12.8.3.18

EE24BTECH11025 - GEEDI HARSHA

Question: The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is:
Solution:
The circle is given by:

$$x^2 + y^2 = 16 \quad \Rightarrow \quad r = 4$$

The parabola is given by:

$$y^2 = 6x$$

Parabola Equation

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V_1} \mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0,$$

Circle Equation

$$g(\mathbf{x}) = \mathbf{x}^\top \mathbf{V_2} \mathbf{x} + 2\mathbf{u_2}^\top \mathbf{x} + f_2 = 0,$$

- $\mu = -\frac{4}{9}$
- $V_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
- $V_2 = \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix}$
- $f_1 = 0$
- $f_2 = -\frac{81}{16}$
- $u_1 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$
- $u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Step 1: Matrix $V_1 + \mu V_2$:

$$V_1 + \mu V_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \left(-\frac{4}{9}\right) \begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Step 2: Vector $u_1 + \mu u_2$:

$$u_1 + \mu u_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + \left(-\frac{4}{9}\right) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Step 3: Scalar $f_1 + \mu f_2$:

$$f_1 + \mu f_2 = 0 + \left(-\frac{4}{9}\right)(-16) = \frac{64}{9}.$$

Step 4: Combined Equation:

$$\mathbf{x}^\top (V_1 + \mu V_2)\mathbf{x} + 2(u_1 + \mu u_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0,$$

$$\begin{pmatrix} x \\ y \end{pmatrix}^\top \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 0 \end{pmatrix}^\top \begin{pmatrix} x \\ y \end{pmatrix} + \frac{64}{9} = 0.$$

Expanding the terms:

$$-\frac{4}{9}x^2 + \frac{5}{9}y^2 - 6x + \frac{64}{9} = 0.$$

Multiplying through by 9:

$$-4x^2 + 5y^2 - 54x + 64 = 0.$$

Substitute $y^2 = 6x$ into the equation:

$$-4x^2 + 5(6x) - 54x + 64 = 0,$$

$$-4x^2 + 30x - 54x + 64 = 0,$$

$$-4x^2 - 24x + 64 = 0.$$

Dividing through by -4:

$$x^2 + 6x - 16 = 0.$$

Solving using the quadratic formula:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(-16)}}{2(1)} = \frac{-6 \pm \sqrt{36 + 64}}{2} = \frac{-6 \pm \sqrt{100}}{2}.$$

$$x = \frac{-6 \pm 10}{2}.$$

Thus, the solutions for x are:

$$x = 2 \quad \text{or} \quad x = -8.$$

For $x = 2$, solve for y:

$$y^2 = 6x = 12 \implies y = \pm\sqrt{12} = \pm 2\sqrt{3}.$$

For $x = -8$, the equation $y^2 = -48$ has no real solutions.
Thus, the intersection points are:

$$(2, 2\sqrt{3}) \quad \text{and} \quad (2, -2\sqrt{3}).$$

The area outside the parabola and inside the circle is given by:

$$\text{Area} = \text{Total Circle Area} - 2 \times \int_0^2 \sqrt{6x}\, dx$$

1. Total circle area:

$$A_{\text{circle}} = \pi r^2 = 16\pi$$

2. Area under the parabola:

$$\begin{aligned}
\int_0^2 \sqrt{6x}\,dx &= \int_0^2 \sqrt{6}\,\sqrt{x}\,dx = \sqrt{6}\int_0^2 x^{1/2}\,dx \\
&= \sqrt{6}\left[\frac{2}{3}x^{3/2}\right]_0^2 \\
&= \sqrt{6}\cdot\frac{2}{3}\cdot(2^{3/2}) = \frac{4\sqrt{6}}{3}
\end{aligned}$$

Thus, the area outside the parabola is:

$$\begin{aligned}
\text{Area} &= 16\pi - \frac{8\sqrt{6}}{3} \\
&= \frac{4}{3}(4\pi - \sqrt{3})
\end{aligned}$$

Computational Solution

The **Trapezoidal** rule for approximating the integral of a function $f(x)$ from a to b is given by:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right]$$

where $h = \frac{b-a}{n}$ is the width of each subinterval
$x_i = a + ih$ for $i = 1, 2, \ldots, n-1$.
Difference equation,

$$A_n = \frac{h}{2}\left(y\left(x_0\right) + y\left(x_1\right)\right) + \frac{h}{2}\left(y\left(x_1\right) + y\left(x_2\right)\right) + \cdots + \frac{h}{2}\left(y\left(x_{n-1}\right) + y\left(x_n\right)\right) \tag{0.1}$$

$$A_n = h\left(\frac{y\left(x_0\right)}{2} + y\left(x_1\right) + y\left(x_2\right)\ldots\frac{y\left(x_n\right)}{2}\right) \tag{0.2}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_{n+1}\right) + y\left(x_n\right)\right),\ x_{n+1} = x_n + h \tag{0.3}$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_n + h\right) + y\left(x_n\right)\right) \tag{0.4}$$

$$\tag{0.5}$$

By the first principle of derivative,

$$y'\left(x\right) = \lim_{h\to 0}\frac{y\left(x+h\right) - y\left(x\right)}{h} \tag{0.6}$$

$$y\left(x+h\right) = y\left(x\right) + h\left(y'\left(x\right)\right),\ h\to 0 \tag{0.7}$$

Rewriting the difference equation, we get,

$$A_{n+1} = A_n + \frac{h}{2}\left(y\left(x_n\right) + hy'\left(x_n\right) + y\left(x_n\right)\right) \tag{0.8}$$

$$A_{n+1} = A_n + h\left(y\left(x_n\right) + \frac{h}{2}y'\left(x_n\right)\right) \tag{0.9}$$

$$A_{n+1} = A_n + hy\left(x_n\right) + \frac{h^2}{2}y'\left(x_n\right) \tag{0.10}$$

In this case, $f(x) = \sqrt{6x}$, and we are integrating from $x = 0$ to $x = 2$.

We will use $n = 4$ subintervals for this approximation. Therefore, the width of each subinterval is:

$$h = \frac{2-0}{4} = 0.5$$

The points x_i are:

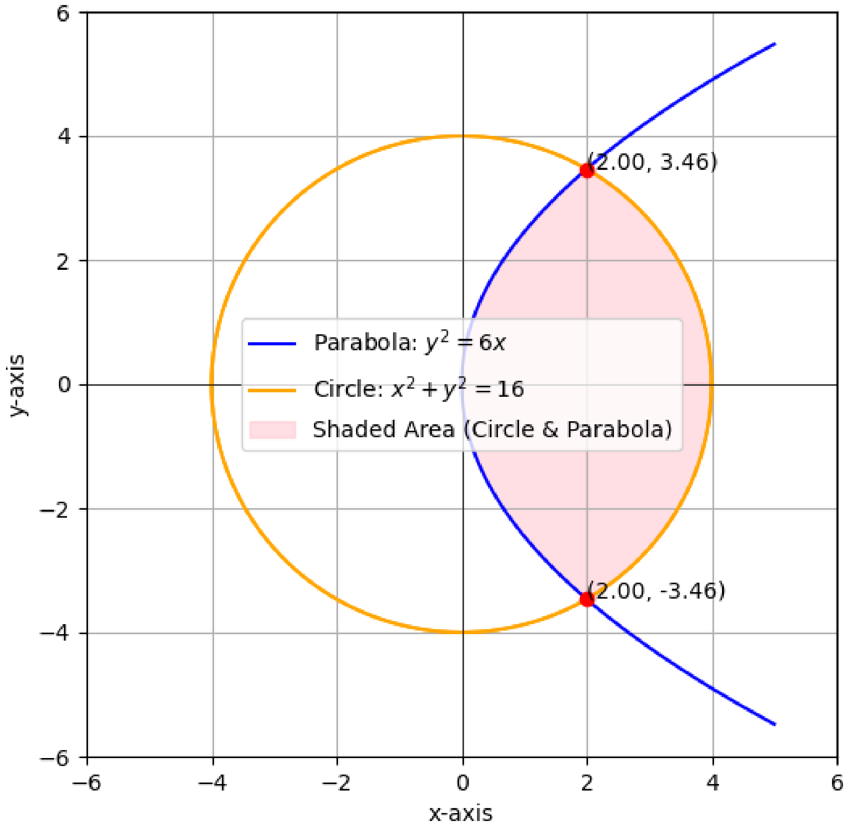
$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2$$

The corresponding function values are:

$$\begin{aligned}
f(x_0) &= \sqrt{6(0)} = 0,\\
f(x_1) &= \sqrt{6(0.5)} = \sqrt{3},\\
f(x_2) &= \sqrt{6(1)} = \sqrt{6},\\
f(x_3) &= \sqrt{6(1.5)} = \sqrt{9} = 3,\\
f(x_4) &= \sqrt{6(2)} = \sqrt{12} = 2\sqrt{3}
\end{aligned}$$

Now, we apply the trapezoidal rule:

$$\begin{aligned}
\int_0^2 \sqrt{6x}\,dx &\approx \frac{0.5}{2}\left[0 + 2\left(\sqrt{3} + \sqrt{6} + 3\right) + 2\sqrt{3}\right]\\
&= 0.25\left[0 + 2\sqrt{3} + 2\sqrt{6} + 6 + 2\sqrt{3}\right]\\
&= 0.25\left[4\sqrt{3} + 2\sqrt{6} + 6\right]
\end{aligned}$$



Parabola: $y^2 = 6x$
Circle: $x^2 + y^2 = 16$
Shaded Area (Circle & Parabola)
(2.00, 3.46)
(2.00, -3.46)
x-axis
y-axis