University of Michigan–Ann Arbor

Department of Electrical Engineering and Computer Science

EECS 475 **Introduction to Cryptography**, Winter 2023

Lecture 22: CPA security continued, El Gamal cryptosystem

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1 CPA Security

In continuation of the previous class, we want to show that one-query CPA implies many-query CPA.

Image a many-query attacker A that makes up to q queries where $q \in poly(n)$. Consider the following worlds:

Hybrid 0 (Left World) : all queries (m_0, m_1) to the LR oracle are answered by $c \leftarrow Enc_{pk}(m_0)$.

Hybrid 1 : First query (m_0, m_1) to the LR oracle is answered by $c \leftarrow Enc_{pk}(m_1)$, then $c \leftarrow Enc_{pk}(m_0)$ thereafter.

Hybrid 2 : First 2 queries (m_0, m_1) to the LR oracle are answered by $c \leftarrow Enc_{pk}(m_1)$, then $c \leftarrow Enc_{pk}(m_0)$ thereafter.

$\vdots$

Hybrid q (Right World) : all queries (m_0, m_1) to the LR oracle are answered by $c \leftarrow Enc_{pk}(m_1)$.

Note here, the only difference between $Hybrid(i-1)$ and $Hybrid(i)$ is how the i^{th} query is answered.
Now, we build a "simulator" $S_i^{LR_{pk,b}(.,.)}(pk)$ that gets **one query** and simulates either $Hybrid(i-1)$ or $Hybrid(i)$ depending on b.

On j^{th} query of A (m_0^j, m_1^j):

- If $j < i$, S_i runs $c \leftarrow Enc_{pk}(m_1^j)$
- If $j > i$, S_i runs $c \leftarrow Enc_{pk}(m_0^j)$
- If $j = i$, S_i queries to LR oracle and gives the result to A

$$\begin{cases} S_i \text{ is in the left world } (b=0)\text{, then we perfectly simulate } Hybrid(i-1) \\ S_i \text{ is in the right world } (b=1)\text{, then we perfectly simulate } Hybrid(i) \end{cases} \tag{1}$$

By triangle inequality,

$$\begin{aligned} Adv_{\pi}^{CPA}(A) &= \big|Pr(A=1 \text{ in } Hybrid(0)) - Pr(A=1 \text{ in } Hybrid(q))\big| \\ &= \big|Pr(A=1 \text{ in } Hybrid(0)) - Pr(A=1 \text{ in } Hybrid(1)) + Pr(A=1 \text{ in } Hybrid(1)) \\ &\quad - Pr(A=1 \text{ in } Hybrid(2)) + Pr(A=1 \text{ in } Hybrid(2)) \cdots - Pr(A=1 \text{ in } Hybrid(q))\big| \\ &\leq \sum_{i=1}^{q} Adv_{\pi}^{single-CPA}(S_i) = q\dot{n}egl(n) = negl(n) \end{aligned}$$

The theorem implies we can encrypt long messages bit-by-bit (or block-by-block) or broken up in any other many calls on "short" messages, which is acceptable by the theorem.

Theorem: Any public key encryption scheme wit deterministic $Enc_{pk}(.)$ can not be CPA secure **even for 1 query**.
Proof: query $c \leftarrow LR_{pk,b}(m_0, m_1)$ for any $m_0 \neq m_1$. Then, run $c' = Enc_{pk}(m_0)$. If $c = c'$ outputs 0, else 1. Because the adversary knows the query (m_0, m_1), the adversary has perfect advantage on distinguishing c and c'.

2 El Gamal Cryptosystem

El Gamal is the public key encryption version of Diffie Hellmen. It works as follows:

$$Alice \underset{B = g^b \in G}{\overset{A = g^a \in G}{\rightleftarrows}} Bob$$

$$\begin{array}{cc} \text{choose random a } \leftarrow Z_q & \text{choose random } b \leftarrow Z_q \\ K = B^a = g^{ab \bmod q} \in G & K = A^b = g^{ba \bmod q} \in G \end{array}$$

where G is a group of order q and g is the generator of G.

K is the secret key derived by two parties. We use the properties of cyclic group to get random number with multiplication.

We can look at El Gamal Cryptosystem interms of (Gen, Enc, Dec) :
Idea: Basically, message is $M \in G$, the "one-time-pad effect" would involve multiplying M wih something random K.

- $Gen(1^n)$: choose random a $\leftarrow Z_q$ output $(pk = A = g^a \in G, sk = a) \Leftarrow$ at Alice computes

- $Enc(pk = A, M \in G)$: choose random $b \leftarrow Z_q$ output ciphertext $(B = g^b \in G, C = M \cdot A^b \in G)$ $\Leftarrow$ what Bob computes

- $Dec(sk = a, (B, C))$: compute $K = B^a$, output $C \cdot K^{-1} \in G$

Correctness: $\forall M \in G, (pk = g^a, sk = a)$

$$Enc(pk, A) = (B = g^b, C = M \cdot (g^a)^b)$$

$$Dec(B, C) = C \cdot (B^a)^{-1} = M \cdot g^{ab} \cdot (g^{(ab)})^{-1} = M$$

CPA Security: Based on the DDH asssumption over $G : (g, g^a, g^b, g^{ab}) \in G^4$, where $a, b \leftarrow Z_q$, is indistinguishable from $(g, g^a, g^b, g^c) \in G^4$, where $a, b, c \leftarrow Z_q$.

Theorem: if DDH holds for G, then El Ganal is CPA-secure.
Proof: Let A be any feasible p.p.t attacker against El Ganal. Use A to construct the distinguisher against DDH.

If (g, A, B, C) is a DH tuple ("real world"), D perfectly simulates the left CPA world because $C = g^{ab}$.
Ideal world: (g, A, B, C) is randomthen D perfectly simulates a "hybrid" CPA world where the ciphertext is two independent random-group elements (regardless if message).
Symmetrically, we can construct D' vs DDH that replies A with $(B, m_1 \cdot C)$

$$Adv^{CPA}(A) \leq Adv^{DDH}(D) + Adv^{DDH}(D') = negl(n) + negl(n) = negl(n)$$