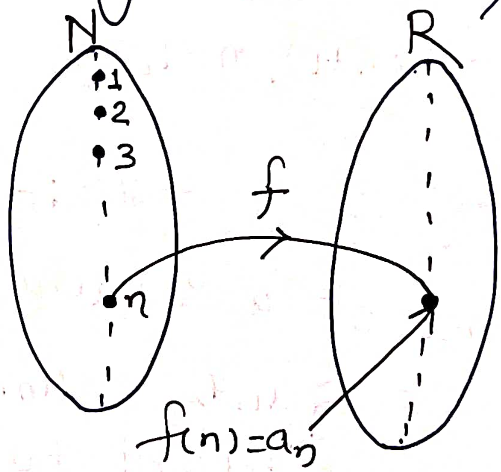
Sequences and Series

Sequences:- A sequence of real numbers is a function whose domain is the set N of natural numbers and whose range is the set R of real numbers. Symbolically, the function $f: N \to R$ which is defined by

$$f(n) = a_n \quad \forall\, n \in N$$

is a sequence of real numbers.

For example, let $f(n) = \frac{1}{n}$, where $n \in N$ then the values of this function corresponding to $n = 1, 2, 3, \ldots$ are respectively $f(1) = 1$, $f(2) = 1/2$, $f(3) = 1/3, \ldots, f(n) = 1/n, \ldots$



Thus, we get an ordered set $S = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$.

Hence the expression

$$\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$$ is a sequence of real numbers.

Examples:-

i) The ordered set of numbers $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$

is a sequence whose nth term is $a_n = f(n) = \frac{1}{n}$, $n = 1, 2, 3, \ldots$.

ii) The ordered set of numbers

$$\left\{ \frac{1}{3}, \frac{2}{5}, \frac{3}{4}, \ldots, \frac{n}{2n+1}, \ldots \right\}$$

is a sequence whose nth term is $a_n = \frac{n}{2n+1}$, $n = 1, 2, 3, \ldots$

iii) The ordered set of numbers $\{-1, 1, -1, 1, \ldots, (-1)^n \ldots\}$ is a sequence whose nth term is $a_n = (-1)^n$, $n = 1, 2, 3, \ldots$

iv) The ordered set of numbers $\{3, 5, 7, 9, \ldots (2n+1), \ldots\}$ is a sequence whose nth term is $a_n = (2n+1)$, $n = 1, 2, 3, \ldots$

<u>Range of a seq^n</u>:- The set of all distinct terms of a sequence is called its range. Thus, if $S = \{a_1, a_2, \ldots a_n, \ldots\}$ be a seq^n then its range $R(S)$ is defined by

$$R(S) = \{a_n : n \in \mathbb{N}\}.$$

Since $n \in \mathbb{N}$ in any seq^n $\{a_n\}_{n=1}^{\infty}$ and $\mathbb{N}$ is an infinite set, so the number of terms in a seq^n is infinite, but the range of a seq^n may be a finite set.

<u>Example</u> - If $a_n = (-1)^n$ $(n \in N)$ then the seq^n is $\{a_n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \cdots\}$, so that the range of the seq^n $\{a_n\}_{n=1}^{\infty} = \{-1, 1\}$ which is a finite set.

<u>Constant seq^n</u> :-

A seq^n $\{a_n\}_{n=1}^{\infty}$ whose nth term is defined by $a_n = c \in \mathbb{R}$ $\forall n \in N$ is called a constant seq^n.

<u>Example</u> - $\{a_n\}_{n=1}^{\infty} = \{1, 1, 1, \cdots\}$ is a constant seq^n whose range is $\{1\}$.

<u>Bounded seq^n</u> :-

A seq^n $\{a_n\}_{n=1}^{\infty}$ is called bounded if there are two real numbers M and m such that

$$m \leq a_n \leq M \quad \forall n \in N.$$

<u>Example</u> - Discuss on the seq^n $\{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{1}{n}$. $\forall n \in N$.

<u>sol^n</u> - Here $\{a_n\}_{n=1}^{\infty} = \{1, 1/2, 1/3, \cdots\}$ clearly 1 is the least number which is greater than or equal to each and every term of the seq^n; also 0 is the greatest number which is less than or equal to each and every term.

Hence the upper bound of the seq is 1 and the lower bound is 0.

Also $0 < a_n \leq 1 \quad \forall n \in N$.

$\therefore \{a_n\}_{n=1}^{\infty}$ is a bounded seq^n.

<u>Monotonic seq^n</u> :-

<u>Monotonic increasing seq^n</u> :-

A seq^n $\{a_n\}_{n=1}^{\infty}$ is said to be monotonic increasing if $a_{n+1} \geq a_n \quad \forall n \in N$.

In other words, a seq^n is said to be monotonic increasing if the values of its terms never decrease.

If $a_{n+1} > a_n \ \forall n \in N$ then the seq^n $\{a_n\}_{n=1}^{\infty}$ is said to be strictly increasing.

<u>Monotonic decreasing seq^n</u> -

A seq^n $\{a_n\}_{n=1}^{\infty}$ is said to be monotonic decreasing if $a_{n+1} \leq a_n \quad \forall n \in N$.

In other words, a seq^n is said to be monotonic decreasing if the values of its term never increase.

If $a_{n+1} > a_n \ \forall n \in N$, then the seq^n $\{a_n\}$ is said to be strictly decreasing.

The word 'monotonic' is used for both, increasing and decreasing sequence.

Examples- 1) The sequence $\{2.1, 2.11, 2.111, \dots\}$ is bounded and monotonic increasing.

2) The seq^n $\{-1, -3/2, -2, -5/2, -3, \dots\}$ is monotonic decreasing. This is bounded above, but not bounded.

3) The seq^n $\{3, 5, 7, 9 \dots\}$ is monotonic increasing. This is bounded below, but not bounded.

4) The sequence $\{1, -1, 1, -1, \dots\}$ is bounded, but this sequence is neither monotonic increasing nor monotonic decreasing.