Beta and Gamma function

Euler's Integral

The first and second ~~in~~ Eulerian Integrals are termed ~~as~~ respectively Beta and Gamma function.

$$\int_0^\infty e^{-x^2} dx$$

Beta function:

The Beta function is, denoted by $B(m,n)$, is defined by

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

Where m, n are positive numbers, integral or fractional. for $m>0$, $n>0$ the integral is convergent.

This integral is called first Eulerian Integral.

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(1) Beta function is symmetric in m and n.

$$B(m,n) = B(n,m)$$

(2) $$B(m,n) = 2\int_0^\infty \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta$$

by putting $x = \sin^2\theta$

∟ Γ

Gamma function

→ The gamma function, denoted by $\Gamma(n)$, is defined by

$$\Gamma(n) = \int_0^\infty e^{-x} . x^{n-1}\, dx$$

Which is convergent for $n>0$. where n is a positive number, integral, or fractional. This integral is called second Eulerian Integral.

(i) $\Gamma(1) = 1$

(ii) $\Gamma(n+1) = n\Gamma(n)$, $n>0$

$$\Gamma(n+1) = \int_0^\infty e^{-x} . x^{n+1-1}\, dx$$

$$= \int_0^\infty e^{-x} . x^n\, dx$$

$$= \left[-x^n . e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\, dx$$

$$= \left[-x^n \cdot e^{-x}\right]_0^{\infty} + \int_0^{\infty} n x^{n-1} e^{-x}\, dx$$

$$= 0 + n\int_0^{\infty} e^{-x} \cdot x^n\, dx$$

$$\Gamma(n+1) = n\Gamma n$$

(iii) $$\Gamma(n+1) = n(n-1)\Gamma(n-2)$$
$$= n(n-1)(n-2) \cdots 3\cdot 2\cdot 1\,\Gamma(1)$$
$$= n(n-1)(n-2) \cdots 3\cdot 2\cdot 1$$
$$\Gamma(n+1) = \lfloor n$$

Relation between Beta and Gamma function

(4) $$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

(5.) $$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$B\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = B\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\, dx$$

$$= \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}\, dx \qquad x = \sin^2\theta$$
$$dx = 2\sin\theta\cdot\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = \cancel{2}\cdot\frac{\pi}{\cancel{2}}$$

$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Prob. $$\int_0^{\infty} e^{-x^2}\, dx$$

$$x^2 = y \Rightarrow 2x\,dx = dy \Rightarrow dx = \frac{dy}{2\sqrt{y}}$$

Prob. $\int_0 e \, dx$

$x^2 = y \Rightarrow 2x\,dx = dy \Rightarrow dx = \frac{dy}{2\sqrt{y}}$

Sol. $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\int_0^\infty e^{-y} \cdot y^{-1/2} dy$

$= \frac{1}{2}\int_0^\infty e^{-y} \cdot y^{\left(\frac{1}{2}-1\right)} dy$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{\sqrt{\pi}}{2}$

$\int_0^\infty e^{-x} \cdot x^{n-1} dx = \Gamma(n)$

$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx$

$\Gamma(m,n)$

Prove that

(i) $\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$

(ii) $\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

Proof: $2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = B(m,n)$

$\Rightarrow \int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{1}{2}B(m,n) = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$= \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$

(ii) Proof: Putting $2m-1 = p$, $2n-1 = q$ $\Rightarrow m = \frac{p+1}{2}$ $n = \frac{q+1}{2}$

$\int_0^{\pi/2} \sin^p$

Prob (1) $\int_0^\infty$

Sol: $\int_0^\infty$

$= 24$

Prob ② $\int_0^1 x^5 (1-x^2)^2 dx$

$\int_0^1 x^{6-1}(1-x)^{3-1} dx = B(6,3) = \cancel{\frac{\Gamma(6)\Gamma(3)}{\Gamma(9)}} = \cancel{\frac{\lfloor 5 \lfloor 2}{\lfloor 8}} = \frac{1}{168}$

Prob. ③ $\int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx =$

Properties ① $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ ✓

② $\Gamma(m)\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$ ✓

③ $\Gamma(n+\frac{1}{2}) = \frac{\Gamma(2n+1)\sqrt{\pi}}{2^{2n}\Gamma(n+1)}$

Prob ② Proof: $\int_0^\infty x^2 e^{-x^4} dx \times \int_0^\infty e^{-x^4} dx$

Putting $x^4 = z$ or $x = z^{1/4}$

$dx = \frac{1}{4} z^{-3/4} dz$

$= \int_0^\infty z^{1/2} e^{-z} \cdot \frac{1}{4} z^{-3/4} dz \times \int_0^\infty e^{-z} \cdot \frac{1}{4} z^{-3/4} dz$

$= \frac{1}{16} \int_0^\infty e^{-z} \cdot z^{-1/4} dz \times \int_0^\infty e^{-z} \cdot z^{-3/4} dz$ $\qquad \int e^{-x} \cdot x^{n-1} dx$

$= \frac{1}{16} \int_0^\infty e^{-z} \cdot z^{\frac{3}{4}-1} dz \times \int_0^\infty e^{-z} \cdot z^{\frac{1}{4}-1} dz$

★ $= \frac{1}{16} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right)$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{8\sqrt{2}}$$

Prob (4) $B(m,n) = B(m+1,n) + B(m,n+1)$

$$= \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} + \frac{\Gamma(m)\cdot n\Gamma(n)}{(m+n)\Gamma(m+n)}$$

$$= \frac{\cancel{(m+n)}\Gamma(m)\Gamma(n)}{\cancel{(m+n)}\Gamma(m+n)} = B(m,n)$$

Prob. (5) $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx$

Sol. $\int_0^{\pi/2} \sin^{-1/2} x\, dx \times \int_0^{\pi/2} \sin^{1/2} x\, dx$

$$= \int_0^{\pi/2} \sin^{-1/2} x \cdot \cos^0 x\, dx \times \int_0^{\pi/2} \sin^{1/2} x \cdot \cos^0 x\, dx$$

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\cancel{\Gamma\left(\frac{3}{4}\right)}} \times \frac{\cancel{\Gamma\left(\frac{3}{4}\right)}\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{4}\right)} = \frac{\cancel{\Gamma\left(\frac{1}{4}\right)}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2\cdot 2\frac{1}{4}\cdot\cancel{\Gamma\left(\frac{1}{4}\right)}}$$

$$= \frac{\left[\sqrt{\pi}\right]^2}{2\times\frac{1}{2}}$$

$$= \pi$$

All important results of Beta and Gamma function

All results of Beta and Gamma function

1. $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

2. $B(m,n) = B(n,m)$

3. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

4. $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$

5. $\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\,dx$

6. $\Gamma(1) = 1$

7. $\Gamma(n+1) = n\Gamma(n)$

8. $\Gamma(n+1) = \underline{|n}$

9. $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$

10. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

11. $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} = \frac{1}{2}B(m,n)$

12. $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

13. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

14. $\Gamma(m)\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$

15.

Rectification — length of an arc
Quadrature — area
Volume —

Rectification

length of arc of a curve between two