10.4.ex.14.2

EE24BTECH11025 - GEEDI HARSHA

Question: Formulate the following problem as a pair of equations, and hence find it's solution:

Ritu can row downstream 20 km in 2 hours, and upstream 4 km in 2 hours. Find her speed of rowing in still water and the speed of the current.

Solution: Let's define:

$$v = \text{speed in still water}$$
$$c = \text{speed of current}$$

From given conditions:

$$(v + c) \times 2 = 20$$
$$(v - c) \times 2 = 4$$

Simplifying:

$$v + c = 10$$
$$v - c = 2$$

The above equations can be written in the form $A\mathbf{x} = \mathbf{b}$ Where,

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$\mathbf{x} = \begin{pmatrix} v \\ c \end{pmatrix}$$
$$\mathbf{b} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

The matrix A can be decomposed into:

$$A = L \cdot U$$

where:

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$U = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}.$$

Factorization of LU: Given a matrix A of size $n \times n$, LU decomposition is performed row by row and column by column. The update equations are as follows:

1) Start by initializing $\mathbf{L}$ as the identity matrix $\mathbf{L} = \mathbf{I}$ and $\mathbf{U}$ as a copy of $\mathbf{A}$

2) For each column $j \geq k$, the entries of U in the k-th row are updated as:

$$U_{k,j} = A_{k,j} - \sum_{m=1}^{k-1} L_{k,m} \cdot U_{m,j} \quad \forall \quad j \geq k$$

3) For each row $i > k$, the entries of L in the k-th column are updated as:

$$L_{i,k} = \frac{1}{U_{k,k}} \left(A_{i,k} - \sum_{m=1}^{k-1} L_{i,m} \cdot U_{m,k} \right) \quad \forall \quad i > k$$

The system $A\mathbf{x} = \mathbf{b}$ is transformed into $L \cdot U \cdot \mathbf{x} = \mathbf{b}$. Let $\mathbf{y}$ satisfy $L\mathbf{y} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

Using forward substitution:

$$\begin{aligned} y_1 &= 10 \\ y_1 + y_2 &= 2 \\ y_2 &= -8 \end{aligned}$$

Thus:

$$\mathbf{y} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$$

Next, solve $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$$

Using backward substitution:

$$\begin{aligned} -2c &= -8 \\ c &= 4 \\ v + c &= 10 \\ v &= 6 \end{aligned}$$

Therefore:

$$\begin{aligned} \text{Speed in still water } (v) &= 6 \text{ km/h} \\ \text{Speed of current } (c) &= 4 \text{ km/h} \end{aligned}$$

Alternative Solution using Orthogonal Matrix Properties:

The system of equations can be written in matrix form $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} v \\ c \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

Note that matrix A is orthogonal ($A^T \cdot A = I$). Multiplying both sides by A^T:

$$A^T A\mathbf{x} = A^T \mathbf{b}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

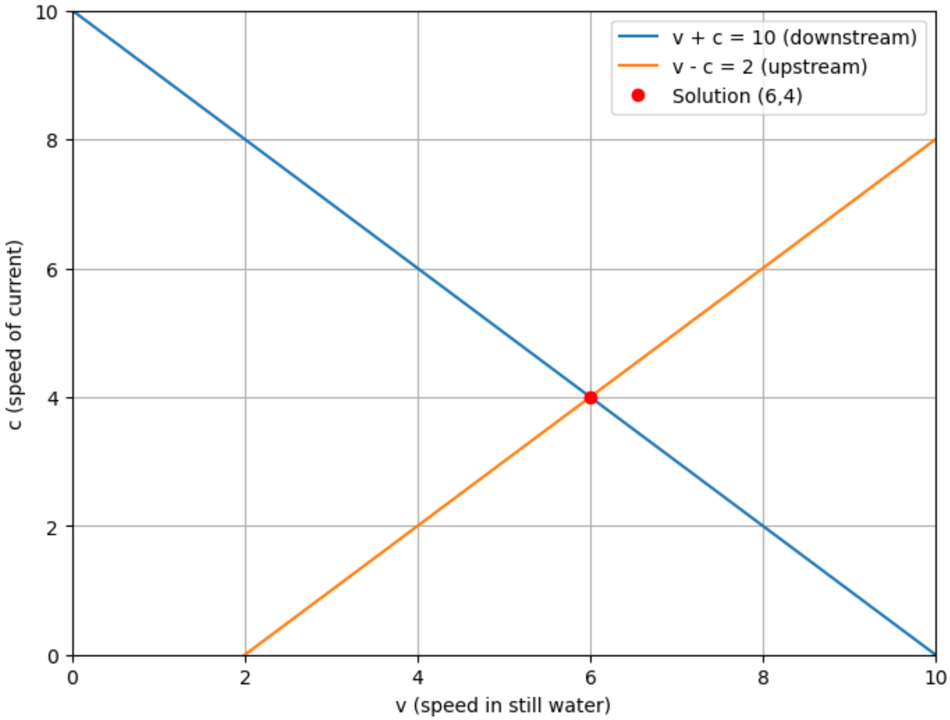
This gives:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} v \\ c \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

Therefore:

$$v = 6 \text{ km/h}$$

$$c = 4 \text{ km/h}$$


v + c = 10 (downstream)
v - c = 2 (upstream)
Solution (6,4)
c (speed of current)
v (speed in still water)
0
2
4
6
8
10