \sum_{n=0}^{N-1} x[n]y[n]e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{4} \sum_{n=0}^3 x[n]y[n]e^{-j\frac{2\pi}{4}kn} \\
&= \frac{1}{4} (x[0]y[0]e^{-j\frac{2\pi}{4}k0} + x[1]y[1]e^{-j\frac{2\pi}{4}k1} + x[2]y[2]e^{-j\frac{2\pi}{4}k2} + x[3]y[3]e^{-j\frac{2\pi}{4}k3}) \\
&= \frac{1}{4} (0 \cdot 2 \cdot e^{-j\frac{2\pi}{4}k0} + 1 \cdot 1 \cdot e^{-j\frac{2\pi}{4}k1} + 0 \cdot 0 \cdot e^{-j\frac{2\pi}{4}k2} + (-1) \cdot 1 \cdot e^{-j\frac{2\pi}{4}k3}) \\
&= \frac{1}{4} (e^{-j\frac{2\pi}{4}k} - e^{-j\frac{2\pi}{4}k3}) \\
c_1 &= 0 \\
c_2 &= \frac{-1}{2j} \\
c_3 &= 0 \\
c_4 &= \frac{1}{2j}
\end{aligned}$$

The results are the same.

6. (a)

$$\begin{aligned}
x[n] &= 1 - \cos\left(\frac{n\pi}{2}\right) \\
&= 1 - \frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2} \\
&= 1 - \frac{1}{2}e^{j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
\alpha_0 &= 1 \\
\alpha_1 &= -\frac{1}{2} \\
\alpha_{-1} &= -\frac{1}{2}
\end{aligned}$$

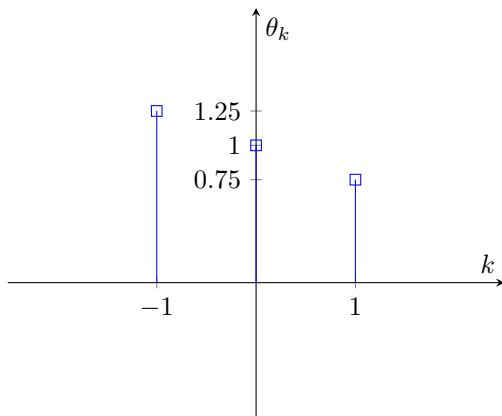
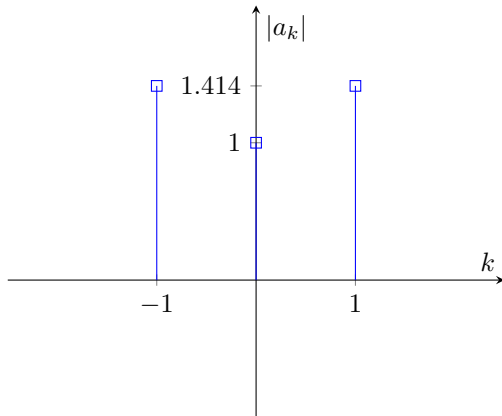




(b) i.

ii.

$$\begin{aligned}
 y[n] &= 1 + \sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right) \\
 &= 1 + \frac{e^{j\frac{n\pi}{2}} - e^{-j\frac{n\pi}{2}}}{2j} - \frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2} \\
 &= 1 + \frac{1}{2j}e^{j\frac{n\pi}{2}} - \frac{1}{2j}e^{-j\frac{n\pi}{2}} - \frac{1}{2}e^{j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
 &= 1 - \frac{1}{2}e^{j\frac{n\pi}{2}} + \frac{1}{2j}e^{j\frac{n\pi}{2}} - \frac{1}{2j}e^{-j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
 \alpha_0 &= 1 \\
 \alpha_1 &= -\frac{1}{2} + \frac{1}{2j} \\
 \alpha_{-1} &= -\frac{1}{2} - \frac{1}{2j}
 \end{aligned}$$



7. (a)

(b)

8. (a) `from numpy import exp, pi`

```
def spectral_coefficients(signal, period, num_coefficients):
    coefficients = []
    for k in range(num_coefficients + 1):
        S = 0
        for n in range(period):
            S += signal[n] * exp(-1j * 2 * pi * n * k / period)
        coefficients.append(S / period)
    return coefficients
```

(b) `from matplotlib import pyplot`  
`from numpy import exp, pi, linspace`

`SAVE_FOLDER = "figures"`

`t = linspace(-0.5, 0.5, 1000)`

```
class SignalFromSpectralCoefficients:
    def __init__(self, coefficients, period):
        self.coefficients = coefficients
        self.period = period

    def __getitem__(self, n):
        S = 0
        for k, coefficient in enumerate(self.coefficients):
            S += coefficient * exp(1j * 2 * pi * n * k / self.period)
        return S

    def __iter__(self):
        for n in range(self.period):
            yield self[n]

    def __len__(self):
        return self.period

    def plot(self, name):
        pyplot.plot(t, self, label="Reconstructed Signal")
        pyplot.legend()
        pyplot.savefig(SAVE_FOLDER + "/" + name + ".svg", format = "svg")
        pyplot.clf()
```

(c) `from matplotlib import pyplot`  
`from scipy.signal import sawtooth`  
`from q8a import spectral_coefficients`  
`from q8b import SignalFromSpectralCoefficients, t`

`square_wave = [-10] * 500 + [10] * 500`

```
for n in (1, 5, 10, 50, 100):
    pyplot.plot(t, square_wave, label="Square Wave")
    coefficients = spectral_coefficients(square_wave, len(square_wave), n)
    reconstructed = SignalFromSpectralCoefficients(coefficients, 1000)
    reconstructed.plot(f"square_wave_{n}")
```

Figure 1: Approximated Square Wave with 1 Spectral Coefficient

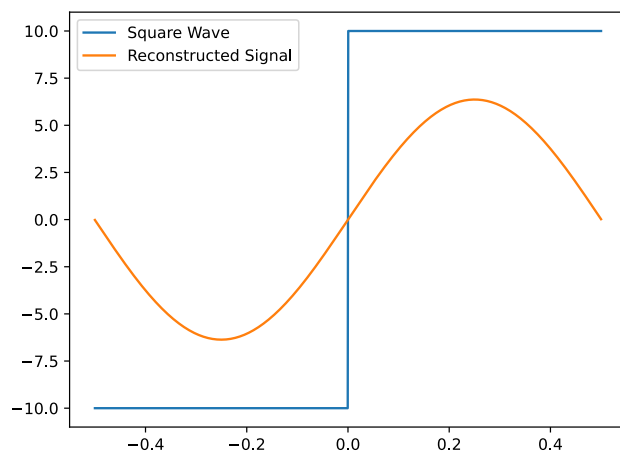


Figure 2: Approximated Square Wave with 5 Spectral Coefficients

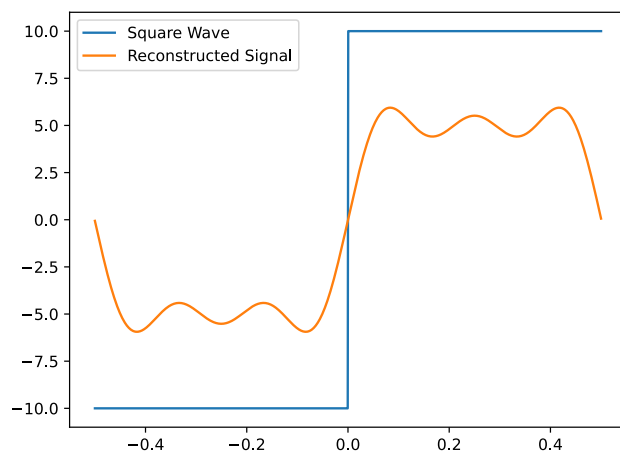


Figure 3: Approximated Square Wave with 10 Spectral Coefficients

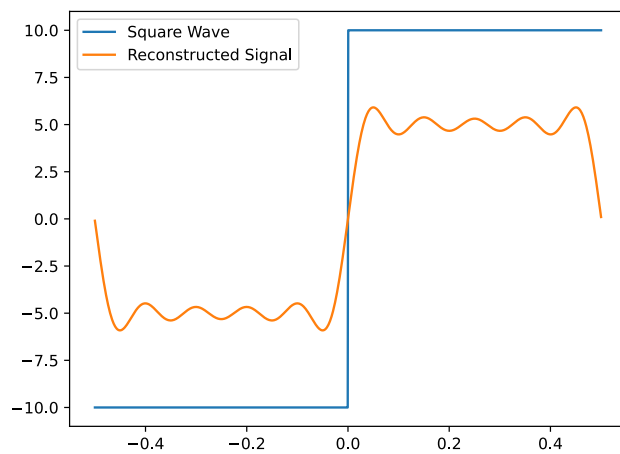




Figure 4: Approximated Square Wave with 50 Spectral Coefficients

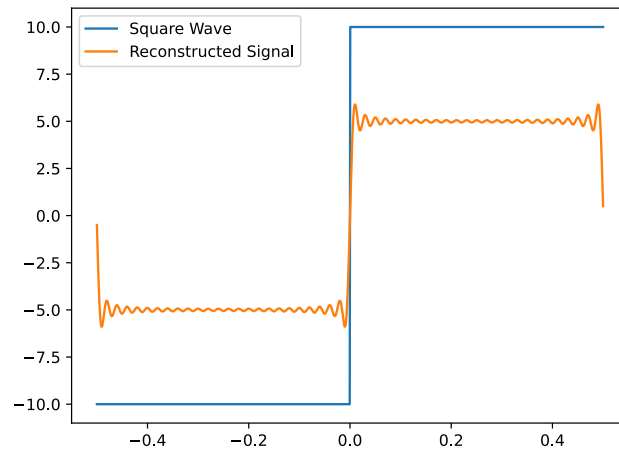
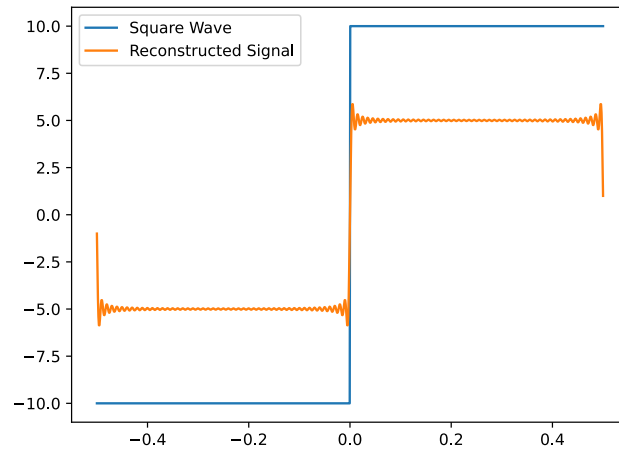


Figure 5: Approximated Square Wave with 100 Spectral Coefficients



```
(d) import numpy
from matplotlib import pyplot
from scipy.signal import sawtooth
from q8a import spectral_coefficients
from q8b import SignalFromSpectralCoefficients, t

sawtooth_wave = sawtooth(2 * numpy.pi * t)
for n in (1, 5, 10, 50, 100):
    pyplot.plot(t, sawtooth_wave, label="Sawtooth Wave")
    coefficients = spectral_coefficients(sawtooth_wave, len(sawtooth_wave), n)
    reconstructed = SignalFromSpectralCoefficients(coefficients, 1000)
    reconstructed.plot(f"sawtooth_wave_{n}")
```

Figure 6: Approximated Sawtooth Wave with 1 Spectral Coefficient

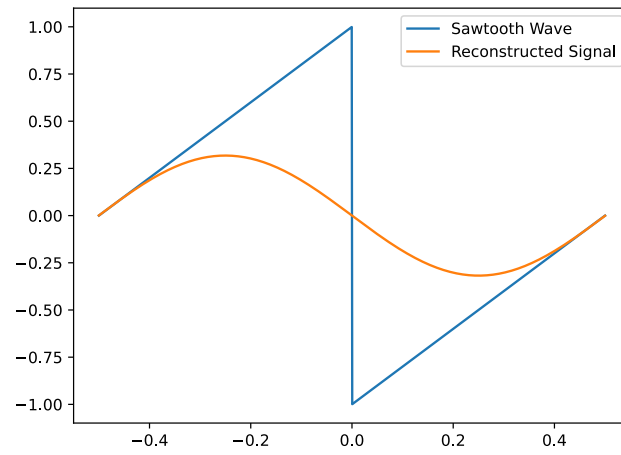


Figure 7: Approximated Sawtooth Wave with 5 Spectral Coefficients

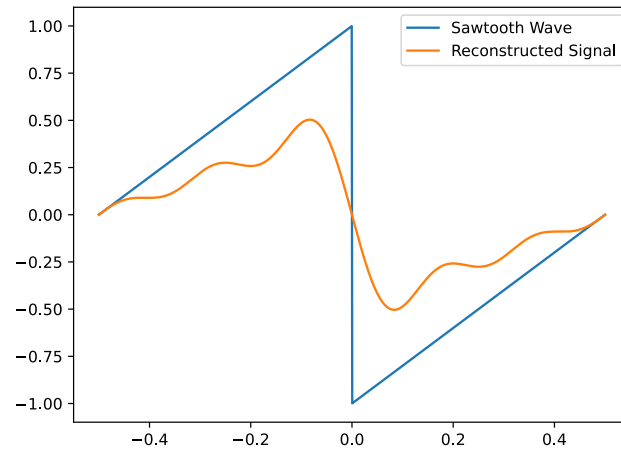


Figure 8: Approximated Sawtooth Wave with 10 Spectral Coefficients

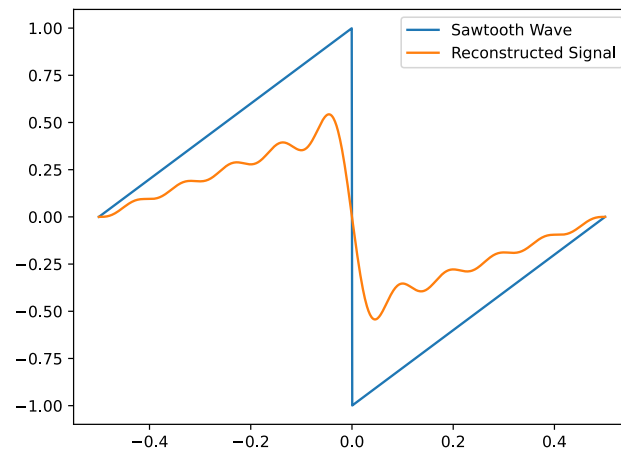


Figure 9: Approximated Sawtooth Wave with 50 Spectral Coefficients

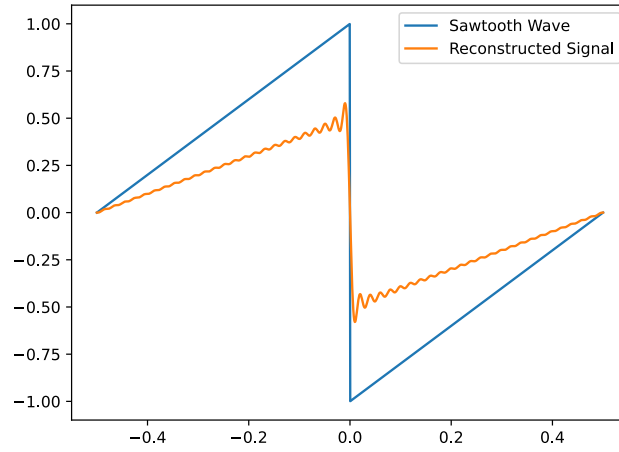
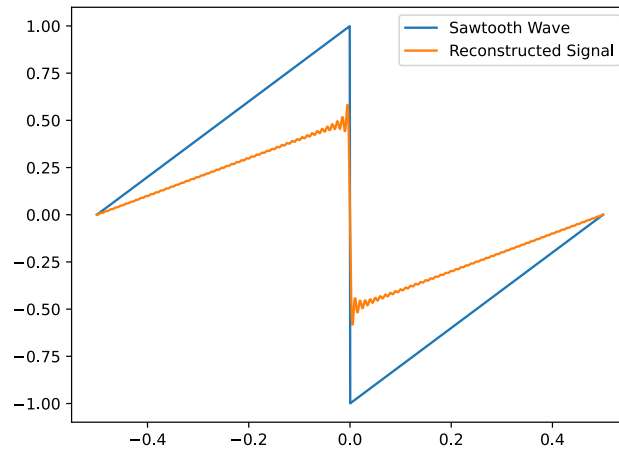


Figure 10: Approximated Sawtooth Wave with 100 Spectral Coefficients



Although, increasing the number of spectral coefficients increases the accuracy of the approximation, there is a scaling difference between the original and the approximated wave. This is because the number of coefficients used is far less than the number of points in the original wave. For an accurate approximation, the number of coefficients should be equal to the number of points in a period of the original wave.