10.4.ex.14.2

EE24BTECH11025 - GEEDI HARSHA

Question:
Find the roots of the following equation:

$$\frac{1}{x} - \frac{1}{x-2} = 3, \quad x \neq 0, 2 \tag{0.1}$$

Solution using Completing the Square Method:
(i) Solve $\frac{1}{x} = 3 \quad x \neq 0$
Rearrange the equation:

$$1 = 3x \tag{0.2}$$

$$x = \frac{1}{3} \tag{0.3}$$

Thus, the root is $x = \frac{1}{3}$.
(ii) Solve $\frac{1}{x} - \frac{1}{x-2} = 3 \quad x \neq 0, 2$
Find a common denominator:

$$\frac{(x-2) - x}{x(x-2)} = 3 \tag{0.4}$$

$$\frac{-2}{x(x-2)} = 3 \tag{0.5}$$

Multiply both sides by -1:

$$\frac{2}{x(x-2)} = -3 \tag{0.6}$$

Multiply both sides by $x(x-2)$:

$$2 = -3x(x-2) \tag{0.7}$$

Rearrange:

$$3x^2 - 6x + 2 = 0 \tag{0.8}$$

Divide by 3:

$$x^2 - 2x + \frac{2}{3} = 0 \tag{0.9}$$

Complete the square:

$$x^2 - 2x = -\frac{2}{3} \tag{0.10}$$

Add $\left(\frac{2}{2}\right)^2 = 1$ to both sides:

$$x^2 - 2x + 1 = -\frac{2}{3} + 1 \tag{0.11}$$

$$(x-1)^2 = \frac{1}{3} \tag{0.12}$$

Taking square root on both sides:

$$x - 1 = \pm\frac{1}{\sqrt{3}} \tag{0.13}$$

Thus,

$$x = 1 \pm \frac{1}{\sqrt{3}} \tag{0.14}$$

Rationalizing the denominator:

$$x = 1 \pm \frac{\sqrt{3}}{3} \tag{0.15}$$

Matrix Method:

$$\text{Matrix A:} \quad A = \begin{bmatrix} 3 & -6 \\ 1 & 0 \end{bmatrix} \tag{0.16}$$

$$\text{Matrix B:} \quad B = \begin{bmatrix} -2 \\ y \end{bmatrix} \tag{0.17}$$

$$\text{Inverse of A:} \quad A^{-1} = \frac{1}{3(0) - (-6)(1)} \begin{bmatrix} 0 & 6 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -\frac{1}{3} & 1 \end{bmatrix} \tag{0.18}$$

$$\text{Solution:} \quad X = A^{-1}B = \begin{bmatrix} 0 & 2 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} -2 \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ \frac{2}{3} + y \end{bmatrix} \tag{0.19}$$

$$\text{Equating } x^2 \text{ and } x: \quad 2y = \frac{2}{3} + y \tag{0.20}$$

$$\text{Solving for } y: \quad y = \frac{2}{3} \tag{0.21}$$

$$\text{Since } y = x^2: \quad x^2 = \frac{2}{3} \tag{0.22}$$

$$\text{Taking square root:} \quad x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3} \tag{0.23}$$

$$\text{Final solution:} \quad x = 1 \pm \frac{\sqrt{3}}{3} \tag{0.24}$$

Solution using Newton's Method:

Newton's method uses the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.25}$$

(i) Solve $\frac{1}{x} = 3, \quad x \neq 0$

Define the function:

$$f(x) = \frac{1}{x} - 3 \tag{0.26}$$

Compute its derivative:

$$f'(x) = -\frac{1}{x^2} \tag{0.27}$$

Newton's iteration formula:

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - 3}{-\frac{1}{x_n^2}} \tag{0.28}$$

Simplify:

$$x_{n+1} = x_n + (3x_n^2 - 1) \tag{0.29}$$

Choosing an initial guess $x_0 = 0.5$, iterating:

$$x_1 = 0.5 + (3(0.5)^2 - 1) = 0.25 \tag{0.30}$$

$$x_2 = 0.25 + (3(0.25)^2 - 1) = 0.1875 \tag{0.31}$$

$$x_3 = 0.1875 + (3(0.1875)^2 - 1) \approx 0.1738 \tag{0.32}$$

Continuing iterations, the root converges to $x = \frac{1}{3}$.

(ii) Solve $\frac{1}{x} - \frac{1}{x-2} = 3, \quad x \neq 0, 2$

Define the function:

$$f(x) = \frac{1}{x} - \frac{1}{x-2} - 3 \tag{0.33}$$

Compute its derivative:

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(x-2)^2} \tag{0.34}$$

Newton's iteration formula:

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - \frac{1}{x_n-2} - 3}{-\frac{1}{x_n^2} + \frac{1}{(x_n-2)^2}} \tag{0.35}$$

Choosing an initial guess $x_0 = 1$, iterating:

$$x_1 = 1 - \frac{\frac{1}{1} - \frac{1}{1-2} - 3}{-\frac{1}{1^2} + \frac{1}{(1-2)^2}} = 1 - \frac{1 + 1 - 3}{-1 + 1} = 1 - \text{undefined} \tag{0.36}$$

Since the denominator is zero at $x_0 = 1$, we choose another guess, say $x_0 = 3$:

$$x_1 = 3 - \frac{\frac{1}{3} - \frac{1}{1} - 3}{-\frac{1}{3^2} + \frac{1}{1^2}} \tag{0.37}$$

$$= 3 - \frac{\frac{1}{3} - 1 - 3}{-\frac{1}{9} + 1} \tag{0.38}$$

$$= 3 - \frac{\frac{1}{3} - 4}{\frac{8}{9}} \tag{0.39}$$

$$= 3 - \frac{-\frac{11}{3}}{\frac{8}{9}} = 3 + \frac{11}{3} \times \frac{9}{8} \tag{0.40}$$

$$= 3 + \frac{99}{24} \approx 3 + 4.125 = 7.125 \tag{0.41}$$

Continuing iterations, the root converges to $x = 1 \pm \frac{\sqrt{3}}{3}$.

Newton's Method Difference Equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{0.42}$$

For the equation $\frac{1}{x} - \frac{1}{x-2} = 3$, we proceed as follows:

Rearrange the equation to standard form $f(x) = 0$

$$f(x) = \frac{1}{x} - \frac{1}{x-2} - 3 = 0 \tag{0.43}$$

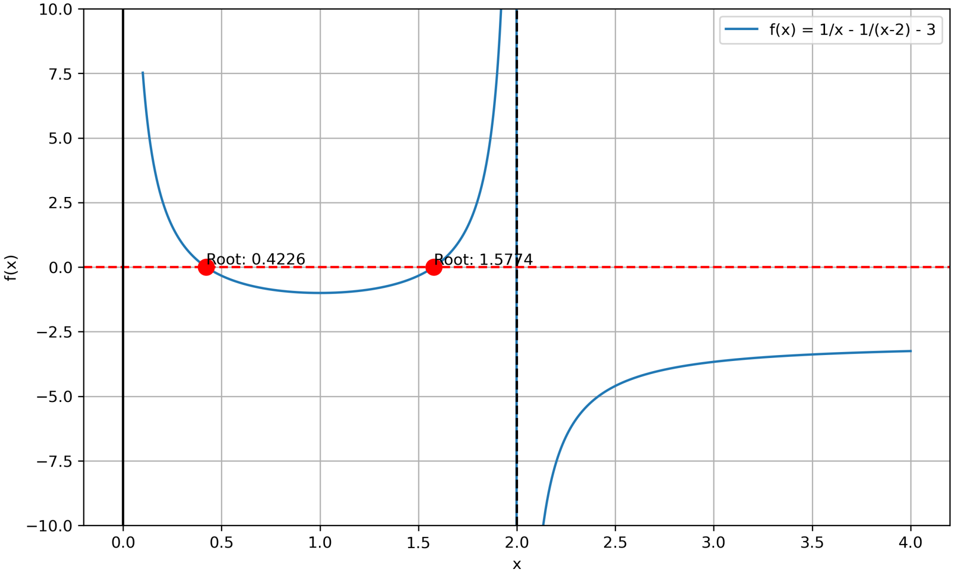
Calculate the derivative $f'(x)$

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(x-2)^2} \tag{0.44}$$

Substitute (0.34) and (0.35) into (0.33)

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - \frac{1}{x_n-2} - 3}{-\frac{1}{x_n^2} + \frac{1}{(x_n-2)^2}} \tag{0.45}$$

Equation (0.36) is the specific difference equation for Newton's method applied to $\frac{1}{x} - \frac{1}{x-2} = 3$.



f(x) = 1/x - 1/(x-2) - 3
Root: 0.4226
Root: 1.5774
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0
f(x)
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
x