MIT Sloan School of Management
Kogan and Wang

15.415.1x Sample Exam

Grade Sheet

1.	________	/	10
2.	________	/	12
3.	________	/	11
4.	________	/	16
6.	________	/	18
6.	________	/	18
7.	________	/	15
8.	________	/	20
Total	________	/	120

1. (10 points) True or false?

   (a) (2 points) When evaluating investment projects, for the projects with higher variance of sales forecasts, firms should apply higher discount rates.

   (b) (2 points) Consider a firm maximizing its market value. When selecting among two mutually exclusive projects with the same initial investment, this firm may optimally select the project with the lower internal rate of return.

   (c) (2 points) When computing the NPV of potential new investments, a firm should discount expected future cash flows attributed to the project using the firm's cost of capital as the discount rate.

   (d) (2 points) If the yield curve is flat, then yield to maturity on a risk-free coupon bond is equal to the expected return (in annualized terms) the investor would collect by holding the bond to its maturity.

   (e) (2 points) Consider a nominal risk-free cash flow. When computing the present value in the presence of inflation risk, it is generally wrong to discount the expected real cash flows at the real risk-free interest rates.

2. (12 points) Consider a state-space model with two periods and three states at time $t = 1$: 1, 2, and 3. All three states are equally likely. Primitive state-contingent claims on each state are traded in the market, and their time-0 prices are:

$$\phi_1 = 0.4;\ \ \phi_2 = 0.3;\ \ \phi_3 = 0.2.$$

   In addition to the primitive claims, the risk-free asset is traded.

   (a) (4 points) Based on absence of arbitrage, what is the risk-free interest rate in this market?

   (b) (4 points) Consider a cash flow C_1 equal to \$1, \$2, and \$4, in states 1, 2, and 3, respectively. What is the expected value of this cash flow (as of time 0)?

   (c) (4 points) Compute the time-0 market value of the cash flow C_1.

3. (11 points) Alice is taking out a bank loan to pay for a new addition to her house. She is comparing two options: a 10-year loan with an annual APR of 6%, compounded monthly; and a 5-year loan, with an annual APR of 6.7%, compounded monthly. The market interest rate is 4% (EAR), and is the same for all maturities. Alice needs to borrow \$50,000.

   (a) (3 points) Compute the EAR on each of the two loans.

   (b) (4 points) Compute the monthly payments on each of the two loans.

   (c) (3 points) Compute the present value of payments on each of the two loans.

   (d) (1 point) Is the ten-year loan a better deal, judging by the present value of the payments that Alice would need to make between now and the maturity of the loan? "Yes" or "no."

4. (17 points) Consider a frictionless market. Several Treasury bonds (with face values of \$100) are traded in the market. Their coupon rates and yield-to-maturity are given in the following table:

Bond name	Maturity	Coupon rate	Yield to maturity
A	1-year	0%	3%
B	2-year	6%	4%
C	3-year	3%	5%

The coupons are paid annually. Now is year 0.

(a) (2 points) What is the 1-year spot interest rate?

(b) (2 points) What is the 2-year spot interest rate?

(c) (2 points) What is the 3-year spot interest rate?

(d) (3 points) Suppose that a new Treasury bond is introduced to the market. It is a zero-coupon bond with 3 years to maturity, and it trades at the 4% yield to maturity. What is the no-arbitrage price of the new bond?

(e) (7 points) Describe explicitly the arbitrage trading strategy involving bonds A, B, C, and the new bond, which pays \$1 at time $t = 0$ and nothing afterwards.

5. (18 points) You are advising a local municipal treasury on a bridge construction project. The project requires an upfront investment (at time/year 0) of \$10M, with an additional investment in year 1 of \$5M. The bridge will become operational two years from now, and will start generating toll revenue. Specifically, the bridge produces no cash flows in year 1, and produces a perpetual stream of cash flows of \$1M per year in subsequent years. Assume that all cash flows are risk-free.

The treasury is financing this project with a ten-year zero-coupon bond. The current term structure of risk-free interest rates is flat at 2%. Assume that the treasury is able to finance this project at the risk-free interest rate.

(a) (3 points) What is the NPV of this project?

(b) (3 points) Suppose that the treasury wants to issue enough bonds to cover the present value of construction costs of this project. Let the face value of each bond be \$1,000. What is the total number of bonds that need to be issued?

(c) (5 points) Compute the modified duration of the bond issued by the treasury.

(d) (5 points) Suppose the treasury goes ahead with your suggestion in (b) and starts the project. Right after its start (at time 0), the yield curve unexpectedly rises by 1% across all maturities. What is the resulting change in the NPV of the project, following the change in interest rates?

(e) (2 points) Using the duration-based approximation, what would be the change in the value of the outstanding bonds following a 1% rise in interest rates?

6. (18 points) A private equity investment fund has firm XYZ in its portfolio. Your task is to estimate the value of this firm, which does not trade publicly. XYZ is 100% equity financed. It is now year 0, and you have the following data on XYZ:

Full-year earnings over year 0	\$100M
Payout ratio in year 0	0%
Cost of capital	10%

Based on your market analysis, you forecast that without any new investments, XYZ is expected to generate \$100M in earnings per year in perpetuity. Investments made in year 0 and 1 are expected to generate \$0.20 per year in perpetuity for each \$1 invested, starting in a year following the investment. Starting in year 2, new investments are expected to generate \$0.10 per year in perpetuity for each \$1 invested, starting in a year following the investment.

The payout ratio of XYZ will stay at zero in year 1, rising permanently to 60% afterward. Assume the cost of capital in the above table applies to all future cash flows generated by XYZ (including its future investments and earnings), and will remain constant.

(a) (2 points) Compute the expected earnings of XYZ in year 1.

(b) (3 points) Compute the expected earnings of XYZ in year 2.

(c) (10 points) Compute the market value of XYZ as of year 0.

(d) (3 points) What is the net present value of growth opportunities (PVGO) of XYZ as of year 0? Do not include the net present value of year-0 investment into the PVGO.

7. (15 points) Suppose that asset returns are described by a 2-factor APT model, which applies exactly to all assets:

$$\tilde{r}_i = \bar{r}_i + b_{i1}\tilde{f}_1 + b_{i2}\tilde{f}_2 + \tilde{u}_i, \quad i = 1, 2, \ldots$$

where both factors have unit variance and are uncorrelated with each other.

The risk premia associated with factors 1 and 2 are 20% and 30%, respectively. The risk-free rate is 2%.

(a) (3 points) We are contemplating investing in a stock, which has the following factor loadings: $b_1 = 0.20$ and $b_2 = 0.10$. According to APT, what should be the expected return on this stock?

(b) (6 points) Consider two stocks, 1 and 2, with the following parameters. For stock 1: $b_{11} = 0.2$, $b_{12} = 0.1$, $SD(\tilde{u}_1) = 0.20$. For stock 2: $b_{21} = 0.1$, $b_{12} = 0.4$, $SD(\tilde{u}_2) = 0.25$. Compute the correlation between returns on stocks 1 and 2.

(c) (6 points) Construct a portfolio with stocks 1 and 2 above, with weights w_1 and w_2, with the expected return equal to the risk-free rate. What is w_1?

8. (20 points) A semiconductor company is considering a purchase of a silicon measurement system costing \$500,000. By reducing wasted Silicon, the system is expected to save approximately \$100,000 per year in raw material costs. The system costs \$25,000 to install. The device would reduce approximately 40 hours of work per week provided by an outsourcing service charged at \$40 per hour. Assume there are 52 weeks in a year. The company is expecting to upgrade the entire plant in 3 years at which time they expect to dispose of the equipment. They estimate that they would receive \$25,000 for the equipment. The equipment is to be depreciated on a straight line basis over 3 years. The discount rate of the firm is 10%.

(a) (2 points) Assume no taxes. What is the total cash flow generated by this project in year 0?

(b) (2 points) Assume no taxes. What is the total cash flow generated by this project in year 1?

(c) (2 points) Assume no taxes. What is the total cash flow generated by this project in year 3?

(d) (2 points) Assume no taxes. What is the Net Present Value of the measurement system?

(e) (3 points) Suppose now the corporate tax rate is 35%. What is the total cash flow generated by this project in year 0?

(f) (3 points) Suppose now the corporate tax rate is 35%. What is the total cash flow generated by this project in year 1?

(g) (3 points) Suppose now the corporate tax rate is 35%. What is the total cash flow generated by this project in year 3?

(h) (3 points) Suppose now the corporate tax rate is 35%. What is the Net Present Value of the measurement system?

Solutions

1. (a) False. Discount rates are equal to the opportunity cost of capital. There is no meaningful relation between that and the variance of cash flow forecasts.

(b) True. A firm should select the project with a higher NPV. Even with the same initial investment, project rankings implied by IRR do not necessarily correspond to the project rankings implied by the NPV rule. Remember that IRR and NPV may rank projects differently, and NPV is always the correct rule.

(c) False. Cash flows of a project should be discounted at the project-specific discount rate (rate of return on projects with the same characteristics). And the discount rate of a project may be different from the discount rate of the firm (as a whole).

(d) False. Investors need to reinvest the coupon payments at the prevailing interest rates, which are not known in advance, and which are not equal in expectation to the bond's yield to maturity. The fact that the yield curve is flat does not imply that expected future interest rates are all the same (the expectation's hypothesis need not hold).

(e) True. Real cash flow is risky due to random inflation, and therefore it should be discounted at the appropriate risk-adjusted rate of return, which in general is not the same as the real interest rate. The key of the question is the 'presence of inflation risk', which makes the real cash flow risky.

2. (a) The risk-free bond pays \$1 in each state. Therefore, the price of the bond is

$$B = \$1 \times \phi_1 + \$1 \times \phi_2 + \$1 \times \phi_3 = \$0.4 + \$0.3 + \$0.2 = \$0.9$$

This is the pricing equation of risk free bond using state prices. The risk free rate, by definition, is the expected return of risk free bond, thus the risk free interest rate is

$$r = \frac{\$1}{\$0.9} - 1 = 11.1111\%.$$

(b) The time-0 expected value of the cash flow is

$$\$1 \times \frac{1}{3} + \$2 \times \frac{1}{3} + \$4 \times \frac{1}{3} = \$2.3333$$

Here the probabilities of all three states are all $\frac{1}{3}$.

(c) The time-0 market value of the cash flow equals the sum of the state prices multiplied by the corresponding state-contingent payoffs

$$\begin{aligned} P =& \$1 \times \phi_1 + \$2 \times \phi_2 + \$4 \times \phi_3 \\ =& \$1 \times 0.4 + \$2 \times 0.3 + \$4 \times 0.2 = \$1.8 \end{aligned}$$

3. (a) With monthly compounding, the EAR on each loan is given by

$$EAR = \left(1 + \frac{APR}{12}\right)^{12} - 1$$

We find

$$EAR^{10} = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 6.1678\%, \quad EAR^{5} = \left(1 + \frac{0.067}{12}\right)^{12} - 1 = 6.9096\%$$

(b) We use the following equation to solve for monthly payments at mortgage rate y:

$$\$50,000 = \sum_{t=1}^{T} \frac{M}{(1+y)^t} = \frac{M}{y}\left(1 - \frac{1}{(1+y)^T}\right),$$

which implies

$$M = \frac{\$50,000y}{\left(1 - \frac{1}{(1+y)^T}\right)},$$

where $y = APR/12$ and T is the maturity of the loan in months.

$$M^{10Y} = \frac{50,000 \times \frac{0.06}{12}}{(1 - \frac{1}{(1+\frac{0.06}{12})^{10\times 12}})} = \frac{50,000 \times .005}{(1 - \frac{1}{(1.005)^{120}})} \approx \$555.1025$$

$$M^{5Y} = \frac{50,000 \times \frac{0.067}{12}}{(1 - \frac{1}{(1+\frac{0.067}{12})^{5\times 12}})} = \frac{50,000 \times .00558333}{(1 - \frac{1}{(1.00558333)^{60}})} \approx \$982.9981$$

(c) The monthly interest rate is

$$r^{monthly} = (1.04)^{1/12} - 1 = 0.003274$$

Then the present value of the 10-year loan is

$$NPV^{10Y} = \sum_{t=1}^{10} \frac{555.1025}{(1+0.003274)^t} = \frac{555.1025}{0.003274}\left[1 - \left(\frac{1}{1.003274}\right)^{120}\right] = \$55,011.24$$

The present value of the 5-year loan is

$$NPV^{5Y} = \sum_{t=1}^{5} \frac{982.9981}{(1+0.003274)^t} = \frac{982.9981}{0.003274}\left[1 - \left(\frac{1}{1.003274}\right)^{60}\right] = \$53,469.12$$

(d) No, the five-year contract is a better deal. For the same amount of money upfront \$50,000, the NPV of payments under the five-year contract is lower.

4. (a) Bond A is a zero-coupon bond; the yield to maturity of Bond A is the one-year spot interest rate: $r_1 = 3\%$.

(b) Bond B's spot rate is $r_2 = 4.03\%$. To get the result, solve for r_2 given $r_1 = 3\%$:

$$\frac{6}{(1+r_1)} + \frac{106}{(1+r_2)^2} = \frac{6}{(1.04)} + \frac{106}{(1.04)^2}.$$

$r_2 = 0.0402973$.

(c) Bond C's spot rate is $r_3 = 5.04\%$. Solve for r_3 given r_1, r_2 :

$$\frac{3}{(1+r_1)} + \frac{3}{(1+r_2)^2} + \frac{103}{(1+r_3)^3} = \frac{3}{(1.05)} + \frac{3}{(1.05)^2} + \frac{103}{(1.05)^3}.$$

$r_3 = 0.0504192$.

(d) Denote the new bond as bond D. The no-arbitrage price of the three-year zero-coupon bond is

$$P_D^{no-arbitrage} = \frac{100}{1.0504^3} = \$86.2804.$$

Note that bond D's yield-to-maturity defines its market price,

$$P_D = \frac{100}{1.04^3} = \$88.90$$

and it is different from the no-arbitrage price. Hence, there arbitrage opportunities in the market.

(e) We construct a portfolio of bonds A, B, C, and D that costs $-\$1$ at time $t = 0$ and pays nothing in later periods. First, compute prices of all the bonds from their yields to maturity:

$$P_A = 97.0874; \; P_B = 103.7722; \; P_C = 94.5535; \; P_D = 88.8996.$$

Next, we solve a system of linear equations relating to portfolio positions x_A, x_B, x_C, and x_D to the payoffs in different periods:

$$\begin{bmatrix} -P_A & -P_B & -P_C & -P_D \\ 100 & 6 & 3 & 0 \\ 0 & 106 & 3 & 0 \\ 0 & 0 & 103 & 100 \end{bmatrix} \begin{bmatrix} x_A \\ x_B \\ x_C \\ x_D \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In the above equation system, the first row corresponds to $t = 0$, the second to $t = 1$, etc. The first column corresponds to cash flows from buying bond A, the second to bond B, etc. We find:

$$X_A = 0.0105, \; X_B = 0.0105, \; X_C = -0.3707, \; X_D = 0.3818$$

5. (a) There are two components to this project. The net present value of the investment, and the net present value of cash flows. The net present value of the investment is

$$-\$10M + \frac{-\$5M}{1 + 2\%}.$$

The net present value of cash flows can be represented as a perpetuity at time $t = 1$

$$\text{t=1:} \quad \frac{\$1M}{2\%}$$

Discounting it to time 0 we get

$$\text{t=0:} \quad \frac{1}{1 + 2\%} \times \frac{\$1M}{.02}$$

The sum of both components is

$$NPV = -\$10M + \frac{-\$5M}{(1.02)} + \frac{1}{(1.02)} \frac{\$1M}{.02} = \$34.12M$$

(b) As we discussed before, the net present value of the construction costs is

$$(10 + \frac{5}{1.02}) \times 10^6.$$

The tresuary should raise this amount of money at time 0.
The price of X ten-year zero coupon bonds with face values of \$1,000 is

$$\frac{1,000X}{(1.02)^{10}}.$$

Therefore, the amount of bonds is defined by the following equation

$$(10 + \frac{5}{1.02}) \times 10^6 = 14.90M = \frac{1,000X}{(1.02)^{10}}.$$

The solution is $X = 18,165.41$ bonds.

(c) Since we have a zero-coupon bond, the duration is equal to the maturity of the bond: $D = 10$. The modified duration of the bond is

$$MD = \frac{D}{1+r} = \frac{10}{1.02} = 9.8039.$$

(d) Recall that you can abstract away from how the project is financed. The net present value of the new project is

$$\begin{aligned} NPV^{old} &= -10 - \frac{5}{1.02} + \frac{1}{(1.02)}\frac{1}{.02} = \$34.12M \\ NPV^{new} &= -\ 10 - \frac{5}{1.03} + \frac{1}{(1.03)}\frac{1}{.03} = \$17.51M \end{aligned}$$

The change is then $\Delta NPV = 17.51 - 34.12 = -\$16.61M$.

(e) Denote the original price of a bond by P_0. Using the delta-based approximation, the approximate change in value of just one bond is

$$\Delta P_0 = -P_0 \times MD \times \Delta y.$$

Multiply this equation by the number of all bonds to get the change in the value of the bonds outstanding:

$$\Delta\text{Value of all Bonds} = \Delta P_0 \times X = -\underbrace{\text{Value of all Bonds}}_{P_0 \times X} \times MD \times \Delta y$$

Hence, the approximate change is

$$\Delta\text{Value of all Bonds} = -14.90M \times \frac{10}{1.02} \times .01 = -\$1.46M.$$

6. All the dollar figures below are in millions.

(a) Since the payout ratio of XYZ will stay at zero in year 0, investment at time 0 is the earnings of the company, $Inv_0 = E_0$. A dollar of the investment generates \$0.20 per year in perpetuity, hence,

$$E_1 = E_0 + Inv_0 \times 0.2 = \$100 + \$100 \times 0.2 = \$120.$$

(b) At $t = 1$ there is no pay-out, therefore, $Inv_1 = E_1$. Then,

$$E_2 = E_1 + Inv_1 \times 0.2 = \$120 + \$120 \times 0.2 = \$144.$$

(c) All investments made starting in year 2 are zero-NPV, because each \$1 of investment generates \$0.10 in perpetuity, and the cost of capital for new investments of XYZ is 10%. Hence, for evaluation purposes we can temporarily assume that $Inv_t \equiv 0$ for $t \geq 2$ and that will not affect the market value of the firm.

To compute the market value of XYZ, excluding time-0 cash flows, note that the value of the firm without any investments after time $t = 0$ is perpetuity with payment $E_1 = \$120$. Hence, using COC as a discount rate,

$$V_{0,\,\text{no investment after t=0}} = \frac{\$120}{0.10} = \$1,200.$$

Firm's investments in year 1 generate extra $\$24 = \$144 - \$120$ dollars in every next year. The NPV of the investment is

$$NPV_1 = -\$120 + \frac{\$24}{0.10} = \$120.$$

The time-0 value of XYZ is then sum of the value without the investment and NPV of the investment discounted to time 0,

$$V_0 = V_{0,\,\text{no investment}} + \frac{NPV_1}{1 + 0.10} = \$1,200 + \frac{\$120}{1.1} = \$1309.091.$$

(d) The present value of growth opportunities of XYZ is the difference between the firm value with and without growth:

$$PVGO_0 = V_0 - V_{0,\,\text{no investment}} = \$1,309.09 - \$1,200 = \$109.09.$$

7. (a) According to APT, expected return on the stocks are

$$\mathrm{E}[\tilde{r}_i] - r_f = b_{i1}\lambda_1 + b_{i2}\lambda_2, \quad i = 1, 2, \ldots$$

where $\lambda_1 = 20\%$ and $\lambda_2 = 30\%$ are risk premia on the factors. Hence, applying it to our stock

$$\mathrm{E}[\tilde{r}] = 0.02 + 0.2 \times 0.2 + 0.1 \times 0.3 = 9\%$$

(b) Let us start from finding volatilities of each stock:

$$\sigma_i = \sqrt{b_{i,1}^2 Var[f_1] + b_{i,2}^2 Var[f_2] + Var[\tilde{u}_i]} = 0.3.$$

Note that we used that the factors are uncorrelated.
Volatility of stock 1 is

$$\sigma_1 = \sqrt{0.2^2 + 0.1^2 + 0.2^2} = 0.3.$$

Volatility of stock 2 is

$$\sigma_2 = \sqrt{0.1^2 + 0.4^2 + 0.25^2} = 0.482183,$$

Next, covariance between the two stocks is

$$\mathrm{Cov}_{1,2} = b_{1,1}b_{2,1}Var[f_1] + b_{1,2}b_{2,2}Var[f_2] = 0.2 \times 0.1 + 0.1 \times 0.4 = 6\%$$

Note that we used that the factors and idiosyncratic components are uncorrelated pairwise. Finally, the correlation between stock 1 and stock 2 is

$$\mathrm{Corr}_{1,2} = \frac{\mathrm{Cov}_{1,2}}{\sigma_1 \sigma_2} = 0.414781$$

(c) Let us firstly find the expected return for every stock using APT pricing:

$$\bar{r}_1 = 0.02 + 0.2 \times 0.2 + 0.1 \times 0.3 = 9\%$$
$$\bar{r}_2 = 0.02 + 0.1 \times 0.2 + 0.4 \times 0.3 = 16\%$$

The weight w_1 satisfies

$$w_1 \times 9\% + (1 - w_1) \times 16\% = r_f = 2\%$$

Solving the equation, we get the weights

$$w_1 = 2 \qquad w_2 = 1 - w_1 = -1.$$

8. (a) The following data summarizes the calculation of the free cash flows and the NPV of the investment (in thousands). Notice that the labor saved every year is $40 \times 40 \times 52 = \$83200$. When computing PV, we have

$$166.5455 = \frac{183.2}{1 + 10\%}$$

$$151.4050 = \frac{183.2}{(1 + 10\%)^2}$$

$$156.4237 = \frac{208.2}{(1 + 10\%)^3}$$

Year:	0	1	2	3
CAPX	-500			
Material Saved		100	100	100
Labor Saved		83.2	83.2	83.2
Installation Cost	-25			
Equipment Sale				25
SUM:	-525	183.2	183.2	208.2
PV:	-525	166.5455	151.4050	156.4237

The cash flow in year 0 is -$525 (thousand).

(b) The cash flow in year 1 is $183.2 (thousand).

(c) The cash flow in year 3 is $208.2 (thousand).

(d) The NPV is $-525 + 166.5455 + 151.4050 + 156.4237 = \$50.6258$ (thousand).

(e) Depreciation per year $= \frac{500}{3}$. So the cash flow from depreciation per year is $\frac{500}{3} * \tau = \frac{500}{3} * .35 = 58.3333$ (thousand). The following table updates the **cash flow numbers** to take into account the corporate tax (in thousand).

Year:	0	1	2	3
CAPX	-500			
Material Saved		100(1-.35)	100 (1-.35)	100 (1-.35)
Labor Saved		83.2(1-.35)	83.2(1-.35)	83.2 (1-.35)
Installation Cost	-25(1-.35)			
Equipment Sale				25(1-.35)
Depreciation		$\frac{500}{3} * .35$	$\frac{500}{3} * .35$	$\frac{500}{3} * .35$
SUM:	-516.25	177.4133	177.4133	193.6633
PV:	-516.25	161.2848	146.6226	145.5021

The cash flow in year 0 is -$516.25 (thousand).

(f) The cash flow in year 1 is $177.4133 (thousand).

(g) The cash flow in year 3 is $193.6633 (thousand).

(h) The NPV is $-516.25 + 161.2848 + 146.6226 + 145.5021 = -62.8405$ (thousand).