

12.9.ex.2

EE24BTECH11025 - GEEDI HARSHA

Question:

Find the roots of the following equation:

$$\frac{1}{x} - \frac{1}{x-2} = 3, \quad x \neq 0, 2 \quad (0.1)$$

Solution using Completing the Square Method:

(i) Solve $\frac{1}{x} = 3 \quad x \neq 0$

Rearrange the equation:

$$1 = 3x \quad (0.2)$$

$$x = \frac{1}{3} \quad (0.3)$$

Thus, the root is $x = \frac{1}{3}$.

(ii) Solve $\frac{1}{x} - \frac{1}{x-2} = 3 \quad x \neq 0, 2$

Find a common denominator:

$$\frac{(x-2) - x}{x(x-2)} = 3 \quad (0.4)$$

$$\frac{-2}{x(x-2)} = 3 \quad (0.5)$$

Multiply both sides by -1 :

$$\frac{2}{x(x-2)} = -3 \quad (0.6)$$

Multiply both sides by $x(x-2)$:

$$2 = -3x(x-2) \quad (0.7)$$

Rearrange:

$$3x^2 - 6x + 2 = 0 \quad (0.8)$$

Divide by 3:

$$x^2 - 2x + \frac{2}{3} = 0 \quad (0.9)$$

Complete the square:

$$x^2 - 2x = -\frac{2}{3} \quad (0.10)$$

Add $\left(\frac{2}{3}\right)^2 = 1$ to both sides:

$$x^2 - 2x + 1 = -\frac{2}{3} + 1 \quad (0.11)$$

$$(x - 1)^2 = \frac{1}{3} \quad (0.12)$$

Taking square root on both sides:

$$x - 1 = \pm \frac{1}{\sqrt{3}} \quad (0.13)$$

Thus,

$$x = 1 \pm \frac{1}{\sqrt{3}} \quad (0.14)$$

Rationalizing the denominator:

$$x = 1 \pm \frac{\sqrt{3}}{3} \quad (0.15)$$

Thus, the roots are $x = 1 + \frac{\sqrt{3}}{3}$, $x = 1 - \frac{\sqrt{3}}{3}$.

Solution using Newton's Method:

Newton's method uses the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (0.16)$$

(i) Solve $\frac{1}{x} = 3$, $x \neq 0$

Define the function:

$$f(x) = \frac{1}{x} - 3 \quad (0.17)$$

Compute its derivative:

$$f'(x) = -\frac{1}{x^2} \quad (0.18)$$

Newton's iteration formula:

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - 3}{-\frac{1}{x_n^2}} \quad (0.19)$$

Simplify:

$$x_{n+1} = x_n + (3x_n^2 - 1) \quad (0.20)$$

Choosing an initial guess $x_0 = 0.5$, iterating:

$$x_1 = 0.5 + (3(0.5)^2 - 1) = 0.25 \quad (0.21)$$

$$x_2 = 0.25 + (3(0.25)^2 - 1) = 0.1875 \quad (0.22)$$

$$x_3 = 0.1875 + (3(0.1875)^2 - 1) \approx 0.1738 \quad (0.23)$$

Continuing iterations, the root converges to $x = \frac{1}{3}$.

(ii) Solve $\frac{1}{x} - \frac{1}{x-2} = 3, \quad x \neq 0, 2$

Define the function:

$$f(x) = \frac{1}{x} - \frac{1}{x-2} - 3 \quad (0.24)$$

Compute its derivative:

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(x-2)^2} \quad (0.25)$$

Newton's iteration formula:

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - \frac{1}{x_n-2} - 3}{-\frac{1}{x_n^2} + \frac{1}{(x_n-2)^2}} \quad (0.26)$$

Choosing an initial guess $x_0 = 1$, iterating:

$$x_1 = 1 - \frac{\frac{1}{1} - \frac{1}{1-2} - 3}{-\frac{1}{1^2} + \frac{1}{(1-2)^2}} = 1 - \frac{1+1-3}{-1+1} = 1 - \text{undefined} \quad (0.27)$$

Since the denominator is zero at $x_0 = 1$, we choose another guess, say $x_0 = 3$:

$$x_1 = 3 - \frac{\frac{1}{3} - \frac{1}{1} - 3}{-\frac{1}{3^2} + \frac{1}{1^2}} \quad (0.28)$$

$$= 3 - \frac{\frac{1}{3} - 1 - 3}{-\frac{1}{9} + 1} \quad (0.29)$$

$$= 3 - \frac{\frac{1}{3} - 4}{\frac{8}{9}} \quad (0.30)$$

$$= 3 - \frac{-\frac{11}{3}}{\frac{8}{9}} = 3 + \frac{11}{3} \times \frac{9}{8} \quad (0.31)$$

$$= 3 + \frac{99}{24} \approx 3 + 4.125 = 7.125 \quad (0.32)$$

Continuing iterations, the root converges to $x = 1 \pm \frac{\sqrt{3}}{3}$.

Newton's Method Difference Equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (0.33)$$

For the equation $\frac{1}{x} - \frac{1}{x-2} = 3$, we proceed as follows:

Rearrange the equation to standard form $f(x) = 0$

$$f(x) = \frac{1}{x} - \frac{1}{x-2} - 3 = 0 \quad (0.34)$$

Calculate the derivative $f'(x)$

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(x-2)^2} \quad (0.35)$$

Substitute (0.34) and (0.35) into (0.33)

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - \frac{1}{x_n-2} - 3}{-\frac{1}{x_n^2} + \frac{1}{(x_n-2)^2}} \quad (0.36)$$

Equation (0.36) is the specific difference equation for Newton's method applied to $\frac{1}{x} - \frac{1}{x-2} = 3$.

