UNIT-④ sequence and series

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1 **sequence:**

Bounded above sequence

Definition: A sequence $\{a_n\}_{n=1}^{\infty}$ is called bounded above sequence if there exist a real number M such that

$$a_n \le M \quad , \quad \forall n \in N$$

Bounded below - $\{a_n\}_{n=1}^{\infty}$ — m

$$m \le \{a_n\}$$

$$m \le a_n \quad \forall n \in N$$

Bounded: if bounded above and bounded below

$f: N \to R$

$$m \le a_n \le M \quad \forall n \in N$$

for example $\{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{1}{n}$ $\forall n \in N$

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \cdots \infty\right\}$$

0.5
0.4
0.3
0.2
0.1
0.001
0.002

≥ 0

0

Here, $UB = 1$
$LB = 0$

$\{a_n\} \to$ Bounded sequence.

→ **Monotonic increasing sequence**

$$a_{n+1} \ge a_n \quad \forall n \in N$$

Monotonic Decreasing sequence

$$a_{n+1} \le a_n \quad \forall n \in N$$

→ **Strictly increasing**

$$a_{n+1} > a_n \quad \forall n \in N$$

strictly decreasing

$$a_{n+1} < a_n \quad , \forall n \in N$$

Limit of a sequence:

Let $\{a_n\}_{n=1}^{\infty}$. If for a given $\varepsilon > 0$ there corresponds a positive number m (dependent on ε)

$$|a_n - l| < \varepsilon \quad \forall n \ge m \quad , n \in N$$

or

$|a_n - l| < \varepsilon \quad \forall\, n \geq m$, $n \in N$

or

$\lim_{n \to \infty} a_n = l$

or

$l - \varepsilon < a_n < l + \varepsilon$

Prob. The limit of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is 0

$\varepsilon > 0$

$$|a_n - 0| < \varepsilon \quad \forall\, n \geq m$$

$$|a_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$$

Provided $n > \frac{1}{\varepsilon}$

$n > m$ (m is positive no.)

Convergence of a sequence

$$\{a_n\}_{n=1}^{\infty} \longrightarrow \text{convergent}$$

$$\lim_{n \to \infty} a_n = l$$

or $|a_n - l| < \varepsilon$, $\quad n \geq m$

Divergence

If $\lim_{n \to \infty} a_n = +\infty$ or $-\infty$

$$\{a_n\} \longrightarrow \text{Divergent.}$$

Cauchy's sequence

Definition. (1) Let $\{a_n\}_{n=1}^{\infty}$. $\{a_n\} \longrightarrow$ cauchy sequence if for each $\varepsilon > 0$ there exist $n_0 \in N$ such that

$$|a_m - a_n| < \varepsilon \quad \forall\, m, n \geq n_0$$

~~Def~~ Definition: $\{a_n\}_{n=1}^{\infty}$ if for each $\varepsilon > 0$ $\exists$ a positive interger m such that

$$|a_{n+p} - a_n| < \varepsilon \quad \forall\, n \geq m, \text{ for each } p > 0$$

... that the sequence $\left\{\frac{n}{\ }\right\}$, $n \in N$ is convergent.

Prob ① show that the sequence $\left\{\frac{n}{n+1}\right\}$, $n \in \mathbb{N}$ is convergent.

$$\{a_n\} = \left\{\frac{n}{n+1}\right\}$$

$$a_n = \frac{n}{n+1}$$

$$a_{n+p} = \frac{n+p}{n+p+1}$$

$$|a_{n+p} - a_n| = \left|\frac{n+p}{n+p+1} - \frac{n}{n+1}\right|$$

$$= \left|\frac{p}{(n+p+1)(n+1)}\right|$$

By cauchy's priniple of convergence if the sequence is convergent then $\exists\, m \in \mathbb{N}$ such that $n \geq m$ for each $p > 0$

$$|a_{n+p} - a_n| < \varepsilon \quad , \quad n \geq m \text{ and } p > 0$$

$$\text{if } \left|\frac{p}{(n+p+1)(n+1)}\right| < \varepsilon$$

$$\Rightarrow \quad |a_{n+p} - a_n| < \varepsilon \qquad n \geq m$$

$$\left|\frac{(n+1)(n+p+1)}{p}\right| > \frac{1}{\varepsilon}$$

$$(n+1) + \frac{(n+1)^2}{p} > \frac{1}{\varepsilon}$$

$$(n+1) > \frac{1}{\varepsilon}$$

$$n > \frac{1}{\varepsilon} - 1$$

Hence for $\varepsilon > 0$ $\exists\, m\left(\frac{1}{\varepsilon} - 1\right) \in \mathbb{N}$ such that

$$|a_{n+p} - a_n| < \varepsilon \quad , \quad n \geq \tilde{m} \quad , \quad p > 0$$

Prob. ②

$\{a_n\} = \frac{3n-1}{4n+5}$ ~~cover~~ converges to $\frac{3}{4}$

$n \geq \tilde{m}$ m is any positive integer.

$|a_n - l| < \varepsilon \quad , n \geq m$

$\exists m \in \mathbb{I}$

$\left| \frac{3n-1}{4n+5} - \frac{3}{4} \right| < \varepsilon \quad , \quad n \geq m$ $\qquad |a_n - l| < \varepsilon , \quad n \geq m$

$$\left| \frac{3n-1}{4n+5} - \frac{3}{4} \right| = \left| \frac{3n-1}{4n+5} - \frac{3}{4} \right|$$

$$= \frac{19}{4(4n+5)} < \varepsilon \qquad n \geq m$$

$$\Rightarrow \quad n > \frac{1}{4}\left(\frac{19}{4\varepsilon} - 5 \right) \Rightarrow n > m$$

Thus for any $\varepsilon > 0$, $\exists\, m \left[\frac{1}{4}\left(\frac{19}{4\varepsilon} - 5 \right) \right] (\in \mathbb{I})$ such that

$|a_n - l| < \varepsilon \quad , \quad n \geq m$ $\qquad |a_n - l| < \varepsilon$ for $n \geq m$

$\{a_n\} \to \frac{3}{4}$

Prob. (3) Prove that $\left\{ \frac{2n-7}{3n+2} \right\}$

(i) is monotonic increasing $\qquad$ (ii) bounded below

$a_{n+1} \geq a_n$ $\qquad n \in \mathbb{N}$

$$-1 \leq \frac{2n-7}{3n+2}$$

Series:

Let $u_1, u_2, u_3 \cdots u_n \cdots$ be a sequence of real no. (first, second, nth)

$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is called an infinite series or simply series.

$$\sum_{n=1}^{\infty} u_n \quad \text{or} \quad \Sigma u_n$$

finite series $\{u_n\}_{n=1}^{\infty}$

$$u_1 + u_2 + u_3 + \cdots + u_n = \Sigma u_n$$

sequence of partial sums of a series

∞

Sequence of partial sums of a series

$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ be a given

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n \qquad \Sigma u_n$$

Then S_n is called the partial sum of first n terms of a given series.

Convergence of a series

Let $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots$. Let S_n be ~~the~~ its nth partial sum.

$S_n \longrightarrow S$ as n is taken infinitely large in such a manner that for sufficiently large n the difference b/w S_n and S can be made as small as.

Σu_n is convergent.

$$|S_n - S| < \varepsilon \quad \forall\ n \geq m$$

An infinite series Σu_n is said to ~~be~~ converge to a sum S if for any given positive number ε, we can find positive number m such that

$$|S_n - S| < \varepsilon, \quad \forall\ n \geq m$$

$$S_n \rightarrow S \text{ as } n \rightarrow \infty$$

For Example: The nth partial sum of the geometric series

$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is convergent.

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right) = 2 - \frac{1}{2^{n-1}} \quad \text{--- (1)}$$

From $\frac{1}{2^{n-1}} \rightarrow$ as $n \rightarrow \infty$ Thus by taking n sufficiently large from $\frac{1}{2^{n-1}} \to$ as $n \to \infty$ Thus by taking n sufficiently large

$\frac{1}{2^{n-1}}$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(2 - \frac{1}{2^{n-1}}\right) = 2$$

$$\lim_{n\to\infty} s_n = 2$$

Prob. (2) $1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$

$$s_n = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} \qquad r > 1$$

$$\frac{1(2^n - 1)}{2-1} = 2^n - 1 \quad \text{———} \quad (1)$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2^n - 1 = \infty$$

Sequence of partial sum $\{s_n\}$ is divergent.

Prob. (3) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} + \cdots$

$$s_n = \frac{1}{1.2} + \frac{1}{2.3} + \cdots + \frac{1}{n.(n+1)}$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = 1 - 0 = 1$$

$$s_n \to 1 \text{ as } n \to \infty$$

Prob. (2) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$

(1) Comparison Test

If Σu_n and Σv_n be two series and

(a) Σv_n is convergent, then Σu_n is also convergent if

$$\lim_{n\to\infty} \frac{u_n}{v_n} = k,$$ where k is the positive, non-zero number.

(b) Σv_n is divergent, then Σu_n is also divergent if

$$\lim_{n\to\infty} \frac{u_n}{v_n} = K \quad , \; K \text{ is finite positive number.}$$

The method of finding v_n

Let

$$u_n = \frac{a n^{q} + b n^{q-1} + c n^{q-2} + \cdots}{\alpha n^{p} + \beta n^{p-1} + \gamma n^{p-2} + \cdots}$$

then $v_n = \dfrac{1}{n^{p-q}}$

If $u_n = \dfrac{a}{n^k} + \dfrac{b}{n^{k+1}} + \cdots$

$$v_n = \frac{1}{n^k}$$

(2) P-series

The infinite series

$$\Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is convergent if $p > 1$ and
divergent if $p \le 1$.

Prob. (1) Test for convergence of the following series

$$u_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n-1} + \cdots$$

so that $u_n = \dfrac{1}{2n-1}$

$\therefore \quad v_n = \dfrac{1}{n}$ ✓ $\qquad \Sigma v_n$ is auxiliary series.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2n-1} \cdot \frac{n}{1} =$$

$$= \lim_{n\to\infty} \frac{1}{2 - 1/n}$$

$$= \frac{1}{2}, \quad \frac{1}{2} \text{ is a finite non-zero number.}$$

Thus, comparison test can be applied. Now by p-series test

the auxiliarly series $\Sigma \frac{1}{n}$ is divergent for $p=1$

$\therefore$ by comparision test

Σv_n is divergent $\Rightarrow$ Σu_n is divergent

Prob. (2) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \cdots$

$$u_n = \frac{2n-1}{n(n+1)(n+2)}$$

$$v_n = \frac{1}{n^3-1} = \frac{1}{n^2}$$

Prob. (3) Test for convergence of the series

(1) $\left(\frac{n^2+2}{2n^2+15}\right)^{1/3}$

(2) $\frac{14}{1^3} + \frac{24}{2^3} + \frac{34}{3^3} + \cdots + \frac{10n+4}{n^3}$

(3) $u_n = \frac{\sqrt{n}}{n^2+1}$

(4) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots \infty$

(5) $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

(6) $u_n = \frac{1}{n} \sin \frac{1}{n}$

D' Alembert's Ratio Test

The series Σu_n of positive term is

(i) convergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$

(ii) Divergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$

(iii) Test fail if $\lim \frac{u_{n+1}}{u_n} = 1$

(ii) Divergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$

(iii) Test fail if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$

Prob. ① $\cancel{2x} + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots + \frac{(n+1)}{n^3} x^n + \cdots \qquad x > 0$

Convergent or divergent?

Sol:
$$u_n = \frac{(n+1)}{n^3} x^n$$

$$u_{n+1} = \frac{n+2}{(n+1)^2} x^{n+1}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{n+2}{(n+1)^4} \cdot n^3 \cdot x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$$

if $x < 1 \longrightarrow C$

$x > 1 \longrightarrow D$

If $x = 1$ then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, so this test is fail,

$$u_n = \frac{n+1}{n^3}$$

Then $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{n+1}{n^3} \cdot \frac{n^2}{1} = \frac{n+1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$$

$\Sigma v_n \to C \Rightarrow \Sigma u_n \Rightarrow C$

$\Sigma v_n \to D \Longrightarrow \Sigma u_n \Rightarrow D$

$v_n = \frac{1}{n^2}$

$P > 1 \therefore \Sigma v_n \to C \Rightarrow \Sigma u_n \Rightarrow C$

Prob. ② $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \cdots \qquad x > 0$

$u_n = \frac{x^n}{n(n+1)} \qquad u_{n+1} = \frac{(x)^{n+1}}{(n+1)(n+2)}$

$$\lim \frac{u_{n+1}}{u_n} = x$$

$n \to \infty \quad u_n$

$\alpha < 1 \longrightarrow C$

$\alpha > 1 \longrightarrow D$

$u_n = \frac{1}{n(n+1)}$

(2) Cauchy's Root Test: $\quad \Sigma u_n = [\;\;]^n$

A series of positive term Σu_n is convergent or divergent according as

$\lim_{n\to\infty} u_n^{1/n} < 1$ —— convergent

$\lim_{n\to\infty} u_n^{1/n} > 1 \longrightarrow$ Divergent

$\lim_{n\to\infty} u_n^{1/n} = 1 \longrightarrow$ case fail.

Prob. (1) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$

Sol. $u_n = \left[\left[\frac{n+1}{n}\right]^{n+1} - \frac{n+1}{n}\right]^{-n}$

$\cancel{n} \times \left(\frac{1}{\cancel{n}}\right) = -1$

$\therefore \quad u_n^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$

$= \left(\frac{n+1}{n}\right)^{-1} \left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-1}$

$= \left(1 + \frac{1}{n}\right)^{-1} \left[\left(1 + \frac{1}{n}\right)^n - 1\right]^{-1}$

$\left\{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right\}$

$\lim_{n\to\infty} u_n^{1/n} = (e-1)^{-1}$

$= \frac{1}{e-1} < 1$

$\lim_{n\to\infty} u_n^{1/n} < 1 \longrightarrow$ convergent.

Prob (2) $\quad u_n = \frac{n^{n^2}}{(1+n)^{n^2}}$

n^2

$$u_n = \left[\frac{a\,n}{n+1}\right]$$

$$\lim_{n\to\infty} u_n^{1/n} = \frac{1}{e} < 1$$

③ Raabe's Test

Σu_n of the positive term is C or D

Comparison Test (Ratio test) → P-Test → D'Alembert → Cauchy → Raabe's → Log.

(i) $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1 \longrightarrow C$

(ii) $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1 \longrightarrow D$

Prob ①

$$x^2 + \frac{2^2}{3\cdot 4}x^4 + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6}x^6 + \frac{2^2\cdot 4^2\cdot 6^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}x^8 + \cdots \quad x > 0$$

$$u_n = \frac{2^2\cdot 4^2\cdot 6^2 \cdots (2n)^2}{3\cdot 4\cdot 5\cdot 6 \cdots (2n+1)(2n+2)}\, x^{2n+2}$$

$$u_{n+1} = \frac{2^2\cdot 4^2\cdot 6^2 \cdots (2n)^2 (2n+2)^2}{3\cdot 4\cdot 5\cdot 6 \cdots (2n+3)(2n+4)}\, x^{2n+4}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n+2)^2}{(2n+3)(2n+4)}\, x^2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{4n^2\left(1+\frac{1}{2n}\right)^2}{4n^2\left(1+\frac{3}{2n}\right)\left(1+\frac{4}{2n}\right)} \cdot x^2$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x^2$$

D'Alembert → if $x^2 < 1 \longrightarrow C$

$x^2 > 1 \longrightarrow D$

If $x^2 = 1$ then test is fail, we use ~~Ra~~ Raabe's Test.

$$\frac{u_{n+1}}{u_n} = \frac{(2n+2)^2}{(2n+3)(2n+4)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)(2n+4)}{\cdot} = \frac{4n^2+14n+12}{4n^2+8n+4}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+3)(2n+4)}{(2n+2)^2} = \frac{4n^2+14n+12}{4n^2+8n+4}$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{6n^2+8n}{4n^2+8n+4} = \frac{3n^2+4n}{2n^2+4n+2}$$

$$= \frac{3+\frac{4}{n}}{2+\frac{4}{n}+\frac{2}{n^2}}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{3}{2} > 1$$

$\rightarrow$ convergent.

Logarithmic Test

$\frac{u_{n+1}}{u_n}$ Raabe log

Σu_n of positive terms

① $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) > 1 \longrightarrow C$

$< 1 \longrightarrow D$

Prob. ① $1+\frac{2x}{\lfloor 2} + \frac{3^2x^2}{\lfloor 3} + \frac{4^3x^3}{\lfloor 4} + \cdots \quad x>0$

D'Alembert's Ratio Test Problems

Prob. ① $\frac{x}{1\cdot2}+\frac{x^2}{2\cdot3}+\frac{x^3}{3\cdot4}+\frac{x^4}{4\cdot5}+\cdots; \quad x>0$

Prob ② $\frac{x}{1\cdot2}+\frac{x^2}{3\cdot4}+\frac{x^3}{5\cdot6}+\frac{x^4}{7\cdot8}+\cdots \quad x>0$

Prob ③ $x+\frac{3}{2}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\cdots \quad x>0$

Prob ④ $1+3x+5x^2+7x^3+9x^4+\cdots \quad x>0$

Prob ⑤ $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots \quad x>0$

Prob ⑥ $2+\frac{3}{2}x+\frac{4}{3}x^2+\frac{5}{4}x^3+\cdots \quad x>0$

Prob ⑦ $\frac{x^n}{\lfloor n}$ Prob ⑧ $\left[\sqrt{n^2+1}-n\right]x^{2n}$ Prob ⑨ $\frac{(nx)^n}{\lfloor n}, \quad x>0$

Cauchy's Root Test

(1.) $\Sigma\left(1+\frac{1}{n}\right)^{-n^2}$ (2.) $\Sigma \frac{x^n}{\lfloor n}$ $x>0$

(3.) $\frac{1^3}{3}+\frac{2^3}{3^2}+\frac{3^3}{3^3}+\frac{4^3}{3^4}+\cdots+\frac{n^3}{3^n}$

~~Rab~~ Raabe's Test

(1.) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^2}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^3}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^5}{7}+\cdots\cdots$

$x>0$

(2.) $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\cdots\cdots\cdots$ $x>0$

Logarithmic Test

(1) $x+\frac{2^2x^2}{\lfloor 2}+\frac{3^3x^3}{\lfloor 3}+\frac{4^4x^4}{\lfloor 4}+\cdots\cdots$ $x>0$

(2) $1+\frac{x}{2}+\frac{\lfloor 2}{3^2}x^2+\frac{\lfloor 3}{4^3}x^3+\cdots\cdots$ $x>0$

Alternative series

A series of the type $u_1-u_2+u_3-u_4+\cdots$ where $u_n>0,\ \forall n\in N$ is called an alternative series. is denoted by

$$\Sigma(-1)^{n-1}u_n = u_1-u_2+u_3-u_4+\cdots+(-1)^{n-1}u_n+\cdots$$

For example

(i) $\Sigma(-1)^{n-1} = 1-1+1-1\cdots\cdots$

(ii) $\Sigma(-1)^{n-1}\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots\cdots$

Leibnitz Test for convergence of an alternative series

If an alternative series

$$\sum_{n=1}^{\infty}(-1)^{n-1}u_n = u_1-u_2+u_3-u_4+\cdots\cdots$$

(i) each term is numerically less than the preceding term i.e.

$$u_{n+1} < u_n \quad \forall n \in N$$

(ii) $\lim\limits_{n\to\infty} u_n = 0$

then the series $\Sigma(-1)^{n-1} u_n$ convergent.

Prob. (1) $1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \cdots$

sol: (i) Each term is numerically less than the preceding term i.e.

$$U_{n+1} < u_n \quad \forall n \in N$$

(ii) $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\lfloor n} = 0$

so, according to Leibnitz's test the given alternative series is convergent.

Absolute convergence

A series Σu_n which contains both positive and negative terms is said to be absolute convergent if the series $\Sigma|u_n|$ is convergent.

For example :-

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots + (-1)^{n-1}\frac{1}{2^{n-1}}$$

is absolutely convergent.

$$\Sigma|u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

$$\frac{a(1-r^n)}{1-r} = \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$$

(ii) $\Sigma(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$$\Sigma|u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is not absolutely convergent.

Test for absolutely convergent

Σu_n — alternative series

$\Sigma |u_n|$

(i) $\lim_{n\to\infty} \left| \frac{u_n}{u_{n+1}} \right| > 1$

(ii) $\lim_{n\to\infty} \left\{ n \cdot \left| \frac{u_n}{u_{n+1}} \right| - 1 \right\} > 1$

(iii) $\lim_{n\to\infty} n \log \left| \frac{u_n}{u_{n+1}} \right| > 1$

Conditional convergent

A series Σu_n is said to be conditionally convergent or semi convergent if Σu_n is convergent but it not absolutely convergent.

For example $\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$
is conditionally convergent because Σu_n is convergent

$\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

is divergent.

Theorem (1) Every absolute convergent series is convergent but not conversely,

Power series:

A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

Where a_n represent the coefficient of the nth term.

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Interval and Radius of convergence

An interval $(-R, R)$ in which a power series converge is called the interval of convergence). The number R is called the radius of convergence.

$(-\infty, \infty)$ $(-1, 1)$

$R = 1$

Test for convergence

$u_n = a_n x^n$

$u_{n+1} = a_{n+1} x^{n+1}$

Applying D'Alembert Ratio test

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1} x^{n+1}}{a_n x^n}\right| = |x| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\ell}$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |x| \cdot \frac{1}{\ell}$$

$$= \left|\frac{x}{\ell}\right|$$

By D'Alembert Ratio test

$\left|\frac{x}{\ell}\right| < 1$ i.e. $|x| < \ell$

i.e. $-\ell < x < \ell$

Interval of the convergence of the series is $(-\ell, \ell)$ and the radius of the convergence is ℓ.

Prob. 1 obtain the range of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{a+\sqrt{n}} \qquad x > 0, \quad a > 0$$

sol: Let $u_n = \frac{x^n}{a+\sqrt{n}}$

$$u_{n+1} = \frac{x^{n+1}}{a+\sqrt{n+1}}$$

$= a + \sqrt{n}\left(1 + \frac{1}{\sqrt{n}}\right)$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \frac{x^{n+1}}{a+\sqrt{n+1}} \times \frac{a+\sqrt{n}}{x^n}$$

$$= x \cdot 1$$

$$= x \qquad x > 0$$

By D'Alembert Ratio test

1. convergent if $x < 1$

(ii) ~~Divergen~~ Divergent if $x > 1$

If $x = 1$

$$u_n = \frac{1}{a + \sqrt{n}}$$

$$v_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{\sqrt{n}}{a + \sqrt{n}}$$

$$= 1$$

According to P-Test

$$\Sigma v_n = \Sigma \frac{1}{n^{1/2}}$$

$$P = \frac{1}{2} < 1$$

$\Sigma v_n \longrightarrow$ Divergent $\Rightarrow$ $\Sigma u_n \Rightarrow$ divergent.

$x > 0$ $\qquad x < 1 \longrightarrow$ convergent

The series is convergent $0 < x < 1$

The range of the convergence is

$$0 < x < 1$$

Prob. show that the following series are convergent.

(a) $1^{-P} - 2^{-P} + 3^{-P} - \ldots\ldots$ when $P > 0$

Prob. (2) Show that the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

is absolutely convergent.

Prob (3) Test for absolute convergence

(a) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots\ldots \infty$

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$

Prob. (4) Test for convergence the series

$$\sum \frac{(-1)^{n+1}}{\sqrt{n}}$$

Prob (5) Test for absolute convergence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Prob (6) show that

$$\sum (-1)^n \left\{ \sqrt{n^2+1} - n \right\}$$

is conditionally convergent.

Prob (7) Test for convergence and absolute convergence

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots \quad \text{for } p>0$$

Prob (8.) Test for absolute convergence

(a) $1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$

(b) $1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} \cdots$

(c) $x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots$

Prob (9) state the value of x for which the following series converges.

(i) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \cdots \infty$

(ii) $\frac{1}{1-x} + \frac{1}{2(1-x^2)} + \frac{1}{3(1-x^3)} + \cdots \infty$