UNIT-④ sequence and series

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1 **sequence:**

Bounded above sequence

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is called bounded above sequence if there exist a real number M such that

$$a_n \leq M \quad , \quad \forall n \in N$$

Bounded below - $\{a_n\}_{n=1}^{\infty}$ — m

$$m \leq \{a_n\}$$

$$m \leq a_n \quad \forall n \in N$$

Bounded: if bounded above and bounded below

$f: N \to R$

$$m \leq a_n \leq M \quad \forall n \in N$$

for example $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{1}{n}$ $\forall n \in N$

$$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots \infty\}$$

0.5
0.4
0.3
0.2
0.1
0.001
0.002

≥ 0

0

Here, $UB = 1$
$LB = 0$

$\{a_n\} \to$ Bounded sequence.

→ **Monotonic increasing sequence**

$$a_{n+1} \geq a_n \quad \forall n \in N$$

Monotonic Decreasing sequence

$$a_{n+1} \leq a_n \quad \forall n \in N$$

→ **Strictly increasing**

$$a_{n+1} > a_n \quad \forall n \in N$$

strictly decreasing

$$a_{n+1} < a_n \quad , \forall n \in N$$

Limit of a sequence:

Let $\{a_n\}_{n=1}^{\infty}$. If for a given $\varepsilon > 0$ there corresponds a positive number m (dependent on ε)

$$\checkmark \; |a_n - l| < \varepsilon \quad \forall n \geq m \; , n \in N$$

or

$|a_n - l| < \varepsilon \quad \forall\, n \geq m$, $n \in N$

or

$\lim_{n\to\infty} a_n = l$

or

$l - \varepsilon < a_n < l + \varepsilon$

Prob. The limit of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is 0

$\varepsilon > 0$

$|a_n - 0| < \varepsilon \quad \forall\, n \geq m$

$|a_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$

Provided $n > \frac{1}{\varepsilon}$

$n > m$ (m is positive no.)

Convergence of a sequence

$\{a_n\}_{n=1}^{\infty} \longrightarrow$ convergent

$\lim_{n\to\infty} a_n = l$

or $|a_n - l| < \varepsilon$, $n \geq m$

Divergence If $\lim_{n\to\infty} a_n = +\infty$ or $-\infty$

$\{a_n\} \longrightarrow$ Divergent.

Cauchy's sequence

Definition. ① Let $\{a_n\}_{n=1}^{\infty}$. $\{a_n\} \longrightarrow$ cauchy sequence if for each $\varepsilon > 0$ there exist $n_0 \in N$ such that

$|a_m - a_n| < \varepsilon \quad \forall\, m, n \geq n_0$

~~Def~~ **Definition**: $\{a_n\}_{n=1}^{\infty}$ if for each $\varepsilon > 0$ $\exists$ a positive interger m such that

$|a_{n+p} - a_n| < \varepsilon \quad \forall\, n \geq m$. for each $p > 0$

[cut off] that the sequence $\{\frac{n}{\ }\}$, $n \in N$ is convergent.

Prob ① show that the sequence $\left\{\frac{n}{n+1}\right\}$, $n \in N$ is convergent.

$$\{a_n\} = \left\{\frac{n}{n+1}\right\}$$

$$a_n = \frac{n}{n+1}$$

$$a_{n+p} = \frac{n+p}{n+p+1}$$

$$|a_{n+p} - a_n| = \left|\frac{n+p}{n+p+1} - \frac{n}{n+1}\right|$$

$$= \left|\frac{p}{(n+p+1)(n+1)}\right|$$

By cauchy's priniple of convergence if the sequence is convergent then $\exists\, m \in N$ such that $n \geq m$ ~~∀~~ for each $p > 0$

$$|a_{n+p} - a_n| < \varepsilon \quad , \quad n \geq m \text{ and } p > 0$$

if $\left|\frac{p}{(n+p+1)(n+1)}\right| < \varepsilon$

$\Rightarrow \quad |a_{n+p} - a_n| < \varepsilon \qquad n \geq m$

$$\left|\frac{(n+1)(n+p+1)}{p}\right| > \frac{1}{\varepsilon}$$

$$(n+1) + \frac{(n+1)^2}{p} > \frac{1}{\varepsilon}$$

$$(n+1) > \frac{1}{\varepsilon}$$

$$n > \frac{1}{\varepsilon} - 1$$

Hence for $\varepsilon > 0$ $\exists\, m\left(\frac{1}{\varepsilon} - 1\right) \in$ ~~■~~ I such that

✓ $|a_{n+p} - a_n| < \varepsilon$, $n \geq m$, $p > 0$

Prob ② $\{a_n\} = \frac{3n-1}{4n+5}$ ~~conver~~ converges to $\frac{3}{4}$

$n \geq m$ $\qquad$ m is any positive integer.

$|a_n - l| < \varepsilon \quad , n \geq m$

$\exists m \in \mathbb{N}$

$\left| \frac{3n-1}{4n+5} - \frac{3}{4} \right| < \varepsilon \quad , \quad n \geq m \qquad |a_n - l| < \varepsilon , \ n \geq m$

$\left| \frac{3n-1}{4n+5} - \frac{3}{4} \right| = \left| \frac{3n-1}{4n+5} - \frac{3}{4} \right|$

$= \frac{19}{4(4n+5)} < \varepsilon \qquad n \geq m$

$\Rightarrow \quad n > \frac{1}{4}\left(\frac{19}{4\varepsilon} - 5 \right) \Rightarrow n > m$

Thus for any $\varepsilon > 0$, $\exists m \left[\frac{1}{4} \left(\frac{19}{4\varepsilon} - 5 \right) \right]$ $(\in \mathbb{N})$ such that

$|a_n - l| < \varepsilon \quad , \quad n \geq m \qquad |a_n - l| < \varepsilon$ for $n \geq m$

$\{a_n\} \to \frac{3}{4}$

Prob. ③ Prove that $\left\{ \frac{2n-7}{3n+2} \right\}$

① is monotonic increasing $\qquad$ ⑪ bounded below

$a_{n+1} \geq a_n \qquad\qquad n \in \mathbb{N}$

$-1 \leq \frac{2n-7}{3n+2}$

Series:

Let $u_1, u_2, u_3 \ldots u_n \ldots$ be a sequence of real no. (first, second, nth)

$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is called an infinite series or simply series.

$\sum_{n=1}^{\infty} u_n$ or $\sum u_n$

finite series $\{u_n\}_{n=1}^{\infty}$

$u_1 + u_2 + u_3 + \cdots + u_n = \sum u_n$

sequence of partial sums of a series

∞

Sequence of partial sums of a series

$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots\cdots$ be a given

$S_n = u_1 + u_2 + u_3 + \cdots + u_n \qquad \sum u_n$

Then S_n is called the partial sum of first n terms of a given series.

Convergence of a series

Let $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots\cdots$. Let S_n be ~~the~~ its nth partial sum.

$S_n \longrightarrow S$ as n is taken infinitely large in such a manner that for sufficiently large n the difference b/w S_n and S can be made as small as.

$\sum u_n$ is convergent.

$|S_n - S| < \varepsilon \quad \forall\, n \geq m$

An infinite series $\sum u_n$ is said to ~~be~~ converge to a sum S if for any given positive number ε, we can find positive number m such that

$|S_n - S| < \varepsilon \;, \quad \forall\, n \geq m$

$S_n \rightarrow S$ as $n \rightarrow \infty$

For Example : The nth partial sum of the geometric series

$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdots$ is convergent.

$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$

$$= \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}} \qquad \text{— (1)}$$

from $\left(\frac{1}{2^{n-1}}\right) \rightarrow$ as $n \rightarrow \infty$ Thus by taking n sufficiently large from $\frac{1}{2^{n-1}} \to$ as $n \to \infty$ Thus by taking n sufficiently large

$\frac{1}{2^{n-1}}$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2 - \frac{1}{2^{n-1}}\right) = 2$$

$$\lim_{n\to\infty} s_n = 2$$

Prob. (2) $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$

$$s_n = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} \qquad r > 1$$

$$\frac{1(2^n - 1)}{2-1} = 2^n - 1 \quad \text{——— (1)}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2^n - 1 = \infty$$

sequence of partial sum $\{s_n\}$ is divergent.

Prob. (3) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} + \cdots$

$$s_n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n.(n+1)}$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = 1 - 0 = 1$$

$s_n \to 1$ as $n \to \infty$

Prob. (2) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$

(1) Comparison Test

If Σu_n and Σv_n be two series and

(a) Σv_n is convergent, then Σu_n is also convergent if

$\lim_{n\to\infty} \frac{u_n}{v_n} = k$, where k is the positive, non-zero number.

(b) Σv_n is divergent, then Σu_n is also divergent if

$$\lim_{n\to\infty} \frac{u_n}{v_n} = k \quad , \quad k \text{ is finite positive number.}$$

The method of finding v_n

Let

$$u_n = \frac{a n^q + b n^{q-1} + c n^{q-2} + \cdots}{\alpha n^p + \beta n^{p-1} + \gamma n^{p-2} + \cdots}$$

then

$$v_n = \frac{1}{n^{p-q}}$$

If $u_n = \frac{a}{n^k} + \frac{b}{n^{k+1}} + \cdots$

$$v_n = \frac{1}{n^k}$$

(2) **P-series**

The infinite series

$$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is convergent if $p>1$ and
divergent if $p \le 1$.

Prob. (1) Test for convergence of the following series

$$u_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1} + \cdots$$

so that $u_n = \frac{1}{2n-1}$

$\therefore \quad v_n = \frac{1}{n}$ ✓ $\qquad\qquad \Sigma v_n$ is auxiliary series.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2n-1} \cdot \frac{n}{1} =$$

$$= \lim_{n\to\infty} \frac{1}{2 - 1/n}$$

$$= \frac{1}{2}, \quad \frac{1}{2} \text{ is a finite non-zero number.}$$

Thus, comparison test can be applied. Now by p-series test the auxiliarly series $\Sigma \frac{1}{n}$ is divergent for $p=1$

$\therefore$ by comparision test

Σv_n is divergent $\Rightarrow$ Σu_n is divergent

Prob. (2) $\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \cdots$

$$u_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$v_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$$

Prob. (3) Test for convergence of the series

(1) $\left(\frac{n^2+2}{2n^2+15}\right)^{1/3}$

(2) $\frac{14}{1^3} + \frac{24}{2^3} + \frac{34}{3^3} + \cdots + \frac{10n+\cancel{4}}{n^3}$

(3) $u_n = \frac{\sqrt{n}}{n^2+1}$

(4) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots \infty$

(5) $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

(6) $u_n = \frac{1}{n} \sin \frac{1}{n}$