Section Nine

Return, Risk, and Arbitrage

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One of the strongest statements that can be made in positive economics is the assertion that if two riskless assets offer rates of return of ρ and ρ', then (in the absence of transactions costs):

$$\rho = \rho'. \tag{9.1}$$

As an arbitrage condition, this equality of rates of return may be expected to hold in all but the most profound disequilibrium. If economic agents can both borrow and lend at the rates ρ and ρ', then infinite profits are envisioned if rates diverge and (9.1) must obtain to prevent such arbitrage possibilities. Furthermore, even when the rates are both only lending rates, for example, if (9.1) did not hold then there would be no (gross) demand for the asset with the lower rate of return. The introduction of risk, however, alters these strong conclusions. Under neoclassical theory if an asset is risky its expected return will equal the riskless rate plus a premium to compensate the holder of the asset for bearing risk. The explanations that have been advanced in an effort to understand this premium have focused on somewhat special forms of equilibrium theories and have essentially abandoned the robust sort of argument that supports (9.1). The intent of this chapter is to develop an arbitrage theory for risky

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assets analogous to that for riskless assets, and, in so doing, to analyze the nature of risk premiums.

There are at present two major theoretical frameworks for the analysis of markets for risky assets; the state space preference approach and, the mean variance model and its variants. The arbitrage model which we will develop is a third approach to capital market theory, empirically distinguishable from the mean variance theory and more directly related to the state space approach. While formally all models may be viewed as special cases of the state space preference framework, it is in the restrictions imposed either on preferences or distributions that the empirical content of the various theories lie. The mean variance equilibrium model of Treynor, Sharpe and Lintner, for example, is derived by either assuming quadratic preferences or that assets are multivariate normally distributed. Neither of these assumptions, however, is particularly appealing on intuitive economic grounds; normality has only an inappropriate (and careless) application of the central limit theorem to recommend it and quadratic utility functions are implausible for any agents, let alone for all of them. We will examine the mean variance model and the assumptions of normality and quadratic preferences in detail in the sections below.

Much recent work has focused on the implications of alternative restrictions on the nature of investor preferences. Yet to obtain useful capital market results by restricting preferences requires severe assumptions on the homogeneity of investors' attitudes towards risk (as well as returns). A priori, it would seem more acceptable to place restrictions on probability structures rather than on preferences if only because there is a single ex post realization and a large body of commonly held information. It is the consequences of this approach that we will explore.[1]

In Section 9.1 a brief but unusual development of the mean variance capital market equilibrium model is given including the role of the zero-beta portfolio as a substitute for a riskless asset. It is to be emphasized that this model is a consequence of special restrictions on the state space preference framework. Section 9.2 introduces and develops the basic alternative model, the arbitrage model. Section 9.3 extends the general state space preference model developed by Arrow and Debreu and extended by Hirschleifer, Diamond, and others. With the aid of some familiar results from the theory of cones, we will examine the full implications of the assertion that a market for risky assets is in equilibrium in the absence of any of the usual constraints on preferences or distributions. In Section 9.4 the arbitrage model's relation to the mean variance model and the underlying state space framework is examined in detail. Section 9.4 also presents a somewhat novel derivation of the mean variance model by the use of the arbitrage theory. Section 9.5 summarizes and concludes the chapter.

SECTION 9.1

The mean variance model of capital market equilibrium represents a very strong restriction on the structure of asset returns across states. In particular, to develop the model we assume that all agents subjectively, and ex ante, view the n assets as being jointly normally distributed with a vector of means, E, and covariance matrix V.[2] It is well known (see, e.g., Sharpe [1970]) and easy to show that the feasible set of means, m, and standard deviations, σ, attainable on portfolios formed from the n assets has the shape illustrated in Figure 9-1. Formally, this feasible set is defined as

$$F \equiv \left\{ <m, \sigma> \mid \text{for some } \alpha \text{ with } \alpha' e = 1, m = \alpha' E, \sigma^2 = \alpha' V \alpha \right\}, \tag{9.2}$$

and its efficient boundary is defined as the southeast boundary of F.[3] In the definition the vector α denotes a portfolio whose i^{th} component, α_i, is the proportion of wealth placed in the i^{th} asset. If a riskless asset with a sure return, ρ, is introduced, the efficient frontier including portfolios formed with this asset is the line tangent to F as shown.

By assumption, agents possess risk averse non-Neumann Morgenstern utility functions and will, therefore, choose efficient points. Figure 9-1 illustrates the familiar separation property (see Markowitz or Tobin) in the presence of a riskless asset. Since the efficient frontier is a line, the market line in $<m, \sigma>$ space, all investors will choose their portfolios as simple combinations of investment (or borrowing) in the riskless asset and a single portfolio of risky assets. In equilibrium, then, this efficient portfolio must consist of risky assets held in proportion to their total dollar value, i.e., it must be the market portfolio obtained by purchasing all variable risky assets.[4] Conversely, the market portfolio must be efficient, it is the minimum variance portfolio of risky assets that attains the market return, E_m.

Let $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$ denote the vector of random returns on the n risky assets (represented as the columns of the state space tableau). In market equilibrium the famous security line equation takes the form

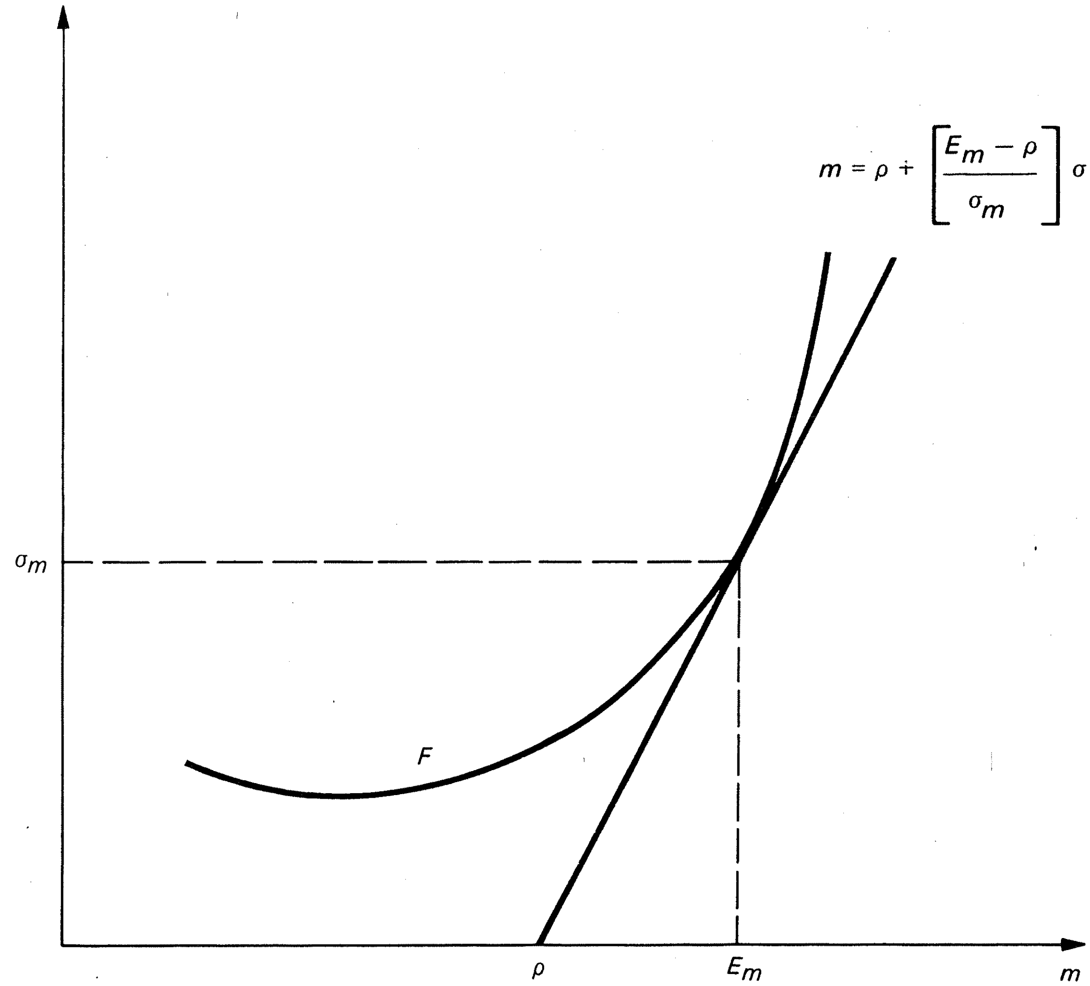

Figure 9-1.

$$E_i = \rho + \lambda\, b_i;\; i = 1, \ldots n, \tag{9.3}$$

where

$$b_i \equiv \frac{\operatorname{cov}\left\{\tilde{x}_i, \tilde{x}_m\right\}}{\sigma_m^2} = (V\alpha_m)_i \tag{9.4}$$

and

$$\lambda \equiv E_m - \rho, \tag{9.5}$$

with α_m denoting the market portfolio, $E_m = \alpha'_m E$ and $\sigma^2_m = \alpha'_m V \alpha_m$ being, respectively, the mean and variance of the random market return, $\tilde{x}_m$. Equation (9.3) is simply the first order condition for the mean variance efficiency of the market portfolio. Once we identify the market portfolio as efficient the security line Equation (9.3) follows immediately—nothing further is required. This observation provides a clue to weakening the underlying assumptions. It is not difficult to show, for example, that since the market portfolio will still be efficient, the security line Equation (9.3), will hold in the absence of a riskless asset. The constant, ρ, is now interpreted as the zero-beta return; it is the return on all assets or portfolios uncorrelated with the market portfolio.

The simple and intuitive equilibrium linear relation between return, E_i, and the beta coefficient, b_i, accounts for much of the popularity of the mean variance model. The risk on an asset is completely described by its covariance with the market portfolio, or by the ratio of its covariance to the variance of the market, the beta coefficient, b_i; the risk premium is simply proportional to the beta coefficient. As asset's risk is priced away in individual portfolios except for that component of risk that is market dependent. As asset that is positively correlated must earn a risk premium, one that is negatively correlated is so valuable as a hedge that it will be priced at a discount from the riskless rate and, perhaps most strikingly, an asset with a zero beta coefficient—despite its uncertainty as measured, for example, by its variance—will be priced at the riskless rate of return.

The exact meaning of the security relation is not always made clear in the literature. It is not important that (9.3) is linear, what is important is that the market portfolio is an observable economic variable, the totality of wealth held at risk, and that it plays a pivotal role in the theory. The linearity of (9.3) is only of interest when it is understood that correlation with the market portfolio is the proper measure of risk in equilibrium. In fact, it is easy to show that unless arbitrage is possible there always exists some portfolio α such that:

$$E_i = \rho + \lambda(V\alpha)_i, \tag{9.6}$$

where λ is a constant chosen so that $\alpha' e = 1$.[5]

However appealing the linear relation (9.6) may appear, then, it is nearly devoid of economic content. In fact, given any collection of random variables with a nonsingular V, (9.6) will be satisfied for some portfolio, α, whether we are in equilibrium or not. As such, (9.6) is empty of any empirical content as well. The result (9.6) only becomes meaningful when something can be said about the nature of α.[6]

When we add the assumption that returns are jointly normal, or that utility functions are quadratic, the portfolio α can be shown to be the market portfolio, and the security line Equation (9.3) constitutes an empirically significant restriction on asset returns.[7] To put the matter somewhat differently, with given expectations, E, and a given covariance matrix, V (or given dependence of E and V on prices) then (9.6) can be solved for the market portfolio and, therefore, relative prices. This is a simultaneous equilibrium system and it would, for example, be antithetical to its spirit to impute to it any directional causality as in a statement such as, "the risk premium is *determined* by covariance with the market."

SECTION 9.2

There have been an enormous number of empirical studies directed either at testing the security line Equation (9.3) or applying it. The commonest approach is to assume that ex post returns are generated by some stochastic relation, most simply chosen as

$$\tilde{x}_i = E_i + \beta_i \tilde{\delta} + \tilde{\epsilon}_i; \quad i = 1, \ldots, n, \tag{9.7}$$

where $\tilde{\epsilon} = \langle \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_n \rangle$ is a mean zero noise vector, E_i is a constant term representing the ex ante expected return, β_i is the ex ante beta coefficient and δ is a mean zero common factor representing the deviations of the market return from its trend. Equation (9.7) is a simple linear regression and is illustrated in Figure 9-2. If a time series (multiple) regression is run on (9.7) the resulting coefficients $\langle \hat{E}_i \rangle$ and $\langle \hat{\beta}_i \rangle$ will be estimates of the ex ante expected return and the beta coefficient respectively. It can be shown that $\tilde{\delta}$ will approximate the market portfolio and the ex ante beta coefficient, β_i, will approximate the ex ante coefficient, $b_i \equiv \sigma_{im}^2 / \sigma_m^2$, in the security market line (9.3). Ignoring some basic and subtle statistical difficulties, it is now possible to consider a test of the linearity of the relation between $\hat{E}_i$ and $\hat{\beta}_i$ as a test of the mean variance capital model.[8] The consequences of such a test, performed by Blume and Friend on NYSE data, is illustrated in Figure 9-3. The relationship is roughly linear at least for low values of $\hat{\beta}_i$, but the intercept term is considerably higher than that of the prevailing riskless rate in the period. According to the mean variance model the intercept should be at the riskless rate. The discrepancy can be explained in a qualitative sense by noting that borrowing rates exceed lending rates

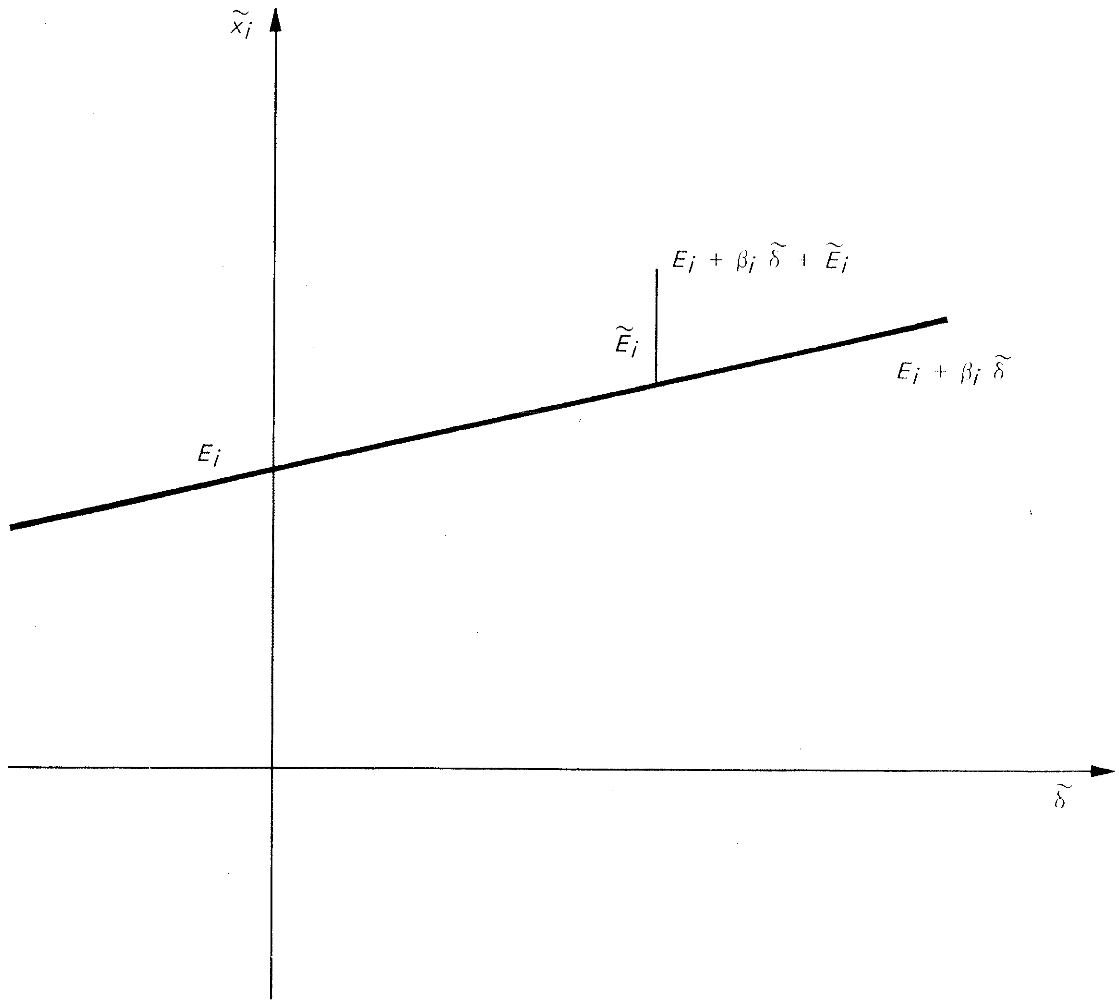

Figure 9-2.

(which, as Black shows, implies an intercept above the riskless lending rate). Alternative forms of the generating function have also been suggested and these seem to offer greater promise, but tests of the mean variance theory become much more difficult and, we shall argue, unnecessary.

In fact, the relation (9.7) of and by itself constitutes a far more satisfactory basis for a capital market theory without the additional baggage of mean variance theory. We will develop such a theory and for reasons that will become apparent refer to it as the arbitrage theory.[9] Throughout we will assume that the number of assets, n, is sufficiently large to permit our arguments to hold. We will, also, assume that the noise vector, $\tilde{\epsilon}$, is sufficiently independent to permit the law of large numbers to work. (Notice that in the generating model of (9.7) this would allow for independent industry effects if there were enough industries.)

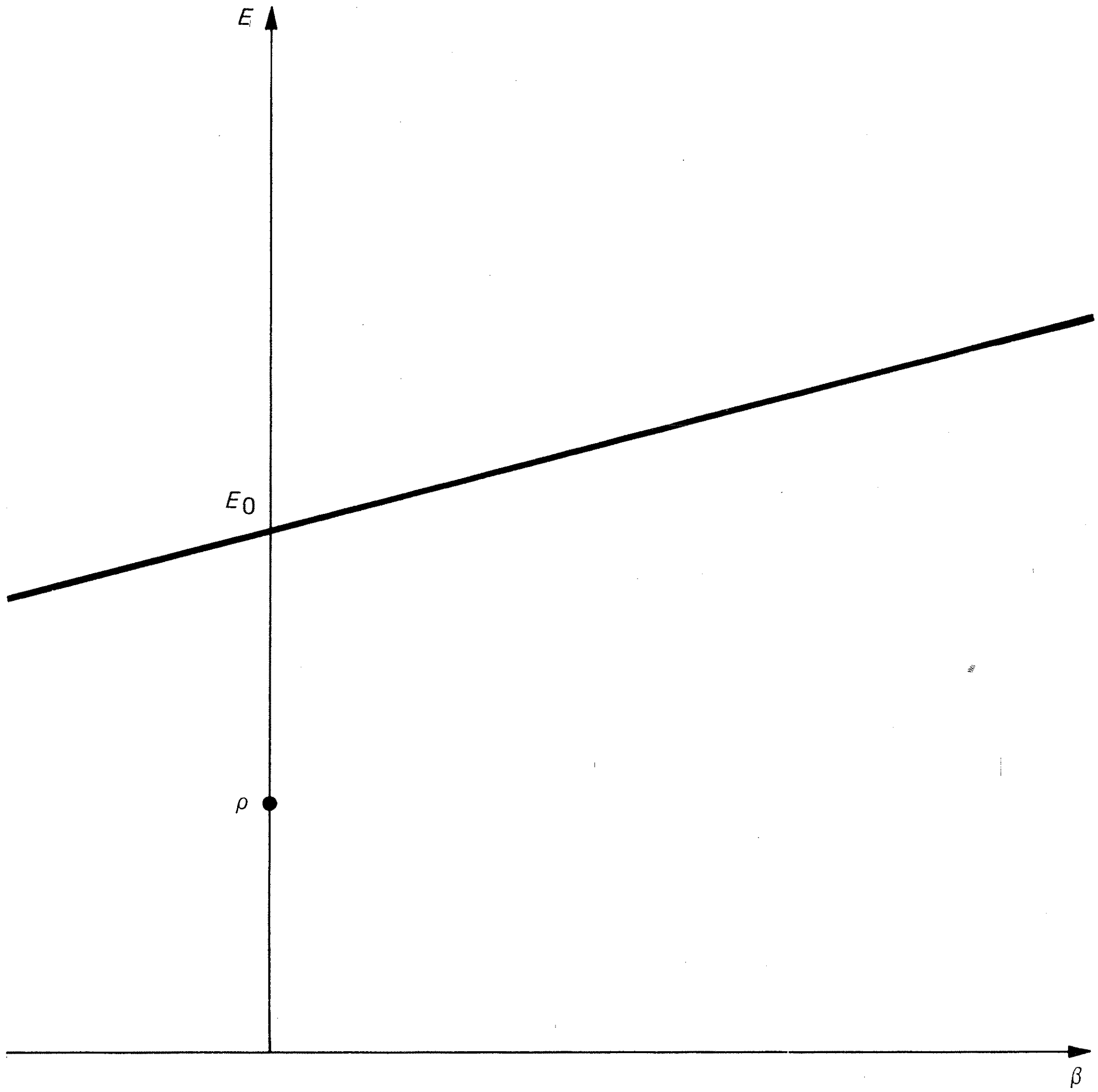

Figure 9-3.

Suppose, first, that we form an arbitrage portfolio, η, of the n assets. As an arbitrage portfolio, η uses no wealth, and

$$\eta' e = 0. \tag{9.8}$$

In other words, the wealth invested long in assets is exactly balanced by the amount borrowed from short sales and, net, the portfolio uses no wealth. Denoting the vector of mean returns by E and the vector of beta coefficients by β, the return on the arbitrage portfolio will be given by

$$\begin{aligned} \tilde{R} \equiv \eta' \tilde{x} &= (\eta' E) + (\eta' \beta) \tilde{\delta} + \eta' \tilde{\epsilon} \\ &\approx (\eta' E) + (\eta' \beta) \tilde{\delta}, \end{aligned} \tag{9.9}$$

where we have, secondly, assumed that the arbitrage portfolio is sufficiently well diversified to permit us to use the law of large numbers to approximately eliminate the noise term, $\eta'\tilde{\epsilon}$.[10] In effect, by using a well diversified portfolio we have been able to eliminate the independent risk from the portfolio return.

Lastly, we can always choose the arbitrage portfolio in such a fashion that it eliminates systematic risk as well, i.e.,

$$\eta'\beta = 0. \tag{9.10}$$

(If an inadequate number of β_i coefficients are negative we will have to engage in short sales to accomplish (9.10) while maintaining a well diversified portfolio.) Having eliminated both components of risk, the portfolio return,

$$\tilde{R} = (\eta' E) + (\eta'\beta)\,\tilde{\delta} = \eta' E, \tag{9.11}$$

risklessly. By choosing the portfolio to be well diversified, to be an arbitrage portfolio satisfying (9.8) and to satisfy (9.10) we have been able to eliminate all risk and realize the return $\eta' E$. Since an arbitrage portfolio uses no wealth, it must now follow that its return,

$$\eta' E = 0. \tag{9.12}$$

If (9.12) did not hold, then by using no wealth we will have been able to obtain a riskless return. Furthermore, we would be able to obtain arbitrarily large returns by simply scaling up the arbitrage portfolio. This is incompatible with the absence of arbitrage, let alone equilibrium, and (9.12) must hold.

In summary, then, any portfolio satisfying (9.8) and (9.10), must also satisfy (9.12).[11] This is simply the algebraic statement that all vectors, η, orthogonal to e and β are orthogonal to E, and it follows that E must be a linear combination of e and β. Hence, there are constants E_0 and a such that for all i

$$E_i = E_0 + a\beta_i.\text{[12]} \tag{9.13}$$

Figure 9-4 illustrates this argument with just three assets. Suppose the assets in Figure 9-4 actually represent clusters (accumulation points) of assets sufficiently dense so that by forming portfolios one can obtain the three return-β points shown without any nonsystematic (ϵ) risk. The algebraic arbitrage argument simply says that the three points must lie on the line given by (9.13). In Figure 9-4, the

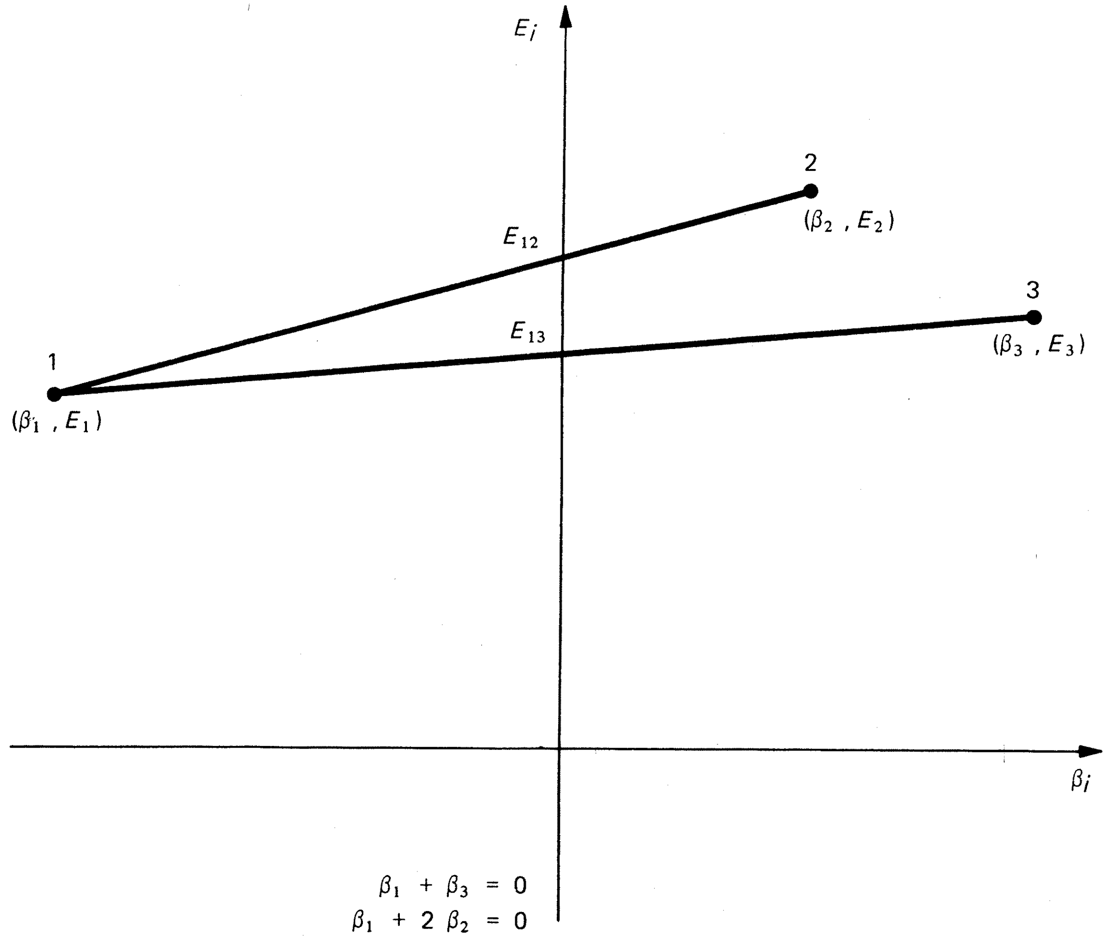

Figure 9-4.

three points illustrated do not lie on a security line. By taking a portfolio that splits wealth evenly between the first and third assets we obtain a return of E_{13} with no systematic, β risk. However, by combining the first and second assets with investments of 1/3 and 2/3 of wealth respectively we obtain the riskless return E_{12} which exceeds E_{13}. Now, by shorting (borrowing against) the E_{13} portfolio to go long in the E_{12} portfolio we can receive arbitrarily high returns without risk. To prevent this we must have $E_{12} = E_{13}$ which means that points 1, 2 and 3 must lie on a line such as (9.13).

If there is a riskless asset with a rate of return, ρ, then the same argument would reveal that $E_0 = \rho$. Alternatively, consider any well-diversified, nonarbitrage zero beta portfolio, i.e., a portfolio satisfying (9.10), and such that $\alpha' e = 1$. From (9.13) the return on such a portfolio will be E_0 and since we have eliminated all risk, we must have $E_0 = \rho$, the riskless rate, to prevent arbitrage. (If all well-diversified portfolios of risky assets have systematic risk then β

will be close to a constant vector and we can again take $E_0 = \rho$ in (9.13)). In the absence of such riskless opportunities, though, E_0 can still be identified as the expected return on all zero beta portfolios. Finally, considering a portfolio of special interest such as the market portfolio (of risky assets), α_m, we can normalize $\tilde{\delta}$ so that $\alpha_m \beta = 1$ and (9.13) becomes

$$E_i = E_0 + (E_m - E_0)\,\beta_i\,, \tag{9.14}$$

where E_m is the expected return on the market portfolio.[13] Equation (9.14) is the arbitrage equivalent of the mean variance security line equation.

Notice that we did not have to assume that the market was in equilibrium to derive our basic arbitrage condition (9.14). It is a stronger result and depends essentially on the absence of arbitrage possibilities rather than on the much more restrictive condition that the market be in equilibrium as is required in the mean variance theory.

The basic arbitrage condition generalizes easily to the l-factor case, as well, provided only that the number of common factors is significantly less than the number of assets.[14] By considering alternative arbitrage portfolios it is possible to show that when the generating model has the form

$$\tilde{x}_i = E_i + \beta_{il}\tilde{\delta}_1 + \ldots + \beta_{ik}\tilde{\delta}_k + \tilde{\epsilon}_i\,, \tag{9.15}$$

the basic arbitrage condition takes the form

$$E_i = E_0 + (E_m - E_0)\,[\gamma_1 \beta_{il} + \ldots + \gamma_k \beta_{ik}]\,, \tag{9.16}$$

where the γ_l are nonnegative constants normalized so that

$$\Sigma\,\gamma_l = 1.\text{[15]} \tag{9.17}$$

In other words, the risk premium on the i^{th} asset, $E_i - E_0$, is a convex combination of its beta weights times the risk premium on the market portfolio.

There is no need, however, for the market portfolio to play any special role in this theory and, in fact, the basic arbitrage condition can be written in a more appealing form than (9.16). The constants in (9.16) arise from the algebraic arbitrage argument and our task is to interpret them. Defining $\theta_l = (E_m - E_0)\,\gamma_1$ permits us to write (9.16) as

$$E_i - E_0 = \theta_1 \beta_{il} + \ldots + \theta_k \beta_{ik} \; . \tag{9.18}$$

Consider a portfolio, α, which is zero beta on all factors except the l^{th}. Formally,

$$\alpha' \beta_j = 0 \text{ if } j \neq l \; , \tag{9.19a}$$

$$\alpha' \beta = 1$$

and

$$\alpha' e = 1 \; , \tag{9.19b}$$

where β_i denotes the vector of i^{th} factor weights, $<\beta_{1i}, \ldots, \beta_{ni}>$. (If such portfolios cannot be formed, then factors can be linearly combined simplifying the generating model.) Using (9.19) the risk premium on such a portfolio, $E_l - E_0$, is simply

$$E - E_0 = \alpha'(E - E_0) = \theta_l . \tag{9.20}$$

(E^l is not to be confused with the return on the l^{th} asset; E^l denotes the return on the l^{th} factor portfolio.) It follows that θ_l is the market risk premium on the l^{th} market factor. Thus, all such portfolios offer the same risk premium, $E^l - E_0$, and we can rewrite the basic arbitrage condition as

$$E_i - E_0 = (E^1 - E_0)\beta_{i1} + \ldots + (E^k - E_0)\beta_{ik} \; . \tag{9.21}$$

In other words, the risk premium on any asset is given by the weighted sum of the factor risk premiums with weights equal to the beta coefficients of the asset. If one of the factors is chosen as the market portfolio, then its risk premium is treated like that for any other factor.

As an example, let us consider the form the basic arbitrage condition (9.21) takes with the following generating model

$$\tilde{x}_i = E_i + (1 - \beta_i)\tilde{\delta}_1 + \beta_i \tilde{\delta}_2 + \tilde{\epsilon}_i \; . \tag{9.22}$$

This model has been used in a number of recent empirical studies. The first factor, $\tilde{\delta}_1$, represents a zero beta factor chosen to be uncorrelated with the second factor, $\tilde{\delta}_2$, the market factor. From the basic arbitrage condition (9.21) we have

$$E_i = E_0 + (E^1 - E_0)(1 - \beta_i) + (E^2 - E_0)\beta_i \qquad (9.23)$$
$$= E^1 + (E^2 - E^1)\beta_i.$$

The model may now possess an ex ante expected return for a zero beta portfolio, E^1, in excess of the riskless rate, ρ (if such an asset exists). As with the single factor generating model (9.7), if $\tilde{\delta}_1$ is uncorrelated with the market factor, the mean variance theory will predict ρ as the intercept in (9.23) when estimated values are used (in large samples). In the absence of a riskless asset, though, mean variance theory will also imply that (9.23) is satisfied.

To empirically distinguish the arbitrage theory from mean variance analysis, then, by the use of the pricing relations alone and with this two factor model requires a demonstration that $E^1 \neq \rho$ if a riskless asset exists. With no riskless asset, however, it is difficult to distinguish between the two theories by such tests. Moving to an independent two factor model or to a three factor model would permit such a test. It must be understood, though, that the mean variance theory has a number of strong implications beyond simply the form of the pricing relation (9.3). As emphasized in Section 9.1, the pivotal role played by the market portfolio is central to mean variance theory and β_i must be the covariance term (divided by the market portfolio's variance) of the i^{th} asset with the market portfolio and not any other. The observation that many well diversified portfolios behave like the market portfolio is a statement in support of a simple factor generating model and *not* in support of mean variance theory. In general, tests of the linear relation (9.13) are better interpreted as studies of the arbitrage theory rather than of mean variance theory.

In the next sections we will explore the theoretical and empirical relations among the arbitrage theory, mean variance theory, and the general state space framework.

SECTION 9.3

The simplest state space framework describes a world with n assets and m discrete, exclusive states, $<\theta_1, \ldots, \theta_m>$, that the world could be in. (Implicitly we will deal with a two-period world—where uncertainty is resolved in the second period—to avoid the problem of intertemporal allocation, but many of the results extend to the multiperiod context in an obvious fashion.) The matrix

$$X \equiv \text{States} \quad \begin{array}{c} \\ \theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \theta_m \end{array} \begin{array}{c} \text{Assets} \\ X^1 \; . \; . \; . \; X^n \\ \left[\begin{array}{ccc} & & \\ & x_{ij} & \\ & & \end{array} \right] \end{array} , \tag{9.24}$$

will denote the commonly agreed upon array of per-dollar (accounting unit) returns, where we define x_{ij} to be the gross return per dollar invested in asset j if state i occurs.[16]

Individuals choose their portfolios as combinations of the n assets so as to maximize the utility of their wealth and the only assumption on preferences is that all utility functions are monotone functions of the form $v(w_1, \ldots, w_m)$, where w_θ is wealth in state θ and for each θ there is some individual who exhibits no satiation in w_θ.[17]

Suppose, now, that the capital market is in equilibrium. It follows that there cannot exist any successful arbitrage strategies. To be specific, it must not be possible to insure a nonnegative return in all states and a positive return in at least one state with no net investment, i.e., without taking a position. Mathematically, if

$$y = X\eta \tag{9.25}$$

where η is an arbitrage portfolio using no wealth, i.e.,

$$\eta' e = 0 \;\; (e' \equiv \langle 1, \ldots, 1 \rangle) \, , \tag{9.26}$$

then we cannot have $y \geqslant 0$.[18] If we did find such a solution, then without risk or cost, the portfolio $\lambda\eta$ would yield arbitrarily high returns in at least one state with no losses in any other states as λ, the scale of operation, diverged. As is shown in a footnote, by applying the duality results of linear programming this is equivalent to requiring the existence of a vector, p, such that

$$p' X = e' \tag{9.27}$$

and

$$p_\theta > 0 \tag{9.28}$$ [19]

The vector, p, is the vector of state space prices, p_θ, representing the current cost of a pure state space or contingent claim asset yielding one dollar if state θ occurs and nothing if θ does not occur.

Since $x_{\theta i}$ is the dollar return per dollar invested in asset i if state θ occurs, $p'X^i = \sum_\theta p_\theta x_{\theta i}$ simply values asset i up across states and, in equilibrium, this must equal its one dollar cost. Of course, if X is of less than full row rank, e.g., if the number of assets, n, is less than the number of states, m, then it will be impossible to form portfolios of the n assets equivalent to pure state space securities for all states and the above interpretation of p_θ is somewhat strained. In fact, it is easy to show that under such circumstances p will not even be unique and will, in general, lie in a subspace of dimension $m - n$. Now, p_θ is properly interpreted as an implicit price and the exact price system, with the property that the same allocation of wealth among the assets would occur if investors faced a market with prices, p_θ for pure state space assets, will depend upon both the subjective state space probabilities and the preferences of the investors. In general, though, no single price system will simultaneously allocate all agents as in the original situation since if not all contingencies are covered the equilibrium is not efficient relative to complete Arrow-Debreu equilibrium. As a consequence not all individual marginal rates of substitution across wealth in different states will be equal.

In the special case explored by Cass and Stiglitz where X is of full row rank a number of interesting results can be obtained. Now, $p' = e'X^{-1}$ is uniquely defined and $p'e = e'X^{-1}e$ is the aggregate cost of obtaining a sure dollar return in all states, or $(e'X^{-1}e)^{-1}$ is the riskless interest factor. In general, though, the existence of a nonnegative state price vector is all that can be ascertained from the assertion that the market is in equilibrium.

To verify this we must show that for any tableau of return, X, with an associated positive price vector, p, for which $p'X = e'$, there exists some structure of preferences such that the market is in equilibrium. To construct such a market consider a single individual who seeks to maximize his expected utility of wealth,

$$E\left\{u(\tilde{w})\right\} = \sum_\theta \pi_\theta u(wx'_\theta \alpha) \tag{9.29}$$

over portfolios α subject to $e'\alpha = 1$, where π_θ is the subjective probability assigned to state θ and is assumed positive, and where X_θ denotes the θ^{th} row of X. The first-order conditions take the form

$$\Sigma_\theta \pi_\theta \; u'(wx'_\theta \alpha) \; x_{\theta i} = \lambda; i = 1, \ldots, n\,, \tag{9.30}$$

where λ is a Langrange multiplier. Since $p'X = e'$, set

$$\pi_\theta \frac{1}{\lambda} u'(wx'_\theta \alpha) = p_\theta \tag{9.31}$$

or

$$u'(wx'_\theta \alpha) = \lambda p_\theta / \pi_\theta > 0. \tag{9.32}$$

Letting u be a concave utility function with everywhere positive marginal utility and α any portfolio, it is clear that λ and a vector of probabilities, π_θ, can be chosen to satisfy (9.32) and that, by concavity, this will be sufficient for an optimum. Notice, too, that information about the market portfolio, α, by itself, provides no further information about X, the equilibrium returns.

We can summarize our findings with the following assertion:

If we impose no a priori restrictions on individual preference structures beyond monotonicity (or, for that matter, on individual subjectively held anticipations) then the assertion that the market is in equilibrium implies the absence of arbitrage possibilities.[20] Conversely, if there are no arbitrage possibilities in a market, then there always exists a structure of preferences such that the market would be in equilibrium.

It follows that to obtain an empirically refutable capital market theory will require the imposition of further assumptions on the structure of the market. This is not to deny that information external to the capital market, savings rates, for example, might provide the required observable inferences on preference structures or distributions.

SECTION 9.4

Explicitly then, an empirically meaningful theory of the pricing of risky assets can be obtained only by placing some restrictions on preference structures or on the underlying subjectively perceived generating mechanism. The easiest way to understand the sense in which the arbitrage theory is based on a restriction of the latter sort is to delete the $\tilde{\epsilon}$ noise term in (9.15). In vector notation, (9.15) now asserts that the random vector of gross returns on the i^{th} asset (across states), the i^{th} column of the state space tableau, is a linear combination of a constant vector, e, and k vectors, $(\delta_1, \ldots, \delta_k)$. This is the same as requiring that the state space tableau be of rank less than or equal to $k + 1$ where the constant vector is in the basis. Equivalently, it implies that all assets when considered as points in an underlying state space lie in a subspace of dimension $k + 1$ spanned by the constant vector and the factor vectors. In this framework the arbitrage condition (9.21) must hold exactly to prevent a pure arbitrage situation from arising. One way to see this, without constructing arbitrage portfolios, is to use the result in Section 9.1 that assures that in the absence of arbitrage a state price vector, p, exists. As shown in a footnote, Equation (9.27) is identical to (9.21) with $E^l - E_0 = -p'\delta_l / p'e$.[21]

Mean variance theory can be thought of as representing a restriction either on preference structures or on distributions. In either case, though, the mean variance analysis can be considered as a special case of arbitrage theory in which the arguments hold exactly, rather than as approximations, despite the fact that all of the randomness has not been removed. To see what we mean, suppose that random returns are governed by the factor model (9.7) where $\tilde{\delta}$ is precisely the market portfolio's excess return, $(\tilde{R}_m - E_m)$ and the $\tilde{\epsilon}_i$ are mean zero and uncorrelated with $\tilde{R}_m$, but are not necessarily independent across assets, i. In fact, such a representation is always possible with any collection of random variables and represents no restriction at all (if second moments exist). Furthermore,

$$\begin{aligned} R_m &= \alpha'_m \tilde{x} \\ &= E_m + \tilde{\delta} + \alpha_m \tilde{\epsilon} \\ &= \tilde{R}_m + \alpha'_m \tilde{\epsilon}\,; \end{aligned} \tag{9.33}$$

and we must have $\alpha'_m \epsilon \equiv 0$.[22]

By the separation theorem given identical subjective anticipations every investor holds the same risky portfolio, α_m. In addition, when only means and variances matter, if any investor is given the opportunity to engage in an arbitrage operation, using no wealth, that offers a return uncorrelated with his own risky portfolio, α_m, he will do so unless its expected return is zero. In other words, the marginal contribution to risk by such an operation is zero.[23] This permits us to use (9.7) and the basic arbitrage argument to derive the mean variance capital pricing relation in a very direct fashion. Let η be an arbitrage portfolio, with $\eta' e = 0$, and chosen to be orthogonal to the market, $\eta'\beta = 0$. From (9.7) its random return is

$$\eta'\tilde{x} = \eta' E + \eta'\tilde{\epsilon}\,, \tag{9.34}$$

and since $\eta'\tilde{\epsilon}$ is orthogonal to the market, the expected return

$$\eta' E = 0\,. \tag{9.12}$$

This is precisely the arbitrage argument used in Section 9.2 and we conclude that

$$E_i = E_0 + (E_m - E_0)\beta_i, \tag{9.14}$$

must hold with $E_0 = \rho$ in the presence of a riskless asset. The difference between this derivation and the original arbitrage argument came at the stage where we concluded that because $\eta'\tilde{\epsilon}$ was uncorrelated with $\tilde{R}_m$ the expected return, $\eta' E = 0$. In the absence of assumptions which justify focusing only on means and (co)variances this conclusion would be unwarranted. This is the sense in which the mean variance model or, at least, its analysis may be considered a special case of arbitrage theory, but it should be emphasized that in a strict sense the underlying assumptions of arbitrage theory and mean variance theory are distinct.

On purely theoretical grounds, then, it cannot be asserted that mean variance theory is a special case of arbitrage theory. In evaluating the theories on such grounds the argument must turn on the a priori appeal of their respective major assumptions. The assumptions that underly mean variance theory have been exhaustively studied in the literature and are outlined above and we will not repeat them. The primary assumption underlying the arbitrage theory is that only a small number of factors is significant.[24] Without presenting a formal argument for why this should be so we can, at least, argue that it is quite plausible.

Essentially, the question turns on whether substitution or income effects in both consumption and production are more important in the short run. Abstracting from capital gains, for the moment, suppose that the returns on investment are the returns on productive activities. In the short run, ideally in a differential time, productive activities will be fixed (as in a putty-clay model) and the returns to capital will be composed of ordinary returns plus (disequilibrium) profits. When the level of aggregate demand changes, sectors will accommodate in a Leontief-like fashion, but the output and profit response of each will be a linear function of the change. If capital receives the residual, after payments to other productive agents, then there will be a specific aggregate demand factor. Of course, if the change in aggregate demand is accompanied by important systematic shifts in demand across sectors and, consequently, in prices then this will not be the case.[25] To the extent to which such secular demand shifts take place and are significant they must be represented by factors in the generating model. In the short run, though, models of habit formation would suggest that price shifts would be of a second order and could be ignored. Finally, to the extent that anticipations of capital gains are extrapolative and based on learning from past experience, the number of factors would be further limited if the slow shift responses postulated on both the consumption and production sides were valid. These should prove to be important areas of future research. The precise determination of what the relevant factors are is crucial to fully understanding the theory and should be the subject of econometric study. While the above arguments suggest the importance of an aggregate demand or GNP or market factor, it will probably be necessary to break such a factor into its real and price components. In addition, other factors that affect long term trends and lead to secular shifts can also be expected to have an impact on returns. Increasing substitution of capital (human and machine) for labor is one example.

Before leaving the theoretical side of the comparison it should also be stressed that the arbitrage theory makes considerably less stringent requirements on the homogeneity of investors' ex ante anticipations than does the mean variance theory. Although, some weakening is possible, to obtain an empirically meaningful form of the pricing relation (9.6) mean variance theory essentially requires that all investors agree about both the expected returns and the covariance matrix.[26] Given agreement on the factor model of (9.15), however, the arbitrage theory is unaltered if agents hold differing subjective views on the distribution either of the market factors, $\tilde{\delta}^{i}$, or of the noise terms, $\tilde{\epsilon}$. As noted in Ross [1972] it is, at least in theory, possible to permit disagreement on the expectations, E, and the beta coefficients of the generating model as well without substantively altering the theory. Suppose, for example, that the wealth weighted sum of ex ante expected risky returns of agents was bounded above as the number of assets is increased. This is like requiring a surrogate ex ante market return to be bounded above. It will follow that each agent's expected return is bounded above and (9.21) will hold for the ex ante values adopted by each agent. If agents agree on factor risk premiums and if the ex post generating model is now a linear combination of the individual ex ante models (e.g., if it was some sort of average), then the basic arbitrage relation will hold ex post and will be testable as it stands. It is only necessary that agents agree on *what* the factors are, not on their impact on asset returns.[27]

In sum, on theoretical grounds we would argue that the arbitrage model may be expected to be quite robust, perhaps more so than the mean variance theory. The empirical implications of the two theories and the interpretation of the available data, however, is equivocal since, as we shall see, appropriately discriminating tests have yet to be attempted.

There are at least two ways to empirically test a theory; we may examine its assumptions or its implications. Neither of the dual assumptions of the mean variance theory, normality of returns or quadratic preferences, have fared well in testing, but there is still considerable room for doubt and no clear refutation is yet available. Blume, for example, finds no clear reason to discard the hypothesis of symmetric Paretian stability for monthly relatives of stock market prices (although this is quite different from accepting normality) and, although quadratic preferences seem at odds with casual observation (e.g., they exhibit increasing absolute risk aversion and, consequently, risky assets are inferior goods (see Arrow)), locally quadratic preferences will be acceptable if returns follow a simple diffusion process and this hypothesis, unfortunately, is untestable.[28] On the other hand, there is considerable evidence to support the basic arbitrage assumption that only a small number of factors matter. Farrar found by performing a factor analysis on New York Stock Exchange data that three factors were sufficient to explain over 93 percent of the covariance matrix in a factor analytic sense.[29] When oblique or nonorthogonal factors are permitted, as in the arbitrage theory itself, Farrar claims that no more than three can be found by any empirical method.

The most intensive effort to empirically test the mean variance theory, however, has focused on the implications of theory and, in particular, on the basic pricing relation. Unfortunately, though, as a test of mean variance theory such analysis is inappropriate and as a means of discriminating between mean variance theory and arbitrage theory, with simple models, it is useless.

All current tests assume that asset returns are generated by a model of the form of (9.15). Given a one or a dependent two-factor model, though, both mean variance theory and arbitrage theory lend to the simple pricing relation of (9.14). It is only with the assumption that the generating model is more complex that the arbitrage theory is distinguishable from mean variance theory on the basis of the pricing relation. The verification of a second factor in the cross section studies of return on the beta coefficient, for example, rather than destroying all of our capital market theory would instead constitute a significant piece of evidence in support of the arbitrage theory as against mean variance theory.

This raises the question of whether in the absence of a second significant factor (with independent beta coefficients not simply equal to one minus the market betas) it is possible to test the arbitrage theory as against mean variance theory. Surely not from the pricing relation, but what then?

In fact, if the number of assets is large such a test will be very difficult. As we have shown in Ross [1971:2], with a one factor generating model and a large number of assets, mean variance theory will be approximately correct, to the same order of approximation as arbitrage theory, irrespective of the underlying distributions (provided that second moments exist) or preference structure. In effect, then, there will be no empirical basis on which to distinguish the two theories in such a world, and any test in support of one will support the other. Of course, it is still open to reject the mean variance theory on the basis of its underlying assumptions.

With an independent two factor model it is at least conceivable that mean variance theory can be rejected while accepting the arbitrage theory. Suppose, that one of the factors is the market factor with beta coefficients $<\beta_i>$ and the other factor has coefficients $<\gamma_i>$. (Both $\tilde{\epsilon}$ and the second factor are uncorrelated with the market factor.) The basic pricing relation given by arbitrage theory asserts that

$$E_i - E_0 = \gamma_i[E_1 - E_0] + \beta_i[E_m - E_0]\,. \tag{9.35}$$

If the factor term, $\gamma_i[E^1 - E_0]$ proves significant with γ_i different from $1 - \beta_i$ this will constitute a rejection of the mean variance theory in favor of the arbitrage theory.

Alternatively, suppose that neither factor is chosen to be the market factor. Now, the pricing relation becomes

$$\begin{aligned} E_i - E_0 &= \beta_{i1}(E^1 - E_0) + \beta_{i2}(E^2 - E_0) \\ &= [\beta_{i1}[\frac{E^1 - E_0}{E_m - E_0}] + \beta_{i2}[\frac{E^2 - E_0}{E_m - E_0}]](E_m - E_0). \end{aligned} \tag{9.36}$$

Mean variance theory would predict that

$$\begin{aligned} E_i - E_0 &= \frac{\mathrm{Cov}\ \tilde{x}_i, \tilde{x}_m}{\sigma_m^2}\ (E_m - E_0) \\ &= [\frac{\sigma_1^2}{\sigma_m^2}\beta_{i1} + \frac{\sigma_2^2}{\sigma_m^2}\beta_{i2}](E_m - E_0), \end{aligned} \tag{9.37}$$

where, for simplicity, we have chosen the factors to be uncorrelated so that $\sigma_m^2 = \sigma_1^2 + \sigma_2^2$ and have normalized so that the market portfolio has $\beta_{m1} = \beta_{m2} = 1$. With the normalization we also have

$$E_m - E_0 = (E^1 - E_0) + (E^2 - E_0). \tag{9.38}$$

In both theories, then, the market coefficient is a convex combination of the two beta coefficients. It follows that mean variance theory will be rejected if the estimated premium ratio

$$\frac{E^1 - E_0}{E^2 - E_0} \neq \frac{\sigma_1^2}{\sigma_2^2}, \tag{9.39}$$

the ratio of the factor variances. It should be clear that similar tests can be constructed when the factors are not orthogonal.

Is it possible to reject the arbitrage theory while accepting mean variance theory? As we have seen, not on the basis of the pricing relation itself. However, if simple factor models were rejected as generating mechanisms while an ex ante covariance matrix and expectations vector could be estimated and the market portfolio efficiency relation (9.3) were accepted (i.e., not rejected), then this would lead us to reject arbitrage theory in favor of mean variance theory while not directly testing the underlying distributional or preference assumptions of mean variance theory. Given the success of factor analytic techniques, however, such a result seems unlikely.

SECTION 9.5

In the above sections we have developed a new theory of the pricing of risky assets. The arbitrage theory was built on the solid foundation of the state space framework and both the theoretical and empirical implications of the theory in contrast to mean variance capital market theory were discussed. It was argued, in particular, that the arbitrage theory follows directly from the generating models that are commonly used to test the mean variance theory and that, as a consequence, in this context the additional assumptions of mean variance theory are not necessary. Furthermore, the arbitrage theory permitted a significant weakening of the assumption that markets were in equilibrium.

Much work, however, remains. The arbitrage theory is constructed in the tradition of Popper and a number of empirical tests have been suggested. On the theoretical side, as footnotes have indicated, it is not difficult to extend the arbitrage theory to an intertemporal context. More pressing is the need to expand, both on the theoretical and empirical fronts, the argument outlined in Section 9.4 supporting the limited dependence of asset returns. For example, if the degree of interdependence in returns is high, it will be very difficult to develop a meaningful theory of competitive (or "small") firm behavior short of assuming that firms can fully assess the complex market valuations of risky assets in the absence of complete price signals. As a final point, neither the arbitrage theory nor any other theory has made a serious attempt to describe the disequilibrium dynamic adjustment of ex post observations to ex ante assumptions. Understanding the impact of information on market adjustment will be a prerequisite for such an analysis and of great interest in its own right.

NOTES TO SECTION NINE

1. The recent work on expanding the number of moments considered seems particularly unsatisfactory; there are an infinity of orthogonal expansions which may be used to split utilities and the Taylor expansion is only one of these.

2. For the full development of the mean variance theory the reader is referred to the work of Treynor, Sharpe and Lintner and to extensions by Black. The assumption of normality rather than quadratic utility functions is in the spirit of the paper, the results are the same in either case. In continuous time, though, utility functions are locally quadratic if returns are governed by a simple diffusion process and this probably constitutes the strongest possible argument for the mean variance model. See Ross [1971:3] or Merton [in this vol.] for development of this model. However, as Merton shows, in a dynamic diffusion model, with an intertemporally stochastic environment even the diffusion assumption will be inadequate to restore the traditional capital market theory. In particular, the presence of intertemporal dynamic programming state dependencies require the incorporation of terms involving covariances with state terms in the basic pricing relation. This necessitates further restrictions of, for example, the arbitrage sort of this paper to obtain empirically interesting results.

3. Notice that here we have imposed no short sales restrictions; such restrictions on all assets, for example, would restrict α to the unit simplex.

4. There are many exceptions to the above argument; e.g., ρ could be too great to allow tangency (although not in equilibrium (see Merton [1972:1])), or F could contain a straight line segment coincident with the market line and the market portfolio would be on that segment but ambiguous (see Fama)).

5. In the absence of arbitrage opportunities, e.g., in equilibrium, all arbitrage portfolios with no net expenditure,

$$\alpha' e = 0,$$

and no risk

$$\alpha' V \alpha = 0\ ,$$

must have no return,

$$\alpha' E = 0.$$

Since V, the covariance matrix of $\tilde{x}$, is positive semidefinite, there exists C such that $V = CC'$, and $\alpha' V \alpha = 0$ is equivalent to $\alpha' C = 0$. If the matrix composed of the columns of C and e, $[C \vdots e]$, is of full rank, then (9.6) always possesses a solution. On the other hand, if $[C \vdots e]$ is of less than full rank then there will exist nontrivial arbitrage portfolios and to insure that such portfolios have no return, E must be a linear combination of e and the columns of C. Since C and $V = CC'$ span the same linear space there is, again, some α for which (9.6) holds. (This is easy to see since $\gamma' C = 0$ if and only if $\gamma' V = \gamma' CC' = 0$. The ranges of C and V are, thus, polar to the same space and since E_n is reflexive they are identical.)

6. This is in contrast to the work of Beja who derives (9.6) assuming complete contingent markets and somehow feels that it is a deep result.

7. The fact that a surrogate for the market portfolio is nearly always used in empirical work might suggest that the exact choice of α is unimportant. Nothing could be further from the truth, and the theoretical justification for the use of surrogates is found in Section 9.2 and not in the mean variance model. The empirical significance of these results is taken up in Section 9.4.

8. An alternative approach is to substitute (9.3) into (9.7) and directly test the mean variance capital market theory with a regression on returns. It is also possible for $\tilde{\delta}$ to exactly be the market portfolio, insuring that $\tilde{\epsilon}_i$ is uncorrelated with $\tilde{\delta}$ and $\beta_i = b_i$ exactly. This is done at the sacrifice of making $\tilde{\epsilon}$ singular (see Equation (9.33)).

9. What follows is not intended to be a truly rigorous exposition of arbitrage theory. Rather, our intent is to develop fully the consequences of this theoretical approach and the intuition behind it. A rigorous study of the argument together with the necessary qualifying assumptions can be found in Ross [1972] with some bounds on the approximation errors available in Ross [1971:2]. In Ross [1972] the key assumptions are that agents have a sufficient homogeneity of anticipations and that expected market return must be bounded above as the number of assets is increased. One further assumption is required. There must be at least one agent with bounded (relative) risk aversion. If not, then as assets are added and agents grow wealthier they might become increasingly risk averse at a rate that just offsets the diminishing unsystematic risk. Put simply, although the unsystematic risk is declining we must assure that at least one agent does not become increasingly concerned about it.

10. The sense of approximation can be taken as in quadratic mean or in some similar L_p metric on distribution functions. More subtly, approximation can be in the sense of any of the large number laws such as the Prohorov metric.

11. We've eliminated the condition that the portfolio be well diversified since the argument is algebraic and if the implication follows for all portfolios in some open neighborhood of a point of the order of $(\pm\frac{1}{n}, \ldots, \pm\frac{1}{n})$ it will hold on the whole space.

12. Of course, the result holds as an approximation with a finite number of assets, but we will not worry about this. See Ross [1971:2] and [1972] for a fuller discussion of the exact nature of the approximation.

13. This normalization is possible whenever $\alpha_m' \beta \neq 0$, and this will always be

true for the market portfolio if agents are risk averse. From (9.7) the return on the risky portfolio held by the ν^{th} agent is given by

$$
\begin{aligned}
\tilde{R}^{\nu} &= \alpha^{\nu}\tilde{x} \\
&= \alpha^{\nu}E + (\alpha^{\nu}\beta)\tilde{\delta} + \alpha^{\nu}\tilde{\epsilon} \\
&= E_0 + (\alpha^{\nu}\beta)[a + \tilde{\delta}] + \alpha^{\nu}\tilde{\epsilon} \\
&= E_0 + (\alpha^{\nu}\beta)[a + \tilde{\delta}],
\end{aligned}
$$

and if the ν^{th} agent is risk averse he will insist on a compensation for bearing risk (more formally it follows from the concavity of his utility function), and, therefore

$$a(\alpha^{\nu}\beta) \geqslant 0 ,$$

with strict inequality in general (except where the utility function is improper at the certain wealth level of wE_0). If ω^{ν} denotes the proportion of wealth held by the ν^{th} agent, then

$$
\begin{aligned}
E_m - \rho &= a(\alpha_m \beta) \\
&= a\sum_{\nu} \omega^{\nu}(\alpha^{\nu}\beta) \\
&> 0,
\end{aligned}
$$

verifying that the normalization is permitted and that the market portfolio earns a premium.

14. In an intertemporal model if E_i or β_i are stochastic it may be necessary to make limited dependence factor assumptions about their movement as well to obtain a useful theory. More generally we can assume that all of the stochastic parameters are governed by a generating model of the form of (9.15).

15. There is also no need to impose any restrictions on the multivariate distribution of the factors.

16. The assumption of agreement on X when states, per se, are unobservable and subjective distributions can differ is not a restriction in any meaningful sense.

17. This is formally more general than the expected utility hypothesis, but not much different in most problems. If $v(\cdot)$ is to be in the form of an expected utility we may assume in a market context that the von-Neuman Morgenstern utility function is concave. As Raiffa has pointed out this is not very restrictive since, if the noncavities of individual utility functions are bounded, individuals could "fill them in" with fair side bets, and the nonconvexities would then be irrelevant for market equilibrium. Finally, with von-Neumann Morgenstern

utilities, individuals may differ in their subjective appraisal of the probabilities that various states will occur, but we will assume that for each state there exists some individual (with positive wealth) who assigns a positive probability to its occurrence.

18. We employ the following vector notation: $x \geqslant y$ if $x_i \geqslant y_i$, $x \geqslant y$ if $x \geqslant y$ and for some $i, x_i > y_i$; and $x > y$ if for all $i, x_i > y_i$. If x and y are vectors $x'y \equiv \Sigma_i x_i y_i$, the inner product. In addition, throughout the paper we will impose no restrictions on short sales.

19. Formally, we require that

$$\begin{bmatrix} X & -I \\ e' & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

not possess a solution with y semipositive. This is equivalent to requiring that

$$\begin{bmatrix} X & -X & -I \\ e' & -e' & 0 \\ 0 & 0 & e' \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

not possess a nonnegative solution, $(\alpha_1, \alpha_2, y) \geqslant 0$. (We have simply set $\alpha = \alpha_1 - \alpha_2$ and without loss of generality required that $(\alpha_1, \alpha_2) \geqslant 0$. In addition, $e'y = 1$ insures that y is semipositive if it is nonnegative.)

By Farkas' Lemma, see, e.g., Gale, the above system will not have a solution only if its dual

$$q'X + ve' = 0,$$

$$-q' + ze' \geqslant 0,$$

$$z < 0,$$

possesses a solution (q, v, z).

To obtain (9.27) and (9.28) we define $p' \equiv \frac{1}{v} q'$. Equation (9.27) follows directly from the definition of p' and the dual equations, and since

$$-q' \geqslant -ze' > 0,$$

it only remains to show that $v > 0$. Noting that X is a semipositive matrix, $-q'X$ is also semipositive and this implies that $v > 0$.

Notice that even if the subjective probability of a state θ occurring is zero, we still have $p_\theta > 0$. If the expected utility hypothesis governs preferences, though, we would be indifferent to wealth in state θ, violating nonsatiation. In this case we can modify the analysis to eliminate states for which all agree there is no probability of occurrence. However, even if the probability is zero, the possibility, as with picking a rational with the uniform measure on the unit interval, remains. In the absence of transactions costs even the most ardent believer in expected utility would demand infinite wealth in state θ if it was free. The lack of upper semicontinuity for the demand correspondence can pose difficulties for the existence of equilibrium in more complex models.

20. Of course, this leaves open the possibility that restrictions on investor risk aversion may have further implications. These results are similar to the more general work of Sonnenschein on the implications of agent optimization for aggregate excess demand functions.

21. A typical column (asset) in (9.27) takes the form

$$E_i(p'e) + \beta_{i1}(p'\delta_1) + \ldots + \beta_{ik}(p'\delta_k) = 1 .$$

Rearranging terms yields (9.21) or the expression in the text. Notice, too, that if there is a riskless asset its rate of return,

$$\rho = 1/p'e .$$

22. We have normalized $\alpha_m'\beta = 1$.

23. It is quite easy to show this formally. Suppose, for simplicity, that there is a riskless asset and that an investor holds a portfolio with returns $(1 - \alpha_0)\rho + \alpha_0\tilde{R}_m$ where α_0 is the proportion of wealth, w, placed at risk. If $\Delta\tilde{z}$ is uncorrelated with $\tilde{R}_m$ its marginal contribution to his expected utility will be given by

$$\left.\begin{aligned} &E\left\{U'[w[(1-\alpha_0)\rho + \alpha_0\tilde{R}_m]]\,\Delta\tilde{z}\right\} \\ &= E\left\{U'[w[(1-\alpha_0)\rho + \alpha_0\tilde{R}_m]]E\left\{\Delta\tilde{z}\right\}\right. \end{aligned}\right\}$$

if $(\tilde{R}_m, \Delta\tilde{z})$ are jointly normal or if $U(\cdot)$ is quadratic.

24. See Footnote (9) and Ross [1972] for a detailed study of the assumptions on which arbitrage theory is based.

25. If the shifts were random and independent then it would not matter, i.e., they would be part of the $\tilde{\epsilon}_i$ terms. It should also be noted that to the extent that the ex ante distributions held by agents differ from the ex post distributions a systematic factor similar to an errors in variable argument can enter. I am indebted to Ferdinand Vaandrager for help on this point.

26. See Lintner [1969] or Ross [1971:1] for some examples.

27. To make this point algebraically, let agents be indexed by "ν" and suppose the ν^{th} agent believes ex ante returns are generated by the model

$$x_i^\nu = E_i^\nu + \beta_i^\nu\delta^\nu + \epsilon_i^\nu , \tag{F1}$$

where the assumptions (9.15) are assumed to hold. Under the conditions described in the text (see Ross [1972]) the boundedness of a market return surrogate will imply that for each ν, the arbitrage condition will hold,

$$E_i^{\nu} = \rho + \lambda^{\nu}\beta_i^{\nu};\ \lambda^{\nu} = E_m^{\ \nu} - \rho\ , \tag{F2}$$

where we have assumed a riskless asset and ignored the approximation. If the true ex post model can be obtained from (F1) by aggregating with weights γ_ν, then true ex post

$$\begin{aligned}\tilde{x}_i &= \sum_{\nu} \gamma_\nu \tilde{x}_i^{\nu} \\ &= \sum_{\nu} \gamma_\nu E_i^{\nu} + \sum_{\nu} \gamma_\nu \beta_i^{\nu} \tilde{\delta}^{\nu} + \Sigma_\nu \gamma_\nu \tilde{\epsilon}_i^{\nu} \\ &\equiv E_i + \beta_i \tilde{\delta} + \tilde{\epsilon}_i,\end{aligned}$$

assuming that $\tilde{\delta}_\theta^{\ \nu} = \tilde{\delta}_\theta$. If the risk premiums, $\lambda^\nu = \lambda$ agree then (F2) aggregates to

$$E_i = \rho + \lambda\beta_i$$

which is now directly testable. To generalize the result, we can scale the $\tilde{\delta}^\nu$ factors, rescaling the β_i^ν variates accordingly, so that $\lambda^\nu = \lambda$ trivially. The requirement that $\tilde{\delta}_\theta^{\ \nu} = \tilde{\delta}_\theta$ is far more subtle, but if the subjectively perceived factor random variables have the same range it will be possible to define them on an underlying state space domain such that $\tilde{\delta}_\theta^{\ \nu} = \tilde{\delta}_\theta$ for all ν; disagreement is entirely captured by the subjective distribution over states. If the state space itself has economic content and is observable this will not be possible.

In all fairness to mean variance theory much the same sort of aggregation is also possible there, but if, as is the spirit of the theory, covariance matrices are specified a priori and not merely covariances with the market, considerably more agreement is required to permit the argument to go through.

28. Although even this is insufficient to derive (9.3). See Footnote 2. In an important work Clark, using cotten futures data, has rejected the hypothesis that these option prices are stable Paretian in favor of a subordinated normal. Such results are not, however, inconsistent with the arbitrage theory.

29. Farrar actually worked with 47 industrial equity groupings prepared by Standard and Poor. We have not, however, performed significance tests on the factor analytic results and, perhaps, by assuming that the covariance matrix follows a Wishart distribution it would be possible to test the simple factor models. In particular, one could see how much of the variance the one, two and so forth best factors could be expected to remove.

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