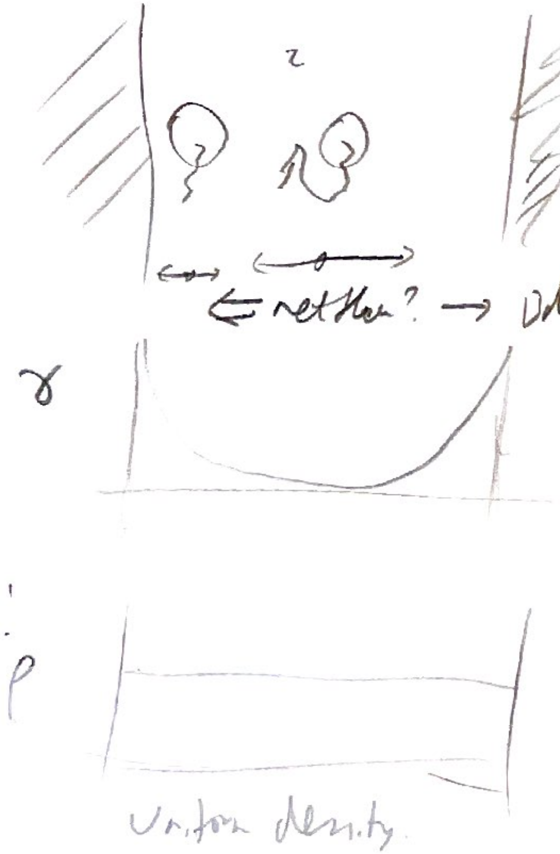
#17

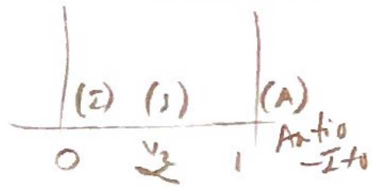
(2)

Or

Overdamped, Multiplicative



Answer depends on stochastic calculus.



→ Must be extra Brownian force →
(Math is tedious)

$$\dot{x} = \frac{1}{\gamma(x)} f(x) + \sqrt{\frac{2}{\beta\gamma}}\,\xi(t) + \frac{1}{\beta\gamma^2}\frac{\partial \gamma}{\partial x} \quad (5)$$

"Spurious force"

$2(\alpha-1)$

But!

Ito also has Extra force term

LAW & LUBENSKY 2007

BERKELEY 2023
#16

Overdamped, Multiplicative, Action

$$P(\xi) = \exp\left(-\frac{1}{2}\int \xi^2 \, dt\right)$$

(unit variance)

(1 degree of freedom)

$$P[x \mid x_0] = \mathcal{N} e^{-A}$$

$$(s) \quad A = \int_0^{\tau} dt \left[\dot{x} - \frac{f(x)}{\gamma} - \frac{1}{\beta \gamma^2} \gamma'(x) \right]^2 \frac{\beta \gamma}{4} + \frac{1}{2} f'(x)$$

$$f(x) = -\frac{\partial V(x)}{\partial x}$$

$$E = U(x) + \frac{1}{\beta \gamma}$$

As if extra energy term

$$\frac{\partial E}{\partial x} = f(x) + \frac{1}{\beta \gamma^2} \gamma'(x)$$

FI (1)

FISHER INFORMATION

Fisher Information

Ronald Fisher 1890-1962

Galton Prof of Eugenics, University College, London

$p(x;\lambda)$

$$I(\lambda) = \int dx\, p(x|\lambda) \underbrace{\left(\frac{\partial \ln p(x|\lambda)}{\partial \lambda}\right)}_{\text{Score}}^2 = \left\langle \left(\frac{\partial \ln p(x|\lambda)}{\partial \lambda}\right)^2 \right\rangle = -\left\langle \frac{\partial^2}{\partial \lambda^2} \ln p(x|\lambda) \right\rangle$$

$$= \mathrm{Var}\left(\frac{\partial \ln p(x|\lambda)}{\partial \lambda}\right)$$

Variance of the Score

Refs: Cover & Thomas
threeplusone.com/fisher

→ Different from previous information measures.
Defined for a family $p(x|\lambda)$

OVERDAMPED LANGEVIN
MULTIPLICATIVE NOISE   LAU & LUBENSKY   (1)

Langevin, Multiplicative Noise

$$m\ddot{x} = f(x) - \gamma(x)\,\dot{x}(t) + \sqrt{\frac{2\gamma(x)}{\beta}}\;\xi(t)$$

"Multiplicative noise"

$$\gamma(x) = \beta\int_0^{\infty}\left\langle \delta f_x(0)\;\delta f_x(t)\right\rangle dt$$

$\frac{\partial F}{\partial x}$ ← free energy

Overdamped, stochastic [illegible] independent

Overdamped, no inertia   Formally $m\ddot{x} = 0$

$$\dot{x} = \frac{1}{\gamma}f(x) + \sqrt{\frac{2}{\beta\gamma}}\;\xi(t)$$

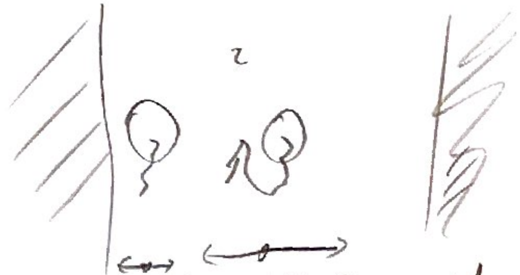
For fixed γ

#16

2

Or

Overdamped, Multiplicative



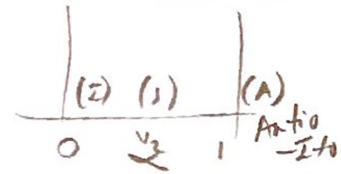
⇐ net flow? → Diffusion force.

γ

But! ρ

Uniform density.

Answer depends on stochastic calculus!



Must be extra ...... force →
(Math is tedious)

$$\dot{x} = \frac{1}{\gamma(x)} f(x) + \sqrt{\frac{2}{\beta\gamma}}\,\xi(t) + \frac{1}{\beta\gamma^2}\frac{\partial\gamma}{\partial x} \quad (S)$$

"Spurious force"

$2(\alpha-1)$

Ito also has Extra force term

LAU & LUBENSKY 2007

BERKELEY 2023
#16

Overdamped, Multiplicative, Action

(1 degree of freedom)

$$P(\xi) = \exp\left(-\frac{1}{2}\int \xi^2 \, dt\right)$$

(unit variance)

$$P[x|x_0] = \mathcal{N} e^{-A}$$

$$(s) \quad A = \int_0^{\tau} dt \left[\dot{x} - \frac{f(x)}{\gamma} - \frac{1}{\beta\gamma^2}\gamma'(x)\right]^2 \frac{\beta\gamma}{4} + \frac{1}{2} f'(x)$$

$$f(x) = -\frac{\partial V(x)}{\partial x} \qquad E = U(x) + \frac{1}{\beta\gamma}$$

As if extra energy term

$$\frac{\partial E}{\partial x} = f(x) + \frac{1}{\beta\gamma^2}\gamma'(x)$$

Underdamped Multiplicative Langevin

<u>Trajectory</u> Fluctuation Theorem

Entropy Production / Heat Flow

Trajectory Fluctuation Theorem

$$\Sigma = A - \tilde{A}$$

$$\frac{P(X \mid x_0)}{P(\tilde{X} \mid x_\tau)} = e^{-(A - \tilde{A})} = e^{\Sigma}$$

$$\Sigma = \int_0^\tau dt \left[\dot{x} \left(f(x) + \overbrace{\frac{1}{\beta \gamma^2} \gamma'(x)}^{? \ \ast} \right) \right] \beta$$

$$\text{Heat } Q = \int_0^\tau \frac{\partial E}{\partial x} \frac{\partial x}{\partial t} \, dt$$

FISHER INFORMATION

FI ①

Fisher Information

Ronald Fisher 1890-1962
Galton Prof of Eugenics, University College, London

$p(x;\lambda)$

$$I(\lambda) = \int dx\, p(x|\lambda) \underbrace{\left(\frac{\partial \ln p(x|\lambda)}{\partial \lambda}\right)^2}_{\text{Score}} = \left\langle \left(\frac{\partial \ln p(x|\lambda)}{\partial \lambda}\right)^2 \right\rangle = -\left\langle \frac{\partial^2}{\partial \lambda^2} \ln p(x|\lambda) \right\rangle$$

$$= \mathrm{Var}\left(\frac{\partial \ln p(x|\lambda)}{\partial \lambda}\right)$$

Variance of the Score

Refs: Cover & Thomas
threeplusone.com/fisher

→ Different from previous information measures.
Defined for a family $p(x|\lambda)$

FI (2)

$$\frac{\partial \ln f(\lambda)}{\partial \lambda} = \frac{1}{f(\lambda)} \frac{\partial f(\lambda)}{\partial \lambda}$$

$$\left\langle \frac{\partial}{\partial \lambda} \ln p(x|\lambda) \right\rangle = \int dx\, p(x|\lambda) \frac{1}{p(x|\lambda)} \frac{\partial}{\partial \lambda} p(x|\lambda)$$

Mean Score = 0

$$= \frac{\partial}{\partial \lambda} \int dx\, p(x|\lambda) \quad = \frac{\partial}{\partial \lambda} 1 = 0$$

$$-\left\langle \frac{\partial^2}{\partial \lambda^2} \ln p(x|\lambda) \right\rangle = -\int dx\, p(x|\lambda) \frac{\partial}{\partial \lambda}\left[\frac{1}{p(x|\lambda)} \frac{\partial p(x|\lambda)}{\partial \lambda} \right]$$

$$= \underbrace{\int dx \frac{1}{p(x|\lambda)} \left[\frac{\partial}{\partial \lambda} p(x|\lambda) \right]^2}_{I(\lambda)} - \int dx \frac{\partial^2 p(x|\lambda)}{\partial \lambda^2}$$

$$= 0 \quad \left(\frac{\partial^2}{\partial \lambda^2} \int dx\, p(x|\lambda) = 0 \right)$$

FI ③

Csiszár f-divergences

"che-sor"

$$C_f(A:B) = \sum_x P_A(x)\, f\left(\frac{P_A(x)}{P_B(x)}\right)$$

f is convex ⌣
$f(1) = 0$

$$D(A\|B) = C_f(A;B) \qquad f(x) := -\ln x$$

$$\text{Jeffreys}(A;B) = D(A\|B) + D(B\|A) \qquad (x-1)\ln x$$

Jensen-Shannon → metric

$$JS(A;B) = \tfrac{1}{2} D(A\|M) + \tfrac{1}{2} D(B\|M)$$

$$P_M(x) = \tfrac{1}{2}\left(P_A(x) + P_B(x)\right)$$

Pearson χ^2-divergence

$$\text{Pearson}(A;B) = \sum \frac{\left[P_A(x) - P_B(x)\right]^2}{P_A(x)} = \sum P_A(x)\left[\frac{P_B(x)}{P_A(x)} - 1\right]^2$$

$$\mathcal{I}(\lambda) = f''(1)\,\lambda^2\, C_f(P_\theta ; P_{\theta+\lambda})$$

$\cancel{g(x)}\ x$

② Θ

FL4

---

f-divergence Relate to Fisher Information

Taylor series
$$g(x) = g(a) + \frac{g'(a)}{1!}(x-a) + \frac{g''(a)}{2!}(x-a)^2 \dots$$

$x = \lambda\Theta \quad a = \lambda$

$\cancel{G_f(P_{\lambda+\Theta}, P_\lambda)}$, $\quad G_f(A||B) =$

$$G_f(P_\lambda : P_{\lambda+\Theta}) = \underbrace{G_f(P_\lambda ; P_\lambda)}_{0} + \underbrace{\Theta \sum P_\lambda(x)\, f'\left(\frac{P_\lambda(x)}{P_\lambda(x)}\right) \frac{1}{P_\lambda(x)} \frac{\partial P_{\lambda+\Theta}}{\partial \Theta}}_{0}$$

$$+ \underbrace{\frac{\Theta^2}{2} \sum P_\lambda(x)\, f''(1) \left(\frac{1}{P_\lambda(x)} \frac{\partial P_{\lambda+\Theta}}{\partial \Theta}\right)^2}_{\frac{\Theta^2}{2} f''(1)\, I(\lambda)} + \underbrace{\frac{\Theta^2}{2} \sum P_\lambda(x)\, f'(1) \frac{1}{P_\lambda(x)} \frac{\partial^2 P_{\lambda+\Theta}}{\partial \Theta^2}}_{0}$$

$$\frac{\Theta^2}{2} f''(1) \quad I(\lambda)$$

~~7~~/5

3

CRAMER - RAO BOUND

$T(x)$ unbiased estimator of λ, i.e. $\langle T \rangle = \lambda$

$$\mathrm{var}(T) \geqslant \frac{1}{I(\lambda)}$$

Bounds variance of unbiased estimator

or if $\langle T \rangle \neq \lambda$

$$\frac{\mathrm{var}(T)}{\left(\frac{\partial \langle T \rangle}{\partial \lambda}\right)^2} \geqslant \frac{1}{I(\lambda)}$$

$$I_{A,B}(\lambda) = I_A(\lambda) + I_B(\lambda)$$

A & B independent

$\therefore$ for N samples, $I = N\, I_A(\lambda)$

~~16~~
4

FI & Equilibrium Stat Mech.

$$P(x|\lambda) = e^{+\beta F(\lambda) - \beta E(x,\lambda)}$$

Score $\dfrac{\partial \ln P(x|\lambda)}{\partial \lambda} = \underbrace{\beta \dfrac{\partial F}{\partial \lambda}} - \beta \dfrac{\partial E}{\partial \lambda} = -\beta\left(\dfrac{\partial E}{\partial \lambda} - \left\langle \dfrac{\partial E}{\partial \lambda} \right\rangle \right)$

$$\beta F = -\ln \sum e^{-\beta E(x,\lambda)}$$

$$\beta \frac{\partial F}{\partial \lambda} = \left\langle \frac{\partial E}{\partial \lambda} \right\rangle$$

$$\mathcal{I}(\lambda) = \left\langle \left(\frac{\partial E}{\partial \lambda} - \left\langle \frac{\partial E}{\partial \lambda} \right\rangle \right)^2 \right\rangle$$

Variance of fluctuations

e.g. Thermodynamic uncertainty Relation

$$\mathcal{I}(\beta) = \langle (E - \langle E \rangle)^2 \rangle \geq \frac{1}{\mathrm{Var}(\hat{\beta})}$$

Jeffreys $(\Omega \,||\, \tilde{\Omega}) = \beta \langle W \rangle_{\Omega} + \beta \langle W \rangle_{\tilde{\Omega}}$ hysteresis.

Jensen-Shannon $(\Omega \,||\, \hat{\Omega})) \overset{?}{=}$ Information gain