EE510 Linear Algebra for Engineering

Week 1 Session 1

Review:

Logical Inference

Logical Statement P and Q

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
1	1	0	0	1	1	1	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	0	0
0	0	1	1	0	0	1	1

$\wedge$ is AND

$\vee$ is OR

$\implies$ If then

$\iff$ If and only if

Conditional: $P \implies Q$

Contrastive: $\neg P \implies \neg Q$

Converse: $Q \implies P$

Predicate: Px means x is P

Quantifier: $\forall x$ (universal) means "for all x"

$\exists x$ (existential) means "for some x"

$\forall x$: Px means "Everything is P"

$Px_1\ AND\ Px_2\ AND\ Px_3\ AND$

$\exists x$: Px means "Something is P"

$Px_1\ OR\ Px_2\ OR\ Px_3\ OR$

Rules of Inference:

- Modus Ponens: Affirming the antecedent

Premise 1: $P \implies Q$

Premise 2: P

Conclusion: Q

- Modus Tollens: Denying the consequent

Premise 1: $P \implies Q$

Premise 2: $\neg Q$

Conclusion: $\neg Q$

- Mathematical Induction

Goal: Proof that $P_n \forall n \geq n_0$ where n_0 is usually 0 or a positive number

1. Basis step: P_{n0}
2. Induction step:

   P_{n0} & $P_{n-1} \implies P_n$

   Assume P_{n0} and P_{n-1} then show P_n

Set Theory

set: a collection of elements

$x \in A$, where x is element, A is set, $\in \equiv$ Element hood

$A = \{a_1, a_2, \ldots, a_n\}$

Subset: $A \subset X$, $B \subset X$

$A \subset X$ if and only if $\forall x \in A$, $x \in A$

$A^c = \{x \in X : X \notin A\}$

$A \bigcup B = \{x \in X : x \in A \, OR \, x \in B\}$

$A \bigcap B = \{x \in X : x \in A \, AND \, x \in B\}$

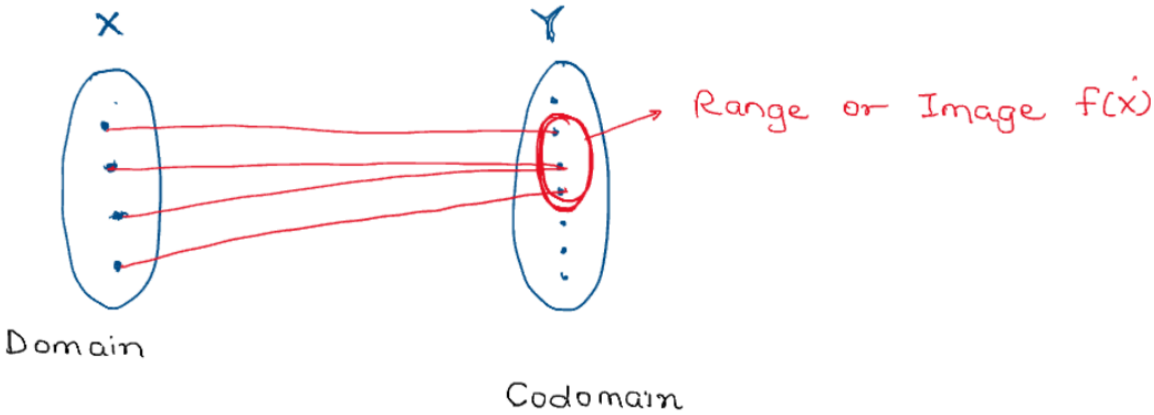
De Morgan's Law:

$A \bigcup B = (A^c \bigcap B^c)^c$

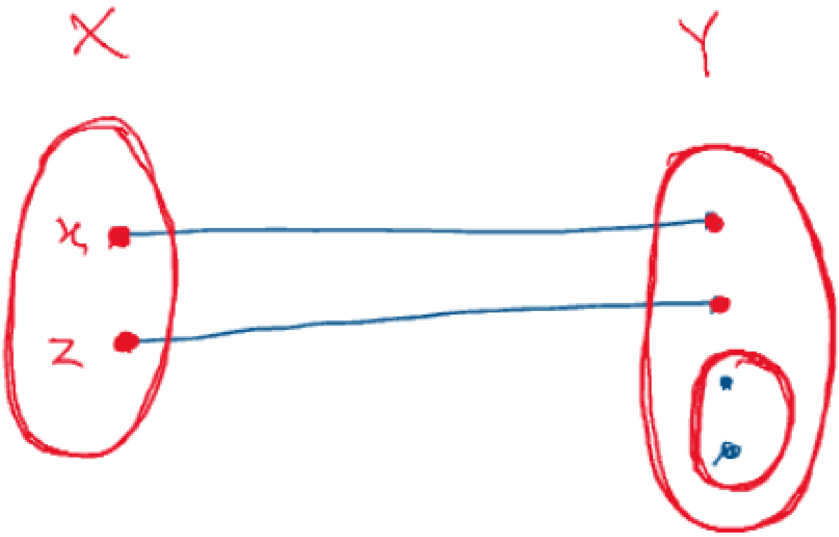
$A \bigcap B = (A^c \bigcup B^c)^c$

Function

$f : X \implies Y$

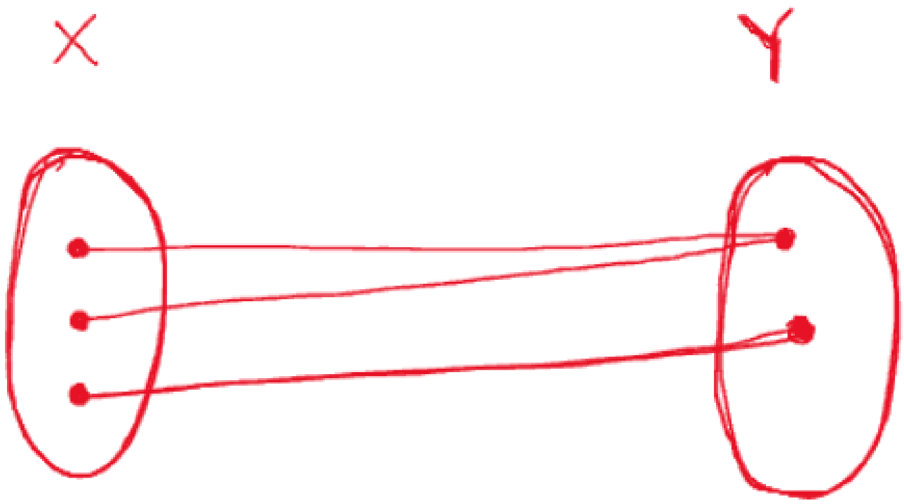


Injective function:



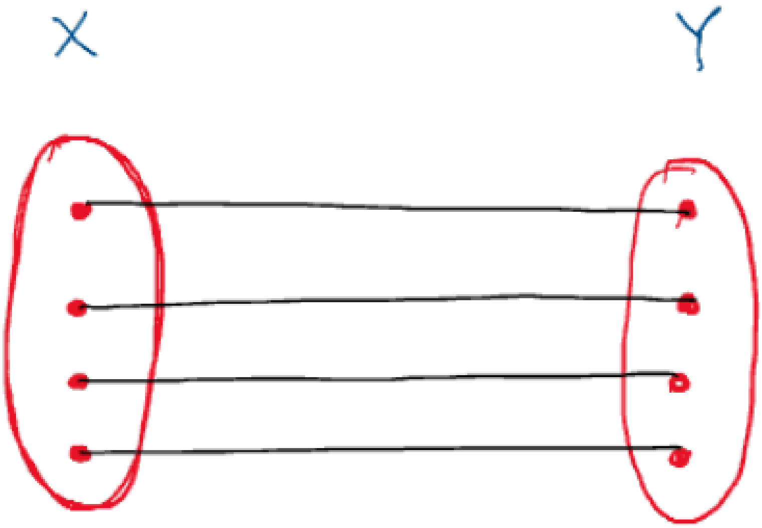
f is injective if and only if $\forall x, z \in X,\ f(x) = f(z) \implies x = z$

Surjective function:



$\forall y \in Y, \exists x \in X : f(x) = y$

Bijective Function (1-1 correspondence)



f is bijective if and only if f is injective and surjective.

Cardinality of a set

Finite set:

$A = \{a_1, \dots, a_n\}$, where $n \in \mathbb{Z}^+$

Infinite set:

1. Uncountably infinite

   $\mathbb{R}$

2. Countably infinite

   $\mathbb{Z}^+$

Example:

$f : \mathbb{Z}^+(1 - 1\, correspondence) \implies \mathbb{Z}^-$

Vectors

A vector is a 1-dimensional array of scalars over a field.

Let $V = \in \mathbb{R}^(n) : v_1, \dots, v_n \in \mathbb{R}$

For $u, v \in \mathbb{R}^n$

- Vector Addition:

$$u + v = \begin{bmatrix} u_1 + v_1 \\ \dots \\ u_n + v_n \end{bmatrix} \in \mathbb{R}^n$$

- Scalar Multiplication:

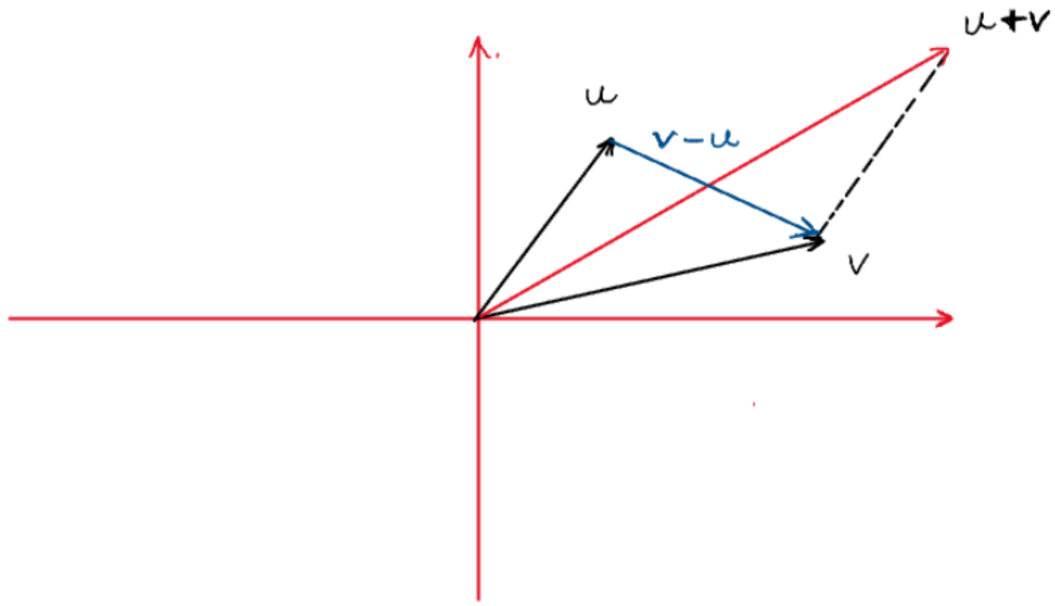
For $a \in \mathbb{R}, v \in R^n$

Then $av = \begin{bmatrix} av_1 \\ \dots \\ av_n \end{bmatrix}$

- Linear Combination:

For $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$

$$au + bv = \begin{bmatrix} au_1 \\ \dots \\ au_n \end{bmatrix} + \begin{bmatrix} bv_1 \\ \dots \\ bv_n \end{bmatrix} = \begin{bmatrix} au_1 + bv_1 \\ \dots \\ au_n + bv_n \end{bmatrix}$$



- Inner Product

$u, v \in \mathbb{R}^n$

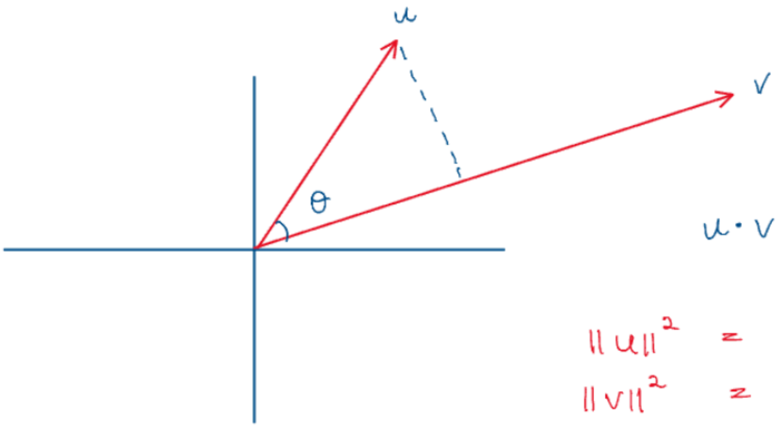
$u \cdot v = \sum_{k=1}^{n} u_k v_k$

Length:

$||u||^2 = u \cdot u = \sum_{k=1}^{n} (u_k)^2$

$||u_k||^2 = \sqrt{(\sum_{k=1}^{n} (u_k)^2)}$

$k = 1$



$u \cdot v = ||u|| \ ||v|| \cos \theta$

$||u||^2 = u \cdot u$
$||v||^2 = v \cdot v$

$u \cdot v = ||u|| \ ||v|| cos(\theta)$

$||v||^2 = v \cdot v$

$cos(\theta) = \frac{u \cdot v}{||u|| \ ||v||}$

Thm: Cauchy Schwartz Inequality

Let $u, v \in \mathbb{R}^n$, $|u \cdot v| \leq ||u|| \ ||v||$

Proof:

Case 1: $||u|| = 0$ or $||v|| = 0$

If $||u|| = 0 : |0 \cdot v| = 0 \leq ||u||\ ||v|| = 0||v|| = 0$

If $||v|| = 0 : |u \cdot 0| = 0 \leq ||u||\ ||v|| = ||u||0 = 0$

Case 2: $||u|| \neq 0$ and $||v|| \neq 0$

Lemma 1: If $a, b \in \mathbb{R}$, then $a^2 + b^2 \geq 2ab$

Proof: $(a-b)^2 \geq 0$ for $a, b \in \mathbb{R}$

$a^2 + b^2 - 2ab \geq 0$

$a^2 + b^2 \geq 2ab$

Lemma 2: If If $a, b \in \mathbb{R}$, then $a^2 + b^2 \geq -2ab$

Proof: $(a+b)^2 \geq 0$ for $a, b \in \mathbb{R}$

$a^2 + b^2 + 2ab \geq 0$

$a^2 + b^2 \geq -2ab$

Let $a_k \equiv \frac{u_k}{||u||}$, $b_k \equiv \frac{v_k}{||v||}$

$(a_k)^2 + (b_k)^2 \geq 2a_k b_k$ using Lemma 1

$\sum_{k=1}^{n} (\frac{(u_k)^2}{(||u||)^2} + \frac{(v_k)^2}{(||v||)^2}) \geq \sum_{k=1}^{n} (2 \frac{u_k}{||u||} \frac{v_k}{||v||})$

$\frac{1}{(||u||)^2} \sum_{k=1}^{n} (u_k)^2 + \frac{1}{(||v||)^2} \sum_{k=1}^{n} (v_k)^2 \geq \frac{2}{||u||||v||} \sum_{k=1}^{n} u_k v_k$

$\frac{(||u||)^2}{(||u||)^2} + \frac{(||v||)^2}{(||v||)^2} \geq \frac{2}{||u||\ ||v||} (u \cdot v)$

$2 \geq \frac{2}{||u||\ ||v||} (u \cdot v)$

$||u||\ ||v|| \geq (u \cdot v)$

Similarly,

$||u||\ ||v|| \geq -(u \cdot v)$ using Lemma 2

Therefore $||u||\ ||v|| = 0$

Week 1 Session 2

Outline

Vectors: Dot Products, Norm, Minkowski Inequality

Matrices: Matrix multiplication ,Transpose, Trace, Block matrices

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$u, v \in \mathbb{R}^n$

Inner Product: $u \cdot v = \sum_{k=1}^{n} u_k v_k$

Length (Norm): $||u||^2 = u \cdot c = \sum_{k=1}^{n} (u_k)^2$

Properties: For $k \in \mathbb{R}$, $u, v, w \in \mathbb{R}^n$

1. $u \cdot v = v \cdot u$
2. $u \cdot (v + w) = (u \cdot v) + (u \cdot w)$
3. $ku \cdot v = k(u \cdot v)$
4. $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if $u = \mathbf{0}$

$|u \cdot v| \leq ||u||\ ||v||$: Cauchy Schwartz Inequality

Minkowski Inequality

$||u + v|| \leq ||u|| + ||v||$

Proof:

$||u + v||^2 = (u + v) \cdot (u + v)$

$= (u \cdot u) + (u \cdot v) + (v \cdot u) + (v \cdot v)$

$= ||u||^2 + 2(u \cdot v) + ||v||^2$

$\leq ||u||^2 + 2|u \cdot v| + ||v||^2 \qquad (u \cdot v) \in \mathbb{R}$

$\leq ||u||^2 + 2||u||\ ||v|| + ||v||^2$ Cauchy Schwartz Inequality

$= (||u|| + ||v||)^2$

Therefore:

$||u + v||^2 \leq (||u|| + ||v||)^2$

$||u + v|| \leq ||u|| + ||v||$

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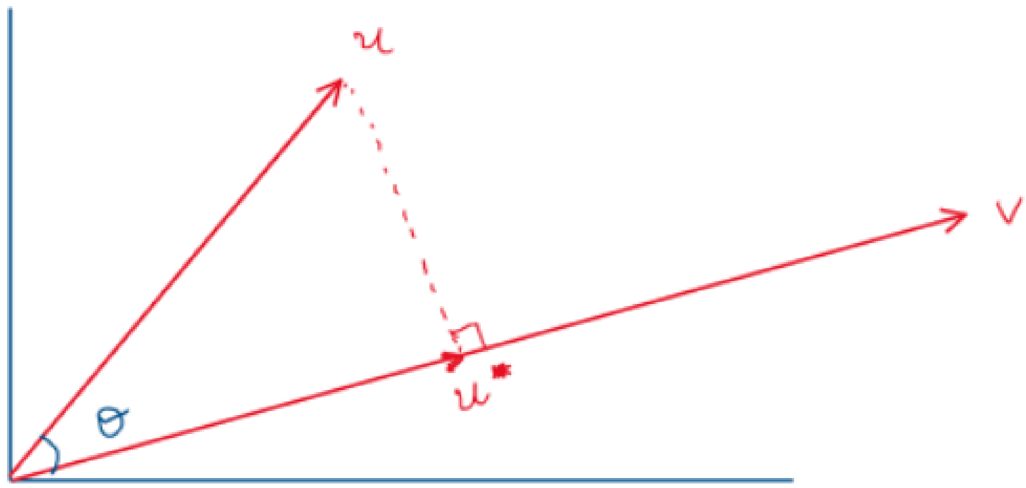
u nd v are orthogonal (perpendicular) $\implies u \cdot v = 0$

Normalizing a vector: $\frac{v}{||v||}$



$u^* \equiv$ Projection of u onto v

$u^* \equiv Proj(u,v) = \frac{u \cdot v}{||v||^2} v$

$u^* \equiv Proj(u,v) = ||u|| \frac{v}{||v||}$, where $||u||$ is the magnitude, $\frac{v}{||v||}$ is the direction

$= ||u|| cos(\theta) \frac{v}{||v||}$

$= ||u|| \, ||v|| cos(\theta) \frac{v}{||v||^2}$

$= \frac{u \cdot v}{||v||^2} v$

Complex Vectors

$u, v \in \mathbb{C}^n$

$u \cdot v = \sum_{k=1}^{n} u_k v_k^\star$

where $v_k \in \mathbb{C}$, $v_k = a_k + jb_k$, where a_k is the real part, and b_k is the imaginary part

Matrices

$$A \equiv [a_i j] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A is $m \times n$ with m rows and n columns

$A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & -1 \\ 3 & 3 & 0 \\ 2 & 4 & 2 \end{bmatrix} \in \mathbb{R}^{4 \times 3}$$

A row vector: $v = [v_1, v_2, \dots, v_n] \in K^{1 \times n}$

A column vector: $v = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_m \end{bmatrix} \in K^{m \times 1}$

Matrix Addition

$$A, B \in K^{m \times n} A + B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Scalar Multiplication

If $k \in K, A \in K^{m \times n}$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Null Matrix

$A \equiv [a_{ij}] = \mathbf{0}$

$\forall i, j, a_{ij} = 0$

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Linear Combination:

$a, b \in K$, $A, B \in K^{M \times N}$

$$aA + bB = \begin{bmatrix} aa_{11} + bb_{11} & aa_{12} + bb_{12} & \dots & aa_{1n} + bb_{1n} \\ \dots & \dots & \dots & \dots \\ aa_{m1} + bb_{m1} & aa_{m2} + bb_{m2} & \dots & aa_{mn} + bb_{mn} \end{bmatrix}$$

Properties:

If $k, k' \in K$ and $A, B, C \in K^{m \times n}$

1. $A + B = B + A$ Commutativity
2. $A + (B + C) = (A + B) + C$ Associativity
3. $k(A + B) = kA + kB$
4. $kk'A = k(k'A)$
5. $A + -A = 0$
6. $A + 0 = A$

Transpose:

If $A \in K^{m \times n}$ and $A = [a_{ij}]$, then

$A^T \in K^{n \times m}$ and $A^T = B = [b_{ij}]$ where $b_{ij} = a_{ji}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$ where dimension is $m \times n$

$$A^T = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{m1} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$ where dimension is $n \times m$

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Properties:

If $A, B \in K^{m \times n}$

1. $(A + B)^T = A^T + B^T$
2. $(A^T)^T = A$

Let $u, v \in K^{m \times 1}$

then $u \cdot v = u^T v$

Square Matrix

$A = [a_{ij}]$ is a square matrix if and only if the number of rows equal the number of columns.

$m = n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Diagonal Matrix:

$A \equiv [a_{ij}]$

A square matrix such that $\forall i \neq j, a_{ij} = 0$

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Triangular Matrices:

Upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

$\forall i > j, a_{ij} = 0$

Lower triangular

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$\forall i < j, a_{ij} = 0$

Matrix Multiplication

A	B	C
$m \times n$	$n \times p$	$m \times p$
$[a_{ij}]$	$[b_{ij}]$	$[c_{ij}]$

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

B	A	D
$n \times m$	$m \times p$	$n \times p$
$[a_{ij}]$	$[b_{ij}]$	$[c_{ij}]$

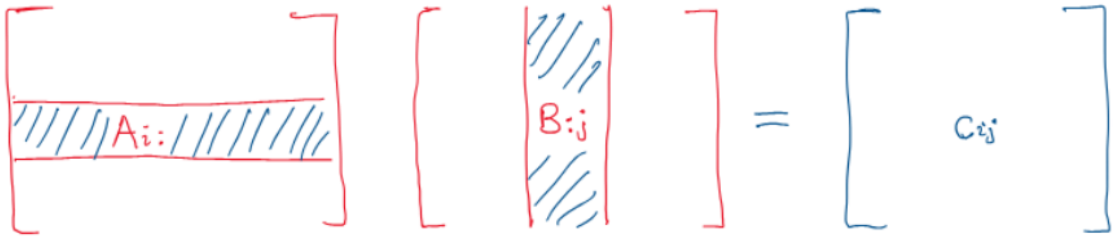
$BA = D$

$d_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$

$c_{ij} = i^{th}\ row\ of\ A \cdot j^{th}\ column\ of\ B$

$i^{th}\ row\ of\ A$: $A_{i:}$

j^{th} $column\ of\ B$: $B_{:j}$



Properties: If A, B, C are conformable for multiplication

1. $(AB)C = A(BC)$ Associativity
2. $A(B + C) = AB + AC$ Left distribution
3. $(A + B)C = AC + BC$ Right distribution
4. $(AB)^T = B^T A^T$
5. $c(AB) = (cA)B = A(cB)$ if c is a scalar
6. $AB \neq BA$

Trace

$A \in K^{n \times n}$

$Tr(A) = \sum_{k=1}^{n} a_{kk}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

$Tr(A) = a_{11} + a_{22} + a_{33} = 1 + 1 + 5 = 7$

Properties:

If A, B, C are conformable for multiplication

1. $Tr(A) = Tr(A^T)$
2. $Tr(BA) = Tr(AB)$
3. $Tr(ABC) = Tr(BCA) = Tr(CAB)$

Cyclic Property of Trace

Thm:

$Tr(A_1\, A_2 \ldots\, A_{n-1}\, A_n) = Tr(A_n\, A_1 \ldots\, A_{n-1})$

If the matrices A_k are conformable for matrix multiplication where T_r is the trace operator:

$Tr(A) = \sum_{k=1}^{p} a_{kk}$ if A is a square matrix

$A_k \in \mathbb{C}^{m_k \times n_k}$

Proof:

Lemma 1:

$Tr(AB) = Tr(BA)$

Lemma 2:

$A \times (B \times C) = (A \times B) \times C$

Lemma 1:

A dimension is $m \times n$

B dimension is $n \times m$

Then $A \times B$ is $m \times m$, $B \times A$ is $n \times n$

$Tr(AB) = \sum_{k=1}^{m} (AB)_{kk}$ def of Tr

$= \sum_{k=1}^{m} (\sum_{l=1}^{n} a_{kl} b_{lk})$ def of matrices, multiplication

$= \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} b_{lk}$ distribution

$= \sum_{l=1}^{n} \sum_{k=1}^{m} a_{kl} b_{lk}$ finite sum

$= \sum_{l=1}^{n} \sum_{k=1}^{m} b_{lk} a_{kl}$ complex number

$= \sum_{l=1}^{n} (\sum_{k=1}^{m} b_{lk} a_{kl})$ distribution

$= \sum_{l=1}^{n} (BA)_{ll}$ def of matrix multiplication

$= Tr(BA)$ def of Tr

Lemma 2:

A dimension is $u \times v$

B dimension is $v \times w$

C dimension is $w \times r$

Then $A \times (B \times C)$ is $u \times r$, $(A \times B) \times C$ is $u \times r$

say $M \equiv [m_{ij}]$, $N \equiv [n_{ij}]$

$m_{ij} = n_{ij}$

$m_{ij} = (A(BC))_{ij}$

$= \sum_{k=1}^{v} a_{ik} (BC)_{kj}$ def of matrix multiplication

$= \sum_{k=1}^{v} a_{ik} (\sum_{l=1}^{w} b_{kl} c_{lj})$ def of matrix multiplication

where $(\sum_{l=1}^{w} b_{kl} c_{lj}) = (BC)_{kj}$

$= \sum_{k=1}^{v} \sum_{l=1}^{w} a_{ik} b_{kl} c_{lj}$ distribution

$= \sum_{l=1}^{w} (\sum_{k=1}^{v} a_{ik} b_{kl}) c_{lj}$ finite sum

where $(\sum_{k=1}^{v} a_{ik} b_{kl}) = (AB)_{il}$

$= \sum_{l=1}^{w} (AB)_{il} c_{lj}$ def of matrix multiplication

$= ((AB)C)_{ij}$ def of matrix multiplication

$= n_{ij}$

$Tr(A_1\, A_2 \ldots\, A_{n-1}\, A_n) = Tr((A_1\, A_2 \ldots\, A_{n-1})\, A_n)$

$= Tr(A_n\, (A_1\, A_2 \ldots\, A_{n-1}))$

$= Tr(A_n\, A_1 \ldots\, A_{n-1})$

A	B	$A + B$
$n \times n$	$n \times n$	$n \times n$
diagonal	diagonal	diagonal
triangular	triangular	triangular
upper	upper	upper
lower	lower	lower

Invertible Matrices

A is invertible if and only if $\exists B : AB = BA = I_n$

$$I = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

Properties:

1. $A^{-1}A = I_n$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

$A \in \mathbb{C}^{m \times n}$

Hermitian

$A^H = (A^*)^T = (A^T)^*$

If $A \in \mathbb{R}^{m \times n}$, $A^H = (A^*)^T = (A^T)^*$

Normal Matrices

$A^T A = AA^H$

Complex:

- Hermitian matrices: $A = A^H$
- Skew Hermitian: $A = -A^H$
- Unitary: $A^{-1} = A^H$

Real:

- Symmetric: $A = A^T$
- Skew symmetric: $A = -A^T$
- Orthogonal: $A^{-1} = A^T$

Block Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Week 2 Session 1

Outlines

Linear System: Lines, Hyperplane, Normal

Equivalent Systems: Elementary row operations

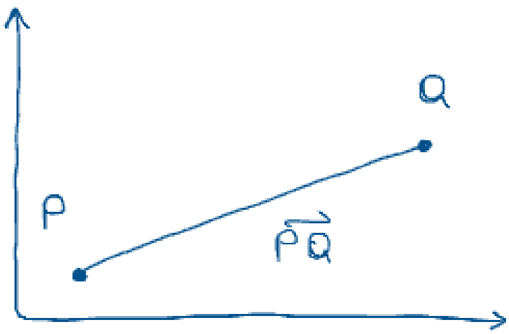
Echelon Form: Gaussian Elimination

Row Canonical Form: Gauss-Jordan

Located Vectors

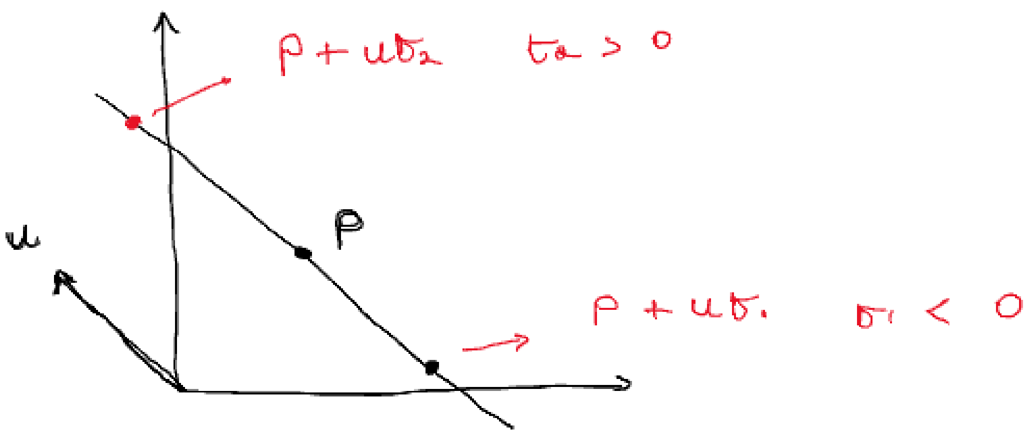
$P(u_1, \dots, u_n)$

$Q(v_1, \dots, v_n)$



$$\overrightarrow{\mathrm{PQ}} = \overrightarrow{\mathrm{Q}} - \overrightarrow{\mathrm{P}} = \begin{bmatrix} v_1 \\ \dots \\ v_n \end{bmatrix} - \begin{bmatrix} u_1 \\ \dots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 - u_1 \\ \dots \\ v_n - u_n \end{bmatrix}$$

Lines



$L = \{x \in \mathbb{R}^n : x = p + ut, t \in \mathbb{R}^n\}$

L is a line that passes through point P with direction $u \in \mathbb{R}^n$

Linear Systems

Linear Equation

$a_1x_1 + \ldots + a_nx_n = b$

$\sum_{j=1}^{n} a_jx_j = b$

where a_j are the coefficients, and x_j are the unknowns

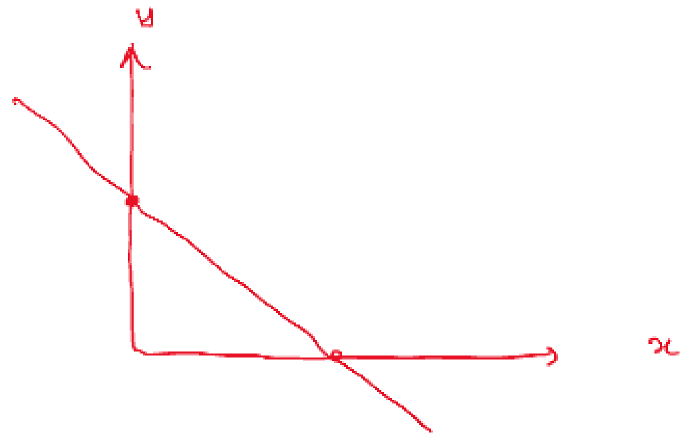
Hyperplane H:

$H = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_jx_j = b\}$

Example:

$6x = 6$, $H = \{1\}$

$x + y = 2$



$x + y + z = 1$

Normal to H: $\sum_{j=1}^{n} a_jx_j = b$

$w \in \mathbb{R}^n$ such that for all any located vector $\overrightarrow{\mathrm{PQ}}$ in H, w is orthogonal to $\overrightarrow{\mathrm{PQ}}$

$$w = \begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix}$$

Proof:

$\sum_{j=1}^{n} a_jx_j = b$

$P(u_1, \ldots, u_n) \in H \implies \sum_{j=1}^{n} a_ju_j = b$

$Q(v_1, \ldots, v_n) \in H \implies \sum_{j=1}^{n} a_jv_j = b$

$w \perp \overrightarrow{\mathrm{PQ}}$

$$w = \begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix}$$

$$w \cdot \overrightarrow{\mathrm{PQ}} = \begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} v_1 - u_1 \\ \ldots \\ v_n - u_n \end{bmatrix}$$

$= \sum_{j=1}^{n} a_j(v_j - u_j)$

$= \sum_{j=1}^{n} a_j v_j - \sum_{j=1}^{n} a_j u_j$

$= b - b$

$= 0$

Linear Systems

A list of linear equations with the same unknowns

m equations and n unkowns

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\ldots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$

- Unique solution
- Infinite solution
- No solution

A	x	b
$m \times n$	$n \times 1$	$m \times 1$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ \ldots \\ b_m \end{bmatrix}$$

Degenerate linear equation:

$0x_1 + \ldots + 0x_n = b$

1. $b = 0$, every $x \in \mathbb{R}^n$ is a solution
2. $b \neq 0$, no solution

Homogenous system: $Ax = b = \mathbf{0}$

Equivalent Systems

$Ax = b$, $A'x = b'$ where x is in dimension $n \times 1$

Theorem:

Let L be a linear combination of the equations m $Ax = b$,, then x is a solution to L
Proof:

$Ax = b$

$\sum_{j=1}^{n} a_{ij}xj = b_i$ where $1 \leq v \leq m$

Let $s = \begin{bmatrix} s_1 \\ \dots \\ s_n \end{bmatrix}$ is a solution to $Ax = b$

Then: $\sum_j \sum_{j=1}^{n} a_{ij}xj = \sum_j b_i$ Integration

$\sum_{i=1}^{m} c_i(\sum_{j=1}^{n} a_{ij}s_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_i a_{ij} s_j$

$= \sum_{j=1}^{n}(\sum_{i=1}^{m} c_i a_{ij}) s_j$

$= \sum_{j=1}^{m} c_i b_i$

x is also a solution to L

$Ax = b$ Linear combination $\rightarrow A'x = b'$

Elementary Row Operations

1. Row swap: $R_i \leftrightarrow R_j$
2. Scalar multiplication: $R_i \rightarrow kR_i$
3. Sum of a row with a scalar multiple of another row: $R_i \rightarrow R_i + kR_j$

Thm:

$Ax = b$ and $A'x = b'$ where A' (b') is obtained form the elementary row operations on $Ax = b$ then they have same solutions.

Geometry: Linear System Solutions

$Ax = b$

Row:

$\sum_{j=1}^{n} a_{ij}x_j = b_i$

Row 1: $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

Row 2: $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\ldots$

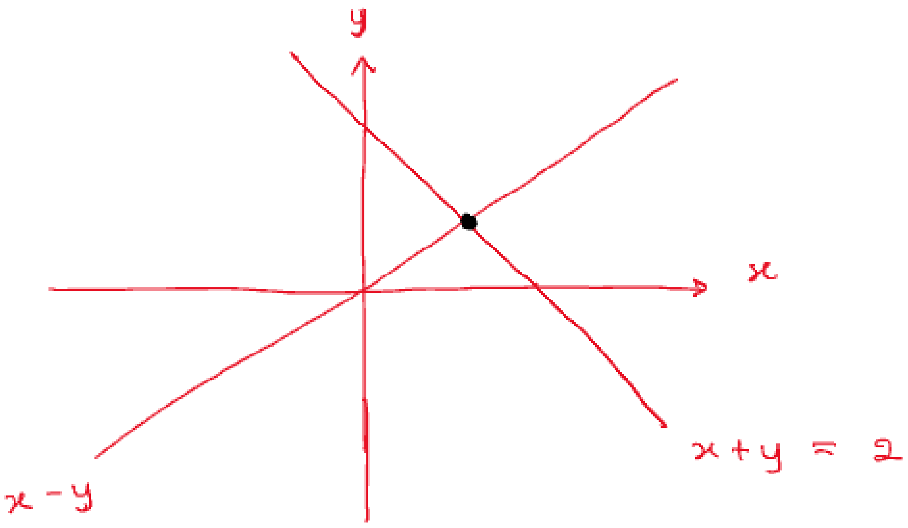
Row m: $a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$

Example 1:

$x + y = 2$

$x - y = 0$

$x = 1, y = 1$ is the unique solution

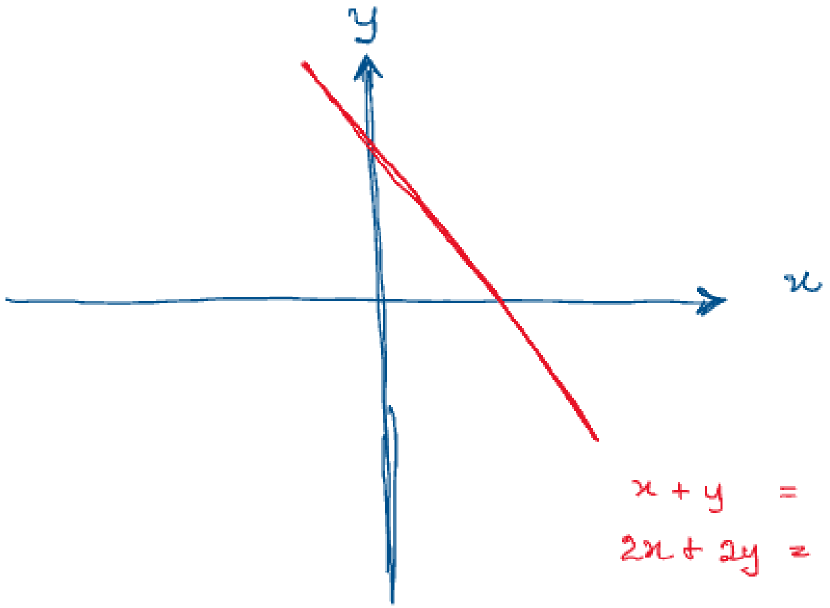


Example 2:

$x + y = 2$

$2x + 2y = 4$

Infinite solution

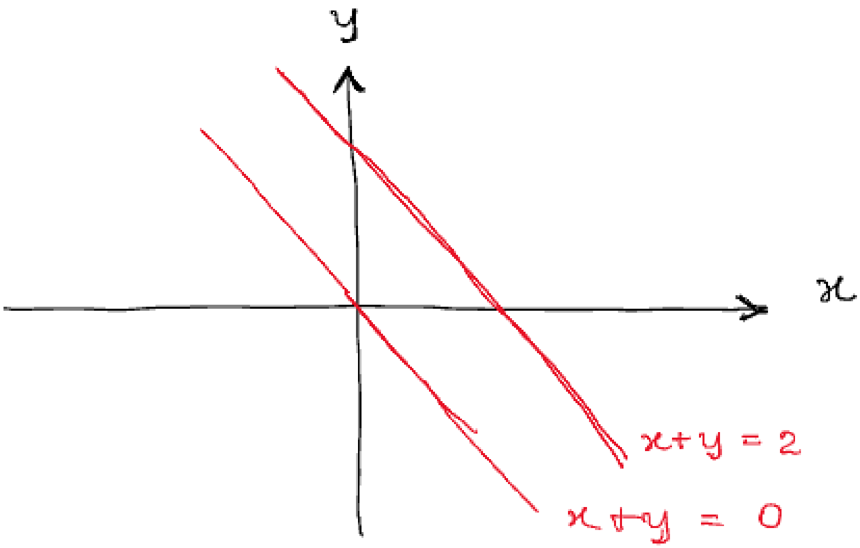


Example 3:

$x + y = 2$

$x + y = 0$

No solution



Column

$Ax = b$

$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ a_{11} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$

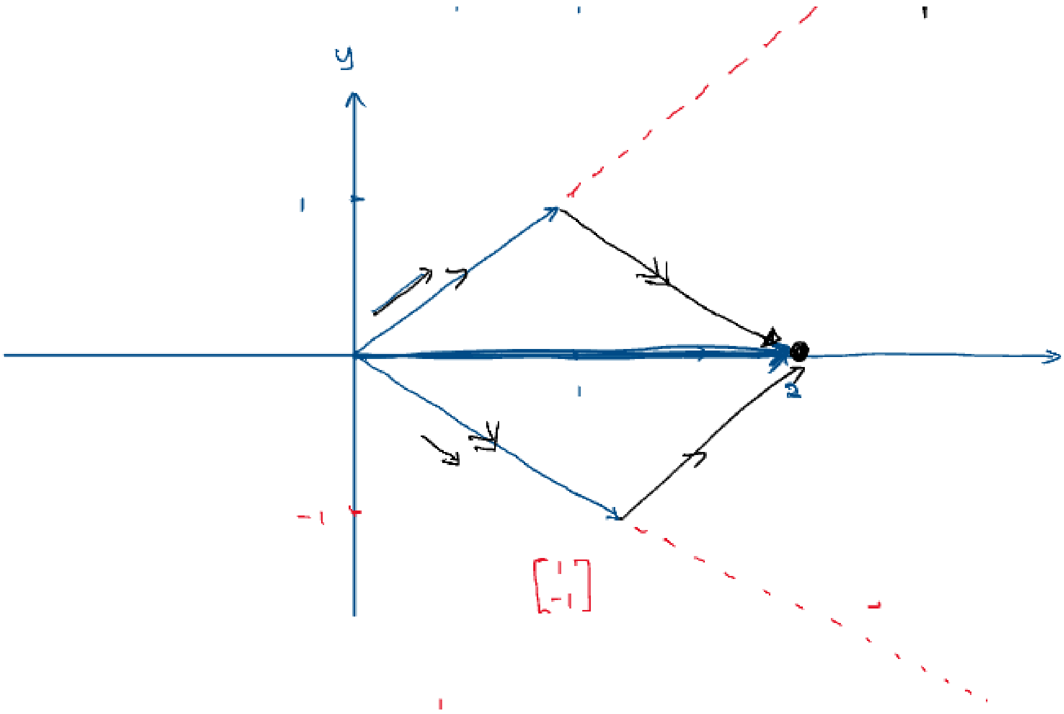
$b = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}$

$\sum_{j=1}^{n} A_{ij} x_j = b$

Example1:
$x + y = 2$

$x - y = 0$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

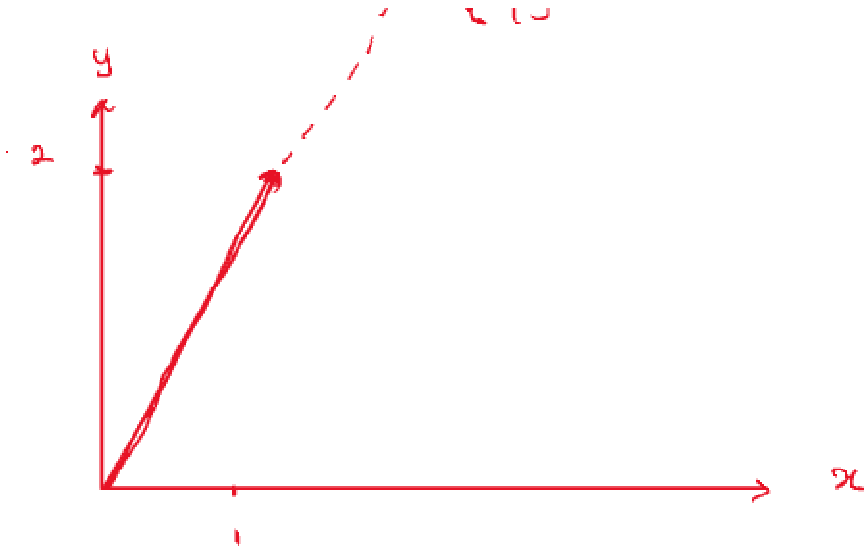


Example 2:

$x + y = 2$

$2x + 2y = 4$

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$



Echelon Form

$2x_1 + 3x_2 + x_3 + x_4 - x_5 = 2$

$x_2 + x_3 + 0x_4 + x_5 = 2$

$x_4 + x_5 = 1$

$m = 3, n = 5$

Pivot variables: x_1, x_2, x_4 (leading variables)

Free variables: x_3, x_5 (non-leading variables)

Special case (Triangular Form)

$2x_1 + 3x_2 + 4x_3 = 5$

$2x_2 + x_3 = 6$

$3x_3 = 1$

$m = 3, n = 3$

Gaussian Elimination

Two step process for solving linear systems of form $Ax = b$

1. Forward elimination: Reduce to Echelon Form
2. Backward substitution

Example 1:

$R1 : 2x + y + z = 5$

$R2 : 4x - 6y = -2$

$R3 : -2x + 7y + 2z = 9$

Forward Elimination:

$R1 : R1$

$R2 : R2 - 2R1$

$R3 : R3 + R1$

$2x + y + z = 5$

$0x - 8y - 2z = -12$

$0x + 8y + 3z = 14$

$R1 : R1$

$R2 : R2$

$R3 : R3 + R2$

$2x + y + z = 5$

$0x - 8y - 2z = -12$

$0x + 0y + z = 2$

Backward Substitution:

$z = 2$

$y = 1$

$x = 1$

Augmented Matrix (M)

A	x	b	M
$m \times n$	$n \times 1$	$m \times 1$	$m \times (n+1)$

$M \equiv [A \mid b]$

$$M = \begin{bmatrix} 2 & 1 & 1 \mid 5 \\ 4 & -6 & 0 \mid -2 \\ -2 & 7 & 2 \mid 9 \end{bmatrix}$$

Where $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

Echelon Matrix:

$$M = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Week 2 Session 2

Outline

Row Canonical Form: Gauss Jordan Elimination

Elementary Matrix Operations

LU Decomposition: LDU

Vector Spaces

Echelon Matrix

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & 2 & 1 & 4 & -1 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Augmented Matrix

$Ax = b$, $M = [A|b]$

Row Canonical Form (Row-reduced Echelon Form)

1. Echelon Form
2. All non zero leading elements must be equal to 1
3. All the other values above and below a leading element must be 0

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$M = [A|b]$

Gauss-Jordan Elimination

$Ax = b$

$M = [A|b]$ - Augmented matrix

Reduce M to its row canonical form

$M' = [A'|b']$ (i.e., $A'x = b'$)

Example:

$2x_1 + x_2 + x_3 = 5$

$4x_1 - 6x_2 = -2$

$-2x_1 + 7x_2 + 2x_3 = 9$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$M \equiv [A|b] = \begin{bmatrix} 2 & 1 & 1|5 \\ 4 & -6 & 0|-2 \\ -2 & 7 & 2|9 \end{bmatrix}$$

$R1 : R1$

$R2 : R2 - 2R1$

$R3 : R3 + R1$

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & 8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 : R3 + R2$

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & 8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ which is the Echelon Form

$R1 : R1 - R3$

$R2 : R2 + 2R3$

$R3 : R3$

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & -8 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R1 : R1$

$R2 : -1/8R2$

$R3 : R3$

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R1 : R1 - R2$

$R2 : R2$

$R3 : R3$

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R1 : 1/2R1$

$R2 : R2$

$R3 : R3$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, which is in row canonical form

$x_1 = 1, x_2 = 2, x_3 = 2$

Linear combination of orthogonal vectors

Let $u_1, u_2, \dots, u_n \in \mathbb{R}^n$ are mutually orthogonal

For any vector $v \in \mathbb{R}$

$v = u_1 x_1 + \dots + u_n x_n$

where $x_i = \frac{v \cdot u_1}{||u_i||^2}$ and $u_i \neq \mathbf{0}$ for $1 \leq i \leq n$

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ u_1 & u_2 & \dots & u_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$Ax = v$ what is x ?

Proof:

$u_i \cdot u_j = \begin{cases} 0, & \text{if } i \neq j \\ ||u_i||^2, & \text{if } i = j \end{cases}$ Equation 1

$Ax = v$

$\sum_{j=1}^{n} x_j u_j = v$ Equation 2

$v \cdot u_i = \sum_{j=1}^{n} x_j u_j \cdot u_i$

$= \sum_{j=1}^{n} x_j (u_j \cdot u_i)$

$= (u_i \cdot u_i) x_i + \sum_{j=1, j \neq i}^{n} x_j (u_i \cdot u_j)$

$= ||u_i||^2 x_i$

Therefore, $v \cdot u_i = ||u_i||^2 x_i$ means that $x_i = \frac{v \cdot u_1}{||u_i||^2}$

$v = \sum_{j=1}^{n} x_i u_i = \sum_{j=1}^{n} \frac{v \cdot u_i}{||u_j||^2} u_i$

Inverse Matrix

Using Gauss Jordan Elimination for A^{-1}

If A $(n \times n)$ is invertible, $\exists A^{-1}$ such that $AA^{-1} = I$

$AA^{-1} = I$

say $B = A^{-1}$

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \dots & \dots & \dots & \dots \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$Ab_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}$$

$M = [A|I]$ Row canonical $\rightarrow$ $[I|A^{-1}]$

Example 1:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$ Find A^{-1}

$$M \equiv \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 = R3 + R2$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$R1 : R1 - R3$

$R2 : R2$

$R3 : R3 + R2$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$R1 : R1 - R3$

$R2 : R2 + 2R3$

$R3 : R3$

$$\begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$R1 : R1$

$R2 = -1/8R2$

$R3 = R3$

$$\begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$R1 : R1 - R2$

$R2 : R2$

$R3 = R3$

$$\begin{bmatrix} 2 & 0 & 0 & 3/2 & -5/8 & -3/4 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$R1 : 1/2R1$

$R2 : R2$

$R3 : R3$

$$\begin{bmatrix} 1 & 0 & 0 & 3/4 & -5/16 & -3/8 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

where $A^{-1} = \begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix}$

Check:

$$AA^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

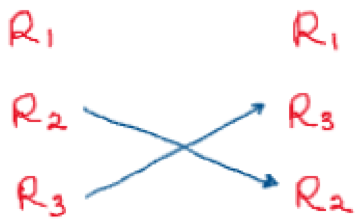
Elementary Matrix Operations

$eA \equiv EA$

where e is the elementary row operation, E is the elementary matrix operation

$e_n \ldots e_1 A = E_n \ldots E_1 A$

1. Row Swap $R_i \leftrightarrow R_j$



$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} \ldots b_1 : \\ \ldots b_2 : \\ \ldots b_3 : \end{bmatrix}$$

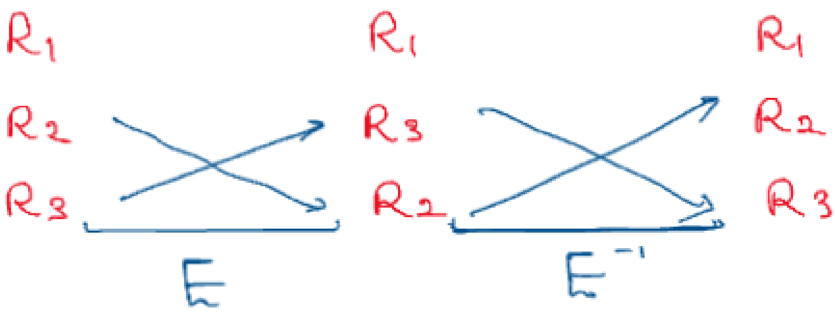
Let $E = I$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$EA \equiv B$ where $B = [b_{ij}]$

$\sum_{k=1}^{n} e_{ik} a_{kj} = b_{ij}$

where $e_{ik} = [e_{i1}, e_{i2}, \ldots, e_{in}]$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Scalar Multiplication of a row

$R_i : kR_i$

$EA = B$

$R1 : R1$

$R2 : kR2$

$R3 : R3$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/k & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Row addition with a scalar multiple of another row

Operation	E	E^{-1}
$R1$	$R1$	$R1$
$R2$	$R2 + kR3$	$R2 + kR3 - kR3$
$R3$	$R3$	$R3$

This is an operation of E and E^{-1}

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$

LU decomposition

$A = LU \equiv LDU$

where A is in dimension $n \times n$, L is the lower triangular, U is the upper triangular, D is the diagonal matrix

A is a nonsingular matrix that can be reduced into triangular from U only row-addition operations

Example:

$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

$e_n ... e_1 A = U = E_n ... E_1 A$

$E_n ... E_1 A = U$

$(E_n ... E_1)^{-1} = E_1^{-1} E_2^{-1} ... E_n^{-1}$

$(E_n ... E_1)^{-1}(E_n ... E_1)A = E_1^{-1} E_2^{-1} ... E_n^{-1} U$

$LHS : A = LU$

$RHS = LU$

$R1 : R1$

$R2 : R2 - 2R1$

$R3 : R3 + R1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Operations	E_1	E_1^{-1}
$R1$	$R1$	$R1$
$R2$	$R2 - 2R1(+2R1)$	$R2$
$R3$	$R3 + R1\,(-R1)$	$R3$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 : R3 + R2$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$(E_2 E1)A = U$

$A = (E_1^{-1} E_2^{-1})U$ and $E_1^{-1} E_2^{-1} = L$

Operations	E_1	E_1^{-1}
$R1$	$R1$	$R1$
$R2$	$R2$	$R2$
$R3$	$R3 + R2\,(-R2)$	$R3$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Check:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Week 3 Session 1

Outlines

LU Decomposition: LDU

Vector Spaces: Fields, Span, Subspaces

Linear Independence: Invertibility

Uniqueness Theorem

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ 2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$A = LU$

$A = LDU$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$$

Vector Spaces

Field:

A field F is a collection of elements such that for binary operations: $+, \times$

We have the following: $\forall a, b, c \in F$

1. $a + b = b + a\,;\, a \cdot b = b \cdot a$
2. $a + (b + c) = (a + b) + c\,;\, a \cdot (b \cdot c) = (a \cdot b) \cdot c$
3. $\exists 0 \in F : a + 0 = a$
   $\exists 1 \in F : a \cdot 1 = a$
4. $\exists a' \in F$: $a + a' = 0$
5. $a \times \frac{1}{a} = 1$ if $a \neq 0$
6. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Example:

$\mathbb{R}, \mathbb{Q}, \mathbb{C}$ - field

$\mathbb{Z}$ not a field ($5^{th} \frac{1}{a} \notin \mathbb{Z}$)

A vector V over field F is a collection of elements $\{\alpha, \beta, \gamma, ...\}$ (typically called vectors) and collection of elements $\{a, b, c, ...\} \in F$ ca;;ed scalars such that:

- Commutative group for $(V, +)$
  1. $\alpha + \beta \in V$
  2. $\alpha + \beta = \beta + \alpha$
  3. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

4. $\forall \alpha, \exists \alpha' \in V : \alpha + \alpha' = \mathbf{0}$
5. $\exists \mathbf{0} \in V : \forall \alpha \in V, 0 + \alpha = \alpha$

- Properties for combination of $+$ and $\times$
  1. $a\alpha \in V$
  2. $a(b\alpha) = (ab)\alpha$
  3. $a(\alpha + \beta) = a\alpha + a\beta$
  4. $(a + b)\alpha = a\alpha + b\alpha$
  5. $\exists 1 \in F : 1\alpha = \alpha$

1

K is field, K^n

$\alpha, \beta \in K^n$

$\alpha = \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix}$, $a_1 \in K$

2

Polynomial Space: $P(t)$

$p(t) \in P(t)$

$p(t) = a_0 + a_1 t^1 + a_2 t^2 + \dots + a_s t^s$

where $s \in \{1, 2, 3, \dots\}$

3

Matrix over a field: $K_{m \times n}$

$A \in K_{m \times n}$
$A \equiv [a_{ij}]$ where $a_{ij} \in K$

Linear Combination:

Let $\alpha_1, \alpha_2, \dots \alpha_n \in V$ where is a vector space over field V

w is a linear combination of the α_i's if:

$w = a_1 \alpha_1 + \dots + a_n \alpha_n$

where $a_1, a_2, \dots, a_n \in F$

Alternatively:

$Ax = b$

$\begin{bmatrix} \dots \\ \alpha_1 \\ \dots \end{bmatrix} x_1 + \begin{bmatrix} \dots \\ \alpha_2 \\ \dots \end{bmatrix} x_2 + \dots + \begin{bmatrix} \dots \\ \alpha_n \\ \dots \end{bmatrix} x_n = w$

Linear Span

Let $S = \{\alpha_1, \dots \alpha_n\} \subset V$ for a vector space V over field F

S spans V means that $\forall w \in V, \exists a_1, \dots, a_n \in F$ such that:

$w = a_1\alpha_1 + \dots + a_n\alpha_n$

Subspace

u is a subspace of vector space V over field F, if

1. $u \subset V$ (u is a subset of V)
2. u is a vector space over F

Thm:

Let Vbe a vector space over field F and u is a subset of v ($u \subset V$), If:

1. $\mathbf{0} \in u$
2. $\forall \alpha, \beta \in u, \forall a, b \in F, a\alpha + b\beta \in u$

Then u is a space of V

Thm:

Let V be a vector space over field F. If u is a subspace of V, and w is a subspace of u, then w is a subspace of V

Thm:

Intersection of any number of subspaces of a vector V over field F is a subspace of V

Proof:

$u_1, u_2, \dots$ are subspaces of V
u_1 is a subspace of V

u_2 is a subspace of $$

$\dots$

If $\bigcap_{i=1}^{n} u_i$ a subspace of V?

Yes.

Example:

$$w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ✅

where $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ ❎

$\mathbb{R}^2 \equiv \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

$\{\mathbf{0}\}$ subspace of $\mathbb{R}^2$ ✅

$ax + by = 1$ ❎

$ax + by = 0$ subspace of $\mathbb{R}^2$

Thm:

Let $S = \{\alpha_1, \dots \alpha_n\} \subset V$ where V is a vector space over F and $L(s)$ be the set of all linear combinations of S with respect to F. Then $L(s)$ is a subspace of V.

1. Vector space V over field F
2. $S = \{\alpha_1, \dots \alpha_n\} \subset V$
3. $L(s) = \{w : w = \sum_{i=1}^n a_i \alpha_1, a_i \in F, \alpha_i \in \S\}$

$\implies L(s)$ (span of S) is a subspace of V

Proof:

1. Show that $L(s) \subset V$

   $v \in L(s) \implies v \in V$

   Assume that $v \in L(s)$

   $v = \sum_{i=1}^n a_i \alpha_i$ - Def of $L(s)$

   $\alpha_1 \in S \implies \alpha_i \in V$ - because $S \subset V$

   $v = \sum_{i=1}^n a_i \alpha_i \in V$ - V is a vector space

   $L(s) \subset V$
2. Show that $\mathbf{0} \in L(s)$

   $\mathbf{0} = 0\alpha_1 + 0\alpha_2 + \dots + 0\alpha_n = \sum_{i=1}^n 0\alpha_i \in L(s)$ - Def of S
3. Show that for $v, w \in L(s)$ and $c, d \in F$, $cv + dw \in L(s)$

   $cv + dw = c\sum_{i=1}^n a_i \alpha_i + d\sum_{i=1}^n b_i \alpha_i$ where $v = \sum_{i=1}^n a_i \alpha_i$ and $w = \sum_{i=1}^n b_i \alpha_i$

   $= \sum_{i=1}^n ca_i \alpha_i + \sum_{i=1}^n db_i \alpha_i$

   $= \sum_{i=1}^n (ca_i + db_i)\alpha_i$ where $ca_1 + db_i \in F$

   Therefore, $cv + dw \in L(s)$

$L(s)$ is a subspace of V

Linear Independence

Let v be a vector space over field F

$S = \{\alpha_1, \dots \alpha_n\} \subset v$

s is a linearly dependent set if there exist a_i's in F such that:

$a_1\alpha_1 + a_2\alpha_2, \ldots, a_n\alpha_n = \mathbf{0}$

and at least one of the a_i's is non-zero

Linearly Independent:

s is linearly independent means that:

$a_1\alpha_1 + a_2\alpha_2, \ldots, a_n\alpha_n = \mathbf{0}$ only holds when:

$a_1 = a_2 = \ldots = a_n = 0$

$Ax = \mathbf{0}$ - Homogenous System

$$A = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}, x = \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}, b = \mathbf{0}$$

Note:

Let $S = \{\alpha_1, \ldots \alpha_n\} \subset v$,then:

1. If $\mathbf{0} \in s$, then s is a linearly dependent set
2. If $s = \{\alpha_1\}$, then s is linearly dependent if and only if $\alpha_1 = 0$

Row Equivalence

A , B are in dimension of $m \times n$
A is row equivalent to B if fB can be obtained from a sequence of elementary row operations of A

Example

A row operations $\implies$ A' (Echelon Form) row operations $\implies$ A'' (Row Canonical Form)

Say A in dimension of $n \times n$

Echelon Form

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$, number of pivots (1, 2, 3, 1) $= n$

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$, number of pivots (1, 2, 2) $< n$ Linearly dependent, 0 row (R4)

Row Canonical Form

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$R = \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I$$

$I^{-1} = I$

$$A \in \mathbb{R}^{n \times n} = \begin{cases} A \sim (Row\ Equivalent)\ I \\ A \nsim I \end{cases}$$

$$B = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(zero row)}, B = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

$$BB^{-1} \neq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

There is no β_4 such that $i_{44} = 1$

So B^{-1} does not exist

Week 3 Session 2

Outlines

Uniqueness Theorem

Basis and Dimension: Dimension Theorem

Subspaces of a matrix

$A \in \mathbb{R}^{n \times n}$

1. Linearly independent rows $\leftrightarrow$ $A \sim I$
2. Linearly dependent rows $\leftrightarrow$ $A \sim B$ such that B^{-1} does not exist

Thm:

Let A be a square matrix, the following statement are equivalent:

1. A is invertible
2. A is row equivalent to I
3. A is a product of elementary matrices

Let P and Q be logical statements

If P then Q ($P \implies Q$)

1. Assume P is TRUE, and then show it logically implies that Q us TRUE
2. Proof by contradiction: $\sim Q \implies \sim P$

If and only if (Equivalence)

P if and only if ($P \leftrightarrow Q$)

- $P \implies Q$
- $Q \implies P$

Proof: $a \implies b, b \implies c, c \implies a$

$a \implies b, b \implies c, c \implies a$

Then, $a \leftrightarrow b$

- $a \implies b$

  A is invertible $\implies$ A is row equivalent to I

  $P \implies Q$

  $\sim Q$: If A if not row equivalent to I, then $A \sim B$ such that B^{-1} does not exist

  so, $B = E_n ... E_1 A$

  $(E_n ... E_1)^{-1} B = (E_n ... E_1)^{-1} E_n ... E_1 A = A$

  Due to $(A_1 A_2)^{-1} = A_2^{-1} A_1^{-1}$

  $A^{-1} = B^{-1}(E_n ... E_1)$

  So, A is not invertible because B^{-1} does not exist

- $b \implies c$

  If A is row equivalent to I then A is a product of elementary matrices

  $P \implies Q$

  $E_n ... E_1 A = I$

  $(E_n ... E_1)^{-1}(E_n ... E_1)A = (E_n ... E_1)^{-1} I$

  so, $A = (E_n ... E_1)^{-1} = E_1^{-1} ... E_n^{-1}$

  For an elementary matrix E_i, E_i^{-1} is also an elementary matrix

- $c \implies a$

  If A is a product of elementary matrices then A is invertible

  $A = (E_1 ... E_n)$

  $A^{-1} = (E_1 ... E_n)^{-1} = E_n^{-1} ... E_1^{-1}$ because E_1^{-1} exists

Therefore, $a \implies b\,, b \implies c\,, c \implies a$

Thm:

Let v be a vector space over F and $S = \{\alpha_1, \ldots, \alpha_n\} \subset V$. Suppose S is a linearly independent set, then for every $w \in V$ there exist at most one representation as a linear combination of vectors in S.

Sketch:

If $S = \{\alpha_1, \ldots, \alpha_n\} \subset V$ (linearly independent set), then $\forall w \in V$, there exist at least one representation: $w = \sum_{i=1}^{n} a_i \alpha_i$

$P \implies Q$

Proof:

$\sim Q$: Assume that $\exists w \in V$, we have two possible representations

$w = \sum_{i=1}^{n} a_i \alpha_i$ and $w = \sum_{i=1}^{n} b_i \alpha_i$, $\exists k : a_k \neq b_k, 1 \leq k \leq n$

So, $\mathbf{0} = w - w = \sum_{i=1}^{n} a_i \alpha_i - \sum_{i=1}^{n} b_i \alpha_i$

$= \sum_{i=1}^{n} (a_i - b_i) \alpha_i$

$(0) = \sum_{i=1, i \neq k}^{n} (a_i - b_i) \alpha_i + (a_k - b_k) \alpha_k$, where $(a_k - b_k) \neq 0$

Therefore, S is linearly dependent set

$S = \{\alpha_1, \ldots, \alpha_n\} \subset V$ (vector space over field F)

If S spans V then $\forall w \in v$, $\exists a_i$'s $\in F : w = \sum_{i=1}^{n} a_i \alpha_i$

$P \implies Q$

Thm:

Let $S = \{\alpha_1, \ldots, \alpha_n\} \subset V$ where V is a vector space over field F

If:

1. S is linearly independent (number of representations ≤ 1)
2. S spans V (number of representations ≥ 1)

then every vector $w \in V$ has a unique representation as a linear combination of vectors in S

Properties

$S = \{\alpha_1, \ldots, \alpha_n\} \subset V$ (vector space over field F)

1. If S is linearly dependent, then any larger set of vectors containing S is linearly dependent
2. If S is linearly independent, then any subset of S is linearly dependent

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Basis and Dimension

Vector space V over field F
Basis of V is a set of vectors $S \in V$ such that:

1. S span V
2. S is a linearly independent set

Dimension of V is the number of vectors in the basis of V

Example:

$V \equiv \mathbb{R}^3$

$V = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R}$

Basis: $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$

$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\}$ ✔

$Dim(\mathbb{R}^3) = 3$

$P_n(t) =$ Polynomial of order $\leq n$

$p(t) \in P_n(t)$

$p(t) = c_0 + c_1 t^1 + \ldots + c_n t^n$

Basis$=$ $\{1, t, t^2, \ldots t^n\}$

$p(t) = \sum_{i=0}^{n} c_0 t^n$

$Dim(P_n(t)) = n + 1$

Thm: Dimension Theorem

All basis of a vector space have the same number of vectors

Proof:

If $T = \{\alpha_1, \ldots \alpha_n\}$ (a basis) and $S = \{\beta_1, \ldots, \beta_m\}$ (a basis) then $n = m$

$P \implies Q$

Proof by contradiction:

$\sim Q : n \neq m \rightarrow (n < m)$ or $(n > m)$

Let $(n < m)$ - (Without Loss Of Generality)

$T = \{\alpha_1, \ldots \alpha_n\}$

$S = \{\beta_1, \ldots, \beta_n, \beta_{n+1}, \ldots \beta_m\}$

$A = \{\alpha_1, \ldots \alpha_n\}, B = \{\beta_1, \ldots, \beta_n\}$

$$B = \begin{bmatrix} \ldots & \beta_1 & \ldots \\ \ldots & \beta_2 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \beta_n & \ldots \end{bmatrix} \in \mathbb{R}^{n\times p}, C = \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ c_{1n} & \ldots & c_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}, A = \begin{bmatrix} \ldots & \alpha_1 & \ldots \\ \ldots & \alpha_2 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \alpha_n & \ldots \end{bmatrix} \in \mathbb{R}^{n\times p}$$

$B = CA$

$$\begin{bmatrix} \ldots & \beta_1 & \ldots \\ \ldots & \beta_2 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \beta_n & \ldots \end{bmatrix} = \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ c_{1n} & \ldots & c_{nn} \end{bmatrix} \begin{bmatrix} \ldots & \alpha_1 & \ldots \\ \ldots & \alpha_2 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \alpha_n & \ldots \end{bmatrix}$$

$\beta_{lj} = \sum_{k=1}^{n} c_{lk}\alpha_{kj}$ - Matrix Multiplication

$\beta_l = \sum_{k=1}^{n} c_{lk}\alpha_k$

Lemma 1:

If A and B have linearly independent rows then C is invertible

$P \implies Q$ or $\sim Q \implies \sim P$

Note: C is invertible $\leftrightarrow$ C has linearly independent rows

$\sim Q : C$ has linearly dependent rows

$c_l = \sum_{i=1, i\neq l} a_i c_i, c_{lk} = \sum_{i=1, i\neq l}^{n} a_i c_{lk}$

$\beta_l = \sum_{k=1}^{n} c_{lk}\alpha_k$

$= \sum_{k=1}^{n} \sum_{i=1, i\neq l}^{n} a_i c_{lk}\alpha_k$

$= \sum_{i=1, i\neq l}^{n} a_i \sum_{k=1}^{n} c_{lk}\alpha_k$

$= \sum_{i=1, i\neq l}^{n} a_i \beta_i$

So for $B = CA$ WITH INVERTIBLE C then,

$A = C^{-1}B$

$C^{-1} = D \equiv [d_{ij}]$

$\alpha_{ij} = \sum_{k=1}^{n} d_{ik}\beta_{kj}$

$\alpha_i = \sum_{k=1}^{n} d_{ik}\beta_k$

$T = \{\alpha_1, \ldots \alpha_n\}$ is a basis of V and $\beta_{m+1} \in V$

$\beta_{n+1} = \sum_{i=1}^{n} e_i\alpha_i$ for some e_i's $\in F$

$\beta_{n+1} = \sum_{i=1}^{n} e_i(\sum_{k=1}^{n} d_{ik}\beta_k)$

$= \sum_{i=1}^{n} \sum_{k=1}^{n} e_i d_{ik}\beta_k$

$= \sum_{i=1}^{n}(\sum_{k=1}^{n} e_i d_{ik})\beta_k$

S is linearly dependent set

so S is a basis

Fundamental subspace of a matrix

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}$$

$Ax = b$ where $x \in \mathbb{R}^{n\times 1}, b \in \mathbb{R}^{m\times 1}$

$T : \mathbb{R}^{n\times 1}$ (Domain) $\rightarrow \mathbb{R}^{m\times 1}$ (Co-domain)

$A^T y = d$ where $A^T \in \mathbb{R}^{n\times m}, y \in \mathbb{R}^{m\times 1}, d \in \mathbb{R}^{n\times 1}$

$T : \mathbb{R}^{m\times 1}$ (Domain) $\rightarrow \mathbb{R}^{n\times 1}$ (Co-domain)

1. Column Space: $C(A)$

   $C(A) = \{b \in \mathbb{R}^{m\times 1} : Ax = b, x \in \mathbb{R}^{n\times 1}\}$
2. Row Space: $C(A^T)$

   $C(A^T) = \{d \in \mathbb{R}^{n\times 1} : A^T y = d, y \in \mathbb{R}^{m\times 1}\}$
3. Null Space: $n(A)$

   $n(A) = \{x \in \mathbb{R}^{n\times 1} : Ax = \mathbf{0}\}$
4. Left Null Space: $n(A^T)$

$n(A) = \{y \in \mathbb{R}^{m\times 1} : A^T y = \mathbf{0}\}$

Subspaces	Dimension
Domain	$n \equiv$ order
$C(A)$	$r \equiv$ rank
$n(A)$	$\zeta \equiv$ nullity

Fact: $n = r + \zeta$

$r \equiv$ rank

$r =$ number of pivots$= Dim(C(A)) = Dim(C(A^T))$

$\zeta =$ number of free variables$= Dim(n(A))$

Example:

$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}$ Find $n(A)$ and its dimension.

$Ax = \mathbf{0}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

$R1 : R1$

$R2 : R2 - 5R1$

$R3 : R3 - 2R1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 : R3 - R2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$r =$ number of pivots $= 2$\$

$\zeta =$ number of free variables $= 0$

$x = 0, y = 0$

$n(A) = \{\mathbf{0}\}$

$Dim(n(A)) = \zeta = 0$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

$R1 : R1$

$R2 : R2 - 5R1$

$R3 : R3 - 2R1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 : R3 - R2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} r = 2, \zeta = 1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $x_3 = z, z \in \mathbb{R}$

$4x_3 + 4x_3 = 0 \implies x_2 = -Z$

$x_1 + x_3 = 0 \implies x_1 = -z$

$$z \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, z \in \mathbb{R}$$

$$n(A) = span\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\}$$

$Dim(C(A)) = 2, Dim(n(A)) = 1$

Thm:

Interchanging the rows of a matrix leaves its rank unchanged.

Thm:

If $Ax = 0$ and $Bx = 0$ have the same solution, then A and B have the same column rank

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Week 4 Session 1

Outlines

Dimension Theorem

Existence and Uniqueness

Inner Product Space

$Ax = b$ where $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n \times 1} = y \in \mathbb{R}^{m \times 1}$, $m \neq n$

Principal Component Analysis (Dimension Reduction)

$X \in \mathbb{R}^{n \times p}$ where n is the number of sample, p is the number of features, $p \gg 1$

$A \in \mathbb{R}^{p \times s}$ where s is a very small dimension

$X \to \bar{X}$ mean$= 0 \implies K_{xx}$ where it is $p \times p \to$ Eigenvalues

$$E \in \mathbb{R}^{p \times p} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots. \\ e_1 & \dots & e_s & \dots & e_p \\ \dots & \dots & \dots & \dots & \dots \\ (\lambda_1) & \dots & (\lambda_s) & \dots & (\lambda_p) \end{bmatrix}$$

$\lambda_1 \gg \lambda_2 \gg \dots \gg \lambda_p$

XE where X is $n \times p$, and E is $p \times p$

$X\bar{E} = \hat{X}$ where X is $n \times p$, $\bar{E}$ is $p \times s$, $\hat{X}$ is $n \times s$

Thm:

If $Ax = 0$ and $Bx = 0$ have the same solution, then A and B have the same column rank.

Proof:

$P \implies Q$

Let s be the column rank of A

Let t be the column rank of B

where $t \neq s$ so, $(t > s)/(s > t)$

Let $t > s$ (WLOG)

$$B \in \mathbb{R}^{m \times n} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots. & \dots & \dots \\ \beta_1 & \dots & \beta_s & \dots & \beta_t & \dots & \beta_n \\ \dots & \dots & \dots & \dots & \dots. & \dots & \dots \end{bmatrix} \text{ so that } Bx = 0$$

where column $1 \dots t$ are linearly independent, $t + 1 \dots n$ are linearly dependent

$$A \in \mathbb{R}^{m \times n} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots. & \dots & \dots \\ \alpha_1 & \dots & \alpha_s & \dots & \alpha_t & \dots & \alpha_n \\ \dots & \dots & \dots & \dots & \dots. & \dots & \dots \end{bmatrix} \text{ so that } Ax = 0$$

Therefore $\exists d_i's \neq 0 : \sum_{i=1}^{\sigma} d_i \alpha_i = \mathbf{0}$ where $t > s$

$\sum_{j=1}^{\sigma} d_j \alpha_j + \sum_{j=t+1}^{n} 0\alpha_j = \mathbf{0}$

$x_1 = d_1, x_2 = d_2, \dots x_t = d_t$, this is the solution to $Ax = 0$

$x_{t+1} = \ldots = x_n = 0$

$\exists d_i's \neq 0 : \sum_{j=1}^{t} d_i \beta_j + \sum_{j=t+1}^{n} 0\beta_j = \mathbf{0}$

$\exists d_i's \neq 0 : \sum_{j=1}^{t} d_i \beta_j = \mathbf{0}$

$\{\beta_1, \ldots \beta_t\}$ is linearly dependent

Contradiction

Thm:

Elementary row operations preserve column rank

$Ax = b$ elementary operations $\implies A'x = b'$

$Ax = 0 \implies A'x = 0$

Thm:

Rank Theorem

Dimension of column space equals the dimension of row space.

$Ax = b$ where $A \in \mathbb{R}^{m \times n}$

Proof:

Let c be the column rank of A

Let r be the row rank of A

$c \leq r$ or $r \leq c$

Case 1: $c \leq r$

$$A = \begin{bmatrix} \ldots & \alpha_1 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \alpha_r & \ldots \\ \ldots & \alpha_{r+1} & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \alpha_m & \ldots \end{bmatrix}$$

where $B \in \mathbb{R}^{r \times n} = \begin{bmatrix} \ldots & \alpha_1 & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \alpha_r & \ldots \end{bmatrix}$ is linearly independent rows

where $D \in \mathbb{R}^{(m-r) \times n} = \begin{bmatrix} \ldots & \alpha_{r+1} & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \alpha_m & \ldots \end{bmatrix}$ is linearly dependent rows

$\forall_j : r+1 \leq j \leq m, \exists t_{ji}'s :$

$\alpha_j = \sum_{i=1}^{r} t_{ji} \alpha_i$, $T \equiv [t_{ji}]$

$D = TB$

$(m-r) \times n = (m-r) \times r(t \times n)$

$$A = \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} B \\ TB \end{bmatrix}$$

If $Ax = \mathbf{0} \implies \begin{bmatrix} B \\ TB \end{bmatrix} x = \begin{bmatrix} Bx \\ TBx \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$

If $Bx = \mathbf{0} \implies Ax = \begin{bmatrix} B \\ TB \end{bmatrix} x = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$

Case 2: $r \leq c$

$A^T y = x$

$A^T \in \mathbb{R}^{n \times m}, y \in \mathbb{R}^{m \times 1}, x \in \mathbb{R}^{n \times 1}$

The column rank of A^T is r

The row rank of A^T is c

$$A^T = \begin{bmatrix} \dots & \beta_1 & \dots \\ \dots & \dots & \dots \\ \dots & \beta_c & \dots \\ \dots & \beta_{c+1} & \dots \\ \dots & \dots & \dots \\ \dots & \beta_n & \dots \end{bmatrix}$$

where $E \in \mathbb{R}^{c \times m} = \begin{bmatrix} \dots & \beta_1 & \dots \\ \dots & \dots & \dots \\ \dots & \beta_c & \dots \end{bmatrix}$ is linearly independent rows

where $F \in \mathbb{R}^{(n-c) \times m} = \begin{bmatrix} \dots & \beta_{c+1} & \dots \\ \dots & \dots & \dots \\ \dots & \beta_n & \dots \end{bmatrix}$ is linearly dependent rows

$\forall_j : c + 1 \leq j \leq n, \exists r'_{ji} s :$

$\beta_j = \sum_{i=1}^{c} r_{ji} \beta_i \, , \; R \equiv [r_{ji}]$

$F = RE$

$$A^T = \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} E \\ RE \end{bmatrix}$$

$A^T y = \mathbf{0}$ if and only if $Ey = 0$

The column rank of $E = r$

$Ey = x$ where $E \in \mathbb{R}^{c \times m}, y \in \mathbb{R}^{m \times 1}, x \in \mathbb{R}^{c \times 1}$

$r \leq c$

Therefore $r = c$

$Bx = y$ where $B \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n \times 1}, y \in \mathbb{R}^{m \times 1}$

x is the domain and y is the co-domain

$Bx \equiv$ column smace or range space

column space $\subset$ co-domain

Counting Theorem

$A \in \mathbb{R}^{m \times n}$

Dimension of column space + Dimension of null sapce = n = number of columns

Proof:

$A \in \mathbb{R}^{m \times n}$

$A \implies R_r$ (row reduced Echelon)

$Ax = 0$ and $R_r x = 0$

Number of pivots in $R_r =$ column rank (A)

$R_r x = 0$ and $Ax = 0$

Dim of null sapce for $R_r = n - r$ where n is the number of columns and r is the number of pivots

Because A and R_r are row equivalent, then

$Ax = 0$ if and only if $R_r x = \mathbf{0}$

Dimension of null space of $A = n - r$

$n - r + r = n$

Dim($n(A)$)+Dim($C(A)$)=number of columns

Thm:

Fundamental Theorem: $A \in \mathbb{R}^{m \times n}$

1. The row space of A and nullsape of A are orthogonal complements in $\mathbb{R}^{n \times 1}$
2. The column space of A and left null sapce of A are orthoginal complements in $\mathbb{R}^{m \times 1}$

Let v be a vector space

u be a subspace of v

w be a subsapce of v

u and w are orthogonal complements means that $\forall \alpha \in u$ and $\forall \beta \in w$, $\alpha \perp \beta$

$\alpha \cdot \beta = 0$

Proof:

$Ax = y$ where $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n \times 1}, y \in \mathbb{R}^{m \times 1}$

1. Row Space: $C(A^T) = \{x \in \mathbb{R}^{n \times 1} : A^T y = x, y \in \mathbb{R}^{m \times 1}\}$
   Null Space: $n(A) = \{x \in \mathbb{R}^{n \times 1} : Ax = \mathbf{0}\}$
   Assume $\alpha \in C(A^T)$ and $\beta = n(A)$
   $\alpha \cdot \beta = \alpha^T \beta = (A^T y)^T x$
   $= y^T Ax$ where $Ax = 0$
   $= 0$
2. Column Space: $C(A) = \{y \in \mathbb{R}^{m \times 1} : Ax = y, x \in \mathbb{R}^{n \times 1}\}$

Left null space $n(A^T) = \{y \in \mathbb{R}^{m\times 1} : A^T y = 0\}$

Assume $\alpha \in C(A)$ and $\beta \in n(A^T)$

$\alpha^T \beta = (Ax)^T y$

$= x^T A^T y$ where $A^T y = 0$

$= 0$

Summary:

$A \in \mathbb{R}^{m\times n}$

column rank =r

dimension of null space= $n - r$

row rank = r

dimension of left null space =$m - r$

$Ax = b$

$m \equiv$ number of equations

$n \equiv$ number of unknowns

$M = [Ab]$

Rank(M)   Rank(A)

Existence and Uniqueness

Thm:

Let $Ax = b$ be a system with n-unknowns m equations and augmented matrix $M = [Ab]$

1. The system has at least one solution if and only if $rank(M) = rank(A)$

   $$M' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} A' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

   $n = 3$

2. The system has a unique solution if and only if $rank(M) = n = rank(A)$

Inner Product Space

Vector space V over a field F

Real Inner Product space:

Let V be a vector space over field $\mathbb{R}$

$< \alpha, \beta >$ assign a real number for $\alpha, \beta \in V$

Then $< \alpha, \beta >$ is an inner product if:

$[I_1]$ Linearity: $< \alpha, a\beta + b\gamma >= a < \alpha, \beta > +b < \alpha, \gamma >$, $\forall \alpha, \beta, \gamma \in V$ and $a, b \in \mathbb{R}$

$[I_2]$ Symmetry: $< \alpha, \beta >=< \beta, \alpha >$, $\forall \alpha, \beta \in V$

$[I_3]$ Positive Definite: $< \alpha, \alpha >\geq 0$ and $< \alpha, \alpha >= 0$ if and only if $\alpha = \mathbf{0}$

Examples:

1. Euclidean $\mathbb{R}^n$

   $< u, v >= u \cdot v = \sum_{i=1}^{n} u_i v_i$
2. Function space $c[a, b]$ and polynomial space $P_n(t)$

   $c[a, b]$ - vector space of all continuous functions on the closed interval $[a, b]$

   $< f, g >= \int_a^b f(x)g(x)dx$
3. Matrix space $M = \mathbb{R}^{m \times n}$

   M - vector space of all real $m \times n$ matrices

   $< A, B >= Tr(B^T A)$

Week 4 Session 2

Outlines

Orthogonality and Inner Products

Gram-Schmidt Process

Inner Product

$< \alpha, \beta >$

Complex Inner Product Space

Vector V over field $\mathbb{C}$

$< \alpha, \beta >= \sum_{i=1}^{n} a_i b_i^*$ where $\alpha = \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix}$, $\beta = \begin{bmatrix} b_1 \\ \dots \\ b_n \end{bmatrix}$

$< \alpha, \beta >$ must satisfy the following properties:

$\forall \alpha, \beta, \gamma \in V; \forall a, b \in \mathbb{C}$

$[I_1]$: Linearity

$< \alpha, a\beta + b\gamma >= a^* < \alpha, \beta > +b^* < \alpha, \gamma >$

$[I_2]$: Conjugate Symmetry

$< \alpha, \beta >=< \beta, \alpha >^*$

$[I_3]$: Positive Definite:

$< \alpha, \alpha >\geq 0$ and $< \alpha, \alpha >= 0$ if and only if $\alpha = \mathbf{0}$

Normed Vector Spaces

Let $V = \{\alpha, \beta, \gamma, \dots\}$ be a vector space over a field F. A norm $||\cdot||$ of V is a function from the elements of v (vectors in V) into the non-negative real number such that:

$[N_1]: ||\alpha|| \geq 0$, $\forall \alpha \in V$ and $||\alpha|| = 0$ if an only if $\alpha = \mathbf{0}$

$[N_2]: ||k\alpha|| = |k|||\alpha||$, $\forall \alpha \in V$ and $\forall k \in F$

$[N_3]: ||\alpha + \beta|| \leq ||\alpha|| + ||\beta||$, $\forall \alpha, \beta \in V$ (triangle inequality)

Example:

1. $v = \mathbb{R}^n, \alpha \in V, \alpha = [a_1, \dots a_n]$
   $||\alpha|| = \sqrt{(a_1)^2 + \dots (a_n)^2}$ - Euclidean Norm
2. $v = \mathbb{C}^n$ Complex field

Metric Space

Vector space V over F

$M(\alpha, \beta)$ - metric

Properties of a matric:

$[M_1]: M(\alpha, \beta) \geq 0$ and $M(\alpha, \beta) = 0$ if and only if $\alpha = \beta$

$[M_2]: M(\alpha, \beta) = M(\beta, \alpha)$

$[M_3]: M(\alpha, \gamma) \leq M(\alpha, \beta) + M(\beta, \gamma)$

Norm

l^p - norm : $\sqrt[p]{\sum_{i=1}^{n} |x_i|^p} = ||x||_p$

l^p - distance : $||x - y||_p$

Volume of an Euclidean ball of radians γ

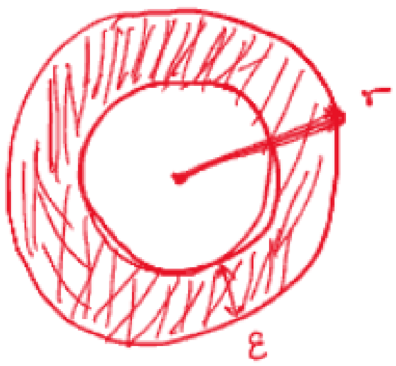
l^2 - norm $r = \sqrt[2]{\sum_{i=1}^{n} x_i^2} = n = 2$

$r = \sqrt{x^2 + y^2}$

Given Conditions:

$V_n(r) = c_n r^n$; $c_n = \frac{2\pi}{n} c_{n-2}$

n	V_n
$c_1 = 2$	$2r$
$c_2 = \pi$	πr^2
$c_3 = \frac{2\pi}{3} c_1 = \frac{4\pi}{3}$	$\frac{4\pi}{3} r^3$
$c_4 = \frac{2\pi}{4} c_2 = \frac{\pi^2}{2}$	$\frac{\pi^2}{2} r^4$



$0 < \epsilon < r$

Volume shell - Entire Volume

$\frac{c_n r^n - c_n (r-\epsilon)^n}{c_n r^n}$

$= \frac{r^n - (r-\epsilon)^n}{r^n}$

$= 1 - (1 - \frac{\epsilon}{r})^n$

$0 < \epsilon < r \implies 0 < \frac{\epsilon}{r} < 1$

$1 > 1 - \frac{\epsilon}{r} > 0$

$lim_{n \to \infty} 1 - (1 - \frac{\epsilon}{r})^n = 1$

Orthogonality

Vector space V over field F

$\alpha, \beta \in V$
$\alpha \perp \beta$ if and only if $< \alpha, \beta >= 0$

Def: Let $S = \{\alpha_1, \dots \alpha_n\} \subset V$ is mutually orthogonal if and only if

$\alpha_i \cdot \alpha_j = 0$ for $i \neq j$

Mutually Orthonormal

A vector is normal if and only if its norm $|| \cdot ||$ is equal to 1

Def: Let $S = \{\beta_1, \dots \beta_n\} \subset V$ is mutually orthonormal if and only if

$$\alpha_i \cdot \alpha_j = \begin{cases} 0, \text{if } i \neq j \\ 1, \text{if } i = j \end{cases}$$

$S = \{\alpha_1, \dots \alpha_n\}$ which is mutually orthogonal $\implies T = \{\frac{\alpha_1}{||\alpha_1||}, \dots, \frac{\alpha_n}{||\alpha_n||}\}$ which is mutually orthonormal

---

S is linearly independent $\not\Longrightarrow$ S is mutually orthogonal

S is mutually orthogonal $\implies$ S is linearly independent

Example.

$S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$v = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$

where $\alpha_1 \cdot \alpha_2 = 0, \alpha_1 \cdot \alpha_3 = 0, \alpha_2 \cdot \alpha_3 = 0$

$c_1 = \frac{v \cdot \alpha_1}{\alpha_1 \cdot \alpha_1} = \frac{3}{1}$

$c_2 = \frac{v \cdot \alpha_2}{\alpha_2 \cdot \alpha_2} = \frac{7}{2}$

$c_3 = \frac{v \cdot \alpha_3}{\alpha_3 \cdot \alpha_3} = \frac{3}{2}$

Therefore: $v = 3\alpha_1 + \frac{7}{2}\alpha_2 + \frac{3}{2}\alpha_3$

Thm:

If $S = \{\alpha_1, \ldots \alpha_n\}$ is in a vector space V and S is mutually orthogonal (with $\alpha_i \neq 0$), then S is linearly independent

Proof:

$c_1\alpha_1 + \ldots c_n\alpha_n = \mathbf{0}$

$(c_1\alpha_1 + \ldots c_n\alpha_n) \cdot \alpha_i = \mathbf{0} \cdot \alpha_i = \mathbf{0}$

$\sum_{j=1}^{n} c_j(\alpha_j \cdot \alpha_i) = 0$

$\sum_{j=1, j\neq i}^{n} c_j(\alpha_j \cdot \alpha_i) + c_i(\alpha_i \cdot \alpha_i) = 0$

$\sum_{j=1, j\neq i}^{n} c_j(\alpha_j \cdot \alpha_i) = 0$ because S is mutually orthogonal

$c_i(\alpha_i \cdot \alpha_i) = 0$

$c_i = \frac{0}{\alpha_i \cdot \alpha_i}$ where $\alpha_i \neq 0$

Therefore, $c_1 = c_2 = \ldots = c_n$ is the only solution

Therefore, S is linearly independent

$S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

It is linearly independent but not mutually orthogonal

where $\alpha_1 \cdot \alpha_2 = 1, \alpha_1 \cdot \alpha_3 = 2, \alpha_2 \cdot \alpha_3 = 1$

Thm:

If $S = \{\alpha_1, \ldots \alpha_n\}$ is in a vector space V, S is a basis of V and S is mutually orthogonal, then $\forall \beta \in V, \exists a_i's$ such that

$a_1\alpha_1 + \ldots + a_n\alpha_n = \beta$

$a_i = \frac{\beta \cdot \alpha_i}{\alpha_i \cdot \alpha_i}$

Proof:

S is a basis for V

$\forall \beta \in V$

$a_1\alpha_1 + \ldots + a_n\alpha_n = \beta$

$(a_1\alpha_1 + \ldots + a_n\alpha_n)\alpha_i = \beta \cdot \alpha_i$

$\sum_{j=1}^{n} a_j(\alpha_j \cdot \alpha_i) = \beta\alpha_i$

$\sum_{j=1, j\neq i}^{n} a_j(\alpha_j \cdot \alpha_i) + a_i(\alpha_i \cdot \alpha_i) = \beta \cdot \alpha_i$

$a_i(\alpha_i \cdot \alpha_i) = \beta \cdot \alpha_i$

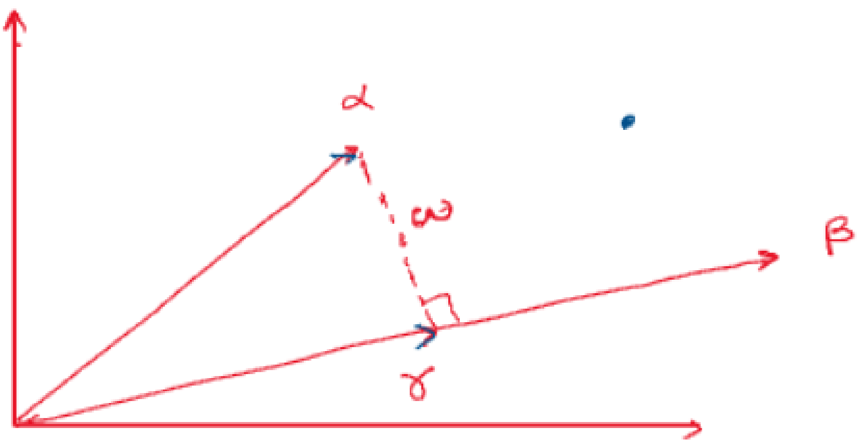
Therefore, $a_i = \frac{\beta \cdot \alpha_i}{\alpha_i \cdot \alpha_i}$

$S = \{\alpha_1, \ldots \alpha_n\}$ is a orthogonal basis of V

1. Basis
2. Mutually orthogonal

Projection

Projection of α onto β



$\gamma = Proj_{\beta}(\alpha) = c\beta$

$\omega = \alpha - \gamma$ and $\omega \perp \beta$

$(\alpha - \gamma) \perp \beta$

$(\alpha - c\beta) \perp \beta$

$(\alpha - c\beta) \cdot \beta = 0$

$\alpha \cdot \beta - c\beta \cdot \beta = 0$

$c = \frac{\alpha \cdot \beta}{\beta \cdot \beta}$

$Proj_{\beta}(\alpha) = c\beta = \frac{\alpha \cdot \beta}{\beta \cdot \beta}\beta$

$Orth_\beta(\alpha) \equiv \alpha - \gamma = \alpha - \frac{\alpha \cdot \beta}{\beta \cdot \beta}\beta$

$Proj_\beta(\alpha) + Orth_\beta(\alpha) = \alpha$

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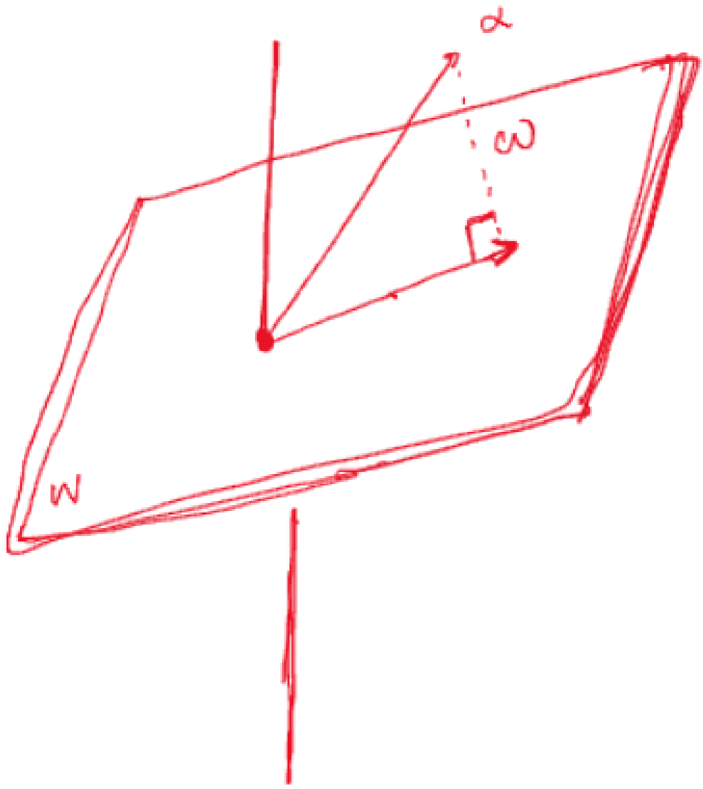
w is a subspace of V

Projection of α onto ω

$S = \{\beta_1, \dots \beta_m\}$ is an orthogonal basis for ω



$\gamma = Proj_w(\alpha) = c_1\beta_1 + \dots + c_m\beta_m$

1. $\gamma \cdot \beta_i = (c_1\beta_1 + \dots + c_m\beta_m) \cdot \beta_i$

   $= c_i(\beta_i \cdot \beta_i) + \sum_{j=1, j\neq i}^{m} c_j(\beta_j \cdot \beta_i)$

2. $\omega = \alpha - \gamma : \omega \perp \beta_i$

   $(\alpha - \gamma) \perp \beta_i$

   $(\alpha - \gamma) \cdot \beta_i = 0$

   $\alpha \cdot \beta_i = \gamma \cdot \beta_i$

Then $\gamma \cdot \beta = \alpha \cdot \beta_i = c_i(\beta_i \cdot \beta_i)$

$c_i = \frac{\alpha \cdot \beta_i}{\beta_i \cdot \beta_i}$

$\gamma = \frac{\alpha \cdot \beta_1}{\beta_1 \cdot \beta_1} + \dots + \frac{\alpha \cdot \beta_m}{\beta_m \cdot \beta_m} = Proj_\omega(\alpha)$

$S = \{\alpha_1, \dots \alpha_n\}$ which is linearly independent "Gram Schmidt" $\implies$ $T = \{\beta_1, \dots \beta_n\}$ which is mutually orthogonal

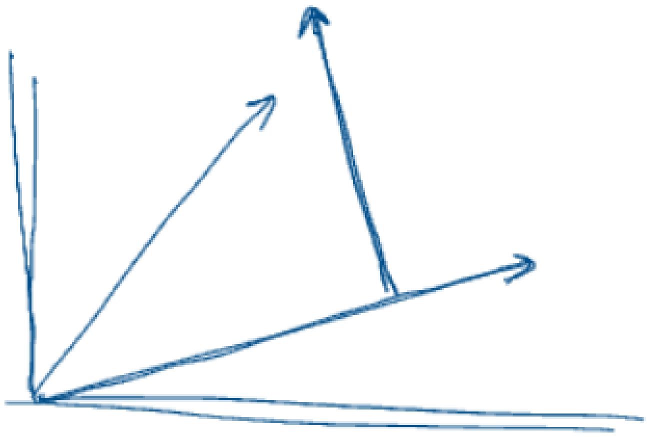
$L(S) = L(T)$

Def:

If V is a vector space and S is a subspace of V, then

$\omega = \{\alpha + \beta : \alpha \in S, \beta \in S^{\perp}\} = V = S \oplus S^{\perp}$

where $S^{\perp}$ is the orthogonal complement of S



Gram-Schmidt Process

Given $S = \{\alpha_1, \ldots, \alpha_n\}$ where S is linearly independent.

Find $T = \{\beta_1, \ldots, \beta_n\}$ where S is mutually orthogonal and $L(s) = L(\tau)$

$\beta_1 = \alpha_1$

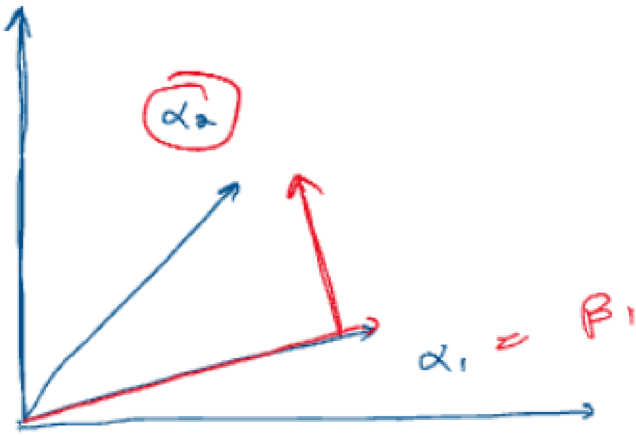
$V_1 = span\{\alpha_1\} = span\{\beta_1\}$

$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1$ where $Proj_{v_1}(\alpha_2) = \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1$

$V_2 = span(\{\alpha_1, \alpha_2\}) = span(\{\beta_1, \beta_2\})$

$\beta_3 = \alpha_3 - [\frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_2]$ where $Proj_{v_2}(\alpha_3) = \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_1$

$\beta_k = \alpha_k - [\frac{\alpha_k \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \ldots + \frac{\alpha_k \cdot \beta_{(k-1)}}{\beta_{(k-1)} \cdot \beta_{(k-1)}}\beta_{(k-1)}]$



$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1$

Ex. $\alpha_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\alpha_2 = \begin{bmatrix}1\\2\\0\\1\end{bmatrix}$, $\alpha_3 = \begin{bmatrix}2\\1\\1\\0\end{bmatrix}$, $\alpha_4 = \begin{bmatrix}0\\0\\3\\1\end{bmatrix}$

$$\beta_1 = \alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\beta_3 = \alpha_3 - [\frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_2] = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\beta_4 = \alpha_4 - [\frac{\alpha_4 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \frac{\alpha_4 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_2 + \frac{\alpha_4 \cdot \beta_3}{\beta_3 \cdot \beta_3}\beta_3] = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} - \frac{4}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-3}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

Let V be a vector where

$V = \{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \text{where } a \in \mathbb{R}\}$

then $\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$ is a basis, and the dimension is 1

Week 4 Session 1 (Messed)

Outlines

- Review
- Dimension

  Rank Theorem

  Counting Theorem

  Fundamental Theorem

  Existence and Uniqueness
- Inner Product, Orthogonality

Review:

Theorem 1:

Interchanging rows of a matrix leaves its row rank unchanged.

Theorem 2:

If $Ax = 0$ and $Bx = 0$ have the same solution then A and B have the same column rank

[Proof: Lecture 6]

Theorem 3:

Elementary row operation does not change the column rank.

Reason: Elementary row operation preserves solution, then apply Theorem 2

$Ax = b$ where $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n \times 1}, b \in \mathbb{R}^{n \times 1}$

Theorem 4: Rank Theorem:

Dimension of column space = Dimension of row space

Proof:

column rank = Dimension of column space

row rank = Dimension of row space

Let $r =$ row rank of A

$c =$ column rank of A

Claim 1: $c \leq r$

Proof:

$A = \begin{bmatrix} \cdots & a_1 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & a_r & \cdots \\ \cdots & a_{r+1} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & a_m & \cdots \end{bmatrix}$, where $\begin{bmatrix} \cdots & a_1 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & a_r & \cdots \end{bmatrix}$ is linearly independent rows, $\begin{bmatrix} \cdots & a_{r+1} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & a_m & \cdots \end{bmatrix}$ is linearly dependent rows

Let $B = \begin{bmatrix} \cdots & a_1 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & a_r & \cdots \end{bmatrix}$, where $B \in \mathbb{R}^{r \times n}$, $D = \begin{bmatrix} \cdots & a_{r+1} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & a_m & \cdots \end{bmatrix}$, where $D \in \mathbb{R}^{(m-r) \times n}$

Note: \$$\forall j : r + 1 \leq j \leq m, \exists t_{ji}$\$' s such that

$a_j = \sum_{i=1}^{r} t_{ji} a_i$ - Linearly dependent rows

Let $T = [t_{ji}]$

$D = TB$

$A = \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} B \\ TB \end{bmatrix}$

So, $Ax = \mathbf{0}$ means $\begin{bmatrix} B \\ TB \end{bmatrix} x = \begin{bmatrix} Bx \\ TBx \end{bmatrix} = \mathbf{0}$

$Ax = 0$ if and only if $Bx = 0$

The column rank of $A = c$

so the column rank of $B = c$

Remember:

$Bx = d \in \mathbb{R}^{r \times 1}$

so the column space of $B \subset R^{r \times 1}$

$Dim(C(B)) \leq D(\mathbb{R}^{r \times 1})$, where $C(B)$ is ye column space

$c \leq r$

Claim 2: $r \leq c$

Proof:

Definition of transpose

$c =$ row rank of A^T

$r =$ column rank of A^T

$A = \begin{bmatrix} \dots & \alpha_1 & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_c & \dots \\ \dots & \alpha_{c+1} & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_n & \dots \end{bmatrix}$, where $\begin{bmatrix} \dots & \alpha_1 & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_c & \dots \end{bmatrix}$ is linearly independent rows, $\begin{bmatrix} \dots & \alpha_{c+1} & \dots \\ \dots & \dots & \dots \\ \dots & a_n & \dots \end{bmatrix}$ is linearly dependent rows

Let $E = \begin{bmatrix} \dots & \alpha_1 & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_c & \dots \end{bmatrix}$, where $B \in \mathbb{R}^{c \times m}$, $F = \begin{bmatrix} \dots & \alpha_{r+1} & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_n & \dots \end{bmatrix}$, where $D \in \mathbb{R}^{(n-c) \times m}$

Note: \$$\forall i : c + 1 \leq i \leq n, \exists r_{ij}$\$' s such that

$\alpha_i = \sum_{j=1}^{c} r_{ij} \alpha_j$

Let $R = [r_{ij}]$

then $F = RE$

$A^T = \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} E \\ RE \end{bmatrix}$

$A^T y = \mathbf{0}$ means $\begin{bmatrix} E \\ RE \end{bmatrix} y = \begin{bmatrix} Ey \\ REy \end{bmatrix} = \mathbf{0}$

$A^T y = \mathbf{0}$ if and only if $Ey = 0$

The column rank of $A^T = r$

So the column rank of $E = r$

Remember:

$Ex = f \in \mathbb{R}^{c \times 1}$

So the column space of $E \subset \mathbb{R}^{c \times 1}$

$Dim(C(E)) \leq Dim(\mathbb{R}^{c \times 1})$, where $C(E)$ is the column space of E

Then $r \leq c$

Therefore $c = r$

Theorem 5: Counting Theorem

Dimension of column space + Dimension of null space =number of columns

Reason: Let R_r be the row-reduced echelon form of A

- Row space of A=row space of R_r

  Because rows of R_r are linear combinations of rows of A and vice versa
- Column Space of $A =$ column space of R_r

  Because same solution for $Ax = 0$ and $R_r x = 0$
- Null space of $A =$ null space of R_r

  Because elementary row operation preserves solution

From R_r

n is the number of variables

r is the number of pivot variables

$n - r$ is the number of free variables

$n = r + (n - r)$

number of columns=$Dim(C(R_r)) + Dim(n(R_r))$

$= Dim(C(A)) + Dim(n(A))$

Similarly,

Dimension of row space + Dimension of left null space = Dimension of rows

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Theorem 6: Fundamental Theorem

1. The row space and null space of A are orthogonal complements in $\mathbb{R}^n$
2. The column space and left null space of A are orthogonal complements in $\mathbb{R}^m$

Proof:

Definition: Let V be a vector space. Let U be a subspace of V and W be a subspace of V. u and w are orthogonal complements in V means that $\forall u \in U$ and $\forall w \in W$, $u \perp w$ ($u \cdot w = 0$)

1. Row space: $C(A^T) = \{A^T y : y \in \mathbb{R}^{m\times 1}\}$

   Null space: $n(A) = \{x \in \mathbb{R}^{n\times 1} : Ax = \mathbf{0}\}$

   Let $A^T y_1 \in C(A^T)$ and $x_0 \in n(A)$

   $(x_0)^T A^T y_1 = ((x_0)^T A^T y_1)^T$ where it is a representation of transpose of scalar

   $= y_1 A x_0$ where $Ax_0 = \mathbf{0}$

   $= \mathbf{0}$
2. Column space: $C(A) = \{Ax : x \in \mathbb{R}^{n\times 1}\}$

   Left null space: $n(A^T) = \{z \in \mathbb{R}^{m\times 1} : A^T z = \mathbf{0}\}$

Let $Ax_1 \in C(A)$ and $z_0 \in n(A^T)$

$(z_0)^T Ax_1 = ((z_0)^T Ax_1)^T$

$= (x_1)^T A^T z_0$ where $A^T z_0 = \mathbf{0}$

$= \mathbf{0}$

Summary

for $A \in \mathbb{R}^{m\times n}$

Column rank = Row rank = Rank = r

Dimension of the null space = $n - r$

Dimension of the left null space = $m - r$

Theorem 1: Existence and Uniqueness

Let $Ax = b$ be a system with n unknowns with augmented matrix $M = [A|b]$ then:

Existence

The system has at least one solution if and only if

rank(A)=rank(M)

Uniqueness

The system has a unique solution if and only if

rank(A)=rank(M)=n

Proof:

1. A has no solution if and only if there exist a degenerate row $[0, 0, \ldots, 0|b]$ in the echelon form of M
2. rank(A)=n if and only if no free variable

Inner product and orthogonality

Real inner product space:

Let V be a vector space over $\mathbb{R}$. Suppose that $\forall \alpha, \beta \in V < \alpha, \beta >$ assigns a real number. Then $< \alpha, \beta >$ is an inner product on V if

$[I_1]$ Linearity: $< \alpha, a\beta + b\gamma >= a < \alpha, \beta > +b < \alpha, \gamma >$, $\forall \alpha, \beta, \gamma \in V$ and $a, b \in \mathbb{R}$

$[I_2]$ Symmetry: $< \alpha, \beta >=< \beta, \alpha >$, $\forall \alpha, \beta \in V$

$[I_3]$ Positive Definite: $< \alpha, \alpha >\geq 0$ and $< \alpha, \alpha >= 0$ if and only if $\alpha = \mathbf{0}$

Examples:

1. Euclidean $\mathbb{R}^n$

$< u, v >= u \cdot v = \sum_{i=1}^{n} u_i v_i$

2. Function space $c[a, b]$ and polynomial space $P_n(t)$

   $c[a, b]$ - vector space of all continuous functions on the closed interval $[a, b]$

   $< f, g >= \int_a^b f(x) g(x) dx$

3. Matrix space $M = \mathbb{R}^{m \times n}$

   M - vector space of all real $m \times n$ matrices

   $< A, B >= Tr(B^T A)$

Complex Inner product Space

Vector space V: $\alpha, \beta, \gamma \in V$

The field is $\mathbb{C}$: $a, b \in \mathbb{C}$

$< u, v >$ must satisfy the following:

$[I_1]$: Linearity

$< \alpha, a\beta + b\gamma >= a^* < \alpha, \beta > +b^* < \alpha, \gamma >$

$[I_2]$: Conjugate Symmetry

$< \alpha, \beta >=< \beta, \alpha >^*$

$[I_3]$: Positive Definite:

$< \alpha, \alpha >\geq 0$ and $< \alpha, \alpha >= 0$ if and only if $\alpha = \mathbf{0}$

Normed Vector Spaces

Let $V = \{\alpha, \beta, \gamma, \ldots\}$ be a vector space over a field F. A norm $|| \cdot ||$ of V is a function from the elements of v (vectors in V) into the non-negative real number such that:

$[N_1]$: $||\alpha|| \geq 0$, $\forall \alpha \in V$ and $||\alpha|| = 0$ if an only if $\alpha = \mathbf{0}$

$[N_2]$: $||k\alpha|| = |k| ||\alpha||$, $\forall \alpha \in V$ and $\forall k \in F$

$[N_3]$: $||\alpha + \beta|| \leq ||\alpha|| + ||\beta||$, $\forall \alpha, \beta \in V$ (triangle inequality)

Example:

1. $v = \mathbb{R}^n, \alpha \in V, \alpha = [a_1, \ldots a_n]$

   $||\alpha|| = \sqrt{(a_1)^2 + \ldots (a_n)^2}$ - Euclidean Norm

2. $v = \mathbb{C}^n$ Complex field

   $\forall \alpha \in V$, $||\alpha|| = \sqrt{(a_1)^2 + \ldots (a_n)^2}$

Definition: A metric $M(\alpha, \beta)$ on pairs of elements $\alpha, \beta \in V$ satisfies the following:

$[M_1]$: $M(\alpha, \beta) = 0$ if and only if $\alpha = \beta$

$[M_2]$: $M(\alpha, \beta) = M(\beta, \alpha)$

$[M_3]$: $M(\alpha, \beta) + M(\beta, \gamma) \geq M(\alpha, \gamma), \forall \alpha, \beta, \gamma \in V$

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l^p - distance

$l^p(x,y) = \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}, 1 \leq p \leq \infty$

$P = 1$

$l^1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ - Absolute

Let $x, y \in B^n = \{0,1\}^n$

consider $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Mid Term 1 Review

Sample 1

$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

Find A^{-1} if the inverse exists otherwise give sufficent reason.

Using Gaussian Jordan-Elimination

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 : R_1$

$R_2 : R_2 - 3R_1$

$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -8 & -3 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{bmatrix}$$

$R_3 : R_3 - 3/4R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -8 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1/12 & -1/4 & 1/3 \end{bmatrix}$$

$R_1 : R_1 - 3R_3$

$R_2 : R_2 + 8R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 3/4 & 3/4 & -1 \\ 0 & -4 & 0 & -7/3 & -1 & 8/3 \\ 0 & 0 & 1 & 1/12 & -1/4 & 1/3 \end{bmatrix}$$

$R_2 : -1/4R_2$

$$\begin{bmatrix} 1 & 2 & 0 & 3/4 & 3/4 & -1 \\ 0 & 1 & 0 & 7/12 & 1/4 & -2/3 \\ 0 & 0 & 1 & 1/12 & -1/4 & 1/3 \end{bmatrix}$$

$R_1 : R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & -5/12 & 1/4 & 1/3 \\ 0 & 1 & 0 & 7/12 & 1/4 & -2/3 \\ 0 & 0 & 1 & 1/12 & -1/4 & 1/3 \end{bmatrix}$$

Sample 2

Find the left null space of matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -2 & -1 & 1 \\ -1 & -8 & -10 & -11 \end{bmatrix}$$

Find the dimension of $C(A)$ and $C(A^T)$

$N(A^T) = \{y \in \mathbb{R}^{3\times1}, A^T y = 0\}$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & -8 \\ 3 & -1 & -10 \\ 4 & 1 & -11 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 - 3R_1$

$R_4 : R_4 - 4R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -6 & -6 \\ 0 & -7 & -7 \\ 0 & -7 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-6y_2 - 6y_3 = 0$

$y_2 = -y_3$

$y_1 + 2y_2 - y_3 = 0$

$y_1 = 3y_3$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3y_3 \\ -y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Basis $(n(A^T)) = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

Null space of A $(Ax = 0, x \in \mathbb{R}^{4\times 1})$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -2 & -1 & 1 \\ -1 & -8 & -10 & -11 \end{bmatrix}$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -7 & -7 \\ 0 & -6 & -7 & -7 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies$ dimension is 2

According to the counting theorem

$D(C(A)) + D(N(A)) =$ number of columns =4

$D(N(A)) = 2$

$D(C(A^T)) + D(N(A^T)) =$ number of rows =3

$D(N(A^T)) = 2$

Sample 3

Use the Gram-Schmidt procedure to construct a set of orthonormal set of $\{[-1, 1, 0], [-1, 0, 1], [0, 1, 1]\}$

Express $v = [2 \quad 3 \quad 5]$ as linear combination of such orthonormal vectors.

$\alpha_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\beta_1 = \alpha_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$||\beta_1||^2 = 2$

$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$||\beta_2||^2 = 3/2$$

$$\beta_3 = \alpha_3 - [\tfrac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \tfrac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_2]$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - -\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1/2}{3/2}\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$||\beta_3||^2 = 12/9$$

The orthonormal basis are:

$$\gamma_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\gamma_2 = \sqrt{2/3}\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \sqrt{1/6}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\gamma_3 = \sqrt{9/12}\begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$$

$$v = (v \cdot \gamma_1)\gamma_1 + (v \cdot \gamma_2)\gamma 2 + (v \cdot \gamma_3)\gamma_3$$

$$v \cdot \gamma_1 = \tfrac{1}{\sqrt{2}}$$

$$v \cdot \gamma_2 = \tfrac{5}{\sqrt{6}}$$

$$v \cdot \gamma_3 = \tfrac{10}{\sqrt{3}}$$

$$v = \tfrac{1}{\sqrt{2}}\gamma_1 + \tfrac{5}{\sqrt{6}}\gamma 2 + \tfrac{10}{\sqrt{3}}\gamma_3$$

$$= 1/2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 5/6\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + 10/3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$