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Modern Statistics: A Computer Based Approach with Python

Solutions

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Contents

Chapter 1
Analyzing Variability: Descriptive Statistics

Import required modules and define required functions

```
import math
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import mistat
from scipy import stats

def trim_std(data, alpha):
  """ Calculate trimmed standard deviation """
  data = np.array(data)
  data.sort()
  n = len(data)
  low = int(n * alpha) + 1
  high = int(n * (1 - alpha))
  return data[low:(high + 1)].std()
```

Solution 1.1 `random.choices` selects k values from the list using sampling with replacement.

```
import random
random.seed(1)
values = random.choices([1, 2, 3, 4, 5, 6], k=50)
```

`Counter` counts the number of occurrences of a given value in a list.

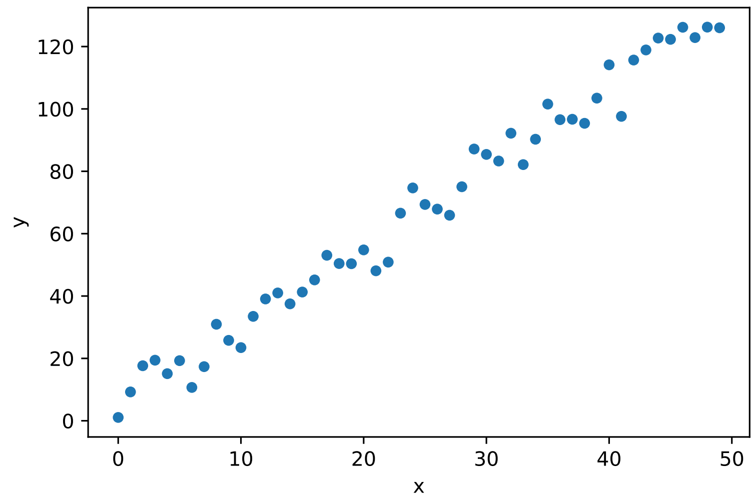
```
from collections import Counter
Counter(values)
```

```
Counter({1: 9, 6: 9, 5: 8, 2: 10, 3: 10, 4: 4})
```

The expected frequency in each cell, under randomness is $50/6 = 8.3$. You will get different numerical results, due to randomness.

Solution 1.2 The Python function `range` is an iterator. As we need a list of values, we need to explicitly convert it.

```
x = list(range(50))
y = [5 + 2.5 * xi for xi in x]
y = [yi + random.uniform(-10, 10) for yi in y]
pd.DataFrame({'x': x, 'y': y}).plot.scatter(x='x', y='y')
plt.show()
```



Solution 1.3 In Python

```
from scipy.stats import binom
np.random.seed(1)

for p in (0.1, 0.3, 0.7, 0.9):
  X = binom.rvs(1, p, size=50)
  print(p, sum(X))
```

```
0.1 4
0.3 12
0.7 33
0.9 43
```

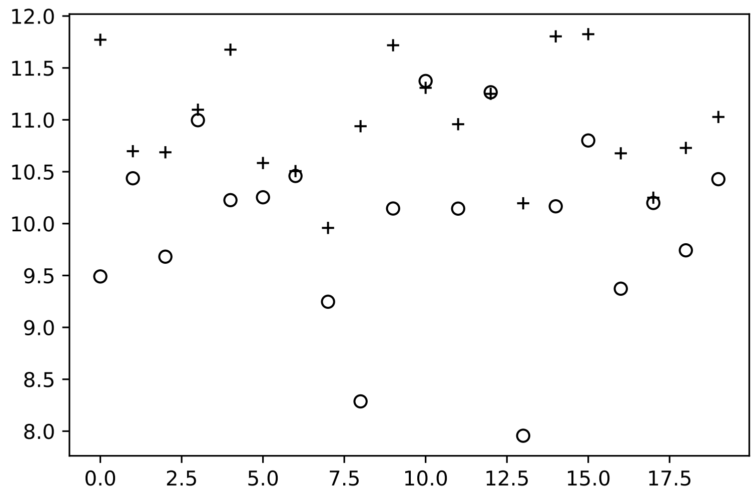
Notice that the expected values of the sums are 5, 15, 35 and 45.

Solution 1.4 We can plot the data and calculate mean and standard deviation.

```
inst1 = [9.490950, 10.436813, 9.681357, 10.996083, 10.226101, 10.253741,
         10.458926, 9.247097, 8.287045, 10.145414, 11.373981, 10.144389,
         11.265351, 7.956107, 10.166610, 10.800805, 9.372905, 10.199018,
         9.742579, 10.428091]
inst2 = [11.771486, 10.697693, 10.687212, 11.097567, 11.676099,
         10.583907, 10.505690, 9.958557, 10.938350, 11.718334,
         11.308556, 10.957640, 11.250546, 10.195894, 11.804038,
         11.825099, 10.677206, 10.249831, 10.729174, 11.027622]
ax = pd.Series(inst1).plot(marker='o', linestyle='none',
                           fillstyle='none', color='black')
pd.Series(inst2).plot(marker='+', linestyle='none', ax=ax,
                      fillstyle='none', color='black')
```

```
plt.show()

print('mean inst1', np.mean(inst1))
print('stdev inst1', np.std(inst1, ddof=1))
print('mean inst2', np.mean(inst2))
print('stdev inst2', np.std(inst2, ddof=1))
```

```
mean inst1 10.03366815
stdev inst1 0.8708144577963102
mean inst2 10.98302505
stdev inst2 0.5685555119253366
```



As shown in the following Figure, the measurements on Instrument 1, ◯, seem to be accurate but less precise than those on Instrument 2, +. Instrument 2 seems to have an upward bias (inaccurate). Quantitatively, the mean of the measurements on Instrument 1 is $\bar{X}_1 = 10.034$ and its standard deviation is $S_1 = 0.871$. For Instrument 2 we have $\bar{X}_2 = 10.983$ and $S_2 = 0.569$.

Solution 1.5 If the scale is inaccurate it will show on the average a deterministic component different than the nominal weight. If the scale is imprecise, different weight measurements will show a high degree of variability around the correct nominal weight. Problems with stability arise when the accuracy of the scale changes with time, and the scale should be recalibrated.

Solution 1.6 The method `random.choices` creates a random selection with replacement. Note that in `range(start, end)` the `end` argument is excluded. We therefore need to set it to 101.

```
import random
random.choices(range(1, 101), k=20)
```

```
[6, 88, 57, 20, 51, 49, 36, 35, 54, 63, 62, 46, 3, 23, 18, 59, 87, 80,
80, 82]
```

Solution 1.7 The method `random.choices` creates a random selection without replacement.

```
import random
random.sample(range(11, 31), 10)
```

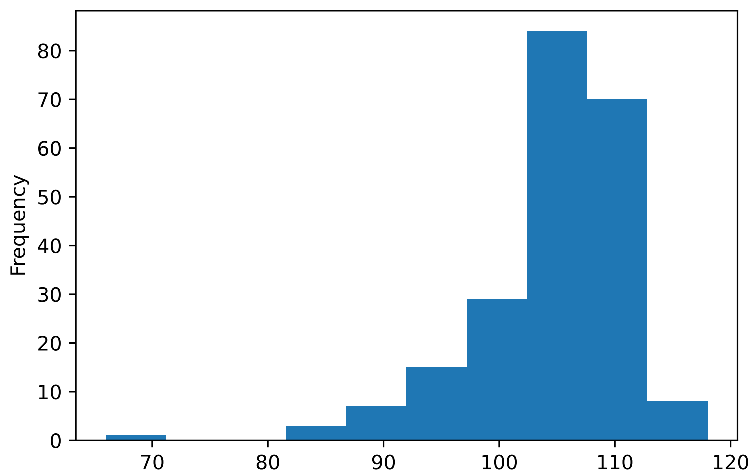
```
[19, 12, 13, 28, 11, 18, 26, 23, 15, 14]
```

Solution 1.8 **(i)** $26^5 = 11,881,376$; **(ii)** $7,893,600$; **(iii)** $26^3 = 17,576$; **(iv)** $2^{10} = 1,024$; **(v)** $\binom{10}{5} = 252$.

Solution 1.9 (i) discrete;
(ii) discrete;
(iii) continuous;
(iv) continuous.

Solution 1.10

```
filmsp = mistat.load_data('FILMSP')
filmsp.plot.hist()
plt.show()
```



Solution 1.11

```
coal = mistat.load_data('COAL')
pd.DataFrame(coal.value_counts(sort=False))
```

```
   COAL
3    17
6     3
4     6
0    33
5     5
2    18
7     1
1    28
```

Solution 1.12 For (1) and (2), we can use the pandas `value_counts` method. e.g.:

```
car = mistat.load_data('CAR')
car['cyl'].value_counts(sort=False)
```

```
4    66
6    30
8    13
Name: cyl, dtype: int64
```

(i) Frequency distribution of number of cylinders:

Cyl	4	6	8	Total
Frequency	66	30	13	109

(ii) Frequency distribution of car's origin:

Origin	U.S.	Europe	Asia	Total
Frequency	58	14	37	109

For (3) to (5), we need to bin the data first. We can use the pandas `cut` method for this.

(iii) Frequency distribution of Turn diameter: We determine the frequency distribution on 8 intervals of length 2, from 28 to 44.

```
pd.cut(car['turn'], bins=range(28, 46, 2)).value_counts(sort=False)
```

```
(28, 30]     3
(30, 32]    16
(32, 34]    16
(34, 36]    26
(36, 38]    20
(38, 40]    18
(40, 42]     8
(42, 44]     2
Name: turn, dtype: int64
```

Note that the bin intervals are open on the left and closed on the right.

(iv) Frequency distribution of Horsepower:

```
pd.cut(car['hp'], bins=range(50, 275, 25)).value_counts(sort=False)
```

```
(50, 75]       7
(75, 100]     34
(100, 125]    22
(125, 150]    18
(150, 175]    17
(175, 200]     6
(200, 225]     4
(225, 250]     1
Name: hp, dtype: int64
```

(v) Frequency Distribution of MPG:

```
pd.cut(car['mpg'], bins=range(9, 38, 5)).value_counts(sort=False)
```

```
(9, 14]      1
(14, 19]    42
(19, 24]    41
(24, 29]    22
(29, 34]     3
Name: mpg, dtype: int64
```

Solution 1.13

```
filmsp = mistat.load_data('FILMSP')
filmsp = filmsp.sort_values(ignore_index=True)  # sort and reset index

print(filmsp.quantile(q=[0, 0.25, 0.5, 0.75, 1.0]))
print(filmsp.quantile(q=[0.8, 0.9, 0.99]))
```

```
0.00     66.0
0.25    102.0
0.50    105.0
0.75    109.0
1.00    118.0
Name: FILMSP, dtype: float64
0.80    109.8
0.90    111.0
0.99    114.0
Name: FILMSP, dtype: float64
```

Here is a solution that uses pure Python. Note that the `pandas` quantile implements different interpolation methods which will lead to differences for smaller datasets. We therefore recommend using the library method and select the method that is most suitable for your use case.

```
def calculate_quantile(x, q):
  idx = (len(x) - 1) * q
  left = math.floor(idx)
  right = math.ceil(idx)
  return 0.5 * (x[left] + x[right])

for q in (0, 0.25, 0.5, 0.75, 0.8, 0.9, 0.99, 1.0):
  print(q, calculate_quantile(filmsp, q))
```

```
0 66.0
0.25 102.0
0.5 105.0
0.75 109.0
0.8 109.5
0.9 111.0
0.99 114.0
1.0 118.0
```

Solution 1.14

```
filmsp = mistat.load_data('FILMSP')
n = len(filmsp)
mean = filmsp.mean()
deviations = [film - mean for film in filmsp]
S = math.sqrt(sum(deviation**2 for deviation in deviations) / n)

skewness = sum(deviation**3 for deviation in deviations) / n / (S**3)
kurtosis = sum(deviation**4 for deviation in deviations) / n / (S**4)
print('Python:\n',
      f'Skewness: {skewness}, Kurtosis: {kurtosis}')

print('Pandas:\n',
      f'Skewness: {filmsp.skew()}, Kurtosis: {filmsp.kurtosis()}')
```

```
Python:
 Skewness: -1.8098727695275856, Kurtosis: 9.014427238360716
Pandas:
 Skewness: -1.8224949285588137, Kurtosis: 6.183511188870432
```

The distribution of film speed is negatively skewed and much steeper than the normal distribution. Note that the calculated values differ between the methods.

Solution 1.15 The pandas `groupby` method groups the data based on the value. We can then calculate individual statistics for each group.

```
car = mistat.load_data('CAR')
car['mpg'].groupby(by=car['origin']).mean()
car['mpg'].groupby(by=car['origin']).std()
# calculate both at the same time
print(car['mpg'].groupby(by=car['origin']).agg(['mean', 'std']))
```

```
             mean       std
origin
1       20.931034  3.597573
2       19.500000  2.623855
3       23.108108  4.280341
```

Solution 1.16 We first create a subset of the data frame that contains only US made cars and then calculate the statistics for this subset only.

```
car = mistat.load_data('CAR')
car_US = car[car['origin'] == 1]
gamma = car_US['turn'].std() / car_US['turn'].mean()
```

Coefficient of variation gamma = 0.084.

Solution 1.17

```
car = mistat.load_data('CAR')

car_US = car[car['origin'] == 1]
car_Asia = car[car['origin'] == 3]
print('US')
print('mean', car_US['turn'].mean())
print('geometric mean', stats.gmean(car_US['turn']))
print('Japanese')
print('mean', car_Asia['turn'].mean())
print('geometric mean', stats.gmean(car_Asia['turn']))
```

```
US
mean 37.203448275862065
geometric mean 37.06877691910792
Japanese
mean 33.04594594594595
geometric mean 32.97599107553825
```

We see that $\bar{X}$ is greater than G. The cars from Asia have smaller mean turn diameter.

Solution 1.18

```
filmsp = mistat.load_data('FILMSP')

Xbar = filmsp.mean()
S = filmsp.std()
```

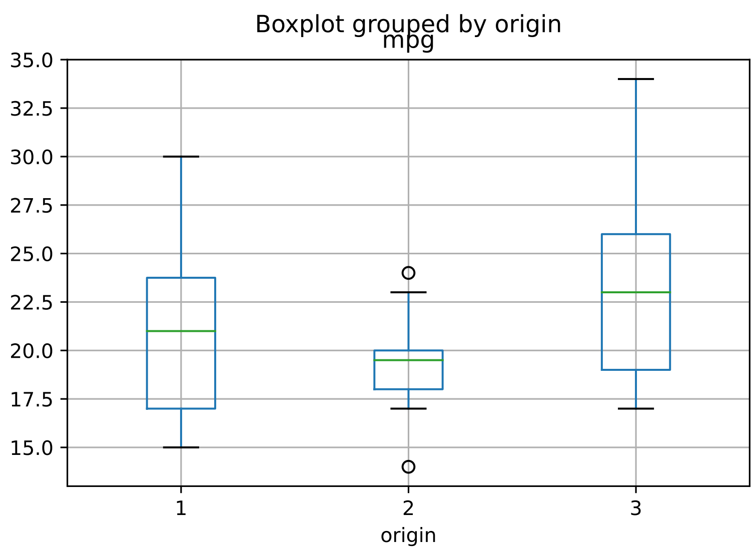
```
print(f'mean: {Xbar}, stddev: {S}')
expected = {1: 0.68, 2: 0.95, 3: 0.997}
for k in (1, 2, 3):
  left = Xbar - k * S
  right = Xbar + k * S
  proportion = sum(left < film < right for film in filmsp)
  print(f'X +/- {k}S: ',
        f'actual freq. {proportion}, ',
        f'pred. freq. {expected[k] * len(filmsp):.2f}')
```

```
mean: 104.59447004608295, stddev: 6.547657682704987
X +/- 1S:  actual freq. 173,  pred. freq. 147.56
X +/- 2S:  actual freq. 205,  pred. freq. 206.15
X +/- 3S:  actual freq. 213,  pred. freq. 216.35
```

The discrepancies between the actual frequencies to the predicted frequencies are due to the fact that the distribution of film speed is neither symmetric nor bell-shaped.

Solution 1.19

```
car = mistat.load_data('CAR')
car.boxplot(column='mpg', by='origin')
plt.show()
```



Solution 1.20

```
oturb = mistat.load_data('OTURB')
mistat.stemLeafDiagram(oturb, 2, leafUnit=0.01)
```

```
   4 2 3444
  18 2 55555666677789
  40 3 0000001111112222333345
(15) 3 566677788899999
  45 4 00022334444
  34 4 566888999
  25 5 0112333
  18 5 6789
  14 6 01122233444
   3 6 788
```

- $X_{(1)} = 0.23$,
- $Q_1 = X_{(25.25)} = X_{(25)} + 0.25(X_{(26)} - X_{(25)}) = 0.31$,
- $M_3 = X_{(50.5)} = 0.385$,
- $Q_3 = X_{(75.75)} = 0.49 + 0.75(0.50 - 0.49) = 0.4975$,
- $X_{(n)} = 0.68$.

Solution 1.21

```
from scipy.stats import trim_mean

oturb = mistat.load_data('OTURB')
print(f'T(0.1) = {trim_mean(oturb, 0.1)}')
print(f'S(0.1) = {trim_std(oturb, 0.1)}')
```

```
T(0.1) = 0.40558750000000005
S(0.1) = 0.09982289003530202
```

$\bar{T}_\alpha = 0.4056$ and $S_\alpha = 0.0998$, where $\alpha = 0.10$.

Solution 1.22

```
germanCars = [10, 10.9, 4.8, 6.4, 7.9, 8.9, 8.5, 6.9, 7.1,
              5.5, 6.4, 8.7, 5.1, 6.0, 7.5]
japaneseCars = [9.4, 9.5, 7.1, 8.0, 8.9, 7.7, 10.5, 6.5, 6.7,
                9.3, 5.7, 12.5, 7.2, 9.1, 8.3, 8.2, 8.5, 6.8, 9.5, 9.7]
# convert to pandas Series
germanCars = pd.Series(germanCars)
japaneseCars = pd.Series(japaneseCars)
# use describe to calculate statistics
comparison = pd.DataFrame({
  'German': germanCars.describe(),
  'Japanese': japaneseCars.describe(),
})
print(comparison)
```

```
          German   Japanese
count  15.000000  20.000000
mean    7.373333   8.455000
std     1.780235   1.589596
min     4.800000   5.700000
25%     6.200000   7.175000
50%     7.100000   8.400000
75%     8.600000   9.425000
max    10.900000  12.500000
```

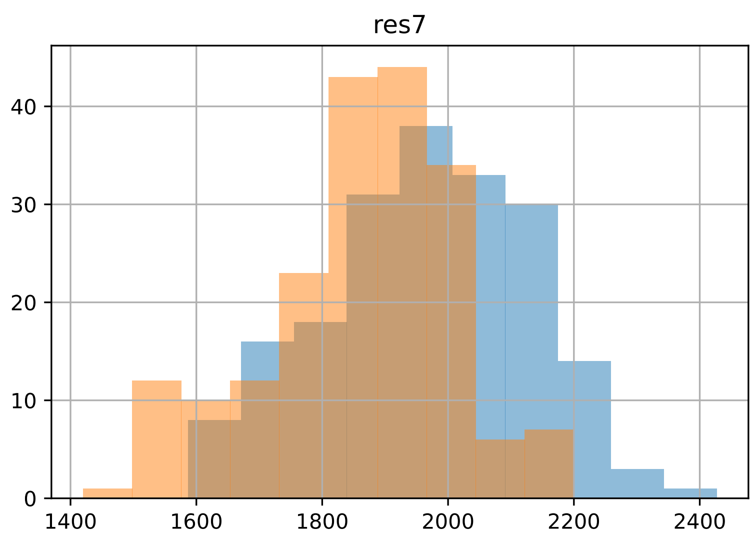
Solution 1.23 Sample statistics:

```
hadpas = mistat.load_data('HADPAS')
sampleStatistics = pd.DataFrame({
  'res3': hadpas['res3'].describe(),
  'res7': hadpas['res7'].describe(),
})
print(sampleStatistics)
```

```
              res3         res7
count   192.000000   192.000000
mean   1965.239583  1857.776042
std     163.528165   151.535930
min    1587.000000  1420.000000
25%    1860.000000  1772.250000
50%    1967.000000  1880.000000
75%    2088.750000  1960.000000
max    2427.000000  2200.000000
```

Histogram:

```
ax = hadpas.hist(column='res3', alpha=0.5)
hadpas.hist(column='res7', alpha=0.5, ax=ax)
plt.show()
```



We overlay both histograms in one plot and make them transparent (`alpha`).

Stem and leaf diagrams:

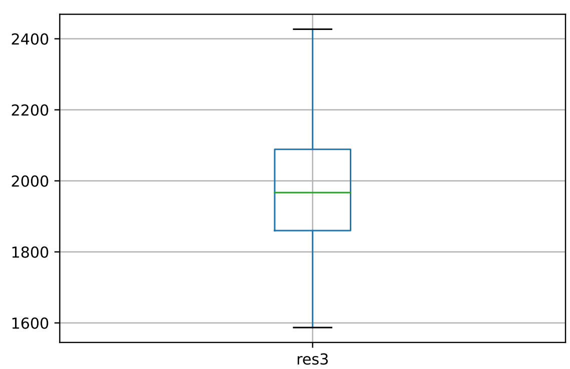
```
print('res3')
mistat.stemLeafDiagram(hadpas['res3'], 2, leafUnit=10)
print('res7')
mistat.stemLeafDiagram(hadpas['res7'], 2, leafUnit=10)
```

```
res3
       1     15   8
       6     16   01124
      14     16   56788889
      22     17   00000234
```

```
      32     17   5566667899
      45     18   0011112233444
      60     18   556666677888899
      87     19   000000111111223333344444444
    (18)     19   566666666667888889
      87     20   00000000122222333333444
      64     20   55556666666677789999
      44     21   0000011222233344444
      25     21   566667788888
      13     22   000111234
       4     22   668
       0     23
       1     24   2
res7
       1     14   2
       9     15   11222244
      15     15   667789
      23     16   00012334
      30     16   5566799
      40     17   0022233334
      54     17   66666666777999
      79     18   0000222222222233344444444
    (28)     18   5555556666666778888888999999
      85     19   00000000111111222222222333333444444
      52     19   566666667777888888889999
      28     20   0000111222333444
      12     20   678
       9     21   1123344
       2     21   8
       1     22   0
```

Solution 1.24

```
hadpas.boxplot(column='res3')
plt.show()
```



Lower whisker starts at $\max(1587, 1511.7) = 1587 = X_{(1)}$; upper whisker ends at $\min(2427, 2440.5) = 2427 = X_{(n)}$. There are no outliers.

Chapter 2
Probability Models and Distribution Functions

Import required modules and define required functions

```
import pandas as pd
from scipy import stats
import matplotlib.pyplot as plt
```

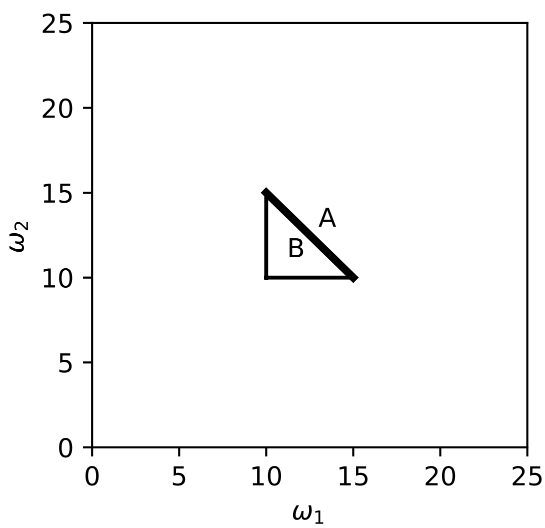
Solution 2.1 **(i)** $\mathcal{S} = \{(w_1, \ldots, w_{20}); w_j = G, D,\ j = 1, \ldots, 20\}$.

(ii) $2^{20} = 1,048,576$.

(iii) $A_n = \left\{(w_1, \ldots, w_{20}) : \sum_{j=1}^{20} I\{w_j = G\} = n\right\}, n = 0, \ldots, 20,$

where $I\{A\} = 1$ if A is true and $I\{A\} = 0$ otherwise. The number of elementary events in A_n is $\binom{20}{n} = \frac{20!}{n!(20-n)!}$.

Solution 2.2 $\mathcal{S} = \{(\omega_1, \ldots, \omega_{10}) : 10 \le \omega_i \le 20, i = 1, \ldots, 10\}$. Looking at the (ω_1, ω_2) components of A and B we have the following graphical representation:



If $(\omega_1, \ldots, \omega_{10}) \in A$ then $(\omega_1, \ldots, \omega_{10}) \in B$. Thus $A \subset B$. $A \cap B = A$.

Solution 2.3 **(i)** $\mathcal{S} = \{(i_1, \ldots, i_{30}) : i_j = 0, 1,\ j = 1, \ldots, 30\}$.

(ii) $A_{10} = \{(1, 1, \ldots, 1, i_{11}, i_{12}, \ldots, i_{30}) : i_j = 0, 1,\ j = 11, \ldots, 30\}$. $|A_{10}| = 2^{20} = 1{,}048{,}576$. ($|A_{10}|$ denotes the number of elements in A_{10}.)

(iii) $B_{10} = \{(i_1, \ldots, i_{30}) : i_j = 0, 1$ and $\sum_{j=1}^{30} i_j = 10\}$, $|B_{10}| = \binom{30}{10} = 30{,}045{,}015$. $A_{10} \not\subset B_{10}$, in fact, A_{10} has only one element belonging to B_{10}.

Solution 2.4 $\mathcal{S} = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c)$, a union of mutually disjoint sets.

(a) $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$. Hence, $(A \cup B)^c = A^c \cap B^c$.

(b)

$$\begin{aligned}(A \cap B)^c &= (A \cap B^c) \cup (A^c \cap B) \cup (A^c \cap B^c)\\ &= (A \cap B^c) \cup A^c\\ &= A^c \cup B^c.\end{aligned}$$

Solution 2.5 As in Exercise **3.1**, $A_n = \left\{(\omega_1, \ldots, \omega_{20}) : \sum_{i=1}^{20} I\{\omega_i = G\} = n\right\}$, $n = 0, \ldots, 20$. Thus, for any $n \neq n'$, $A_n \cap A_{n'} = \emptyset$, moreover $\bigcup_{n=0}^{20} A_n = \mathcal{S}$. Hence $\{A_0, \ldots, A_{20}\}$ is a partition.

Solution 2.6 $\bigcup_{i=1}^{n} A_i = \mathcal{S}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

$$\begin{aligned}B = B \cap \mathcal{S} &= B \cap \left(\bigcup_{i=1}^{n} A_i\right)\\ &= \bigcup_{i=1}^{n} A_i B.\end{aligned}$$

Solution 2.7

$$\begin{aligned}\Pr\{A \cup B \cup C\} &= \Pr\{(A \cup B) \cup C\}\\ &= \Pr\{(A \cup B)\} + \Pr\{C\} - \Pr\{(A \cup B) \cap C\}\\ &= \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} + \Pr\{C\}\\ &\quad - \Pr\{A \cap C\} - \Pr\{B \cap C\} + \Pr\{A \cap B \cap C\}\\ &= \Pr\{A\} + \Pr\{B\} + \Pr\{C\} - \Pr\{A \cap B\}\\ &\quad - \Pr\{A \cap C\} - \Pr\{B \cap C\} + \Pr\{A \cap B \cap C\}.\end{aligned}$$

{exc:partition-union} **Solution 2.8** We have shown in Exercise 2.6 that $B = \bigcup_{i=1}^{n} A_i B$. Moreover, since $\{A_1, \ldots, A_n\}$ is a partition, $A_i B \cap A_j B = (A_i \cap A_j) \cap B = \emptyset \cap B = \emptyset$ for all $i \neq j$. Hence, from Axiom 3

$$\Pr\{B\} = \Pr\left\{\bigcup_{i=1}^{n} A_i B\right\} = \sum_{i=1}^{n} \Pr\{A_i B\}.$$

Solution 2.9

$$\begin{aligned}
\mathcal{S} &= \{(i_1, i_2) : i_j = 1, \ldots, 6,\ j = 1, 2\} \\
A &= \{(i_1, i_2) : i_1 + i_2 = 10\} = \{(4,6), (5,5), (6,4)\} \\
\Pr\{A\} &= \frac{3}{36} = \frac{1}{12}.
\end{aligned}$$

Solution 2.10

$$\Pr\{B\} = \Pr\{A_{150}\} - \Pr\{A_{280}\} = \exp\left(-\frac{150}{200}\right) - \exp\left(-\frac{280}{200}\right) = 0.2258.$$

Solution 2.11

$$\frac{\binom{10}{2}\binom{10}{2}\binom{15}{2}\binom{5}{2}}{\binom{40}{8}} = 0.02765$$

Solution 2.12 **(i)** $100^5 = 10^{10}$; **(ii)** $\binom{100}{5} = 75,287,520$.

Solution 2.13 $N = 1,000$, $M = 900$, $n = 10$.

$$\begin{aligned}
(\mathbf{i})\ \Pr\{X \geq 8\} &= \sum_{j=8}^{10} \binom{10}{j} (0.9)^j (0.1)^{10-j} = 0.9298. \\
(\mathbf{ii})\ \Pr\{X \geq 8\} &= \sum_{j=8}^{10} \frac{\binom{900}{j}\binom{100}{10-j}}{\binom{1000}{10}} = 0.9308.
\end{aligned}$$

Solution 2.14 $1 - (0.9)^{10} = 0.6513$.

Solution 2.15 $\Pr\{T > 300 \mid T > 200\} = 0.6065$

Solution 2.16 **(i)** $\Pr\{D \mid B\} = \frac{1}{4}$; **(ii)** $\Pr\{C \mid D\} = 1$.

Solution 2.17 Since A and B are independent, $\Pr\{A \cap B\} = \Pr\{A\}\Pr\{B\}$. Using this fact and DeMorgan's Law,

$$\begin{aligned}
\Pr\{A^c \cap B^c\} &= \Pr\{(A \cup B)^c\} \\
&= 1 - \Pr\{A \cup B\} \\
&= 1 - (\Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}) \\
&= 1 - \Pr\{A\} - \Pr\{B\} + \Pr\{A\}\Pr\{B\} \\
&= \Pr\{A^c\} - \Pr\{B\}(1 - \Pr\{A\}) \\
&= \Pr\{A^c\}(1 - \Pr\{B\}) \\
&= \Pr\{A^c\}\Pr\{B^c\}.
\end{aligned}$$

Since $\Pr\{A^c \cap B^c\} = \Pr\{A^c\}\Pr\{B^c\}$, A^c and B^c are independent.

Solution 2.18 We assume that $\Pr\{A\} > 0$ and $\Pr\{B\} > 0$. Thus, $\Pr\{A\}\Pr\{B\} > 0$. On the other hand, since $A \cap B = \emptyset$, $\Pr\{A \cap B\} = 0$.

Solution 2.19

$$\begin{aligned}\Pr\{A \cup B\} &= \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \\ &= \Pr\{A\} + \Pr\{B\} - \Pr\{A\}\Pr\{B\} \\ &= \Pr\{A\}(1 - \Pr\{B\}) + \Pr\{B\} \\ &= \Pr\{B\}(1 - \Pr\{A\}) + \Pr\{A\}.\end{aligned}$$

Solution 2.20 By Bayes' theorem,

$$\Pr\{D \mid A\} = \frac{\Pr\{A \mid D\}\Pr\{D\}}{\Pr\{A \mid D\}\Pr\{D\} + \Pr\{A \mid G\}\Pr\{G\}} = \frac{0.10 \times 0.01}{0.10 \times 0.01 + 0.95 \times 0.99} = 0.0011.$$

Additional problems in combinatorial and geometric probabilities

Solution 2.21 Let n be the number of people in the party. The probability that all their birthdays fall on different days is $\Pr\{D_n\} = \prod_{j=1}^{n} \left(\frac{365 - j + 1}{365}\right)$.

(i) If $n = 10$, $\Pr\{D_{10}\} = 0.8831$.

(ii) If $n = 23$, $\Pr\{D_{23}\} = 0.4927$. Thus, the probability of at least 2 persons with the same birthday, when $n = 23$, is $1 - \Pr\{D_{23}\} = 0.5073 > \frac{1}{2}$.

Solution 2.22 $\left(\frac{7}{10}\right)^{10} = 0.02825$.

Solution 2.23 $\prod_{j=1}^{10} \left(1 - \frac{1}{24 - j + 1}\right) = 0.5833$.

Solution 2.24 **(i)** $\frac{\binom{4}{1}\binom{86}{4}}{\binom{100}{5}} = 0.1128$; **(ii)** $\frac{\binom{4}{1}\binom{10}{1}\binom{86}{3}}{\binom{100}{5}} = 0.0544$; **(iii)** $1 - \frac{\binom{86}{5}}{\binom{100}{5}} = 0.5374$.

Solution 2.25 $\frac{4}{\binom{10}{2}} = 0.0889$.

Solution 2.26 The sample median is $X_{(6)}$, where $X_{(1)} < \cdots < X_{(11)}$ are the ordered sample values. $\Pr\{X_{(6)} = k\} = \frac{\binom{k-1}{5}\binom{20-k}{5}}{\binom{20}{11}}$, $k = 6, \ldots, 15$. This is the probability distribution of the sample median. The probabilities are

k	6	7	8	9	10	11	12	13	14	15
Pr	.01192	.04598	.09902	.15404	.18905	.18905	.15404	.09902	.04598	.01192

Solution 2.27 Without loss of generality, assume that the stick is of length 1. Let x, y and $(1 - x - y)$, $0 < x, y < 1$, be the length of the 3 pieces. Obviously, $0 < x + y < 1$. All points in $\mathcal{S} = \{(x, y) : x, y > 0, x + y < 1\}$ are uniformly distributed. In order

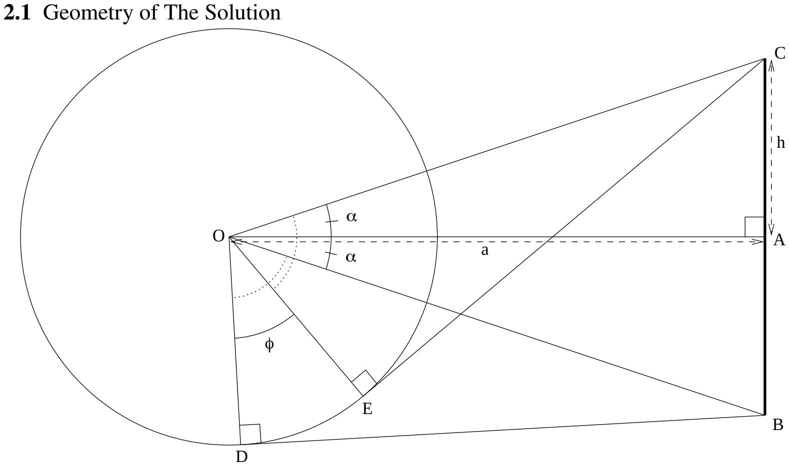
that the three pieces can form a triangle, the following three conditions should be satisfied:

(i) $x + y > (1 - x - y)$
(ii) $x + (1 - x - y) > y$
(iii) $y + (1 - x - y) > x$.

The set of points (x, y) satisfying (i), (ii) and (iii) is bounded by a triangle of area 1/8. $\mathcal{S}$ is bounded by a triangle of area 1/2. Hence, the required probability is 1/4.

Solution 2.28 Consider Figure 2.1. Suppose that the particle is moving along the circumference of the circle in a counterclockwise direction. Then, using the notation in the diagram, $\Pr\{\text{hit}\} = \phi/2\pi$. Since $OD = 1 = OE$, $OB = \sqrt{a^2 + h^2} = OC$ and the lines $\overline{DB}$ and $\overline{EC}$ are tangential to the circle, it follows that the triangles ΔODB and ΔOEC are congruent. Thus $m(\angle\, DOB) = m(\angle\, EOC)$, and it is easily seen that $\phi = 2\alpha$. Now $\alpha = \tan^{-1}\left(\frac{h}{a}\right)$, and hence, $\Pr\{\text{hit}\} = \frac{1}{\pi}\tan^{-1}\left(\frac{h}{a}\right)$. {exc:geometrySolution}

Fig. 2.1 Geometry of The Solution {exc:geometrySolution}



Solution 2.29 $1-(0.999)^{100}-100\times(0.001)\times(0.999)^{99}-\binom{100}{2}\times(0.001)^2(0.999)^{98} = 0.0001504$.

Solution 2.30 The probability that n tosses are required is $p(n) = \binom{n-1}{1}\left(\frac{1}{2}\right)^n, n \geq 2$. Thus, $p(4) = 3 \cdot \frac{1}{2^4} = \frac{3}{16}$.

Solution 2.31 $\mathcal{S} = \{(i_1, \ldots, i_{10}) : i_j = 0, 1, j = 1, \ldots, 10\}$. One random variable is the number of 1's in an element, i.e., for $\omega = (i_1, \ldots, i_{10})$ $X_1(\omega) = \sum_{j=1}^{10} i_j$. Another random variable is the number of zeros to the left of the 1st one, i.e., $X_2(\omega) = \sum_{j=1}^{10}\prod_{k=1}^{j}(1 - i_k)$. Notice that $X_2(\omega) = 0$ if $i_1 = 1$ and $X_2(\omega) = 10$ if

$i_1 = i_2 = \ldots = i_{10} = 0$. The probability distribution of X_1 is $\Pr\{X_1 = k\} = \binom{10}{k}/2^{10}$, $k = 0, 1, \ldots, 10$. The probability distribution of X_2 is

$$\Pr\{X_2 = k\} = \begin{cases} \left(\dfrac{1}{2}\right)^{k+1}, & k = 0, \ldots, 9 \\ \left(\dfrac{1}{2}\right)^{10}, & k = 10. \end{cases}$$

Solution 2.32 **(i)** Since $\sum_{x=0}^{\infty} \dfrac{5^x}{x!} = e^5$, we have $\sum_{x=0}^{\infty} p(x) = 1.$; **(ii)** $\Pr\{X \le 1\} = e^{-5}(1 + 5) = 0.0404$; **(iii)** $\Pr\{X \le 7\} = 0.8666$.

Solution 2.33 **(i)** $\Pr\{X = -1\} = 0.3$; **(ii)** $\Pr\{-0.5 < X < 0\} = 0.1$; **(iii)** $\Pr\{0 \le X < 0.75\} = 0.425$; **(iv)** $\Pr\{X = 1\} = 0$; **(v)** $E\{X\} = -0.25$, $V\{X\} = 0.4042$.

Solution 2.34

$$E\{X\} = \int_0^{\infty} (1 - F(x))\,\mathrm{d}x = \int_0^{\infty} e^{-x^2/2\sigma^2}\,\mathrm{d}x = \sigma\sqrt{\frac{\pi}{2}}.$$

Solution 2.35

$$E\{X\} = \frac{1}{N}\sum_{i=1}^{N} i = \frac{N+1}{2}; \quad E\{X^2\} = \frac{1}{N}\sum_{i=1}^{N} i^2 = \frac{(N+1)(2N+1)}{6}$$

$$V\{X\} = E\{X^2\} - (E\{X\})^2 = \frac{2(N+1)(2N+1) - 3(N+1)^2}{12} = \frac{N^2 - 1}{12}.$$

Solution 2.36 $\Pr\{8 < X < 12\} = \Pr\{|X - 10| < 2\} \ge 1 - \dfrac{V\{X\}}{4} = 1 - \dfrac{0.25}{4} = 0.9375$.

Solution 2.37 Notice that $F(x)$ is the standard Cauchy distribution. The p-th quantile, x_p, satisfies the equation $\dfrac{1}{2} + \dfrac{1}{\pi}\tan^{-1}(x_p) = p$, hence $x_p = \tan\left(\pi\left(p - \dfrac{1}{2}\right)\right)$. For $p = 0.25, 0.50, 0.75$ we get $x_{.25} = -1$, $x_{.50} = 0$, $x_{.75} = 1$, respectively.

Solution 2.38 $\mu_l^* = E\{(X - \mu_1)^l\} = \sum_{j=0}^{l} (-1)^j \binom{l}{j} \mu_1^j \mu_{l-j}$.

When $j = l$ the term is $(-1)^l \mu_1^l$. When $j = l - 1$ the term is $(-1)^{l-1} l \mu_1^{l-1} \mu_1 = (-1)^{l-1} l \mu_1^l$. Thus, the sum of the last 2 terms is $(-1)^{l-1}(l-1)\mu_1^l$ and we have
$\mu_l^* = \sum_{j=0}^{l-2} (-1)^j \binom{l}{j} \mu_1^j \mu_{l-j} + (-1)^{l-1}(l-1)\mu_1^l$.

{exc:mixed-cdf-example} **Solution 2.39** We saw in the solution of Exercise 2.33 that $\mu_1 = -0.25$. Moreover, $\mu_2 = V\{X\} + \mu_1^2 = 0.4667$.

Solution 2.40 $M_X(t) = \dfrac{1}{t(b-a)}(e^{tb} - e^{ta}), \quad -\infty < t < \infty, \quad a < b.$

$E\{X\} = \dfrac{a+b}{2}$; $V\{X\} = \dfrac{(b-a)^2}{12}$.

Solution 2.41 **(i)** For $t < \lambda$ we have $M(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$,

$$M'(t) = \frac{1}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-2}, \qquad \mu_1 = M'(0) = \frac{1}{\lambda}$$
$$M''(t) = \frac{2}{\lambda^2}\left(1 - \frac{t}{\lambda}\right)^{-3}, \qquad \mu_2 = M''(0) = \frac{2}{\lambda^2}$$
$$M^{(3)}(t) = \frac{6}{\lambda^3}\left(1 - \frac{t}{\lambda}\right)^{-4}, \qquad \mu_3 = M^{(3)}(0) = \frac{6}{\lambda^3}$$
$$M^{(4)}(t) = \frac{24}{\lambda^4}\left(1 - \frac{t}{\lambda}\right)^{-5}, \qquad \mu_4 = M^{(4)}(0) = \frac{24}{\lambda^4}.$$

(ii) The central moments are

$$\mu_1^* = 0,$$
$$\mu_2^* = \frac{1}{\lambda^2},$$
$$\mu_3^* = \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3 = \frac{6}{\lambda^3} - \frac{6}{\lambda^3} + \frac{2}{\lambda^3} = \frac{2}{\lambda^3},$$
$$\mu_4^* = \mu_4 - 4\mu_3\mu_1 + 6\mu_2(\mu_1)^2 - 3\mu_1^4 = \frac{1}{\lambda^4}(24 - 4 \cdot 6 + 6 \cdot 2 - 3) = \frac{9}{\lambda^4}.$$

(iii) The index of kurtosis is $\beta_4 = \dfrac{\mu_4^*}{(\mu_2^*)^2} = 9$.

Solution 2.42 `scipy.stats.binom` provides the distribution information.

```
x = list(range(15))
table = pd.DataFrame({
  'x': x,
  'p.d.f.': [stats.binom(20, 0.17).pmf(x) for x in x],
  'c.d.f.': [stats.binom(20, 0.17).cdf(x) for x in x],
})
print(table)
```

```
     x        p.d.f.    c.d.f.
0    0  2.407475e-02  0.024075
1    1  9.861947e-02  0.122694
2    2  1.918921e-01  0.314586
3    3  2.358192e-01  0.550406
4    4  2.052764e-01  0.755682
5    5  1.345426e-01  0.890224
6    6  6.889229e-02  0.959117
7    7  2.822094e-02  0.987338
8    8  9.392812e-03  0.996731
9    9  2.565105e-03  0.999296
10  10  5.779213e-04  0.999874
11  11  1.076086e-04  0.999981
12  12  1.653023e-05  0.999998
13  13  2.083514e-06  1.000000
14  14  2.133719e-07  1.000000
```

Solution 2.43 $Q_1 = 2,\ \text{Med} = 3,\ Q_3 = 4.$

Solution 2.44 $E\{X\} = 15.75,\ \sigma = 3.1996$.

Solution 2.45 Pr{no defective chip on the board} $= p^{50}$. Solving $p^{50} = 0.99$ yields $p = (0.99)^{1/50} = 0.999799$.

Solution 2.46 Notice first that $\lim_{\substack{n\to\infty\\ np\to\lambda}} b(0; n, p) = \lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$. Moreover, for all $j = 0, 1, \cdots, n-1$, $\frac{b(j+1; n, p)}{b(j; n, p)} = \frac{n-j}{j+1}\cdot\frac{p}{1-p}$. Thus, by induction on j, for $j > 0$

$$\begin{aligned}
\lim_{\substack{n\to\infty\\ np\to\lambda}} b(j; n, p) &= \lim_{\substack{n\to\infty\\ np\to\lambda}} b(j-1; n, p)\frac{n-j+1}{j}\cdot\frac{p}{1-p}\\
&= e^{-\lambda}\frac{\lambda^{j-1}}{(j-1)!}\lim_{\substack{n\to\infty\\ np\to\lambda}}\frac{(n-j+1)p}{j\left(1-\frac{\lambda}{n}\right)}\\
&= e^{-\lambda}\frac{\lambda^{j-1}}{(j-1)!}\cdot\frac{\lambda}{j} = e^{-\lambda}\frac{\lambda^j}{j!}.
\end{aligned}$$

Solution 2.47 Using the Poisson approximation, $\lambda = n\cdot p = 1000\cdot 10^{-3} = 1$. $\Pr\{X < 4\} = e^{-1}\sum_{j=0}^{3}\frac{1}{j!} = 0.9810$.

Solution 2.48 $E\{X\} = 20\cdot\frac{350}{500} = 14;\quad V\{X\} = 20\cdot\frac{350}{500}\cdot\frac{150}{500}\left(1 - \frac{19}{499}\right) = 4.0401$.

Solution 2.49 Let X be the number of defective items observed. $\Pr\{X > 1\} = 1 - \Pr\{X \le 1\} = 1 - H(1; 500, 5, 50) = 0.0806$.

Solution 2.50

$$\begin{aligned}
\Pr\{R\} &= 1 - H(3; 100, 10, 20) + \sum_{i=1}^{3} h(i; 100, 10, 20)\left[1 - H(3-i; 80, 10-i, 40)\right]\\
&= 0.87395.
\end{aligned}$$

Solution 2.51 The m.g.f. of the Poisson distribution with parameter λ, $P(\lambda)$, is

$$\begin{aligned}
M(t) &= \exp\{-\lambda(1-e^t)\},\ -\infty < t < \infty. \text{ Accordingly,}\\
M'(t) &= \lambda M(t)e^t\\
M''(t) &= (\lambda^2 e^{2t} + \lambda e^t)M(t)\\
M^{(3)}(t) &= (\lambda^3 e^{3t} + 3\lambda^2 e^{2t} + \lambda e^t)M(t)\\
M^{(4)}(t) &= (\lambda^4 e^{4t} + 6\lambda^3 e^{3t} + 7\lambda^2 e^{2t} + \lambda e^t)M(t).
\end{aligned}$$

The moments and central moments are

$$\begin{aligned} \mu_1 &= \lambda & \mu_1^* &= 0 \\ \mu_2 &= \lambda^2 + \lambda & \mu_2^* &= \lambda \\ \mu_3 &= \lambda^3 + 3\lambda^2 + \lambda & \mu_3^* &= \lambda \\ \mu_4 &= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda & \mu_4^* &= 3\lambda^2 + \lambda. \end{aligned}$$

Thus, the indexes of skewness and kurtosis are $\beta_3 = \lambda^{-1/2}$ and $\beta_4 = 3 + \frac{1}{\lambda}$. For $\lambda = 10$ we have $\beta_3 = 0.3162$ and $\beta_4 = 3.1$.

Solution 2.52 Let X be the number of blemishes observed. $\Pr\{X > 2\} = 0.1912$.

Solution 2.53 Using the Poisson approximation with $N = 8000$ and $p = 380 \times 10^{-6}$, we have $\lambda = 3.04$ and $\Pr\{X > 6\} = 0.0356$, where X is the number of insertion errors in 2 hours of operation.

Solution 2.54 The distribution of N is geometric with $p = 0.00038$. $E\{N\} = 2631.6$, $\sigma_N = 2631.08$.

Solution 2.55 Using Python we obtain that for the $NB(p, k)$ with $p = 0.01$ and $k = 3$, $Q_1 = 170$, $Me = 265$, and $Q_3 = 389$.

```
stats.nbinom.ppf([0.25, 0.5, 0.75], 3, 0.01)
```

```
array([170., 265., 389.])
```

Solution 2.56 By definition, the m.g.f. of $NB(p, k)$ is

$$M(t) = \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} p^k ((1-p)e^t)^i,$$

for $t < -\log(1-p)$.

Thus $M(t) = \dfrac{p^k}{(1-(1-p)e^t)^k} \sum_{i=0}^{\infty} \binom{k+i-1}{k-1} (1-(1-p)e^t)^k ((1-p)e^t)^i$.

Since the last infinite series sums to one, $M(t) = \left[\dfrac{p}{1-(1-p)e^t}\right]^k$, $t < -\log(1-p)$.

Solution 2.57 $M(t) = \dfrac{pe^t}{1-(1-p)e^t}$, for $t < -\log(1-p)$. The derivatives of $M(t)$ are

$$\begin{aligned} M'(t) &= M(t)(1-(1-p)e^t)^{-1} \\ M''(t) &= M(t)(1-(1-p)e^t)^{-2}(1+(1-p)e^t) \\ M^{(3)}(t) &= M(t)(1-(1-p)e^t)^{-3} \cdot [(1+(1-p)e^t)^2 + 2(1-p)e^t] \\ M^{(4)}(t) &= M(t)(1-(1-p)e^t)^{-4}[1+(1-p)^3e^{3t} + 11(1-p)e^t + 11(1-p)^2e^{2t}]. \end{aligned}$$

The moments are

$$\mu_1 = \frac{1}{p}$$
$$\mu_2 = \frac{2-p}{p^2}$$
$$\mu_3 = \frac{(2-p)^2 + 2(1-p)}{p^3} = \frac{6 - 6p + p^2}{p^3}$$
$$\mu_4 = \frac{11(1-p)(2-p) + (1-p)^3 + 1}{p^4} = \frac{24 - 36p + 14p^2 - p^3}{p^4}.$$

The central moments are

$$\mu_1^* = 0$$
$$\mu_2^* = \frac{1-p}{p^2},$$
$$\mu_3^* = \frac{1}{p^3}(1-p)(2-p),$$
$$\mu_4^* = \frac{1}{p^4}(9 - 18p + 10p^2 - p^3).$$

Thus the indices of skewness and kurtosis are $\beta_3^* = \dfrac{2-p}{\sqrt{1-p}}$ and $\beta_4^* = \dfrac{9 - 9p + p^2}{1-p}$.

Solution 2.58 If there are n chips, $n > 50$, the probability of at least 50 good ones is $1 - B(49; n, 0.998)$. Thus, n is the smallest integer > 50 for which $B(49; n, 0.998) < 0.05$. It is sufficient to order 51 chips.

Solution 2.59 If X has a geometric distribution then, for every j, $j = 1, 2, \ldots$ $\Pr\{X > j\} = (1-p)^j$. Thus,

$$\begin{aligned}\Pr\{X > n+m \mid X > m\} &= \frac{\Pr\{X > n+m\}}{\Pr\{X > m\}} \\ &= \frac{(1-p)^{n+m}}{(1-p)^m} \\ &= (1-p)^n \\ &= \Pr\{X > n\}.\end{aligned}$$

Solution 2.60 For $0 < y < 1$, $\Pr\{F(X) \le y\} = \Pr\{X \le F^{-1}(y)\} = F(F^{-1}(y)) = y$. Hence, the distribution of $F(X)$ is uniform on $(0, 1)$. Conversely, if U has a uniform distribution on $(0, 1)$, then

$$\Pr\{F^{-1}(U) \le x\} = \Pr\{U \le F(x)\} = F(x).$$

Solution 2.61 $E\{U(10,50)\} = 30$; $V\{U(10,50)\} = \dfrac{1600}{12} = 133.33$;
$\sigma\{U(10,50)\} = \dfrac{40}{2\sqrt{3}} = 11.547$.

Solution 2.62 Let $X = -\log(U)$ where U has a uniform distribution on (0,1).

$$\begin{aligned}
\Pr\{X \le x\} &= \Pr\{-\log(U) \le x\} \\
&= \Pr\{U \ge e^{-x}\} \\
&= 1 - e^{-x}.
\end{aligned}$$

Therefore X has an exponential distribution $E(1)$.

Solution 2.63 **(i)** $\Pr\{92 < X < 108\} = 0.4062$; **(ii)** $\Pr\{X > 105\} = 0.3694$;
(iii) $\Pr\{2X + 5 < 200\} = \Pr\{X < 97.5\} = 0.4338$.

```
rv = stats.norm(100, 15)
print('(i)', rv.cdf(108) - rv.cdf(92))
print('(ii)', 1 - rv.cdf(105))
print('(iii)', rv.cdf((200 - 5)/2))
```

```
(i) 0.4061971427922976
(ii) 0.36944134018176367
(iii) 0.43381616738909634
```

Solution 2.64 Let z_α denote the α quantile of a $N(0,1)$ distribution. Then the two equations $\mu + z_{.9}\sigma = 15$ and $\mu + z_{.99}\sigma = 20$ yield the solution $\mu = 8.8670$ and $\sigma = 4.7856$.

Solution 2.65 Due to symmetry, $\Pr\{Y > 0\} = \Pr\{Y < 0\} = \Pr\{E < v\}$, where $E \sim N(0,1)$. If the probability of a bit error is $\alpha = 0.01$, then $\Pr\{E < v\} = \Phi(v) = 1 - \alpha = 0.99$.

Thus $v = z_{.99} = 2.3263$.

Solution 2.66 Let X_p denote the diameter of an aluminum pin and X_h denote the size of a hole drilled in an aluminum plate. If $X_p \sim N(10, 0.02)$ and $X_h \sim N(\mu_d, 0.02)$ then the probability that the pin will not enter the hole is $\Pr\{X_h - X_p < 0\}$. Now $X_h - X_p \sim N(\mu_d - 10, \sqrt{0.02^2 + 0.02^2})$ and for $\Pr\{X_h - X_p < 0\} = 0.01$, we obtain $\mu_d = 10.0658$ mm. (The fact that the sum of two independent normal random variables is normally distributed should be given to the student since it has not yet been covered in the text.)

Solution 2.67 For $X_1, \ldots, X_n$ i.i.d. $N(\mu, \sigma^2)$, $Y = \sum_{i=1}^n iX_i \sim N(\mu_Y, \sigma_Y^2)$ where
$\mu_Y = \mu \sum_{i=1}^n i = \mu\dfrac{n(n+1)}{2}$ and $\sigma_Y^2 = \sigma^2 \sum_{i=1}^n i^2 = \sigma^2\dfrac{n(n+1)(2n+1)}{6}$.

Solution 2.68 $\Pr\{X > 300\} = \Pr\{\log X > 5.7038\} = 1 - \Phi(0.7038) = 0.24078$.

Solution 2.69 For $X \sim e^{N(\mu,\sigma)}$, $X \sim e^Y$ where $Y \sim N(\mu,\sigma)$, $M_Y(t) = e^{\mu t + \sigma^2 t^2/2}$.

$$\xi = E\{X\} = E\{e^Y\} = M_Y(1) = e^{\mu+\sigma^2/2}.$$

Since $E\{X^2\} = E\{e^{2Y}\} = M_Y(2) = e^{2\mu+2\sigma^2}$ we have

$$\begin{aligned} V\{X\} &= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} \\ &= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1) \\ &= \xi^2(e^{\sigma^2} - 1). \end{aligned}$$

Solution 2.70 The quantiles of $E(\beta)$ are $x_p = -\beta \log(1-p)$. Hence, $Q_1 = 0.2877\beta,\ Me = 0.6931\beta,\ Q_3 = 1.3863\beta.$

Solution 2.71 If $X \sim E(\beta)$, $\Pr\{X > \beta\} = e^{-\beta/\beta} = e^{-1} = 0.3679$.

Solution 2.72 The m.g.f. of $E(\beta)$ is

$$\begin{aligned} M(t) &= \frac{1}{\beta}\int_0^\infty e^{tx - x/\beta}\,\mathrm{d}x \\ &= \frac{1}{\beta}\int_0^\infty e^{-\frac{(1-t\beta)}{\beta}x}\,\mathrm{d}x \\ &= (1-t\beta)^{-1}, \quad \text{for } t < \frac{1}{\beta}. \end{aligned}$$

Solution 2.73 By independence,

$$\begin{aligned} M_{(X_1+X_2+X_3)}(t) &= E\{e^{t(X_1+X_2+X_3)}\} \\ &= \prod_{i=1}^{3} E\{e^{tX_i}\} \\ &= (1-\beta t)^{-3}, \quad t < \frac{1}{\beta}. \end{aligned}$$

{exc:mgf-ind-exp-rv} Thus $X_1 + X_2 + X_3 \sim G(3,\beta)$, (see Exercise 2.76).
Using the formula of the next exercise,

$$\begin{aligned} \Pr\{X_1 + X_2 + X_3 \geq 3\beta\} &= \Pr\{\beta G(3,1) \geq 3\beta\} \\ &= \Pr\{G(3,1) \geq 3\} \\ &= e^{-3}\sum_{j=0}^{2}\frac{3^j}{j!} \\ &= 0.4232. \end{aligned}$$

Solution 2.74

$$
\begin{aligned}
G(t;k,\lambda) &= \frac{\lambda^k}{(k+1)!}\int_0^t x^{k-1}e^{-\lambda x}\,\mathrm{d}x \\
&= \frac{\lambda^k}{k!}t^k e^{-\lambda t} + \frac{\lambda^{k+1}}{k!}\int_0^t x^k e^{-\lambda x}\,\mathrm{d}x \\
&= \frac{\lambda^k}{k!}t^k e^{-\lambda t} + \frac{\lambda^{k+1}}{(k+1)!}t^{k+1}e^{-\lambda t} + \frac{\lambda^{k+2}}{(k+1)!}\int_0^t x^{k+1}e^{-\lambda x}\,\mathrm{d}x \\
&= \cdots \\
&= e^{-\lambda t}\sum_{j=k}^{\infty}\frac{(\lambda t)^j}{j!} \\
&= 1 - e^{-\lambda t}\sum_{j=0}^{k-1}\frac{(\lambda t)^j}{j!}.
\end{aligned}
$$

Solution 2.75 $\Gamma(1.17) = 0.9267$, $\Gamma\left(\frac{1}{2}\right) = 1.77245$, $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = 0.88623$.

```
from scipy.special import gamma
print(gamma(1.17), gamma(1 / 2), gamma(3 / 2))
```

```
0.9266996106177159 1.7724538509055159 0.8862269254527579
```

Solution 2.76 The moment generating function of the sum of independent random variables is the product of their respective m.g.f.'s. Thus, if $X_1, \cdots, X_k$ are i.i.d. $E(\beta)$, using the result of Exercise 2.73, $M_S(t) = \prod_{i=1}^k (1-\beta t)^{-1} = (1-\beta t)^{-k}$, {exc:prob-ind-exp-rv} $t < \dfrac{1}{\beta}$, where $S = \sum_{i=1}^k X_i$. On the other hand, $(1-\beta t)^{-k}$ is the m.g.f. of $G(k,\beta)$.

Solution 2.77 The expected value and variance of $W(2, 3.5)$ are

$$
\begin{aligned}
E\{W(2,3.5)\} &= 3.5 \times \Gamma\left(1+\frac{1}{2}\right) = 3.1018, \\
V\{W(2,3.5)\} &= (3.5)^2\left[\Gamma\left(1+\frac{2}{2}\right) - \Gamma^2\left(1+\frac{1}{2}\right)\right] = 2.6289.
\end{aligned}
$$

Solution 2.78 Let T be the number of days until failure. $T \sim W(1.5, 500) \sim 500W(1.5, 1)$.

$$\Pr\{T \geq 600\} = \Pr\left\{W(1.5,1) \geq \frac{6}{5}\right\} = e^{-(6/5)^{1.5}} = 0.2686.$$

Solution 2.79 Let $X \sim \text{Beta}\left(\frac{1}{2}, \frac{3}{2}\right)$.

$$E\{X\} = \frac{1/2}{\frac{1}{2}+\frac{3}{2}} = \frac{1}{4},\quad V\{X\} = \frac{\frac{1}{2}\cdot\frac{3}{2}}{2^2\cdot 3} = \frac{1}{16} \text{ and } \sigma\{X\} = \frac{1}{4}.$$

Solution 2.80 Let $X \sim \text{Beta}(\nu, \nu)$. The first four moments are

$$\begin{aligned}
\mu_1 &= \nu/2\nu = \frac{1}{2} \\
\mu_2 &= \frac{B(\nu+2,\nu)}{B(\nu,\nu)} = \frac{\nu+1}{2(2\nu+1)} \\
\mu_3 &= \frac{B(\nu+3,\nu)}{B(\nu,\nu)} = \frac{(\nu+1)(\nu+2)}{2(2\nu+1)(2\nu+2)} \\
\mu_4 &= \frac{B(\nu+4,\nu)}{B(\nu,\nu)} = \frac{(\nu+1)(\nu+2)(\nu+3)}{2(2\nu+1)(2\nu+2)(2\nu+3)}.
\end{aligned}$$

The variance is $\sigma^2 = \dfrac{1}{4(2\nu+1)}$ and the fourth central moment is $\mu_4^* = \mu_4 - 4\mu_3 \cdot \mu_1 + 6\mu_2 \cdot \mu_1^2 - 3\mu_1^4 = \dfrac{3}{16(3+8\nu+4\nu^2)}$.

Finally, the index of kurtosis is $\beta_2 = \dfrac{\mu_4^*}{\sigma^4} = \dfrac{3(1+2\nu)}{3+2\nu}$.

Solution 2.81 Let (X, Y) have a joint p.d.f.

$$f(x, y) = \begin{cases} \frac{1}{2}, & (x, y) \in S \\ 0, & \text{otherwise.} \end{cases}$$

(i) The marginal distributions of X and Y have p.d.f.'s
$f_X(x) = \dfrac{1}{2}\int_{-1+|x|}^{1-|x|} \mathrm{d}y = 1 - |x|, -1 < x < 1,$ and by symmetry,
$f_Y(y) = 1 - |y|, -1 < y < 1.$
(ii) $E\{X\} = E\{Y\} = 0,\quad V\{X\} = V\{Y\} = 2\int_0^1 y^2(1-y)\,\mathrm{d}y = 2B(3,2) = \dfrac{1}{6}.$

Solution 2.82 The marginal p.d.f. of Y is $f(y) = e^{-y}$, $y > 0$, that is, $Y \sim E(1)$. The conditional p.d.f. of X, given $Y = y$, is $f(x \mid y) = \frac{1}{y}e^{-x/y}$ which is the p.d.f. of an exponential with parameter y. Thus, $E\{X \mid Y = y\} = y$, and $E\{X\} = E\{E\{X \mid Y\}\} = E\{Y\} = 1$. Also,

$$\begin{aligned}
E\{XY\} &= E\{YE\{X \mid Y\}\} \\
&= E\{Y^2\} \\
&= 2.
\end{aligned}$$

Hence, $\text{cov}(X, Y) = E\{XY\} - E\{X\}E\{Y\} = 1$. The variance of Y is $\sigma_Y^2 = 1$. The variance of X is

$$\begin{aligned}\sigma_X^2 &= E\{V\{X \mid Y\}\} + V\{E\{X \mid Y\}\} \\ &= E\{Y^2\} + V\{Y\} \\ &= 2 + 1 = 3.\end{aligned}$$

The correlation between X and Y is $\rho_{XY} = \dfrac{1}{\sqrt{3}}$.

Solution 2.83 Let (X, Y) have joint p.d.f. $f(x, y) = \begin{cases} 2, & \text{if } (x, y) \in T \\ \\ 0, & \text{otherwise.} \end{cases}$

The marginal densities of X and Y are

$f_X(x) = 2(1 - x), \quad 0 \le x \le 1$

$f_Y(y) = 2(1 - y), \quad 0 \le y \le 1.$

Notice that $f(x, y) \ne f_X(x) f_Y(y)$ for $x = \dfrac{1}{2}$, $y = \dfrac{1}{4}$. Thus, X and Y are dependent.

$\operatorname{cov}(X, Y) = E\{XY\} - E\{X\}E\{Y\} = E\{XY\} - \dfrac{1}{9}$.

$$\begin{aligned}E\{XY\} &= 2\int_0^1 x \int_0^{1-x} y \,\mathrm{d}y \,\mathrm{d}x \\ &= \int_0^1 x(1 - x)^2 \,\mathrm{d}x \\ &= B(2, 3) = \frac{1}{12}.\end{aligned}$$

Hence, $\operatorname{cov}(X, Y) = \dfrac{1}{12} - \dfrac{1}{9} = -\dfrac{1}{36}$.

Solution 2.84 $J \mid N \sim B(N, p)$; $N \sim P(\lambda)$. $E\{N\} = \lambda$, $V\{N\} = \lambda$, $E\{J\} = \lambda p$.

$$\begin{aligned}V\{J\} &= E\{V\{J \mid N\}\} + V\{E\{J \mid N\}\} \\ &= E\{Np(1 - p)\} + V\{Np\} \\ &= \lambda p(1 - p) + p^2\lambda = \lambda p.\end{aligned}$$

$$\begin{aligned}E\{JN\} &= E\{NE\{J \mid N\}\} \\ &= pE\{N^2\} \\ &= p(\lambda + \lambda^2)\end{aligned}$$

Hence, $\operatorname{cov}(J, N) = p\lambda(1 + \lambda) - p\lambda^2 = p\lambda$ and $\rho_{JN} = \dfrac{p\lambda}{\lambda\sqrt{p}} = \sqrt{p}$.

Solution 2.85 Let $X \sim G(2, 100) \sim 100G(2, 1)$ and $Y \sim W(1.5, 500) \sim 500W(1.5, 1)$. Then $XY \sim 5 \times 10^4 G(2, 1) \cdot W(1.5, 1)$ and $V\{XY\} = 25 \times 10^8 \cdot V\{GW\}$, where $G \sim G(2, 1)$ and $W \sim W\left(\dfrac{3}{2}, 1\right)$.

$$\begin{aligned}V\{GW\} &= E\{G^2\}V\{W\} + E^2\{W\}V\{G\} \\ &= 6\left(\Gamma\left(1 + \frac{4}{3}\right) - \Gamma^2\left(1 + \frac{2}{3}\right)\right) + 2 \cdot \Gamma^2\left(1 + \frac{2}{3}\right) \\ &= 3.88404.\end{aligned}$$

Thus $V\{XY\} = 9.7101 \times 10^9$.

Solution 2.86 Using the notation of Example 2.33, {ex:insert-machine-trinomial}

(i) $\Pr\{J_2 + J_3 \le 20\} = B(20; 3500, 0.005) = 0.7699$.

(ii) $J_3 \mid J_2 = 15 \sim$ Binomial $B\left(3485, \dfrac{0.004}{0.999}\right)$.

(iii) $\lambda = 3485 \times \dfrac{0.004}{0.999} = 13.954$, $\Pr\{J_2 \le 15 \mid J_3 = 15\} \approx P(15; 13.954) = 0.6739$.

{ex:pdf-hypergeom-joint} **Solution 2.87** Using the notation of Example 2.34, the joint p.d.f. of J_1 and J_2 is

$$p(j_1, j_2) = \frac{\binom{20}{j_1}\binom{50}{j_2}\binom{30}{20-j_1-j_2}}{\binom{100}{20}}, \quad 0 \le j_1, j_2;\ j_1 + j_2 \le 20.$$

The marginal distribution of J_1 is $H(100, 20, 20)$. The marginal distribution of J_2 is $H(100, 50, 20)$. Accordingly,

$$V\{J_1\} = 20 \times 0.2 \times 0.8 \times \left(1 - \frac{19}{99}\right) = 2.585859,$$

$$V\{J_2\} = 20 \times 0.5 \times 0.5 \times \left(1 - \frac{19}{99}\right) = 4.040404.$$

The conditional distribution of J_1, given J_2, is $H(50, 20, 20 - J_2)$. Hence,

$$\begin{aligned} E\{J_1 J_2\} &= E\{E\{J_1 J_2 \mid J_2\}\} \\ &= E\left\{J_2(20 - J_2) \times \frac{2}{5}\right\} \\ &= 8E\{J_2\} - 0.4E\{J_2^2\} \\ &= 80 - 0.4 \times 104.040404 = 38.38381 \end{aligned}$$

and $\mathrm{cov}(J_1, J_2) = -1.61616$.

Finally, the correlation between J_1 and J_2 is $\rho = \dfrac{-1.61616}{\sqrt{2.585859 \times 4.040404}} = -0.50$.

Solution 2.88 $V\{Y \mid X\} = 150$, $V\{Y\} = 200$, $V\{Y \mid X\} = V\{Y\}(1-\rho^2)$. Hence $|\rho| = 0.5$. The sign of ρ cannot be determined.

Solution 2.89 **(i)** $X_{(1)} \sim E\left(\dfrac{100}{10}\right)$, $E\{X_{(1)}\} = 10.$; **(ii)** $E\{X_{(10)}\} = 100\sum_{i=1}^{10} \dfrac{1}{i} = 292.8968$.

Solution 2.90 $J \sim B(10, 0.95)$. If $\{J = j\}$, $j > 1$, $X_{(1)}$ is the minimum of a sample of j i.i.d. $E(10)$ random variables. Thus $X_{(1)} \mid J = j \sim E\left(\dfrac{10}{j}\right)$.

(i) $\Pr\{J = k, X_{(1)} \le x\} = b(k; 10, 0.95)(1 - e^{-\frac{kx}{10}})$, $k = 1, 2, \cdots, 10$.

(ii) First note that $\Pr\{J \ge 1\} = 1 - (0.05)^{10} \approx 1$.

$$
\begin{aligned}
\Pr\{X_{(1)} \le x \mid J \ge 1\} &= \sum_{k=1}^{10} b(k; 10, 0.95)(1 - e^{-\frac{kx}{10}}) \\
&= 1 - \sum_{k=1}^{10} \binom{10}{k} (0.95 e^{-\frac{x}{10}})^k (0.05)^{(10-k)} \\
&= 1 - [0.05 + 0.95 e^{-x/10}]^{10} + (0.05)^{10} \\
&= 1 - (0.05 + 0.95 e^{-x/10})^{10}.
\end{aligned}
$$

Solution 2.91 The median is $Me = X_{(6)}$.

(a) The p.d.f. of Me is $f_{(6)}(x) = \dfrac{11!}{5!5!} \lambda (1 - e^{-\lambda x})^5 e^{-6\lambda x}$, $x \ge 0$.

(b) The expected value of Me is

$$
\begin{aligned}
E\{X_{(6)}\} &= 2772\lambda \int_0^\infty x(1 - e^{-\lambda x})^5 e^{-6\lambda x} \, \mathrm{d}x \\
&= 2772\lambda \sum_{j=0}^{5} (-1)^j \binom{5}{j} \int_0^\infty x e^{-\lambda x(6+j)} \, \mathrm{d}x \\
&= 2772 \sum_{j=0}^{5} (-1)^j \binom{5}{j} \frac{1}{\lambda(6+j)^2} \\
&= 0.73654/\lambda.
\end{aligned}
$$

Solution 2.92 Let X and Y be i.i.d. $E(\beta)$, $T = X + Y$ and $W + X - Y$.

$$
\begin{aligned}
V\left\{T + \frac{1}{2}W\right\} &= V\{\frac{3}{2}X + \frac{1}{2}Y\} \\
&= \beta^2 \left(\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \right) \\
&= 2.5\beta^2.
\end{aligned}
$$

Solution 2.93 $\mathrm{cov}(X, X + Y) = \mathrm{cov}(X, X) + \mathrm{cov}(X, Y) = V\{X\} = \sigma^2$.

Solution 2.94 $V\{\alpha X + \beta Y\} = \alpha^2 \sigma_X^2 + \beta^2 \sigma_Y^2 + 2\alpha\beta \mathrm{cov}(X, Y) = \alpha^2 \sigma_X^2 + \beta^2 \sigma_Y^2 + 2\alpha\beta\rho_{XY}\sigma_X\sigma_Y$.

Solution 2.95 Let $U \sim N(0, 1)$ and $X \sim N(\mu, \sigma)$. We assume that U and X are independent. Then $\Phi(X) = \Pr\{U < X \mid X\}$ and therefore

$$
\begin{aligned}
E\{\Phi(X)\} &= E\{\Pr\{U < X \mid X\}\} \\
&= \Pr\{U < X\} \\
&= \Pr\{U - X < 0\} \\
&= \Phi\left(\frac{\mu}{\sqrt{1 + \sigma^2}}\right).
\end{aligned}
$$

The last equality follows from the fact that $U - X \sim N(-\mu, \sqrt{1 + \sigma^2})$.

Solution 2.96 Let U_1, U_2, X be independent random variables; U_1, U_2 i.i.d. $N(0,1)$. Then

$\Phi^2(X) = \Pr\{U_1 \le X, U_2 \le X \mid X\}$. Hence

$E\{\Phi^2(X)\} = \Pr\{U_1 \le X, U_2 \le X\} = \Pr\{U_1 - X \le 0, U_2 - X \le 0\}$.

Since $(U_1 - X, U_2 - X)$ have a bivariate normal distribution with means $(-\mu, -\mu)$ and variance-covariance matrix $V = \begin{bmatrix} 1+\sigma^2 & \sigma^2 \\ \sigma^2 & 1+\sigma^2 \end{bmatrix}$, it follows that

$$E\{\Phi^2(X)\} = \Phi_2\left(\frac{\mu}{\sqrt{1+\sigma^2}}, \frac{\mu}{\sqrt{1+\sigma^2}}; \frac{\sigma^2}{1+\sigma^2}\right).$$

Solution 2.97 Since X and Y are independent, $T = X + Y \sim P(12)$, $\Pr\{T > 15\} = 0.1556$.

Solution 2.98 Let $F_2(x) = \int_{-\infty}^{x} f_2(z)\,\mathrm{d}z$ be the c.d.f. of X_2. Since $X_1 + X_2$ are independent

$$\begin{aligned}\Pr\{Y \le y\} &= \int_{-\infty}^{\infty} f_1(x)\Pr\{X_2 \le y - x\}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} f_1(x) F_2(y-x)\,\mathrm{d}x.\end{aligned}$$

Therefore, the p.d.f. of Y is

$$\begin{aligned}g(y) &= \frac{d}{dy}\Pr\{Y \le y\} \\ &= \int_{-\infty}^{\infty} f_1(x) f_2(y-x)\,\mathrm{d}x.\end{aligned}$$

Solution 2.99 Let $Y = X_1 + X_2$ where X_1, X_2 are i.i.d. uniform on (0,1). Then the p.d.f.'s are

$f_1(x) = f_2(x) = I\{0 < x < 1\}$

$$g(y) = \begin{cases} \int_0^y \mathrm{d}x = y, & \text{if } 0 \le y < 1 \\ \int_{y-1}^1 \mathrm{d}x = 2 - y, & \text{if } 1 \le y \le 2. \end{cases}$$

Solution 2.100 X_1, X_2 are i.i.d. $E(1)$. $U = X_1 - X_2$. $\Pr\{U \le u\} = \int_0^\infty e^{-x}\Pr\{X_1 \le u + x\}\,\mathrm{d}x$. Notice that $-\infty < u < \infty$ and $\Pr\{X_1 \le u + x\} = 0$ if $x + u < 0$. Let $a^+ = \max(a, 0)$. Then

$$\begin{aligned}\Pr\{U \le u\} &= \int_0^\infty e^{-x}(1 - e^{-(u+x)^+})\,\mathrm{d}x \\ &= 1 - \int_0^\infty e^{-x-(u+x)^+}\,\mathrm{d}x \\ &= \begin{cases} 1 - \frac{1}{2}e^{-u}, & \text{if } u \ge 0 \\ \frac{1}{2}e^{-|u|}, & \text{if } u < 0 \end{cases}\end{aligned}$$

Thus, the p.d.f. of U is $g(u) = \dfrac{1}{2}e^{-|u|}$, $-\infty < u < \infty$.

Solution 2.101 $T = X_1 + \cdots + X_{20} \dot{\sim} N(20\mu, \sqrt{20\sigma^2})$. $\Pr\{T \le 50\} \approx \Phi\left(\frac{50-40}{44.7214}\right) = 0.5885$.

Solution 2.102 $X \sim B(200, 0.15)$. $\mu = np = 30$, $\sigma = \sqrt{np(1-p)} = 5.0497$.

$$\Pr\{25 < X < 35\} \approx \Phi\left(\frac{34.5-30}{5.0497}\right) - \Phi\left(\frac{25.5-30}{5.0497}\right) = 0.6271.$$

Solution 2.103 $X \sim P(200)$. $\Pr\{190 < X < 210\} \approx 2\Phi\left(\frac{9.5}{\sqrt{200}}\right) - 1 = 0.4983$.

Solution 2.104 $X \sim \text{Beta}(3, 5)$. $\mu = E\{X\} = \frac{3}{8} = 0.375$. $\sigma = \sqrt{V\{X\}} = \left(\frac{3 \cdot 5}{8^2 \cdot 9}\right)^{1/2} = 0.161374$.

$$\Pr\{|\bar{X}_{200} - 0.375| < 0.2282\} \approx 2\Phi\left(\frac{\sqrt{200} \cdot 0.2282}{0.161374}\right) - 1 = 1.$$

Solution 2.105 $t_{.95}[10] = 1.8125$, $t_{.95}[15] = 1.7531$, $t_{.95}[20] = 1.7247$.

```
print(stats.t.ppf(0.95, 10))
print(stats.t.ppf(0.95, 15))
print(stats.t.ppf(0.95, 20))
```

```
1.8124611228107335
1.7530503556925547
1.7247182429207857
```

Solution 2.106 $F_{.95}[10, 30] = 2.1646$, $F_{.95}[15, 30] = 2.0148$, $F_{.95}[20, 30] = 1.9317$.

```
print(stats.f.ppf(0.95, 10, 30))
print(stats.f.ppf(0.95, 15, 30))
print(stats.f.ppf(0.95, 20, 30))
```

```
2.164579917125473
2.0148036912954885
1.931653475236928
```

Solution 2.107 The solution to this problem is based on the fact, which is not discussed in the text, that $t[\nu]$ is distributed like the ratio of two independent random variables, $N(0, 1)$ and $\sqrt{\chi^2[\nu]/\nu}$. Accordingly, $t[n] \sim \frac{N(0,1)}{\sqrt{\frac{\chi^2[n]}{n}}}$, where $N(0, 1)$ and $\chi^2[n]$ are independent. $t^2[n] \sim \frac{(N(0,1))^2}{\frac{\chi^2[n]}{n}} \sim F[1, n]$. Thus, since $\Pr\{F[1, n] \le F_{1-\alpha}[1, n]\} = 1 - \alpha$.

$$\Pr\{-\sqrt{F_{1-\alpha}[1,n]} \le t[n] \le \sqrt{F_{1-\alpha}[1,n]}\} = 1-\alpha.$$

It follows that $\sqrt{F_{1-\alpha}[1,n]} = t_{1-\alpha/2}[n]$,

or $F_{1-\alpha}[1,n] = t^2_{1-\alpha/2}[n]$. (If you assign this problem, please inform the students of the above fact.)

Solution 2.108 A random variable $F[\nu_1,\nu_2]$ is distributed like the ratio of two independent random variables $\chi^2[\nu_1]/\nu_1$ and $\chi^2[\nu_2]/\nu_2$. Accordingly, $F[\nu_1,\nu_2] \sim \dfrac{\chi^2[\nu_1]/\nu_1}{\chi^2[\nu_2]/\nu_2}$ and

$$\begin{aligned}
1-\alpha &= \Pr\{F[\nu_1,\nu_2] \le F_{1-\alpha}[\nu_1,\nu_2]\}\\
&= \Pr\left\{\frac{\chi_1^2[\nu_1]/\nu_1}{\chi_2^2[\nu_2]/\nu_2} \le F_{1-\alpha}[\nu_1,\nu_2]\}\right\}\\
&= \Pr\left\{\frac{\chi_2^2[\nu_2]/\nu_2}{\chi_1^2[\nu_1]/\nu_1} \ge \frac{1}{F_{1-\alpha}[\nu_1,\nu_2]}\right\}\\
&= \Pr\left\{F[\nu_2,\nu_1] \ge \frac{1}{F_{1-\alpha}[\nu_1,\nu_2]}\right\}\\
&= \Pr\left\{F[\nu_2,\nu_1] \ge F_\alpha[\nu_2,\nu_1]\right\}.
\end{aligned}$$

Hence $F_{1-\alpha}[\nu_1,\nu_2] = \dfrac{1}{F_\alpha[\nu_2,\nu_1]}$.

Solution 2.109 Using the fact that $t[\nu] \sim \dfrac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}$, where $N(0,1)$ and $\chi^2[\nu]$ are independent,

$$\begin{aligned}
V\{t[\nu]\} &= V\left\{\frac{N(0,1)}{\sqrt{\chi^2[\nu]/\nu}}\right\}\\
&= E\left\{V\left\{\frac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}\,\middle|\,\chi^2[\nu]\right\}\right\} + V\left\{E\left\{\frac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}\,\middle|\,\chi^2[\nu]\right\}\right\}.
\end{aligned}$$

By independence, $V\left\{\dfrac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}\,\middle|\,\chi^2[\nu]\right\} = \dfrac{\nu}{\chi^2[\nu]}$, and $E\left\{\dfrac{N(0,1)}{\sqrt{\frac{\chi^2[\nu]}{\nu}}}\,\middle|\,\chi^2[\nu]\right\} = 0$.

Thus, $V\{t[\nu]\} = \nu E\left\{\dfrac{1}{\chi^2[\nu]}\right\}$. Since $\chi^2[\nu] \sim G\left(\dfrac{\nu}{2},2\right) \sim 2G\left(\dfrac{\nu}{2},1\right)$,

$$\begin{aligned}
E\left\{\frac{1}{\chi^2[\nu]}\right\} &= \frac{1}{2}\cdot\frac{1}{\Gamma\left(\frac{\nu}{2}\right)}\int_0^\infty x^{\nu-2}e^{-x}\,\mathrm{d}x\\
&= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{\nu}{2}-1\right)}{\Gamma\left(\frac{\nu}{2}\right)} = \frac{1}{2}\cdot\frac{1}{\frac{\nu}{2}-1} = \frac{1}{\nu-2}.
\end{aligned}$$

Finally, $V\{t[\nu]\} = \dfrac{\nu}{\nu - 2}$, $\nu > 2$.

Solution 2.110 $E\{F[3, 10]\} = \dfrac{10}{8} = 1.25$, $\quad V\{F[3, 10]\} = \dfrac{2 \cdot 10^2 \cdot 11}{3 \cdot 8^2 \cdot 6} = 1.9097$.

Chapter 3
Statistical Inference and Bootstrapping

Import required modules and define required functions

```
import random
import math
import numpy as np
import pandas as pd
from scipy import stats
import matplotlib.pyplot as plt
import pingouin as pg
import mistat
import os
os.environ['OUTDATED_IGNORE'] = '1'
```

Solution 3.1 By the WLLN, for any $\epsilon > 0$, $\lim_{n\to\infty} \Pr\{|M_l - \mu_l| < \epsilon\} = 1$. Hence, M_l is a consistent estimator of the l-th moment.

Solution 3.2 Using the CLT, $\Pr\{|\bar{X}_n - \mu| < 1\} \approx 2\Phi\left(\frac{\sqrt{n}}{\sigma}\right) - 1$. To determine the sample size n so that this probability is 0.95 we set $2\Phi\left(\frac{\sqrt{n}}{\sigma}\right) - 1 = 0.95$ and solve for n. This gives $\frac{\sqrt{n}}{\sigma} = z_{.975} = 1.96$. Thus $n \geq \sigma^2(1.96)^2 = 424$ for $\sigma = 10.5$.

Solution 3.3 $\hat{\xi}_p = \bar{X}_n + z_p\hat{\sigma}_n$, where $\hat{\sigma}_n^2 = \frac{1}{n}\sum(X_i - \bar{X}_n)^2$.

Solution 3.4 Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from a bivariate normal distribution with density $f(x, y; \mu, \eta, \sigma_X, \sigma_Y, \rho)$ as given in Eq. (4.6.6). Let $Z_i = X_iY_i$ for $i = 1, \ldots, n$. Then the first moment of Z is given by

$$\begin{aligned}\mu_1(F_Z) &= E\{Z\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y; \mu, \eta, \sigma_X, \sigma_Y, \rho)\,\mathrm{d}x\,\mathrm{d}y \\ &= \mu\eta + \rho\sigma_X\sigma_Y.\end{aligned}$$

Using this fact, as well as the first 2 moments of X and Y, we get the following moment equations:

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \mu \qquad\qquad \frac{1}{n}\sum_{i=1}^{n} Y_i = \eta$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \sigma_X^2 + \mu^2 \qquad\qquad \frac{1}{n}\sum_{i=1}^{n} Y_i^2 = \sigma_Y^2 + \eta^2$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i Y_i = \mu\eta + \rho\sigma_X\sigma_Y .$$

Solving these equations for the 5 parameters gives

$$\hat{\rho}_n = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \bar{X}\bar{Y}}{\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \bar{Y}^2\right)\right]^{1/2}}, \qquad \text{or equivalently,}$$

$$\hat{\rho}_n = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \cdot \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right)^{1/2}}.$$

Solution 3.5 The two first moments are

$$\mu_1 = \frac{\nu_1}{\nu_1 + \nu_2}, \qquad \mu_2 = \frac{\nu_1(\nu_1 + 1)}{(\nu_1 + \nu_2)(\nu_1 + \nu_2 + 1)}.$$

Equating the theoretical moments to the sample moments $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$, we otain with $\hat{\sigma}_n^2 = M_2 - M_1^2$ the moment equation estimators.

$$\hat{\nu}_1 = M_1(M_1 - M_2)/\hat{\sigma}_n^2 \quad \text{and} \quad \hat{\nu}_2 = (M_1 - M_2)(1 - M_1)/\hat{\sigma}_n^2.$$

Solution 3.6 $V\{\bar{Y}_w\} = \left(\sum_{i=1}^{k} w_i^2 \frac{\sigma_i^2}{n_i}\right) \Big/ \left(\sum_{i=1}^{k} w_i\right)^2$. Let $\lambda_i = \frac{w_i}{\sum_{j=1}^{k} w_j}$, $\sum_{i=1}^{k}\lambda_i = 1$. We find weights λ_i which minimize $V\{\bar{Y}_w\}$, under the constraint $\sum_{i=1}^{k}\lambda_i = 1$. The Lagrangian is $L(\lambda_1, \ldots, \lambda_k, \rho) = \sum_{i=1}^{k}\lambda_i^2\frac{\sigma_i^2}{n_i} + \rho\left(\sum_{i=1}^{k}\lambda_i - 1\right)$. Differentiating with respect to λ_i, we get

$$\frac{\partial}{\partial\lambda_i}L(\lambda_1, \ldots, \lambda_k, \rho) = 2\lambda_i\frac{\sigma_i^2}{n_i} + \rho, i = 1, \ldots, k \quad \text{and} \quad \frac{\partial}{\partial\rho}L(\lambda_1, \ldots, \lambda_k, \rho) = \sum_{i=1}^{k}\lambda_i - 1.$$

Equating the partial derivatives to zero, we get $\lambda_i^0 = -\frac{\rho}{2}\frac{n_i}{\sigma_i^2}$ for $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda_i^0 = -\frac{\rho}{2}\sum_{i=1}^k \frac{n_i}{\sigma_i^2} = 1$.

Thus, $-\dfrac{\rho}{2} = \dfrac{1}{\sum_{i=1}^k n_i/\sigma_i^2}$, $\lambda_i^0 = \dfrac{n_i/\sigma_i^2}{\sum_{j=1}^k n_j/\sigma_j^2}$, and therefore $w_i = n_i/\sigma_i^2$.

Solution 3.7 Since the Y_i are uncorrelated,
$V\{\hat{\beta}_1\} = \sum_{i=1}^n w_i^2 V\{Y_i\} = \sigma^2 \sum_{i=1}^n \dfrac{(x_i - \bar{x}_n)^2}{SS_x^2} = \dfrac{\sigma^2}{SS_x}$, where $SS_x = \sum_{i=1}^n (x_i - \bar{x}_n)^2$.

Solution 3.8 Let $w_i = \dfrac{x_i - \bar{x}_n}{SS_x}$ for $i = 1, \cdots, n$ where $SS_x = \sum_{i=1}^n (x_i - \bar{x}_n)^2$. Then we have

$$\sum_{i=1}^n w_i = 0, \quad \text{and} \quad \sum_{i=1}^n w_i^2 = \frac{1}{SS_x}.$$

Hence,

$$\begin{aligned}
V\{\hat{\beta}_0\} &= V\{\bar{Y}_n - \hat{\beta}_1 \bar{x}_n\} \\
&= V\left\{\bar{Y}_n - \left(\sum_{i=1}^n w_i Y_i\right)\bar{x}_n\right\} \\
&= V\left\{\sum_{i=1}^n \left(\frac{1}{n} - w_i \bar{x}_n\right) Y_i\right\} \\
&= \sum_{i=1}^n \left(\frac{1}{n} - w_i \bar{x}_n\right)^2 \sigma^2 \\
&= \sigma^2 \sum_{i=1}^n \left(\frac{1}{n^2} - \frac{2w_i \bar{x}_n}{n} + w_i^2 \bar{x}_n^2\right) \\
&= \sigma^2 \left(\frac{1}{n} - \frac{2}{n}\bar{x}_n \sum_{i=1}^n w_i + \bar{x}_n^2 \sum_{i=1}^n w_i^2\right) \\
&= \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}_n^2}{SS_x}\right).
\end{aligned}$$

Also

$$\begin{aligned}
\mathrm{cov}(\hat{\beta}_0, \hat{\beta}_1) &= \mathrm{cov}\left(\sum_{i=1}^n \left(\frac{1}{n} - w_i \bar{x}_n\right) Y_i, \sum_{i=1}^n w_i Y_i\right) \\
&= \sigma^2 \sum_{i=1}^n \left(\frac{1}{n} - w_i \bar{x}_n\right) w_i \\
&= -\sigma^2 \frac{\bar{x}_n}{SS_x}.
\end{aligned}$$

Solution 3.9 The correlation between $\hat{\beta}_0$ and $\hat{\beta}_1$ is

$$\rho_{\hat{\beta}_0,\hat{\beta}_1} = -\frac{\sigma^2 \bar{x}_n}{\sigma^2 SS_x \left[\left(\frac{1}{n} + \frac{\bar{x}_n^2}{SS_x}\right)\frac{1}{SS_x}\right]^{1/2}}$$
$$= -\frac{\bar{x}_n}{\left(\frac{1}{n}\sum_{i=1}^n x_i^2\right)^{1/2}}.$$

Solution 3.10 $X_1, X_2, \ldots, X_n$ i.i.d., $X_1 \sim P(\lambda)$. The likelihood function is

$$L(\lambda; X_1, \ldots, X_n) = e^{-n\lambda}\frac{\lambda^{\sum_{i=1}^n X_i}}{\prod_{i=1}^n X_i!}$$

Thus, $\frac{\partial}{\partial\lambda}\log L(\lambda; X_1, \ldots, X_n) = -n + \frac{\sum_{i=1}^n X_i}{\lambda}$. Equating this to zero and solving for λ, we get $\hat{\lambda}_n = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X}_n$.

Solution 3.11 Since ν is known, the likelihood of β is $L(\beta) = C_n \frac{1}{\beta^{n\nu}} e^{-\sum_{i=1}^n X_i/\beta}$, $0 < \beta < \infty$ where C_n does not depend on β. The log-likelihood function is

$$l(\beta) = \log C_n - n\nu\log\beta - \frac{1}{\beta}\sum_{i=1}^n X_i.$$

The score function is $l'(\beta) = -\frac{n\nu}{\beta} + \frac{\sum_{i=1}^n X_i}{\beta^2}$. Equating the score to 0 and solving for β, we obtain the MLE $\hat{\beta} = \frac{1}{n\nu}\sum_{i=1}^n X_i = \frac{1}{\nu}\bar{X}_n$. The variance of the MLE is $V\{\hat{\beta}\} = \frac{\beta^2}{n\nu}$.

{exc:mgf-nb} **Solution 3.12** We proved in Exercise 2.56 that the m.g.f. of $NB(2, p)$ is

$$M_X(t) = \frac{p^2}{(1-(1-p)e^t)^2}, \quad t < -\log(1-p).$$

Let $X_1, X_2, \ldots, X_n$ be i.i.d. $NB(2, p)$, then the m.g.f. of $T_n = \sum_{i=1}^n X_i$ is

$$M_{T_n}(t) = \frac{p^{2n}}{(1-(1-p)e^t)^{2n}}, \quad t < -\log(1-p).$$

{ex:neg-binom-dist} Thus, $T_n \sim NB(2n, p)$. According to Example 3.4, the MLE of p, based on T_n (which is a sufficient statistic) is $\hat{p}_n = \frac{2n}{T_n + 2n} = \frac{2}{\bar{X}_n + 2}$, where $\bar{X}_n = T_n/n$ is the sample mean.

(i) According to the WLLN, $\bar{X}_n$ converges in probability to $E\{X_1\} = \dfrac{2(1-p)}{p}$. Substituting $2\dfrac{1-p}{p}$ for $\bar{X}_n$ in the formula of $\hat{p}_n$ we obtain $p^* = \dfrac{2}{2+2\frac{1-p}{p}} = p$. This shows that the limit in probability as $n \to \infty$, of $\hat{p}_n$ is p.

(ii) Substituting $k = 2n$ in the formulas of Example 3.4 we obtain {ex:neg-binom-dist}

$$\text{Bias}(\hat{p}_n) \approx \frac{3p(1-p)}{4n} \quad \text{and} \quad V\{\hat{p}_n\} \approx \frac{p^2(1-p)}{2n}.$$

Solution 3.13 The likelihood function of μ and β is

$$L(\mu,\beta) = I\{X_{(1)} \ \geq \ \mu\}\frac{1}{\beta^n}\exp\left\{-\frac{1}{\beta}\sum_{i=1}^{n}(X_{(i)} - X_{(1)}) - \frac{n}{\beta}(X_{(1)} - \mu)\right\},$$

for $-\infty < \mu \leq X_{(1)}, \quad 0 < \beta < \infty$.

(i) $L(\mu,\beta)$ is maximized by $\hat{\mu} = X_{(1)}$, that is,

$$L^*(\beta) = \sup_{\mu \leq X_{(1)}} L(\mu,\beta) = \frac{1}{\beta^n}\exp\left\{-\frac{1}{\beta}\sum_{i=1}^{n}(X_{(i)} - X_{(1)})\right\}$$

where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the ordered statistics.

(ii) Furthermore $L^*(\beta)$ is maximized by $\hat{\beta}_n = \dfrac{1}{n}\sum_{i=2}^{n}(X_{(i)} - X_{(1)})$. The MLEs are $\hat{\mu} = X_{(1)}$, and $\hat{\beta}_n = \dfrac{1}{n}\sum_{i=2}^{n}(X_{(i)} - X_{(1)})$.

(iii) $X_{(1)}$ is distributed like $\mu + E\left(\dfrac{\beta}{n}\right)$, with p.d.f.

$$f_{(1)}(x;\mu,\beta) = I\{x \geq \mu\}\frac{n}{\beta}e^{-\frac{n}{\beta}(x-\mu)}.$$

Thus, the joint p.d.f. of $(X_1, \ldots, X_n)$ is factored to a product of the p.d.f. of $X_{(1)}$ and a function of $\hat{\beta}_n$, which does not depend on $X_{(1)}$ (nor on μ). This implies that $X_{(1)}$ and $\hat{\beta}_n$ are independent. $V\{\hat{\mu}\} = V\{X_{(1)}\} = \dfrac{\beta^2}{n^2}$. It can be shown that $\hat{\beta}_n \sim \dfrac{1}{n}G(n-1,\beta)$. Accordingly, $V\{\hat{\beta}_n\} = \dfrac{n-1}{n^2}\beta^2 = \dfrac{1}{n}\left(1 - \dfrac{1}{n}\right)\beta^2$.

Solution 3.14 In sampling with replacement, the number of defective items in the sample, X, has the binomial distribution $B(n, p)$. We test the hypotheses $H_0 : p \leq 0.03$ against $H_1 : p > 0.03$. H_0 is rejected if $X > B^{-1}(1-\alpha, 20, 0.03)$. For $\alpha = 0.05$ the rejection criterion is $k_\alpha = B^{-1}(0.95, 20, 0.03) = 2$. Since the number of defective items in the sample is $X = 2$, H_0 is not rejected at the $\alpha = 0.05$ significance level.

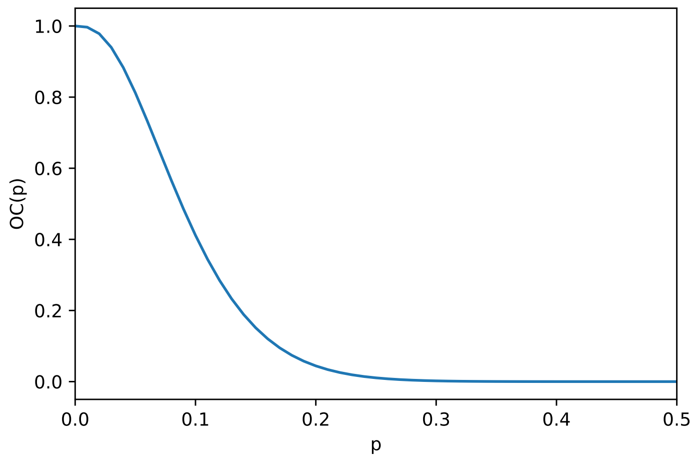


Fig. 3.1 The OC Function $B(2; 30, p)$. {fig:PlotOCcurveBinomial_2_30}

Solution 3.15 The OC function is $OC(p) = B(2; 30, p)$, $0 < p < 1$. A plot of this OC function is given in Figure 3.1. {fig:PlotOCcurveBinomial_2_30}

Solution 3.16 Let $p_0 = 0.01$, $p_1 = 0.03$, $\alpha = 0.05$, $\beta = 0.05$. According to Eq. (3.3.12), the sample size n should satisfy

$$1 - \Phi\left(\frac{p_1 - p_0}{\sqrt{p_1 q_1}}\sqrt{n} - z_{1-\alpha}\sqrt{\frac{p_0 q_0}{p_1 q_1}}\right) = \beta$$

or, equivalently,

$$\frac{p_1 - p_0}{\sqrt{p_1 q_1}}\sqrt{n} - z_{1-\alpha}\sqrt{\frac{p_0 q_0}{p_1 q_1}} = z_{1-\beta}.$$

This gives

$$\begin{aligned} n &\approx \frac{(z_{1-\alpha}\sqrt{p_0 q_0} + z_{1-\beta}\sqrt{p_1 q_1}\,)^2}{(p_1 - p_0)^2} \\ &= \frac{(1.645)^2(\sqrt{0.01 \times 0.99} + \sqrt{0.03 \times 0.97})^2}{(0.02)^2} = 494. \end{aligned}$$

For this sample size, the critical value is $k_\alpha = np_0 + z_{1-\alpha}\sqrt{np_0 q_0} = 8.58$. Thus, H_0 is rejected if there are more than 8 "successes" in a sample of size 494.

Solution 3.17 $\bar{X}_n \sim N(\mu, \frac{\sigma}{\sqrt{n}})$.

(i) If $\mu = \mu_0$ the probability that $\bar{X}_n$ will be outside the control limits is

$$\Pr\left\{\bar{X}_n < \mu_0 - \frac{3\sigma}{\sqrt{n}}\right\} + \Pr\left\{\bar{X}_n > \mu_0 + \frac{3\sigma}{\sqrt{n}}\right\} = \Phi(-3) + 1 - \Phi(3) = 0.0027.$$

(ii) $(1 - 0.0027)^{20} = 0.9474$.

(iii) If $\mu = \mu_0 + 2(\sigma/\sqrt{n})$, the probability that $\bar{X}_n$ will be outside the control limits is

$$\Phi(-5) + 1 - \Phi(1) = 0.1587.$$

(iv) $(1 - 0.1587)^{10} = 0.1777$.

Solution 3.18 We can run the 1-sample t-test in Python as follows:

```
socell = mistat.load_data('SOCELL')
t1 = socell['t1']

statistic, pvalue = stats.ttest_1samp(t1, 4.0)
# divide pvalue by two for one-sided test
pvalue = pvalue / 2
print(f'pvalue {pvalue:.2f}')
```

```
pvalue 0.35
```

The hypothesis $H_0 : \mu \geq 4.0$ amps is not rejected.

Solution 3.19 We can run the 1-sample t-test in Python as follows:

```
socell = mistat.load_data('SOCELL')
t2 = socell['t2']

statistic, pvalue = stats.ttest_1samp(t2, 4.0)
# divide pvalue by two for one-sided test
pvalue = pvalue / 2
print(f'pvalue {pvalue:.2f}')
```

```
pvalue 0.03
```

The hypothesis $H_0 : \mu \geq 4.0$ amps is rejected at a 0.05 level of significance.

Solution 3.20 Let $n = 30$, $\alpha = 0.01$. The $OC(\delta)$ function for a one-sided t-test is

$$\begin{aligned} OC(\delta) &= 1 - \Phi\left(\frac{\delta\sqrt{30} - 2.462 \times (1 - \frac{1}{232})}{(1 + \frac{6.0614}{58})^{1/2}}\right) \\ &= 1 - \Phi(5.2117\delta - 2.3325). \end{aligned}$$

```
delta = np.linspace(0, 1.0, 11)

a = np.sqrt(30)
b = 2.462 * ( 1 - 1/232)
f = np.sqrt(1 + 6.0614 / 58)
OC_delta = 1 - stats.norm.cdf((a * delta - b) / f)
```

Values of $OC(\delta)$ for $\delta = 0, 1(0.1)$ are given in the following table.

δ	OC(δ)
0.0	0.990164
0.1	0.964958
0.2	0.901509
0.3	0.779063
0.4	0.597882
0.5	0.392312
0.6	0.213462
0.7	0.094149
0.8	0.033120
0.9	0.009188
1.0	0.001994

Solution 3.21 Let $n = 31$, $\alpha = 0.10$. The OC function for testing $H_0 : \sigma^2 \leq \sigma_0^2$ against $H_1 : \sigma^2 > \sigma_0^2$ is

$$\begin{aligned} OC(\sigma^2) &= \Pr\left\{S^2 \leq \frac{\sigma_0^2}{n-1}\chi^2_{0.9}[n-1]\right\} \\ &= \Pr\left\{\chi^2[30] \leq \frac{\sigma_0^2}{\sigma^2}\chi^2_{0.9}[30]\right\} \\ &= 1 - P\left(\frac{30}{2} - 1; \frac{\sigma_0^2}{\sigma^2} \cdot \frac{\chi^2_{0.9}[30]}{2}\right). \end{aligned}$$

```
sigma2 = np.linspace(1, 2, 11)
OC_sigma2 = 1 - stats.poisson.cdf(30 / 2 - 1,
                                  stats.chi2(30).ppf(0.90) / (2 * sigma2))
```

The values of $OC(\sigma^2)$ for $\sigma^2 = 1, 2(0.1)$ are given in the following table: (Here $\sigma_0^2 = 1$.)

σ^2	OC(σ^2)
1.0	0.900000
1.1	0.810804
1.2	0.700684
1.3	0.582928
1.4	0.469471
1.5	0.368201
1.6	0.282781
1.7	0.213695
1.8	0.159540
1.9	0.118063
2.0	0.086834

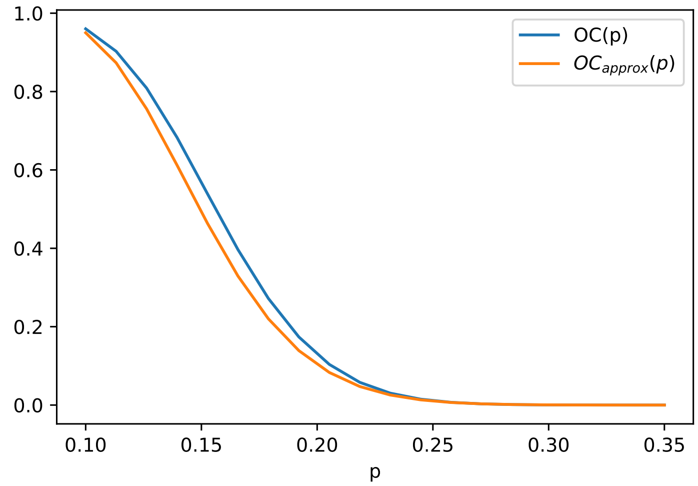


Fig. 3.2 Comparison exact to normal approximation of $OC(p)$ {fig:OC_OCapprox}

Solution 3.22 The OC function, for testing $H_0 : p \leq p_0$ against $H_1 : p > p_0$ is approximated by

$$OC(p) = 1 - \Phi\left(\frac{p - p_0}{\sqrt{pq}}\sqrt{n} - z_{1-\alpha}\sqrt{\frac{p_0 q_0}{pq}}\right),$$

for $p \geq p_0$. In Figure 3.2 we present the graph of $OC(p)$, for $p_0 = 0.1$, $n = 100$, $\alpha = 0.05$ both using the exact solution and the normal approximation. {fig:OC_OCapprox}

```
n = 100
p0 = 0.1
alpha = 0.05

c_alpha = stats.binom(n, p0).ppf(1 - alpha)
p = np.linspace(0.1, 0.35, 20)
OC_exact = stats.binom(n, p).cdf(c_alpha)

z_ma = stats.norm.ppf(1 - alpha)
q0 = 1- p0
pq = p * (1 - p)
OC_approx = 1 - stats.norm.cdf((p - p0) * np.sqrt((n / pq)) -
                               z_ma * np.sqrt(p0 * q0 / pq))
df = pd.DataFrame({
  'p': p,
  'OC(p)': OC_exact,
  '$OC_{approx}(p)$': OC_approx,
})
df.plot.line(x='p', y=['OC(p)', '$OC_{approx}(p)$'])
plt.show()
```

Solution 3.23 The power function is

$$\psi(\sigma^2) = \Pr\left\{S^2 \geq \frac{\sigma_0^2}{n-1}\chi^2_{1-\alpha}[n-1]\right\}$$
$$= \Pr\left\{\chi^2[n-1] \geq \frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}[n-1]\right\}.$$

Solution 3.24 The power function is $\psi(\rho) = \Pr\left\{F[n_1-1, n_2-1] \geq \frac{1}{\rho}F_{1-\alpha}[n_1-1, n_2-1]\right\}$, for $\rho \geq 1$, where $\rho = \frac{\sigma_1^2}{\sigma_2^2}$.

Solution 3.25 **(i)** Using the following Python commands we get a 99% C.I. for μ:

```
data = [20.74, 20.85, 20.54, 20.05, 20.08, 22.55, 19.61, 19.72,
        20.34, 20.37, 22.69, 20.79, 21.76, 21.94, 20.31, 21.38,
        20.42, 20.86, 18.80, 21.41]
alpha = 1 - 0.99

df = len(data) - 1
mean = np.mean(data)
sem = stats.sem(data)

print(stats.t.interval(1 - alpha, df, loc=mean, scale=sem))
```

```
(20.136889216656858, 21.38411078334315)
```

Confidence Intervals

Variable	N	Mean	StDev	SE Mean	99.0% C.I.
Sample	20	20.760	0.975	0.218	(20.137, 21.384)

(ii) A 99% C.I. for σ^2 is $(0.468, 2.638)$.

```
var = np.var(data, ddof=1)
print(df * var / stats.chi2(df).ppf(1 - alpha/2))
print(df * var / stats.chi2(df).ppf(alpha/2))
```

```
0.46795850248657883
2.6380728125212016
```

(iii) A 99% C.I. for σ is $(0.684, 1.624)$.

Solution 3.26 Let $(\underline{\mu}_{.99}, \bar{\mu}_{.99})$ be a confidence interval for μ, at level 0.99. Let $(\underline{\sigma}_{.99}, \bar{\sigma}_{.99})$ be a confidence interval for σ at level 0.99. Let $\underline{\xi} = \underline{\mu}_{.99} + 2\underline{\sigma}_{.99}$ and $\bar{\xi} = \bar{\mu}_{.99} + 2\bar{\sigma}_{.99}$. Then

$$\Pr\{\underline{\xi} \leq \mu + 2\sigma \leq \bar{\xi}\} \geq \Pr\{\underline{\mu}_{.99} \leq \mu \leq \bar{\mu}_{.99},\ \underline{\sigma}_{.99} \leq \sigma \leq \bar{\sigma}_{.99}\} \geq 0.98.$$

Thus, $(\underline{\xi}, \bar{\xi})$ is a confidence interval for $\mu + 2\sigma$, with confidence level greater or equal to 0.98. Using the data of the previous problem, a 98% C.I. for $\mu + 2\sigma$ is $(21.505, 24.632)$.

Solution 3.27 Let $X \sim B(n, \theta)$. For $X = 17$ and $n = 20$, a confidence interval for θ, at level 0.95, is (0.6211, 0.9679).

```
alpha = 1 - 0.95
X = 17
n = 20
F1 = stats.f(2*(n-X+1), 2*X).ppf(1 - alpha/2)
F2 = stats.f(2*(X+1), 2*(n-X)).ppf(1 - alpha/2)
pL = X / (X + (n-X+1) * F1)
pU = (X+1) * F2 / (n-X + (X+1) * F2)
print(pL, pU)
```

```
0.6210731734546859 0.9679290628145363
```

Solution 3.28 From the data we have $n = 10$ and $T_{10} = 134$. For $\alpha = 0.05$, $\lambda_L = 11.319$ and $\lambda_U = 15.871$.

```
X = [14, 16, 11, 19, 11, 9, 12, 15, 14, 13]
alpha = 1 - 0.95
T_n = np.sum(X)

# exact solution
print(stats.chi2(2 * T_n + 2).ppf(alpha/2) / (2 * len(X)))
print(stats.chi2(2 * T_n + 2).ppf(1 - alpha/2) / (2 * len(X)))

# approximate solution
nu = 2 * T_n + 2
print((nu + stats.norm.ppf(alpha/2) * np.sqrt(2 * nu)) / (2 * len(X)))
print((nu + stats.norm.ppf(1-alpha/2) * np.sqrt(2 * nu)) / (2 * len(X)))
```

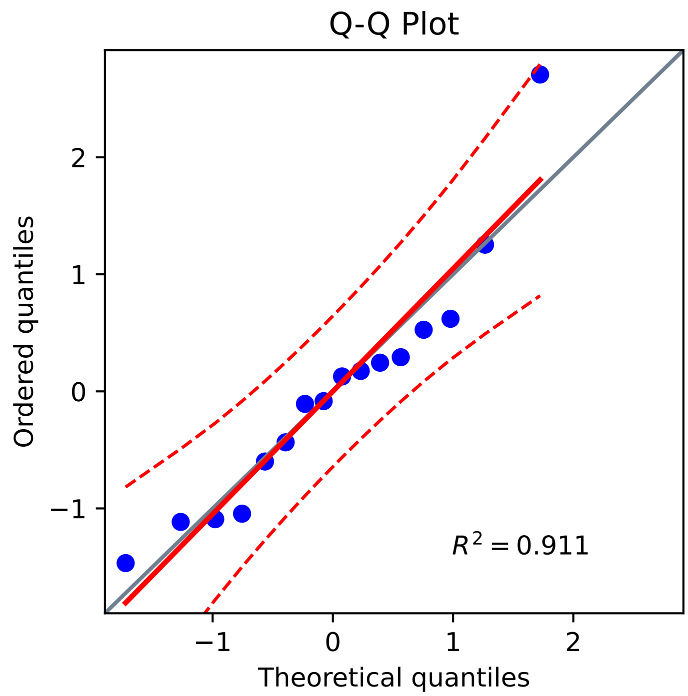
```
11.318870163746238
15.870459268116013
11.222727638613012
15.777272361386988
```

Solution 3.29 For $n = 20$, $\sigma = 5$, $\bar{Y}_{20} = 13.75$, $\alpha = 0.05$ and $\beta = 0.1$, the tolerance interval is (3.33, 24.17).

Solution 3.30 Use the following Python code:

```
yarnstrg = mistat.load_data('YARNSTRG')
n = len(yarnstrg)
Ybar = yarnstrg.mean()
S = yarnstrg.std()

alpha, beta = 0.025, 0.025
z_1a = stats.norm.ppf(1-alpha)
z_1b = stats.norm.ppf(1-beta)
z_a = stats.norm.ppf(alpha)
z_b = stats.norm.ppf(beta)

t_abn = (z_1b/(1 - z_1a**2/(2*n)) +
         z_1a*np.sqrt(1 + z_b**2/2 - z_1a**2/(2*n)) /
             (np.sqrt(n)*(1 - z_1a**2/(2*n))))

print(Ybar - t_abn*S, Ybar + t_abn*S)
```

```
0.7306652047594424 5.117020795240556
```



{fig:qqplotISCT1Socell} **Fig. 3.3** Q-Q plot of ISC-t_1 (**SOCELL.csv**)

From the data we have $\bar{X}_{100} = 2.9238$ and $S_{100} = 0.9378$.

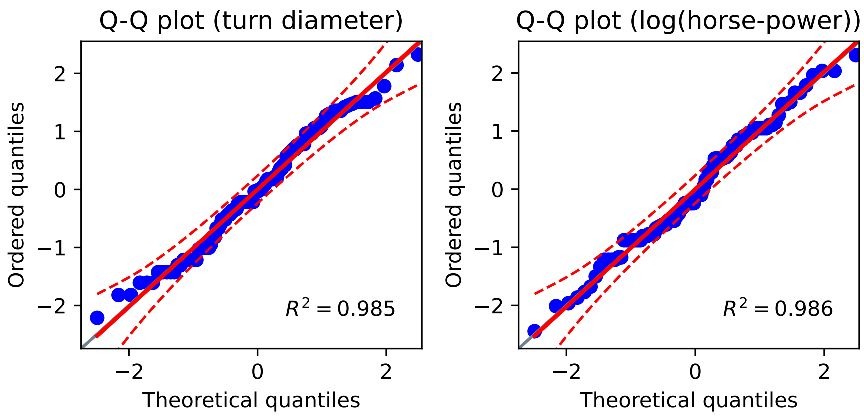
$$t(0.025, 0.025, 100) = \frac{1.96}{1 - 1.96^2/200} + \frac{1.96(1 + \frac{1.96^2}{2} - \frac{1.96^2}{200})^{1/2}}{10(1 - \frac{1.96^2}{200})}$$
$$= 2.3388.$$

The tolerance interval is $(0.7306, 5.1171)$.

Solution 3.31 From the data, $Y_{(1)} = 1.151$ and $Y_{(100)} = 5.790$. For $n = 100$ and $\beta = 0.10$ we have $1 - \alpha = 0.988$. For $\beta = 0.05$, $1 - \alpha = 0.847$, the tolerance interval is $(1.151, 5.790)$. The nonparametric tolerance interval is shorter and is shifted to the right with a lower confidence level.

Solution 3.32 The following is a normal probability plot of ISC-t_1:
{fig:qqplotISCT1Socell} According to Figure 3.3, the hypothesis of normality is not rejected.

{fig:qqplotCar} **Solution 3.33** As is shown in the normal probability plots in Figure 3.4, the hypothesis of normality is not rejected in either case.



{fig:qqplotCar} **Fig. 3.4** *Q-Q* plot of **CAR.csv** data

Solution 3.34 Frequency distribution for turn diameter:

Interval	Observed	Expected	$(0 - E)^2/E$
- 31	11	8.1972	0.9583
31 - 32	8	6.3185	0.4475
32 - 33	9	8.6687	0.0127
33 - 34	6	10.8695	2.1815
34 - 35	18	12.4559	2.4677
35 - 36	8	13.0454	1.9513
36 - 37	13	12.4868	0.0211
37 - 38	6	10.9234	2.2191
38 - 39	9	8.7333	0.0081
39 - 40	8	6.3814	0.4106
40 -	13	9.0529	1.7213
Total	109	—	12.399

The expected frequencies were computed for $N(35.5138, 3.3208)$. Here $\chi^2 = 12.4$, d.f. = 8 and the P value is 0.134. The differences from normal are not significant.

```
          observed   expected  (O-E)^2/E
[28, 31)        11   8.197333   0.958231
[31, 32)         8   6.318478   0.447500
[32, 33)         9   8.668695   0.012662
[33, 34)         6  10.869453   2.181487
[34, 35)        18  12.455897   2.467673
[35, 36)         8  13.045357   1.951317
[36, 37)        13  12.486789   0.021093
[37, 38)         6  10.923435   2.219102
[38, 39)         9   8.733354   0.008141
[39, 40)         8   6.381395   0.410550
[40, 44)        13   9.052637   1.721231
chi2-statistic of fit 12.398987638400024
```

```
chi2[8] for 95% 15.50731305586545
p-value of observed statistic 0.1342700576126994
```

Solution 3.35 In Python:

```
KstestResult(statistic=0.07019153486614366, pvalue=0.6303356787948367)
k_alpha 0.08514304524687971
```

$D^*_{109} = 0.0702$. For $\alpha = 0.05$ $k^*_\alpha = 0.895/\left(\sqrt{109} - 0.01 + \dfrac{0.85}{\sqrt{109}}\right) = 0.0851$. The deviations from the normal distribution are not significant.

Solution 3.36 For $X \sim P(100)$, $n^0 = P^{-1}\left(\frac{0.2}{0.3}, 100\right) = 100 + z_{.67} \times 10 = 105$.

Solution 3.37 Given $X = 6$, the posterior distribution of p is Beta(9,11).

Solution 3.38 $E\{p \mid X = 6\} = \dfrac{9}{20} = 0.45$ and $V\{p \mid X = 6\} = \dfrac{99}{20^2 \times 21} = 0.0118$ so $\sigma_{p|X=6} = 0.1086$.

Solution 3.39 Let $X \mid \lambda \sim P(\lambda)$ where $\lambda \sim G(2, 50)$.

(i) The posterior distribution of $\lambda \mid X = 82$ is $G\left(84, \dfrac{50}{51}\right)$.

(ii) $G_{.025}\left(84, \dfrac{50}{51}\right) = 65.6879$ and $G_{.975}\left(84, \dfrac{50}{51}\right) = 100.873$.

Solution 3.40 The posterior probability for $\lambda_0 = 70$ is

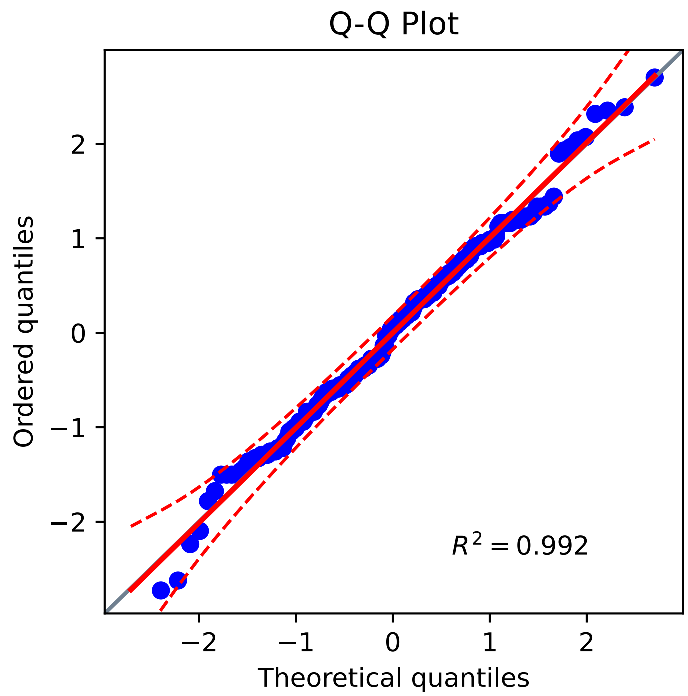
$$\pi(72) = \frac{\frac{1}{3}p(72; 70)}{\frac{1}{3}p(72; 70) + \frac{2}{3}p(72; 90)} = 0.771.$$

$H_0 : \lambda = \lambda_0$ is accepted if $\pi(X) > \dfrac{r_0}{r_0 + r_1} = 0.4$. Thus, H_0 is accepted.

Solution 3.41 The credibility interval for μ is (43.235,60.765). Since the posterior distribution of μ is symmetric, this credibility interval is also a HPD interval.

Solution 3.42 In Python:

```
random.seed(1)
car = mistat.load_data('CAR')

def mean_of_sample_with_replacement(x, n):
  sample = random.choices(x, k=n)
  return np.mean(sample)

means = [mean_of_sample_with_replacement(car['mpg'], 64)
         for _ in range(200)]

fig, ax = plt.subplots(figsize=[4, 4])
pg.qqplot(means, ax=ax)
plt.show()
```



{fig:qqplotSampleMeansMPG} **Fig. 3.5** *Q*-*Q* Plot of mean of samples from `mpg` data

{fig:qqplotSampleMeansMPG} As Figure 3.5 shows, the resampling distribution is approximately normal.

```
S = np.std(car['mpg'], ddof=1)
print('standard deviation of mpg', S)
print('S/8', S/8)
S_resample = np.std(means, ddof=1)
print('S.E.\{X\}', S_resample)
```

```
standard deviation of mpg 3.9172332424696052
S/8 0.48965415530870066
S.E.\{X\} 0.44718791755315535
```

Executing the macro 200 times, we obtained S.E.$\{\bar{X}\} = \frac{S}{8} = 0.48965$. The standard deviation of the resampled distribution is 0.4472. This is a resampling estimate of S.E.$\{\bar{X}\}$.

Solution 3.43 In our particular execution with $M = 500$, we have a proportion $\hat{\alpha} = 0.07$ of cases in which the bootstrap confidence intervals do not cover the mean of `yarnstrg`, $\mu = 2.9238$. This is not significantly different from the nominal $\alpha = 0.05$. The determination of the proportion $\hat{\alpha}$ can be done by using the following commands:

```
random.seed(1)
yarnstrg = mistat.load_data('YARNSTRG')

def confidence_interval(x, nsigma=2):
  sample_mean = np.mean(x)
  sigma = np.std(x, ddof=1) / np.sqrt(len(x))
  return (sample_mean - 2 * sigma, sample_mean + 2 * sigma)

mean = np.mean(yarnstrg)
outside = 0
for _ in range(500):
  sample = random.choices(yarnstrg, k=30)
  ci = confidence_interval(sample)
  if mean < ci[0] or ci[1] < mean:
    outside += 1

hat_alpha = outside / 500

ci = confidence_interval(yarnstrg)
print(f' Mean: {mean}')
print(f' 2-sigma-CI: {ci[0]:.1f} - {ci[1]:.1f}')
print(f' proportion outside: {hat_alpha:.3f}')
```

```
Mean: 2.9238429999999993
2-sigma-CI: 2.7 - 3.1
proportion outside: 0.068
```

Solution 3.44 In Python:

```
random.seed(1)
car = mistat.load_data('CAR')
us_cars = car[car['origin'] == 1]
us_turn = list(us_cars['turn'])

sample_means = []
for _ in range(100):
  x = random.choices(us_turn, k=58)
  sample_means.append(np.mean(x))

is_larger = sum(m > 37.406 for m in sample_means)
ratio = is_larger / len(sample_means)
print(ratio)
```

```
0.23
```

We obtained $\tilde{P} =$ 0.23. The mean $\bar{X} = 37.203$ is **not** significantly larger than 37.

Solution 3.45 Let X_{50} be the number of non-conforming units in a sample of $n = 50$ items. We reject H_0, at level of $\alpha = 0.05$, if $X_{50} > B^{-1}(0.95, 50, 0.03) = 4$. The criterion k_α is obtained by using the Python commands:

```
stats.binom(50, 0.03).ppf(0.95)
```

```
4.0
```

Solution 3.46 i Calculation of 95% confidence intervals:

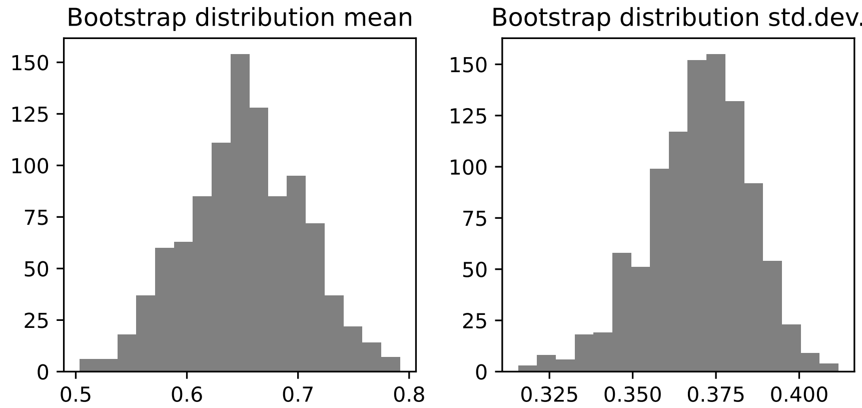


Fig. 3.6 Histograms of EBD for **CYCLT.csv** data {fig:histEBD_CYCLT}

```
random.seed(1)
cyclt = mistat.load_data('CYCLT')

B = pg.compute_bootci(cyclt, func=np.mean, n_boot=1000,
                      confidence=0.95, return_dist=True, seed=1)

ci_mean, dist_mean = B
print(f' Mean: {np.mean(dist_mean):.3f}')
print(f' 95%-CI: {ci_mean[0]:.3f} - {ci_mean[1]:.3f}')

B = pg.compute_bootci(cyclt, func=lambda x: np.std(x, ddof=1), n_boot=1000,
                      confidence=0.95, return_dist=True, seed=1)

ci_std, dist_std = B
print(f' Mean: {np.mean(dist_std):.3f}')
print(f' 95%-CI: {ci_std[0]:.3f} - {ci_std[1]:.3f}')
```

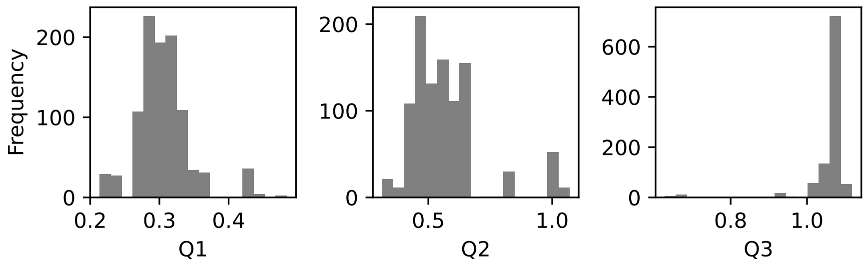
```
Mean: 0.652
95%-CI: 0.550 - 0.760
Mean: 0.370
95%-CI: 0.340 - 0.400
```

ii Histograms of the EBD, see Figure 3.6. {fig:histEBD_CYCLT}

```
fig, axes = plt.subplots(figsize=[6, 3], ncols=2)
axes[0].hist(dist_mean, color='grey', bins=17)
axes[1].hist(dist_std, color='grey', bins=17)
axes[0].set_title('Bootstrap distribution mean')
axes[1].set_title('Bootstrap distribution std.dev.')
plt.tight_layout()
plt.show()
```

Solution 3.47 In Python:

```
cyclt = mistat.load_data('CYCLT')
ebd = {}
```



{fig:histEBD_CYCLT_quartiles} **Fig. 3.7** Histograms of EBD for quartiles for **CYCLT.csv** data

```
for quantile in (1, 2, 3):
  B = pg.compute_bootci(cyclt, func=lambda x: np.quantile(x, 0.25 * quantile),
                        n_boot=1000, confidence=0.95, return_dist=True, seed=1)
  ci, dist = B
  ebd[quantile] = dist
  print(f'Quantile {quantile}: {np.mean(dist):.3f} 95%-CI: {ci[0]:.3f} - {ci[1]:.3f}')
```

```
Quantile 1: 0.306 95%-CI: 0.230 - 0.420
Quantile 2: 0.573 95%-CI: 0.380 - 1.010
Quantile 3: 1.060 95%-CI: 0.660 - 1.090
```

{fig:histEBD_CYCLT_quartiles} **ii** Histograms of the EBD, see Figure 3.7.

```
fig, axes = plt.subplots(figsize=[6, 2], ncols=3)
axes[0].hist(ebd[1], color='grey', bins=17)
axes[1].hist(ebd[2], color='grey', bins=17)
axes[2].hist(ebd[3], color='grey', bins=17)
axes[0].set_xlabel('Q1')
axes[1].set_xlabel('Q2')
axes[2].set_xlabel('Q3')
axes[0].set_ylabel('Frequency')
plt.tight_layout()
plt.show()
```

Solution 3.48 In Python:

```
socell = mistat.load_data('SOCELL')
t1 = socell['t1']
t2 = socell['t2']

# use the index
idx = list(range(len(socell)))
def sample_correlation(x):
  return stats.pearsonr(t1[x], t2[x])[0]

B = pg.compute_bootci(idx, func=sample_correlation,
                      n_boot=1000, confidence=0.95, return_dist=True, seed=1)
ci, dist = B
print(f'rho_XY: {np.mean(dist):.3f} 95%-CI: {ci[0]:.3f} - {ci[1]:.3f}')
```

```
rho_XY: 0.975 95%-CI: 0.940 - 0.990
```

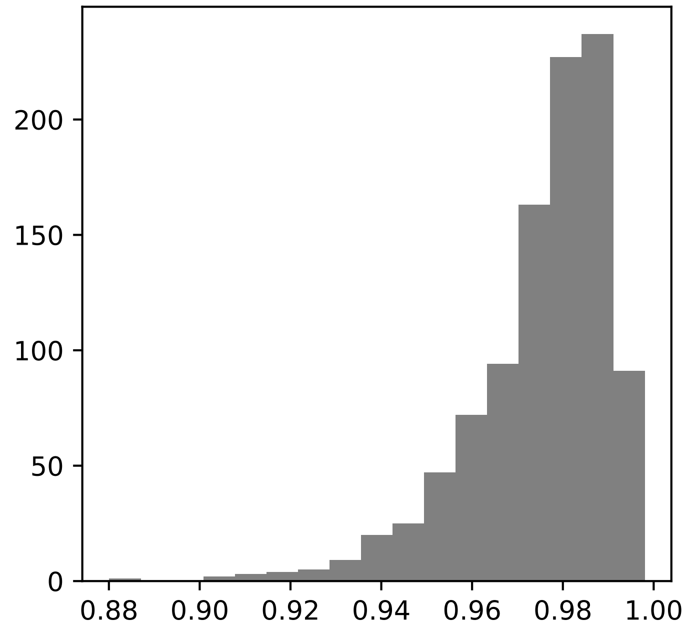


Fig. 3.8 Histograms of EBD for correlation for **SOCELL.csv** data {fig:histEBD_SOCELL_correlation}

{fig:histEBD_SOCELL_correlation} Histogram of bootstrap correlations, see Figure 3.8.

```
fig, ax = plt.subplots(figsize=[4, 4])
ax.hist(dist, color='grey', bins=17)
plt.show()
```

Solution 3.49 (i) and (ii)

```
car = mistat.load_data('CAR')
mpg = car['mpg']
hp = car['hp']

idx = list(range(len(mpg)))
sample_intercept = []
sample_slope = []
for _ in range(1000):
  x = random.choices(idx, k=len(idx))
  result = stats.linregress(hp[x], mpg[x])
  sample_intercept.append(result.intercept)
  sample_slope.append(result.slope)

ci = np.quantile(sample_intercept, [0.025, 0.975])
print(f'intercept (a): {np.mean(sample_intercept):.3f} ' +
      f'95%-CI: {ci[0]:.3f} - {ci[1]:.3f}')
ci = np.quantile(sample_slope, [0.025, 0.975])
print(f'slope (b): {np.mean(sample_slope):.4f} ' +
      f'95%-CI: {ci[0]:.4f} - {ci[1]:.4f}')
```

```
reg = stats.linregress(hp, mpg)
hm = np.mean(hp)

print(np.std(sample_intercept))
print(np.std(sample_slope))
```

```
intercept (a): 30.724 95%-CI: 28.766 - 32.691
slope (b): -0.0741 95%-CI: -0.0891 - -0.0599
1.0170449375724464
0.0074732552885114645
```

(iii) The bootstrap S.E. of *slope* and *intercept* are 1.017 and 0.00747, respectively. The standard errors of a and b, according to the formulas of Section 4.3.2.1 are 0.8099 and 0.00619, respectively. The bootstrap estimates are quite close to the correct values.

{sec:least-squares-single}

Solution 3.50 In Python:

```
cyclt = mistat.load_data('CYCLT')
X50 = np.mean(cyclt)
SD50 = np.std(cyclt)
result = stats.ttest_1samp(cyclt, 0.55)
print(f'Xmean_50 = {X50:.3f}')
print(result)

B = pg.compute_bootci(cyclt, func=np.mean, n_boot=1000,
                      confidence=0.95, return_dist=True, seed=1)
ci_mean, dist = B
pstar = sum(dist < 0.55) / len(dist)
print(f'p*-value: {pstar}')
```

```
Xmean_50 = 0.652
Ttest_1sampResult(statistic=1.9425149510299369,
pvalue=0.057833259176805)
p*-value: 0.024
```

The mean of the sample is $\bar{X}_{50} = 0.652$. The studentized difference from $\mu_0 = 0.55$ is $t = 1.943$.

(i) The t-test obtained a P-level of 0.058 and the bootstrap resulted in $P^* = 0.024$.

(ii) Yes, but μ is very close to the lower bootstrap confidence limit (0.540). The null hypothesis $H_0 : \mu = 0.55$ is accepted.

(iii) No, but since P^* is close to 0.05, we expect that the bootstrap confidence interval will be very close to μ_0.

Solution 3.51 In Python:

```
almpin = mistat.load_data('ALMPIN')
diam1 = almpin['diam1']
diam2 = almpin['diam2']

# calculate the ratio of the two variances:
var_diam1 = np.var(diam1)
var_diam2 = np.var(diam2)
F = var_diam2 / var_diam1
print(f'Variance diam1: {var_diam1:.5f}')
print(f'Variance diam2: {var_diam2:.5f}')
print(f'Ratio: {F:.4f}')
```

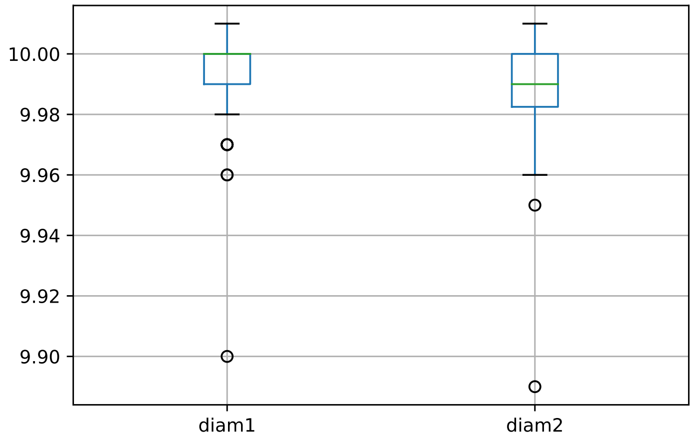


Fig. 3.9 Box plots of `diam1` and `diam2` measurements of the ALMPIN dataset {fig:boxplot-almpin}

```
# Calculate the p-value
p_value = stats.f.cdf(F, len(diam1) - 1, len(diam2) - 1)
print(f'p-value: {p_value:.3f}')
```

```
Variance diam1: 0.00027
Variance diam2: 0.00032
Ratio: 1.2016
p-value: 0.776
```

The variances are therefore not significantly different.
(ii) The box plots of the two measurements are shown in Figure 3.9. {fig:boxplot-almpin}

```
almpin.boxplot(column=['diam1', 'diam2'])
plt.show()
```

Solution 3.52 The variances were already compared in the previous exercise. To compare the means use:

```
almpin = mistat.load_data('ALMPIN')
diam1 = almpin['diam1']
diam2 = almpin['diam2']

# Compare means
mean_diam1 = np.mean(diam1)
mean_diam2 = np.mean(diam2)
print(f'Mean diam1: {mean_diam1:.5f}')
print(f'Mean diam2: {mean_diam2:.5f}')

# calculate studentized difference and p-value
se1, se2 = stats.sem(diam1), stats.sem(diam2)
sed = np.sqrt(se1**2.0 + se2**2.0)
t_stat = (mean_diam1 - mean_diam2) / sed
print(f'Studentized difference: {t_stat:.3f}')
df = len(diam1) + len(diam2) - 2
p = (1 - stats.t.cdf(abs(t_stat), df)) * 2
```

```
print(f'p-value: {p:.3f}')

# or use any of the available implementations of the t-test
print(stats.ttest_ind(diam1, diam2))
```

```
Mean diam1: 9.99286
Mean diam2: 9.98729
Studentized difference: 1.912
p-value: 0.058
Ttest_indResult(statistic=1.9119658005133064,
pvalue=0.05795318184124417)
```

The bootstrap based p-value for the comparison of the means is:

```
random.seed(1)

# return studentized distance between random samples from diam1 and diam2
def stat_func():
    d1 = random.choices(diam1, k=len(diam1))
    d2 = random.choices(diam2, k=len(diam2))
    return stats.ttest_ind(d1, d2).statistic

dist = np.array([stat_func() for _ in range(1000)])

pstar = sum(dist < 0) / len(dist)
print(f'p*-value: {pstar}')
```

```
p*-value: 0.014
```

The bootstrap based p-value for the comparison of the variances is:

```
columns = ['diam1', 'diam2']
# variance for each column
S2 = almpin[columns].var(axis=0, ddof=0)
F0 = max(S2) / min(S2)
print('S2', S2)
print('F0', F0)

# Step 1: sample variances of bootstrapped samples for each column
seed = 1
B = {}
for column in columns:
    ci = pg.compute_bootci(almpin[column], func='var', n_boot=500,
                           confidence=0.95, seed=seed, return_dist=True)
    B[column] = ci[1]
Bt = pd.DataFrame(B)

# Step 2: compute Wi
Wi = Bt / S2

# Step 3: compute F*
FBoot = Wi.max(axis=1) / Wi.min(axis=1)
FBoot95 = np.quantile(FBoot, 0.95)
print('FBoot 95%', FBoot95)
pstar = sum(FBoot >= F0)/len(FBoot)
print(f'p*-value: {pstar}')
```

```
S2 diam1    0.000266
diam2    0.000320
dtype: float64
F0 1.2016104294478573
FBoot 95% 1.1855457165968324
p*-value: 0.04
```

The variance of Sample 1 is $S_1^2 = 0.00027$. The variance of Sample 2 is $S_2^2 = 0.00032$. The variance ratio is $F = S_2^2/S_1^2 = 1.202$. The bootstrap level for variance ratios is $P^* = 0.04$.

Solution 3.53 In Python:

```
mpg = mistat.load_data('MPG')
columns = ['origin1', 'origin2', 'origin3']
# variance for each column
S2 = mpg[columns].var(axis=0, ddof=1)
F0 = max(S2) / min(S2)
print('S2', S2)
print('F0', F0)

# Step 1: sample variances of bootstrapped samples for each column
seed = 1
B = {}
for column in columns:
    ci = pg.compute_bootci(mpg[column].dropna(), func='var', n_boot=500,
                           confidence=0.95, seed=seed, return_dist=True)
    B[column] = ci[1]
Bt = pd.DataFrame(B)

# Step 2: compute Wi
Wi = Bt / S2

# Step 3: compute F*
FBoot = Wi.max(axis=1) / Wi.min(axis=1)
FBoot95 = np.quantile(FBoot, 0.95)
print('FBoot 95%', FBoot95)
pstar = sum(FBoot >= F0)/len(FBoot)
print(f'p*-value: {pstar}')
```

```
S2 origin1    12.942529
origin2     6.884615
origin3    18.321321
dtype: float64
F0 2.6611975103595213
FBoot 95% 2.6925366761838987
p*-value: 0.058
```

With $M = 500$ we obtained the following results:

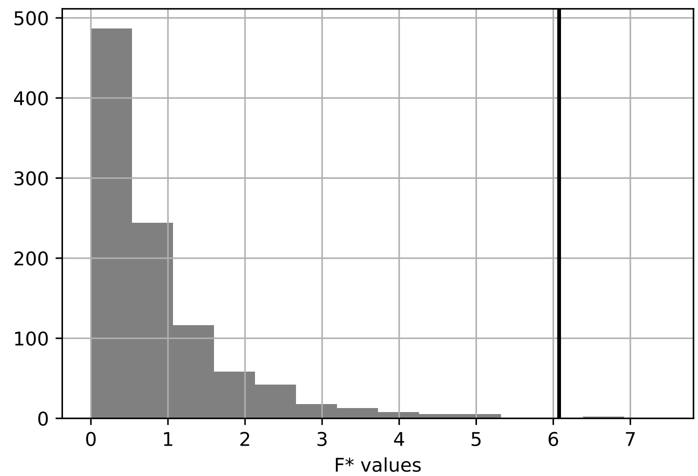
- 1^{st} sample variance = 12.9425,
- 2^{nd} sample variance = 6.8846,
- 3^{rd} sample variance = 18.3213,

$F_{\text{max/min}} = 2.6612$ and the bootstrap P value is $P^* = 0.058$. The bootstrap test does not reject the hypothesis of equal variances at the 0.05 significance level.

Solution 3.54 With $M = 500$ we obtained

$$\begin{aligned} \bar{X}_1 &= 20.931 & S_1^2 &= 12.9425 \\ \bar{X}_2 &= 19.5 & S_2^2 &= 6.8846 \\ \bar{X}_3 &= 23.1081 & S_3^2 &= 18.3213 \end{aligned}$$

Using the approach shown in Section 3.11.5.2, we get: {sec:comp-means-one-way-anova}



{fig:mpg-equal-mean} **Fig. 3.10** Distribution of EBD for exercise 3.54. exc:mpg-equal-mean

```
mpg = mistat.load_data('MPG.csv')
samples = [mpg[key].dropna() for key in ['origin1', 'origin2', 'origin3']]

def test_statistic_F(samples):
    return stats.f_oneway(*samples).statistic

# Calculate sample shifts
Ni = np.array([len(sample) for sample in samples])
N = np.sum(Ni)
XBni = np.array([np.mean(sample) for sample in samples])
XBB = np.sum(Ni * XBni) / N
DB = XBni - XBB

F0 = test_statistic_F(samples)
Ns = 1000
Fstar = []
for _ in range(Ns):
    Ysamples  = []
    for sample, DBi in zip(samples, DB):
        Xstar = np.array(random.choices(sample, k=len(sample)))
        Ysamples.append(Xstar - DBi)
    Fs = test_statistic_F(Ysamples)
    Fstar.append(Fs)
Fstar = np.array(Fstar)

print(f'F = {F0:.3f}')
print('ratio', sum(Fstar > F0)/len(Fstar))

ax = pd.Series(Fstar).hist(bins=14, color='grey')
ax.axvline(F0, color='black', lw=2)
ax.set_xlabel('F* values')
plt.show()
```

```
F = 6.076
ratio 0.003
```

{fig:mpg-equal-mean} $F = 6.076$, $P^* = 0.003$ and the hypothesis of equal means is rejected. See Figure 3.10 for the calculated EBD.

Solution 3.55 In Python:

```
np.random.seed(1)

def qbinomBoot(x, p):
    return stats.binom.ppf(p, 50, p=x.mean())

for p_real in (0.2, 0.1, 0.05):
    defects = stats.bernoulli.rvs(p_real, size=50)
    B_025 = pg.compute_bootci(defects, func=lambda x: qbinomBoot(x, p=0.025),
                              n_boot=500, seed=1, return_dist=True)
    B_975 = pg.compute_bootci(defects, func=lambda x: qbinomBoot(x, p=0.975),
                              n_boot=500, seed=1, return_dist=True)
    tol_int = [np.quantile(B_025[1], 0.025),np.quantile(B_975[1], 0.975)]
    print(f'Tolerance interval p={p_real}: ({tol_int[0]}, {tol_int[1]})')
```

```
Tolerance interval p=0.2: (1.0, 22.0)
Tolerance interval p=0.1: (0.0, 17.0)
Tolerance interval p=0.05: (0.0, 9.0)
```

The tolerance intervals of the number of defective items in future batches of size $N = 50$, with $\alpha = 0.05$ and $\beta = 0.05$ are

	Limits	
p	Lower	Upper
0.2	1	23
0.1	0	17
0.05	0	9

Solution 3.56 In Python:

```
np.random.seed(1)

def getQuantile(x, p):
    return np.quantile(x, p)

oturb = mistat.load_data('OTURB.csv')
B_025 = pg.compute_bootci(oturb, func=lambda x: getQuantile(x, p=0.025),
                          n_boot=500, seed=1, return_dist=True)
B_975 = pg.compute_bootci(oturb, func=lambda x: getQuantile(x, p=0.975),
                          n_boot=500, seed=1, return_dist=True)
tol_int = [np.quantile(B_025[1], 0.025),np.quantile(B_975[1], 0.975)]
print(f'Tolerance interval ({tol_int[0]}, {tol_int[1]})')
```

```
Tolerance interval (0.2399, 0.68305)
```

A $(0.95, 0.95)$ tolerance interval for OTURB.csv is $(0.24, 0.683)$.

Solution 3.57 In Python:

```
cyclt = mistat.load_data('CYCLT.csv')
# make use of the fact that a True value is interpreted as 1 and False as 0
print('Values greater 0.7:', sum(cyclt>0.7))
```

```
Values greater 0.7: 20
```

We find that in the sample of $n = 50$ cycle times, there are $X = 20$ values greater than 0.7. If the hypothesis is $H_0 : \xi_{.5} \leq 0.7$, the probability of observing a value smaller than 0.7 is $p \geq \frac{1}{2}$. Thus, the sign test rejects H_0 if $X < B^{-1}(\alpha; 50, \frac{1}{2})$. For $\alpha = 0.10$ the critical value is $k_\alpha = 20$. H_0 is not rejected.

Solution 3.58 We apply the `wilcoxon` test from `scipy` on the differences of `oelect` from 220.

```
oelect = mistat.load_data('OELECT.csv')
print(stats.wilcoxon(oelect-220))
```

```
WilcoxonResult(statistic=1916.0, pvalue=0.051047599707252124)
```

The null hypothesis is rejected with P value equal to 0.051.

Solution 3.59 In Python

```
car = mistat.load_data('CAR.csv')
fourCylinder = car[car['cyl'] == 4]
uscars = fourCylinder[fourCylinder['origin'] == 1]
foreign = fourCylinder[fourCylinder['origin'] != 1]

print(f'Mean of Sample 1 (U.S. made) {np.mean(uscars["turn"]):.3f}')
print(f'Mean of Sample 2 (foreign) {np.mean(foreign["turn"]):.3f}')

_ = mistat.randomizationTest(uscars['turn'], foreign['turn'], np.mean,
                             aggregate_stats=lambda x: x[0] - x[1],
                             n_boot=1000, seed=1)
```

```
Mean of Sample 1 (U.S. made) 36.255
Mean of Sample 2 (foreign) 33.179
Original stat is 3.075758
Original stat is at quantile 1001 of 1001 (100.00%)
Distribution of bootstrap samples:
 min: -2.12, median: 0.01,  max: 2.68
```

The original stat 3.08 is outside of the distribution of the bootstrap samples. The difference between the means of the turn diameters is therefore significant. Foreign cars have on the average a smaller turn diameter.

Chapter 4
Variability in Several Dimensions and Regression Models

Import required modules and define required functions

```
import random
import numpy as np
import pandas as pd
import pingouin as pg
from scipy import stats
import statsmodels.api as sm
import statsmodels.formula.api as smf
import statsmodels.stats as sms
import seaborn as sns
import matplotlib.pyplot as plt

import mistat
```

Solution 4.1 In Figure 4.1 one sees that horsepower and miles per gallon are inversely proportional. Turn diameter seems to increase with horsepower. {fig:ex_car_pairplot}

```
car = mistat.load_data('CAR')
sns.pairplot(car[['turn', 'hp', 'mpg']])
plt.show()
```

Solution 4.2 The box plots in Figure 4.2 show that cars from Asia generally have the smallest turn diameter. The maximal turn diameter of cars from Asia is smaller than the median turn diameter of U.S. cars. European cars tend to have larger turn diameter than those from Asia, but smaller than those from the U.S. {fig:ex_car_boxplots}

```
car = mistat.load_data('CAR')

ax = car.boxplot(column='turn', by='origin')
ax.set_title('')
ax.get_figure().suptitle('')
ax.set_xlabel('origin')
plt.show()
```

Solution 4.3 **(i)** The multiple box plots (see Figure 4.3) show that the conditional distributions of `res3` at different hybrids are different. {fig:ex_hadpas_plot_i}

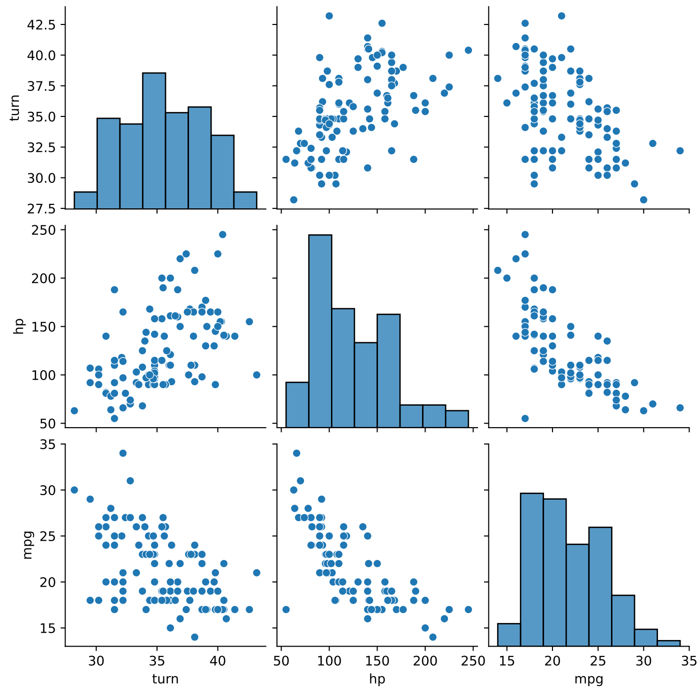


Fig. 4.1 Scatterplot matrix for CAR dataset {fig:ex_car_pairplot}

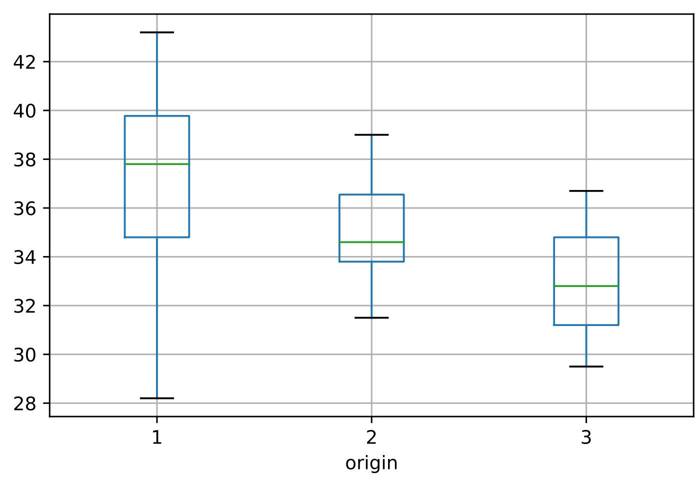


Fig. 4.2 Boxplots of turn diameter by origin for CAR dataset {fig:ex_car_boxplots}

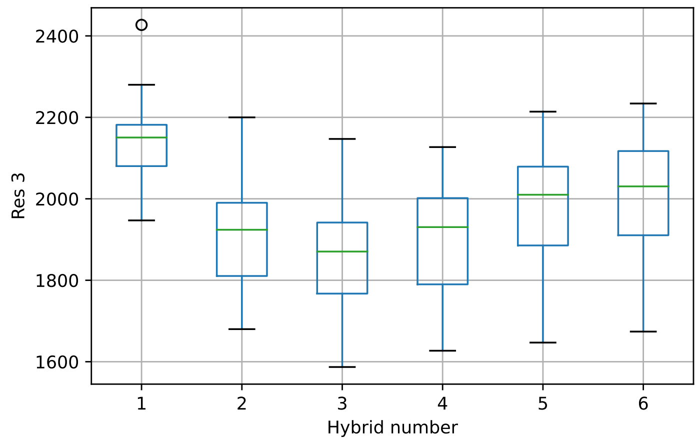


Fig. 4.3 Multiple box plots of Res 3 grouped by hybrid number

{fig:ex_hadpas_plot_i}

```
hadpas = mistat.load_data('HADPAS')
ax = hadpas.boxplot(column='res3', by='hyb')
ax.set_title('')
ax.get_figure().suptitle('')
ax.set_xlabel('Hybrid number')
ax.set_ylabel('Res 3')
plt.show()
```

(ii) The matrix plot of all the Res variables (see Figure 4.4) reveals that Res 3 and Res 7 are positively correlated. Res 20 is generally larger than the corresponding Res 14. Res 18 and Res 20 seem to be negatively associated. {fig:ex_hadpas_plot_ii}

```
sns.pairplot(hadpas[['res3', 'res7', 'res18', 'res14', 'res20']])
plt.show()
```

Solution 4.4 The joint frequency distribution of horsepower versus miles per gallon is

```
car = mistat.load_data('CAR')
binned_car = pd.DataFrame({
  'hp': pd.cut(car['hp'], bins=np.arange(50, 275, 25)),
  'mpg': pd.cut(car['mpg'], bins=np.arange(10, 40, 5)),
})
freqDist = pd.crosstab(binned_car['hp'], binned_car['mpg'])
print(freqDist)
# You can get distributions for hp and mpg by summing along an axis
print(freqDist.sum(axis=0))
print(freqDist.sum(axis=1))
```

```
mpg         (10, 15]  (15, 20]  (20, 25]  (25, 30]  (30, 35]
hp
(50, 75]           0         1         0         4         2
(75, 100]          0         0        23        11         0
```

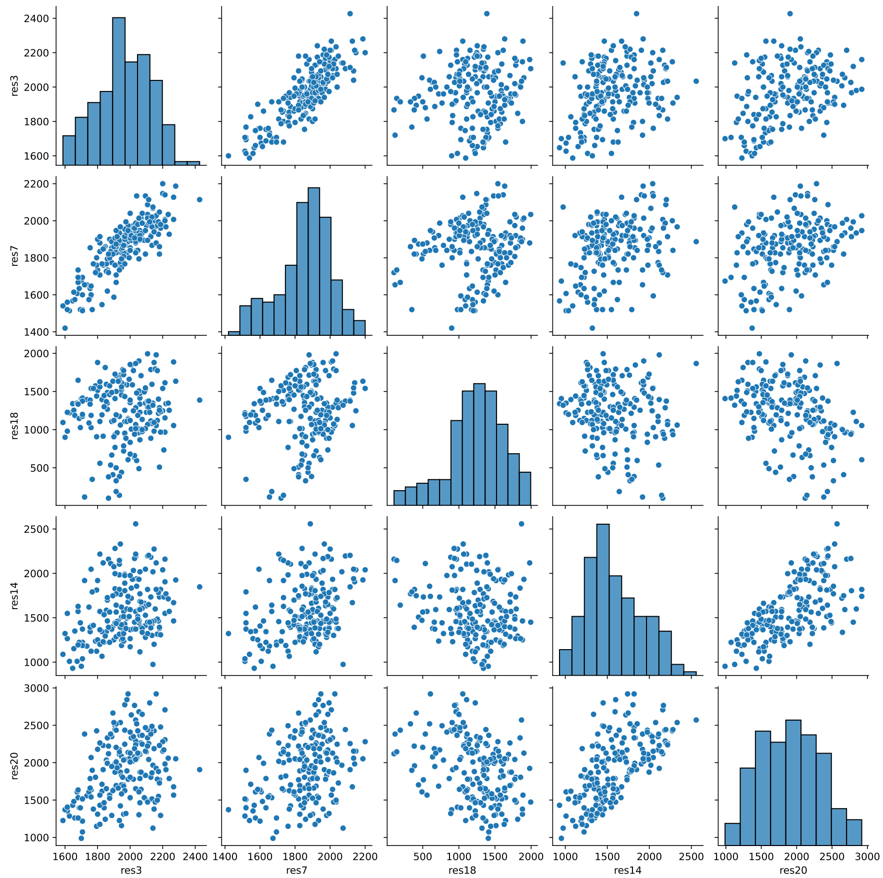


Fig. 4.4 Scatterplot matrix Res variables for HADPAS dataset {fig:ex_hadpas_plot_ii}

```
(100, 125]          0          10          11          1          0
(125, 150]          0          14           3          1          0
(150, 175]          0          17           0          0          0
(175, 200]          1           5           0          0          0
(200, 225]          1           3           0          0          0
(225, 250]          0           1           0          0          0
mpg
(10, 15]      2
(15, 20]     51
(20, 25]     37
(25, 30]     17
(30, 35]      2
dtype: int64
hp
(50, 75]       7
(75, 100]     34
(100, 125]    22
(125, 150]    18
(150, 175]    17
```

```
(175, 200]     6
(200, 225]     4
(225, 250]     1
dtype: int64
```

The intervals for HP are from 50 to 250 at fixed length of 25. The intervals for MPG are from 10 to 35 at length 5. Students may get different results by defining the intervals differently.

Solution 4.5 The joint frequency distribution of Res 3 and Res 14 is given in the following table:

```
hadpas = mistat.load_data('HADPAS')
binned_hadpas = pd.DataFrame({
  'res3': pd.cut(hadpas['res3'], bins=np.arange(1580, 2780, 200)),
  'res14': pd.cut(hadpas['res14'], bins=np.arange(900, 3000, 300)),
})
pd.crosstab(binned_hadpas['res14'], binned_hadpas['res3'])
```

```
res3          (1580, 1780]  (1780, 1980]  (1980, 2180]  (2180, 2380]  \
res14
(900, 1200]             11             3             3             0
(1200, 1500]            11            33            28             2
(1500, 1800]             5            16            24             6
(1800, 2100]             2            11            12             5
(2100, 2400]             0             9             8             1
(2400, 2700]             0             0             1             0

res3          (2380, 2580]
res14
(900, 1200]              0
(1200, 1500]             0
(1500, 1800]             0
(1800, 2100]             1
(2100, 2400]             0
(2400, 2700]             0
```

The intervals for Res 3 start at 1580 and end at 2580 with length of 200. The intervals of Res 14 start at 900 and end at 2700 with length of 300.

Solution 4.6 The following is the conditional frequency distribution of Res 3, given that Res 14 is between 1300 and 1500 ohms:

```
hadpas = mistat.load_data('HADPAS')
in_range = hadpas[hadpas['res14'].between(1300, 1500)]
pd.cut(in_range['res3'], bins=np.arange(1580, 2780, 200)).value_counts(sort=False)
```

```
(1580, 1780]     8
(1780, 1980]    21
(1980, 2180]    25
(2180, 2380]     2
(2380, 2580]     0
Name: res3, dtype: int64
```

Solution 4.7 Following the instructions in the question we obtained the following results:

```
hadpas = mistat.load_data('HADPAS')
bins = [900, 1200, 1500, 1800, 2100, 3000]
binned_res14 = pd.cut(hadpas['res14'], bins=bins)

results = []
for group, df in hadpas.groupby(binned_res14):
  res3 = df['res3']
  results.append({
    'res3': group,
    'N': len(res3),
    'mean': res3.mean(),
    'std': res3.std(),
  })
pd.DataFrame(results)
```

```
          res3   N         mean         std
0   (900, 1200]  17  1779.117647  162.348730
1  (1200, 1500]  74  1952.175676  154.728251
2  (1500, 1800]  51  1997.196078  151.608841
3  (1800, 2100]  31  2024.774194  156.749845
4  (2100, 3000]  19  1999.736842  121.505758
```

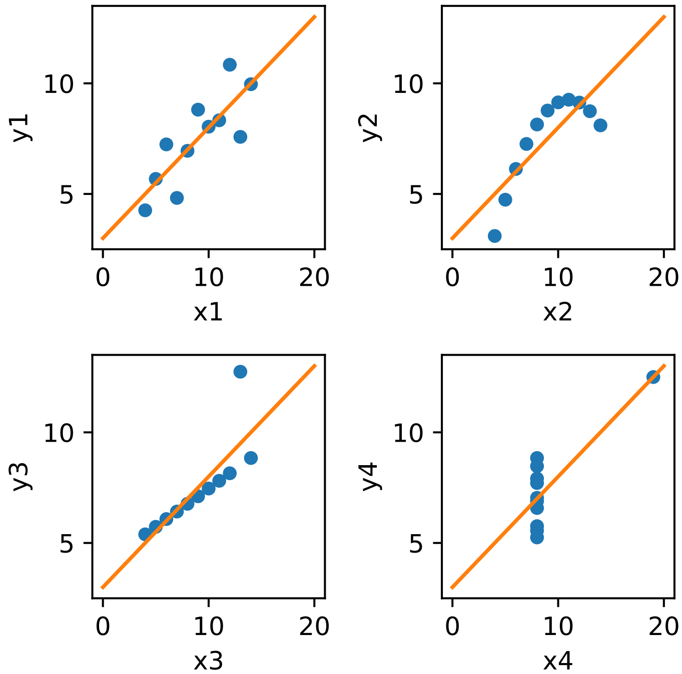
Solution 4.8 In Python

```
df = pd.DataFrame([
  [10.0, 8.04, 10.0, 9.14, 10.0, 7.46, 8.0, 6.58],
  [8.0, 6.95, 8.0, 8.14, 8.0, 6.77, 8.0, 5.76],
  [13.0, 7.58, 13.0, 8.74, 13.0, 12.74, 8.0, 7.71],
  [9.0, 8.81, 9.0, 8.77, 9.0, 7.11, 8.0, 8.84],
  [11.0, 8.33, 11.0, 9.26, 11.0, 7.81, 8.0, 8.47],
  [14.0, 9.96, 14.0, 8.10, 14.0, 8.84, 8.0, 7.04],
  [6.0, 7.24, 6.0, 6.13, 6.0, 6.08, 8.0, 5.25],
  [4.0, 4.26, 4.0, 3.10, 4.0, 5.39, 19.0, 12.50],
  [12.0, 10.84, 12.0, 9.13, 12.0, 8.15, 8.0, 5.56],
  [7.0, 4.82, 7.0, 7.26, 7.0, 6.42, 8.0, 7.91],
  [5.0, 5.68, 5.0, 4.74, 5.0, 5.73, 8.0, 6.89],
], columns=['x1', 'y1', 'x2', 'y2', 'x3', 'y3', 'x4', 'y4'])

results = []
for i in (1, 2, 3, 4):
  x = df[f'x{i}']
  y = df[f'y{i}']
  model = smf.ols(formula=f'y{i} ~ 1 + x{i}', data=df).fit()
  results.append({
    'Data Set': i,
    'Intercept': model.params['Intercept'],
    'Slope': model.params[f'x{i}'],
    'R2': model.rsquared,
  })
pd.DataFrame(results)
```

```
   Data Set  Intercept     Slope        R2
0         1   3.000091  0.500091  0.666542
1         2   3.000909  0.500000  0.666242
2         3   3.002455  0.499727  0.666324
3         4   3.001727  0.499909  0.666707
```

Notice the influence of the point (19,12.5) on the regression in Data Set 4. Without this point the correlation between x and y is zero.

{fig:AnscombeQuartet} The dataset is known as Anscombe's quartet (see Figure 4.5). It not only has identical linear regression, but it has also identical means and variances of x and y,



{fig:AnscombeQuartet} **Fig. 4.5** Anscombe's quartet

and correlation between x and y. The dataset clearly demonstrates the importance of visualization in data analysis.

Solution 4.9 The correlation matrix:

```
car = mistat.load_data('CAR')
car[['turn', 'hp', 'mpg']].corr()
```

```
          turn        hp       mpg
turn  1.000000  0.507610 -0.541061
hp    0.507610  1.000000 -0.754716
mpg  -0.541061 -0.754716  1.000000
```

Solution 4.10 $SSE = \sum_{i=1}^{n}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2})^2$

(i)

$$\frac{\partial}{\partial \beta_0} SSE = -2 \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2})$$
$$\frac{\partial}{\partial \beta_1} SSE = -2 \sum_{i=1}^{n} X_{i1}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2})$$
$$\frac{\partial}{\partial \beta_2} SSE = -2 \sum_{i=1}^{n} X_{i2}(Y_i - \beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2}).$$

Equating these partial derivatives to zero and arranging terms, we arrive at the following set of linear equations:

$$\begin{bmatrix} n & \sum_{i=1}^n X_{i1} & \sum_{i=1}^n X_{i2} \\ \sum_{i=1}^n X_{i1} & \sum_{i=1}^n X_{i1}^2 & \sum_{i=1}^n X_{i1}X_{i2} \\ \sum_{i=1}^n X_{i2} & \sum_{i=1}^n X_{i1}X_{i2} & \sum_{i=1}^n X_{i2}^2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n Y_i \\ \sum_{i=1}^n X_{i1}Y_i \\ \sum_{i=1}^n X_{i2}Y_i \end{bmatrix}$$

(ii) Let b_0, b_1, b_2 be the (unique) solution. From the first equation we get, after dividing by n, $b_0 = \bar{Y} - \bar{X}_1 b_1 - \bar{X}_2 b_2$, where $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$, $\bar{X}_1 = \frac{1}{n}\sum_{i=1}^n X_{i1}$, $\bar{X}_2 = \dfrac{1}{n}\sum_{i=1}^n X_{i2}$. Substituting b_0 in the second and third equations and arranging terms, we obtain the reduced system of equations:

$$\begin{bmatrix} (Q_1 - n\bar{X}_1^2) & (P_{12} - n\bar{X}_1\bar{X}_2) \\ (P_{12} - n\bar{X}_1\bar{X}_2) & (Q_2 - n\bar{X}_2^2) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} P_{1y} - n\bar{X}_1\bar{Y} \\ P_{2y} - n\bar{X}_2\bar{Y} \end{bmatrix}$$

where $Q_1 = \sum X_{i1}^2$, $Q_2 = \sum X_{i2}^2$, $P_{12} = \sum X_{i1}X_{i2}$ and $P_{1y} = \sum X_{i1}Y_i$, $P_{2y} = \sum X_{i2}Y_i$. Dividing both sides by $(n-1)$ we obtain Eq. (4.4.3), and solving we get b_1 and b_2.

Solution 4.11 We get the following result using `statsmodels`

```
car = mistat.load_data('CAR')

model = smf.ols(formula='mpg ~ 1 + hp + turn', data=car).fit()
print(model.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.596
Dependent Variable: mpg              AIC:                511.2247
Date:               2022-08-04 18:38 BIC:                519.2988
No. Observations:   109              Log-Likelihood:     -252.61
Df Model:           2                F-statistic:        80.57
Df Residuals:       106              Prob (F-statistic): 5.29e-22
R-squared:          0.603            Scale:              6.2035
------------------------------------------------------------------
               Coef.   Std.Err.     t     P>|t|    [0.025   0.975]
------------------------------------------------------------------
Intercept     38.2642    2.6541  14.4167  0.0000  33.0020  43.5263
hp            -0.0631    0.0069  -9.1070  0.0000  -0.0768  -0.0493
```

```
turn          -0.2510    0.0838  -2.9965  0.0034  -0.4171  -0.0849
-----------------------------------------------------------------
Omnibus:              12.000       Durbin-Watson:          2.021
Prob(Omnibus):        0.002        Jarque-Bera (JB):       25.976
Skew:                 -0.335       Prob(JB):               0.000
Kurtosis:             5.296        Condition No.:          1507
=================================================================
* The condition number is large (2e+03). This might indicate
strong multicollinearity or other numerical problems.
```

The regression equation is MPG = $38.3 - 0.251 \times \text{turn} - 0.0631 \times \text{hp}$.

We see that only 60% of the variability in MPG is explained by the linear relationship with Turn and HP. Both variables contribute significantly to the regression.

Solution 4.12 The partial correlation is -0.70378.

```
car = mistat.load_data('CAR')

# y: mpg, x1: cyl, x2: hp
model_1 = smf.ols(formula='mpg ~ cyl + 1', data=car).fit()
e_1 = model_1.resid

model_2 = smf.ols(formula='hp ~ cyl + 1', data=car).fit()
e_2 = model_2.resid

print(f'Partial correlation {stats.pearsonr(e_1, e_2)[0]:.5f}')
```

```
Partial correlation -0.70378
```

Solution 4.13 In Python:

```
car = mistat.load_data('CAR')

# y: mpg, x1: hp, x2: turn
model_1 = smf.ols(formula='mpg ~ hp + 1', data=car).fit()
e_1 = model_1.resid
print('Model mpg ~ hp + 1:\n', model_1.params)

model_2 = smf.ols(formula='turn ~ hp + 1', data=car).fit()
e_2 = model_2.resid
print('Model turn ~ hp + 1:\n', model_2.params)

print('Partial correlation', stats.pearsonr(e_1, e_2)[0])
df = pd.DataFrame({'e1': e_1, 'e2': e_2})
model_partial = smf.ols(formula='e1 ~ e2 - 1', data=df).fit()
# print(model_partial.summary2())
print('Model e1 ~ e2:\n', model_partial.params)
```

```
Model mpg ~ hp + 1:
 Intercept    30.663308
hp           -0.073611
dtype: float64
Model turn ~ hp + 1:
 Intercept    30.281255
hp            0.041971
dtype: float64
Partial correlation -0.27945246615045016
Model e1 ~ e2:
 e2   -0.251008
dtype: float64
```

The partial regression equation is $\hat{e}_1 = -0.251\hat{e}_2$.

Solution 4.14 The regression of MPG on HP is MPG = 30.6633 − 0.07361 HP. The regression of TurnD on HP is TurnD = 30.2813 + 0.041971 HP. The regression of the residuals $\hat{e}_1$ on $\hat{e}_2$ is $\hat{e}_1 = -0.251 \cdot \hat{e}_2$. Thus,

$$\begin{aligned} \text{Const.}: &\quad b_0 = 30.6633 + 30.2813 \times 0.251 = 38.2639 \\ \text{HP}: &\quad b_1 = -0.07361 + 0.041971 \times 0.251 = -0.063076 \\ \text{TurnD}: &\quad b_2 = -0.251. \end{aligned}$$

Solution 4.15 The regression of Cap Diameter on Diam2 and Diam3 is

```
almpin = mistat.load_data('ALMPIN')
model = smf.ols('capDiam ~ 1 + diam2 + diam3', data=almpin).fit()
model.summary2()
```

```
<class 'statsmodels.iolib.summary2.Summary'>
"""
                  Results: Ordinary least squares
=================================================================
Model:               OLS              Adj. R-squared:     0.842
Dependent Variable:  capDiam          AIC:                -482.1542
Date:                2022-08-04 18:38 BIC:                -475.4087
No. Observations:    70               Log-Likelihood:     244.08
Df Model:            2                F-statistic:        184.2
Df Residuals:        67               Prob (F-statistic): 5.89e-28
R-squared:           0.846            Scale:              5.7272e-05
-------------------------------------------------------------------
                Coef.    Std.Err.     t      P>|t|    [0.025   0.975]
-------------------------------------------------------------------
Intercept       4.7565     0.5501   8.6467   0.0000   3.6585   5.8544
diam2           0.5040     0.1607   3.1359   0.0025   0.1832   0.8248
diam3           0.5203     0.1744   2.9830   0.0040   0.1722   0.8684
-----------------------------------------------------------------
Omnibus:               1.078        Durbin-Watson:           2.350
Prob(Omnibus):         0.583        Jarque-Bera (JB):        0.976
Skew:                  -0.071       Prob(JB):                0.614
Kurtosis:              2.439        Condition No.:           8689
=================================================================
* The condition number is large (9e+03). This might indicate
strong multicollinearity or other numerical problems.
"""
```

The dependence of CapDiam on Diam2, without Diam1 is significant. This is due to the fact that Diam1 and Diam2 are highly correlated ($\rho = 0.957$). If Diam1 is in the regression, then Diam2 does not furnish additional information on CapDiam. If Diam1 is not included then Diam2 is very informative.

Solution 4.16 The regression of yield ($Yield$) on the four variables is:

```
gasol = mistat.load_data('GASOL')
# rename column 'yield' to 'Yield' as 'yield' is a special keyword in Python
gasol = gasol.rename(columns={'yield': 'Yield'})
model = smf.ols(formula='Yield ~ x1 + x2 + astm + endPt',
                data=gasol).fit()
print(model.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.957
Dependent Variable: Yield            AIC:                146.8308
Date:               2022-08-04 18:38 BIC:                154.1595
No. Observations:   32               Log-Likelihood:     -68.415
Df Model:           4                F-statistic:        171.7
Df Residuals:       27               Prob (F-statistic): 8.82e-19
R-squared:          0.962            Scale:              4.9927
------------------------------------------------------------------
              Coef.   Std.Err.     t     P>|t|    [0.025    0.975]
------------------------------------------------------------------
Intercept    -6.8208   10.1232  -0.6738  0.5062  -27.5918  13.9502
x1            0.2272    0.0999   2.2739  0.0311    0.0222   0.4323
x2            0.5537    0.3698   1.4976  0.1458   -0.2049   1.3124
astm         -0.1495    0.0292  -5.1160  0.0000   -0.2095  -0.0896
endPt         0.1547    0.0064  23.9922  0.0000    0.1414   0.1679
-----------------------------------------------------------------
Omnibus:               0.635       Durbin-Watson:          1.402
Prob(Omnibus):         0.728       Jarque-Bera (JB):       0.719
Skew:                  0.190       Prob(JB):               0.698
Kurtosis:              2.371       Condition No.:          10714
=================================================================
* The condition number is large (1e+04). This might indicate
strong multicollinearity or other numerical problems.
```

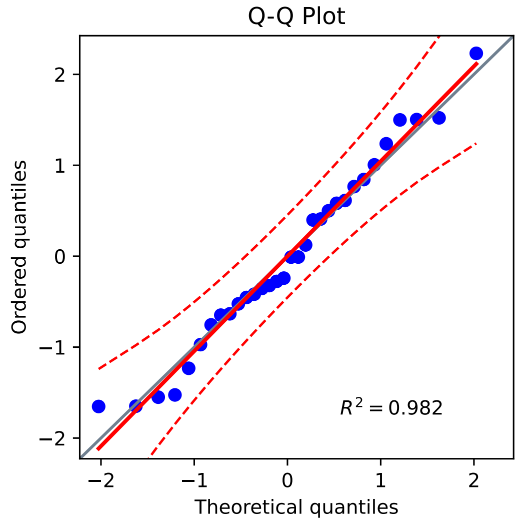
(i) The regression equation is

$$\hat{y} = -6.8 + 0.227x_1 + 0.554x_2 - 0.150\,\text{astm} + 0.155\,\text{endPt}.$$

(ii) $R^2 = 0.962$.
(iii) The regression coefficient of x_2 is not significant.
(iv) Running the multiple regression again, without x_2, we obtain the equation

```
model = smf.ols(formula='Yield ~ x1 + astm + endPt', data=gasol).fit()
print(model.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.955
Dependent Variable: Yield            AIC:                147.3842
Date:               2022-08-04 18:38 BIC:                153.2471
No. Observations:   32               Log-Likelihood:     -69.692
Df Model:           3                F-statistic:        218.5
Df Residuals:       28               Prob (F-statistic): 1.59e-19
R-squared:          0.959            Scale:              5.2143
------------------------------------------------------------------
             Coef.   Std.Err.     t      P>|t|    [0.025    0.975]
------------------------------------------------------------------
Intercept    4.0320    7.2233    0.5582  0.5811  -10.7643  18.8284
x1           0.2217    0.1021    2.1725  0.0384    0.0127   0.4308
astm        -0.1866    0.0159  -11.7177  0.0000   -0.2192  -0.1540
endPt        0.1565    0.0065   24.2238  0.0000    0.1433   0.1698
-----------------------------------------------------------------
Omnibus:               0.679       Durbin-Watson:          1.150
Prob(Omnibus):         0.712       Jarque-Bera (JB):       0.738
Skew:                  0.174       Prob(JB):               0.692
Kurtosis:              2.343       Condition No.:          7479
=================================================================
* The condition number is large (7e+03). This might indicate
strong multicollinearity or other numerical problems.
```



{fig:qqPlotGasolRegResid} **Fig. 4.6** Q-Q plot of gasol regression residuals.

$$\hat{y} = 4.03 + 0.222x_1 + 0.554x_2 - 0.187\,\text{astm} + 0.157\,\text{endPt},$$

with $R^2 = 0.959$. Variables x_1, astm and endPt are important.

(v) Normal probability plotting of the residuals $\hat{e}$ from the equation of **(iv)** shows that they are normally distributed. (see Figure 4.6) {fig:qqPlotGasolRegResid}

Solution 4.17 (i)

$$\begin{aligned}
(H) &= (X)(B) = (X)[(X)'(X)]^{-1}(X)' \\
H^2 &= (X)[(X)'(X)]^{-1}(X)'(X)[(X)'(X)]^{-1}(X)' \\
&= (X)[(X)'(X)]^{-1}(X)' \\
&= H.
\end{aligned}$$

(ii)

$$\begin{aligned}
(Q) &= I - (H) \\
(Q)^2 &= (I - (H))(I - (H)) \\
&= I - (H) - (H) + (H)^2 \\
&= I - (H) \\
&= Q. \\
s_e^2 &= \mathbf{y}'(Q)(Q)\mathbf{y}/(n - k - 1) \\
&= \mathbf{y}'(Q)\mathbf{y}/(n - k - 1).
\end{aligned}$$

Solution 4.18 We have $\hat{\mathbf{y}} = (X)\hat{\boldsymbol{\beta}} = (X)(B)\mathbf{y} = (H)\mathbf{y}$ and $\hat{\mathbf{e}} = Q\mathbf{y} = (I - (H))\mathbf{y}$.

$$\begin{aligned}\hat{\mathbf{y}}'\hat{\mathbf{e}} &= \mathbf{y}'(H)(I - (H))\mathbf{y} \\ &= \mathbf{y}'(H)\mathbf{y} - \mathbf{y}'(H)^2\mathbf{y} \\ &= 0.\end{aligned}$$

Solution 4.19 $1 - R^2_{y(x)} = \dfrac{SSE}{SSD_y}$ where $SSE = \hat{\mathbf{e}}'\hat{\mathbf{e}} = ||\hat{\mathbf{e}}||^2$.

Solution 4.20 From the basic properties of the $\text{cov}(X, Y)$ operator,

$$\begin{aligned}\text{cov}\left(\sum_{i=1}^{n} \beta_i X_i, \sum_{j=1}^{n} \gamma_j X_j\right) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \gamma_j \text{cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \gamma_j \; \Sigma_{ij} \\ &= \boldsymbol{\beta}'(\boldsymbol{\Sigma})\boldsymbol{\gamma}.\end{aligned}$$

Solution 4.21 $\mathbf{W} = (W_1, \ldots, W_m)'$ where $W_i = \mathbf{b}_i'\mathbf{X}$ $(i = 1, \ldots, m)$. $\mathbf{b}_i'$ is the i-th row vector of B. Thus, by the previous exercise $\text{cov}(W_i, W_j) = \mathbf{b}_i'(\boldsymbol{\Sigma})\mathbf{b}_j$. This is the (i, j) element of the covariance matrix of $\mathbf{W}$. Hence, the covariance matrix of $\mathbf{W}$ is $C(\mathbf{W}) = (B)(\boldsymbol{\Sigma})(B)'$.

Solution 4.22 From the model, $\boldsymbol{\Sigma}(\mathbf{Y}) = \sigma^2 I$ and $\mathbf{b} = (B)\mathbf{Y}$.

$$\begin{aligned}\boldsymbol{\Sigma}(\mathbf{b}) &= (B)\boldsymbol{\Sigma}(\mathbf{Y})(B)' = \sigma^2(B)(B)' \\ &= \sigma^2[(\mathbf{X})'(\mathbf{X})]^{-1} \cdot \mathbf{X}'\mathbf{X}[(\mathbf{X})'(\mathbf{X})]^{-1} \\ &= \sigma^2[(\mathbf{X})'(\mathbf{X})]^{-1}.\end{aligned}$$

Solution 4.23 Rearrange the dataset into the format suitable for the test outlines in the multiple linear regression section.

```
socell = mistat.load_data('SOCELL')

# combine the two datasets and add the additional columns z and w
socell_1 = socell[['t3', 't1']].copy()
socell_1.columns = ['t3', 't']
socell_1['z'] = 0
socell_2 = socell[['t3', 't2']].copy()
socell_2.columns = ['t3', 't']
socell_2['z'] = 1
combined = pd.concat([socell_1, socell_2])
combined['w'] = combined['z'] * combined['t']

# multiple linear regression model
model_test  = smf.ols(formula='t3 ~ t + z + w + 1',
                      data=combined).fit()
print(model_test.summary2())
```

```
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.952
Dependent Variable: t3               AIC:                -58.2308
Date:               2022-08-04 18:38 BIC:                -52.3678
No. Observations:   32               Log-Likelihood:     33.115
Df Model:           3                F-statistic:        205.8
Df Residuals:       28               Prob (F-statistic): 3.55e-19
R-squared:          0.957            Scale:              0.0084460
-------------------------------------------------------------------
             Coef.    Std.Err.      t      P>|t|     [0.025   0.975]
-------------------------------------------------------------------
Intercept    0.5187     0.2144    2.4196   0.0223    0.0796   0.9578
t            0.9411     0.0539   17.4664   0.0000    0.8307   1.0515
z           -0.5052     0.3220   -1.5688   0.1279   -1.1648   0.1545
w            0.0633     0.0783    0.8081   0.4259   -0.0971   0.2237
-----------------------------------------------------------------
Omnibus:               1.002        Durbin-Watson:           1.528
Prob(Omnibus):         0.606        Jarque-Bera (JB):        0.852
Skew:                  -0.378       Prob(JB):                0.653
Kurtosis:              2.739        Condition No.:           111
=================================================================
```

Neither of the z nor w The P-values corresponding to z and w are 0.128 and 0.426 respectively. Accordingly, we can conclude that the slopes and intercepts of the two simple linear regressions given above are not significantly different. Combining the data we have the following regression line for the combined dataset:

```
model_combined  = smf.ols(formula='t3 ~ t + 1',
                          data=combined).fit()
print(model_combined.summary2())
```

```
                 Results: Ordinary least squares
================================================================
Model:              OLS              Adj. R-squared:     0.870
Dependent Variable: t3               AIC:                -28.1379
Date:               2022-08-04 18:38 BIC:                -25.2064
No. Observations:   32               Log-Likelihood:     16.069
Df Model:           1                F-statistic:        208.3
Df Residuals:       30               Prob (F-statistic): 4.86e-15
R-squared:          0.874            Scale:              0.022876
------------------------------------------------------------------
             Coef.    Std.Err.      t      P>|t|    [0.025   0.975]
------------------------------------------------------------------
Intercept    0.6151     0.2527    2.4343   0.0211   0.0990   1.1311
t            0.8882     0.0615   14.4327   0.0000   0.7625   1.0139
----------------------------------------------------------------
Omnibus:               1.072       Durbin-Watson:           0.667
Prob(Omnibus):         0.585       Jarque-Bera (JB):        0.883
Skew:                  0.114       Prob(JB):                0.643
Kurtosis:              2.219       Condition No.:           41
================================================================
```

Solution 4.24 Load the data frame

```
df = mistat.load_data('CEMENT.csv')
```

(a) Regression of Y on X_1 is

```
model1 = smf.ols('y ~ x1 + 1', data=df).fit()
print(model1.summary().tables[1])
r2 = model1.rsquared
print(f'R-sq: {r2:.3f}')

anova = sms.anova.anova_lm(model1)
print('Analysis of Variance\n', anova)

F = anova.F['x1']
SSE_1 = anova.sum_sq['Residual']
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     81.4793      4.927     16.536      0.000      70.634      92.324
x1             1.8687      0.526      3.550      0.005       0.710       3.027
==============================================================================
R-sq: 0.534
Analysis of Variance
             df       sum_sq      mean_sq          F    PR(>F)
x1         1.0  1450.076328  1450.076328  12.602518  0.004552
Residual  11.0  1265.686749   115.062432        NaN       NaN
```

- $R^2_{Y|(X_1)}$ =0.534.
- $SSE_1 =$ 1265.7,
- $F =$ 12.60 (In the 1st stage F is equal to the partial$-F$.)

(b) The regression of Y on X_1 and X_2 is

```
model2 = smf.ols('y ~ x1 + x2 + 1', data=df).fit()
r2 = model2.rsquared
print(model2.summary().tables[1])
print(f'R-sq: {r2:.3f}')
anova = sms.anova.anova_lm(model2)
print('Analysis of Variance\n', anova)
SEQ_SS_X2 = anova.sum_sq['x2']
SSE_2 = anova.sum_sq['Residual']
s2e2 = anova.mean_sq['Residual']
partialF = np.sum(anova.sum_sq) * (model2.rsquared - model1.rsquared) / s2e2

anova = sms.anova.anova_lm(model1, model2)
print('Comparing models\n', anova)
partialF = anova.F[1]
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     52.5773      2.286     22.998      0.000      47.483      57.671
x1             1.4683      0.121     12.105      0.000       1.198       1.739
x2             0.6623      0.046     14.442      0.000       0.560       0.764
==============================================================================
R-sq: 0.979
Analysis of Variance
             df       sum_sq      mean_sq           F        PR(>F)
x1         1.0  1450.076328  1450.076328  250.425571  2.088092e-08
x2         1.0  1207.782266  1207.782266  208.581823  5.028960e-08
Residual  10.0    57.904483     5.790448         NaN           NaN
Comparing models
    df_resid          ssr  df_diff      ss_diff           F        Pr(>F)
0      11.0  1265.686749      0.0          NaN         NaN           NaN
1      10.0    57.904483      1.0  1207.782266  208.581823  5.028960e-08
```

- $R^2_{Y|(X_1,X_2)}$ =0.979.
- $SSE_2 =$ 57.9,
- $s^2_{e_2} =$ 5.79,
- $F =$ 12.60
- Partial-$F =$ 208.582

Notice that SEQ SS for $X_2 = 2716.9(0.974 - 0.529) =$ 1207.782.

(c) The regression of Y on X_1, X_2, and X_3 is

```
model3 = smf.ols('y ~ x1 + x2 + x3 + 1', data=df).fit()
r2 = model3.rsquared
print(model3.summary().tables[1])
print(f'R-sq: {r2:.3f}')
anova = sms.anova.anova_lm(model3)
print('Analysis of Variance\n', anova)
SEQ_SS_X3 = anova.sum_sq['x3']
SSE_3 = anova.sum_sq['Residual']
s2e3 = anova.mean_sq['Residual']

anova = sms.anova.anova_lm(model2, model3)
print('Comparing models\n', anova)
partialF = anova.F[1]
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     48.1936      3.913     12.315      0.000      39.341      57.046
x1             1.6959      0.205      8.290      0.000       1.233       2.159
x2             0.6569      0.044     14.851      0.000       0.557       0.757
x3             0.2500      0.185      1.354      0.209      -0.168       0.668
==============================================================================
R-sq: 0.982
Analysis of Variance
             df       sum_sq      mean_sq           F        PR(>F)
x1        1.0  1450.076328  1450.076328  271.264194  4.995767e-08
x2        1.0  1207.782266  1207.782266  225.938509  1.107893e-07
x3        1.0     9.793869     9.793869    1.832128  2.088895e-01
Residual  9.0    48.110614     5.345624         NaN           NaN
Comparing models
    df_resid        ssr  df_diff   ss_diff         F    Pr(>F)
0      10.0  57.904483      0.0       NaN       NaN       NaN
1       9.0  48.110614      1.0  9.793869  1.832128  0.208889
```

- $R^2_{Y|(X_1,X_2,X_3)}$ =0.982.
- Partial-$F =$ 1.832

The SEQ SS of X_3 is 9.79. The .95-quantile of $F[1, 9]$ is 5.117. Thus, the contribution of X_3 is not significant.

(d) The regression of Y on X_1, X_2, X_3, and X_4 is

```
model4 = smf.ols('y ~ x1 + x2 + x3 + x4 + 1', data=df).fit()
r2 = model4.rsquared
print(model4.summary().tables[1])
print(f'R-sq: {r2:.3f}')
anova = sms.anova.anova_lm(model4)
print('Analysis of Variance\n', anova)
SEQ_SS_X4 = anova.sum_sq['x4']
SSE_4 = anova.sum_sq['Residual']
s2e4 = anova.mean_sq['Residual']
```

```
anova = sms.anova.anova_lm(model3, model4)
print('Comparing models\n', anova)
partialF = anova.F[1]
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept     62.4054     70.071      0.891      0.399     -99.179     223.989
x1             1.5511      0.745      2.083      0.071      -0.166       3.269
x2             0.5102      0.724      0.705      0.501      -1.159       2.179
x3             0.1019      0.755      0.135      0.896      -1.638       1.842
x4            -0.1441      0.709     -0.203      0.844      -1.779       1.491
==============================================================================
R-sq: 0.982
Analysis of Variance
            df       sum_sq      mean_sq           F         PR(>F)
x1        1.0  1450.076328  1450.076328  242.367918  2.887559e-07
x2        1.0  1207.782266  1207.782266  201.870528  5.863323e-07
x3        1.0     9.793869     9.793869    1.636962  2.366003e-01
x4        1.0     0.246975     0.246975    0.041280  8.440715e-01
Residual  8.0    47.863639     5.982955         NaN           NaN
Comparing models
    df_resid        ssr  df_diff   ss_diff        F    Pr(>F)
0       9.0  48.110614      0.0       NaN      NaN       NaN
1       8.0  47.863639      1.0  0.246975  0.04128  0.844071
```

- $R^2_{Y|(X_1,X_2,X_3)}$ =0.982.
- Partial-F = 0.041

The effect of X_4 is not significant.

Solution 4.25 Using the step-wise regression method from the `mistat` package, we get:

```
outcome = 'y'
all_vars = ['x1', 'x2', 'x3', 'x4']

included, model = mistat.stepwise_regression(outcome, all_vars, df)

formula = ' + '.join(included)
formula = f'{outcome} ~ 1 + {formula}'
print()
print('Final model')
print(formula)
print(model.params)
```

```
Step 1 add - (F: 22.80)  x4
Step 2 add - (F: 108.22)  x1 x4
Step 3 add - (F: 5.03)  x1 x2 x4

Final model
y ~ 1 + x4 + x1 + x2
Intercept    71.648307
x4           -0.236540
x1            1.451938
x2            0.416110
dtype: float64
```

Solution 4.26 Build the regression model using `statsmodels`.

```
car = mistat.load_data('CAR')
car_3 = car[car['origin'] == 3]
print('Full dataset shape', car.shape)
print('Origin 3 dataset shape', car_3.shape)
model = smf.ols(formula='mpg ~ hp + 1', data=car_3).fit()
print(model.summary2())
```

```
Full dataset shape (109, 5)
Origin 3 dataset shape (37, 5)
                 Results: Ordinary least squares
=================================================================
Model:              OLS              Adj. R-squared:     0.400
Dependent Variable: mpg              AIC:                195.6458
Date:               2022-08-04 18:38 BIC:                198.8676
No. Observations:   37               Log-Likelihood:     -95.823
Df Model:           1                F-statistic:        25.00
Df Residuals:       35               Prob (F-statistic): 1.61e-05
R-squared:          0.417            Scale:              10.994
------------------------------------------------------------------
               Coef.   Std.Err.     t     P>|t|    [0.025   0.975]
------------------------------------------------------------------
Intercept     31.8328    1.8282  17.4117  0.0000  28.1213  35.5444
hp            -0.0799    0.0160  -4.9996  0.0000  -0.1123  -0.0474
-----------------------------------------------------------------
Omnibus:               7.375        Durbin-Watson:          1.688
Prob(Omnibus):         0.025        Jarque-Bera (JB):       6.684
Skew:                  -0.675       Prob(JB):               0.035
Kurtosis:              4.584        Condition No.:          384
=================================================================
```

Compute the additional properties

```
influence = model.get_influence()
df = pd.DataFrame({
  'hp': car_3['hp'],
  'mpg': car_3['mpg'],
  'resi': model.resid,
  'sres': influence.resid_studentized_internal,
  'hi': influence.hat_matrix_diag,
  'D': influence.cooks_distance[0],
})
print(df.round(4))
```

```
     hp  mpg     resi    sres      hi       D
0    118   25   2.5936  0.7937  0.0288  0.0093
1    161   18  -0.9714 -0.3070  0.0893  0.0046
51    55   17 -10.4392 -3.3101  0.0953  0.5770
52    98   23  -1.0041 -0.3075  0.0299  0.0015
53    92   27   2.5166  0.7722  0.0339  0.0105
54    92   29   4.5166  1.3859  0.0339  0.0337
55   104   20  -3.5248 -1.0781  0.0277  0.0165
56    68   27   0.5993  0.1871  0.0665  0.0012
57    70   31   4.7591  1.4826  0.0627  0.0736
58   110   20  -3.0455 -0.9312  0.0270  0.0120
62   121   19  -3.1668 -0.9699  0.0303  0.0147
63    82   24  -1.2823 -0.3956  0.0442  0.0036
64   110   22  -1.0455 -0.3197  0.0270  0.0014
65   158   19  -0.2110 -0.0664  0.0823  0.0002
71    92   26   1.5166  0.4654  0.0339  0.0038
72   102   22  -1.6846 -0.5154  0.0282  0.0039
73    81   27   1.6378  0.5056  0.0455  0.0061
74   142   18  -2.4892 -0.7710  0.0520  0.0163
75   107   18  -5.2852 -1.6161  0.0271  0.0364
76   160   19  -0.0513 -0.0162  0.0869  0.0000
```

```
77    90   24  -0.6432 -0.1975  0.0356  0.0007
78    90   26   1.3568  0.4167  0.0356  0.0032
79    97   21  -3.0840 -0.9446  0.0305  0.0140
80   106   18  -5.3650 -1.6406  0.0273  0.0377
81   140   20  -0.6489 -0.2007  0.0490  0.0010
82   165   18  -0.6518 -0.2071  0.0993  0.0024
94    66   34   7.4396  2.3272  0.0704  0.2051
95    97   23  -1.0840 -0.3320  0.0305  0.0017
96   100   25   1.1557  0.3537  0.0290  0.0019
97   115   24   1.3539  0.4141  0.0278  0.0025
98   115   26   3.3539  1.0259  0.0278  0.0151
99    90   27   2.3568  0.7238  0.0356  0.0097
100  190   19   2.3453  0.7805  0.1786  0.0662
101  115   25   2.3539  0.7200  0.0278  0.0074
102  200   18   2.1441  0.7315  0.2184  0.0748
103   78   28   2.3982  0.7419  0.0497  0.0144
108   64   28   1.2798  0.4012  0.0745  0.0065
```

Notice that points 51 and 94 have residuals with large magnitude (-10.4, 7.4). Points 51 and 94 have also the largest Cook's distance (0.58, 0.21) Points 100 and 102 have high HI values (leverage; 0.18, 0.22).

Solution 4.27 This solution uses version 2 of the Piston simulator.

Run piston simulation for different piston weights and visualize variation of times (see Figure 4.7). {fig:anovaWeightPiston}

```
np.random.seed(1)
settings = {'s': 0.005, 'v0': 0.002, 'k': 1000, 'p0': 90_000,
            't': 290, 't0': 340}
results = []
n_simulation = 5
for m in [30, 40, 50, 60]:
  simulator = mistat.PistonSimulator(m=m, n_simulation=n_simulation,
                                     **settings)
  sim_result = simulator.simulate()
  results.extend([m, s] for s in sim_result['seconds'])
results = pd.DataFrame(results, columns=['m', 'seconds'])

group_std = results.groupby('m').std()
pooled_std = np.sqrt(np.sum(group_std**2) / len(group_std))[0]
print('Pooled standard deviation', pooled_std)

group_mean = results.groupby('m').mean()
ax = results.plot.scatter(x='m', y='seconds', color='black')
ax.errorbar(group_mean.index, results.groupby('m').mean().values.flatten(),
            yerr=[pooled_std] * 4, color='grey')
plt.show()
```

```
Pooled standard deviation 0.4948439561427665
```

Perform ANOVA of data.

```
model = smf.ols(formula='seconds ~ C(m)', data=results).fit()
aov_table = sm.stats.anova_lm(model)
aov_table
```

```
            df    sum_sq   mean_sq         F    PR(>F)
C(m)       3.0  0.076379  0.025460  0.103972  0.956549
Residual  16.0  3.917929  0.244871       NaN       NaN
```

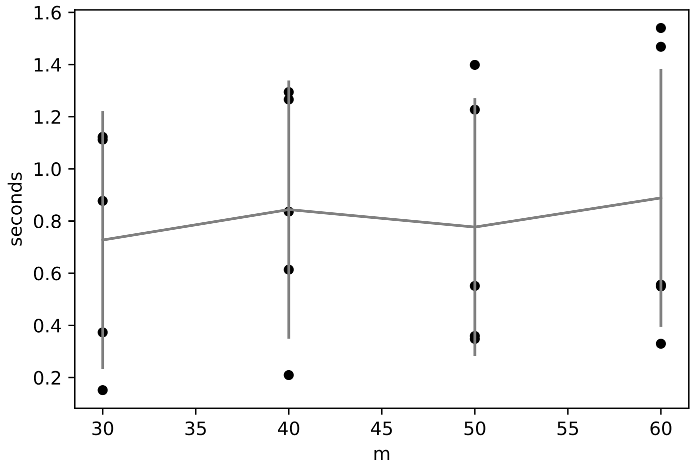


Fig. 4.7 ANOVA of effect of changing weight in piston simulation {fig:anovaWeightPiston}

We see that the differences between the sample means are not significant in spite of the apparent upward trend in cycle times.

Solution 4.28 Prepare dataset and visualize distributions (see Figure 4.8). {fig:boxplotIntegratedCircuits}

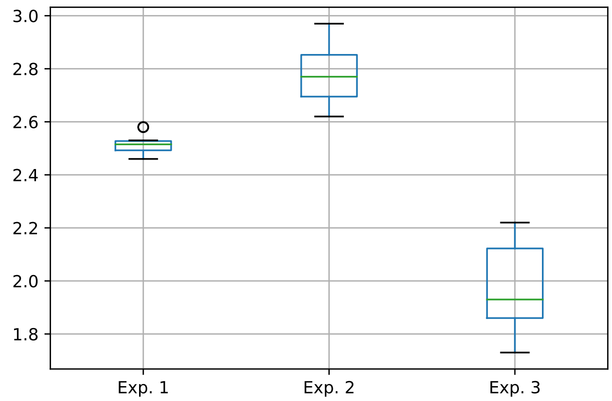
```
df = pd.DataFrame([
  [2.58, 2.62, 2.22],
  [2.48, 2.77, 1.73],
  [2.52, 2.69, 2.00],
  [2.50, 2.80, 1.86],
  [2.53, 2.87, 2.04],
  [2.46, 2.67, 2.15],
  [2.52, 2.71, 2.18],
  [2.49, 2.77, 1.86],
  [2.58, 2.87, 1.84],
  [2.51, 2.97, 1.86]
], columns=['Exp. 1', 'Exp. 2', 'Exp. 3'])
df.boxplot()

# Convert data frame to long format using melt
df = df.melt(var_name='Experiment', value_name='mu')
```

Analysis using ANOVA:

```
model = smf.ols(formula='mu ~ C(Experiment)', data=df).fit()
aov_table = sm.stats.anova_lm(model)
aov_table
```

```
                 df    sum_sq   mean_sq           F        PR(>F)
C(Experiment)   2.0  3.336327  1.668163  120.917098  3.352509e-14
Residual       27.0  0.372490  0.013796         NaN           NaN
```

The difference in the experiments is significant.
Bootstrap test:



{fig:boxplotIntegratedCircuits} **Fig. 4.8** Box plot of pre-etch line width from integrated circuits fabrication process

```
experiment = df['Experiment']
mu = df['mu']
def onewayTest(x, verbose=False):
    df = pd.DataFrame({
        'value': x,
        'variable': experiment,
    })
    aov = pg.anova(dv='value', between='variable', data=df)
    return aov['F'].values[0]

B = pg.compute_bootci(mu, func=onewayTest, n_boot=1000,
    seed=1, return_dist=True)

Bt0 = onewayTest(mu)
print('Bt0', Bt0)
print('ratio', sum(B[1] >= Bt0)/len(B[1]))
```

```
Bt0 120.91709844559576
ratio 0.0
```

The bootstrap test also shows that the difference in means is significant.

Solution 4.29 Create dataset and visualize distribution (see Figure 4.9). {fig:filmSpeedData}

```
df = pd.DataFrame({
  'Batch A': [103, 107, 104, 102, 95, 91, 107, 99, 105, 105],
  'Batch B': [104, 103, 106, 103, 107, 108, 104, 105, 105, 97],
})
df.boxplot()
plt.show()
```

(i) Randomization test (see Section 3.13.2) {sec:randomizaton-test}

```
dist = mistat.randomizationTest(df['Batch A'], df['Batch B'], np.mean,
                                aggregate_stats=lambda x: x[0] - x[1],
                                n_boot=10000, seed=1)
# ax = sns.distplot(dist)
# ax.axvline(np.mean(df['Batch A']) - np.mean(df['Batch B']))
```

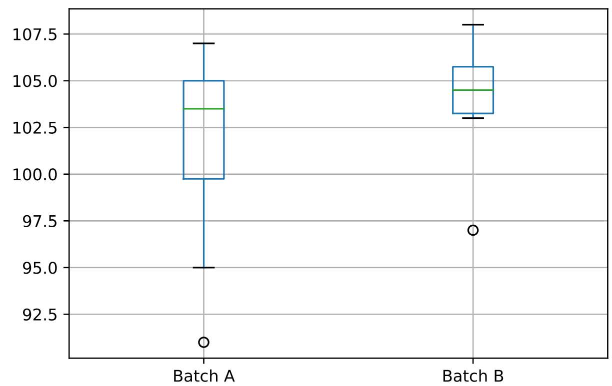


Fig. 4.9 Box plot of film speed data

{fig:filmSpeedData}

```
Original stat is -2.400000
Original stat is at quantile 1062 of 10001 (10.62%)
Distribution of bootstrap samples:
 min: -5.40, median: 0.00,  max: 5.60
```

The randomization test gave a P value of 0.106. The difference between the means is not significant.

(ii)

```
# Convert data frame to long format using melt
df = df.melt(var_name='Batch', value_name='film_speed')

model = smf.ols(formula='film_speed ~ C(Batch)', data=df).fit()
aov_table = sm.stats.anova_lm(model)
aov_table
```

```
            df  sum_sq    mean_sq         F    PR(>F)
C(Batch)   1.0    28.8  28.800000  1.555822  0.228263
Residual  18.0   333.2  18.511111       NaN       NaN
```

The ANOVA also shows no significant difference in the means. The P value is 0.228. Remember that the F-test in the ANOVA is based on the assumption of normality and equal variances. The randomization test is nonparametric.

Solution 4.30 Define function that calculates the statistic and execute bootstrap.

```
def func_stats(x):
    m = pd.Series(x).groupby(df['Experiment']).agg(['mean', 'count'])
    top = np.sum(m['count'] * m['mean'] ** 2) - len(x)*np.mean(x)**2
    return top / np.std(x) ** 2

Bt = []
mu = list(df['mu'])
for _ in range(1000):
    mu_star = random.sample(mu, len(mu))
    Bt.append(func_stats(mu_star))

Bt0 = func_stats(mu)
```

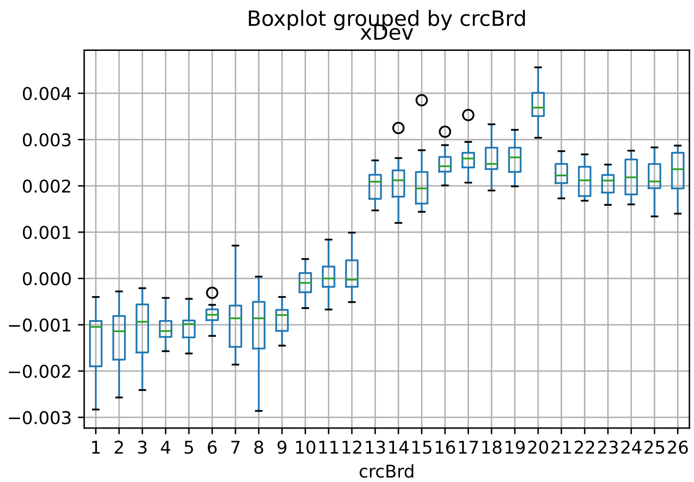


Fig. 4.10 Box plot visualisation of xDev distribution by crcBrd for the PLACE dataset {fig:boxplotXdevCrcBrdPlace}

```
print('Bt0', Bt0)
print('ratio', sum(Bt >= Bt0)/len(Bt))
```

```
Bt0 26.986990459670288
ratio 0.0
```

The result demonstrates that the differences between the results is significant.

Solution 4.31 Load the data and visualize the distributions (see Figure 4.10). {fig:boxplotXdevCrcBrdPlace}

```
place = mistat.load_data('PLACE')
place.boxplot('xDev', by='crcBrd')
plt.show()
```

(a) ANOVA for the fulldataset

```
model = smf.ols(formula='xDev ~ C(crcBrd)', data=place).fit()
aov_table = sm.stats.anova_lm(model)
aov_table
```

```
              df    sum_sq       mean_sq           F         PR(>F)
C(crcBrd)   25.0  0.001128  4.512471e-05  203.292511  2.009252e-206
Residual   390.0  0.000087  2.219694e-07         NaN            NaN
```

(b) There seem to be four homogeneous groups: $G_1 = \{1, 2, \ldots, 9\}$, $G_2 = \{10, 11, 12\}$, $G_3 = \{13, \ldots, 19, 21, \ldots, 26\}$, $G_4 = \{20\}$.

In multiple comparisons we use the Scheffé coefficient $S_{.05} = (25 \times F_{.95}[25, 390])^{1/2} = (25 \times 1.534)^{1/2} = 6.193$. The group means and standard errors are:

```
G1 = [1, 2, 3, 4, 5, 6, 7, 8, 9]
G2 = [10, 11, 12]
```

```
G3 = [13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26]
G4 = [20]
place['group'] = 'G1'
place.loc[place['crcBrd'].isin(G2), 'group'] = 'G2'
place.loc[place['crcBrd'].isin(G3), 'group'] = 'G3'
place.loc[place['crcBrd'].isin(G4), 'group'] = 'G4'

statistics = place['xDev'].groupby(place['group']).agg(['mean', 'sem', 'count'])
statistics = statistics.sort_values(['mean'], ascending=False)
print(statistics.round(8))
statistics['Diff'] = 0
n = len(statistics)
print(statistics['mean'][:-1].values - statistics['mean'][1:].values)
print(statistics['mean'][:(n-1)].values - statistics['mean'][1:].values)
statistics.loc[1:, 'Diff'] = (statistics['mean'][:-1].values -
                              statistics['mean'][1:].values)
statistics['CR'] = 6.193 * statistics['Diff']
print(statistics.round(8))
# 0.001510 0.0022614 0.0010683
# 0.000757 0.000467 0.000486

sem = statistics['sem'].values
sem = sem**2
sem = np.sqrt(sem[:-1] + sem[1:])
print(sem * 6.193)
print(757/644, 467/387, 486/459)
```

```
           mean       sem  count
group
G4     0.003778  0.000100     16
G3     0.002268  0.000030    208
G2     0.000006  0.000055     48
G1    -0.001062  0.000050    144
[0.00151029 0.00226138 0.00106826]
[0.00151029 0.00226138 0.00106826]
           mean       sem  count      Diff        CR
group
G4     0.003778  0.000100     16  0.000000  0.000000
G3     0.002268  0.000030    208  0.001510  0.009353
G2     0.000006  0.000055     48  0.002261  0.014005
G1    -0.001062  0.000050    144  0.001068  0.006616
[0.00064435 0.00038718 0.00045929]
1.1754658385093169 1.20671834625323 1.0588235294117647
```

The differences between the means of the groups are all significant.

```
from statsmodels.stats.multicomp import pairwise_tukeyhsd
m_comp = pairwise_tukeyhsd(endog=place['xDev'], groups=place['group'],
                           alpha=0.05)
print(m_comp)
```

```
Multiple Comparison of Means - Tukey HSD, FWER=0.05
===============================================
group1 group2 meandiff p-adj lower  upper  reject
-----------------------------------------------
    G1     G2   0.0011   0.0 0.0009 0.0013   True
    G1     G3   0.0033   0.0 0.0032 0.0035   True
    G1     G4   0.0048   0.0 0.0045 0.0052   True
    G2     G3   0.0023   0.0 0.0021 0.0025   True
    G2     G4   0.0038   0.0 0.0034 0.0041   True
    G3     G4   0.0015   0.0 0.0012 0.0018   True
-----------------------------------------------
```

Solution 4.32

```
df = pd.DataFrame({
    'US': [33, 25],
    'Europe': [7, 7],
    'Asia': [26, 11],
  })

print(df)

col_sums = df.sum(axis=0)
row_sums = df.sum(axis=1)
total = df.to_numpy().sum()

expected_frequencies = np.outer(row_sums, col_sums) / total

chi2 = (df - expected_frequencies) ** 2 / expected_frequencies
chi2 = chi2.to_numpy().sum()
print(f'chi2: {chi2:.3f}')
print(f'p-value: {1 - stats.chi2.cdf(chi2, 2):.3f}')
```

```
   US  Europe  Asia
0  33       7    26
1  25       7    11
chi2: 2.440
p-value: 0.295
```

The chi-square test statistic is $X^2 = 2.440$ with d.f. = 2 and P value = 0.295. The null hypothesis that the number of cylinders a car has is independent of the origin of the car is not rejected.

We can also use the `scipy` function `chi2_contingency`.

```
chi2 = stats.chi2_contingency(df)
print(f'chi2-statistic: {chi2[0]:.3f}')
print(f'p-value: {chi2[1]:.3f}')
print(f'd.f.: {chi2[2]}')
```

```
chi2-statistic: 2.440
p-value: 0.295
d.f.: 2
```

Solution 4.33 In Python:

```
car = mistat.load_data('CAR')
binned_car = pd.DataFrame({
  'turn': pd.cut(car['turn'], bins=[27, 30.6, 34.2, 37.8, 45]), #np.arange(27, 50, 3.6)),
  'mpg': pd.cut(car['mpg'], bins=[12, 18, 24, 100]),
})
freqDist = pd.crosstab(binned_car['mpg'], binned_car['turn'])
print(freqDist)

chi2 = stats.chi2_contingency(freqDist)
print(f'chi2-statistic: {chi2[0]:.3f}')
print(f'p-value: {chi2[1]:.3f}')
print(f'd.f.: {chi2[2]}')
```

```
turn       (27.0, 30.6]  (30.6, 34.2]  (34.2, 37.8]  (37.8, 45.0]
mpg
(12, 18]              2             4            10            15
(18, 24]              0            12            26            15
(24, 100]             4            15             6             0
```

```
chi2-statistic: 34.990
p-value: 0.000
d.f.: 6
```

The dependence between turn diameter and miles per gallon is significant.

Solution 4.34 In Python:

```
question_13 = pd.DataFrame({
  '1': [0,0,0,1,0],
  '2': [1,0,2,0,0],
  '3': [1,2,6,5,1],
  '4': [2,1,10,23,13],
  '5': [0,1,1,15,100],
  }, index = ['1', '2', '3', '4', '5']).transpose()
question_23 = pd.DataFrame({
  '1': [1,0,0,3,1],
  '2': [2,0,1,0,0],
  '3': [0,4,2,3,0],
  '4': [1,1,10,7,5],
  '5': [0,0,1,30,134],
  }, index = ['1', '2', '3', '4', '5']).transpose()

chi2_13 = stats.chi2_contingency(question_13)
chi2_23 = stats.chi2_contingency(question_23)

msc_13 = chi2_13[0] / question_13.to_numpy().sum()
tschuprov_13 = np.sqrt(msc_13 / (2 * 2)) # (4 * 4))
cramer_13 = np.sqrt(msc_13 / 2) # min(4, 4))

msc_23 = chi2_23[0] / question_23.to_numpy().sum()
tschuprov_23 = np.sqrt(msc_23 / 4) # (4 * 4))
cramer_23 = np.sqrt(msc_23 / 2) # min(4, 4))

print('Question 1 vs 3')
print(f'  Mean squared contingency : {msc_13:.3f}')
print(f'  Tschuprov : {tschuprov_13:.3f}')
print(f"  Cramer's index : {cramer_13:.3f}")
print('Question 2 vs 3')
print(f'  Mean squared contingency : {msc_23:.3f}')
print(f'  Tschuprov : {tschuprov_23:.3f}')
print(f"  Cramer's index : {cramer_23:.3f}")
```

```
Question 1 vs 3
  Mean squared contingency : 0.629
  Tschuprov : 0.397
  Cramer's index : 0.561
Question 2 vs 3
  Mean squared contingency : 1.137
  Tschuprov : 0.533
  Cramer's index : 0.754
```

Chapter 5
Sampling for Estimation of Finite Population Quantities

Import required modules and define required functions

```
import random
import numpy as np
import pandas as pd
import pingouin as pg
from scipy import stats
import matplotlib.pyplot as plt
import mistat
```

Solution 5.1 Define the binary random variables

$$I_{ij} = \begin{cases} 1, & \text{if the } j\text{-th element is selected at the } i\text{-th sampling} \\ \\ 0, & \text{otherwise.} \end{cases}$$

The random variables $X_1, \cdots, X_n$ are given by $X_i = \sum_{j=1}^{N} x_j I_{ij}$, $i = 1, \cdots, n$. Since sampling is RSWR, $\Pr\{X_i = x_j\} = \dfrac{1}{N}$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, N$. Hence, $\Pr\{X_i \le x\} = F_N(x)$ for all x, and all $i = 1, \ldots, n$. Moreover, by definition of RSWR, the vectors $\mathbf{I}_i = (I_{i1}, \ldots, I_{iN})$, $i = 1, \ldots, n$ are mutually independent. Therefore $X_1, \ldots, X_n$ are i.i.d., having a common c.d.f. $F_N(x)$.

Solution 5.2 In continuation of the previous exercise, $E\{X_i\} = \dfrac{1}{N} \sum_{i=1}^{N} x_i = \mu_N$. Therefore, by the weak law of large numbers, $\lim_{n\to\infty} P\{|\bar{X}_n - \mu_N| < \epsilon\} = 1$.

Solution 5.3 By the CLT ($0 < \sigma_N^2 < \infty$),

$$\Pr\{\sqrt{n}|\bar{X}_n - \mu_N| < \delta\} \approx 2\Phi\left(\frac{\delta}{\sigma_N}\right) - 1,$$

as $n \to \infty$.

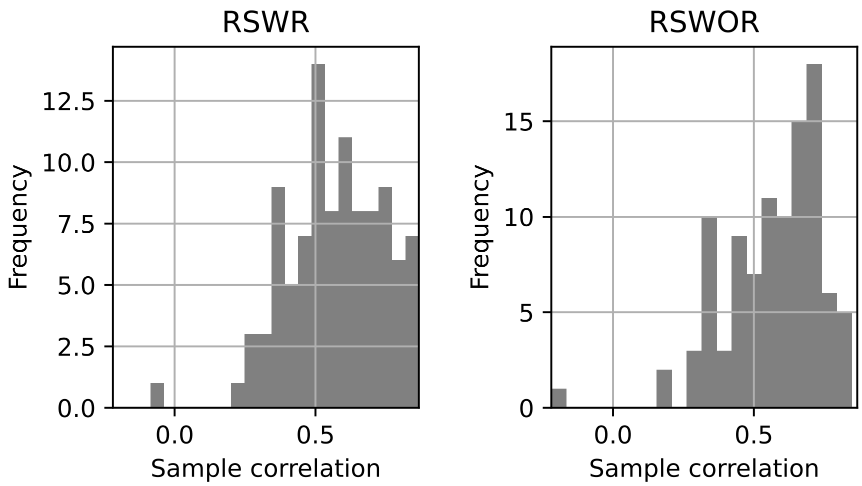


Fig. 5.1 Distribution of correlation between xDev and yDev for sampling with and without distribution.

{fig:exDistCorrPlace}

Solution 5.4 We create samples of size $k = 20$ with and without replacement, determine the correlation coefficient and finally create the two histograms (see Figure 5.1).

{fig:exDistCorrPlace}

```
random.seed(1)
place = mistat.load_data('PLACE')

# calculate correlation coefficient based on a sample of rows
def stat_func(idx):
    return stats.pearsonr(place['xDev'][idx], place['yDev'][idx])[0]

rswr = []
rswor = []
idx = list(range(len(place)))
for _ in range(100):
    rswr.append(stat_func(random.choices(idx, k=20)))
    rswor.append(stat_func(random.sample(idx, k=20)))

corr_range = (min(*rswr, *rswor), max(*rswr, *rswor))

def makeHistogram(title, ax, data, xrange):
  ax = pd.Series(data).hist(color='grey', ax=ax, bins=20)
  ax.set_title(title)
  ax.set_xlabel('Sample correlation')
  ax.set_ylabel('Frequency')
  ax.set_xlim(*xrange)

fig, axes = plt.subplots(figsize=[5, 3], ncols=2)
makeHistogram('RSWR', axes[0], rswr, corr_range)
makeHistogram('RSWOR', axes[1], rswor, corr_range)
plt.tight_layout()
plt.show()
```

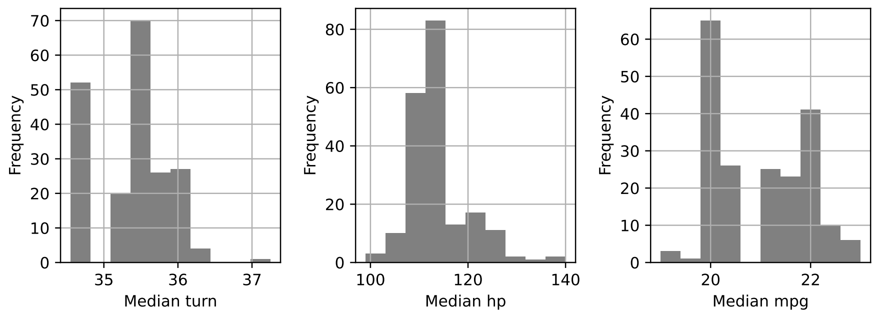


Fig. 5.2 Distribution of median of turn-diameter, horsepower, and mpg of the CAR dataset using random sampling without replacement. {fig:exDistMedianCAR}

{fig:exDistMedianCAR} **Solution 5.5** The Python code creates the histograms shown in Figure 5.2.

```
random.seed(1)
car = mistat.load_data('CAR')
columns = ['turn', 'hp', 'mpg']

# calculate correlation coefficient based on a sample of rows
def stat_func(idx):
    sample = car[columns].loc[idx,]
    return sample.median()

idx = list(range(len(car)))
result = []
for _ in range(200):
    result.append(stat_func(random.sample(idx, k=50)))
result = pd.DataFrame(result)

fig, axes = plt.subplots(figsize=[8, 3], ncols=3)
for ax, column in zip(axes, columns):
    result[column].hist(color='grey', ax=ax)
    ax.set_xlabel(f'Median {column}')
    ax.set_ylabel('Frequency')
plt.tight_layout()
plt.show()
```

Solution 5.6 For RSWOR,

$$\text{S.E.}\{\bar{X}_i\} = \frac{\sigma}{\sqrt{n}}\left(1 - \frac{n-1}{N-1}\right)^{1/2}$$

Equating the standard error to δ we get $n_1 = 30$, $n_2 = 116$, $n_3 = 84$.

Solution 5.7 The required sample size is a solution of the equation $0.002 = 2 \cdot 1.96 \cdot \sqrt{\frac{P(1-P)}{n}\left(1 - \frac{n-1}{N}\right)}$. The solution is $n = 1611$.

Solution 5.8 The following are Python commands to estimate the mean of all $N = 416$ x-dev values by stratified sampling with proportional allocation. The total sample size is $n = 200$ and the weights are $W_1 = 0.385$, $W_2 = 0.115$, $W_3 = 0.5$. Thus, $n_1 = 77$, $n_2 = 23$, and $n_3 = 100$.

```
# load dataset and split into strata
place = mistat.load_data('PLACE')
strata_1 = list(place['xDev'][:160])
strata_2 = list(place['xDev'][160:208])
strata_3 = list(place['xDev'][208:])
N = len(place)
w_1 = 0.385
w_2 = 0.115
w_3 = 0.5
n_1 = int(w_1 * 200)
n_2 = int(w_2 * 200)
n_3 = int(w_3 * 200)

sample_means = []
for _ in range(500):
    m_1 = np.mean(random.sample(strata_1, k=n_1))
    m_2 = np.mean(random.sample(strata_2, k=n_2))
    m_3 = np.mean(random.sample(strata_3, k=n_3))
    sample_means.append(w_1*m_1 + w_2*m_2 + w_3*m_3)
std_dev_sample_means = np.std(sample_means)
print(std_dev_sample_means)
print(stats.sem(place['xDev'], ddof=0))
```

```
3.442839155174113e-05
8.377967188860638e-05
```

The standard deviation of the estimated means is an estimate of S.E.$(\hat{\mu}_N)$. The true value of this S.E. is 0.000034442.

Solution 5.9 $L(n_1, \ldots, n_k; \lambda) = \sum_{i=1}^k W_i^2 \frac{\tilde{\sigma}_{N_i}^2}{n_i} - \lambda \left(n - \sum_{i=1}^k n_i\right)$. Partial differentiation of L w.r.t. $n_1, \ldots, n_k$ and λ and equating the result to zero yields the following equations:

$$\frac{W_i^2 \tilde{\sigma}_{N_i}^2}{n_i^2} = \lambda, \quad i = 1, \ldots, k$$
$$\sum_{i=1}^k n_i = n.$$

Equivalently, $n_i = \frac{1}{\sqrt{\lambda}} W_i \tilde{\sigma}_{N_i}$, for $i = 1, \ldots, k$ and $n = \frac{1}{\sqrt{\lambda}} \sum_{i=1}^k W_i \tilde{\sigma}_{N_i}$. Thus $n_i^0 = n \frac{W_i \tilde{\sigma}_{N_i}}{\sum_{j=1}^k W_j \tilde{\sigma}_{N_j}}, \quad i = 1, \ldots, k.$

Solution 5.10 The prediction model is $y_i = \beta + e_i$, $i = 1, \ldots, N$, $E\{e_i\} = 0$, $V\{e_i\} = \sigma^2$, $\text{cov}(e_i, e_j) = 0$ for all $i \neq j$.

$$
\begin{aligned}
E\{\bar{Y}_n - \mu_N\} &= E\left\{\beta + \frac{1}{n}\sum_{i=1}^{N} I_i e_i - \beta - \frac{1}{N}\sum_{i=1}^{N} e_i\right\} \\
&= \frac{1}{n}\sum_{i=1}^{N} E\{I_i e_i\},
\end{aligned}
$$

where $I_i = \begin{cases} 1, & \text{if } i\text{-th population element is sampled} \\ 0, & \text{otherwise.} \end{cases}$

Notice that $I_1, \ldots, I_N$ are independent of $e_1, \ldots, e_N$. Hence, $E\{I_i e_i\} = 0$ for all $i = 1, \ldots, N$. This proves that $\bar{Y}_n$ is prediction unbiased, irrespective of the sample strategy. The prediction MSE of $\bar{Y}_n$ is

$$
\begin{aligned}
PMSE\{\bar{Y}_n\} &= E\{(\bar{Y}_n - \mu_N)^2\} \\
&= V\left\{\frac{1}{n}\sum_{i=1}^{N} I_i e_i - \frac{1}{N}\sum_{i=1}^{N} e_i\right\} \\
&= V\left\{\left(\frac{1}{n} - \frac{1}{N}\right)\sum_{i=1}^{N} I_i e_i - \frac{1}{N}\sum_{i=1}^{N}(1 - I_i)e_i\right\}.
\end{aligned}
$$

Let $\mathbf{s}$ denote the set of units in the sample. Then

$$
\begin{aligned}
PMSE\{\bar{Y}_n \mid \mathbf{s}\} &= \frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right)^2 + \frac{1}{N}\left(1 - \frac{n}{N}\right)\sigma^2 \\
&= \frac{\sigma^2}{n}\left(1 - \frac{n}{N}\right).
\end{aligned}
$$

Notice that $PMSE\{\bar{Y}_n \mid \mathbf{s}\}$ is independent of $\mathbf{s}$, and is equal for all samples.

Solution 5.11 The model is $y_i = \beta_0 + \beta_1 x_i + e_i$, $i = 1, \ldots, N$. $E\{e_i\} = 0$, $V\{e_i\} = \sigma^2 x_i$, $i = 1, \ldots, N$. Given a sample $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$, we estimate β_0 and β_1 by the weighted LSE because the variances of y_i depend on x_i, $i = 1, \ldots, N$. These weighted LSE values $\hat{\beta}_0$ and $\hat{\beta}_1$ minimizing $Q = \sum_{i=1}^{N} \dfrac{1}{X_i}(Y_i - \beta_0 - \beta_1 X_i)^2$, are given by

$$
\hat{\beta}_1 = \frac{\bar{Y}_n \cdot \frac{1}{n}\sum_{i=1}^{N}\frac{1}{X_i} - \frac{1}{n}\sum_{i=1}^{N}\frac{Y_i}{X_i}}{\bar{X}_n \cdot \frac{1}{n}\sum_{i=1}^{N}\frac{1}{X_i} - 1} \quad \text{and} \quad \hat{\beta}_0 = \frac{1}{\sum_{i=1}^{N}\frac{1}{X_i}}\left(\sum_{i=1}^{N}\frac{Y_i}{X_i} - n\hat{\beta}_1\right).
$$

It is straightforward to show that $E\{\hat{\beta}_1\} = \beta_1$ and $E\{\hat{\beta}_0\} = \beta_0$. Thus, an unbiased predictor of μ_N is $\hat{\mu}_N = \hat{\beta}_0 + \hat{\beta}_1 \bar{x}_N$.

Solution 5.12 The predictor $\hat{Y}_{RA}$ can be written as $\hat{Y}_{RA} = \bar{x}_N \cdot \dfrac{1}{n}\sum_{i=1}^{N} I_i \frac{y_i}{x_i}$, where

$$I_i = \begin{cases} 1, & \text{if } i\text{-th population element is in the sample } \mathbf{s} \\ \\ 0, & \text{otherwise.} \end{cases}$$

Recall that $y_i = \beta x_i + e_i$, $i = 1, \ldots, N$, and that for any sampling strategy, $e_1, \ldots, e_N$ are independent of $I_1, \ldots, I_N$. Hence, since $\sum_{i=1}^N I_i = n$,

$$\begin{aligned} E\{\hat{Y}_{RA}\} &= \bar{x}_N \cdot \frac{1}{n} \sum_{i=1}^N E\left\{ I_i \frac{\beta x_i + e_i}{x_i} \right\} \\ &= \bar{x}_N \left(\beta + \frac{1}{n} \sum_{i=1}^N E\left\{ I_i \frac{e_i}{x_i} \right\} \right) \\ &= \bar{x}_N \beta, \end{aligned}$$

because $E\left\{ I_i \dfrac{e_i}{x_i} \right\} = E\left\{ \dfrac{I_i}{x_i} \right\} E\{e_i\} = 0$, $i = 1, \ldots, N$. Thus, $E\{\hat{Y}_{RA} - \mu_N\} = 0$ and $\hat{Y}_{RA}$ is an unbiased predictor.

$$\begin{aligned} PMSE\{\hat{Y}_{RA}\} &= E\left\{ \left(\frac{\bar{x}_N}{n} \sum_{i=1}^N I_i \frac{e_i}{x_i} - \frac{1}{N} \sum_{i=1}^N e_i \right)^2 \right\} \\ &= V\left\{ \sum_{i \in \mathbf{s}_n} \left(\frac{\bar{x}_N}{n x_i} - \frac{1}{N} \right) e_i - \sum_{i' \in \mathbf{r}_n} \frac{e_{i'}}{N} \right\} \end{aligned}$$

where $\mathbf{s}_n$ is the set of elements in the sample and $\mathbf{r}_n$ is the set of elements in $\mathcal{P}$ but not in $\mathbf{s}_n$, $\mathbf{r}_n = \mathcal{P} - \mathbf{s}_n$. Since $e_1, \ldots, e_N$ are uncorrelated,

$$\begin{aligned} PMSE\{\hat{Y}_{RA} \mid \mathbf{s}_n\} &= \sigma^2 \sum_{i \in \mathbf{s}_n} \left(\frac{\bar{x}_N}{n x_i} - \frac{1}{N} \right)^2 + \sigma^2 \frac{N-n}{N^2} \\ &= \frac{\sigma^2}{N^2} \left[(N-n) + \sum_{i \in \mathbf{s}_n} \left(\frac{N \bar{x}_N}{n x_i} - 1 \right)^2 \right]. \end{aligned}$$

A sample $\mathbf{s}_n$ which minimizes $\sum_{i \in \mathbf{s}_n} \left(\frac{N \bar{x}_N}{n x_i} - 1 \right)^2$ is optimal.

The predictor $\hat{Y}_{RG}$ can be written as

$$\hat{Y}_{RG} = \bar{x}_N \frac{\sum_{i=1}^N I_i x_i y_i}{\sum_{i=1}^N I_i x_i^2} = \bar{x}_N \left(\frac{\sum_{i=1}^N I_i x_i (\beta x_i + e_i)}{\sum_{i=1}^N I_i x_i^2} \right) = \beta \bar{x}_N + \bar{x}_N \frac{\sum_{i=1}^N I_i x_i e_i}{\sum_{i=1}^N I_i x_i^2}.$$

Hence, $E\{\hat{Y}_{RG}\} = \beta \bar{x}_N$ and $\hat{Y}_{RG}$ is an unbiased predictor of μ_N. The conditional prediction MSE, given $\mathbf{s}_n$, is

$$PMSE\{\hat{Y}_{RG} \mid \mathbf{s}_n\} = \frac{\sigma^2}{N^2}\left[N + \frac{N^2\bar{x}_N^2}{\sum_{i\in\mathbf{s}_n} x_i^2} - 2N\bar{x}_N\,\frac{n\bar{X}_n}{\sum_{i\in\mathbf{s}_n} x_i^2}\right].$$

Chapter 6
Time Series Analysis and Prediction

Import required modules and define required functions

```
import math
import mistat
import numpy as np
import pandas as pd
from scipy import stats
import matplotlib.pyplot as plt
from statsmodels.tsa.stattools import acf, pacf
from statsmodels.graphics.tsaplots import plot_acf, plot_pacf
from statsmodels.tsa import tsatools
import statsmodels.formula.api as smf
```

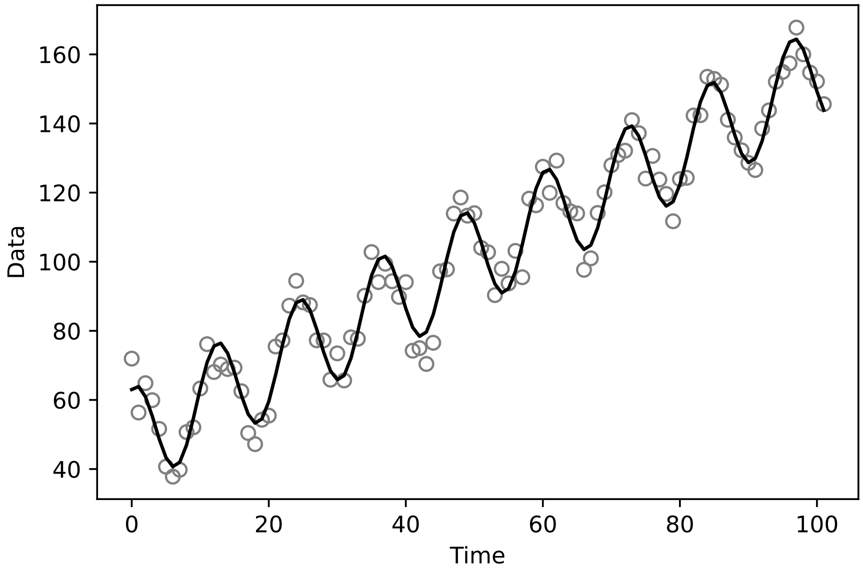
Solution 6.1 TODO: Provide a sample solution

Solution 6.2 **(i)** Figure 6.1 shows the change of demand over time {fig:seascom-timeline}

(ii) We are first fitting the seasonal trend to the data in SeasCom data set. We use the linear model$Y = X\beta + \varepsilon$, where the Y vector has 102 elements, which are in data set. X is a 102x4 matrix of four column vectors. We combined X and Y in a data frame. In addition to the SeasCom data, we add a column of 1's (`const`), a column with numbers 1 to 102 (`trend`) and two columns to describe the seasonal pattern. The column `season_1` consists of $\cos(\pi \times \text{trend}/6)$, and the column `season_2` is $\sin(\pi \times \text{trend}/6)$.

```
seascom = mistat.load_data('SEASCOM.csv')
df = tsatools.add_trend(seascom, trend='ct')
df['season_1'] = [np.cos(math.pi * tx/6) for tx in df['trend']]
df['season_2'] = [np.sin(math.pi * tx/6) for tx in df['trend']]
print(df.head())

model = smf.ols(formula='SeasCom ~ trend + 1 + season_1 + season_2',
                data=df).fit()
print(model.params)
print(f'r2-adj: {model.rsquared_adj:.3f}')
```

```
    SeasCom  const  trend      season_1  season_2
0  71.95623    1.0    1.0  8.660254e-01  0.500000
1  56.36048    1.0    2.0  5.000000e-01  0.866025
2  64.85331    1.0    3.0  6.123234e-17  1.000000
```



{fig:seascom-timeline} **Fig. 6.1** Seasonal trend model of SeasCom data set

```
3  59.93460    1.0    4.0 -5.000000e-01  0.866025
4  51.62297    1.0    5.0 -8.660254e-01  0.500000
Intercept    47.673469
trend         1.047236
season_1     10.653968
season_2     10.130145
dtype: float64
r2-adj: 0.981
```

The least squares estimates of β is

$$b = (47.67347, 1.04724, 10.65397, 10.13015)'$$

{fig:seascom-timeline} The fitted trend is $Y_t = Xb$. (see Figure 6.1).

```
seascom = mistat.load_data('SEASCOM.csv')
fig, ax = plt.subplots()
ax.scatter(seascom.index, seascom, facecolors='none', edgecolors='grey')
model.predict(df).plot(ax=ax, color='black')
ax.set_xlabel('Time')
ax.set_ylabel('Data')
plt.show()
```

{fig:seascom-model-deviation} **(iii)** Calculate the residuals and plot them (see Figure 6.2).

```
U = df['SeasCom'] - model.predict(df)
fig, ax = plt.subplots()
ax.scatter(U.index, U, facecolors='none', edgecolors='black')
```

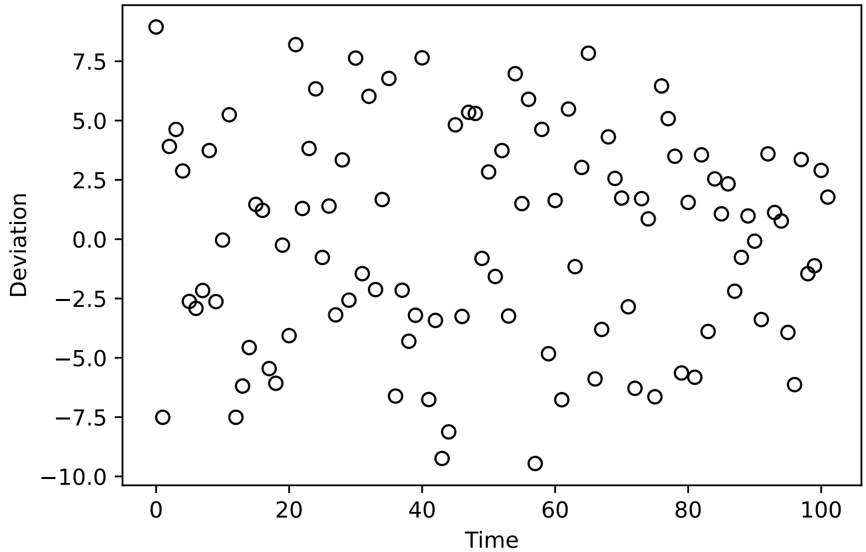


Fig. 6.2 Deviation of SeasCom data from cyclical trend. {fig:seascom-model-deviation}

```
ax.set_xlabel('Time')
ax.set_ylabel('Deviation')
plt.show()
```

(iv) Calculate the correlations

```
# use slices to get sublists
corr_1 = np.corrcoef(U[:-1], U[1:])[0][1]
corr_2 = np.corrcoef(U[:-2], U[2:])[0][1]
print(f'Corr(Ut,Ut-1) = {corr_1:.3f}')
print(f'Corr(Ut,Ut-2) = {corr_2:.3f}')
```

```
Corr(Ut,Ut-1) = -0.191
Corr(Ut,Ut-2) = 0.132
```

Indeed the correlations between adjacent data points are $corr(X_t, X_{t-1}) = -0.191$, and $corr(X_t, X_{t-2}) = 0.132$.

(iv) A plot of the deviations and the low autocorrelations shows something like randomness.

```
# keep some information for later exercises
seascom_model = model
seascom_df = df
```

Solution 6.3 According to Equation 6.2.2, the auto-correlation in a stationary MA(q) is {eqn:ma-covariance}

$$\rho(h) = \frac{K(h)}{K(0)} = \frac{\sum_{j=0}^{q-h} \beta_j \beta_{j+h}}{\sum_{j=0}^{q} \beta_j^2}$$

for $0 \le h \le q$).

Notice that $\rho(h) = \rho(-h)$, and $\rho(h) = 0$ for all $h > q$.

Solution 6.4 In Python

```
beta = np.array([1, 1.05, 0.76, -0.35, 0.45, 0.55])
data = []
n = len(beta)
sum_0 = np.sum(beta * beta)
for h in range(6):
  sum_h = np.sum(beta[:n-h] * beta[h:])
  data.append({
    'h': h,
    'K(h)': sum_h,
    'rho(h)': sum_h / sum_0,
  })
```

	0	1	2	3	4	5
K(h)	3.308	1.672	0.542	0.541	1.028	0.550
rho(h)	1.000	0.506	0.164	0.163	0.311	0.166

Solution 6.5 We consider the $AQ(\infty)$, given by coefficients $\beta_j = q^j$, with $0 < q < 1$. In this case,

(i) $E\{X_t\} = 0$, and

(ii) $V\{X_t\} = \sigma^2 \sum_{j=0}^{\infty} q^{2j} = \sigma^2/(1 - q^2)$.

Solution 6.6 We consider the AR(1), $X_t = 0.75X_{t-1} + \varepsilon_t$.

(i) This time-series is equivalent to $X_t = \sum_{j=0}^{\infty}(-0.75)^j \varepsilon_{t-j}$, hence it is covariance stationary.

(ii) $E\{X_t\} = 0$

(iii) According to the Yule-Walker equations,

$$\begin{bmatrix} 1 & -0.75 \\ -0.75 & 1 \end{bmatrix} \begin{bmatrix} K(0) \\ K(1) \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}$$

It follows that $K(0) = 2.285714\,\sigma^2$ and $K(1) = 1.714286\,\sigma^2$.

Solution 6.7 The given AR(2) can be written as $\mathbf{X}_t - 0.5\mathbf{X}_{t-1} + 0.3\mathbf{X}_{t-2} = \boldsymbol{\varepsilon}_t$.

(i) The corresponding characteristic polynomial is $\mathbf{P}_2(\mathbf{z}) = 0.3 - 0.5\mathbf{z} + \mathbf{z}^2$. The two characteristic roots are $\mathbf{z}_{1,2} = \frac{1}{4} \pm \mathrm{i}\frac{\sqrt{95}}{20}$. These two roots belong to the unit circle. Hence this AR(2) is covariance stationary.

(ii) We can write $A_2(z)X_t = \varepsilon_t$, where $A_2(z) = 1 - 0.5z^{-1} + 0.3z^{-2}$. Furthermore, $\phi_{1,2}$ are the two roots of $A_2(z) = 0$.

It follows that

$$\begin{aligned} X_t &= (A_2(z))^{-1}\varepsilon_t \\ &= \varepsilon_t + 0.5\varepsilon_{t-1} - 0.08\varepsilon_{t-2} - 0.205\varepsilon_{t-3} - 0.0761\varepsilon_{t-4} + 0.0296\varepsilon_{t-5} + \dots \end{aligned}$$

Solution 6.8 The Yule-Walker equations are:

$$\begin{bmatrix} 1 & -0.5 & 0.3 & -0.2 \\ -0.5 & 1.3 & -0.2 & 0 \\ 0.3 & -0.7 & 1 & 0 \\ -0.2 & 0.3 & -0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} K(0) \\ K(1) \\ K(2) \\ K(3) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $K(0) = 1.2719$, $K(1) = 0.4825$, $K(2) = -0.0439$, $K(3) = 0.0877$.

Solution 6.9 The Toeplitz matrix is

$$R_4 = \begin{bmatrix} 1.0000 & 0.3794 & -0.0235 & 0.0690 \\ 0.3794 & 1.0000 & 0.3794 & -0.0235 \\ -0.0235 & 0.3794 & 1.0000 & 0.3794 \\ 0.0690 & -0.0235 & 0.3794 & 1.0000 \end{bmatrix}$$

Solution 6.10 This series is an ARMA(2,2), given by the equation

$$(1 - z^{-1} + 0.25z^{-2})X_t = (1 + 0.4z^{-1} - 0.45z^{-2})\varepsilon_t.$$

Accordingly,

$$\begin{aligned} X_t &= (1 + 0.4z^{-1} - 0.45z^{-2})(1 - z^{-1} + 0.25z^{-2})^{-1}\varepsilon_t \\ &= \varepsilon_t + 1.4\varepsilon_{t-1} + 0.7\varepsilon_{t-2} + 0.35\varepsilon_{t-3} + 0.175\varepsilon_{t-4} \\ &\quad + 0.0875\varepsilon_{t-5} + 0.0438\varepsilon_{t-6} + 0.0219\varepsilon_{t-7} + 0.0109\varepsilon_{t-8} + \ldots \end{aligned}$$

Solution 6.11 Let X denote the DOW1941 data set. We create a new set, Y of second order difference, i.e. $Y_t = X_t - 2X_{t-1} + X_{t-2}$.

```
dow1941 = mistat.load_data('DOW1941.csv')

X = dow1941.values # extract values to remove index for calculations
Y = X[2:] - 2 * X[1:-1] + X[:-2]
```

(i) In the following table we present the autocorrelations, acf, and the partial autocorrelations, pacf, of Y. For a visualisation see Figure 6.3. {fig:acf-pacf-dow-second-order}

```
# use argument alpha to return confidence intervals
y_acf, ci_acf = acf(Y, nlags=11, fft=True, alpha=0.05)
y_pacf, ci_pacf = pacf(Y, nlags=11, alpha=0.05)

# determine if values are significantly different from zero
def is_significant(y, ci):
    return not (ci[0] < 0 < ci[1])

s_acf = [is_significant(y, ci) for y, ci in zip(y_acf, ci_acf)]
s_pacf = [is_significant(y, ci) for y, ci in zip(y_pacf, ci_pacf)]
```

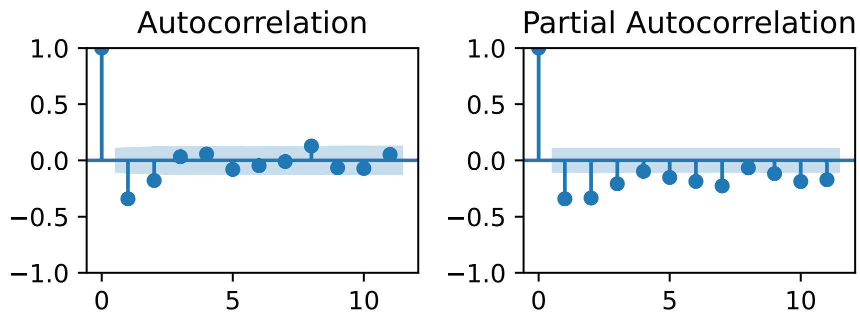


Fig. 6.3 Autocorrelations, acf, and the partial autocorrelations, pacf, of the second order differences of the DOW1941 dataset. {fig:acf-pacf-dow-second-order}

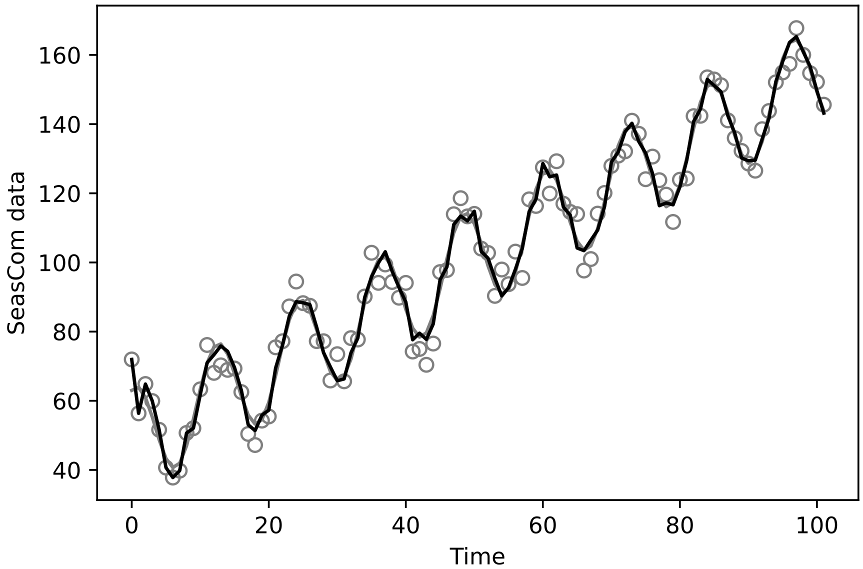
h	acf	S/NS	pacf	S/NS
1	-0.342	S	-0.343	S
2	-0.179	S	-0.337	S
3	0.033	NS	-0.210	S
4	0.057	NS	-0.100	NS
5	-0.080	NS	-0.155	S
6	-0.047	NS	-0.193	S
7	-0.010	NS	-0.237	S
8	0.128	NS	-0.074	NS
9	-0.065	NS	-0.127	S
10	-0.071	NS	-0.204	S
11	0.053	NS	-0.193	S

S denotes significantly different from 0. NS denotes not significantly different from 0.

(ii) All other correlations are not significant. It seems that the ARIMA(1,2,2) is a good approximation.

{fig:seascom-one-day-ahead-model} **Solution 6.12** In Fig. 6.4 we present the seasonal data SeasCom, and the one-day ahead predictions, We see an excellent prediction.

```
predictedError = mistat.optimalLinearPredictor(seascom_model.resid,
                      10, nlags=9)
predictedTrend = seascom_model.predict(seascom_df)
correctedTrend = predictedTrend + predictedError

fig, ax = plt.subplots()
ax.scatter(seascom_df.index, seascom_df['SeasCom'],
           facecolors='none', edgecolors='grey')
predictedTrend.plot(ax=ax, color='grey')
correctedTrend.plot(ax=ax, color='black')
ax.set_xlabel('Time')
ax.set_ylabel('SeasCom data')
plt.show()
```



{fig:seascom-one-day-ahead-model}

Fig. 6.4 One-day ahead prediction model of SeasCom data set

Chapter 7
Modern analytic methods: Part I

Import required modules and define required functions

```
import warnings
import mistat
import random
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from sklearn.metrics import confusion_matrix, ConfusionMatrixDisplay
from sklearn.metrics import mean_squared_error
from sklearn.tree import DecisionTreeClassifier, DecisionTreeRegressor
from sklearn.ensemble import RandomForestClassifier
from sklearn.preprocessing import KBinsDiscretizer
from sklearn.preprocessing import OneHotEncoder
from sklearn.preprocessing import LabelEncoder
from sklearn.model_selection import train_test_split
from sklearn.naive_bayes import MultinomialNB
from sklearn.metrics import accuracy_score
from xgboost import XGBClassifier
from sklearn.cluster import KMeans
from sklearn.pipeline import make_pipeline

# Uncomment the following if xgboost crashes
import os
os.environ['KMP_DUPLICATE_LIB_OK'] = 'TRUE'
```

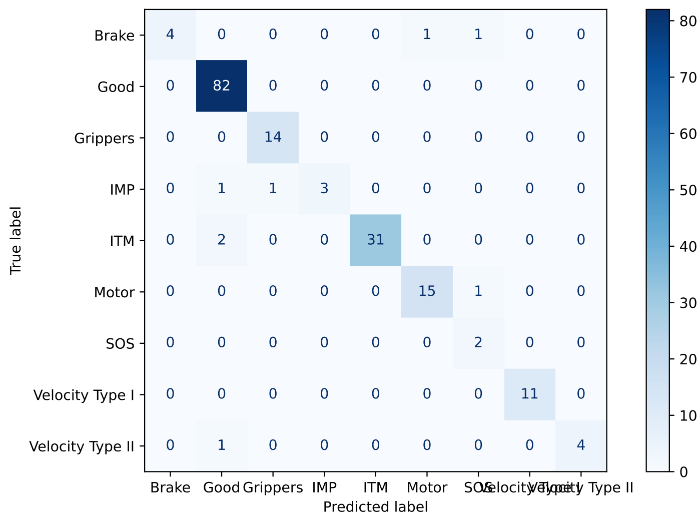
Solution 7.1 Articles reviewing the application of supervised and unsupervised methods can be found online (see e.g. doi:10.1016/j.chaos.2020.110059)

An example for supervised applications is the classification of a COVID-19 based on diagnostic data (see e.g. doi:10.1155/2021/4733167 or doi:10.3390/s21103322)

An example of unsupervised learning is hierarchical clustering to evaluate COVID-19 pandemic preparedness and performance in 180 countries (see doi:10.1016/j.rinp.2021.104639)

Solution 7.2 The decision tree model for `testResult` results in the confusion matrix shown in Figure 7.1. {fig:ex-confusion-matrix-testResult}

```
sensors = mistat.load_data('SENSORS.csv')
predictors = [c for c in sensors.columns if c.startswith('sensor')]
outcome = 'testResult'
X = sensors[predictors]
```



{fig:ex-confusion-matrix-testResult} **Fig. 7.1** Decision tree model to predict testResult from sensor data (Exc. 7.2) exc:sensors-test-result

```
y = sensors[outcome]

# Train the model
clf = DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
clf.fit(X, y)

fig, ax = plt.subplots(figsize=(10, 6))
ConfusionMatrixDisplay.from_estimator(clf, X, y, ax=ax, cmap=plt.cm.Blues)

plt.show()
```

The models classification performance is very good. The test result ‘Good’, which corresponds to status ‘Pass’ is correctly predicted. Most of the other individual test results have also low missclassification rates. The likely reason for this is that each test result is associated with a specific subset of the sensors.

Solution 7.3 In Python

```
# convert the status information into numerical labels
outcome = 'status'
y = sensors[outcome].replace({'Pass': 1, 'Fail': 0})

# Train the model
xgb = XGBClassifier(objective='binary:logistic', subsample=.63,
                    eval_metric='logloss', use_label_encoder=False,
                    seed=1)
xgb.fit(X, y)

# actual in rows / predicted in columns
print('Confusion matrix')
print(confusion_matrix(y, xgb.predict(X)))
```

```
Confusion matrix
[[92  0]
 [ 0 82]]
```

The models confusion matrix is perfect.

```
var_imp = pd.DataFrame({
  'importance': xgb.feature_importances_,
  }, index=predictors)
var_imp = var_imp.sort_values(by='importance', ascending=False)
var_imp['order'] = range(1, len(var_imp) + 1)
print(var_imp.head(10))
var_imp.loc[var_imp.index.isin(['sensor18', 'sensor07', 'sensor21'])]
```

```
          importance  order
sensor18    0.290473      1
sensor54    0.288680      2
sensor53    0.105831      3
sensor55    0.062423      4
sensor61    0.058112      5
sensor48    0.040433      6
sensor07    0.026944      7
sensor12    0.015288      8
sensor03    0.013340      9
sensor52    0.013160     10
```

```
          importance  order
sensor18    0.290473      1
sensor07    0.026944      7
sensor21    0.000000     50
```

The decision tree model uses sensors 18, 7, and 21. The xgboost model identifies sensor 18 as the most important variable. Sensor 7 is ranked 7th, sensor 21 has no importance.

Solution 7.4 Create the random forest classifier model.

```
y = sensors['status']

# Train the model
model = RandomForestClassifier(ccp_alpha=0.012, random_state=0)
model.fit(X, y)

# actual in rows / predicted in columns
print('Confusion matrix')
print(confusion_matrix(y, model.predict(X)))
```

```
Confusion matrix
[[92  0]
 [ 0 82]]
```

The models confusion matrix is perfect.

```
var_imp = pd.DataFrame({
  'importance': model.feature_importances_,
  }, index=predictors)
var_imp = var_imp.sort_values(by='importance', ascending=False)
var_imp['order'] = range(1, len(var_imp) + 1)
print(var_imp.head(10))
var_imp.loc[var_imp.index.isin(['sensor18', 'sensor07', 'sensor21'])]
```

```
          importance  order
sensor61    0.138663      1
sensor18    0.100477      2
sensor53    0.079890      3
sensor52    0.076854      4
sensor46    0.052957      5
sensor50    0.051970      6
sensor44    0.042771      7
sensor48    0.037087      8
sensor24    0.036825      9
sensor21    0.035014     10
```

```
          importance  order
sensor18    0.100477      2
sensor21    0.035014     10
sensor07    0.023162     17
```

The decision tree model uses sensors 18, 7, and 21. Sensor 18 has the second largest importance value, sensor 21 ranks 10th, and sensor 7 is on rank 17.

Solution 7.5 In Python:

```
# convert outcome values from strings into numerical labels
# use le.inverse_transform to convert the predictions to strings
le = LabelEncoder()
y = le.fit_transform(sensors['status'])

train_X, valid_X, train_y, valid_y = train_test_split(X, y,
  test_size=0.4, random_state=2)

dt_model = DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
dt_model.fit(train_X, train_y)

xgb_model = XGBClassifier(objective='binary:logistic', subsample=.63,
                  eval_metric='logloss', use_label_encoder=False,
                  seed=1)
xgb_model.fit(train_X, train_y)

rf_model = RandomForestClassifier(ccp_alpha=0.014, random_state=0)
rf_model.fit(train_X, train_y)

print('Decision tree model')
print(f'Accuracy {accuracy_score(valid_y, dt_model.predict(valid_X)):.3f}')
print(confusion_matrix(valid_y, dt_model.predict(valid_X)))

print('Gradient boosting model')
print(f'Accuracy {accuracy_score(valid_y, xgb_model.predict(valid_X)):.3f}')
print(confusion_matrix(valid_y, xgb_model.predict(valid_X)))

print('Random forest model')
print(f'Accuracy {accuracy_score(valid_y, rf_model.predict(valid_X)):.3f}')
print(confusion_matrix(valid_y, rf_model.predict(valid_X)))
```

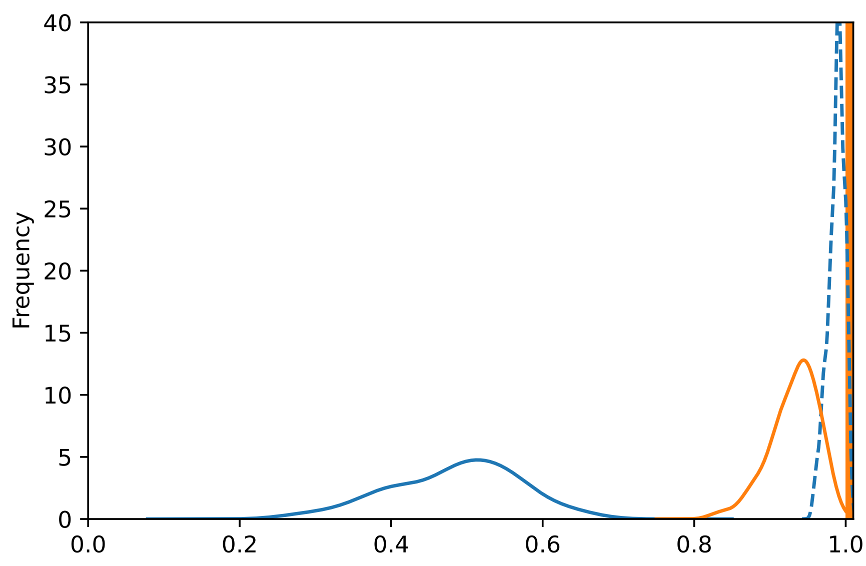
```
Decision tree model
Accuracy 0.900
[[37  2]
 [ 5 26]]
Gradient boosting model
Accuracy 0.957
[[36  3]
 [ 0 31]]
Random forest model
Accuracy 0.986
[[38  1]
 [ 0 31]]
```

The accuracy for predicting the validation set is very good for all three models, with random forest giving the best model with an accuracy of 0.986. The xgboost model has a slightly lower accuracy of 0.957 and the accuracy for the decision tree model is 0.900.

If you change the random seeds or remove them for the various commands, you will see that the accuracies vary and that the order can change.

Solution 7.6 In Python:

```
dt_model = DecisionTreeClassifier(ccp_alpha=0.012)

random_valid_acc = []
random_train_acc = []
org_valid_acc = []
org_train_acc = []
for _ in range(100):
  train_X, valid_X, train_y, valid_y = train_test_split(X, y,
    test_size=0.4)
  dt_model.fit(train_X, train_y)
  org_train_acc.append(accuracy_score(train_y, dt_model.predict(train_X)))
  org_valid_acc.append(accuracy_score(valid_y, dt_model.predict(valid_X)))

  random_y = random.sample(list(train_y), len(train_y))
  dt_model.fit(train_X, random_y)
  random_train_acc.append(accuracy_score(random_y, dt_model.predict(train_X)))
  random_valid_acc.append(accuracy_score(valid_y, dt_model.predict(valid_X)))

ax = pd.Series(random_valid_acc).plot.density(color='C0')
pd.Series(random_train_acc).plot.density(color='C0', linestyle='--',
                                         ax=ax)
pd.Series(org_valid_acc).plot.density(color='C1', ax=ax)
pd.Series(org_train_acc).plot.hist(color='C1', linestyle='--',
                                   ax=ax)
ax.set_ylim(0, 40)
ax.set_xlim(0, 1.01)
plt.show()
```



Solution 7.7 Load data

```
distTower = mistat.load_data('DISTILLATION-TOWER.csv')
predictors = ['Temp1', 'Temp2', 'Temp3', 'Temp4', 'Temp5', 'Temp6',
              'Temp7', 'Temp8', 'Temp9', 'Temp10', 'Temp11', 'Temp12']
outcome = 'VapourPressure'
Xr = distTower[predictors]
yr = distTower[outcome]
```

(i) Split dataset into train and validation set

```
train_X, valid_X, train_y, valid_y = train_test_split(Xr, yr,
  test_size=0.2, random_state=2)
```

{fig:exc-cpp-pruning} **(ii)** Determine model performance for different tree complexity along the dependence of tree depth on ccp parameter; see Figure 7.2.

```
# Code to analyze tree depth vs alpha
model = DecisionTreeRegressor(random_state=0)
path = model.cost_complexity_pruning_path(Xr, yr)
ccp_alphas, impurities = path.ccp_alphas, path.impurities
mse = []
mse_train = []
for ccp_alpha in ccp_alphas:
    model = DecisionTreeRegressor(random_state=0, ccp_alpha=ccp_alpha)
    model.fit(train_X, train_y)
    mse.append(mean_squared_error(valid_y, model.predict(valid_X)))
    mse_train.append(mean_squared_error(train_y, model.predict(train_X)))
ccp_alpha = ccp_alphas[np.argmin(mse)]
```

{fig:exc-cpp-pruning} The smallest validation set error is obtained for `ccp_alpha` = 0.372. The dependence of training and validation error on `ccp_alpha` is shown in Figure 7.2.

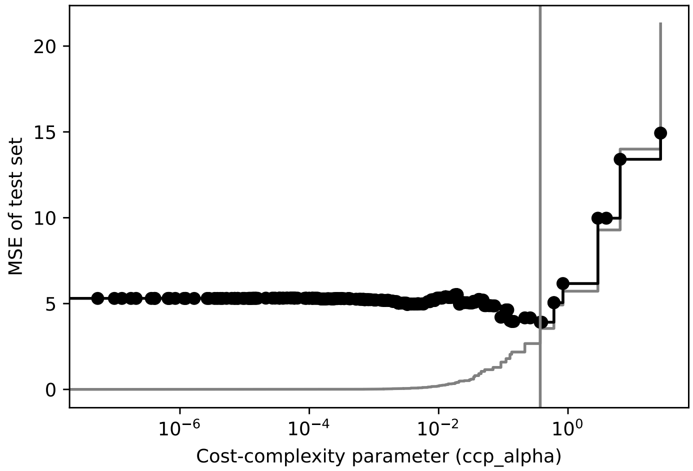
{fig:exc-dtreeviz-visualization} **(iii)** The final model is visualized using `dtreeviz` in Figure 7.3.

```
# the dtreeviz methods requires the classifier to be trained with a numerical
# representation of the classes
# Train the model
model = DecisionTreeRegressor(ccp_alpha=ccp_alpha, random_state=0)
model.fit(Xr, yr)

viz = dtreeviz(model, Xr, yr,
               target_name=outcome,
               feature_names=Xr.columns)
viz2pdf(viz, 'compiled/figures/Chap008_ex_dtreeviz_regressor.pdf')
```

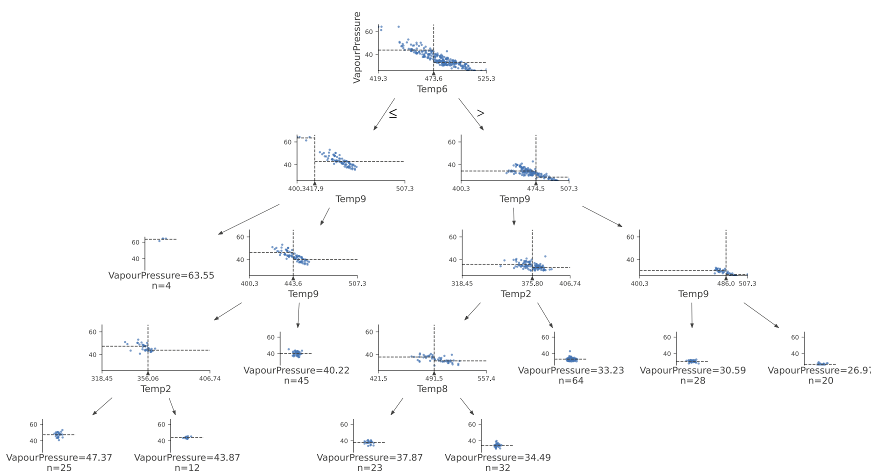
```
findfont: Font family ['Arial'] not found. Falling back to DejaVu
Sans.
```

{fig:nb-binning-performance} **Solution 7.8** Vary the number of bins and binning strategy. The influence of the two model parameter on model performance is shown in Figure 7.4.

```
y = sensors['status']

results = []
for strategy in 'uniform', 'quantile', 'kmeans':
  for n_bins in range(2, 11):
```

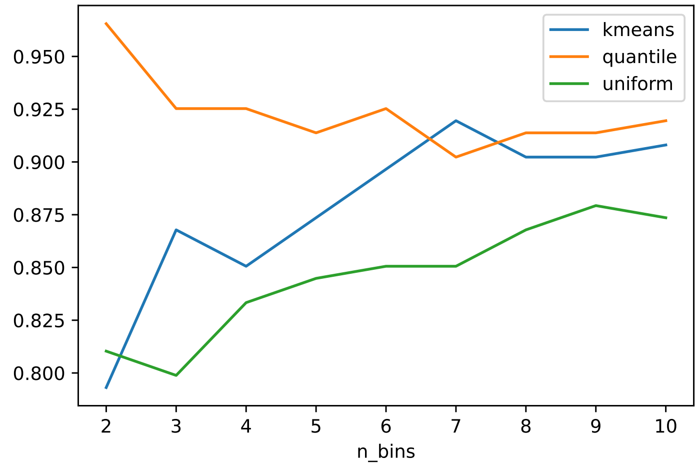


{fig:exc-cpp-pruning} **Fig. 7.2** Decision tree regressor complexity as a function of `ccp_alpha`. The validation set error is shown in black, the training set error in grey (Exercise exc:dec-tree-reg-distillation 7.7)



{fig:exc-dtreeviz-visualization} **Fig. 7.3** Decision tree visualization of regression tree (Exercise exc:dec-tree-reg-distillation 7.7)

```
    kbinsDiscretizer = KBinsDiscretizer(encode='ordinal',
      strategy=strategy, n_bins=n_bins)
    X_binned = kbinsDiscretizer.fit_transform(X)
    nb_model = MultinomialNB()
    nb_model.fit(X_binned, y)
    results.append({'strategy': strategy, 'n_bins': n_bins,
       'accuracy': accuracy_score(y, nb_model.predict(X_binned))})
results = pd.DataFrame(results)
fig, ax = plt.subplots()
for key, group in results.groupby('strategy'):
  group.plot(x='n_bins', y='accuracy', label=key, ax=ax)
```



{fig:nb-binning-performance} **Fig. 7.4** Influence of number of bins and binning strategy on model performance for the sensors data with status as outcome

---

The `quantile` binning strategy (for each feature, each bin has the same number of data points) with splitting each column into two bins leads to the best performing model. With this strategy, performance declines with increasing number of bins. The `uniform` (for each feature, each bin has the same width) and the `kmeans` (for each feature, a k-means clustering is used to bin the feature) strategies on the other hand, show increasing performance with increasing number of bins.

The confusion matrix for the best performing models is:

```
kbinsDiscretizer = KBinsDiscretizer(encode='ordinal',
  strategy='quantile', n_bins=2)
X_binned = kbinsDiscretizer.fit_transform(X)
nb_model = MultinomialNB()
nb_model.fit(X_binned, y)
print('Confusion matrix')
print(confusion_matrix(y, nb_model.predict(X_binned)))
```

```
Confusion matrix
[[87  5]
 [ 1 81]]
```

The decision tree model missclassified three of the ‘Pass’ data points as ‘Fail’. The Naïve Bayes model on the other hand missclassifies six data points. However, five of these are ‘Pass’ and predicted as ‘Fail’. Depending on your use case, you may prefer a model with more false negatives or false positives.

Solution 7.9 In Python:

```
from sklearn.cluster import AgglomerativeClustering
from sklearn.preprocessing import StandardScaler
```

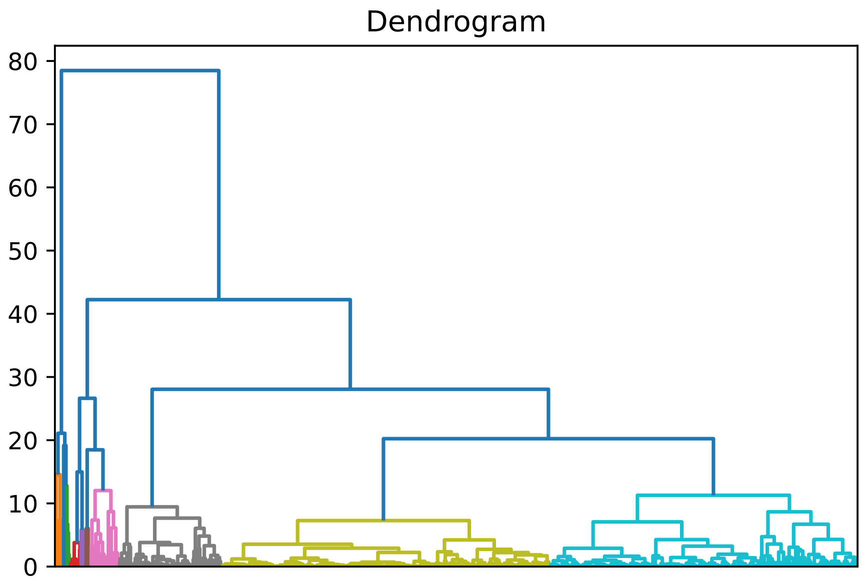
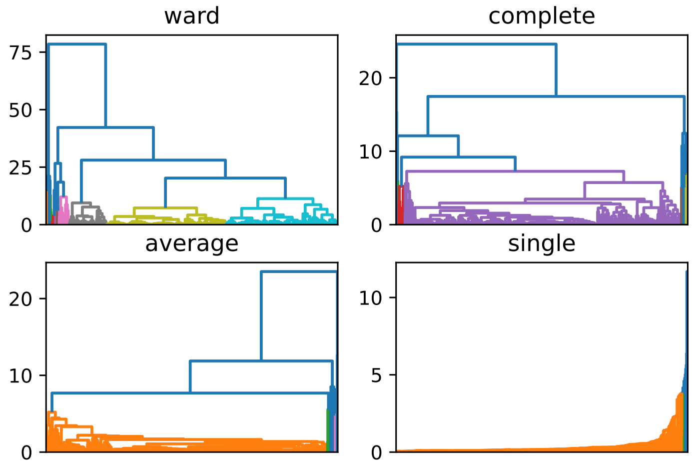


Fig. 7.5 Hierarchical clustering of food data set using Ward clustering {fig:food-ward-10-clusters}

```
from mistat import plot_dendrogram

food = mistat.load_data('FOOD.csv')

scaler = StandardScaler()
model = AgglomerativeClustering(n_clusters=10, compute_distances=True)

X = scaler.fit_transform(food)
model = model.fit(X)
fig, ax = plt.subplots()
plot_dendrogram(model, ax=ax)
ax.set_title('Dendrogram')
ax.get_xaxis().set_ticks([])
plt.show()
```

Solution 7.10 In Python:

```
food = mistat.load_data('FOOD.csv')
scaler = StandardScaler()
X = scaler.fit_transform(food)

fig, axes = plt.subplots(ncols=2, nrows=2)

for linkage, ax in zip(['ward', 'complete', 'average', 'single'], axes.flatten()):
  model = AgglomerativeClustering(n_clusters=10, compute_distances=True,
    linkage=linkage)
  model = model.fit(X)
  plot_dendrogram(model, ax=ax)
  ax.set_title(linkage)
```



{fig:food-compare-linkage} **Fig. 7.6** Comparison of different linkage methods for hierarchical clustering of food data set

```
  ax.get_xaxis().set_ticks([])
plt.show()
```

{fig:food-compare-linkage} The comparison of the different linkage methods is shown in Figure 7.6. We can see that Ward's clustering gives the most balanced clusters; three bigger clusters and seven small clusters. Complete, average, and single linkage lead to one big cluster.

Solution 7.11 We determine 10 clusters using K-means clustering.

```
sensors = mistat.load_data('SENSORS.csv')
predictors = [c for c in sensors.columns if c.startswith('sensor')]
outcome = 'status'
X = sensors[predictors]

scaler = StandardScaler()
X = scaler.fit_transform(X)
model = KMeans(n_clusters=10, random_state=1).fit(X)
```

Combine the information and analyse cluster membership by status.

```
df = pd.DataFrame({
  'status': sensors['status'],
  'testResult': sensors['testResult'],
  'cluster': model.predict(X),
})

for status, group in df.groupby('status'):
  print(f'Status {status}')
  print(group['cluster'].value_counts())
```

```
Status Fail
8    19
0    15
```

```
4    13
1    13
9    13
3    10
5     5
7     2
6     1
2     1
Name: cluster, dtype: int64
Status Pass
1    48
8    34
Name: cluster, dtype: int64
```

There are several clusters that contain only ‘Fail’ data points. They correspond to specific sensor value combinations that are very distinct to the sensor values during normal operation. The ‘Pass’ data points are found in two clusters. Both of these clusters contain also ‘Fail’ data points.

Analyse cluster membership by testResult.

```
print('Number of clusters by testResult')
for cluster, group in df.groupby('cluster'):
  print(f'Cluster {cluster}')
  print(group['testResult'].value_counts())
  print()
```

```
Number of clusters by testResult
Cluster 0
ITM    15
Name: testResult, dtype: int64

Cluster 1
Good        48
Brake        6
IMP          4
Grippers     1
Motor        1
ITM          1
Name: testResult, dtype: int64

Cluster 2
SOS    1
Name: testResult, dtype: int64

Cluster 3
Velocity Type I    10
Name: testResult, dtype: int64

Cluster 4
Grippers    10
ITM          3
Name: testResult, dtype: int64

Cluster 5
Velocity Type II    5
Name: testResult, dtype: int64

Cluster 6
SOS    1
Name: testResult, dtype: int64

Cluster 7
Grippers    2
Name: testResult, dtype: int64
```

```
Cluster 8
Good               34
Motor              15
Grippers            1
IMP                 1
ITM                 1
Velocity Type I     1
Name: testResult, dtype: int64

Cluster 9
ITM    13
Name: testResult, dtype: int64
```

We can see that some of the test results are only found in one or two clusters.

Solution 7.12 The `scikit-learn` K-means clustering method can return either the cluster centers or the distances of a data point to all the cluster centers. We evaluate both as features for classification.

```
# Data preparation
sensors = mistat.load_data('SENSORS.csv')
predictors = [c for c in sensors.columns if c.startswith('sensor')]
outcome = 'status'
X = sensors[predictors]
scaler = StandardScaler()
X = scaler.fit_transform(X)
y = sensors[outcome]
```

First, classifying data points based on the cluster center. In order to use that information in a classifier, we transform the cluster center information using on-hot-encoding.

```
# Iterate over increasing number of clusters
results = []
clf = DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
for n_clusters in range(2, 20):
  # fit a model and assign the data to clusters
  model = KMeans(n_clusters=n_clusters, random_state=1)
  model.fit(X)
  Xcluster = model.predict(X)
  # to use the cluster number in a classifier, use one-hot encoding
  # it's necessary to reshape the vector of cluster numbers into a column vector
  Xcluster = OneHotEncoder().fit_transform(Xcluster.reshape(-1, 1))

  # create a decision tree model and determine the accuracy
  clf.fit(Xcluster, y)
  results.append({'n_clusters': n_clusters,
                  'accuracy': accuracy_score(y, clf.predict(Xcluster))})
ax = pd.DataFrame(results).plot(x='n_clusters', y='accuracy')
ax.set_ylim(0.5, 1)
plt.show()
pd.DataFrame(results).round(3)
```

```
    n_clusters  accuracy
0            2     0.529
1            3     0.805
2            4     0.805
3            5     0.799
4            6     0.793
5            7     0.897
```

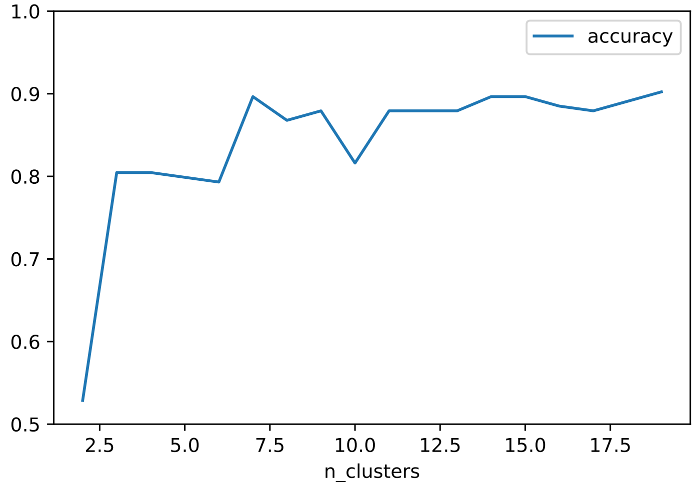


Fig. 7.7 Dependence of accuracy on number of clusters using cluster membership as feature (Exc. 7.12)

```
6          8     0.868
7          9     0.879
8         10     0.816
9         11     0.879
10        12     0.879
11        13     0.879
12        14     0.897
13        15     0.897
14        16     0.885
15        17     0.879
16        18     0.891
17        19     0.902
```

The accuracies are visualized in Figure 7.7. We see that splitting the dataset into 7 clusters gives a classification model with an accuracy of about 0.9.

Next we use the distance to the cluster centers as variable in the classifier.

```
results = []
clf = DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
for n_clusters in range(2, 20):
  # fit a model and convert data to distances
  model = KMeans(n_clusters=n_clusters, random_state=1)
  model.fit(X)
  Xcluster = model.transform(X)

  # create a decision tree model and determine the accuracy
  clf.fit(Xcluster, y)
  results.append({'n_clusters': n_clusters,
                  'accuracy': accuracy_score(y, clf.predict(Xcluster))})
ax = pd.DataFrame(results).plot(x='n_clusters', y='accuracy')
ax.set_ylim(0.5, 1)
plt.show()
pd.DataFrame(results).round(3)
```

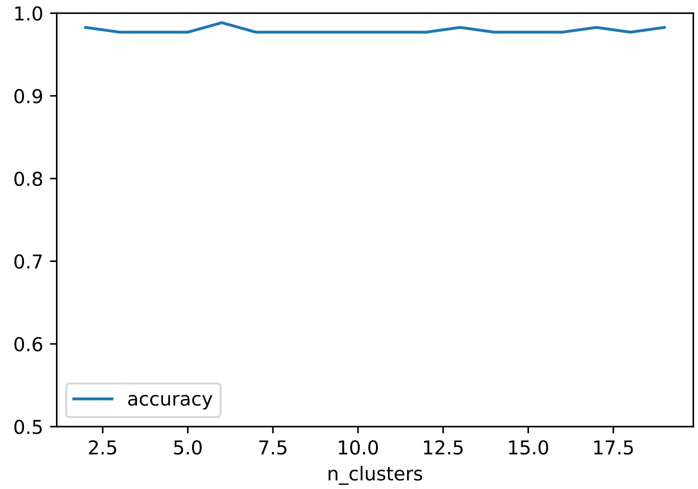


Fig. 7.8 Dependence of accuracy on number of clusters using distance to cluster center as feature (Exc. 7.12)

	n_clusters	accuracy
0	2	0.983
1	3	0.977
2	4	0.977
3	5	0.977
4	6	0.989
5	7	0.977
6	8	0.977
7	9	0.977
8	10	0.977
9	11	0.977
10	12	0.977
11	13	0.983
12	14	0.977
13	15	0.977
14	16	0.977
15	17	0.983
16	18	0.977
17	19	0.983

The accuracies of all models are very high. The largest accuracy is achived for 6 clusters.

Based on these results, we would design the procedure using the decision tree classifier combined with K-means clustering into six clusters. Using `scikit-learn`, we can define the full procedure as a single pipeline as follows:

```
pipeline = make_pipeline(
  StandardScaler(),
  KMeans(n_clusters=6, random_state=1),
  DecisionTreeClassifier(ccp_alpha=0.012, random_state=0)
)
X = sensors[predictors]
y = sensors[outcome]
```

```
process = pipeline.fit(X, y)
print('accuracy', accuracy_score(y, process.predict(X)))
print('Confusion matrix')
print(confusion_matrix(y, process.predict(X)))
```

```
accuracy 0.9885057471264368
Confusion matrix
[[91  1]
 [ 1 81]]
```

The final model has two missclassified data points.

Chapter 8
Modern analytic methods: Part II

Import required modules and define required functions

```
import mistat
import networkx as nx
from pgmpy.estimators import HillClimbSearch
import pandas as pd
import numpy as np
from matplotlib import pyplot as plt
```

Solution 8.1 Load the data and convert to `FDataGrid`.

```
from skfda import FDataGrid
from skfda.representation.interpolation import SplineInterpolation

dissolution = mistat.load_data('DISSOLUTION.csv')

# convert the data to FDataGrid
data = []
labels = []
names = []
for label, group in dissolution.groupby('Label'):
  data.append(group['Data'].values)
  labels.append('Reference' if label.endswith('R') else 'Test')
  names.append(label)
labels = np.array(labels)
grid_points = np.array(sorted(dissolution['Time'].unique()))
fd = FDataGrid(np.array(data), grid_points,
       dataset_name='Dissolution',
       argument_names=['Time'],
       coordinate_names=['Dissolution'])
```

```
warning in stationary: failed to import cython module: falling back to
numpy
warning in coregionalize: failed to import cython module: falling back
to numpy
warning in choleskies: failed to import cython module: falling back to
numpy
```

Use shift registration to align the dissolution data with spline interpolation of order 1, 2, and 3.

{fig:fdaDissolutionSplineComparison}
Fig. 8.1 Mean dissolution curves for reference and test tablets derived using spline interpolation of order 1, 2, and 3.

```
from skfda.preprocessing.registration import ShiftRegistration
shift_registration = ShiftRegistration()

fd_registered = {}
for order in (1, 2, 3):
    fd.interpolation = SplineInterpolation(interpolation_order=order)
    fd_registered[order] = shift_registration.fit_transform(fd)
```

For each of the three registered datasets, calculate the mean dissolution curves for reference and test tablets and plot the results.

```
from skfda.exploratory import stats

group_colors = {'Reference': 'grey', 'Test': 'black'}

fig, axes = plt.subplots(ncols=3, figsize=(8, 3))
for ax, order in zip(axes, (1, 2, 3)):
    mean_ref = stats.mean(fd_registered[order][labels=='Reference'])
    mean_test = stats.mean(fd_registered[order][labels=='Test'])
    means = mean_ref.concatenate(mean_test)
    means.plot(axes=[ax], group=['Reference', 'Test'], group_colors=group_colors)
    ax.set_title(f'Order {order}')
plt.tight_layout()
```

{fig:fdaDissolutionSplineComparison}
The dissolution curves are shown in Figure 8.1. We can see in all three graphs, that the test tablets show a slightly faster dissolution than the reference tablets. If we compare the shape of the curves, the curves for the linear splines interpolation shows a levelling off with time. In the quadratic spline interpolation result, the dissolution curves go through a maximum. This behaviour is unrealistic. The cubic spline interpolation also leads to unrealistic curves that first level of and then start to increase again.

Solution 8.2 (i)

```
import skfda
from skfda import FDataGrid

pinchraw = skfda.datasets.fetch_cran('pinchraw', 'fda')['pinchraw']
pinchtime = skfda.datasets.fetch_cran('pinch', 'fda')['pinchtime']
```

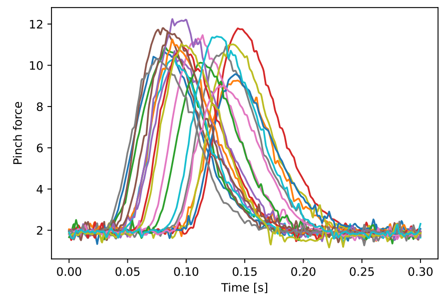


Fig. 8.2 Twenty measurements of pinch force {fig:fdaPinchforceOriginal}

```
fd = FDataGrid(pinchraw.transpose(), pinchtime)
```

Note that the measurement data need to be transposed.
(ii)

```
fig = fd.plot()
ax = fig.axes[0]
ax.set_xlabel('Time [s]')
ax.set_ylabel('Pinch force')
plt.show()
```

Figure 8.2 shows the measure pinch forces. We can see that the start of the force varies from 0.025 to 0.1 seconds. This makes it difficult to compare the shape of the curves. The shapes of the individual curves is not symmetric with a faster onset of the force and slower decline. {fig:fdaPinchforceOriginal}

(iii) We create a variety of smoothed version of the dataset to explore the effect of varying the `smoothing_parameter`.

```
import itertools
from skfda.preprocessing.smoothing.kernel_smoothers import NadarayaWatsonSmoother

def plotSmoothData(fd, smoothing_parameter, ax):
    smoother = NadarayaWatsonSmoother(smoothing_parameter=smoothing_parameter)
    fd_smooth = smoother.fit_transform(fd)
    _ = fd_smooth.plot(axes=[ax])
    ax.set_title(f'Smoothing parameter {smoothing_parameter}')
    ax.set_xlabel('Time')
    ax.set_ylabel('Pinch force')

fig, axes = plt.subplots(ncols=2, nrows=2)
axes = list(itertools.chain(*axes))  # flatten list of lists
for i, sp in enumerate([0.03, 0.01, 0.001, 0.0001]):
    plotSmoothData(fd, sp, axes[i])
plt.tight_layout()
```

Figure 8.3 shows smoothed measurement curves for a variety of `smoothing_parameter` {fig:fdaPinchforceSmoothing} values. If values are too large, the data are oversmoothed and the asymmetric shape of the curves is lost. With decreasing values, the shape is reproduced better but the curves are getting noisier again. We select 0.005 as the smoothing parameter.

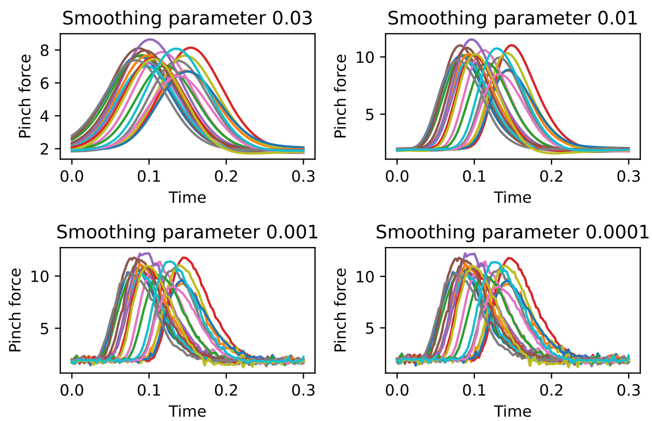


Fig. 8.3 Effect of smoothing parameter on measurement curves {fig:fdaPinchforceSmoothing}

```
smoother = NadarayaWatsonSmoother(smoothing_parameter=0.005)
fd_smooth = smoother.fit_transform(fd)
```

(iii) We first determine the maxima of the smoothed curves:

```
max_idx = fd_smooth.data_matrix.argmax(axis=1)
landmarks = [pinchtime[idx] for idx in max_idx]
```

Use the landmarks to shift register the measurements:

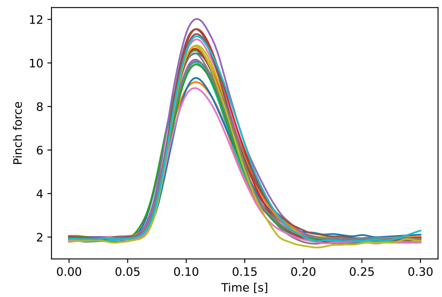
```
from skfda.preprocessing.registration import landmark_shift
fd_landmark = landmark_shift(fd_smooth, landmarks)
```

{fig:fdaPinchforceRegistered} **(iv)** Figure 8.4 shows the measurements after smoothing and landmark shift registration.

```
fig = fd_landmark.plot()
ax = fig.axes[0]
ax.set_xlabel('Time [s]')
ax.set_ylabel('Pinch force')
plt.show()
```

Solution 8.3 **(i)** Load the data

```
import skfda

moisturespectrum = skfda.datasets.fetch_cran('Moisturespectrum', 'fds')
moisturevalues = skfda.datasets.fetch_cran('Moisturevalues', 'fds')

frequencies = moisturespectrum['Moisturespectrum']['x']
spectra = moisturespectrum['Moisturespectrum']['y']
moisture = moisturevalues['Moisturevalues']
```



{fig:fdaPinchforceRegistered} **Fig. 8.4** Registered measurement curves of the **Pinch** dataset

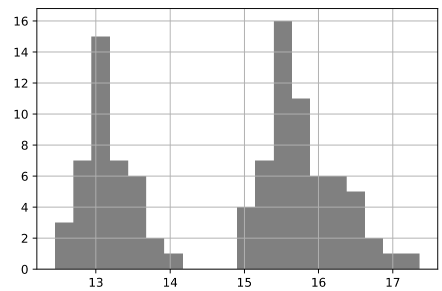


Fig. 8.5 Histogram of moisture content {fig:fdaMoistureDistribution}

(ii) We can use a histogram to look at the distribution of the moisture values.

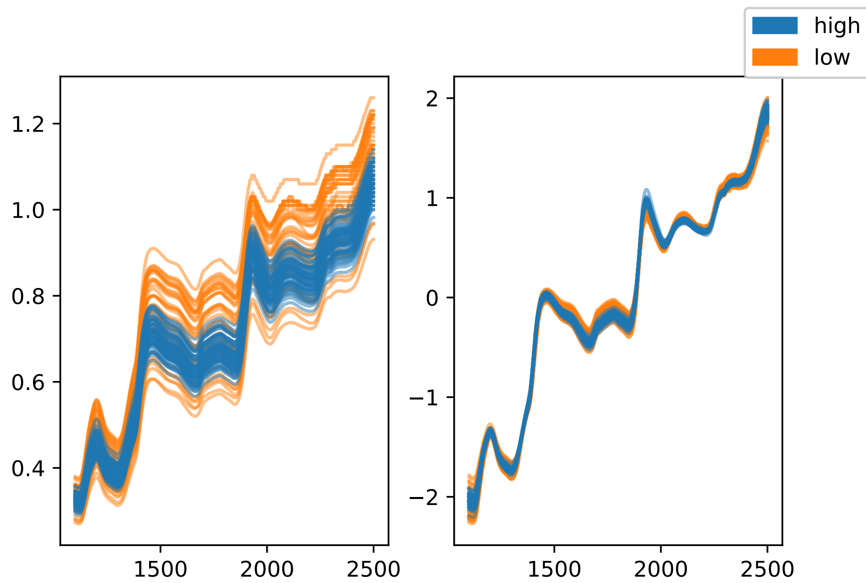
```
_ = pd.Series(moisture).hist(bins=20, color='grey', label='Moisture content')
```

Figure 8.5 shows a bimodal distribution of the moisture content with a clear separation of the two peaks. Based on this, we select 14.5 as the threshold to separate into high and low moisture content. {fig:fdaMoistureDistribution}

```
moisture_class = ['high' if m > 14.5 else 'low' for m in moisture]
```

(iii - iv) The `spectrum` information is already in the array format required for the `FDataGrid` class. In order to do this, we need to transpose the spectrum information. As we can see in the left graph of Figure 8.6, the spectra are not well aligned but show a large spread in the intensities. This is likely due to the difficulty in having a clearly defined concentration between samples. In order to reduce this variation, we can transform the intensities by dividing the intensities by the mean of each sample. {fig:fdaMoistureSpectrum}

```
intensities = spectra.transpose()
fd = skfda.FDataGrid(intensities, frequencies)

# divide each sample spectrum by it's mean intensities
intensities_normalized = (intensities - intensities.mean(dim='dim_0')) / intensities.std(dim='dim_0')
fd_normalized = skfda.FDataGrid(intensities_normalized, frequencies)
```



{fig:fdaMoistureSpectrum} **Fig. 8.6** Near-infrared spectra of the moisture dataset. Left: raw spectra, Right: normalized spectra

Code for plotting the spectra:

```
fig, axes = plt.subplots(ncols=2)
_ = fd.plot(axes=axes[0], alpha=0.5,
            # color lines by moisture class
            group=moisture_class, group_names={'high': 'high', 'low': 'low'})
_ = fd_normalized.plot(axes=axes[1], alpha=0.5,
            group=moisture_class, group_names={'high': 'high', 'low': 'low'})
```

{fig:fdaMoistureSpectrum} As we can see in right graph of Figure 8.6, the normalized spectra are now better aligned. We also see that the overall shape of the spectra is fairly consistent between samples.

(v) We repeat the model building both for the original and normalized spectra 50 times. At each iteration, we split the data set into training and test sets (50-50), build the model with the training set and measure accuracy using the test set. By using the same random seed for splitting the original and the normalized dataset, we can better compare the models. The accuracies from the 50 iteration are compared in
{fig:fdaMoistureAccuracies} Figure 8.7.

```
from skfda.ml.classification import KNeighborsClassifier
from sklearn.model_selection import train_test_split
from sklearn.metrics import accuracy_score, confusion_matrix

accuracies = []
for rs in range(50):
    X_train, X_test, y_train, y_test = train_test_split(fd,
```

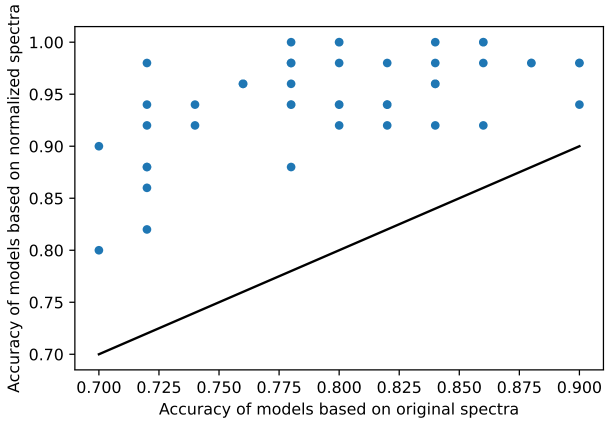


Fig. 8.7 Accuracies of classification models based original and normalized spectra. The line indicates equal performance. {fig:fdaMoistureAccuracies}

```
        moisture_class, random_state=rs, test_size=0.5)
    knn_original = KNeighborsClassifier()
    knn_original.fit(X_train, y_train)
    acc_original = accuracy_score(y_test, knn_original.predict(X_test))

    X_train, X_test, y_train, y_test = train_test_split(fd_normalized,
        moisture_class, random_state=rs, test_size=0.5)
    knn_normalized = KNeighborsClassifier()
    knn_normalized.fit(X_train, y_train)
    acc_normalized = accuracy_score(y_test, knn_normalized.predict(X_test))
    accuracies.append({
        'original': acc_original,
        'normalized': acc_normalized,
    })
accuracies = pd.DataFrame(accuracies)
ax  = accuracies.plot.scatter(x='original', y='normalized')
_ = ax.plot([0.7, 0.9], [0.7, 0.9], color='black')
ax.set_xlabel('Accuracy of models based on original spectra')
ax.set_ylabel('Accuracy of models based on normalized spectra')
plt.show()

# mean of accuracies
mean_accuracy = accuracies.mean()
mean_accuracy
```

```
original      0.7976
normalized    0.9468
dtype: float64
```

Figure 8.7 clearly shows that classification models based on the normalized spectra achieve better accuracies. The mean accuracy increases from 0.8 to 0.95. {fig:fdaMoistureAccuracies}

Solution 8.4 **(i)** See solution for Exercise 8.3. {exc:fda-moisture-classification}

(ii) We use the method `skfda.ml.regression.KNeighborsRegressor` to build the regression models.

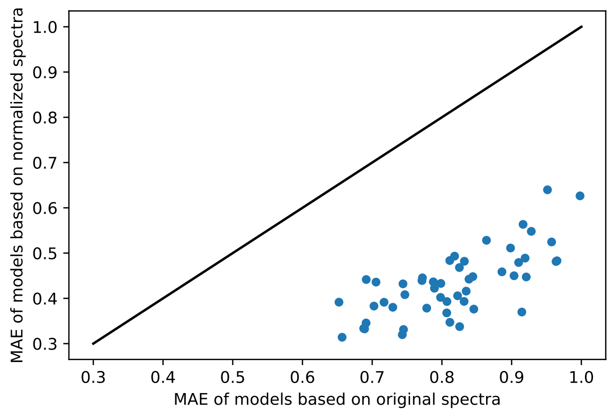


Fig. 8.8 Mean absolute error of regression models using original and normalized spectra. The line indicates equal performance.

{fig:fdaMoistureMAE}

```
from skfda.ml.regression import KNeighborsRegressor
from sklearn.model_selection import train_test_split
from sklearn.metrics import mean_absolute_error

mae = []
for rs in range(50):
    X_train, X_test, y_train, y_test = train_test_split(fd,
        moisture, random_state=rs, test_size=0.5)
    knn_original = KNeighborsRegressor()
    knn_original.fit(X_train, y_train)
    mae_original = mean_absolute_error(y_test, knn_original.predict(X_test))

    X_train, X_test, y_train, y_test = train_test_split(fd_normalized,
        moisture, random_state=rs, test_size=0.5)
    knn_normalized = KNeighborsRegressor()
    knn_normalized.fit(X_train, y_train)
    mae_normalized = mean_absolute_error(y_test, knn_normalized.predict(X_test))
    mae.append({
        'original': mae_original,
        'normalized': mae_normalized,
    })
mae = pd.DataFrame(mae)
ax  = mae.plot.scatter(x='original', y='normalized')
ax.plot([0.3, 1.0], [0.3, 1.0], color='black')
ax.set_xlabel('MAE of models based on original spectra')
ax.set_ylabel('MAE of models based on normalized spectra')
plt.show()

# mean of MAE
mean_mae = mae.mean()
mean_mae
```

```
original      0.817016
normalized    0.433026
dtype: float64
```

{fig:fdaMoistureMAE} Figure 8.8 clearly shows that regression models based on the normalized spectra achieve better performance. The mean absolute error descrease from 0.82 to 0.43.

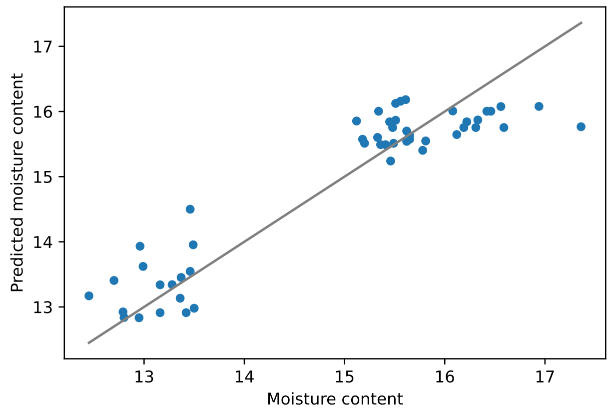


Fig. 8.9 Actual versus predicted moisture content

{fig:fdaMoisturePredictions}

(iii) We use the last regression model from **(ii)** to create a plot of actual versus predicted moisture content for the test data.

```
y_pred = knn_normalized.predict(X_test)
predictions = pd.DataFrame({'actual': y_test, 'predicted': y_pred})
minmax = [min(*y_test, *y_pred), max(*y_test, *y_pred)]

ax = predictions.plot.scatter(x='actual', y='predicted')
ax.set_xlabel('Moisture content')
ax.set_ylabel('Predicted moisture content')
ax.plot(minmax, minmax, color='grey')
plt.show()
```

Figure 8.9 shows two clusters of points. One cluster contains the samples with the high moisture content and the other cluster the samples with low moisture content. The clusters are well separated and the predictions are in the typical range for each cluster. However, within a cluster, predictions and actual values are not highly correlated. In other words, while the regression model can distinguish between samples with a high and low moisture content, the moisture content is otherwise not well predicted. There is therefore no advantage of using the regression model compared to the classification model. {fig:fdaMoisturePredictions}

Solution 8.5 **(i)** See solution for Exercise 8.3. {exc:fda-moisture-classification}

(ii) In Python:

```
from skfda.preprocessing.dim_reduction.projection import FPCA

fpca_original = FPCA(n_components=2)
_ = fpca_original.fit(fd)

fpca_normalized = FPCA(n_components=2)
_ = fpca_normalized.fit(fd_normalized)
```

The projections of the spectra can now be visualized:

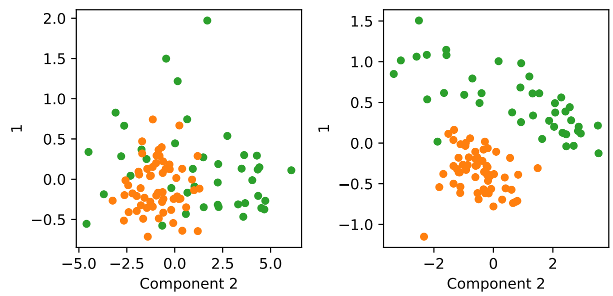


Fig. 8.10 Projection of spectra onto first two principal components. Left: original spectra, right: normalized spectra

{fig:fdaMoisturePCA}

```
def plotFPCA(fpca, fd, ax):
  fpca_df = pd.DataFrame(fpca.transform(fd))
  fpca_df.plot.scatter(x=0, y=1,
      c=['C1' if mc == 'high' else 'C2' for mc in moisture_class], ax=ax)
  ax.set_xlabel('Component 1')
  ax.set_xlabel('Component 2')

fig, axes = plt.subplots(ncols=2, figsize=[6, 3])
plotFPCA(fpca_original, fd, axes[0])
plotFPCA(fpca_normalized, fd_normalized, axes[1])
plt.tight_layout()
```

{fig:fdaMoisturePCA}

Figure 8.10 compares the PCA projections for the original and normalized data. We can see that the second principal component clearly separates the two moisture content classes for the normalized spectra. This is not the case for the original spectra.

Solution 8.6 (i) We demonstrate a solution to this exercise using two blog posts preprocessed and included in the `mistat` package. The content of the posts was converted text with each paragraph on a line. The two blog posts can be loaded as follows:

```
from mistat.nlp import globalWarmingBlogs
blogs = globalWarmingBlogs()
```

The variable `blogs` is a dictionary with labels as keys and text as values. We next split the data into a list of labels and a list of non-empty paragraphs.

```
paragraphs = []
labels = []
for blog, text in blogs.items():
  for paragraph in text.split('\n'):
    paragraph = paragraph.strip()
    if not paragraph: # ignore empty paragraphs
      continue
    paragraphs.append(paragraph)
    labels.append(blog)
```

(ii) Using `CountVectorizer`, transform the list of paragraphs into a document-term matrix (DTM).

```
import re
from sklearn.feature_extraction.text import CountVectorizer

def preprocessor(text):
    text = text.lower()
    text = re.sub(r'\d[\d,]*', '', text)
    text = '\n'.join(line for line in text.split('\n')
                     if not line.startswith('ntsb'))
    return text

vectorizer = CountVectorizer(preprocessor=preprocessor,
                             stop_words='english')
counts = vectorizer.fit_transform(paragraphs)

print('shape of DTM', counts.shape)
print('total number of terms', np.sum(counts))
```

```
shape of DTM (123, 1025)
total number of terms 2946
```

The ten most frequenly occurring terms are:

```
termCounts = np.array(counts.sum(axis=0)).flatten()
topCounts = termCounts.argsort()
terms = vectorizer.get_feature_names_out()
for n in reversed(topCounts[-10:]):
  print(f'{terms[n]} & {termCounts[n]} \\\\')
```

```
global & 63 \\
climate & 59 \\
warming & 57 \\
change & 55 \\
ice & 35 \\
sea & 34 \\
earth & 33 \\
ocean & 29 \\
temperatures & 28 \\
heat & 25 \\
```

(iii) Conversion of counts using TF-IDF.

```
from sklearn.feature_extraction.text import TfidfTransformer

tfidfTransformer = TfidfTransformer(smooth_idf=False, norm=None)
tfidf = tfidfTransformer.fit_transform(counts)
```

(iv)

```
from sklearn.decomposition import TruncatedSVD
from sklearn.preprocessing import Normalizer
svd = TruncatedSVD(5)
norm_tfidf = Normalizer().fit_transform(tfidf)
lsa_tfidf = svd.fit_transform(norm_tfidf)
```

We can analyze the loadings to get an idea of topics.

```
terms = vectorizer.get_feature_names_out()
data = {}
for i, component in enumerate(svd.components_, 1):
  compSort = component.argsort()
  idx = list(reversed(compSort[-10:]))
  data[f'Topic {i}'] = [terms[n] for n in idx]
  data[f'Loading {i}'] = [component[n] for n in idx]
df = pd.DataFrame(data)
```

```
print("{\\tiny")
print(df.style.format(precision=2).hide(axis='index').to_latex(hrules=True))
print("}")
```

Topic 1	Loading 1	Topic 2	Loading 2	Topic 3	Loading 3	Topic 4	Loading 4	Topic 5	Loading 5
change	0.24	ice	0.39	sea	0.25	extreme	0.54	snow	0.42
climate	0.24	sea	0.35	earth	0.24	events	0.32	cover	0.26
global	0.23	sheets	0.27	light	0.21	heat	0.22	sea	0.16
sea	0.22	shrinking	0.19	energy	0.21	precipitation	0.21	climate	0.12
warming	0.21	level	0.17	gases	0.19	earth	0.13	decreased	0.12
ice	0.20	arctic	0.15	ice	0.17	light	0.12	level	0.12
temperature	0.18	ocean	0.13	infrared	0.17	energy	0.12	temperatures	0.11
ocean	0.16	declining	0.10	greenhouse	0.16	gases	0.11	hurricanes	0.11
earth	0.16	levels	0.08	level	0.15	greenhouse	0.10	increase	0.10
extreme	0.15	glaciers	0.07	atmosphere	0.13	infrared	0.10	temperature	0.09

We can identify topics related to sea warming, ice sheets melting, greenhouse effect, extreme weather events, and hurricanes.

(v) Repeat the analysis requesting 10 components in the SVD.

```
svd = TruncatedSVD(10)
norm_tfidf = Normalizer().fit_transform(tfidf)
lsa_tfidf = svd.fit_transform(norm_tfidf)
```

We now get the following topics.

```
terms = vectorizer.get_feature_names_out()
data = {}
for i, component in enumerate(svd.components_, 1):
  compSort = component.argsort()
  idx = list(reversed(compSort[-10:]))
  data[f'Topic {i}'] = [terms[n] for n in idx]
  data[f'Loading {i}'] = [component[n] for n in idx]
df = pd.DataFrame(data)
```

Topic 1	Loading 1	Topic 2	Loading 2	Topic 3	Loading 3	Topic 4	Loading 4	Topic 5	Loading 5
change	0.24	ice	0.39	sea	0.25	extreme	0.54	snow	0.39
climate	0.24	sea	0.35	earth	0.24	events	0.31	cover	0.23
global	0.23	sheets	0.27	energy	0.21	heat	0.24	sea	0.17
sea	0.22	shrinking	0.19	light	0.21	precipitation	0.20	climate	0.12
warming	0.21	level	0.17	gases	0.19	earth	0.12	hurricanes	0.12
ice	0.20	arctic	0.15	ice	0.18	light	0.12	temperatures	0.12
temperature	0.18	ocean	0.13	infrared	0.17	energy	0.11	level	0.12
ocean	0.16	declining	0.10	greenhouse	0.16	gases	0.11	decreased	0.11
earth	0.16	levels	0.08	level	0.15	greenhouse	0.10	temperature	0.10
extreme	0.15	glaciers	0.07	atmosphere	0.12	infrared	0.10	increase	0.10

Topic 6	Loading 6	Topic 7	Loading 7	Topic 8	Loading 8	Topic 9	Loading 9	Topic 10	Loading 10
sea	0.38	snow	0.37	ocean	0.44	glaciers	0.37	responsibility	0.32
level	0.36	cover	0.27	acidification	0.22	retreat	0.25	authorities	0.24
rise	0.17	extreme	0.20	hurricanes	0.19	water	0.22	pollution	0.20
extreme	0.14	decreased	0.15	waters	0.14	glacial	0.22	wildfires	0.19
global	0.11	ocean	0.13	water	0.13	months	0.16	heat	0.19
hurricanes	0.11	warming	0.12	temperatures	0.12	summer	0.13	personal	0.14
events	0.11	events	0.12	coral	0.12	plants	0.13	carbon	0.14
temperature	0.10	acidification	0.10	reefs	0.12	going	0.12	reduce	0.12
impacts	0.10	point	0.10	glaciers	0.11	animals	0.12	arctic	0.11
coastal	0.08	level	0.09	pollution	0.10	stream	0.11	risk	0.10

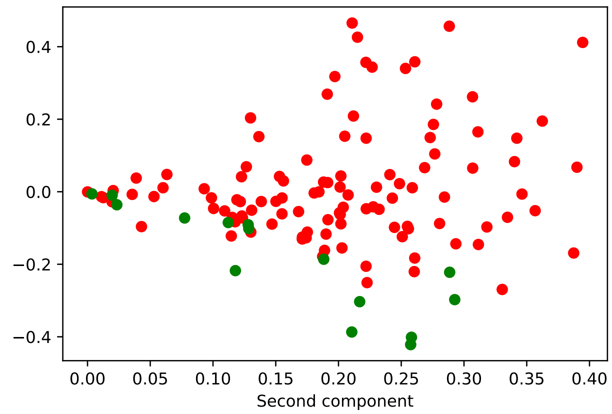


Fig. 8.11 Projection of the documents onto first two singular values. Red: blog post 1, green: blog post 2 {fig:nlpSVD}

The first five topics are identical to the result in **(iv)**. This is an expected property of the SVD.

(vi) Figure 8.11 shows the individual documents projected onto the first two singular values of the LSA. Based on this visualization, we can say that the two documents discuss different aspects of global warming and that blog post 1 contains more details about the area. {fig:nlpSVD}

```
fig, ax = plt.subplots()
blog1 = [label == 'blog-1' for label in labels]
blog2 = [label == 'blog-2' for label in labels]
ax.plot(lsa_tfidf[blog1, 0], lsa_tfidf[blog1, 1], 'ro')
ax.plot(lsa_tfidf[blog2, 0], lsa_tfidf[blog2, 1], 'go')
ax.set_xlabel('First component')
ax.set_xlabel('Second component')
plt.show()
```

Solution 8.7 We follow the same procedure as in Exercise 8.6 using a set of three articles preprocessed and included in the `mistat` package. {exc:nlp-topic-1}

```
from mistat.nlp import covid19Blogs
blogs = covid19Blogs()
```

Determine DTM using paragraphs as documents:

```
paragraphs = []
labels = []
for blog, text in blogs.items():
  for paragraph in text.split('\n'):
    paragraph = paragraph.strip()
    if not paragraph:
      continue
    paragraphs.append(paragraph)
```

```
    labels.append(blog)

def preprocessor(text):
  text = text.lower()
  text = re.sub(r'\d[\d,]*', '', text)
  text = '\n'.join(line for line in text.split('\n')
                   if not line.startswith('ntsb'))
  return text

vectorizer = CountVectorizer(preprocessor=preprocessor, stop_words='english')
counts = vectorizer.fit_transform(paragraphs)
```

TF-IDF transformation

```
tfidfTransformer = TfidfTransformer(smooth_idf=False, norm=None)
tfidf = tfidfTransformer.fit_transform(counts)
```

Latent semantic analysis (LSA)

```
svd = TruncatedSVD(10)
tfidf = Normalizer().fit_transform(tfidf)
lsa_tfidf = svd.fit_transform(tfidf)
```

Topics:

```
terms = vectorizer.get_feature_names_out()
data = {}
for i, component in enumerate(svd.components_, 1):
  compSort = component.argsort()
  idx = list(reversed(compSort[-10:]))
  data[f'Topic {i}'] = [terms[n] for n in idx]
  data[f'Loading {i}'] = [component[n] for n in idx]
df = pd.DataFrame(data)
```

Topic 1	Loading 1	Topic 2	Loading 2	Topic 3	Loading 3	Topic 4	Loading 4	Topic 5	Loading 5
labour	0.29	south	0.28	economic	0.24	capacity	0.21	agenda	0.15
covid	0.22	labour	0.27	percent	0.22	financial	0.20	household	0.14
impact	0.19	north	0.22	covid	0.19	firms	0.18	firms	0.14
market	0.19	differences	0.20	gdp	0.17	household	0.17	post	0.14
south	0.18	americas	0.17	impact	0.15	international	0.17	capacity	0.13
america	0.16	channel	0.16	social	0.13	largely	0.14	labour	0.13
pandemic	0.15	covid	0.13	imf	0.13	depends	0.14	financial	0.13
channel	0.15	agenda	0.13	pre	0.12	access	0.13	global	0.12
economic	0.14	welfare	0.10	growth	0.10	state	0.13	international	0.12
north	0.14	impact	0.09	lac	0.10	markets	0.13	largely	0.11

Topic 6	Loading 6	Topic 7	Loading 7	Topic 8	Loading 8	Topic 9	Loading 9	Topic 10	Loading 10
economic	0.30	occupations	0.23	crisis	0.16	poverty	0.17	need	0.14
social	0.17	covid	0.22	labor	0.14	international	0.14	differences	0.12
channel	0.16	asymmetric	0.17	deepen	0.13	inequality	0.14	work	0.12
consequences	0.12	consequences	0.16	levels	0.12	self	0.13	north	0.11
recovery	0.11	transition	0.14	inequality	0.12	informal	0.13	outsiders	0.10
impact	0.11	differences	0.14	poverty	0.11	financial	0.12	insiders	0.10
market	0.10	north	0.13	persistent	0.10	employed	0.12	precarious	0.10
asymmetric	0.10	americas	0.13	adopt	0.10	unbearable	0.12	majority	0.10
labour	0.10	economic	0.13	deleterious	0.10	lightness	0.12	americas	0.10
preservation	0.10	occupation	0.13	facing	0.10	social	0.12	workers	0.09

Looking at the differnt loadings, we can see different topics emerging.

{fig:nlpSVD-covid} In Figure 8.12, we can see that the paragraphs in the article show more overlap
{exc:nlp-topic-1} compared to what we've observed in the Exercise 8.6.

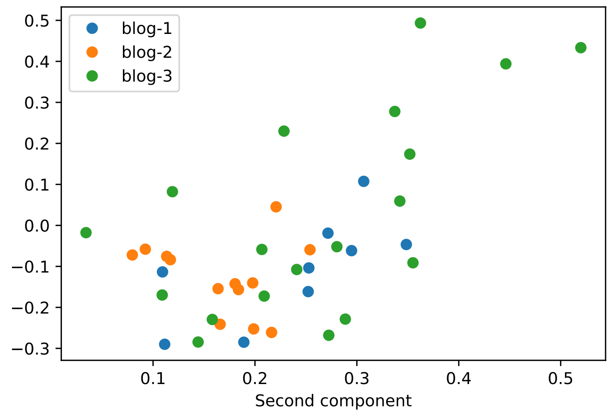


Fig. 8.12 Projection of the documents onto first two singular values. Red: blog post 1, green: blog post 2

{fig:nlpSVD-covid}

```
fig, ax = plt.subplots()
for blog in blogs:
    match = [label == blog for label in labels]
    ax.plot(lsa_tfidf[match, 0], lsa_tfidf[match, 1], 'o', label=blog)
ax.legend()
ax.set_xlabel('First component')
ax.set_xlabel('Second component')
plt.show()
```

Solution 8.8 **(i)** Load and preprocess the data

```
data = mistat.load_data('LAPTOP_REVIEWS')
data['Review'] = data['Review title'] + ' ' + data['Review content']
reviews = data.dropna(subset=['User rating', 'Review title', 'Review content'])
```

(ii) Convert the text representation into a document term matrix (DTM).

```
import re
from sklearn.feature_extraction.text import CountVectorizer
def preprocessor(text):
    text = text.lower()
    text = re.sub(r'\d[\d,]*', '', text)
    return text

vectorizer = CountVectorizer(preprocessor=preprocessor,
                             stop_words='english')
counts = vectorizer.fit_transform(reviews['Review'])
print('shape of DTM', counts.shape)
print('total number of terms', np.sum(counts))
```

```
shape of DTM (7433, 12823)
total number of terms 251566
```

(iii) Convert the counts in the document term matrix (DTM) using TF-IDF.

```
from sklearn.feature_extraction.text import TfidfTransformer

tfidfTransformer = TfidfTransformer(smooth_idf=False, norm=None)
tfidf = tfidfTransformer.fit_transform(counts)
```

(iv) Using scikit-learn's `TruncatedSVD` method, we convert the sparse tfidf matrix to a denser representation.

```
from sklearn.decomposition import TruncatedSVD
from sklearn.preprocessing import Normalizer
svd = TruncatedSVD(20)
tfidf = Normalizer().fit_transform(tfidf)
lsa_tfidf = svd.fit_transform(tfidf)
print(lsa_tfidf.shape)
```

```
(7433, 20)
```

(v) We use logistic regression to classify reviews with a user rating of five as positive and negative otherwise.

```
from sklearn.linear_model import LogisticRegression
from sklearn.metrics import accuracy_score, confusion_matrix
from sklearn.model_selection import train_test_split

outcome = ['positive' if rating == 5 else 'negative'
           for rating in reviews['User rating']]

# split dataset into 60% training and 40% test set
Xtrain, Xtest, ytrain, ytest = train_test_split(lsa_tfidf, outcome,
                                                test_size=0.4, random_state=1)

# run logistic regression model on training
logit_reg = LogisticRegression(solver='lbfgs')
logit_reg.fit(Xtrain, ytrain)

# print confusion matrix and accuracty
accuracy = accuracy_score(ytest, logit_reg.predict(Xtest))
print(accuracy)
confusion_matrix(ytest, logit_reg.predict(Xtest))
```

```
0.7696704774714189
```

```
array([[ 859,  387],
       [ 298, 1430]])
```

The predicted accuracy of the classification model is 0.77.