CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 3

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1.

$$
\begin{aligned}
\int_{-\infty}^{t} x(s)ds &= \int_{-\infty}^{t} \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi s} ds \\
&= \sum_{k=-\infty}^{\infty} (a_k \cdot \frac{e^{jkw_0 t}}{jkw_0}\Big|_{-\infty}^{t}) \\
&= \sum_{k=-\infty}^{\infty} (a_k \cdot \frac{e^{jkw_0 t}}{jkw_0} - a_k \cdot \frac{e^{jkw_0(-\infty)}}{jkw_0}) \\
&= \sum_{k=-\infty}^{\infty} (a_k \cdot \frac{e^{jkw_0 t}}{jkw_0} - a_k \cdot \frac{0}{jkw_0}) \\
&= \sum_{k=-\infty}^{\infty} (a_k \cdot \frac{e^{jkw_0 t}}{jkw_0})
\end{aligned}
$$

This equation is in the synthesis equation form where $a_k \frac{1}{jkw_0}$ is the Fourier series coefficients of the integrated signal.

Since w_0 is the frequency of the signal, $w_0 = \frac{2\pi}{T}$ where T is the period of the signal.

Substituting w_0 in the equation above, we prove the integration property of the Fourier series.

2. (a) $x(t)x(t) \leftrightarrow a_k * a_k$ (Multiplication Property)

   (b) $\mathcal{E}v\{x(t)\} \leftrightarrow b_k$ (Even Property)

$$
b_k = \begin{cases} a_k & k \geq 0 \\ a_{-k} & k < 0 \end{cases}
$$

   (c) $x(t+t_0) + x(t-t_0) \leftrightarrow a_k e^{jkw_0 t_0} + a_{-k} e^{-jkw_0 t_0}$ (Shifting and Linearity Properties)

3.

$$
x(t) = \begin{cases} 2 & x \in (0,1) \\ 0 & x \in (1,2) \\ -2 & x \in (2,3) \\ 0 & x \in (3,4) \\ \text{Periodic} & x \notin (0,4) \end{cases}
$$

$$\begin{aligned}
a_k &= \frac{1}{T}\int_0^T x(t)e^{-jkw_0t}dt \\
&= \frac{1}{4}(\int_0^1 2e^{-jkw_0t}dt + \int_1^2 0dt + \int_2^3 -2e^{-jkw_0t}dt + \int_3^4 0dt) \\
&= \frac{1}{4}(2\frac{e^{-jkw_0t}}{-jkw_0}\Big|_0^1 - 2\frac{e^{-jkw_0t}}{-jkw_0}\Big|_2^3) \\
&= \frac{1}{4}(2\frac{e^{-jkw_0}}{-jkw_0} - \frac{2}{-jkw_0} - 2\frac{e^{-3jkw_0}}{-jkw_0} + 2\frac{e^{-2jkw_0}}{-jkw_0}) \\
&= \frac{1}{-2jkw_0}(e^{-jkw_0} - 1 - e^{-3jkw_0} + e^{-2jkw_0})
\end{aligned}$$

Substitute $w_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\begin{aligned}
a_k &= \frac{1}{-2jk\frac{\pi}{2}}(e^{-jk\frac{\pi}{2}} - 1 - e^{-3jk\frac{\pi}{2}} + e^{-2jk\frac{\pi}{2}}) \\
&= \frac{1}{-jk\pi}(e^{-jk\frac{\pi}{2}} - 1 - e^{-3jk\frac{\pi}{2}} + e^{-jk\pi}) \\
&= \frac{1}{-jk\pi}(cos(-k\frac{\pi}{2}) + jsin(-k\frac{\pi}{2}) - 1 - cos(-3k\frac{\pi}{2}) - jsin(-3k\frac{\pi}{2}) + cos(-k\pi) + jsin(-k\pi)) \\
&= \frac{1}{-jk\pi}(-2jsin(k\frac{\pi}{2}) - 1 + cos(-k\pi))
\end{aligned}$$

4. (a)

$$\begin{aligned}
x(t) &= 1 + \sin(2\pi t) + 2\cos(2\pi t) + \cos(4\pi t + \frac{\pi}{4}) \\
&= 1 + \sin(2\pi t) + 2\cos(2\pi t) + \cos(4\pi t)\cos(\frac{\pi}{4}) - \sin(4\pi t)\sin(\frac{\pi}{4}) \\
&= 1 + \sin(2\pi t) + 2\cos(2\pi t) + \frac{\sqrt{2}}{2}\cos(4\pi t) - \frac{\sqrt{2}}{2}\sin(4\pi t) \\
&= 1 + \frac{e^{j2\pi t} - e^{-j2\pi t}}{2j} + 2\frac{e^{j2\pi t} + e^{-j2\pi t}}{2} + \frac{\sqrt{2}}{2}\frac{e^{j4\pi t} + e^{-j4\pi t}}{2} - \frac{\sqrt{2}}{2}\frac{e^{j4\pi t} - e^{-j4\pi t}}{2j} \\
&= 1 + \frac{1}{2j}e^{j2\pi t} - \frac{1}{2j}e^{-j2\pi t} + e^{j2\pi t} + e^{-j2\pi t} + \frac{\sqrt{2}}{4}e^{j4\pi t} + \frac{\sqrt{2}}{4}e^{-j4\pi t} - \frac{\sqrt{2}}{4j}e^{j4\pi t} + \frac{\sqrt{2}}{4j}e^{-j4\pi t}
\end{aligned}$$

$$\begin{aligned}
\alpha_0 &= 1 \\
\alpha_1 &= 1 + \frac{1}{2j} \\
\alpha_{-1} &= 1 - \frac{1}{2j} \\
\alpha_2 &= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4j} \\
\alpha_{-2} &= \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4j}
\end{aligned}$$

(b)

$$\begin{aligned}
x(t) &= e^{jk2\pi t} \\
y(t) &= H(jk2\pi)e^{jk2\pi t} \\
y'(t) + y(t) &= x(t) \\
jk2\pi H(jk2\pi)e^{jk2\pi t} + H(jk2\pi)e^{jk2\pi t} &= e^{jk2\pi t} \\
H(jk2\pi) &= \frac{1}{1 + jk2\pi}
\end{aligned}$$

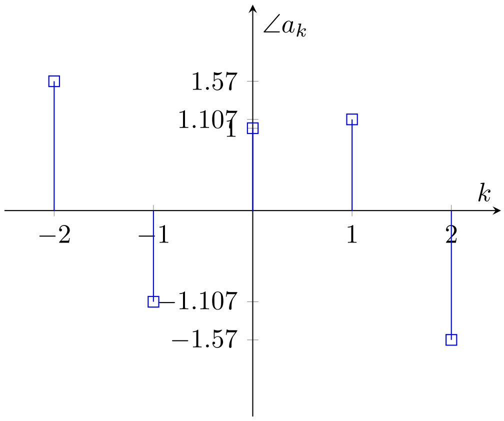

Figure 1: Graph of $\angle a_k$

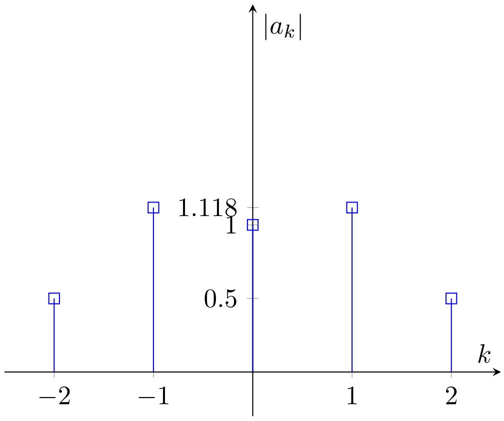

Figure 2: Graph of $|a_k|$

Our transfer function is $H(jk2\pi)$ and our kth eigenvalue is $\frac{1}{1+jk2\pi}$.

(c)

$$\begin{aligned}
y(t) = H(jk2\pi)e^{jk2\pi t} &= \frac{1}{1+jk2\pi}e^{jk2\pi t} \\
N &= 4 \\
b_k &= a_k \frac{1}{1+jk2\pi} \\
b_0 &= 1 \\
b_1 &= \frac{1+\frac{1}{2j}}{1+j2\pi} \\
b_{-1} &= \frac{1-\frac{1}{2j}}{1-j2\pi} \\
b_2 &= \frac{\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{4j}}{1+4j\pi} \\
b_{-2} &= \frac{\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4j}}{1-4j\pi}
\end{aligned}$$

(d)

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk2\pi t} \\
&= b_1 e^{j2\pi t} + b_{-1}e^{-j2\pi t} + b_2 e^{j4\pi t} + b_{-2}e^{-j4\pi t} \\
&= \frac{1+\frac{1}{2j}}{1+j2\pi}e^{j2\pi t} + \frac{1-\frac{1}{2j}}{1-j2\pi}e^{-j2\pi t} + \frac{\frac{\sqrt{2}}{4}-\frac{\sqrt{2}}{4j}}{1+4j\pi}e^{j4\pi t} + \frac{\frac{\sqrt{2}}{4}+\frac{\sqrt{2}}{4j}}{1-4j\pi}e^{-j4\pi t}
\end{aligned}$$

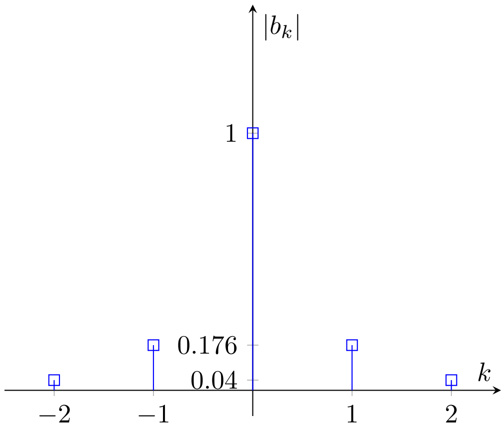

Figure 3: Graph of $|b_k|$

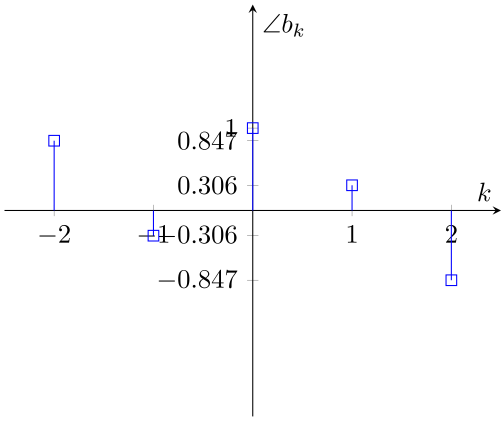

Figure 4: Graph of $\angle b_k$

5. (a)

$$\begin{aligned}
x[n] &= sin(\frac{\pi}{2}n) \\
&= \frac{e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}}{2j} \\
&= \frac{1}{2j}e^{j\frac{\pi}{2}n} - \frac{1}{2j}e^{-j\frac{\pi}{2}n} \\
\alpha_1 &= \frac{1}{2j} \\
\alpha_{-1} &= -\frac{1}{2j}
\end{aligned}$$

(b)

$$\begin{aligned}
y[n] &= 1 + cos(\frac{\pi}{2}n) \\
&= 1 + \frac{e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}}{2} \\
&= 1 + \frac{1}{2}e^{j\frac{\pi}{2}n} + \frac{1}{2}e^{-j\frac{\pi}{2}n} \\
\alpha_0 &= 1 \\
\alpha_1 &= \frac{1}{2} \\
\alpha_{-1} &= \frac{1}{2}
\end{aligned}$$

(c)

$$\begin{aligned}
x[n]y[n] &\leftrightarrow \alpha_k * \beta_k \\
&= \sum_{k=0}^{N-1} \alpha_l \beta_{k-l} \\
&= \sum_{k=0}^{3} \alpha_l \beta_{k-l} \\
&= \alpha_0\beta_{k-0} + \alpha_1\beta_{k-1} + \alpha_2\beta_{k-2} + \alpha_3\beta_{k-3} \\
c_k &= \frac{1}{2}\beta_{k-1} + \frac{1}{2}\beta_{k-3}
\end{aligned}$$

$$\begin{aligned}
c_1 &= 0 \\
c_2 &= \frac{-1}{2j} \\
c_3 &= 0 \\
c_4 &= \frac{1}{2j}
\end{aligned}$$

(d)

$$\begin{aligned}
c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]y[n]e^{-j\frac{2\pi}{N}kn} \\
&= \frac{1}{4}\sum_{n=0}^{3} x[n]y[n]e^{-j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}(x[0]y[0]e^{-j\frac{2\pi}{4}k0} + x[1]y[1]e^{-j\frac{2\pi}{4}k1} + x[2]y[2]e^{-j\frac{2\pi}{4}k2} + x[3]y[3]e^{-j\frac{2\pi}{4}k3}) \\
&= \frac{1}{4}(0\cdot 2\cdot e^{-j\frac{2\pi}{4}k0} + 1\cdot 1\cdot e^{-j\frac{2\pi}{4}k1} + 0\cdot 0\cdot e^{-j\frac{2\pi}{4}k2} + (-1)\cdot 1\cdot e^{-j\frac{2\pi}{4}k3}) \\
&= \frac{1}{4}(e^{-j\frac{2\pi}{4}k} - e^{-j\frac{2\pi}{4}k3}) \\
c_1 &= 0 \\
c_2 &= \frac{-1}{2j} \\
c_3 &= 0 \\
c_4 &= \frac{1}{2j}
\end{aligned}$$

The results are the same.

6. (a)

$$\begin{aligned}
x[n] &= 1 - \cos(\frac{n\pi}{2}) \\
&= 1 - \frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2} \\
&= 1 - \frac{1}{2}e^{j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
\alpha_0 &= 1 \\
\alpha_1 &= -\frac{1}{2} \\
\alpha_{-1} &= -\frac{1}{2}
\end{aligned}$$

(b) i.

$$y[n] = x[n] - \Sigma_{k=-\infty}^{\infty}\delta[n-(3+4k)]$$

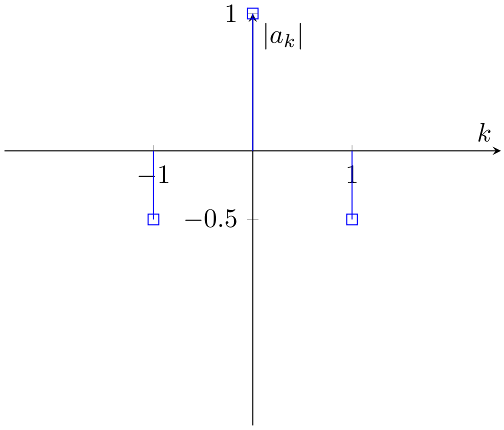

Figure 5: Graph of $|a_k|$

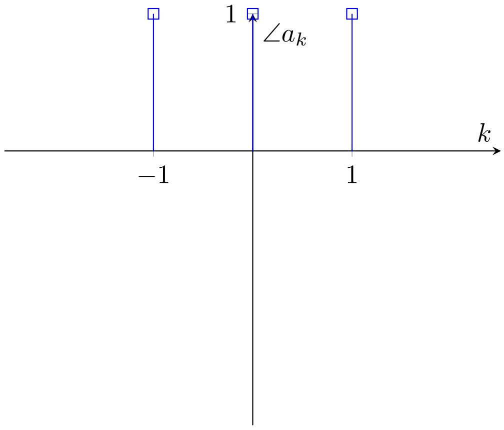

Figure 6: Graph of $\angle a_k$

ii.

$$
\begin{aligned}
y[n] &= 1 + \sin(\frac{n\pi}{2}) - \cos(\frac{n\pi}{2}) \\
&= 1 + \frac{e^{j\frac{n\pi}{2}} - e^{-j\frac{n\pi}{2}}}{2j} - \frac{e^{j\frac{n\pi}{2}} + e^{-j\frac{n\pi}{2}}}{2} \\
&= 1 + \frac{1}{2j}e^{j\frac{n\pi}{2}} - \frac{1}{2j}e^{-j\frac{n\pi}{2}} - \frac{1}{2}e^{j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
&= 1 - \frac{1}{2}e^{j\frac{n\pi}{2}} + \frac{1}{2j}e^{j\frac{n\pi}{2}} - \frac{1}{2j}e^{-j\frac{n\pi}{2}} - \frac{1}{2}e^{-j\frac{n\pi}{2}} \\
\alpha_0 &= 1 \\
\alpha_1 &= -\frac{1}{2} + \frac{1}{2j} \\
\alpha_{-1} &= -\frac{1}{2} - \frac{1}{2j}
\end{aligned}
$$

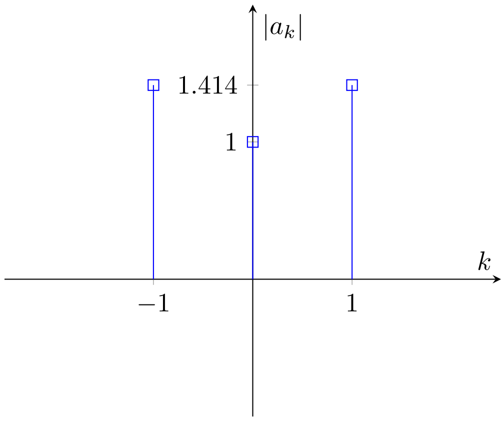

Figure 7: Graph of $|a_k|$

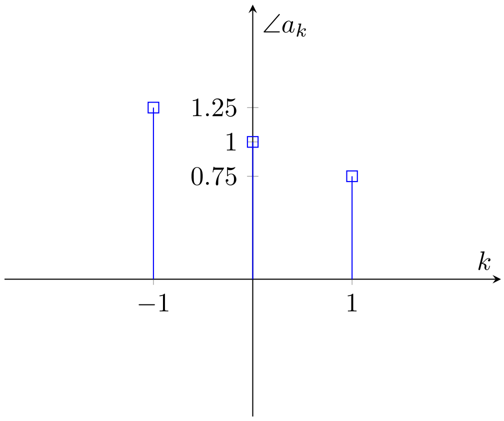

Figure 8: Graph of $\angle a_k$

7. (a)

$$y(t) = H(jk\omega_0)e^{jk\omega_0 t} \qquad \text{Lecture Notes 6.8}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \Rightarrow y(t) = \sum_{k=-\infty}^{+\infty} H(jk\omega_0)a_k e^{jk\omega_0 t} \qquad \text{Linearity Property}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi/K} = 2K$$

$$y(t) = \sum_{k=-\infty}^{+\infty} H(j2Kk)a_k e^{jk2Kt} \qquad \text{Substitute } \omega_0$$

$$y(t) = \sum_{k=-40/K}^{+40/K} a_k e^{jk2Kt} \qquad |\omega| \leq 80 \Rightarrow H(j\omega) = 1$$

$$\sum_{k=-40/K}^{+40/K} a_k e^{jk2Kt} = \sum_{k=-\infty}^{+\infty} a_k e^{jk2Kt} \qquad \text{x(t) = y(t)}$$

In conclusion, $a_k = 0$ for $k \notin [-40/K, 40/K]$.

(b) Nothing can be concluded if $x(t) \neq y(t)$.

8. (a) Python function for finding the spectral coefficients:

```
from numpy import exp, pi

def spectral_coefficients(signal, period, num_coefficients):
    coefficients = []
    for k in range(num_coefficients + 1):
        S = 0
        for n in range(period):
            S += signal[n] * exp(-1j * 2 * pi * n * k / period)
        coefficients.append(S / period)
    return coefficients
```

(b) Python class for reconstructing the approximated signal:

```python
from matplotlib import pyplot
from numpy import exp, pi, linspace

SAVE_FOLDER = "figures"

t = linspace(-0.5, 0.5, 1000)

class SignalFromSpectralCoefficients:
    def __init__(self, coefficients, period):
        self.coefficients = coefficients
        self.period = period

    def __getitem__(self, n):
        S = 0
        for k, coefficient in enumerate(self.coefficients):
            S += coefficient * exp(1j * 2 * pi * n * k / self.period)
        return S

    def __iter__(self):
        for n in range(self.period):
            yield self[n]

    def __len__(self):
        return self.period

    def plot(self, name):
        pyplot.plot(t, self, label="Reconstructed Signal")
        pyplot.legend()
        pyplot.savefig(SAVE_FOLDER + "/" + name + ".svg", format = "svg")
        pyplot.clf()
```

(c) Python code for approximating the square wave signal:

```python
from matplotlib import pyplot
from scipy.signal import sawtooth
from q8a import spectral_coefficients
from q8b import SignalFromSpectralCoefficients, t

square_wave = [-10] * 500 + [10] * 500
for n in (1, 5, 10, 50, 100):
    pyplot.plot(t, square_wave, label="Square Wave")
    coefficients = spectral_coefficients(square_wave, len(square_wave), n)
    reconstructed = SignalFromSpectralCoefficients(coefficients, 1000)
    reconstructed.plot(f"square_wave_{n}")
```

Figure 9: Approximated Square Wave with 1 Spectral Coefficient

Figure 10: Approximated Square Wave with 5 Spectral Coefficients

Figure 11: Approximated Square Wave with 10 Spectral Coefficients

Figure 12: Approximated Square Wave with 50 Spectral Coefficients

Figure 13: Approximated Square Wave with 100 Spectral Coefficients

(d) Python code for approximating the sawtooth wave signal:

```python
import numpy
from matplotlib import pyplot
from scipy.signal import sawtooth
```

```python
from q8a import spectral_coefficients
from q8b import SignalFromSpectralCoefficients, t

sawtooth_wave = sawtooth(2 * numpy.pi * t)
for n in (1, 5, 10, 50, 100):
    pyplot.plot(t, sawtooth_wave, label="Sawtooth Wave")
    coefficients = spectral_coefficients(sawtooth_wave, len(sawtooth_wave), n)
    reconstructed = SignalFromSpectralCoefficients(coefficients, 1000)
    reconstructed.plot(f"sawtooth_wave_{n}")
```

Figure 14: Approximated Sawtooth Wave with 1 Spectral Coefficient

Figure 15: Approximated Sawtooth Wave with 5 Spectral Coefficients

Figure 16: Approximated Sawtooth Wave with 10 Spectral Coefficients

Figure 17: Approximated Sawtooth Wave with 50 Spectral Coefficients

Figure 18: Approximated Sawtooth Wave with 100 Spectral Coefficients

Although, increasing the number of spectral coefficients increases the accuracy of the approximation, there is a scaling difference between the original and the approximated wave. This is because the number of coefficients used is far less than the number of points in the original wave. For an accurate approximation, the number of coefficients should be equal to the number of points in a period of the original wave.