12.8.3.18

EE24BTECH11025 - GEEDI HARSHA

Question: The area of the circle $x^2 + y^2 = 16$ exterior to the parabola $y^2 = 6x$ is:
Solution:
The circle is given by:

$$x^2 + y^2 = 16 \quad \Rightarrow \quad r = 4$$

The parabola is given by:

$$y^2 = 6x$$

Intersection Points

Equating $x^2 + y^2 = 16$ and $y^2 = 6x$, we substitute $y^2 = 6x$ into the circle equation:

$$x^2 + 6x = 16 \quad \Rightarrow \quad x^2 + 6x - 16 = 0$$

Solving the quadratic equation:

$$x = \frac{-6 \pm \sqrt{6^2 - 4(1)(-16)}}{2(1)} = \frac{-6 \pm \sqrt{100}}{2} = \frac{-6 \pm 10}{2}$$
$$x = 2 \quad \text{and} \quad x = -8$$

Thus, the intersection points are $(2, \pm\sqrt{12})$.

Area Calculation

The area outside the parabola and inside the circle is given by:

$$\text{Area} = \text{Total Circle Area} - 2 \times \int_0^2 \sqrt{6x}\, dx$$

1. Total circle area:

$$A_{\text{circle}} = \pi r^2 = 16\pi$$

2. Area under the parabola:

$$\int_0^2 \sqrt{6x}\, dx = \int_0^2 \sqrt{6}\sqrt{x}\, dx = \sqrt{6}\int_0^2 x^{1/2}\, dx$$
$$= \sqrt{6}\left[\frac{2}{3}x^{3/2}\right]_0^2$$
$$= \sqrt{6} \cdot \frac{2}{3} \cdot (2^{3/2}) = \frac{4\sqrt{6}}{3}$$

Thus, the area outside the parabola is:

$$\begin{aligned} \text{Area} &= 16\pi - \frac{8\sqrt{6}}{3} \\ &= \frac{4}{3}(4\pi - \sqrt{3}) \end{aligned}$$

Computational Solution

The **Trapezoidal** rule for approximating the integral of a function $f(x)$ from a to b is given by:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right]$$

where $h = \frac{b-a}{n}$ is the width of each subinterval
$x_i = a + ih$ for $i = 1, 2, \ldots, n-1$.
Difference equation,

$$A_n = \frac{h}{2}\left(y(x_0) + y(x_1)\right) + \frac{h}{2}\left(y(x_1) + y(x_2)\right) + \cdots + \frac{h}{2}\left(y(x_{n-1}) + y(x_n)\right) \quad (0.1)$$

$$A_n = h\left(\frac{y(x_0)}{2} + y(x_1) + y(x_2)\ldots\frac{y(x_n)}{2}\right) \quad (0.2)$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y(x_{n+1}) + y(x_n)\right),\ x_{n+1} = x_n + h \quad (0.3)$$

$$A_{n+1} = A_n + \frac{h}{2}\left(y(x_n + h) + y(x_n)\right) \quad (0.4)$$

$$(0.5)$$

By the first principle of derivative,

$$y'(x) = \lim_{h\to 0}\frac{y(x+h) - y(x)}{h} \quad (0.6)$$

$$y(x+h) = y(x) + h\left(y'(x)\right),\ h \to 0 \quad (0.7)$$

Rewriting the difference equation, we get,

$$A_{n+1} = A_n + \frac{h}{2}\left(y(x_n) + hy'(x_n) + y(x_n)\right) \quad (0.8)$$

$$A_{n+1} = A_n + h\left(y(x_n) + \frac{h}{2}y'(x_n)\right) \quad (0.9)$$

$$A_{n+1} = A_n + hy(x_n) + \frac{h^2}{2}y'(x_n) \quad (0.10)$$

In this case, $f(x) = \sqrt{6x}$, and we are integrating from $x = 0$ to $x = 2$.

We will use $n = 4$ subintervals for this approximation. Therefore, the width of each subinterval is:

$$h = \frac{2 - 0}{4} = 0.5$$

The points x_i are:

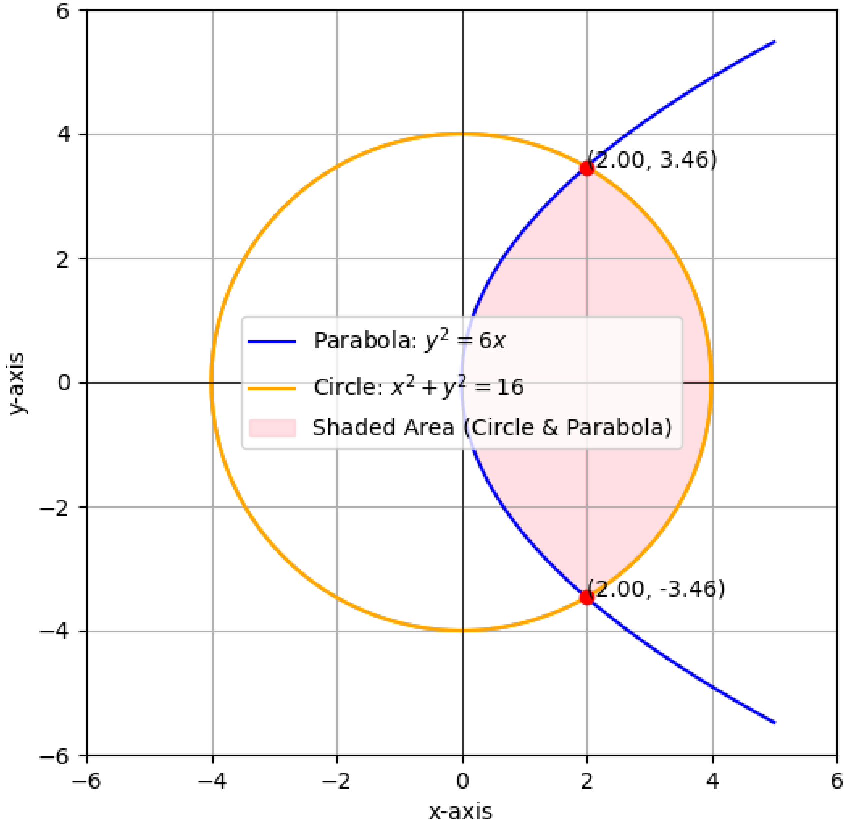
$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2$$

The corresponding function values are:

$$\begin{aligned}
f(x_0) &= \sqrt{6(0)} = 0, \\
f(x_1) &= \sqrt{6(0.5)} = \sqrt{3}, \\
f(x_2) &= \sqrt{6(1)} = \sqrt{6}, \\
f(x_3) &= \sqrt{6(1.5)} = \sqrt{9} = 3, \\
f(x_4) &= \sqrt{6(2)} = \sqrt{12} = 2\sqrt{3}
\end{aligned}$$

Now, we apply the trapezoidal rule:

$$\begin{aligned}
\int_0^2 \sqrt{6x}\, dx &\approx \frac{0.5}{2}\left[0 + 2\left(\sqrt{3} + \sqrt{6} + 3\right) + 2\sqrt{3}\right] \\
&= 0.25\left[0 + 2\sqrt{3} + 2\sqrt{6} + 6 + 2\sqrt{3}\right] \\
&= 0.25\left[4\sqrt{3} + 2\sqrt{6} + 6\right]
\end{aligned}$$


(2.00, 3.46)
(2.00, -3.46)
Parabola: $y^2 = 6x$
Circle: $x^2 + y^2 = 16$
Shaded Area (Circle & Parabola)
x-axis
y-axis
−6
−4
−2
0
2
4
6