Differential Calculus

Monday, December 07, 2020 10:00 AM

Limit:- $f(x) = \dfrac{x^2-4}{x-2} \longrightarrow 4 = \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = x+2$

$x \longrightarrow 2$

$\lim_{x\to 2} f(x) = \lim_{x\to 2} (x+2) = 4$

$\lim_{x\to a} f(x) = l$

x	$f(x)$	$f(x) = x+2$
1.9	3.9	$\lvert 3.9-4 \rvert$
1.99	3.99	$\lvert 3.99-4 \rvert$
1.999	3.999	$\lvert 3.999-4 \rvert$
$a=2$	$f(x)=4=l$	
2.0001	4.0001	
2.001	4.001	
2.01	4.01	
2.1	4.1	

$\delta > 0$

$|1.9-2|$

$|1.99-2|$

$|1.999-2|$

$\varepsilon > 0$ δ

$f(x) \longrightarrow l$

$x \longrightarrow a$

$\lim_{x\to a} f(x) = l$

$|x-a| \longrightarrow 0$ as $|f(x)-l| \longrightarrow 0$

$|f(x)-l| \longrightarrow 0$ as $|x-a| \longrightarrow 0$

$|f(x)-l| < \varepsilon$ as $|x-a| < \delta$

Limit:- A function $f(x)$ is said to tend to limit l as x is tending to a.

$\lim_{x\to a} f(x) = l$

or $|f(x)-l| \longrightarrow 0$ as $|x-a| \longrightarrow 0$

or $|f(x)-l| < \varepsilon$ as $|x-a| < \delta$

$\langle \varepsilon \rangle \longrightarrow \delta$

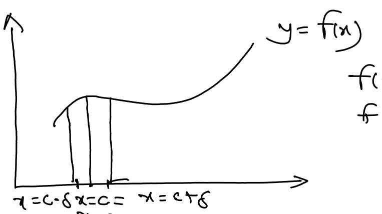
Continuity: A function $\lim_{x\to c} f(x) = f(c)$

$x = 1.99999\ldots$

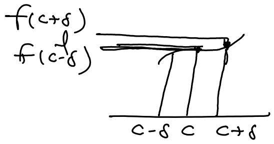
$x = 2$

$x = 2.0000001\ldots$

0.00000



$\lim_{x\to c} f(x) = f(c) = l$



$x = c-\delta$

$x = c+\delta$

$x = c$

$x = 1.99999\ldots$

$x = 2$

$x = 2.000001\ldots$

$|f(x)-l|$

$|f(x)-f(c)| \longrightarrow 0$ as $|x-c| \longrightarrow 0$

or $|f(x)-f(c)| < \varepsilon$ as $|x-c| < \delta$

(1) Removable discontinuity

$\rightarrow$ $f(x)$ is said to have removable discontinuity at $x=a$

$\lim f(x)$ exists but not equal to $f(a)$

$x \to a$

$$f(a+0) = f(a-0) \neq f(a)$$

(2) Discontinuity of first kind:

A function $f(x)$ is said to have a discontinuity of first kind at $x=a$ if $f(a+0)$ and $f(a-0)$ both exists and $f(a+0) \neq f(a-0)$

from Left

If $f(a-0) \neq f(a)$ and $f(a+0) = f(a)$

from Right If $f(a+0) \neq f(a)$ and $f(a-0) = f(a)$

(3) Discontinuity of second kind

$f(x)$ have a discontinuity of second kind at $x=a$ iff $f(a+0)$ and $f(a-0)$ both does not exist.

ε-δ definition of limit of a function

Let $f(x)$ be a function of variable x. A number l is called the limit of $f(x)$ at $x=a$ if for any arbitrary chosen positive number ε, there exists a corresponding positive number δ such that,

$$0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

$\lim\limits_{x \to a} f(x) = l$

$|x-a| \to 0 \Rightarrow |f(x) - l| \to 0$

for all values of x, where $|x-a|$ means the numerical value of $x-a$.

Meaning of $0 < |x-a| < \delta$

since $|x-a|$ represents the numerical value [i.e. absolute value of $(x-a)$ without regard of sign of $(x-a)$

$$|x-a| < \delta$$

$$x-a < \delta$$

$$x < a+\delta \quad \text{—— (I)}$$

$$-x+a < \delta$$

$$x > a-\delta \quad \text{—— (II)}$$

$$\Rightarrow \quad a-\delta < x < a+\delta$$

$(a-\delta, a+\delta)$

$(1, 2)$

Prob(1) Using ε-δ technique, verify that $\lim\limits_{x \to 2} f(x) = 10$ where,

$$f(x) = \frac{2(x^2+x-6)}{x-2}$$

Solution:- To verify that 10 is the limit of f at $x=2$, we show that for any given $\varepsilon > 0$, there exists $(\exists)$ $\delta > 0$ such that

Now,

$$|f(x) - 10| = \left| \frac{2(x^2+x-6)}{x-2} - 10 \right|$$

$$= \left| \frac{2(x-2)(x+3)}{x-2} - 10 \right|$$

$\lim\limits_{x \to 2} \frac{2(x^2+x-6)}{x-2} = 10$

$|x-2| < \delta \Rightarrow |f(x) - l| < \varepsilon$

$$= \left| \frac{2(x-2)(x+3)}{x-2} - 10 \right|$$

$|x-2| < \delta \Rightarrow |f(x)-4| < \varepsilon$

$$= |2x-4|$$

$|x-2| < \delta$

$$= 2|x-2|$$

$$< 2\cdot\delta = \varepsilon$$

Thus, for any given $\varepsilon > 0$, there exists $\delta \left(= \frac{\varepsilon}{2}\right) > 0$ such that

$0 < |x-2| < \delta \Rightarrow |f(x) - 10| < \varepsilon$

$\therefore \lim_{x \to 2} f(x) = 10$

Prob. (2) Using ε-δ technique, verify that $\lim_{x \to 2} x^2 = 4$

Sol. To verify that $\lim_{x \to 2} x^2 = 4$, we must show that for any given $\varepsilon > 0$ we can produce $\delta > 0$ such that

$|x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$

$0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$

$\varepsilon > 0$

$\exists\, \delta =$

$\Rightarrow |x-2|\,|x+2| < \varepsilon$

As a first choice let $\delta = 1$. Then

$0 < |x-2| < 1$

$|x-2| < 1$

$1 < x < 3$

$|x-2|\,|x+2| < \varepsilon$

$3 < x+2 < 5$

$3 < \underline{|x+2| < 5}$

$\therefore |x-2|\,|x+2| < 5|x-2| \quad \text{——— (1)}$

$5|x-2| < \varepsilon$

$|x-2| < \delta$

$|x-2| < \frac{\varepsilon}{5}$

$\frac{\varepsilon}{5} > 0 \quad \delta > 0$

$|x-2|\,|x+2| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$

Thus, we take $\delta = \min\left(1, \frac{\varepsilon}{5}\right)$

$\delta > 0$

$\therefore 0 < |x-2| < \delta \Rightarrow |f(x) - 4| < \varepsilon$

$0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$

Prob. (3) Using ε-δ, verify that $\lim_{x \to 5} \frac{1}{x} = \frac{1}{5}$

Sol.: To verify that $\lim_{x \to 5} \frac{1}{x} = \frac{1}{5}$, we must show that for any given $\varepsilon > 0$, we can produce $\delta > 0$ such that

$\varepsilon > 0$

$0 < |x-5| < \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{5} \right| < \varepsilon > 0$

$\delta > 0$

$\delta > 0$

$\delta > 0$

$0 < |x-5| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{5}\right| < \varepsilon > 0$ $\quad \delta > 0$

$\varepsilon > 0$
$\delta > 0$
$\delta > 0$

$\Rightarrow \frac{|x-5|}{5|x|} < \varepsilon$ — (1)

$\to 0 \quad 1$

$\frac{|x-5|}{5 \cdot |x|} < \frac{|x-5|}{5 \cdot 4}$
$< \frac{|x-5|}{20} < \varepsilon$
$|x-5| < 20\varepsilon$
$< \delta = 20\varepsilon$
$<$

As a first choice take $\delta = 1$. Then

$0 < |x-5| < \delta$

$|x-5| < 1$

$4 < x < 6$

$\Rightarrow \quad 4 < |x| < 6$

$\Rightarrow \quad \frac{1}{6} < \frac{1}{|x|} < \frac{1}{4}$

$x \to 5$
$|x-5| < \delta$

$\varepsilon > 0$
$20\varepsilon > 0$
$\delta > 0$

$$\frac{|x-5|}{5|x|} < \frac{|x-5|}{20} < \varepsilon$$

$\Rightarrow |x-5| < 20\varepsilon$

Then, we take $\delta = \min(1, 20\varepsilon)$

then $\quad 0 < |x-5| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{5}\right| < \varepsilon$

$\delta > 0$
$|x-a| \to 0 \quad 0.5 \quad \delta \neq 1$
0.6
$0.7 \quad 1$
0.8

Prob. (4) Using ε-δ, $\lim_{x \to 0} x \sin\frac{1}{x} = 0$

Prob (5) $\lim_{x \to 3} (x^2 + 2x) = 15$

Prob (6) $\lim_{x \to -1} (2x^2 + 3) = 5$

Prob (7) $\lim_{x \to 1} (x^2 + 4x) = 5$

Prob (8) $\lim_{x \to 1} \sqrt{x^2 + 8} = 3$

Prob (4) solution $\lim_{x \to 0} x \sin\frac{1}{x} = 0$

To prove that $\lim_{x \to 0} x \sin\frac{1}{x} = 0$, for any given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$0 < |x-0| < \delta \Rightarrow \left|x\sin\frac{1}{x} - 0\right| < \varepsilon$

$\Rightarrow \quad |x-0| < \delta \Rightarrow \left|x \sin\frac{1}{x}\right| < \varepsilon$

Now $\quad \left|x \sin\frac{1}{x}\right| = |x| \cdot \left|\sin\frac{1}{x}\right|$ $\quad \left[\because \left|\sin\frac{1}{x}\right| \leq 1\right] \forall x \neq 0$

$\leq |x| \cdot 1$

$< \delta \cdot 1$

$< \delta = \varepsilon$

Thus, for any given $\varepsilon > 0$, $\exists\, \delta (= \varepsilon) > 0$ such that

$|x-0| < \delta \Rightarrow \left|x \sin\frac{1}{x} - 0\right| < \varepsilon$

Prob. 5 Solution:- $\lim_{x \to 3} (x^2 + 2x) = 15$

For any given $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$0 < |x-3| < \delta \Rightarrow |x^2 + 2x - 15| < \varepsilon$

Now $|x^2+2x-15| = |(x-3)(x+5)|$
$= |(x-3)|\,|(x+5)|$

As a first choice, we take $\delta = 1$, then we have

$|x-3| < 1 \Rightarrow 2 < x < 4$
$\Rightarrow x \in (2, 4)$
$\Rightarrow x+5 \in (7, 9)$
$\Rightarrow |x+5| \in (7, 9)$
$\Rightarrow |(x+5)| < 9$

$\Rightarrow \quad |x-3|\,|x+5| \leq \delta\, 9 = \varepsilon$
$\delta = \frac{\varepsilon}{9} \qquad \because \varepsilon > 0$
$\therefore \delta > 0$

We take $\delta = \min\left(1, \frac{\varepsilon}{9}\right)$

Then for any given $\varepsilon > 0$, $\exists$ a $\delta = \left[\min\left(1, \frac{\varepsilon}{9}\right)\right] > 0$

$\therefore \quad 0 < |x-3| < \delta \Rightarrow |x^2+2x-15| < \varepsilon$

Prob:- $\lim_{x \to 1} \sqrt{x^2+8} = 3$ $\qquad |x+2|$ $\downarrow$ x

Sol. Given $\varepsilon > 0$, we must find a $\delta > 0$ such that

$0 < |x-1| < \delta \Rightarrow |\sqrt{x^2+8} - 3| < \varepsilon$

Now, $|\sqrt{x^2+8} - 3| = \left| \frac{(\sqrt{x^2+8} - 3)(\sqrt{x^2+8} + 3)}{\sqrt{x^2+8} + 3} \right|$

$= \left| \frac{x^2+8-9}{\sqrt{x^2+8} + 3} \right|$

$= \left| \frac{(x-1)(x+1)}{\sqrt{x^2+8} + 3} \right|$

$= \frac{|x-1|\,|x+1|}{\sqrt{x^2+8} + 3}$ —— (1)

As a first choice $\delta = 1$

$$|x-1| < 1 \Rightarrow 0 < x < 2$$
$$\Rightarrow x \in (0,2)$$
$$\Rightarrow x+1 \in (1,3)$$
$$\Rightarrow |x+1| \in (1,3)$$
$$\Rightarrow |x+1| < 3$$

Now $\sqrt{x^2+8}+3$

$$\Rightarrow x+1 > 1$$
$$\Rightarrow (x+1)^2 > 1$$
$$\Rightarrow x^2+2x+1 > 1$$
$$\Rightarrow x^2+2x+1+1 > 1+1$$
$$\Rightarrow x^2+2x+2 > 2$$
$$\Rightarrow x^2+2(x+1) > 2$$
$$\Rightarrow x^2+2\cdot 3 > 2$$
$$\Rightarrow x^2+6 > 2$$
$$\Rightarrow x^2+8 > 2+2=4$$
$$\Rightarrow \sqrt{x^2+8} > 2$$
$$\Rightarrow \sqrt{x^2+8}+3 > 5$$
$$\Rightarrow \frac{1}{\sqrt{x^2+8}+3} < \frac{1}{5}$$

$$\Rightarrow \frac{|x-1|\,|x+1|}{\sqrt{x^2+8}+3} < \frac{\delta \cdot 3}{5} = \varepsilon$$

$$\frac{3\delta}{5} = \varepsilon$$

$$\delta = \frac{5\varepsilon}{3} \qquad \because \varepsilon > 0$$

$$\delta > 0$$

Now, $|x-1| < \delta \Rightarrow \left|\sqrt{x^2+8} - 3\right| < \frac{3\delta}{5} = \varepsilon$

$\begin{cases} \varepsilon > 0 \\ \delta > 0 \end{cases}$

$\delta > 0$

$0 < x < 2$

$\sqrt{x^2+8}+3$

$$\delta = \min\left(1, \frac{5\varepsilon}{3}\right)$$

Prob $\lim_{x \to 1} x^2+4x = 5$

Given $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|x-1| < \delta \Rightarrow |x^2+4x-5| < \varepsilon$$

Now, $|(x-1)(x+5)| \Rightarrow x \in (0,2)$

$$|x+5| < 7$$

$$|(x-1)(x-5)| < \delta \cdot 7 = \varepsilon$$

$$\delta = \frac{\varepsilon}{7}$$

Continuity:- A function $f(x)$ is called continuous at the point $x = a$, if for any arbitrary chosen positive number ε, there exists a corresponding positive number δ (depending upon ε) such that,

$$|f(x) - f(a)| < \varepsilon$$

for all values of x for which $|x-a| < \delta$

$$\lim_{x \to a} f(x) = f(a)$$

Prob. ① Show that $\sin^2 x$ is continuous for all finite values of x.

Sol. Let a be any arbitrary finite value of x

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

$x \to a$
$|x-a| < \delta$

$$|f(x) - f(a)| = |\sin^2 x - \sin^2 a|$$

$$= |\sin x - \sin a||\sin x + \sin a|$$

$$= 2\left|\cos\frac{x+a}{2} \cdot \sin\frac{x-a}{2}\right| |\sin x + \sin a|$$

$$\leq 4 \left|\sin\frac{x-a}{2}\right| \qquad \because \sin\theta < \theta$$

$$< 4\left|\frac{x-a}{2}\right|$$

$$< 2|x-a|$$

$$< 2\delta = \varepsilon \Rightarrow \delta = \frac{\varepsilon}{2} > 0$$

Therefore, $0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Prob ② Test the following function

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & \text{When } x \neq 0 \\ 0, & \text{When } x = 0 \end{cases}$$

Sol.: $0 < |x-0| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$$\left|x \sin\frac{1}{x} - 0\right| = \left|x \sin\frac{1}{x}\right|$$

$$= |x| \left|\sin\frac{1}{x}\right|$$

$$< \delta \cdot 1 = \varepsilon \qquad \varepsilon > 0$$

$$< \delta = \varepsilon \qquad \delta > 0$$

Prob. ③ Show that $|x|$, $\cos x$, $\sin x$, $\sin^2 x$, $\cos^2 x$ are continuous.

Prob. (4) Determine the constant a and b so the function f defined below is continuous everywhere,

$$f(x) = \begin{cases} 2x+1 , & \text{for } x \le 1 \\ ax^2+b , & \text{for } 1 < x < 3 \\ 5x+2a , & \text{for } x \ge 3 \end{cases}$$

— (I)

Differentiation :- $\underset{\text{Dependent}}{y} = f(\underset{\text{independent}}{x})$ be function of variable x.

$\frac{dy}{dx}$ rate of change of variable y w.r.to variable x

$$y + \delta y = f(x+\delta x) \quad \text{— (II)}$$

$$\delta y = f(x+\delta x) - f(x)$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\Rightarrow \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx}$$

Prob. Prove that the function $f(x) = |x-1|$ is continuous at $x=1$ but not derivable at $x=1$.

Prob. The function f defined by

$$f(x) = \begin{cases} x^2+3x+a , & \text{for } x \le 1 \\ bx+2 , & \text{for } x > 1 \end{cases}$$

is given to be derivable for every x. Find a and b.

Mean Value Theorems

1. Roller Theorem
2. Lagrange mean value theorem
3. Cauchy mean value Theorem

→ Rolle's theorem

If a function f with domain $[a,b]$ is such that it is

(I) continuous in the closed interval $[a,b]$

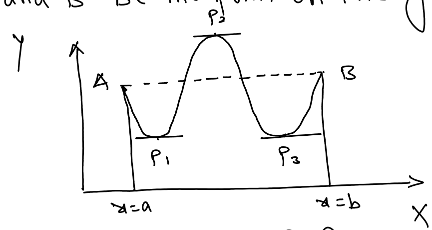
(II) derivable in the open interval (a,b)

(III) $f(a) = f(b)$

then there exists $c \in (a,b)$ such that $f'(c) = 0$

Geometrical interpretation of Rolle's Theorem

Let A and B be the point on the graph



on the graph $AP_1P_2P_3B$ of fun. $y=f(x)$ corresponding to $x=a$ and $x=b$.
Then geometrically Rolle's theorem asserts that there is at least one point between $x=a$ and $x=b$, at which tangent to the curve of the function, is parallel to x axis

Prob. (1) Verify ~~we~~ whether the fun $f(x)=\sin x$ in $[0,\pi]$ satisfies the conditions of Rolle's theorem and hence find c as prescribed in the theorem

Sol. Given $f(x)=\sin x$ in $[0,\pi]$

Here, $f(0)=0=f(\pi)$. Also, the fun. $\sin x$ is continuous in $[0,\pi]$ and differentiable in the $(0,\pi)$. Hence f satisfies all the conditions of Rolle's theorem, therefore $\exists$ at least one 'c' $\in (a,b)$, $c\in(0,\pi)$ such that

$$f'(c)=0$$

$$f(x)=\sin x$$

$$f'(x)=\cos x=0$$

$$f'(c)=\cos c=0$$

$$\cos c=\cos\frac{\pi}{2}$$

$$c=\pi/2$$

$$c\in(0,\pi)$$

Hence, Rolle's theorem satisfied.

Prob. (2) Verify Rolle's theorem for

$$f(x)=|x| \quad \text{in } [-1,1]$$

since $Rf'(0)\neq Lf'(0)$, $f(x)$ is not differentiable at $x=0$
$\quad\;(1)\qquad(-1)$

Hence, Rolle's theorem is not applicable to $f(x)=|x|$ in $[-1,1]$

→ Verify Rolle's theorem

Prob. $f(x)=(x-a)^m(x-b)^n$, $x\in[a,b]$, m,n being positive integer.

Prob. $f(x)=2+(x-1)^{2/3}$ in $[0,2]$

Prob. $f(x)=\log\{(x^2+ab)/(a+b)x\}$ in the interval $[a,b]$, $0<a<b$

Prob. $f(x)=x^3-6x^2+11x-6$ in $[1,3]$.

P.b1 : $f(x) = x^3 - 6x^2 + 11x - 6$ in $[1, 3]$