Recitation 4

Fall 2020

Question 1

There are three bonds available in the market, A, B, and C, all with face value of \$100. Bond A is a zero-coupon bond, with maturity of 1 year, and current price of \$97.50. Bond B pays an annual coupon of 3%, has 2 years to maturity, and is currently traded at \$96.00. Bond C pays an annual coupon of 3.5%, has 3 years to maturity, and is currently traded at \$98.00.

Compute 1-year, 2-year, and 3-year spot interest rates.

Solutions:

Let us write the prices of all bonds as a function of the spot interest rates and coupon rate. Start with bond A:

$$P_A = \frac{\$100}{1 + r_1}.$$

Its price is just a function of the face value, \$100, and the 1-year spot price. Price of bond B is the discounted coupons paid in Year 1 and Year 2 and discounted \$100 of the face value. The appropriate discounting rates are the spot interest rates r_1 and r_2, the last is squared since we get the respective payments only in two years:

$$P_B = \frac{\$3}{1 + r_1} + \frac{\$3 + \$100}{(1 + r_2)^2}.$$

Similarly, price of bond C is

$$P_C = \frac{\$3.5}{1 + r_1} + \frac{\$3.5}{(1 + r_2)^2} + \frac{\$3.5 + \$100}{(1 + r_3)^3}.$$

Given that prices of bonds, P_A, P_B, and P_C, are known, we efficiently have a system of three equations with three unknowns. From first equation we immediately get $r_1 = 2.56\%$[1]. Plug r_1 into the second equation to get another equation

[1] Note that this number, like many others in the recitation, is rounded up to the second digit. Nevertheless, we do not round it when plug it in further equation to avoid accumulation of roundoff error.

with just one unknown:

$$96 = \frac{\$3}{1 + 2.56\%} + \frac{\$3 + \$100}{(1 + r_2)^2}.$$

Solving it get the 2-year spot interest rate $r_2 = 5.20\%$. Plugging both found interest rates into third equation we get

$$98 = \frac{\$3.5}{1 + 2.56\%} + \frac{\$3.5}{(1 + 5.20\%)^2} + \frac{\$3.5 + \$100}{(1 + r_3)^3}.$$

we receive $r_3 = 4.22\%$.

Question 2: Arbitrage

You are a bond trader, and see the below bonds and prices on your screen (these are risk-free bonds):

Bond	Face Value	Maturity	Coupon rate	Price
A	\$100	1	0%	\$96.90
B	\$100	2	2.75%	\$99.00
C	\$100	3	0%	\$88.55

There is another bond, D, that just became available. Bond D has a face value of \$100, 3 years to maturity, and 3% coupon rate. Its current price is \$99.50.

(a) Is there an arbitrage opportunity in this market? If yes, explain how you would take an advantage of this opportunity.

(b) Construct an arbitrage trading strategy that pays off \$100 today and nothing in the future.

Solutions:

(a) Is there an arbitrage opportunity in this market? If yes, explain how you would take an advantage of this opportunity.

First, we should find term structure of interest rates from. We should use prices of bonds A, B, and C to find r_1, r_2, and r_3.

$$\$96.90 = P_A = \frac{\$100}{1 + r_1},$$

$$\$99 = P_B = \frac{\$2.75}{1 + r_1} + \frac{\$2.75 + \$100}{(1 + r_2)^2},$$

$$\$88.55 = P_C = \frac{\$100}{(1 + r_3)^3}.$$

The first and third equation can be solved to $r_1 = 3.20\%$ and $r_3 = 4.14\%$. Plugging r_1 into the second equation solve $r_2 = 3.28\%$.

Given the term structure we can conclude what the price of bond D **should** be

$$P_D = \frac{\$3}{1 + 3.20\%} + \frac{\$3}{(1 + 3.28\%)^2} + \frac{\$3 + \$100}{(1 + 4.14\%)^3} = \$96.93.$$

But the quoted price is \$99.50. Therefore, bond D is overpriced relative to its fair price. This presents an arbitrage opportunity.

To take advantage of this situation, we should sell the overpriced asset or buy the underpriced asset. More specifically, sell the bond D and buy bonds A, B, and D in the amount such that they guarantee cash-flows \$3, \$3, and \$103 in years 1,2, and 3 respectively.

In the next item we will show find the exact amount of each type of bonds to make a given amount of profit at time 0.

(b) Construct an arbitrage trading strategy that pays off \$100 today and nothing in the future.

Suppose we buy x_A of bond A, x_B of bond B, x_C of bond C, and x_D of bond D.

Note that we will allow amounts x_A, x_B, x_C, and x_D to be negative. This would imply taking selling the corresponding bond, or taking the short position. We want to construct a trading strategy that gives us \$100 today. Hence bond prices multiplied by our positions should add up to \$100:

$$-\$96.90x_A - \$99.00x_B - \$88.55x_C - \$99.5x_D = \$100$$

Notice the negative signs. This is because we assumed that we are buying x of each bond, which represents cash **outflow**. Our positions should be such that bond payoffs times positions are zero in subsequent years. Hence, payoff in Year 1 defines equation

$$\$100x_A + \$2.75x_B + \$3.00x_D = \$0,$$

since bond A pays \$100, bond B pays \$2.75, bond C pays \$0, bond D pays \$3.00, and the bond portfolio must pay us zero. Notice the positive signs. This is because payoffs from the bought bonds represent cash **inflow**. Similarly, payoffs in Year 2 and Year 3 return us

$$\$102.75x_B + \$3.00x_D = \$0$$

$$\$100.00x_C + \$103.00x_D = \$0$$

Finally, combining those equations we get the following system of four equations with four unknowns

$$\begin{cases} -\$96.90x_A - \$99.00x_B - \$88.55x_C - \$99.5x_D & = \$100 \\ \$100x_A + \$2.75x_B + \$3.00x_D & = \$0 \\ \$102.75x_B + \$3.00x_D & = \$0 \\ \$100.00x_C + \$103.00x_D & = \$0 \end{cases}$$

It is a linear system and it can be solved in many ways. One of the options is to start from last two equations and express x_B and x_C as functions of x_D:

$$x_B = -\tfrac{3}{102.75}x_D$$

$$x_C = -\tfrac{103}{100}x_D$$

Plugging the result into the second equation the system simplifies to

$$\begin{cases} -\$96.90x_A - \$99.00x_B - \$88.55x_C - \$99.5x_D = \$100 \\ x_A = \frac{\left(\frac{2.75\times 3}{102.75} - 3\right)}{100}x_D \\ x_B = -\frac{3}{102.75}x_D \\ x_C = -\frac{103}{100}x_D \end{cases}$$

Plug x_A, x_B, x_C into the first equation to solve for $x_D = -38.85$, what means we sell (short) 38.85 units of bond D. Plugging X_D into equations for x_A, x_B, and x_C we get

$$x_A = 1.13$$
$$x_B = 1.13$$
$$x_C = 40.02$$

This means we buy (long) the corresponding number of units for each bond A, B, and C.

We can also verify that our trading strategy works. In Year 0, 1, 2 and 3 we get:

$$\begin{cases} -\$96.90 \times 1.13 - \$99.00 \times 1.13 - \$88.55 \times 40.02 - \$99.5 \times x_D = \$100 \\ \$100 \times 1.13 + \$2.75 \times 1.13 + \$3.00 \times -38.85 = \$0 \\ \$102.75 \times 1.13 + \$3.00 \times -38.85 = \$0 \\ \$100.00 \times 40.02 + \$103.00 \times -38.85 = \$0 \end{cases}$$

Let us discuss how to automate solving the arbitrage question. The system of equations can be rewritten in the matrix form.

$$\begin{pmatrix} -96.90 & -99.00 & -88.55 & -99.50 \\ 100.00 & 2.75 & 0.00 & 3.00 \\ 0.00 & 102.75 & 0.00 & 3.00 \\ 0.00 & 0.00 & 100.00 & 103.00 \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \\ x_D \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Denote the 4×4 matrix by A. Then the system is equivalent to

$$A \times x = b,$$

and can be easily solved by inverting the matrix.

$$x = A^{-1}b.$$

To see how to invert the matrix in Excel and solve a system of linear equations watch the video.

Question 3: Arbitrage

Consider a 5-year bond with annual payments, the principal payment of \$100 at maturity, and the coupon rate of 4.5%. The yield to maturity (YTM) on this bond is 3.15%. What is the current price of the bond?

Solutions:

To price a bond with given YTM, all what we need to do is to take all of the cash flows that this bond produces and discount them to present.

$$\text{Price} = \frac{\$4.5}{1+y} + \frac{\$4.5}{(1+y)^2} + \frac{\$4.5}{(1+y)^3} + \frac{\$4.5}{(1+y)^4} + \frac{\$4.5+\$100}{(1+y)^5}.$$

Plugging the number for YTM into the equation we get

$$\text{Price} = \frac{\$4.5}{1+3.15\%} + \frac{\$4.5}{(1+3.15\%)^2} + \frac{\$4.5}{(1+3.15\%)^3} + \frac{\$4.5}{(1+3.15\%)^4} + \frac{\$4.5+\$100}{(1+3.15)^5} = \$106.16.$$

This problem is a good illustration for the following irregularity. Notice that the price of the bond is higher than the face value of the bond. It is possible because the coupon rate is higher than YTM. In this case, we are saying that the bond is **priced at a premium** because coupon rate is higher.

If the coupon rate was the same as YTM, i.e. $y = 4.5\%$, the price of the bond would be exactly its face value, i.e. \$100,

$$\text{Price} = \frac{\$4.5}{1+4.5\%} + \frac{\$4.5}{(1+4.5\%)^2} + \frac{\$4.5}{(1+4.5\%)^3} + \frac{\$4.5}{(1+4.5\%)^4} + \frac{\$4.5+\$100}{(1+4.5)^5} = \$100.$$

In this case, we say that the bond is **priced at par**.

If YTM was higher than the coupon rate then the bond's price would be less than face value, and we would say that the bond is **priced at discount**.

Question 4: Computing YTM

Consider a 3-year bond with annual payments, the principal payment of \$100 at maturity, and the coupon rate of 5.25%. Current spot interest rates are as follows: the 1-year rate is 1.1%, the 2-year rate is 1.15%, and the 3-year rate is 1.50%. Compute the yield to maturity on this bond.

Solutions:

At first step, find the price of the bond similarly to the previous questions

$$P = \frac{\$5.25}{(1+1.1\%)} + \frac{\$5.25}{(1+1.15\%)^2} + \frac{\$5.25+\$100}{(1+1.50\%)^3} = \$110.98.$$

Given the price of the bond, we can find YTM y as the solution of the following equation

$$P = \$110.98 = \frac{\$5.25}{(1+y\%)} + \frac{\$5.25}{(1+y\%)^2} + \frac{\$5.25+\$100}{(1+y\%)^3}.$$

The only issue with equation is that it not linear, so to solve it we will need to use some soft with a numerical solver. In the video we show how to solve it using either RATE function or IRR function. The correct answer is $y = 1.48\%$.

Question 5: Computing duration

Assume that the yield curve is flat at 2%. Compute the Macaulay duration and modified duration for the following bonds:

(a) A zero-coupon \$100 face value bond, maturing in 20 years.

(b) A 5% coupon \$100 face value perpetual bond. Assume that coupons are paid annually.

(c) A 5% coupon \$100 face value bond, maturing in 20 years. Assume that coupons are paid annually.

Solutions:

(a) A zero-coupon \$100 face value bond, maturing in 20 years.

Denote $T = 20$. By definition Macaulay duration:

$$D = \frac{1}{P}\sum_{t=1}^{T} \frac{CF_t}{(1+y)^t} \times t$$

$$D = \frac{1}{P} \times \frac{CF_1}{(1+y)^1} \times 1 + \frac{1}{P} \times \frac{CF_2}{(1+y)^2} \times 2 + \cdots + \frac{1}{P} \times \frac{CF_T}{(1+y)^T} \times T$$

For a zero-coupon bond, all $CF_t = 0$ for $t < T$, and $CF_T = \$100$. Therefore, Macaulay duration for zero-coupon bond simplifies to

$$D = \frac{1}{P} \times \frac{\$100}{(1+y)^T} \times T,$$

i.e. the only term of the original sum which survives is the terminal term.

Notice that bond price for a zero-coupon bond is:

$$P = \frac{\$100}{(1+y)^T}.$$

Hence, Macaulay duration for zero-coupon bond is

$$D = \frac{1}{P} \times P \times T = T.$$

Therefore, **for a zero-coupon bond, Macaulay duration equals maturity of the bond.**

The Macaulay duration of the bond in (a) is 20 years.

Next, modified duration by definition is

$$MD = \frac{D}{1+y}.$$

The modified duration of the bond in (a) is:

$$MD = \frac{20}{1+2\%} = 19.61.$$

(b) A 5% coupon \$100 face value perpetual bond. Assume that coupons are paid annually.

Let us start by deriving modified duration for the perpetual bond. By definition,

$$MD = -\frac{1}{P} \times \frac{dP}{dy}$$

The price of a perpetual bond that pays C coupon annually is

$$P = \sum_{t=1}^{\infty} \frac{C}{(1+y)^t} = \frac{C}{y}.$$

The formula comes from Lecture 3 as the present value of the perpetuity. First, let us find the derivative of the price of the bond with respect to y:

$$\frac{dP}{dy} = \frac{d\frac{C}{y}}{dy} = -\frac{C}{y^2}$$

Plugging the result into the modified duration we get

$$MD = -\frac{1}{P} \times \left(-\frac{C}{y^2}\right) = -\frac{1}{\frac{C}{y}} \times \left(-\frac{C}{y^2}\right) = \frac{1}{y}$$

Using $y = 2\%$, we get the MD of the perpetual bond is 50.

The Macaulay duration for perpetual bond:

$$D = MD \times (1 + y) = \frac{1 + y}{y}$$

In our case,

$$D = \frac{1 + 2\%}{2\%} = 51.$$

Note that coupon rate does not show up in the duration formulas.

(c) A 5% coupon \$100 face value bond, maturing in 20 years. Assume that coupons are paid annually.

Start with Macaulay duration

$$D = \frac{1}{P} \sum_{t=1}^{T} \frac{\$5}{(1 + y)^t} \times t.$$

One can calculate the sum by hand but it is not necessary for the purposes of the class.[2]In the video we show the Excel solution which is based on calculating the sum on the RHS of the equation numerically. Macaulay duration is $D = 14.43$. Note that for the calculations we also need to find the price of bond which equals to sum of the discounted future cash flows

$$P = \sum_{t=1}^{20} \frac{\$5}{(1 + 2\%)^t} + \frac{\$100}{(1 + 2\%)^{20}} = \$149.05.$$

Modified duration can be found in the usual way by discounting Macaulay duration:

$$MD = \frac{D}{(1 + y)} = \frac{14.43}{1 + 2\%} = 14.15.$$

[2]Use the formula for increasing geometric series, $\sum_{t=1}^{T} tX^t = \frac{(Tx - T - 1)x^{T+1} + x}{(1-x)^2}$, to get $D = \frac{1}{P} \sum_{t=1}^{T} \frac{\$5}{(1+y)^t} \times t = \frac{\$5}{P} \times \frac{\left(\frac{1}{y+1}\right)^T + y\left(T\left(\frac{1}{y+1}\right)^T + \left(\frac{1}{y+1}\right)^T - 1\right) - 1}{y^2}$

Question 6: Interest rate risk

You manage a pension fund and your liabilities consist of three payments as follows:

Time (Years)	Payment
10	\$40 million
15	\$75 million
20	\$105 million

The term structure of interest rates is flat at 5.0%.

(a) Use modified duration to determine what happens to the present value of your liabilities when interest rates increase by 0.20%.

(b) Compute the actual change in the present value of your liabilities when interest rates increase by 0.20%.

Solutions:

(a) Use modified duration to determine what happens to the present value of your liabilities when interest rates increase by 0.20%.

Let us compute the present value of cash flows:

$$P = \frac{\$40}{(1+5\%)^{10}} + \frac{\$75}{(1+5\%)^{15}} + \frac{\$105}{(1+5\%)^{20}}$$

The present value of cash flows (our liabilities) is \$100.21 million

Recall definition of Macaulay duration:

$$D = \frac{1}{P}\sum_{t=1}^{T} \frac{CF_t}{(1+y)^t} \times t.$$

Hence,

$$D = \frac{1}{100.21}\left(\frac{40}{(1+5\%)^{10}} \times 10 + \frac{75}{(1+5\%)^{15}} \times 15 + \frac{105}{(1+5\%)^{20}} \times 20\right)$$

The Macaulay duration of liabilities is 15.75 year.

The modified duration is linked to Macaulay duration and,

$$MD = \frac{D}{1+y} = \frac{15.75}{1+5\%} = 15$$

Using definition of modified duration,

$$MD = -\frac{1}{P} \times \frac{dP}{dy}$$

we can get the approximate change in the value of liabilities:

$$\Delta P = -P \times MD \times \Delta y.$$

In our case, $\Delta y = 0.2\%$ therefore:

$$\Delta P = -\$100.21 \times 15 \times 0.2\% = -\$3.00.$$

Therefore, the value of our liabilities decreases by approximately \$3 million if the yield curve goes up by 0.2%. Note that the answer is received by using approximation only. We will compare it with the real change in the next item.

(b) Compute the actual change in the present value of your liabilities when interest rates increase by 0.20%.

New discount rate is $y = 5.2\%$.

$$P = \frac{\$40}{(1+5.2\%)^{10}} + \frac{\$75}{(1+5.2\%)^{15}} + \frac{\$105}{(1+5.2\%)^{20}}$$

The present value of liabilities is \$97.25 million. Therefore, the actual decrease in value is

$$\$100.21 - \$97.25 = \$2.96 \text{ million}$$

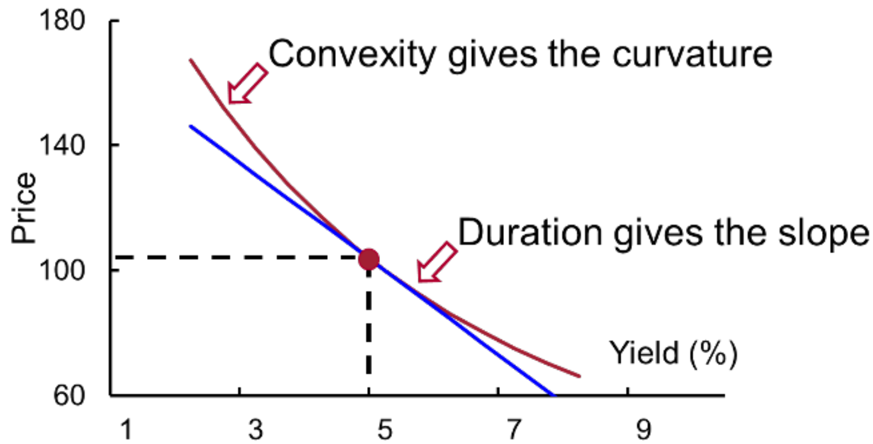
Note that the approximation in (a) did a good job approximating actual decrease. Though the values are not identically the same. The actual change is less than what duration approximation gives us.

Let us understand why the actual change is less. The red line of the picture below shows how the price of a bond depends on the yield. Since every discounted cash-flow term in the value of a bond is a convex function in the yield, the bond value is also the convex function in the yield.[3] The actual decrease in the value of the liabilities can be found by moving along the red line. Modified duration helps to find the approximate change along the blue line which is the tangent line with the slope proportional to modified duration. Since the tangent line is always below the convex red line we get that approximation will give us always larger decrease in the price than the actual one in response to raise of the discounting rate.

Question 7: Computing duration

Assume that the yield curve is flat at 3.5%. There is a \$100 face value bond that pays annual coupons and has 5 years to maturity. The coupon rate is 5%.

[3] Note that $\frac{a}{(1+y)^t}$ is a convex function in variable y. Hence $\sum_{t=1}^{T} \frac{a_t}{(1+y)^t}$ is a convex function in variable y.



(a) Compute modified duration of the bond.

(b) Compute the convexity of the bond.

(c) Suppose that interest rates drop by 1.5% at all maturities (i.e., there is a parallel shift in the yield curve downwards). Use modified duration to determine a first-order approximate change in the bond price.

(d) Suppose that interest rates drop by 1.5% at all maturities. Use modified duration and convexity to determine a second-order approximate change in the bond price.

(e) Suppose that interest rates drop by 1.5% at all maturities. Compute the exact change in the bond price. Compare it approximations obtained in (c) and (d).

Solution:

(a) Compute modified duration of the bond.

The solution repeats calculations we did in question 5. The price of the bond is

$$P = \sum_{t=1}^{5} \frac{CF_t}{(1+y)^t} = \sum_{t=1}^{5} \frac{\$5}{(1+3.5\%)^t} + \frac{\$100}{(1+3.5\%)^5} = \$106.77.$$

Macaulay Duration is

$$D = \frac{1}{P} \sum_{t=1}^{5} \frac{CF_t}{(1+y)^t} \times t = \frac{1}{\$106.77} \sum_{t=1}^{5} \frac{CF_t}{(1+3.5\%)^t} \times t = 4.56$$

Finally, the modified duration is linked to Macaulay Duration:

$$MD = \frac{D}{1+y} = \frac{4.56}{1+3.5\%} = 4.41.$$

(b) Compute the convexity of the bond.

Let us start with definition of bond convexity:

$$CX = \frac{1}{2}\frac{1}{P}\frac{\mathrm{d}^2 P}{\mathrm{d}y^2}$$

Let's derive convexity formula. Bond price is the sum of discounted cash flows:

$$P = \sum_{t=1}^{T} \frac{CF_t}{(1+y)^t}$$

Thus, first derivative of the bond price is

$$\frac{dP}{dy} = -\sum_{t=1}^{T} \frac{t \times CF_t}{(1+y)^{t+1}},$$

and second derivative is

$$\frac{d^2 P}{dy^2} = \sum_{t=1}^{T} \frac{t \times (t+1) \times CF_t}{(1+y)^{t+2}}$$

$$\frac{d^2 P}{dy^2} = \frac{1}{(1+y)^2} \sum_{t=1}^{T} \frac{t \times (t+1) \times CF_t}{(1+y)^t}$$

Convexity of the bond is thus

$$CX = \frac{1}{2}\frac{1}{P}\frac{1}{(1+y)^2} \sum_{t=1}^{T} \frac{t \times (t+1) \times CF_t}{(1+y)^t}.$$

$$CX = \frac{1}{2}\frac{1}{P}\frac{1}{(1+y)^2} \sum_{t=1}^{T} PV\left(CF_t\right) \times t \times (t+1)$$

To make the calculations we use again Excel (watch the video).

$$CX = 12.38$$

(c) Suppose that interest rates drop by 1.5% at all maturities (i.e., there is a parallel shift in the yield curve downwards). Use modified duration to determine a first-order approximate change in the bond price.

Similar to Question 6, approximation of price change can be found as product of modified duration with current price of the bond and the change in the interest rate.

$$\Delta P = -P \times MD \times \Delta y$$

Plugging the numbers into the equation we get

$$\Delta P = -\$106.77 \times 4.41 \times (-1.5\%)$$

$$\Delta P = \$7.06$$

(d) Suppose that interest rates drop by 1.5% at all maturities. Use modified duration and convexity to determine a second-order approximate change in the bond price.

The second-order approximate change in the bond price is given by the following expression,

$$\Delta \mathrm{P} = \mathrm{P} \times \left(-MD \times \Delta \mathrm{y} + \mathrm{CX} \times \Delta \mathrm{y}^2\right)$$

Note that this is the approximate change we found in part (c) with small additional term $\mathrm{P} \times \mathrm{CX} \times \Delta \mathrm{y}^2$ which increases the approximation.

$$\Delta \mathrm{P} = \$106.77 \times \left(-4.41 \times (-1.5\%) + 12.38 \times (-1.5\%)^2\right)$$

$$\Delta P = \$7.36$$

(e) Suppose that interest rates drop by 1.5% at all maturities. Compute the exact change in the bond price. Compare it approximations obtained in (c) and (d).

New YTM is 2%. We can calculate the new bond price,

$$P = \sum_{t=1}^{5} \frac{\$5}{(1+2\%)^t} + \frac{\$100}{(1+2\%)^5} = \$114.14,$$

and get the actual change in value of the bond:

$$\Delta P = \$114.14 - \$106.77 = \$7.37$$

Note that approximate change in (d) is much closer to the actual change than the on we got in (c).