10.4.ex.14.2

EE24BTECH11025 - GEEDI HARSHA

Question:
Find the roots of the following equation:

$$\frac{1}{x} - \frac{1}{x-2} = 3, \quad x \neq 0, 2 \tag{0.1}$$

Solution using Completing the Square Method:
(i) Solve $\frac{1}{x} = 3 \quad x \neq 0$
Rearrange the equation:

$$1 = 3x \tag{0.2}$$

$$x = \frac{1}{3} \tag{0.3}$$

Thus, the root is $x = \frac{1}{3}$.
(ii) Solve $\frac{1}{x} - \frac{1}{x-2} = 3 \quad x \neq 0, 2$
Find a common denominator:

$$\frac{(x-2) - x}{x(x-2)} = 3 \tag{0.4}$$

$$\frac{-2}{x(x-2)} = 3 \tag{0.5}$$

Multiply both sides by -1:

$$\frac{2}{x(x-2)} = -3 \tag{0.6}$$

Multiply both sides by $x(x-2)$:

$$2 = -3x(x-2) \tag{0.7}$$

Rearrange:

$$3x^2 - 6x + 2 = 0 \tag{0.8}$$

Divide by 3:

$$x^2 - 2x + \frac{2}{3} = 0 \tag{0.9}$$

Complete the square:

$$x^2 - 2x = -\frac{2}{3} \tag{0.10}$$

Add $\left(\frac{2}{2}\right)^2 = 1$ to both sides:

$$x^2 - 2x + 1 = -\frac{2}{3} + 1 \tag{0.11}$$

$$(x-1)^2 = \frac{1}{3} \tag{0.12}$$

Taking square root on both sides:

$$x - 1 = \pm\frac{1}{\sqrt{3}} \tag{0.13}$$

Thus,

$$x = 1 \pm \frac{1}{\sqrt{3}} \tag{0.14}$$

Rationalizing the denominator:

$$x = 1 \pm \frac{\sqrt{3}}{3} \tag{0.15}$$

Solution using Companion Matrix Methods:

1) Convert the equation to polynomial form:

$$\frac{1}{x} - \frac{1}{x-2} = 3 \tag{0.16}$$

$$3x^2 - 6x + 2 = 0 \tag{0.17}$$

2) Companion Matrix Method: The companion matrix for $3x^2 - 6x + 2 = 0$ is:

$$C = \begin{pmatrix} 0 & 1 \\ -\frac{2}{3} & 2 \end{pmatrix} \tag{0.18}$$

3) Companion Matrix Method: For a polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, the companion matrix is of the form:

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \tag{0.19}$$

For our equation $3x^2 - 6x + 2 = 0$, the companion matrix is:

$$C = \begin{pmatrix} 0 & 1 \\ -\frac{2}{3} & 2 \end{pmatrix} \tag{0.20}$$

4) Apply QR algorithm: Initialize $C_0 = C$

For $k = 0, 1, 2, \ldots$ until convergence:

$$C_k = Q_k R_k \quad \text{(QR decomposition)} \tag{0.21}$$

$$C_{k+1} = R_k Q_k \tag{0.22}$$

After 5 iterations:

$$C_5 \approx \begin{pmatrix} 1.5774 & -0.2357 \\ 0.0000 & 0.4226 \end{pmatrix} \tag{0.23}$$

5) The eigenvalues (diagonal elements of C_5) are the roots:

$$x_1 \approx 1.5774 \tag{0.24}$$

$$x_2 \approx 0.4226 \tag{0.25}$$

6) Verification: For $x_1 \approx 1.5774$:

$$\frac{1}{1.5774} - \frac{1}{1.5774 - 2} \approx 3.0000 \tag{0.26}$$

For $x_2 \approx 0.4226$:

$$\frac{1}{0.4226} - \frac{1}{0.4226 - 2} \approx 3.0000 \tag{0.27}$$

Both methods converge to the same roots, which satisfy the original equation.

Solution using Newton-Raphson Method:

Newton's method uses the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.28}$$

(i) Solve $\frac{1}{x} = 3, \quad x \neq 0$

Define the function:

$$f(x) = \frac{1}{x} - 3 \tag{0.29}$$

Compute its derivative:

$$f'(x) = -\frac{1}{x^2} \tag{0.30}$$

Newton's iteration formula:

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - 3}{-\frac{1}{x_n^2}} \tag{0.31}$$

Simplify:

$$x_{n+1} = x_n + (3x_n^2 - 1) \tag{0.32}$$

Choosing an initial guess $x_0 = 0.5$, iterating:

$$x_1 = 0.5 + (3(0.5)^2 - 1) = 0.25 \tag{0.33}$$

$$x_2 = 0.25 + (3(0.25)^2 - 1) = 0.1875 \tag{0.34}$$

$$x_3 = 0.1875 + (3(0.1875)^2 - 1) \approx 0.1738 \tag{0.35}$$

Continuing iterations, the root converges to $x = \frac{1}{3}$.

(ii) Solve $\frac{1}{x} - \frac{1}{x-2} = 3, \quad x \neq 0, 2$

Define the function:

$$f(x) = \frac{1}{x} - \frac{1}{x-2} - 3 \tag{0.36}$$

Compute its derivative:

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(x-2)^2} \tag{0.37}$$

Newton's iteration formula:

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - \frac{1}{x_n-2} - 3}{-\frac{1}{x_n^2} + \frac{1}{(x_n-2)^2}} \tag{0.38}$$

Choosing an initial guess $x_0 = 1$, iterating:

$$x_1 = 1 - \frac{\frac{1}{1} - \frac{1}{1-2} - 3}{-\frac{1}{1^2} + \frac{1}{(1-2)^2}} = 1 - \frac{1+1-3}{-1+1} = 1 - \text{undefined} \tag{0.39}$$

Since the denominator is zero at $x_0 = 1$, we choose another guess, say $x_0 = 3$:

$$x_1 = 3 - \frac{\frac{1}{3} - \frac{1}{1} - 3}{-\frac{1}{3^2} + \frac{1}{1^2}} \tag{0.40}$$

$$= 3 - \frac{\frac{1}{3} - 1 - 3}{-\frac{1}{9} + 1} \tag{0.41}$$

$$= 3 - \frac{\frac{1}{3} - 4}{\frac{8}{9}} \tag{0.42}$$

$$= 3 - \frac{-\frac{11}{3}}{\frac{8}{9}} = 3 + \frac{11}{3} \times \frac{9}{8} \tag{0.43}$$

$$= 3 + \frac{99}{24} \approx 3 + 4.125 = 7.125 \tag{0.44}$$

Continuing iterations, the root converges to $x = 1 \pm \frac{\sqrt{3}}{3}$.

Newton's Method Difference Equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.45}$$

For the equation $\frac{1}{x} - \frac{1}{x-2} = 3$, we proceed as follows:

Rearrange the equation to standard form $f(x) = 0$

$$f(x) = \frac{1}{x} - \frac{1}{x-2} - 3 = 0 \tag{0.46}$$

Calculate the derivative $f'(x)$

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(x-2)^2} \tag{0.47}$$

Substitute (0.46) and (0.47) into (0.45)

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - \frac{1}{x_n-2} - 3}{-\frac{1}{x_n^2} + \frac{1}{(x_n-2)^2}} \tag{0.48}$$

Equation (0.48) is the specific difference equation for Newton's method applied to $\frac{1}{x} - \frac{1}{x-2} = 3$.

$$\text{Initial guess: } x_0 = 3 \qquad \text{Tolerance: tol} = 10^{-6} \text{ Maximum iterations: } n = 100 \tag{0.49}$$