CENG 384 - Signals and Systems for Computer Engineers
Spring 2023
Homework 2

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1. (a)

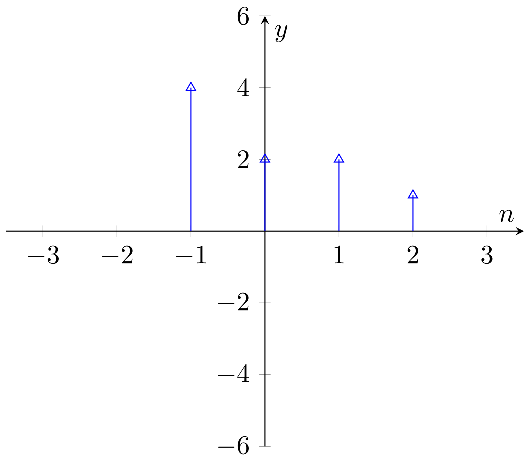
$$y(t) = x(t) - 5\dot{y}(t)$$

(b)

$$\begin{aligned}
y(t) &= (e^{-t} + e^{-3t})u(t) - 5\dot{y}(t) \\
y(t) + 5\dot{y}(t) &= (e^{-t} + e^{-3t})u(t) \\
y(t) &= y_p(t) + y_h(t) \\
y_p(t) &= Ke^{-t}u(t) + Le^{-3t}u(t) \\
Ke^{-t}u(t) + Le^{-3t}u(t) + 5(-Ke^{-t}u(t) - 3Le^{-3t}u(t)) &= (e^{-t} + e^{-3t})u(t) \\
Ke^{-t}u(t) + Le^{-3t}u(t) - 5Ke^{-t}u(t) - 15Le^{-3t}u(t) &= (e^{-t} + e^{-3t})u(t) \\
e^{-t}u(t)(K - 5K) + e^{-3t}u(t)(L - 15L) &= (e^{-t} + e^{-3t})u(t) \\
K - 5K &= 1 \\
K &= -1/4 \\
L - 15L &= 1 \\
L &= -1/14 \\
y_p(t) &= \frac{-1}{4}e^{-t}u(t) + \frac{-1}{14}e^{-3t}u(t) \\
y_h(t) &= c_1e^{\alpha t} \\
c_1e^{\alpha t} + 5\alpha c_1e^{\alpha t} &= 0 \\
c_1 + 5\alpha c_1 &= 0 \\
\alpha &= \frac{-1}{5} \\
y_h(t) &= c_1e^{\frac{-1}{5}t} \\
y(t) &= y_p(t) + y_h(t) \\
&= \frac{-1}{4}e^{-t}u(t) + \frac{-1}{14}e^{-3t}u(t) + c_1e^{\frac{-1}{5}t} \\
y(0) &= 0 \\
0 &= \frac{-1}{4} + \frac{-1}{14} + c_1 \\
c_1 &= \frac{9}{28} \\
y(t) &= \frac{-1}{4}e^{-t}u(t) + \frac{-1}{14}e^{-3t}u(t) + \frac{9}{28}e^{\frac{-1}{5}t}
\end{aligned}$$

2. (a)

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=0}^{n} x[k]h[n-k] \\
&= \sum_{k=0}^{n} (2\delta[k] + \delta[k+1]) (\delta[n-(1+k)] + 2\delta[n+1-k]) \\
&= 2\sum_{k=0}^{n} \delta[k]\delta[n-(1+k)] + 4\sum_{k=0}^{n} \delta[k]\delta[n+1-k] + \sum_{k=0}^{n} \delta[k+1]\delta[n-(1+k)] + 2\sum_{k=0}^{n} \delta[k+1]\delta[n+1-k] \\
&= 2\delta\left[\frac{n-1}{2}\right] + 4\delta\left[\frac{n+1}{2}\right] + \delta\left[\frac{n-2}{2}\right] + 2\delta\left[\frac{n}{2}\right]
\end{aligned}
$$



(b)

$$
\begin{aligned}
y(t) &= \frac{dx(t)}{dt} * h(t) \\
&= \frac{d}{dt}\left(u(t-1) + u(t+1)\right) * e^{-t}\sin(t)u(t) \\
&= \left(\delta(t-1) + \delta(t+1)\right) * e^{-t}\sin(t)u(t) \\
&= \int_{-\infty}^{\infty} \left(\delta(\tau-1) + \delta(\tau+1)\right) e^{-t-\tau}\sin(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \delta(\tau-1)e^{-t-\tau}\sin(t-\tau)d\tau + \int_{-\infty}^{\infty} \delta(\tau+1)e^{-t-\tau}\sin(t-\tau)d\tau \\
&= e^{-t-1}\sin(t-1)u(t) + e^{-t+1}\sin(t+1)u(t)
\end{aligned}
$$

3. (a)

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{-(t-\tau)}e^{-2\tau}d\tau \\
&= \int_{0}^{t} e^{-(t-\tau)}e^{-2\tau}d\tau \\
&= e^{-t}\int_{0}^{t} e^{\tau}d\tau \\
&= e^{-t}\left(1 - e^{-t}\right)u(t)
\end{aligned}
$$

(b)

$$\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} \left(u(t-\tau) - u(t-(\tau+1))\right) e^{3\tau} d\tau \\
&= \int_{-\infty}^{\infty} u(t-\tau)e^{3\tau}d\tau - \int_{-\infty}^{\infty} u(t-(\tau+1))e^{3\tau}d\tau \\
&= \int_{-\infty}^{t} e^{3\tau}d\tau - \int_{-\infty}^{t-1} e^{3\tau}d\tau \\
&= \frac{e^{3t}}{3} - \frac{e^{3t-3}}{3}
\end{aligned}$$

4. (a)

$$\begin{aligned}
y[n] - y[n-1] - y[n-2] = 0 \\
y[n] &= y[n-1] + y[n-2] \\
y[2] &= y[1] - y[0] = 2 \\
y[3] &= y[2] - y[1] = 3 \\
y[4] &= y[3] - y[2] = 5 \\
y[5] &= y[4] - y[3] = 8 \\
&\ldots
\end{aligned}$$

It is the Fibonacci sequence.

$$y[n] = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

(b)

$$\begin{aligned}
y^{(3)}(t) - 6y''(t) + 13y'(t) - 10y(t) &= 0 \\
K^3 - 6K^2 + 13K - 10 &= 0 \\
K(K-5)(K-2) + (K+5)(K-2) &= 0 \\
(K-2)(K^2 - 4K + 5) &= 0
\end{aligned}$$

$$K = 2, 2-j, 2+j$$

$$\begin{aligned}
y_h(t) &= c_1e^{2t} + c_2e^{(2+j)t} + c_3e^{(2-j)t} \\
y_h(t) &= c_1e^{2t} + c_2\left(e^{2t}\cos(t) + je^{2t}\sin(t)\right) + c_3\left(e^{2t}\cos(t) - je^{2t}\sin(t)\right) \\
y_h(t) &= c_1e^{2t} + c_2e^{2t}\cos(t) + c_3e^{2t}\cos(t) + c_2je^{2t}\sin(t) - c_3je^{2t}\sin(t) \\
y_h(t) &= c_1e^{2t} + (c_2+c_3)e^{2t}\cos(t) + j(c_2-c_3)e^{2t}\sin(t) \\
y_h(t) &= C_1e^{2t} + C_2e^{2t}\cos(t) + C_3e^{2t}\sin(t)
\end{aligned}$$

$$\begin{aligned} y''(0) &= 3 \\ &= 4C_1 + 3C_2 + 4C_3 \\ y'(0) &= 1.5 \\ &= 2C_1 + 2C_2 + C_3 \\ y(0) &= 1 \\ &= C_1 + C_2 \end{aligned}$$

$$\begin{aligned} C_1 &= 2 \\ C_2 &= -1 \\ C_3 &= -0.5 \end{aligned}$$

$$y_h(t) = 2e^{2t} - e^{2t}\cos(t) - \frac{1}{2}e^{2t}\sin(t)$$

5. (a)

$$\begin{aligned} y''(t) + 5y'(t) + 6y(t) &= \cos(5t) \\ &= \frac{e^{j5t} - e^{-j5t}}{2} \\ &= \frac{e^{j5t}}{2} - \frac{e^{-j5t}}{2} \end{aligned}$$

$$y_p(t) = c_1e^{j5t} + c_2e^{-j5t}$$

$$\begin{aligned} -25c_1e^{j5t} - 25c_2e^{-j5t} + 25jc_1e^{j5t} - 25jc_2e^{-j5t} + 6c_1e^{j5t} + 6c_2e^{-j5t} &= \frac{e^{j5t}}{2} - \frac{e^{-j5t}}{2} \\ e^{j5t}\left(-19c_1 + 25jc_1\right) + e^{-j5t}\left(-19c_2 - 25jc_2\right) &= \frac{e^{j5t}}{2} - \frac{e^{-j5t}}{2} \end{aligned}$$

$$(-19c_1 + 25jc_1) = (-19c_2 - 25jc_2) = \frac{1}{2}$$

$$\begin{aligned} c_1 &= \frac{1}{50j - 38} \\ c_2 &= \frac{-1}{50j + 38} \\ y_p(t) &= \frac{1}{50j - 38}e^{j5t} - \frac{1}{50j + 38}e^{-j5t} \end{aligned}$$

(b)

$$\begin{aligned} y''(t) + 5y'(t) + 6y(t) &= 0 \\ K^2 + 5K + 6 &= 0 \\ K &= -3, -2 \end{aligned}$$

$$y_h(t) = c_1e^{-3t} + c_2e^{-2t}$$

(c)

$$\begin{aligned} y(t) &= y_p(t) + y_h(t) \\ &= \frac{1}{50j - 38}e^{j5t} - \frac{1}{50j + 38}e^{-j5t} + c_1e^{-3t} + c_2e^{-2t} \end{aligned}$$

$$\begin{aligned} y(0) = y'(0) &= 0 \\ \frac{1}{50j-38} - \frac{1}{50j+38} + c_1 + c_2 &= 0 \\ \frac{5j}{50j-38} + \frac{5j}{50j+38} - 3c_1 - 2c_2 &= 0 \end{aligned}$$

$$\begin{aligned} c_1 &= \frac{3}{34} \\ c_2 &= \frac{-2}{29} \end{aligned}$$

$$y(t) = \frac{1}{50j-38}e^{j5t} - \frac{1}{50j+38}e^{-j5t} + \frac{3}{34}e^{-3t} - \frac{2}{29}e^{-2t}$$

6. (a)

$$\begin{aligned} w[n] - \frac{1}{2}w[n-1] &= x[n] \\ w[n] &= 0, \forall n < 0 \\ w[0] - \frac{1}{2}w[-1] &= x[0] \\ w[0] &= x[0] \\ w[1] - \frac{1}{2}w[0] &= x[1] \\ w[1] &= x[1] + \frac{1}{2}x[0] \\ w[2] - \frac{1}{2}w[1] &= x[2] \\ w[2] &= x[2] + \frac{1}{2}x[1] + \frac{1}{4}x[0] \\ &\ldots \\ w[n] &= \sum_{k=0}^{n} 2^{-k}x[n-k] \end{aligned}$$

$$\begin{aligned} h_0[n] &= \sum_{k=0}^{n} 2^{-k}\delta[n-k] \\ &= 2^{-n} && (\delta[n-k] = 0 \text{ for } k! = n) \end{aligned}$$

(b) If we feed the first system with the unit impulse, we get $h_0[n]$. If we feed the second system with $h_0[n]$, we get

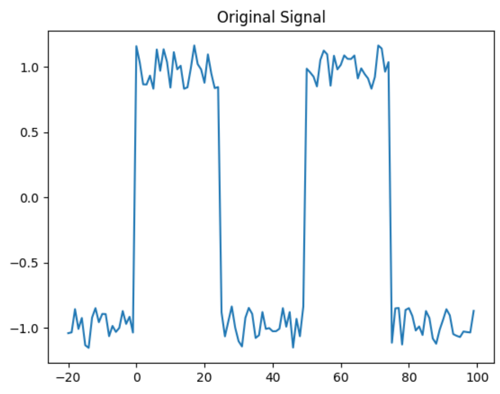
$$\begin{aligned} h[n] &= \sum_{k=0}^{n} 2^{-k}2^{-n+k} \\ &= \sum_{k=0}^{n} 2^{-n} \\ &= (n+1) * 2^{-n} \end{aligned}$$

(c)

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[n-k]h[k] \\
&= \sum_{k=0}^{\infty} x[n-k]h[k] && (h[k] = 0 \text{ for } k < 0) \\
&= \sum_{k=0}^{\infty} x[k] * (n+1) * 2^{k-n}
\end{aligned}$$

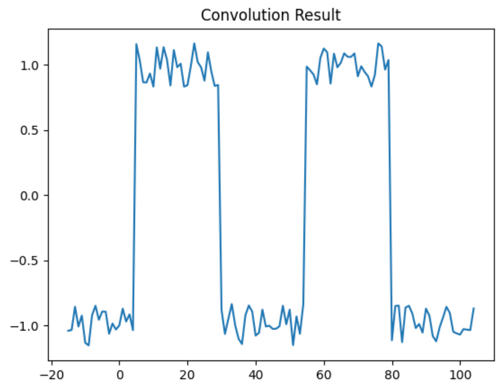
7. The code used to generate the plots is at the end of the answer.

Figure 1: Original Signal



(a) The effect of convoluting with $\delta[n-5]$ is to shift the signal by 5 units to the right.

Figure 2: Convolution with $h[n] = \delta[n-5]$



(b) The effect is unknown ???????.

Figure 3: Convolution with the moving average filter, N = 3

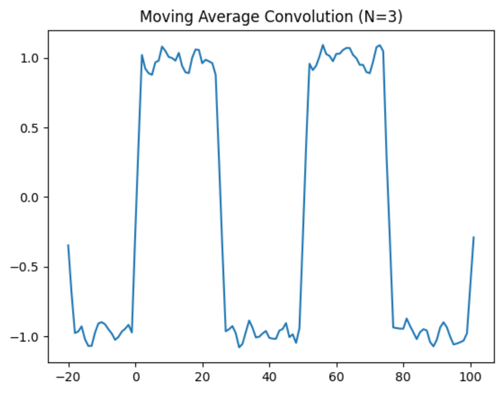


Figure 4: Convolution with the moving average filter, N = 5

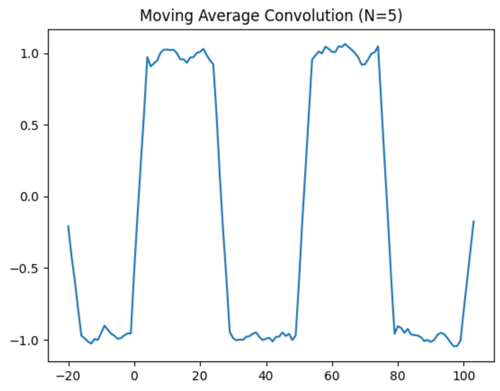


Figure 5: Convolution with the moving average filter, N = 10

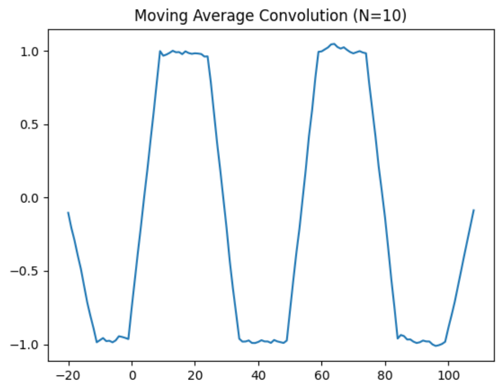


Figure 6: Convolution with the moving average filter, N = 20

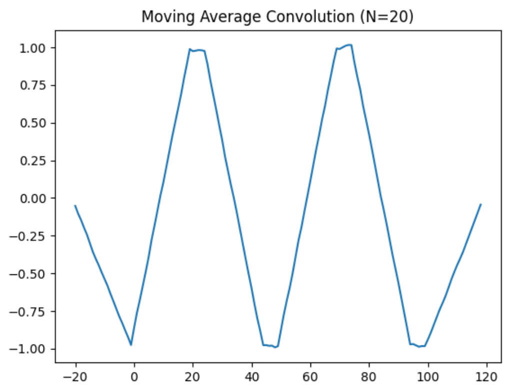


Figure 7: Python Code

```python
from matplotlib import pyplot

class Signal:
    def __init__(self, data, start):
        self.data = data
        self.start = start
        self.end = start + len(data) - 1

    def __getitem__(self, index):
        if index < self.start or index > self.end:
            return 0
        return self.data[index - self.start]

    def __mul__(self, other):
        return ConvolutionResult(self, other)

    def plot(self, title, save_path):
        pyplot.title(title)
        pyplot.plot(range(self.start, self.end + 1), self.data)
        pyplot.savefig(save_path)
        pyplot.clf()


class ConvolutionResult(Signal):
    def __init__(self, signal1, signal2):
        self.signal1 = signal1
        self.signal2 = signal2
        self.start = signal1.start + signal2.start
        self.end = signal1.end + signal2.end
        self.data = [self[i] for i in range(self.start, self.end + 1)]

    def __getitem__(self, index):
        if index < self.start or index > self.end:
            return 0
        result = 0
        for i in range(self.start, self.end + 1):
            result += self.signal1[i] * self.signal2[index - i]
        return result

with open("hw2_signal.csv", "r", encoding="ascii") as file:
    raw_data = [float(item) for item in file.read().split(",")]

x = Signal(raw_data[1:], int(raw_data[0]))
x.plot("Original Signal", "images/original.png")

h = Signal([1], 5)
convolution = h * x
convolution.plot("Convolution Result", "images/convolution.png")

for N in (3, 5, 10, 20):
    m = Signal([1 / N] * N, 0)
    moving_average = m * x
    moving_average.plot(f"Moving Average Convolution (N={N})", f"images/moving_average_{N}.png")
```