Recitation 7

Fall 2020

Question 1

Consider a two-factor model

$$r_i = E[r_i] + \beta_{i,1} f_1 + \beta_{i,2} f_2 + \epsilon_i$$

with the following properties:

$$E[f_1] = 0, E[f_2] = 0, \mathrm{Cov}(f_1, f_2) = 0$$

Assume that the standard deviation of Factor 1 is 7% and that of Factor 2 is 5%. There are two stocks with the following properties:

Stock	Expected return	Standard deviation	Loading on Factor 1	Loading on Factor 2
A	10%	25%	1	1.5
B	10%	10%	2	3

Assume that the idiosyncratic component of returns of stocks A and B is uncorrelated with each other, as well as with both factor returns.

(a) Compute standard deviation of returns on Stocks A and B.

(b) Compute the correlation between returns on Stock A and Stock B.

Solutions:

(a) Compute standard deviation of returns on Stocks A and B.

Let us start with stock A. The return on stock A is given by the factor model:

$$r_A = E[r_A] + \beta_{A,1} f_1 + \beta_{A,2} f_2 + \epsilon_A.$$

The variance of sum of random variables is sum of the variance plus pair-wise covariances.[1] Note that $E[r_A]$ is a constant and it does not affect

[1] $\mathrm{Var}[X_1 + X_2 \cdots + X_N] = \sum_{i=1}^{N} \mathrm{Var}[X_i] + \sum_{i=1}^{N} \sum_{j \neq i} \mathrm{Cov}[X_i, X_j]$ for any random variables X_1, X_2, $\ldots X_N$. In particular for 3 variables as in the problem

$\mathrm{Var}[X_1 + X_2 + X_3] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \mathrm{Var}[X_3] + 2\mathrm{Cov}[X_1, X_2] + 2\mathrm{Cov}[X_2, X_3] + 2\mathrm{Cov}[X_1, X_3]$

the variance[2]. Hence,

$$\begin{aligned}\mathrm{Var}[r_A] &= \mathrm{Var}[\beta_{A,1}f_1] \ + \ \mathrm{Var}[\beta_{A,2}f_2] \ + \ \mathrm{Var}[\epsilon_A] \ + \\ &+ \ 2\mathrm{Cov}[\beta_{A,1}f_1, \beta_{A,2}f_2] \ + \ 2\mathrm{Cov}[\beta_{A,1}f_1, \epsilon_A] \ + \ 2\mathrm{Cov}[\beta_{A,2}f_2, \epsilon_A].\end{aligned}$$

Use the properties of variance[3] and covariance[4] operators to get

$$\begin{aligned}\mathrm{Var}[r_A] &= \beta_{A,1}^2\mathrm{Var}[f_1] \ + \ \beta_{A,2}^2\mathrm{Var}[f_2] \ + \ \mathrm{Var}[\epsilon_A] \ + \\ &+ \ 2\beta_{A,1}\beta_{A,2}\mathrm{Cov}[f_1, f_2] \ + \ 2\beta_{A,1}\mathrm{Cov}[f_1, \epsilon_A] \ + \ 2\beta_{A,2}\mathrm{Cov}[f_2, \epsilon_A].\end{aligned}$$

Since this is a factor model, by its definition $\mathrm{Cov}[f_1, \epsilon_A] = 0$ and $\mathrm{Cov}[f_2, \epsilon_A] = 0$. Next, by assumption of the problem, $\mathrm{Cov}[f_1, f_2]$. Hence,

$$\mathrm{Var}[r_A] \ = \beta_{A,1}^2\mathrm{Var}[f_1] \ + \ \beta_{A,2}^2\mathrm{Var}[f_2] \ + \ \mathrm{Var}[\epsilon_A].$$

Plugging the numbers from the table into the equation, we get

$$\mathrm{Var}[r_A] \ = 1^2 \cdot 0.07^2 \ + \ 1.5^2 \cdot 0.05^2 \ + \ 0.25^2 = 0.073.$$

$$\mathrm{SD}[r_A] = \sqrt{\mathrm{Var}[r_A]} = 27.02\%$$

Doing same calculations for stock B we get,

$$\mathrm{Var}[r_B] \ = 2^2 \cdot 0.07^2 \ + \ 3^2 \cdot 0.05^2 \ + \ 0.10^2 = 0.0521.$$

$$\mathrm{SD}[r_B] = \sqrt{\mathrm{Var}[r_B]} = 22.83\%$$

(b) Compute the correlation between returns on Stock A and Stock B.

Let us start from getting the covariance between the returns. To get it we will employ the following linear property of covariance:

$$\mathrm{Cov}[\sum_{i=1}^{I} X_i, \sum_{j=1}^{J} Y_j] = \sum_{i=1}^{I}\sum_{j=1}^{J} \mathrm{Cov}[X_i, Y_j]$$

For example, if $I = 2$ and $J = 3$,

$$\begin{aligned}\mathrm{Cov}[X_1 + X_2, Y_1 + Y_2 + Y_3] = \mathrm{Cov}[X_1, Y_1] \ + \ \mathrm{Cov}[X_1, Y_2] \ + \ \mathrm{Cov}[X_1, Y_3] + \\ + \ \mathrm{Cov}[X_2, Y_1] \ + \ \mathrm{Cov}[X_2, Y_2] \ + \ \mathrm{Cov}[X_2, Y_3]\end{aligned}$$

Using the property, and that covariance ignores constants[5] we get

$$\mathrm{Cov}[r_A, \ r_B] = \mathrm{Cov}[\underbrace{E\left[r_A\right] + \beta_{A,1}f_1 + \beta_{A,2}f_2 + \epsilon_A}_{r_A}, \ r_B] =$$

$$\mathrm{Cov}[\underbrace{\beta_{A,1}f_1 + \beta_{A,2}f_2 + \epsilon_A}_{r_A}, \ \underbrace{\beta_{B,1}f_1 + \beta_{B,2}f_2 + \epsilon_B}_{r_B}] =$$

[2] $\mathrm{Var}[a + X] = \mathrm{Var}[X]$ for any constant a and random variable X

[3] $\mathrm{Var}[aX] = a^2\mathrm{Var}[X]$ for any constant a and random variable X

[4] $\mathrm{Cov}[aX, bY] = a \cdot b \cdot \mathrm{Cov}[X, Y]$ for any constants a and b and random variables X and Y

[5] Cov[a + X, b + Y] = Cov[X,Y] for any constants a and b and random variables X and Y

The idiosyncratic components ϵ-s are not going to matter here because, by definition, the covariance between idiosyncratic components of stock A and B is 0, i.e., they are uncorrelated. Moreover, as we discussed in Part (a), they are not correlated with factors. Formally, use that $\text{Cov}[\epsilon_A, \epsilon_B] = 0$ and $\text{Cov}[\epsilon_i, f_j] = 0$ for $i = A, B$ and $j = A, B$ to simplify the last expression to to[6]

$$\text{Cov}[\beta_{A,1}f_1,\ \beta_{B,1}f_1] + \text{Cov}[\beta_{A,2}f_2,\ \beta_{B,1}f_1] + \text{Cov}[\beta_{A,1}f_1,\ \beta_{B,2}f_2] + \text{Cov}[\beta_{A,2}f_2,\ \beta_{B,2}f_2] =$$
$$\beta_{A,1}\beta_{B,1}\text{Cov}[f_1,\ f_1] + \beta_{A,2}\beta_{B,1}\text{Cov}[f_2,\ f_1] + \beta_{A,1}\beta_{B,2}\text{Cov}[f_1,\ f_2] + \beta_{A,2}\beta_{B,2}\text{Cov}[f_2,\ f_2]$$

Finally, use that according to assumption of the problem the factors are uncorrelated, $\text{Cov}[f_1, f_2] = 0$[7],

$$\begin{aligned} \text{Cov}[r_A,\ r_B] =&\ \beta_{A,1}\beta_{B,1}\text{Cov}[f_1,\ f_1]\ +\ \beta_{A,2}\beta_{B,2}\text{Cov}[f_2,\ f_2] = \\ =&\ \beta_{A,1}\beta_{B,1}\text{Var}[f_1]\ +\ \beta_{A,2}\beta_{B,2}\text{Var}[f_2] = \\ =&\ 1 \cdot 2 \cdot 0.07^2\ +\ 1.5 \cdot 3 \cdot 0.05^2 = 0.02105 \end{aligned}$$

The final step is to take the covariance and compute the correlation. By definition, correlation equals the covariance of returns on A and B divided by their standard deviations.

$$\text{Corr}[r_A,\ r_B] = \frac{\text{Cov}[r_A,\ r_B]}{SD[r_A] \times SD[r_B]} = \frac{0.021}{27.02\% \times 22.83\%} = 0.3413 = 34.13\%$$

Question 2

Consider a two-factor model

$$r_i = E\left[r_i\right] + \beta_{i,1}f_1 + \beta_{i,2}f_2 + \epsilon_i$$

Suppose that there are

$$N_1 = 30$$

stocks in the market with the following properties: Factor loadings:

$$\beta_{i,1} = 1,\ \text{for all } i = 1, 2, \cdots, N_1$$

$$\beta_{i,2} = \begin{cases} 1,\ \text{for } i = 1, 2, \cdots, 10 \\ 1.6,\ \text{for } i = 11, 2, \cdots, 20 \\ 2.1,\ \text{for } i = 21, 2, \cdots, 30 \end{cases}$$

[6]otherwise, there would be 3×3 terms, since we are "lazy" to write all of them, let us remove the 5 epsilon-terms which will be zero in any case

[7]Note that if the assumption does not hold, then in this expression, we will have to take into account the other two covariance terms as well!

Idiosyncratic volatility:

$$\mathrm{SD}(\epsilon_i) = \begin{cases} 5\%, & \text{for } i = 1, 2, \cdots, 10 \\ 6\%, & \text{for } i = 11, 2, \cdots, 20 \\ 8\%, & \text{for } i = 21, 2, \cdots, 30 \end{cases}$$

Part (a)

(a) Compute factor loadings of an equally-weighted portfolio that consists of all 30 stocks.

Solution:

Denote the equally-weighted portfolio by P. Let us start with the factor loading 1 or beta 1 of portfolio P (same for factor 2). By definition, this is just the weighted average of betas of individual assets in our portfolio. The weights are the weights of each of the individual assets in the portfolio.

$$\beta_{P,1} = \sum_{i=1}^{30} w_i \cdot \beta_{i,1} \qquad \beta_{P,2} = \sum_{i=1}^{30} w_i \cdot \beta_{i,2}$$

In this problem, since P is equally-weighted, the weights of all assets are the same and equal to $\frac{1}{30}$, $w_i \equiv \frac{1}{30}$. Next, $\beta_{i,1}$ are all the same. Hence,

$$\beta_{P,1} = \sum_{i=1}^{30} \frac{1}{30} \cdot 1 = 1.$$

That must be intuitive because each individual stock's beta (also called **loading to factor** or **exposure to the factor**) is one. For an equally weighted portfolio, like for any other portfolio that we can form out of these assets[8], the beta of this portfolio will always be equal to 1, just because individual assets, all of them have the same exposure to the factor.

Next, for factor 2, $\beta_{i,2}$ form three different groups. Splitting the sum based on the groups we get,

$$\beta_{P,2} = \sum_{i=1}^{10} \frac{1}{30} \cdot 1 + \sum_{i=11}^{20} \frac{1}{30} \cdot 1.6 + \sum_{i=21}^{30} \frac{1}{30} \cdot 2.1 = \frac{10}{30} + \frac{16}{30} + \frac{21}{30} = 1.5667.$$

Part (b)

Now assume that instead of 30 stocks, we have

$$N_1 = 300$$

[8] maintaining the weights adding up to one

stocks with the following properties: Factor loadings:

$$\beta_{i,1} = 1, \text{ for all } i = 1, 2, \cdots, N_1$$

$$\beta_{i,2} = \begin{cases} 1, \text{ for } i = 1, 2, \cdots, 100 \\ 1.6, \text{ for } i = 101, 2, \cdots, 200 \\ 2.1, \text{ for } i = 201, 2, \cdots, 300 \end{cases}$$

Idiosyncratic volatility:

$$\mathrm{SD}(\epsilon_i) = \begin{cases} 5\%, \text{ for } i = 1, 2, \cdots, 100 \\ 6\%, \text{ for } i = 101, 2, \cdots, 200 \\ 8\%, \text{ for } i = 201, 2, \cdots, 300 \end{cases}$$

Compute factor loadings of an equally-weighted portfolio that consists of these 300 stocks.

Solution

Denote the equally-weighted portfolio by P.

As in the previous problem the weights of all assets are the same and equal to $\frac{1}{300}$, $w_i \equiv \frac{1}{300}$. For factor 1, we get the same $\beta_{i,1}$ as in Part (a):

$$\beta_{P,1} = \sum_{i=1}^{300} \frac{1}{300} \cdot 1 = 1.$$

Since individual exposures of all stocks to the factor are the same, these stocks' portfolio maintains the same exposure to the factor.

Next, for factor 2, $\beta_{i,2}$ still forms three different groups. Splitting the sum based on the groups we get,

$$\beta_{P,2} = \sum_{i=1}^{100} \frac{1}{300} \cdot 1 + \sum_{i=101}^{200} \frac{1}{300} \cdot 1.6 + \sum_{i=201}^{300} \frac{1}{300} \cdot 2.1 = \frac{100}{300} + \frac{160}{300} + \frac{210}{300} = 1.5667.$$

Therefore, the result here is that as we increase the number of stocks in this portfolio from 30 to 300, the factor loadings stay the same. The main reason is that even increasing the number of stocks, we maintained the same fraction of assets with different portfolio loadings.

A takeaway from part (b) is that by increasing diversification of the portfolio, you might not be able to hedge the **systematic risk**, i.e., reduce your exposure to the systematic factors.

Part (c)

Now go back to the original 30 stocks. Compute idiosyncratic volatility of an equally-weighted portfolio that consists of these 30 stocks.

Solution

To solve the problem recall that for any random variables $X_1, X_2 \ldots, X_I$,

$$\text{Var}[\sum_{i=1}^{I} X_i] = \sum_{i=1}^{I} \text{Var}[X_i] + \sum_{i=1}^{I} \sum_{j \neq i} \text{Cov}[X_i, X_j]$$

Note that the formula simplifies to

$$\text{Var}[\sum_{i=1}^{I} X_i] = \sum_{i=1}^{I} \text{Var}[X_i]$$

if $X_1, X_2 \ldots, X_I$ are pairwise uncorrelated.

The idiosyncratic component of the assets portfolio is the weighted average of idiosyncratic components of individual assets entering into the portfolio. The weights are the weights of each of the individual assets in the portfolio.

$$\epsilon_P = \sum_{i=1}^{30} w_i \epsilon_i = \sum_{i=1}^{30} \frac{1}{30} \epsilon_i$$

Taking the variance of the idiosyncratic component and using the variance property discussed above, we get[9]

$$\begin{aligned} \text{Var}[\epsilon_P] &= \text{Var}[\sum_{i=1}^{I} \frac{1}{30}\epsilon_i] = \left(\frac{1}{30}\right)^2 \text{Var}[\sum_{i=1}^{30} \epsilon_i] = \left(\frac{1}{30}\right)^2 \sum_{i=1}^{30} \text{Var}[\epsilon_i] = \\ &= \left(\frac{1}{30}\right)^2 \times \left(10 \cdot (5\%)^2 + 10 \cdot (6\%)^2 + 10 \cdot (8\%)^2\right) = 0.00014 \end{aligned}$$

Find the volatility of the idiosyncratic component of portfolio of assets,

$$\text{SD}[\epsilon_P] = \sqrt{\text{Var}[\epsilon_P]} = 1.1785\%.$$

Note that the variance is very low. That means that even by taking 30 stocks, we can diversify our portfolio's idiosyncratic risk quite well.

Part (d)

Compute idiosyncratic volatility of an equally-weighted portfolio that consists of 300 stocks given in Part (B).

Solution

Repeat calculation from Part (c)

$$\begin{aligned} \text{Var}[\epsilon_P] &= \text{Var}[\sum_{i=1}^{I} \frac{1}{300}\epsilon_i] = \left(\frac{1}{300}\right)^2 \text{Var}[\sum_{i=1}^{300} \epsilon_i] = \left(\frac{1}{300}\right)^2 \sum_{i=1}^{300} \text{Var}[\epsilon_i] = \\ &= \left(\frac{1}{300}\right)^2 \times \left(100 \cdot (5\%)^2 + 100 \cdot (6\%)^2 + 100 \cdot (8\%)^2\right) = 0.000014 \end{aligned}$$

[9] also use $\text{Var}[aX] = a^2\text{Var}[X]$ for any constant a and random variable X

Note that the idiosyncratic component's variance decreased ten times compared to the variance in Part (c). In other words, the diversification helped to reduce the idiosyncratic risk of the portfolio even more! Let us find the volatility which decreased by $\sqrt{10}$ times:

$$\mathrm{SD}[\epsilon_P] = \sqrt{\mathrm{Var}[\epsilon_P]} = 0.37\%.$$

The volatility of our portfolio is minimal. It could be reduced even smaller if one took more stocks. For example, if the number of stock were $N = 3000$ and we kept the same structure, then $\mathrm{SD}[\epsilon_P] = 0.12\%$, which is a tiny number.

Question 3

Consider a two-factor model

$$r_i = E[r_i] + \beta_{i,1} f_1 + \beta_{i,2} f_2 + \epsilon_i$$

There are three well-diversified portfolios with the following properties:

Portfolio	Expected return	Loading on Factor 1	Loading on Factor 2
A	8%	0.95	1.15
B	6%	0.85	0.70
C	10.5%	1.20	1.50

Assume that the current risk-free rate is 1.5%. Construct an arbitrage strategy that generates \$1,000 today and zero payoff in the future.

Solutions:

Let us start building our arbitrage strategy by assuming that we buy (go long) portfolios A, B, and C and the risk-free bond in the following (dollar) amounts: $x_A, x_B,\ x_C, x_{r_f}$. We will allow these amounts to be negative, which would indicate that we are shorting these securities.

Summarise all the data in the table:

Portfolio	$E[r_i]$	$\beta_{i,1}$	$\beta_{i,2}$	Amount
A	8%	0.95	1.15	x_A
B	6%	0.85	0.70	x_B
C	10.5%	1.20	1.50	x_C
Risk-free bond	1.5%	0	0	x_r

We want to construct a trading strategy that gives us \$1,000 today. That can be reformulated as buying 4 securities in the amounts of x_A, x_B, x_C, x_{r_f} should give us \$1,000 today, i.e.,

$$-x_A - x_B - x_C - x_{r_f} = \$1,000$$

Notice the negative signs. This is because we assumed that we are buying x of each security, which represents cash outflow.

Let us discuss what arbitrage means in the context. The key to arbitrage:

1. Zero expected return
2. No exposure to systematic risk
3. since we are working with well-diversified portfolios, we are not exposed to idiosyncratic risk

Let us start with zero expected return:

$$x_A\left(1+E\left[r_A\right]\right)+x_B\left(1+E\left[r_B\right]\right)+$$
$$+x_C\left(1+E\left[r_C\right]\right)+x_{r_f}\left(1+r_f\right)=\$0$$

Plugging the numbers from the table, we get

$$x_A(1+8\%)+x_B(1+6\%)+x_C(1+10.5\%)+x_{r_f}(1+1.5\%)=\$0$$

The next assumption is that there is no exposure to systematic risk factor 1.

$$\beta_{P,1}=0$$

We can expand the equation since the loading of the portfolio on factor 1 is just a weighted sum of its components' loadings, where the weights are the same as the weights of the components in the portfolio.

$$\frac{x_A}{V_P}\beta_{A,1}+\frac{x_B}{V_P}\beta_{B,1}+\frac{x_C}{V_P}\beta_{C,1}=0,$$

where

$$V_P=x_A+x_B+x_C+x_{r_f}.$$

Plugging the numbers we get

$$0.95x_A+0.85x_B+1.20x_C=0$$

Similarly, the exposure of the portfolio to factor 2 must be zero:

$$\frac{x_A}{V_P}\beta_{A,2}+\frac{x_B}{V_P}\beta_{B,2}+\frac{x_C}{V_P}\beta_{C,2}=0.$$

$$1.15x_A+0.70x_B+1.50x_C=0$$

All the equations can be summarized in the following system,

$$\begin{cases}-x_A-x_B-x_C-x_{r_f}=1000\\ 1.08x_A+1.06x_B+1.105x_C+1.015x_{r_f}=0\\ 0.95x_A+0.85x_B+1.20x_C=0\\ 1.15x_A+0.70x_B+1.50x_C=0\end{cases}$$

In the matrix form, the system of equations is

$$\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1.08 & 1.06 & 1.105 & 1.015 \\ 0.95 & 0.85 & 1.20 & 0 \\ 1.15 & 0.70 & 1.50 & 0 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \\ x_{r_f} \end{pmatrix} = \begin{pmatrix} 1000 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Recall that the matrix equation can be solved by inversion of the matrix:

$$A \times x = b \quad \Rightarrow \quad x = A^{-1}b$$

Doing this in Excel or any other soft will give you the answer:

$$\begin{pmatrix} x_A \\ x_B \\ x_C \\ x_{r_f} \end{pmatrix} = \begin{pmatrix} -\$235,480 \\ \$24,360 \\ \$169,167 \\ \$40,953 \end{pmatrix}$$

Looking at our solution, we can conclude that we short portfolio A and go long into other portfolios. This means that if portfolio A and B are correctly priced by the factors, then the portfolio A promised to low return given its systematic risk, or alternatively, portfolio A is too expensive.

Question 4

Consider a three-factor model

$$r_i = E[r_i] + \beta_{i,1} f_1 + \beta_{i,2} f_2 + \beta_{i,3} f_3 + \epsilon_i$$

There are four well-diversified portfolios with the following properties:

Portfolio	Expected return	Loading on Factor 1	Loading on Factor 2	Loading on Factor 3
A	21.65%	0.95	1.20	0.85
B	19.23%	1.00	0.75	1.35
C	21.48%	1.10	1.05	0.95
D	23.15%	1.25	1.05	1.15

Compute factor risk premia (also known as factor risk prices) as well as the risk-free rate.

Solutions:

$$E[r_P] - r_f = \lambda_1 \beta_{P,1} + \lambda_2 \beta_{P,2} + \lambda_3 \beta_{P,3}$$

For solving this problem, we will use the APT pricing equation that tells us that the expected excess return on a portfolio is the sum of products of factor loadings with the risk-premia of the factors:

$$E[r_P] - r_f = \lambda_1 \beta_{P,1} + \lambda_2 \beta_{P,2} + \lambda_3 \beta_{P,3}$$

We can write down this pricing equation for all four portfolios. For example, for portfolio A:

$$E[r_A] - r_f = \lambda_1 \beta_{A,1} + \lambda_2 \beta_{A,2} + \lambda_3 \beta_{A,3}$$

or

$$21.65\% - r_f = \lambda_1 0.95 + \lambda_2 1.20 + \lambda_3 0.85$$

Hence, the APT pricing relationship implies restrictions on the factors risk-premia and the risk-free rate. Similarly, using portfolio B, C, and D, we get

$$\begin{aligned}
19.23\% - r_f &= \lambda_1 1.00 + \lambda_2 0.75 + \lambda_3 1.35 \\
21.48\% - r_f &= \lambda_1 1.10 + \lambda_2 1.05 + \lambda_3 0.95 \\
23.15\% - r_f &= \lambda_1 1.25 + \lambda_2 1.05 + \lambda_3 1.15
\end{aligned}$$

The system of equations in the matrix form is

$$\begin{pmatrix} 0.95 & 1.20 & 0.85 & 1 \\ 1.00 & 0.75 & 1.35 & 1 \\ 1.10 & 1.05 & 0.95 & 1 \\ 1.25 & 1.05 & 1.15 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ r_f \end{pmatrix} = \begin{pmatrix} 21.65\% \\ 19.23\% \\ 21.48\% \\ 23.15\% \end{pmatrix}$$

Solving the matrix equation we get

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ r_f \end{pmatrix} = \begin{pmatrix} 6.51\% \\ 9.95\% \\ 3.47\% \\ 0.57\% \end{pmatrix}$$

This gives us factor risk-premia on factors $1, 2$, and 3 as well as the risk-free rate and we can price all the other well-diversified portfolios as long as we know their factor loadings.

Question 5 (Implementation of APT: a Portfolio Factor Model)

In the question, we will learn how to implement the arbitrage pricing theory (APT) model in practice. We will work with the data contained in the file posted on Edx. Please download it before starting working on this question.

The data contains historical monthly returns on eight firms from January 2005 through December 2019. Four of these firms are from the gold mining industry. And the other four are from the technology industry.

We will consider a two-factor model that has the market and the price of gold as the risk. To proxy those factors, we will use factor-mimicking portfolios. For the market factor, we will use the returns on the market portfolio, and for the price of gold, we will use the returns on the Spider Gold Shares ETF. The data with the returns of those factors and the risk-free rates are also in the file.

The following equation describes the two-factor model:

$$r_i - r_f = \alpha_i + \beta^i_{mkt}(r_{mkt} - r_f) + \beta^i_{gld}(r_{xle} - r_f) + \varepsilon_i.$$

(a) Using the OLS regression over the full sample, estimate factor loadings $\beta^i_{\mathrm{mit}\,t}$ and β^i_{xle} and the alphas for each of the eight stocks.

Report, separately, the **arithmetic average** of the factor loadings and alphas for the gold mining stocks and for the technology stocks.

(b) Assume that the returns on all stocks satisfy the exact 2 -factor APT model with the stock market and the gold price as the two factors. Take your estimated factor loadings as exact. Suppose you are given a forecast for the expected excess monthly returns going forward: 0.75% for the stock market portfolio, and 0.15% for the gold price. Using the average factor loadings you've computed in part (a), compute the expected excess return for the gold Mining sector, and the technology sector. Report both numbers as monthly, in %.

Solutions:

This problem is solved in Excel and the solution is provided in the video.

(a) Using the OLS regression over the full sample, estimate factor loadings $\beta^i_{\mathrm{mit}\,t}$ and β^i_{xle} and the alphas for each of the eight stocks.

Report, separately, the **arithmetic average** of the factor loadings and alphas for the gold mining stocks and for the technology stocks.

The coefficients in the regression

$$r_i - r_f = \alpha_i + \beta^i_{mkt}\left(r_{mkt} - r_f\right) + \beta^i_{gld}\left(r_{xle} - r_f\right) + \varepsilon_i$$

are

	β_{GLD}	β_{MKT}	α
Kinross Gold Corporation	1.920	0.300	−0.88%
AngloGold Ashanti	1.824	0.095	−0.73%
Barrick Gold Corporation	1.811	0.161	−0.82%
Newmont Corporation	1.449	0.202	−0.64%
Apple Inc.	−0.072	1.079	2.10%
Microsoft Corporation	−0.229	0.923	0.88%
Oracle Corporation	−0.075	0.979	0.36%
Intel Corporation	−0.297	0.946	0.53%

The average coefficients by industry are

	β_{GLD}	β_{MKT}	α
Gold Mining Stocks	1.751	0.189	−0.8%
Technology Stocks	−0.168	0.982	1.0%

(b) Assume that the returns on all stocks satisfy the exact 2 -factor APT model with the stock market and the gold price as the two factors. Take your estimated factor loadings as exact. Suppose you are given a forecast for the expected excess monthly returns going forward: 0.75% for the stock market portfolio, and 0.15% for the gold price. Using the average factor loadings you've computed in part (a), compute the expected excess return for the gold Mining sector, and the technology sector. Report both numbers as monthly, in %.

Solutions:

Use the following pricing relationship.

$$\bar{r}_p - r_f = \lambda_{MKT} b_{MKT} + \lambda_{GLD} b_{GLD}$$

to get that the expected return on technology sector is 0.71%. And the expected return on the gold mining sector 0.40%.