Econometría Financiera

Karoll GOMEZ
kgomezp@unal.edu.co
https://sites.google.com/unal.edu.co/karollgomezportilla

Primer semestre 2021

UNIVERSIDAD NACIONAL DE COLOMBIA

2. Volatility Modeling

Introduction I

A. Main characteristics of volatility:

- Volatility is a statistical measure of dispersion of returns for a given security or market index.
- It is not directly observable.

Crucially, variances, covariances, and **all higher order moments** (skewness, kurtosis, etc.) **are not directly observable**—unlike market prices, they are **latent variables**

- E.g., if prices fluctuate a lot, we know volatility is high, but we cannot ascertain precisely how high
- Can't distinguish if a large shock to prices is transitory or permanent

Introduction II

Stylized facts of volatility:

① Financial data tend to be leptokurtic, their unconditional density is characterized by tails that are "thicker" than a normal well as by more probability mass collected around the mean (or the mode)

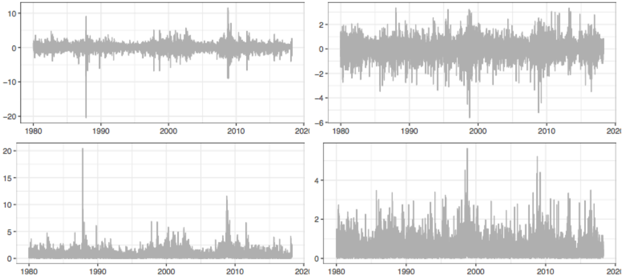
② Data are characterized by clusters in higher order moments, especially volatility: large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes

- One typical way in which volatility clustering is analyzed is by conducting Box-Jenkins analysis on the squared residuals of some conditional mean function (or on returns themselves)...
- ... or by studying the cross-serial correlation of powers of the data

Introduction III

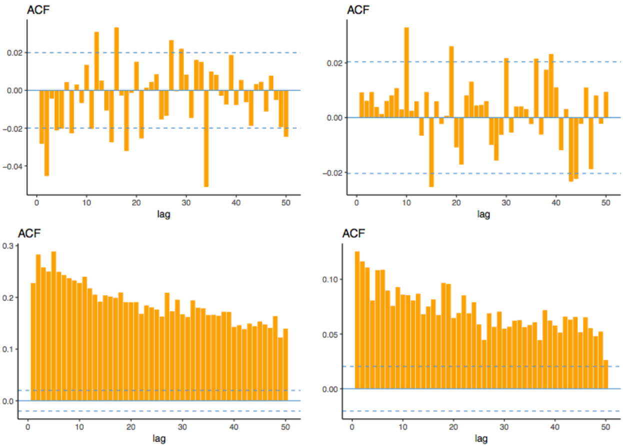
The Figure below shows the *returns* and *absolute returns* starting in Janaury 1980 at the daily frequency for:

- S&P 500 Index returns (left)
- The JPY/USD exchange rate (right)



Introduction IV

The ACF of the returns and absolute returns up to lag 50
(left: S&P; right: JPY/USD)

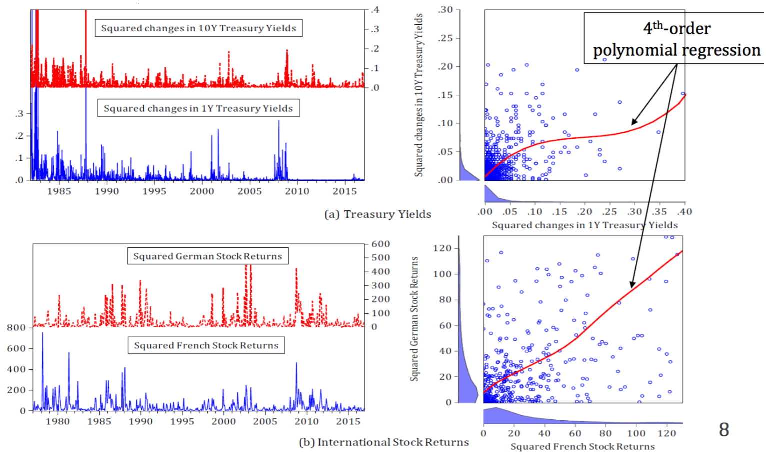


Introduction V

③ The so-called leverage effect, first reported by Black (1976), the tendency for changes in (stock) prices to be negatively correlated with changes in subsequent (stock) volatility

④ Co-movements in volatilities across different stocks, maturities of bonds issued by the same company or institution, different pairs of currencies, and even across alternative markets or asset classes

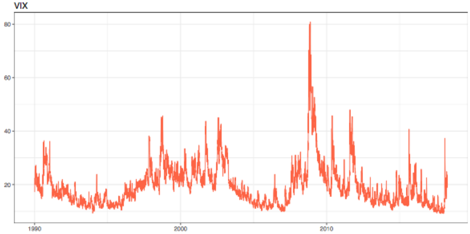
Introduction VI



Introduction VII

The CBOE VIX Index:

- The *CBOE Volatility Index* (`VIX`) is a measure of market uncertainty that is constructed from implied volatilities of S&P 500 Index options
- It is widely used as a proxy for market volatility



- How can we construct a measure of volatility (similar to `VIX`) for any asset we are interested in?

Introduction VIII

A Return Model:

- The AR(1) assumes that $R_{t+1} = \beta_0 + \beta_1 R_t + \epsilon_{t+1}$, where:
  - $E_t(R_{t+1}) = \beta_0 + \beta_1 R_t$
  - $Var_t(R_{t+1}) = \sigma^2$
- The $E_t(\cdot)$ and $Var_t(\cdot)$ represent the expected return of R_{t+1} *conditional* on the information available at time t
- The evidence in the previous graph indicates that this might not be a realistic assumption because returns are *heteroskedastic* (i.e., the variance and the standard deviation change over time)
- In other words, we need $Var(\epsilon_t)$ to be σ_t^2 instead of σ^2
- Why should we care about modeling the time variation of the standard deviation σ_t^2? What is the usefulness of forecasting volatility instead of or in addition to forecasting returns? Numerous applications in finance:
  - pricing derivatives
  - measuring risk (Value-at-Risk)
  - portfolio allocation

Introduction IX

A Volatility Model:

- We assume that returns follow the model

$$R_{t+1} = \mu_{t+1} + \eta_{t+1}$$

  that decomposes the return in two components:

  - *Expected return* μ_{t+1}: the predictable componenent of the return; it can be assumed equal to 0, equal to a constant μ, or following an AR(1) process $\mu_{t+1} = \phi_0 + \phi_1 * R_t$

  - *Shock* η_{t+1}: the unpredictable component of the return; we can assume that it is equal to $\eta_{t+1} = \sigma_{t+1}\epsilon_{t+1}$ where:

    - *Volatility*: σ_{t+1} is the standard deviation of the error at time t which we assume varies over time; for example, $\sigma_{t+1} = \omega + \alpha R_t^2$
    - *Standardized shock*: ϵ_{t+1} represents an unpredictable error term or a shock with mean zero and variance 1 (additionally, we can assume that it is normally distributed)

Introduction X

- Both μ_{t+1} and σ_{t+1} can be assumed to be functions of past values
- At the market closing in day t I can then calculate the forecast of the return the following day, μ_{t+1}, and the expected volatility the following day, σ_{t+1}

- At high frequencies (daily or intra-daily) there is typically very little predictability in the mean but significant predictability in the conditional variance; it is thus reasonable to assume that $\mu_{t+1} = 0$ (of course, we can statistically test this hypothesis ...)

Introduction XI

How to model the volatility process σ_t?

We consider different approaches

I. Unconditional measures
   - B.1 Quadratic and absolute returns
   - B.2 Realised volatility
   - B.3 Moving Average (MA)
   - B.4 Exponential Moving Average (EMA)

II. Conditional volatility measures
   - C.1 Generalized Auto-Regressive Conditional Heteroskedasticity (GARCH) model
   - C.2 Extension to GARCH model

Historical Volatility I

I. Unconditional Measures:

B. Historical Volatility Measures:

- Historical volatility is a statistical measure of the dispersion of returns for a given security or market index over a given period of time.
- The higher the historical volatility value, the riskier the security.
- Historical volatility is backward looking rather than forward looking.

Historical Volatility II

B.1 Squared (absolute) returns

$$\sigma_t^2 = \frac{1}{\tau_j - 1} \sum_{i=1}^{\tau_j} (r_{t+i} - \mu)^2 \quad j = 1, ..., n \quad i = 1, ..., \tau_j$$

where:
$\tau = T - t$ is the period between T and t,
$T - t = (T_n - Tn - 1, T_{n-1} - Tn - 2..., T_1 - t) =$
$\tau_n + \tau_{n-1} + ... + \tau_1 = \sum_{j=1}^{n} \tau_j$, so τ_j it is the period to which the volatility estimate is applied.
and μ is the average return over the τ period.

- However, Figlewski (1997) noted that the sample mean μ is a very inaccurate estimate of the true mean especially for small samples; taking deviations around zero instead of the sample mean typically increases volatility forecast accuracy.

Historical Volatility III

- Many financial time series are available at the daily interval, while the volatility reference period, τ_j , could vary from 1 to 10 days s (for risk management), months (for option pricing) and years (for investment).
- When monthly volatility is required and daily data is available, volatility can simply be calculated based on equation above, where τ_j is one month, and r_{t+i} for $i = 1, ..., \tau_j$ are the daily observations in that month.
- Since variance is linear in time and can be aggregated but not standard deviation. So, under the assumption that volatility remains constant we can compute $\sigma_w^2 = 5 * \sigma_d^2$, with a multiplier of 5 since there are 5 trading days in a week.
- To derive volatility, which is often linked to the standard deviation, we have the weekly volatility $\sigma_w = \sqrt{\sigma_w^2}$
- The use of daily absolute return to proxy daily volatility will produce a very noisy volatility estimator.

Historical Volatility IV

B.2 Realized volatility

$$RV_t = 100 * \sqrt{\frac{252}{n} \sum_{t=1}^{T} (r_t)^2}$$

- However, more recently and with the increased availability of tick data (i.e. prices recorded at transaction level), the term realised volatility is now used to refer to volatility estimates calculated as the sum of intraday squared returns at short intervals such as 5 or 15 minutes (see Fung and Hsieh (1991) and Andersen and Bollerslev (1998)).

Historical Volatility V

- Intraday trading data allows results to come closer to capturing continuous time volatility

$$RV_t = \sum_{j=1}^{m} r^2_{m,t+j/m}$$

  where m is the sampling frequency within each period t; i.e. there are m continuously compounded returns between $t-1$ and t so that $r_{m,t+1/m} = p_{t+1/m} - p_t$; $r_{m,t+2/m} = p_{t+2/m} - p_{t+1/m}$
- The underlying assumption is that the discretely sampled returns are serially uncorrelated, i.e for a series that has zero mean and no jumps.
- If there is a jump Barndorff-Nielsen and Shephard (2003) show that volatility can be measures using the standardised realised bipower variation measure.

Historical Volatility VI

- Characteristics of financial market data suggest that returns measured at an interval shorter than 5 minutes are plagued by spurious serial correlation caused by various market microstructure effects including nonsynchronous trading, discrete price observations, intraday periodic volatility patterns and bid-ask bounce.
- Bollen and Inder (2002), Ait-Sahalia, Mykland and Zhang (2003) and Bandi and Russell (2004) gave suggestions on how to isolate microstructure noise from realised volatility estimator.
- The forecast of multi-period volatility $\sigma_{t,t+\tau} = \sum_{s=1}^{\tau} h_{t+\tau|t}$ is taken to be the sum of individual multi-step point forecasts, where $h_{t+\tau|t}$ denotes a volatility forecast formulated at time t for volatility over the period from t to τ.
- If returns are iid (independent and identically distributed, or strict white noise), then variance of returns over a long horizon can be derived as a simple multiple of single period variance.

B.3 Moving Average (MA) Volatility

- The **MA model** consists of averaging the square returns on a window of the latest M days (where M denotes the window size)

- The estimate of the variance in day t is σ_t^2 and it is given by

$$\sigma_{t+1}^2 = \frac{1}{M}\left(R_t^2 + R_{t-1}^2 + \cdots + R_{t-M+1}^2\right) = \frac{1}{M}\sum_{j=1}^{M} R_{t-j+1}^2$$

- Dependence on M:
  - $M = 1$: $\sigma_{t+1}^2 = R_t^2$
  - $M = t$: $\sigma_{t+1}^2 = \sigma^2$
- Using the squared returns in the summation relies on the assumption that $\mu_{t+1} = 0$ so that $R_{t+1} = \eta_{t+1}$
- If $\mu_{t+1} \neq 0$ then we apply MA on $\eta_{t+1} = R_{t+1} - \mu_{t+1}$ that is $\sigma_{t+1}^2 = \left(\sum_{j=1}^{M} \eta_{t-j+1}^2\right)/M$

Historical Volatility VIII

- The MA estimate of volatility can be interpreted as a weighted average of the square returns

$$\sigma^2_{t+1} = \sum_{j=1}^{t} w_j R_j^2$$

where the weight w_j of day j takes the following form:

  - $w_j = 1/M$ if $(t - M + 1) \leq j \leq t$
    (ex: for $M = 25$ $w_j = 1/25 = 0.04$ or 4%)
  - $w_j = 0$ if $j < (t - M + 1)$

- The MA approach can be implemented in **R** using the **rollmean()** function in package **zoo** which takes as arguments:
  - the window size M
  - if the estimate should be aligned to the left, center, or right of the window

- The effect of increasing the window size M is to provide a smoother (slowly changing) estimate of volatility σ_{t+1} since each daily return receives a smaller weight

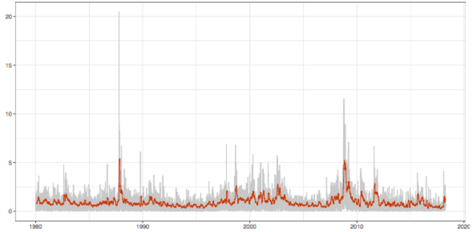
Historical Volatility IX

- The MA approach is very simple to calculate and implement which is convenient when dealing with a large number of assets
- A drawback of the approach is that it is sensitive to large returns: when a large return enters/exits the estimation window M the estimate σ^2_{t+1} has a upward/downward jump
- How to choose M? Practitioners use $M = 25$ (one trading month) as an ad hoc value rather than being chosen optimally

Historical Volatility X

Example:

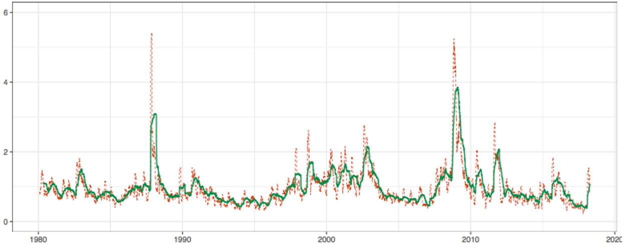
```
GSPC              <- getSymbols("^GSPC", from="1980-01-01", auto.assign=FALSE)
sp500daily        <- 100 * ClCl(GSPC) %>% na.omit
names(sp500daily) <- "RET"
sigma25           <- zoo::rollmean(sp500daily^2, 25, align="right")
names(sigma25)    <- "MA25"
ggplot(merge(sigma25, sp500daily)) + geom_line(aes(time(sp500daily), abs(RET)),color="gray80") +
  theme_bw() + geom_line(aes(index(sp500daily), MA25^0.5), color="orangered3")  +
  labs(x=NULL, y=NULL)
```



Historical Volatility XI

- Below a comparison of $M = 25$ and 100

```
sigma100 <- zoo::rollmean(sp500daily^2, 100, align="right")
names(sigma100) <- "MA100"
ggplot(merge(sigma25, sigma100)) +
  geom_line(aes(time(sigma25), MA25^0.5),color="orangered3", size=0.3, linetype="dashed") +
  theme_bw() +  geom_line(aes(index(sigma25), MA100^0.5), color="springgreen4", size=0.6) +
  labs(x=NULL, y=NULL) + ylim(c(0, 6))
```



Historical Volatility XII

B.4 Exponential Moving Average (EMA) or JP Morgan's RiskMetrics Volatility

- An alternative approach is the EMA approach which solves the problem of the sensitivity to large returns
- EMA is calculated as follows:

$$\sigma_{t+1}^2 = \lambda * R_t^2 + \lambda(1-\lambda) * R_{t-1}^2 + \lambda(1-\lambda)^2 * R_{t-2}^2 + \ldots$$

$$= \lambda \sum_{j=0}^{\infty} (1-\lambda)^j R_{t-j}^2$$

where λ is a smoothing parameter (equivalent of M for MA)

Historical Volatility XIII

- Based on the formula for σ^2_{t+1}, its lagged value σ^2_t is given by

$$\sigma^2_t = \lambda * R^2_{t-1} + \lambda(1-\lambda) * R^2_{t-2} + ...$$

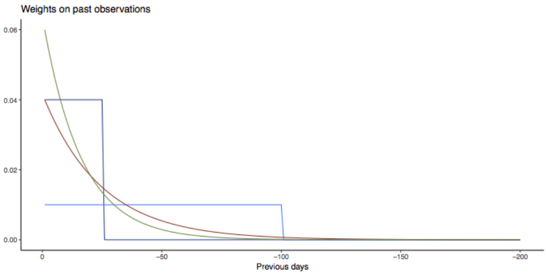
- This expression for σ^2_{t-1} can be used to rewrite σ^2_t as:

$$\begin{aligned}\sigma^2_{t+1} &= \lambda * R^2_t + \lambda(1-\lambda) * R^2_{t-1} + \lambda(1-\lambda)^2 * R^2_{t-2} + \cdots \\ &= \lambda * R^2_t + (1-\lambda) * \left(\lambda R^2_{t-1} + \lambda(1-\lambda) R^2_{t-2} + \cdots\right) \\ &= \lambda * R^2_t + (1-\lambda)\sigma^2_t\end{aligned}$$

  which shows that the current volatility estimate is given by a weighted average of the previous day estimate (σ^2_t) and the current day square return (R^2_t)

- The weight function w_j for EMA is $w_j = \lambda * (1-\lambda)^j$; compared to the MA weight function:
  - It weighs all observations (not only the M most recent)
  - The weighs are (smoothly) declining the farther in the past

Historical Volatility XIV



- Can you match the color to the method?
  - MA(25), MA(100), EMA(0.06), and EMA(0.04)
  - **`darkolivegreen4` `royalblue1`**, **`tomato4`**, and **`royalblue4`**

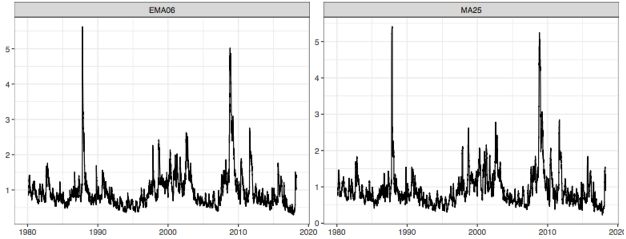
- Practitioners have found that a value of λ of 0.06 for EMA *works well* for a large set of assets

Historical Volatility XV

Example:

- The package **TTR** (part of **quantmod**) provides functions to calculate moving averages, both simple and of the exponential type as in code below:

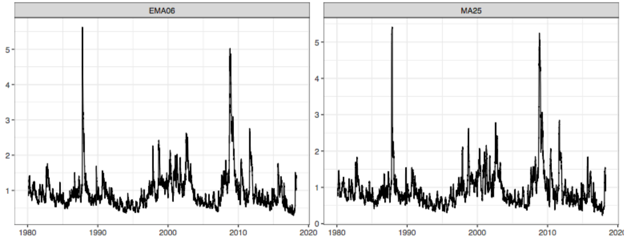
```
library(TTR)
# SMA() function for Simple Moving Average; n = number of days
ma25  <- SMA(sp500daily^2, n=25)
names(ma25) <- "MA25"
# EMA() function for Exponential Moving Average; ratio = lambda
ema06 <- EMA(sp500daily^2, ratio=0.06)
names(ema06) <- "EMA06"
autoplot(merge(ma25, ema06)^0.5, ncol=2) + theme_bw()
```



Historical Volatility XVI

- The package **TTR** (part of **quantmod**) provides functions to calculate moving averages, both simple and of the exponential type as in code below:

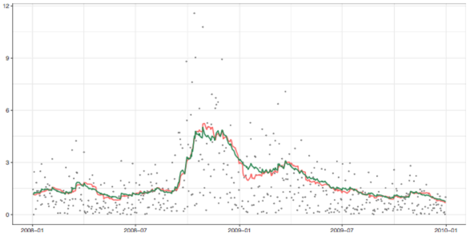
```
library(TTR)
# SMA() function for Simple Moving Average; n = number of days
ma25  <- SMA(sp500daily^2, n=25)
names(ma25) <- "MA25"
# EMA() function for Exponential Moving Average; ratio = lambda
ema06 <- EMA(sp500daily^2, ratio=0.06)
names(ema06) <- "EMA06"
autoplot(merge(ma25, ema06)^0.5, ncol=2) + theme_bw()
```



Historical Volatility XVII

- Comparison of the MA and EMA volatility estimators between 2008 and 2010 (gray lines are the absolute returns)

```
temp <- merge(sp500daily, ma25, ema06) %>% window(., start="2008-01-01", end="2009-12-31")
ggplot(temp) + geom_point(aes(time(temp), abs(RET)), color="gray45", size=0.4) +
  geom_line(aes(time(temp), MA25^0.5), color="indianred1", size = 0.8) +
  geom_line(aes(time(temp), EMA06^0.5), color="seagreen4", size = 0.8) +
  theme_bw() + labs(x=NULL, y=NULL)
```



Historical Volatility XVIII

Advantages of RiskMetrics

- It tracks variance in a way is broadly consistent with observe returns: recent returns matter more for tomorrow's variance than distant returns
- It contains only one unknown parameter
- Relative little data needs to be stored in order to calculate tomorrow's variance. We only need about 100 daily lags of returns
- However, it does not allow for leverage effects for instance.

Problems:

- The MA and EMA are simple tools that work in many situations of practical relevance, in particular forecasting volatility at short horizons
- However, they require to introduce some assumptions that might be questionable:
  - ad hoc choice of the parameters (M and λ) rather than being estimated
  - volatility is non-stationary (random walk)

II. Conditional Measures:

C. Conditional Heteroskedasticity models:

- GARCH models were introduced by Engle (1982) and Bollerslev (1986) under the goal to specify and estimate models for risk.
- Until 30 years ago the focus of time series analysis centered on conditional first moment, with any dependence in higher order moments treated as a nuisance that required at best adjustment to estimation.
- But data display patterns of time-varying variances and covariances, so they are said to be conditionally heteroskedastic.
- The first method used to identified risk was based on unconditional moments (standard deviation $\Rightarrow$ volatility)
- However, variance becomes a proper measure of risk only when coupled with an assumption on predictive densities:
  - When returns are normal

Conditional Volatility II

C.1 ARCH and GARCH Models

- Assume for now that $\mu_{t+1} = 0$ so that $R_{t+1} = \eta_{t+1}$
- The **ARCH** model assumes that the conditional variance σ^2_{t+1} is a function of the current squared return, that is,

$$\sigma^2_{t+1} = \omega + \alpha * R^2_t$$

  where ω and α are parameters to be estimated.
- The model above is called ARCH(1) since only one lag is included, but it can be generalized to include p lagged returns
- A more general specification is the Generalized ARCH (**GARCH**) model which is characterized by the following Equation for the conditional variance:

$$\sigma^2_{t+1} = \omega + \alpha * R^2_t + \beta * \sigma^2_t$$

  where, the previous variance forecast σ^2_t is included in addition to the current return R^2_t
- We typically refer to the previous model as ARCH(1) and GARCH(1,1) since we included one lag of R^2_t and σ^2_t; this can be generalized to more lags

Conditional Volatility III

- We make a distinction between:
  - σ^2: the *unconditional* variance (i.e. the long-run variance)
  - σ^2_{t+1}: the *conditional* variance (i.e. based on information available at time t)
- They measure the volatility that you expect by holding the asset for a long-period (e.g., 20 years) vs holding the asset for a short period (e.g., one day)
- The unconditional mean of the conditional variance, $E(\sigma^2_{t+1}) = \sigma^2$, for the GARCH(1,1) is given by:

$$\sigma^2 = \frac{\omega}{1 - (\alpha + \beta)}$$

- The unconditional variance of the returns is finite if $\alpha + \beta < 1$.
- On the other hand, if $\alpha + \beta = 1$ the long-run variance is not finite and variance is non-stationary (as for EMA)

Conditional Volatility IV

- What does it mean that variance (or volatility) is non-stationary? Do we expect the variance to be mean reverting?
- If volatility is mean-reverting it means that:
  - σ^2_{t+1} oscillates (more/less persistently) around its unconditional mean σ^2
  - Periods of high volatility (higher than σ^2) will be followed by periods of low volatility (lower than σ^2)
- This is consistent with the discussion in the first slide in which we observe returns alternating between periods of high and low volatility (volatility clustering)

Conditional Volatility V

- On the other hand, non-stationary volatility means that we **expect** variance (or volatility) to stay at the current level *forever*
- This implication seems at odds with the observation that volatility alternates between periods of high and low volatility
- Although the forecasts assuming $\alpha + \beta = 1$ might be as accurate as those from models assuming $\alpha + \beta < 1$ in the short-run, they might differ significantly in the long-run
- When do we need accurate long-run forecast of volatility? Pricing long-dated options, investment allocation, etc.

Conditional Volatility VI

Example:

$$\begin{aligned} R_{t+1} &= \mu_t + \sigma_{t+1} z_{t+1} \qquad z_{t+1} \sim \text{IID}\, \mathcal{N}(0,1), \\ \sigma_{t+1}^2 &= \omega + \alpha \left(R_t - \mu_t\right)^2 + \beta \sigma_t^2 \\ \alpha + \beta &< 1 \end{aligned}$$

where returns have a constant mean (that is usually zero) and a time varying GARCH(1,1) structure.

Conditional Volatility VII

R_{t+1} has a finite unconditional long-run variance of $\frac{\omega}{1-\alpha-\beta}$

$$\begin{aligned}
\sigma^2 &= E\left(\sigma_{t+1}^2\right) = \omega + \alpha E\left(R_t - \mu\right)^2 + \beta\sigma^2 \\
&= \omega + \alpha\sigma^2 + \beta\sigma^2 \\
&= \frac{\omega}{1-\alpha-\beta}
\end{aligned}$$

Substituting ω out of the GARCH expression:

$$\begin{aligned}
\sigma_{t+1}^2 &= \left(1-\alpha-\beta\right)\sigma^2 + \alpha R_t^2 + \beta\sigma_t^2 \\
&= \sigma^2 + \alpha\left(\left(R_t-\mu\right)^2 - \sigma^2\right) + \beta\left(\sigma_t^2 - \sigma^2\right)
\end{aligned}$$

which illustrates the relation between predicted variance and long-run variance in a GARCH model.

Conditional Volatility VIII

A GARCH(1,1) model can be considered as the equivalent of an ARMA(1,1) model for the variance, More generally, in the ARMA(q,p) case, we have:

$$\begin{aligned} \sigma_{t+1}^2 &= \omega + \sum_{i=1}^{q} \alpha_i \left(R_{t+1-i}^2 - \mu\right)^2 + \sum_{j=1}^{p} \beta_j \sigma_{t+1-j}^2. && (1) \\ \bar{\sigma}^2 &= \frac{\omega}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{j=1}^{p} \beta_j} && (2) \end{aligned}$$

Because unconditional variance exists only if $\bar{\sigma}^2 > 0$, the equation above implies that when $\omega > 0$, the condition

$$1 - \sum_{i=1}^{q} \alpha_i - \sum_{j=1}^{p} \beta_j > 0 \Longrightarrow \sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$$

must hold. When the long-run variance of a GARCH process exists, we say that the GARCH process is stationary and we refer to the condition $\sum_{i=1}^{q} \alpha_i + \sum_{j=1}^{p} \beta_j < 1$ as a stationarity condition.

Conditional Volatility IX

MLE estimation:

The assumption of IID normal shocks (z_t),

$$R_{t+1} = \sigma_{t+1} z_{t+1} \qquad z_{t+1} \sim \text{IID } \mathcal{N}(0,1),$$

implies (from normality and identical distribution of z_{t+1}) that the density of the time t observation is:

$$l_t \equiv \Pr(R_t; \boldsymbol{\theta}) = \frac{1}{\sigma_t(\boldsymbol{\theta})\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{R_t^2}{\sigma_t^2(\boldsymbol{\theta})}\right),$$

where the notation $\sigma_t^2(\boldsymbol{\theta})$ emphasizes that conditional variance depends on $\boldsymbol{\theta} \in \Theta$.

Because each shock is independent of the others (from independence over time of z_{t+1}), the total probability density function (PDF) of the entire sample is then the product of T such densities:

$$L(R_1, R_2, ..., R_T; \boldsymbol{\theta}) \equiv \prod_{t=1}^{T} l_t = \prod_{t=1}^{T} \frac{1}{\sigma_t(\boldsymbol{\theta})\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{R_t^2}{\sigma_t^2(\boldsymbol{\theta})}\right).$$

taking logs

$$\mathcal{L}(R_1, R_2, ..., R_T; \boldsymbol{\theta}) = -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log\sigma_t^2(\boldsymbol{\theta}) - \frac{1}{2}\sum_{t=1}^{T}\frac{R_t^2}{\sigma_t^2(\boldsymbol{\theta})}$$

Conditional Volatility XI

Substituting an expression for $\sigma_t^2(\boldsymbol{\theta})$ (given by the chosen GARCH specification) given the observations on the returns and given an initial observation for variance

$$
\begin{aligned}
\mathcal{L}(R_1, R_2, ..., R_T; \boldsymbol{\theta}) &= -\frac{T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log\left[\omega + \alpha R_{t-1}^2 + \beta\sigma_{t-1}^2\right] \\
&\quad -\frac{1}{2}\sum_{t=1}^{T}\frac{R_t^2}{\omega + \alpha R_{t-1}^2 + \beta\sigma_{t-1}^2}, \\
\sigma_0^2 &= \frac{\omega}{1-\alpha-\beta}
\end{aligned}
$$

maximizing the log-likelihood to select the unknown parameters will deliver the MLE, denoted as $\hat{\boldsymbol{\theta}}_T^{ML}$

Conditional Volatility XII

QMLE estimation:

- The QMLE result says that we can still use MLE estimation *based on normality assumptions* even when the shocks are not normally distributed, if our choices of conditional mean and variance function are defendable, at least in empirical terms (i.e. conditional mean and conditional variance are correctly specified).
- However, because the maintained model still has that $R_{t+1} = \sigma_{t+1} z_{t+1}$ with $z_{t+1} \sim$ IID $\mathcal{D}(0,1)$, the shocks will have to be anyway IID: you can just do without normality, but the convenience of $z_{t+1} \sim$ IID $\mathcal{D}(0,1)$ needs to be preserved.

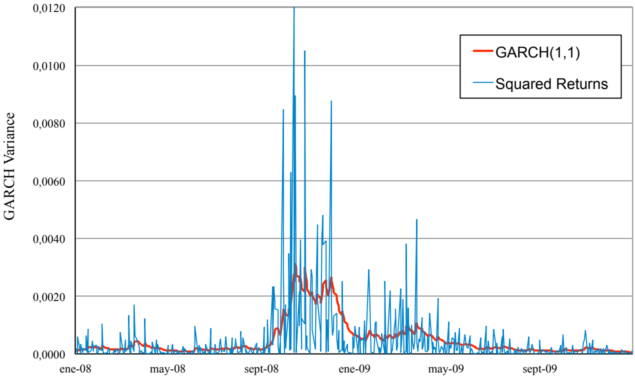
Conditional Volatility XIII

Forecasting:

$$
\begin{aligned}
\sigma^2_{t+1|t} &= \bar{\sigma}^2 + \alpha\left[(R_t-\mu_t)^2 - \bar{\sigma}^2\right] + \beta\left(\sigma_t^2 - \bar{\sigma}^2\right), \\
\sigma^2_{t+2|t} &= \bar{\sigma}^2 + (\alpha+\beta)\,\sigma^2_{t+1|t} \\
\sigma^2_{t+n+1|t} &= \bar{\sigma}^2 + (\alpha+\beta)^n\,\sigma^2_{t+1|t}
\end{aligned}
$$

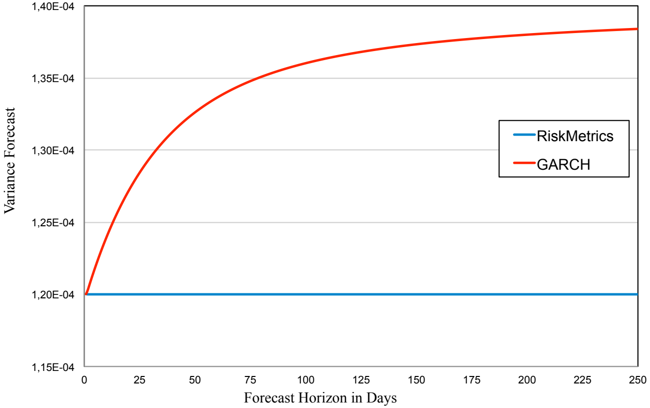
Conditional Volatility XIV

Squared S&P500 Returns vs GARCH



Conditional Volatility XV

Variance forecast for 1-250 days Cumulative Returns



Conditional Volatility XVI

GARCH in R: An exercise

- The package **fGarch** provides functionalities to estimate a wide array of GARCH models
- The function **garchFit** estimates by ML a model specified by the user:
  - **arma(p,q)** for the conditional mean (typically **arma(0,0)** or **arma(1,0)**)
  - **garch(r,s)** for the conditional variance (typically **garch(1,1)**)
- The code below estimates a GARCH(1,1) model:

```
require(fGarch)
fit <- garchFit(~garch(1,1), data=sp500daily, trace=FALSE)
round(fit@fit$matcoef, 3)
```

```
       Estimate  Std. Error  t value Pr(>|t|)
mu        0.060       0.008    7.431        0
omega     0.014       0.002    7.184        0
alpha1    0.084       0.006   14.389        0
beta1     0.905       0.007  135.880        0
```

- $\alpha + \beta = 0.988$ is very close to one and indicates that the EMA assumptions is likely to hold
- However, while practitioners use λ equal to 0.06 the estimation on our sample suggests a higher value of 0.084

Conditional Volatility XVII

- We can add an AR(1) term to the conditional mean, while the conditional variance remains of the GARCH(1,1) type:

```
fit <- garchFit(~ arma(1,0) + garch(1,1), data=sp500daily, trace=FALSE)
round(fit@fit$matcoef, 3)
```

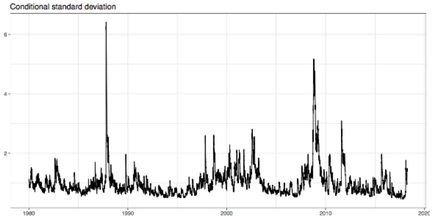
```
       Estimate  Std. Error  t value Pr(>|t|)
mu        0.060       0.008    7.395    0.000
ar1       0.003       0.011    0.244    0.807
omega     0.014       0.002    7.184    0.000
alpha1    0.084       0.006   14.389    0.000
beta1     0.905       0.007  135.842    0.000
```

- the AR(1) term is not significant
- The estimates of α and β are the same as in the case of just the intercept

Conditional Volatility XVIII

- Once we have the coefficient estimates we can calculate the estimate of the conditional variance as $\sigma^2_{t+1} = \hat{\omega} + \hat{\alpha} R^2_t + \hat{\beta}\sigma^2_t$
- The function calculates σ_t as output of the estimation step

```
sigma <- fit@sigma.t  # class is numeric
qplot(time(sp500daily), sigma, geom="line", xlab=NULL, ylab="") +
  theme_bw() + labs(title="Conditional standard deviation")
```



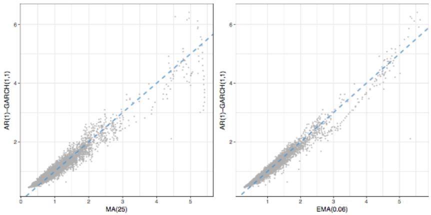
Conditional Volatility XIX

- If we set $\omega = 0$, $\alpha = \lambda$ and $\beta = 1 - \lambda$ in the GARCH conditional variance equation we obtain
$$\sigma_{t+1}^2 = \lambda R_t^2 + (1 - \lambda)\sigma_t^2$$
which is the conditional variance for EMA
- So EMA is a restricted version of a GARCH model in which α and β are restricted to be equal to λ and $1 - \lambda$ so that ...
- ... $\alpha + \beta = \lambda + 1 - \lambda = 1$ means that EMA volatility is non-stationary
- Empirically, this hypothesis can be tested by assuming the null hypothesis $\alpha + \beta = 1$ versus the one sided hypothesis that $\alpha + \beta < 1$.

Conditional Volatility XX

- We considered three methods for modeling and forecasting volatility: MA, EMA, and GARCH
- How different are the volatility forecasts obtained from these three methods?
- Correlations: cor(MA,GARCH) = 0.967 and cor(EMA, GARCH) = 0.981
- Scatter plots:

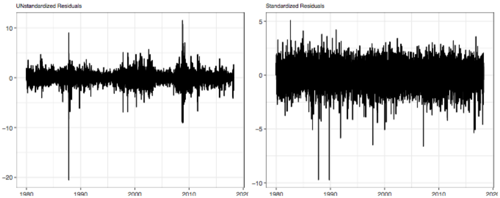
```
p1 <- ggplot() +
  geom_point(aes(ma25^0.5, sigma), color="gray70", size=0.3) +
  geom_abline(intercept=0, slope=1, color="steelblue3", linetype="dashed", size=0.8) +
  theme_bw() + labs(x="MA(25)",y="AR(1)-GARCH(1,1)")
p2 <- ggplot() +
  geom_point(aes(ema06^0.5, sigma), color="gray70", size=0.3) +
  geom_abline(intercept=0, slope=1, color="steelblue3", linetype="dashed", size=0.8) +
  theme_bw() + labs(x="EMA(0.06)",y="AR(1)-GARCH(1,1)")
gridExtra::grid.arrange(p1,p2, ncol=2)
```



Conditional Volatility XXI

- The residuals of the GARCH model are $\eta_{t+1} = \sigma_{t+1}\epsilon_{t+1}$
- According to our assumptions they should be:
  - independent over time (i.e., no auto-correlation)
  - normally distributed (i.e., we made this assumption)
- Two types of residuals:
  1. *unstandardized residuals*: η_{t+1}
  2. *standardized residuals*: $\epsilon_{t+1} = \eta_{t+1}/\sigma_{t+1}$

```
res <- fit@residuals  # class is numeric
p1  <- qplot(time(sp500daily), res, geom="line", main="UNstandardized Residuals") +
  labs(x=NULL, y=NULL) + theme_bw() + theme(plot.title = element_text(size = 8))
p2  <- qplot(time(sp500daily), res/sigma, geom="line", main="Standardized Residuals") +
  labs(x=NULL, y=NULL) + theme_bw() + theme(plot.title = element_text(size = 8))
gridExtra::grid.arrange(p1, p2, ncol=2)
```

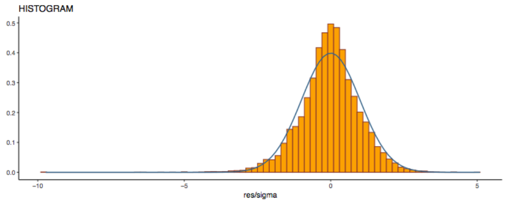


Conditional Volatility XXII

- Are the standardized residuals approximately normal?
- Figure below shows the histogram of ϵ_{t+1} and the standard normal distribution
- The std. residuals seem to be more peaked at the center and with longer tails (not very visible at this scale)
- But: skewness = -0.47189 and excess kurtosis = 3.32609
- Conclusion:

  1. normal distribution for ϵ_{t+1} is inaccurate
  2. the GARCH(1,1) conditional variance equation is inaccurate

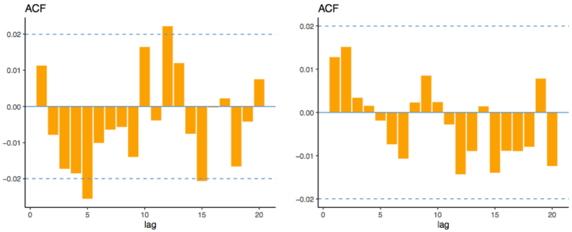
```
ggplot() +
  geom_histogram(aes(x=res/sigma,y = ..density..), binwidth=0.2, fill="orange", color="tomato4") +
  stat_function(aes(res/sigma), fun=dnorm, color="steelblue4", args=list(mean=0, sd=1), size=0.7) +
  theme_classic() + labs(y=NULL, title="HISTOGRAM")
```



Conditional Volatility XXIII

- We started off with the ACF of the squared returns R^2_{t+1} showing significant and persistent autocorrelation
- We now filter the residuals with the conditional standard deviation, η_{t+1}/σ_{t+1} and we find all this autocorrelation has disappeared
- This indicates that filtering the returns with the GARCH(1,1) conditional standard deviation takes care of the dependence in the squared returns

```
p1 <- ggacf((res/sigma), lag=20)
p2 <- ggacf((res/sigma)^2, lag=20)
gridExtra::grid.arrange(p1, p2, ncol=2)
```



Conditional Volatility XXIV

Another example:

- Attempt to use ARCH leads to a large, possibly ARCH(11) specification
- GARCH(1,1) offers best trade-off between simplicity and in-sample fit

p-values

$$R_{t+1} = \underset{(0.169)}{0.003} + \underset{(0.000)}{0.230} R_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \ \text{IID} \ \text{N}(0, \sigma^2_{t+1|t})$$

$$\sigma^2_{t+1|t} = \underset{(0.003)}{0.0002} + \underset{(0.000)}{0.073} \varepsilon_t^2 + \underset{(0.000)}{0.910} \sigma^2_{t|t-1},$$

- The sum of the coefficients is 0.983 $\Longrightarrow$ (covariance) stationarity
- Evidence in favor of GARCH(1,1) is strong: SACF of squared stdz residuals is characterized by absence of additional structure
- Regression that tests whether GARCH(1,1) can forecast squared residuals gives (standard errors in parentheses):

$$\varepsilon^2_{t+1} = \underset{(0.0013)}{0.0017} + \underset{(0.067)}{0.878} (\sigma^{GARCH}_{t+1|t})^2 + e_{t+1|t} \qquad R^2 = 8.69\%$$

- Intercept is not significant, while

$$t^{GARCH}_{b=1} = \frac{0.878 - 1}{0.067} = -\frac{0.122}{0.067} = -1.82$$

ACF and PACF of Squared Std. Residuals from GARCH(1,1)

	Autocorrelation	Partial Correlation	AC	PAC	Q-Stat	Prob
1			-0.017	-0.017	0.5367	0.464
2			-0.013	-0.013	0.8249	0.662
3			0.043	0.043	4.1959	0.241
4			0.000	0.002	4.1963	0.380
5			-0.004	-0.003	4.2248	0.518
6			0.013	0.011	4.5334	0.605
7			0.018	0.018	5.1207	0.645
8			0.035	0.036	7.3509	0.499
9			0.009	0.010	7.5054	0.585
10			0.012	0.012	7.7847	0.650
11			-0.028	-0.031	9.2487	0.599
12			-0.009	-0.011	9.4020	0.668

- F-test of hypothesis of *a* = 0, *b* = 1 gives 1.687 that with (2, 1822) d.f. implies a p-value of 0.185 and leads to a failure to reject

Conditional Volatility XXV

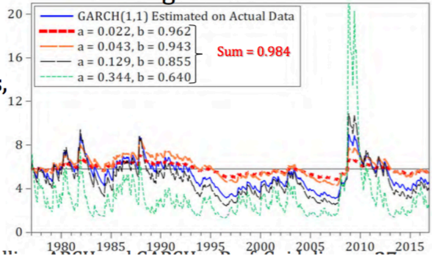
The persistence of Shocks in GARCH models:

- Although the persistence index of a GARCH(p,q) model is given by $\boxed{\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j}$ different coefficients contribute to increase $\sigma^2_{t+1|t}$ in different ways
    - The larger are the α_is, the larger is the response of $\sigma^2_{t+1|t}$ to new information; the larger are the β_js, the longer and stronger is the memory of conditional variance to past (forecasts of) variance

Conditional Volatility XXVI

Simulation study:

- For any given persistence index, it is possible for different stationary GARCH models to behave rather differently and therefore yield heterogeneous economic insights
  - This plot performs simulations on a baseline estimate on monthly UK stock returns, sample period 1977-2016
  - The volatility scenarios different from solid blue fix the persistence but impute it to alternative α and β



Conditional Volatility XXVII

C.2 Extension of the GARCH Model

C.2.1 The Integrated GARCH(p,q) model

The IGARCH model, both with or without trend, are therefore part of a wider class of models with a property called persistent variance in which the current information remains important for the forecasts of the conditional variances for all horizon.

$$\sigma_{t+1}^2 = \omega + \sigma_t^2(1-\alpha_1) + \alpha_1(R_t - \mu_t)^2$$

- It is a GARCH(1,1) with $\alpha_1 + \beta_1 = 1$.
- In the previous case a shock to the conditional variance is infinitely persistent, i.e it remains equally important at all horizons.
- In fact, from law of iterated expectations we have $E_t[\sigma_{t+s}^2] = (s-1)\omega + \sigma_{t+1}^2$ (it is similar to a random walk with deriva).

Conditional Volatility XXVIII

- GARCH(1,1) can be expresses as:

$$\sigma_{t+1}^2 = \frac{\omega}{1-\beta} + \alpha \sum_{k=1}^{\infty} \beta^{k-1} \varepsilon_{t-k}^2$$

  Notice that the second term has a logistic expression, which implies that this term is going very fast to zero (given that β has to be less than one)

- This makes the autocrrelation function goes to zero fastly, however $\alpha + \beta$ is very close to 1, which implies a long memory process.

$$\rho^k = (\alpha + \frac{\alpha^2 \beta}{1 - 2\alpha\beta - \beta^2})(\alpha + \beta)^{k-1}$$

- So that Granger and Ding (1995) introduce the FIGARCH model, which allows us to model short and long term volatility effects in the GARCH model.

Conditional Volatility XXIX

C.2.2 GARCH-in-means Model

- The GARCH-in-mean (GARCH-M) proposed by Engle, Lilien and Robins (1987) consists of the system:

$$R_{t+1} = \mu_t + \sigma_{t+1} z_{t+1} + \gamma g(\sigma^2_{t+1})$$

$$\sigma^2_{t+1} = \omega + \alpha_1 (R_t - \mu_t)^2 + \beta \sigma^2_t$$

- Engle, et. al (1987) extend the Engle's ARCH model to allow the conditional variance to be a determinant of the conditional mean of the process, i.e., the expected risk premium.
- When $R_{t+1} = r_t - r_f$ is the risk premium on holding the asset then the GARCH-M represents a simple way to model the relation between risk premium and its conditional variance.
- They consider an economy where risk averse economic agents choose among two kind of financial investment in order to maximize their expected utility.

i. The first possibility is represented by a risky asset with normally distributed returns, i.e., the risky is measured by the asset return variance and the compensation by a rise in the expected returns.

ii. The second investment choice is represented by a riskless asset.

Conditional Volatility XXX

- The agents utility function maximization subject to the market clearing conditions lead to the traditional positive relation between the mean and the variance of the risky asset return.
- Engle, et. al (1987) investigate the previous relation when the risky asset variance changes over time and therefore the risky asset price will change as well.
- The above assumptions determine a relation between the mean and the variance of asset return that is still positive but not constant.
- So that, the GARCH-M model therefore allows to analize the possibility of time-varying risk premium.
- This model characterizes the evolution of the mean and the variance of a time series simultaneously.
- The GARCH-in-mean model can be used to estimate the conditional CAPM.

Conditional Volatility XXXI

C.2.3 Asymmetric Models

- Exponential GARCH (Nelson(1991)
- GJR GARCH (Glosten, Jagannathan and Runkle (1993))
- Asymmetric Power GARCH (Ding, Engle and Granger (1993))
- Threshold GARCH (Zakoian(1994))
- Quadratic GARCH Model (Sentana (1995))
- Volatility Switching Model (Fonari and Mele (1996))
- Logistic smooth transition GARCH (Hagerud(1996) and Gonzales-Rivera(1996))
- ...

Conditional Volatility XXXII

Exponential GARCH Model ($\mu_t = 0$):

$$log\sigma^2_{t+1} = \omega + \alpha(r_t) + \gamma(\mid r_t \mid - E[\mid r_t \mid]) + \beta log\sigma^2_t$$

- GARCH models assume that only the magnitude and not the positivity or negativity of unanticipated excess returns determines feature σ^2_{t+1}
- BUT there exists a negative correlation between stock returns and changes in returns volatility, i.e. volatility tends to rise more in response to bad news, (excess returns lower than expected) and to fall less in response to good news (excess returns higher than expected).
- This asymmetry used to be called leverage effect because the increase in risk was believed to come from the increased leverage induced by a negative shock, but nowadays we know that this channel is just too small.

Conditional Volatility XXXIII

- Nelson (1991) write the model allowing σ^2_{t+1} depends on both size and the sign of lagged residuals considering the following:
- Three terms: (i) $\mid r_t \mid - E[\mid r_t \mid]$, (ii) r_t and (iii) $log \sigma^2_t$ (captures persistence in variance) .
- Terms (i) and (ii) captures shock to the returns, but while term (i) produces symmetric responses in log variance, term (ii) creates an asymmetrical responses to rises and falls in asset prices.
- Since negative returns have a more pronounced effect on volatility than positive returns of the same magnitude, the parameter α usually takes negative values.
- In fact, a negative shock to the returns which would increase the debt to equity ratio and therefore increase uncertainty of future returns could be accounted when $\alpha < 0$. This is leverage effect !
- If $\alpha < 0$, then term (i) can be decomposed as:
  $\alpha + \gamma$ when $0 < r_t < \infty$ and
  $\alpha - \gamma$ when $-\infty < r_t < 0$.

Conditional Volatility XXXIV

- The log-specification ensures a positive variance. This implies that α, γ, β are not restricted to be non-negative.

  Mathematically,

- ▶ The equation includes leverage term to capture the asymmetric effects between positive and negative asset returns.

$$g(r_t) = \alpha r_t + \gamma(| r_t | - E[| r_t |])$$

- By construction $g(r_t)_{t=-\infty}^{\infty}$ is a zero-mean, i.i.d. random sequence.
- Asymmetric response: Over the range $0 < r_t < \infty$ $g(r_{t+1})$ is linear with slope $\alpha + \gamma$ and $-\infty < r_t < 0$ $g(r_t)$ is linear with slope $\alpha - \gamma$
- The volatility process display leverage effect if $\alpha < 0$
- $\gamma(| z_t | - E[| z_t |])$ represents a magnitude effect.

Conditional Volatility XXXV

- If $\alpha = 0$ and $\gamma > 0$ the innovation in $log\sigma^2_{t+1}$ is positive (negative) when the magnitude of r_t is larger (smaller) than its expected value
- If $\alpha < 0$ and $\gamma = 0$ the innovation in conditional variance is now positive (negative) when returns innovations are negative (positive).

Example:

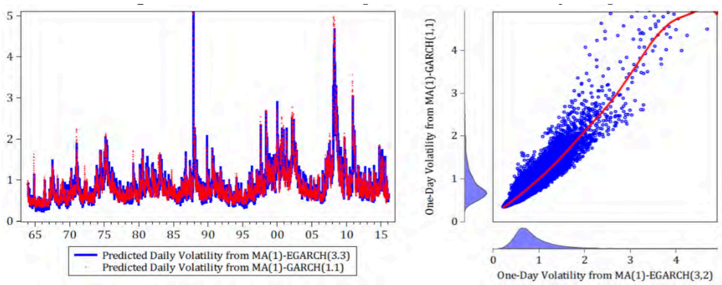
$$x_{t+1} = \underset{(0.000)}{0.023} + \underset{(0.000)}{0.121}\varepsilon_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \text{ IID N}(0, \sigma^2_{t+1|t})$$

EGARCH(3,3)

$$\begin{aligned}\ln\sigma^2_{t+1|t} = &-\underset{(0.000)}{0.006} + \underset{(0.000)}{0.108}|z_t| - \underset{(0.000)}{0.063}|z_{t-1}| - \underset{(0.000)}{0.037}|z_{t-2}| - \underset{(0.000)}{0.171}z_t + \underset{(0.000)}{0.180}z_{t-1} + \\ &-\underset{(0.208)}{0.011}z_{t-2} + \underset{(0.000)}{1.792}\ln\sigma^2_{t|t-1} - \underset{(0.000)}{0.693}\ln\sigma^2_{t-1|t-2} - \underset{(0.000)}{0.099}\ln\sigma^2_{t-2|t-3}\end{aligned}$$

- This process implies an odd, mixed leverage effect, because negative returns from the previous business day increase predicted variance, but negative returns from two previous business days depress it

Conditional Volatility XXXVI



Conditional Volatility XXXVII

- Although variance forecasts are not radically different, the scatter plot shows that when volatility is predicted to be high, often GARCH(1,1) predicts a higher level than EGARCH(3,3) does
- We have tested the two models for their ability to predict squared realized residuals, obtaining:

$$\text{GARCH}(1,1):\quad \varepsilon^2_{t+1} = \underset{(0.037)}{0.131} + \underset{(0.018)}{0.841}(\sigma^{GARCH}_{t+1|t})^2 + e_{t+1|t} \qquad R^2 = 13.6\%$$

$$EGARCH(3,3) = \varepsilon^2_{t+1} = \underset{(0.039)}{-0.221} + \underset{(0.025)}{1.303}(\sigma^{EGARCH}_{t+1|t})^2 + e_{t+1|t} \qquad R^2 = 16.5\%$$

- While in the case of GARCH(1,1) we obtain the same result as before, in the case of EGARCH the R^2 increases but the results on the intercept and slope point towards a rejection of model accuracy

Conditional Volatility XXXVIII

GJR-GARCH Model:

- Another GARCH specification that has become popular was proposed by Glosten, Jagganathan and Runkle (hence, **GJR-GARCH**) with the conditional variance given by

$$\sigma_{t+1}^2 = \omega + \alpha_1 * R_t^2 + \gamma_1 R_t^2 * I(R_t \leq 0) + \beta * \sigma_t^2$$

  where the effect of the unexpected return depends on its sign:

  - if positive its effect on the conditional variance is α_1
  - if negative the effect is $\alpha_1 + \gamma_1$

- Testing the hypothesis that $\gamma_1 = 0$ thus provides a test of the **asymmetric** effect of current and past squared returns on volatility.

Conditional Volatility XXXIX

- The GJR-GARCH model can be easily estimated by simply setting the `model` option to `gjrGARCH` in the model specification function `ugarchspec()`

```
spec   = ugarchspec(variance.model=list(model="gjrGARCH",garchOrder=c(1,1)),
                    mean.model=list(armaOrder=c(1,0)))
fitgjr = ugarchfit(spec = spec, data = sp500daily)
knitr::kable(data.frame(Estimate=coef(fitgjr), SE=fitgjr@fit$tval))
```

	Estimate	SE
mu	0.03723	4.52342
ar1	0.00753	0.68824
omega	0.01875	8.48399
alpha1	0.01946	4.46062
beta1	0.90219	132.58377
gamma1	0.12181	12.04780

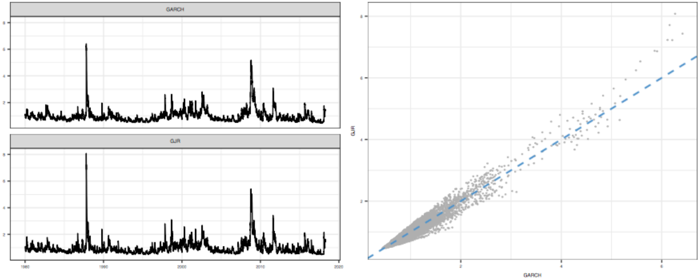
- Is the asymmetry coefficient significant?
- What do we learn about the S&P 500 volatility dynamics?

Conditional Volatility XL

GJR in R: An example

- Comparison of GARCH(1,1) and GJR-GARCH(1,1) volatility estimates

```
p1 <- autoplot(merge(GARCH = sigma(fitgarch), GJR = sigma(fitgjr)), scales="fixed") + theme_bw() +
  theme(strip.text = element_text(size=5), text = element_text(size=5))
p2 <- ggplot(data=merge(GARCH = sigma(fitgarch), GJR = sigma(fitgjr))) +
  geom_point(aes(x = GARCH, y=GJR), color="gray70", size=0.3) +
  geom_abline(intercept=0, slope=1, color="steelblue3", linetype="dashed", size=0.8) + theme_bw() +
  theme(text = element_text(size=5))
gridExtra::grid.arrange(p1, p2, ncol=2)
```



Conditional Volatility XLI

GJR on FX returns?

- Is the asymmetric effect of returns on volatility also relevant for exchange rate volatility?
- Estimation of GJR-GARCH(1,1) on JPY/USD daily returns:

```
fitgjr.fx = ugarchfit(spec = spec, data = DEXJPUS)
data.frame(Estimate=coef(fitgjr.fx), SE=fitgjr.fx@fit$tval)
```

```
          Estimate       SE
mu      -0.0042226 -0.65700
ar1      0.0040482  0.36879
omega    0.0078450  2.55350
alpha1   0.0393570  4.08780
beta1    0.9390349 56.07169
gamma1   0.0077128  1.39328
```

- Is the asymmetry coefficient significant?
- What do we learn about the FX volatility dynamics?

Conditional Volatility XLII

How do we compare volatility models?

- The Akaike Information Criterion (AIC) discussed for AR models can also be used to select GARCH models
- The model with the smallest criterion value is selected

```r
fit.ic <- cbind(infocriteria(fitgarch), infocriteria(fitgjr))
colnames(fit.ic) <- c("GARCH","GJR")
```

```
             GARCH    GJR
Akaike      2.6571 2.6339
Bayes       2.6608 2.6383
Shibata     2.6571 2.6339
Hannan-Quinn 2.6583 2.6354
```

- Which model shall we select: GARCH or GJR-GARCH?

- For FX returns:

```r
fit.ic.fx <- cbind(infocriteria(fitgarch.fx), infocriteria(fitgjr.fx))
```

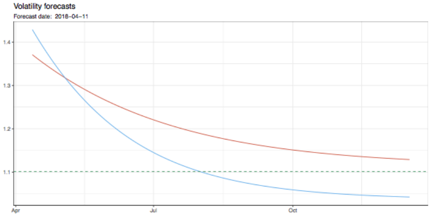
```
             GARCH    GJR
Akaike      1.9383 1.9382
Bayes       1.9421 1.9429
Shibata     1.9383 1.9382
Hannan-Quinn 1.9396 1.9398
```

- Which model shall we select: GARCH or GJR-GARCH?

Forecasts from models

- The estimated σ_t at the end of the sample (2018-04-11) is 1.423 for GARCH and 1.478 for GJR-GARCH
- We then forecast volatility for the following 250 days as shown below:

```
nforecast = 250
garchforecast <- ugarchforecast(fitgarch, n.ahead = nforecast)
gjrforecast   <- ugarchforecast(fitgjr, n.ahead = nforecast)
temp          <- data.frame(Date = end(sp500daily) + 1:nforecast,
                            GJR   = as.numeric(sigma(gjrforecast)),
                            GARCH = as.numeric(sigma(garchforecast)))
ggplot(temp) + geom_line(aes(Date, GARCH), color="tomato3") +
  geom_line(aes(Date, GJR), color="steelblue2") +
  geom_hline(yintercept = sd(sp500daily), color="seagreen4", linetype="dashed") +
  labs(x=NULL, y=NULL, title="Volatility forecasts",
       subtitle=paste("Forecast date: ", end(sp500daily))) + theme_bw()
```



Conditional Volatility XLIV

News impact curve

- The news have asymmetric effects on volatility.
- In the asymmetric volatility models good news and bad news have different predictability for future volatility.
- The news impact curve characterizes the impact of past return shocks on the return volatility which is implicit in a volatility model.
- Specifically, NIC is the relationship in which today's shock to return z_t impacts tomorrow variance σ^2_{t+1}

$$\sigma^2_{t+1} = \omega + \alpha_1 \sigma^2_t NIF(z_t) + \beta \sigma^2_t$$

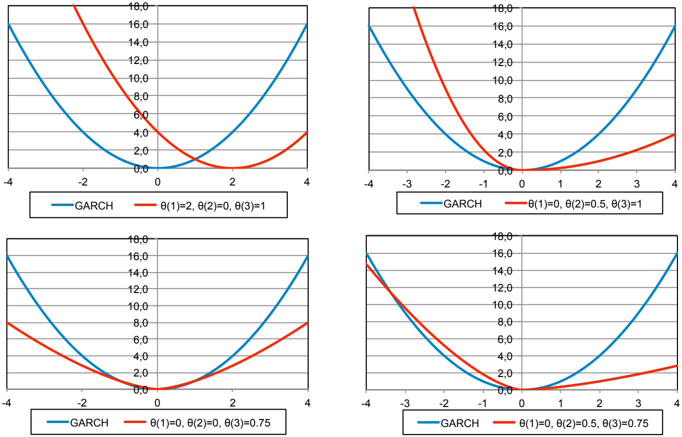
- The NIF equation is

$$NIC(z_t) = (\mid z_t - \theta_1 \mid - \theta_2 (z_t - \theta_1))^{2\theta_3}$$

- This curve measures how new information is incorporated into volatility estimates

Conditional Volatility XLV

- In the simple GARCH model we have that $NIF(z_t) = z_t^2$ ($\theta_1 = \theta_2 = 0$ y $\theta_3 = 1$)
- So that the NIF is a symmetric parabola that takes the minimum value 0 when z_t is zero.



Conditional Volatility XLVI

continued...

- The GJR-GARCH implies an asymmetric response to shocks (ϵ_{t-1}) relative to the GARCH model which is evident when plotting the **news impact curve** (the response of conditional variance σ_t^2 to a shock ϵ_{t-1} 1)

```
newsgarch <- newsimpact(fitgarch)
newsgjr   <- newsimpact(fitgjr)
p1 <- qplot(newsgarch$zx, newsgarch$zy, geom="line", main="GARCH") + labs(x="shock",y="volatility") +
  geom_vline(xintercept = 0, linetype="dashed") + theme_bw() + theme(text = element_text(size=5))
p2 <- qplot(newsgjr$zx, newsgjr$zy, geom="line", main="GJR")  + labs(x="shock",y="volatility") +
  geom_vline(xintercept = 0, linetype="dashed") + theme_bw() + theme(text = element_text(size=5))
gridExtra::grid.arrange(p1, p2, ncol=2)
```