EE510 Linear Algebra for Engineering

Week 1 Session 1

Review:

Logical Inference

Logical Statement P and Q

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
1	1	0	0	1	1	1	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	0	0
0	0	1	1	0	0	1	1

$\wedge$ is AND

$\vee$ is OR

$\implies$ If then

$\iff$ If and only if

Conditional: $P \implies Q$

Contrastive: $\neg P \implies \neg Q$

Converse: $Q \implies P$

Predicate: Px means x is P

Quantifier: $\forall x$ (universal) means "for all x"

$\exists x$ (existential) means "for some x"

$\forall x$: Px means "Everything is P"

$Px_1 \; AND \; Px_2 \; AND \; Px_3 \; AND$

$\exists x$: Px means "Something is P"

$Px_1 \; OR \; Px_2 \; OR \; Px_3 \; OR$

<u>Rules of Inference:</u>

- Modus Ponens: Affirming the antecedent

Premise 1: $P \implies Q$

Premise 2: P

Conclusion: Q

- Modus Tollens: Denying the consequent

Premise 1: $P \implies Q$

Premise 2: $\neg Q$

Conclusion: $\neg Q$

- Mathematical Induction

Goal: Proof that $P_n \forall n \geq n_0$ where n_0 is usually 0 or a positive number

1. Basis step: P_{n0}
2. Induction step:

   P_{n0} & $P_{n-1} \implies P_n$

   Assume P_{n0} and P_{n-1} then show P_n

Set Theory

set: a collection of elements

$x \in A$, where x is element, A is set, $\in \equiv$ Element hood

$A = \{a_1, a_2, \ldots, a_n\}$

Subset: $A \subset X$, $B \subset X$

$A \subset X$ if and only if $\forall x \in A$, $x \in A$

$A^c = \{x \in X : X \notin A\}$

$A \bigcup B = \{x \in X : x \in A \, OR \, x \in B\}$

$A \bigcap B = \{x \in X : x \in A \, AND \, x \in B\}$

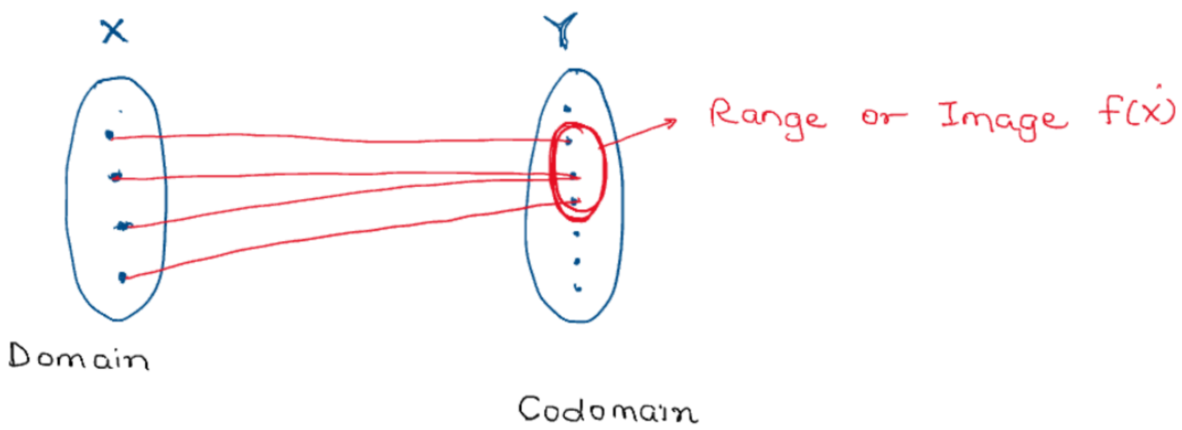
De Morgan's Law:

$A \bigcup B = (A^c \bigcap B^c)^c$

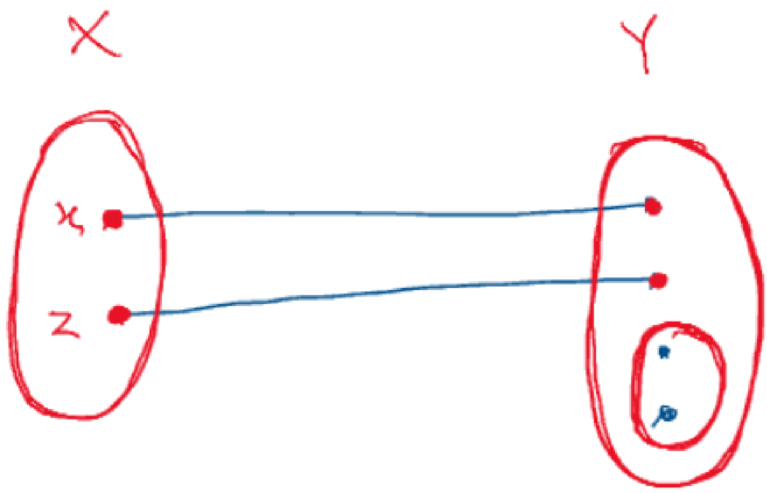
$A \bigcap B = (A^c \bigcup B^c)^c$

Function

$f : X \implies Y$

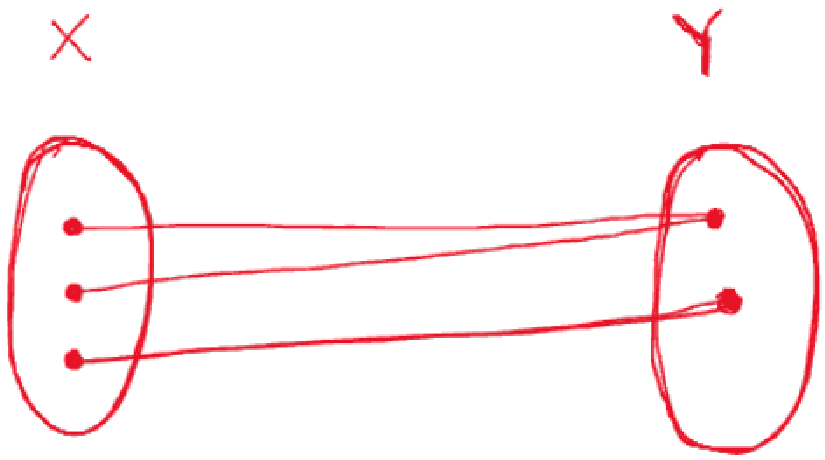


Injective function:



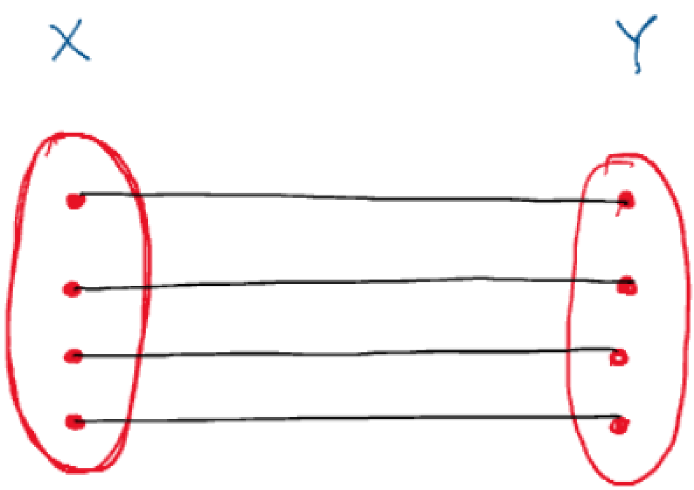
f is injective if and only if $\forall x, z \in X,\ f(x) = f(z) \implies x = z$

Surjective function:



$\forall y \in Y, \exists x \in X : f(x) = y$

Bijective Function (1-1 correspondence)



f is bijective if and only if f is injective and surjective.

Cardinality of a set

Finite set:

$A = \{a_1, \ldots, a_n\}$, where $n \in \mathbb{Z}^+$

Infinite set:

1. Uncountably infinite

   $\mathbb{R}$

2. Countably infinite

   $\mathbb{Z}^+$

Example:

$f : \mathbb{Z}^+(1 - 1\, correspondence) \implies \mathbb{Z}^-$

Vectors

A vector is a 1-dimensional array of scalars over a field.

Let $V = \in \mathbb{R}^(n) : v_1, \ldots, v_n \in \mathbb{R}$

For $u, v \in \mathbb{R}^n$

- Vector Addition:

$$u + v = \begin{bmatrix} u_1 + v_1 \\ \ldots \\ u_n + v_n \end{bmatrix} \in \mathbb{R}^n$$

- Scalar Multiplication:

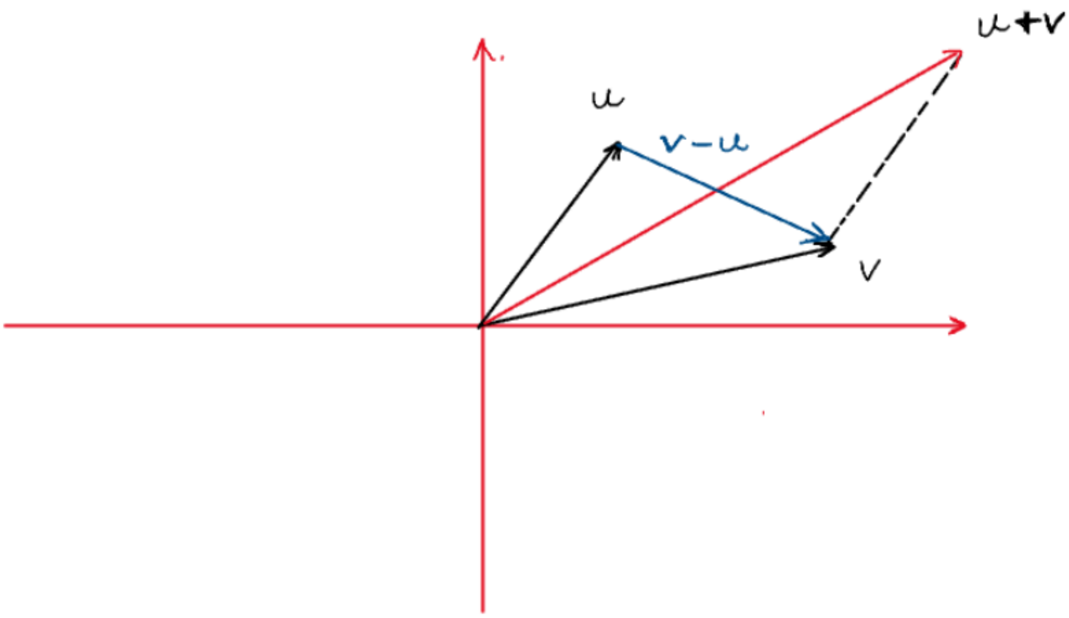
For $a \in \mathbb{R}, v \in R^n$

Then $av = \begin{bmatrix} av_1 \\ \ldots \\ av_n \end{bmatrix}$

- Linear Combination:

For $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$

$$au + bv = \begin{bmatrix} au_1 \\ \ldots \\ au_n \end{bmatrix} + \begin{bmatrix} bv_1 \\ \ldots \\ bv_n \end{bmatrix} = \begin{bmatrix} au_1 + bv_1 \\ \ldots \\ au_n + bv_n \end{bmatrix}$$



- Inner Product

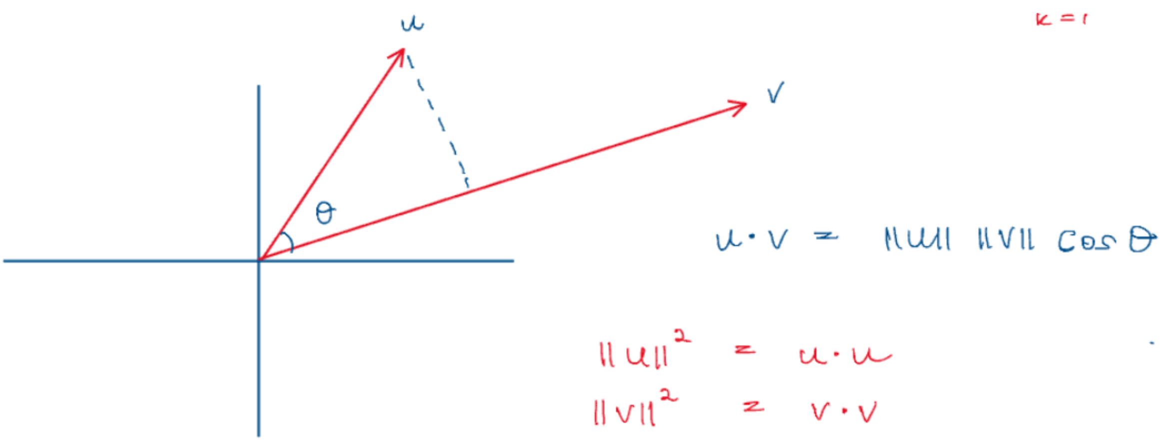
$u, v \in \mathbb{R}^n$

$u \cdot v = \sum_{k=1}^{n} u_k v_k$

Length:

$||u||^2 = u \cdot u = \sum_{k=1}^{n} (u_k)^2$

$||u_k||^2 = \sqrt{(\sum_{k=1}^{n} (u_k)^2)}$



$u \cdot v = ||u||\ ||v|| cos(\theta)$

$||v||^2 = v \cdot v$

$cos(\theta) = \frac{u \cdot v}{||u||\ ||v||}$

Thm: Cauchy Schwartz Inequality

Let $u, v \in \mathbb{R}^n$, $|u \cdot v| \leq ||u||\ ||v||$

Proof:

Case 1: $||u|| = 0$ or $||v|| = 0$

If $||u|| = 0$: $|0 \cdot v| = 0 \leq ||u||\ ||v|| = 0||v|| = 0$

If $||v|| = 0$: $|u \cdot 0| = 0 \leq ||u||\ ||v|| = ||u||0 = 0$

Case 2: $||u|| \neq 0$ and $||v|| \neq 0$

Lemma 1: If $a, b \in \mathbb{R}$, then $a^2 + b^2 \geq 2ab$

Proof: $(a-b)^2 \geq 0$ for $a, b \in \mathbb{R}$

$a^2 + b^2 - 2ab \geq 0$

$a^2 + b^2 \geq 2ab$

Lemma 2: If If $a, b \in \mathbb{R}$, then $a^2 + b^2 \geq -2ab$

Proof: $(a+b)^2 \geq 0$ for $a, b \in \mathbb{R}$

$a^2 + b^2 + 2ab \geq 0$

$a^2 + b^2 \geq -2ab$

Let $a_k \equiv \frac{u_k}{||u||}$, $b_k \equiv \frac{v_k}{||v||}$

$(a_k)^2 + (b_k)^2 \geq 2a_k b_k \qquad \text{using Lemma 1}$

$\sum_{k=1}^{n} \left(\frac{(u_k)^2}{(||u||)^2} + \frac{(v_k)^2}{(||v||)^2} \right) \geq \sum_{k=1}^{n} \left(2 \frac{u_k}{||u||} \frac{v_k}{||v||} \right)$

$\frac{1}{(||u||)^2} \sum_{k=1}^{n} (u_k)^2 + \frac{1}{(||v||)^2} \sum_{k=1}^{n} (v_k)^2 \geq \frac{2}{||u||||v||} \sum_{k=1}^{n} u_k v_k$

$\frac{(||u||)^2}{(||u||)^2} + \frac{(||v||)^2}{(||v||)^2} \geq \frac{2}{||u|| \, ||v||} (u \cdot v)$

$2 \geq \frac{2}{||u|| \, ||v||} (u \cdot v)$

$||u|| \, ||v|| \geq (u \cdot v)$

Similarly,

$||u|| \, ||v|| \geq -(u \cdot v) \qquad \text{using Lemma 2}$

Therefore $||u|| \, ||v|| = 0$

Week 1 Session 2

Outline

Vectors: Dot Products, Norm, Minkowski Inequality

Matrices: Matrix multiplication ,Transpose, Trace, Block matrices

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$u, v \in \mathbb{R}^n$

Inner Product: $u \cdot v = \sum_{k=1}^{n} u_k v_k$

Length (Norm): $||u||^2 = u \cdot c = \sum_{k=1}^{n} (u_k)^2$

Properties: For $k \in \mathbb{R}$, $u, v, w \in \mathbb{R}^n$

1. $u \cdot v = v \cdot u$
2. $u \cdot (v + w) = (u \cdot v) + (u \cdot w)$
3. $ku \cdot v = k(u \cdot v)$
4. $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if $u = \mathbf{0}$

$|u \cdot v| \leq ||u|| \, ||v||$: Cauchy Schwartz Inequality

Minkowski Inequality

$||u+v|| \leq ||u|| + ||v||$

Proof:

$||u+v||^2 = (u+v)\cdot(u+v)$

$= (u\cdot u) + (u\cdot v) + (v\cdot u) + (v\cdot v)$

$= ||u||^2 + 2(u\cdot v) + ||v||^2$

$\leq ||u||^2 + 2|u\cdot v| + ||v||^2 \qquad (u\cdot v) \in \mathbb{R}$

$\leq ||u||^2 + 2||u||\ ||v|| + ||v||^2$ Cauchy Schwartz Inequality

$= (||u|| + ||v||)^2$

Therefore:

$||u+v||^2 \leq (||u|| + ||v||)^2$

$||u+v|| \leq ||u|| + ||v||$

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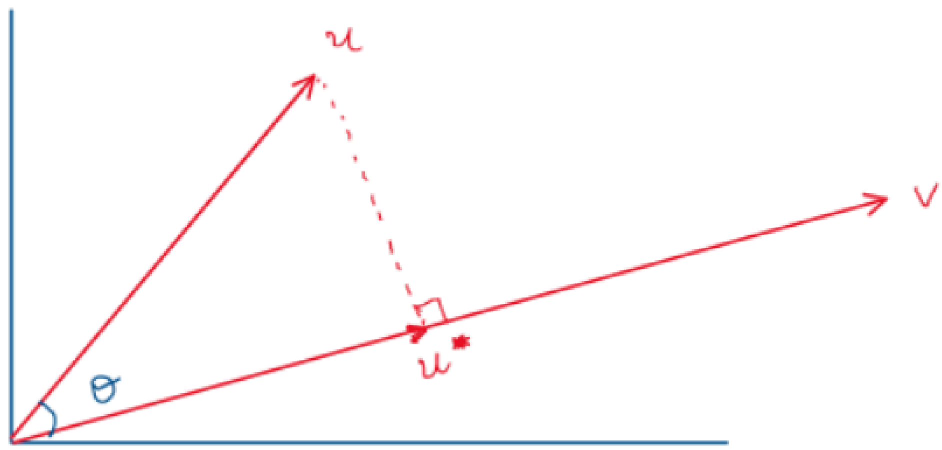
u nd v are orthogonal (perpendicular) $\implies \quad u\cdot v = 0$

Normalizing a vector: $\frac{v}{||v||}$



$u^* \equiv$ Projection of u onto v

$u^* \equiv Proj(u,v) = \frac{u\cdot v}{||v||^2}v$

$u^* \equiv Proj(u,v) = ||u||\frac{v}{||v||}$, where $||u||$ is the magnitude, $\frac{v}{||v||}$ is the direction

$= ||u||cos(\theta)\frac{v}{||v||}$

$= ||u||\ ||v||cos(\theta)\frac{v}{||v||^2}$

$= \frac{u\cdot v}{||v||^2}v$

Complex Vectors

$u, v \in \mathbb{C}^n$

$u\cdot v = \sum_{k=1}^{n} u_k v_k^\star$

where $v_k \in \mathbb{C}$, $v_k = a_k + jb_k$, where a_k is the real part, and b_k is the imaginary part

Matrices

$$A \equiv [a_i j] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A is $m \times n$ with m rows and n columns

$A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & -1 \\ 3 & 3 & 0 \\ 2 & 4 & 2 \end{bmatrix} \in \mathbb{R}^{4 \times 3}$$

A row vector: $v = [v_1, v_2, \dots, v_n] \in K^{1 \times n}$

A column vector: $v = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_m \end{bmatrix} \in K^{m \times 1}$

Matrix Addition

$$A, B \in K^{m \times n} A + B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Scalar Multiplication

If $k \in K, A \in K^{m \times n}$

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Null Matrix

$A \equiv [a_{ij}] = \mathbf{0}$

$\forall i, j, a_{ij} = 0$

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Linear Combination:

$a, b \in K$, $A, B \in K^{M \times N}$

$$aA + bB = \begin{bmatrix} aa_{11} + bb_{11} & aa_{12} + bb_{12} & \dots & aa_{1n} + bb_{1n} \\ \dots & \dots & \dots & \dots \\ aa_{m1} + bb_{m1} & aa_{m2} + bb_{m2} & \dots & aa_{mn} + bb_{mn} \end{bmatrix}$$

Properties:

If $k, k' \in K$ and $A, B, C \in K^{m \times n}$

1. $A + B = B + A$ Commutativity
2. $A + (B + C) = (A + B) + C$ Associativity
3. $k(A + B) = kA + kB$
4. $kk'A = k(k'A)$

5. $A + -A = 0$
6. $A + 0 = A$

Transpose:

If $A \in K^{m \times n}$ and $A = [a_{ij}]$, then

$A^T \in K^{n \times m}$ and $A^T = B = [b_{ij}]$ where $b_{ij} = a_{ji}$

$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ where dimension is $m \times n$

$A^T = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{m1} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$ where dimension is $n \times m$

Example.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

then

$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Properties:

If $A, B \in K^{m \times n}$

1. $(A + B)^T = A^T + B^T$
2. $(A^T)^T = A$

Let $u, v \in K^{m \times 1}$

then $u \cdot v = u^T v$

Square Matrix

$A = [a_{ij}]$ is a square matrix if and only if the number of rows equal the number of columns.

$m = n$

$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

Diagonal Matrix:

$A \equiv [a_{ij}]$

A square matrix such that $\forall i \neq j, a_{ij} = 0$

$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$

Triangular Matrices:

Upper triangular

$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$

$\forall i > j, a_{ij} = 0$

Lower triangular

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$\forall i < j, a_{ij} = 0$

Matrix Multiplication

A	B	C
$m \times n$	$n \times p$	$m \times p$
$[a_{ij}]$	$[b_{ij}]$	$[c_{ij}]$

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

B	A	D
$n \times m$	$m \times p$	$n \times p$
$[a_{ij}]$	$[b_{ij}]$	$[c_{ij}]$

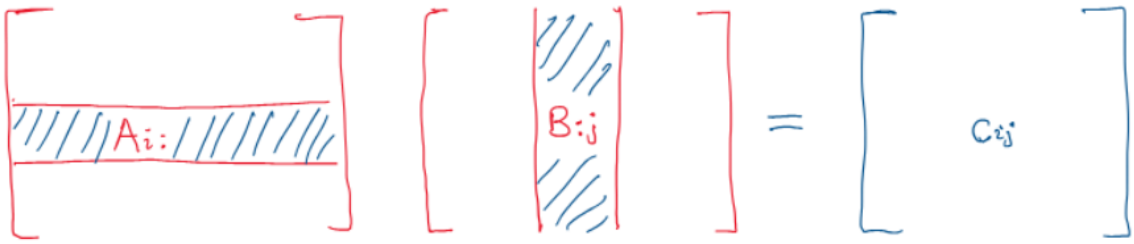
$BA = D$

$d_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj}$

$c_{ij} = i^{th}\ row\ of\ A\ \cdot j^{th}\ column\ of\ B$

$i^{th}\ row\ of\ A$: $A_{i:}$

$j^{th}\ column\ of\ B$: $B_{:j}$



Properties: If A, B, C are conformable for multiplication

1. $(AB)C = A(BC)$ Associativity
2. $A(B + C) = AB + AC$ Left distribution
3. $(A + B)C = AC + BC$ Right distribution
4. $(AB)^T = B^T A^T$
5. $c(AB) = (cA)B = A(cB)$ if c is a scalar
6. $AB \neq BA$

Trace

$A \in K^{n \times n}$

$Tr(A) = \sum_{k=1}^{n} a_{kk}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

$Tr(A) = a_{11} + a_{22} + a_{33} = 1 + 1 + 5 = 7$

Properties:

If A, B, C are conformable for multiplication

1. $Tr(A) = Tr(A^T)$
2. $Tr(BA) = Tr(AB)$
3. $Tr(ABC) = Tr(BCA) = Tr(CAB)$

Cyclic Property of Trace

Thm:

$Tr(A_1\ A_2 \ldots\ A_{n-1}\ A_n) = Tr(A_n\ A_1 \ldots\ A_{n-1})$

If the matrices A_k are conformable for matrix multiplication where T_r is the trace operator:

$Tr(A) = \sum_{k=1}^{p} a_{kk}$ if A is a square matrix

$A_k \in \mathbb{C}^{m_k \times n_k}$

Proof:

Lemma 1:

$Tr(AB) = Tr(BA)$

Lemma 2:

$A \times (B \times C) = (A \times B) \times C$

Lemma 1:

A dimension is $m \times n$

B dimension is $n \times m$

Then $A \times B$ is $m \times m$, $B \times A$ is $n \times n$

$Tr(AB) = \sum_{k=1}^{m} (AB)_{kk}$ def of Tr

$= \sum_{k=1}^{m} (\sum_{l=1}^{n} a_{kl} b_{lk})$ def of matrices, multiplication

$= \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} b_{lk}$ distribution

$= \sum_{l=1}^{n} \sum_{k=1}^{m} a_{kl} b_{lk}$ finite sum

$= \sum_{l=1}^{n} \sum_{k=1}^{m} b_{lk} a_{kl}$ complex number

$= \sum_{l=1}^{n} (\sum_{k=1}^{m} b_{lk} a_{kl})$ distribution

$= \sum_{l=1}^{n} (BA)_{ll}$ def of matrix multiplication

$= Tr(BA)$ def of Tr

Lemma 2:

A dimension is $u \times v$

B dimension is $v \times w$

C dimension is $w \times r$

Then $A \times (B \times C)$ is $u \times r$, $(A \times B) \times C$ is $u \times r$

say $M \equiv [m_{ij}]$, $N \equiv [n_{ij}]$

$m_{ij} = n_{ij}$

$m_{ij} = (A(BC))_{ij}$

$= \sum_{k=1}^{v} a_{ik}(BC)_{kj}$ def of matrix multiplication

$= \sum_{k=1}^{v} a_{ik}(\sum_{l=1}^{w} b_{kl}c_{lj})$ def of matrix multiplication

where $(\sum_{l=1}^{w} b_{kl}c_{lj}) = (BC)_{kj}$

$= \sum_{k=1}^{v} \sum_{l=1}^{w} a_{ik}b_{kl}c_{lj}$ distribution

$= \sum_{l=1}^{w}(\sum_{k=1}^{v} a_{ik}b_{kl})c_{lj}$ finite sum

where $(\sum_{k=1}^{v} a_{ik}b_{kl}) = (AB)_{il}$

$= \sum_{l=1}^{w}(AB)_{il}c_{lj}$ def of matrix multiplication

$= ((AB)C)_{ij}$ def of matrix multiplication

$= n_{ij}$

$Tr(A_1\, A_2 \dots\, A_{n-1}\, A_n) = Tr((A_1\, A_2 \dots\, A_{n-1})\, A_n)$

$= Tr(A_n\, (A_1\, A_2 \dots\, A_{n-1}))$

$= Tr(A_n\, A_1 \dots\, A_{n-1})$

A	B	$A + B$
$n \times n$	$n \times n$	$n \times n$
diagonal	diagonal	diagonal
triangular	triangular	triangular
upper	upper	upper
lower	lower	lower

Invertible Matrices

A is invertible if and only if $\exists B : AB = BA = I_n$

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Properties:

1. $A^{-1}A = I_n$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

$A \in \mathbb{C}^{m \times n}$

Hermitian

$A^H = (A^*)^T = (A^T)^*$

If $A \in \mathbb{R}^{m \times n}$, $A^H = (A^*)^T = (A^T)^*$

Normal Matrices

$A^T A = AA^H$

Complex:

- Hermitian matrices: $A = A^H$
- Skew Hermitian: $A = -A^H$

- Unitary: $A^{-1} = A^H$

Real:

- Symmetric: $A = A^T$
- Skew symmetric: $A = -A^T$
- Orthogonal: $A^{-1} = A^T$

Block Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Week 2 Session 1

Outlines

Linear System: Lines, Hyperplane, Normal

Equivalent Systems: Elementary row operations

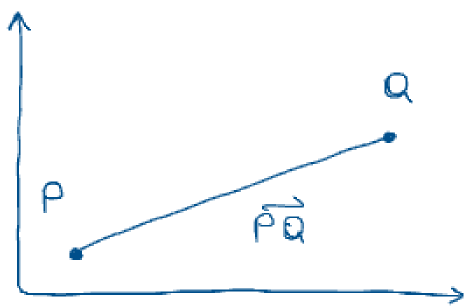
Echelon Form: Gaussian Elimination

Row Canonical Form: Gauss-Jordan

Located Vectors

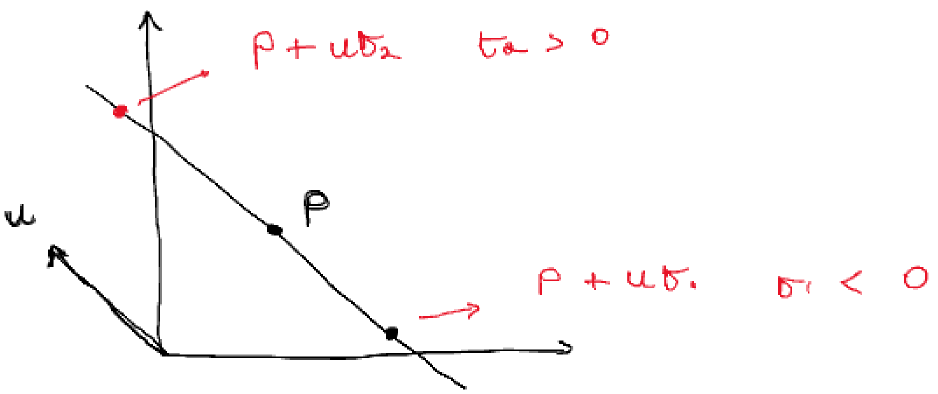
$P(u_1, \ldots, u_n)$

$Q(v_1, \ldots, v_n)$



$$\overrightarrow{\mathrm{PQ}} = \vec{\mathrm{Q}} - \vec{\mathrm{P}} = \begin{bmatrix} v_1 \\ \ldots \\ v_n \end{bmatrix} - \begin{bmatrix} u_1 \\ \ldots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 - u_1 \\ \ldots \\ v_n - u_n \end{bmatrix}$$

Lines



$L = \{x \in \mathbb{R}^n : x = p + ut, t \in \mathbb{R}^n\}$

L is a line that passes through point P with direction $u \in \mathbb{R}^n$

Linear Systems

Linear Equation

$a_1 x_1 + \ldots + a_n x_n = b$

$\sum_{j=1}^{n} a_j x_j = b$

where a_j are the coefficients, and x_j are the unknowns

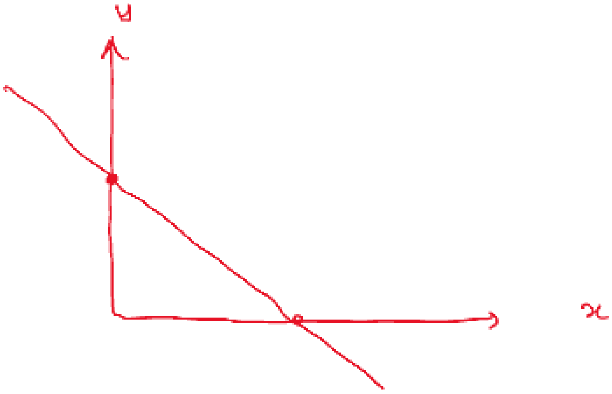
Hyperplane H:

$H = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} a_j x_j = b\}$

Example:

$6x = 6$, $H = \{1\}$

$x + y = 2$



$x + y + z = 1$

Normal to H: $\sum_{j=1}^{n} a_j x_j = b$

$w \in \mathbb{R}^n$ such that for all any located vector $\overrightarrow{\mathrm{PQ}}$ in H, w is orthogonal to $\overrightarrow{\mathrm{PQ}}$

$w = \begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix}$

Proof:

$\sum_{j=1}^{n} a_j x_j = b$

$P(u_1, \ldots, u_n) \in H \implies \sum_{j=1}^{n} a_j u_j = b$

$Q(v_1, \ldots, v_n) \in H \implies \sum_{j=1}^{n} a_j v_j = b$

$w \perp \overrightarrow{\mathrm{PQ}}$

$w = \begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix}$

$w \cdot \overrightarrow{\mathrm{PQ}} = \begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} v_1 - u_1 \\ \ldots \\ v_n - u_n \end{bmatrix}$

$= \sum_{j=1}^{n} a_j (v_j - u_j)$

$= \sum_{j=1}^{n} a_j v_j - \sum_{j=1}^{n} a_j u_j$

$= b - b$

$= 0$

Linear Systems

A list of linear equations with the same unknowns

m equations and n unkowns

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\ldots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$

- Unique solution
- Infinite solution
- No solution

A	x	b
$m \times n$	$n \times 1$	$m \times 1$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ \ldots \\ b_m \end{bmatrix}$$

Degenerate linear equation:

$0x_1 + \ldots + 0x_n = b$

1. $b = 0$, every $x \in \mathbb{R}^n$ is a solution
2. $b \neq 0$, no solution

Homogenous system: $Ax = b = \mathbf{0}$

Equivalent Systems

$Ax = b$, $A'x = b'$ where x is in dimension $n \times 1$

Theorem:

Let L be a linear combination of the equations m $Ax = b$,, then x is a solution to L
Proof:

$Ax = b$

$\sum_{j=1}^{n} a_{ij} xj = b_i$ where $1 \leq v \leq m$

Let $s = \begin{bmatrix} s_1 \\ \ldots \\ s_n \end{bmatrix}$ is a solution to $Ax = b$

Then: $\sum_j \sum_{j=1}^{n} a_{ij} xj = \sum_j b_i$ Integration

$\sum_{i=1}^m c_i(\sum_{j=1}^n a_{ij}s_j) = \sum_{i=1}^m \sum_{j=1}^n c_i a_{ij} s_j$

$= \sum_{j=1}^n (\sum_{i=1}^m c_i a_{ij}) s_j$

$= \sum_{j=1}^m c_i b_i$

x is also a solution to L

$Ax = b$ Linear combination $\rightarrow A'x = b'$

Elementary Row Operations

1. Row swap: $R_i \leftrightarrow R_j$
2. Scalar multiplication: $R_i \rightarrow kR_i$
3. Sum of a row with a scalar multiple of another row: $R_i \rightarrow R_i + kR_j$

Thm:

$Ax = b$ and $A'x = b'$ where A' (b') is obtained form the elementary row operations on $Ax = b$ then they have same solutions.

Geometry: Linear System Solutions

$Ax = b$

Row:

$\sum_{j=1}^n a_{ij} x_j = b_i$

Row 1: $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$

Row 2: $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$

$\ldots$

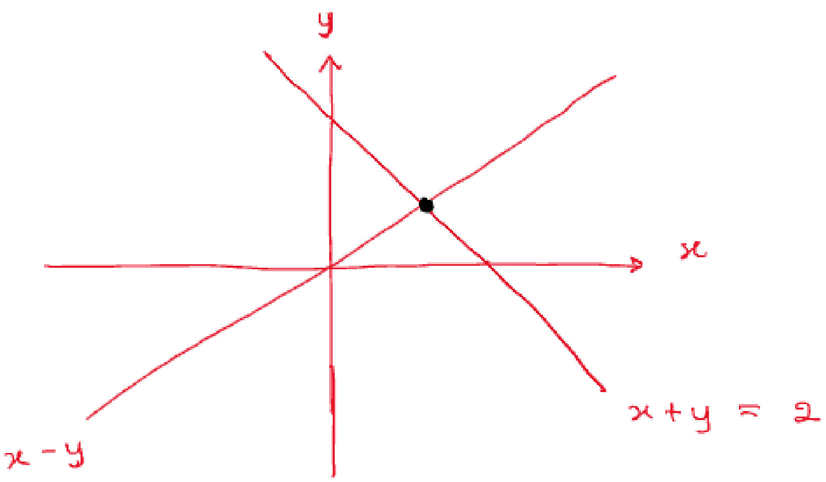
Row m: $a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$

Example 1:

$x + y = 2$

$x - y = 0$

$x = 1, y = 1$ is the unique solution

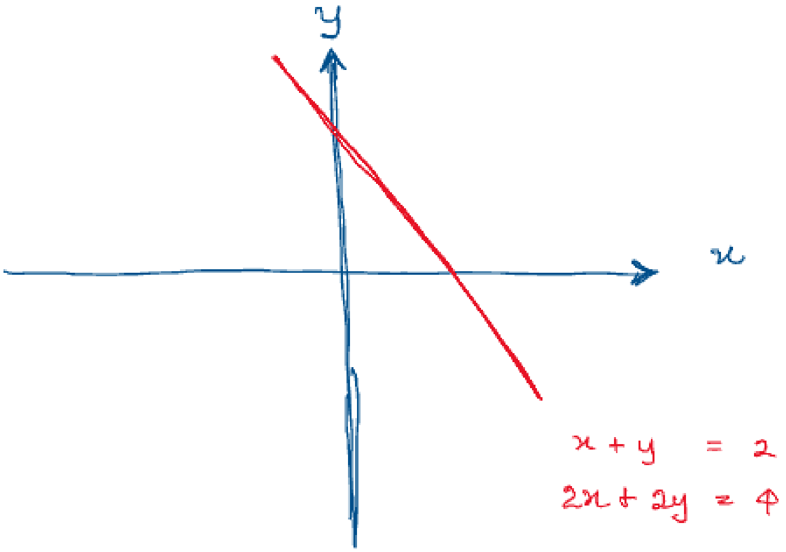


Example 2:

$x + y = 2$

$2x + 2y = 4$

Infinite solution

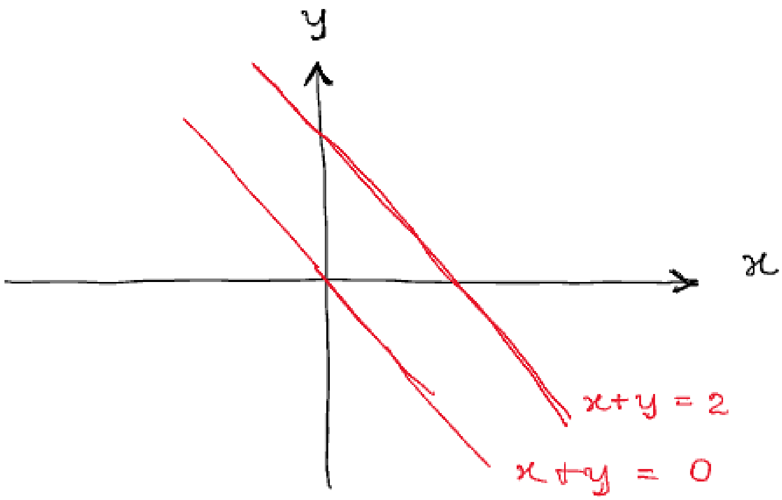


Example 3:

$x + y = 2$

$x + y = 0$

No solution



Column

$Ax = b$

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ a_{11} & a_{i2} & \dots & a_{in} \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}$$

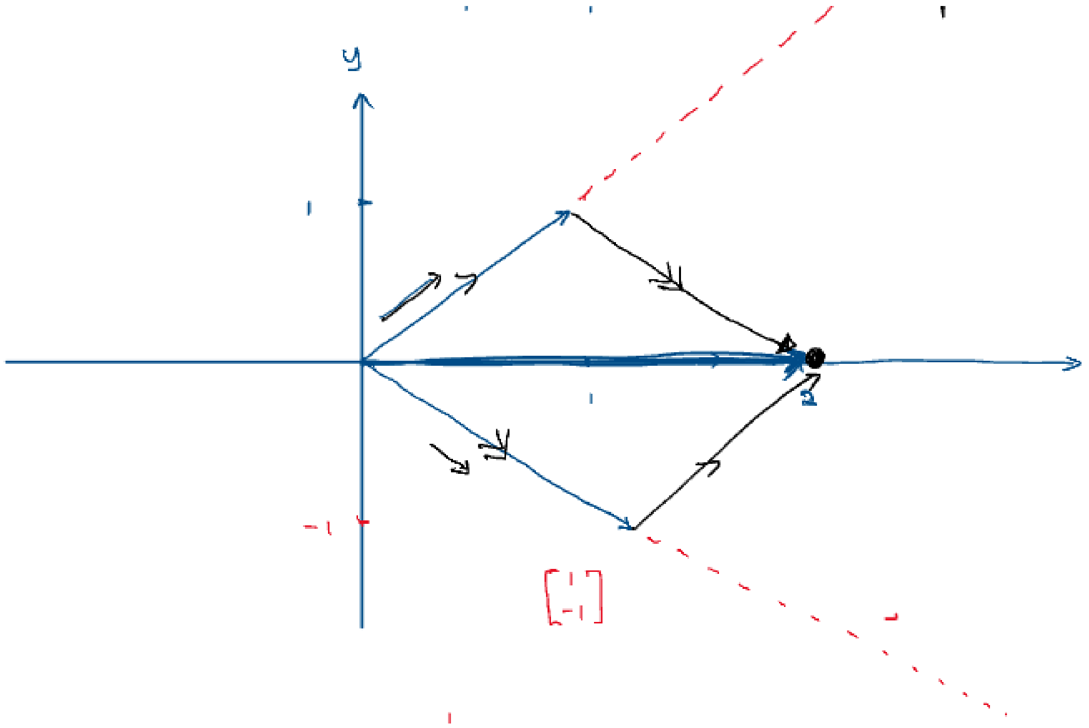
$\sum_{j=1}^{n} A_{ij}x_j = b$

Example1:

$x + y = 2$

$x - y = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

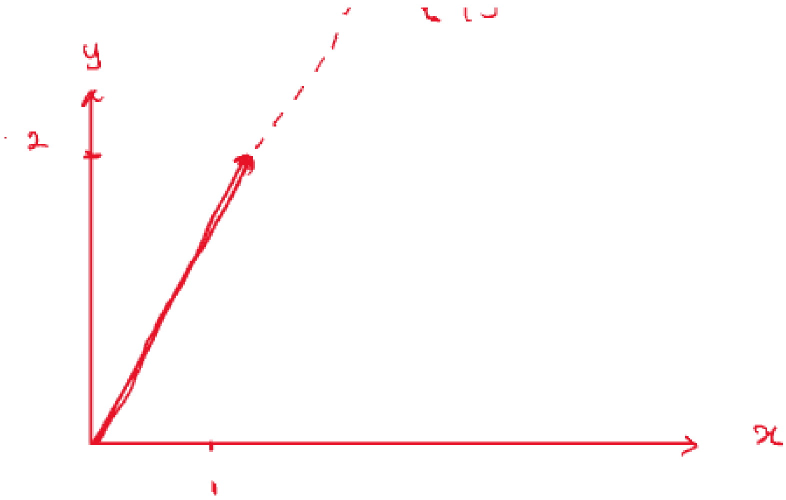


Example 2:

$x + y = 2$

$2x + 2y = 4$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



Echelon Form

$2x_1 + 3x_2 + x_3 + x_4 - x_5 = 2$

$x_2 + x_3 + 0x_4 + x_5 = 2$

$x_4 + x_5 = 1$

$m = 3, n = 5$

Pivot variables: x_1, x_2, x_4 (leading variables)

Free variables: x_3, x_5 (non-leading variables)

Special case (Triangular Form)

$2x_1 + 3x_2 + 4x_3 = 5$

$2x_2 + x_3 = 6$

$3x_3 = 1$

$m = 3, n = 3$

Gaussian Elimination

Two step process for solving linear systems of form $Ax = b$

1. Forward elimination: Reduce to Echelon Form
2. Backward substitution

Example 1:

$R1 : 2x + y + z = 5$

$R2 : 4x - 6y = -2$

$R3 : -2x + 7y + 2z = 9$

Forward Elimination:

$R1 : R1$

$R2 : R2 - 2R1$

$R3 : R3 + R1$

$2x + y + z = 5$

$0x - 8y - 2z = -12$

$0x + 8y + 3z = 14$

$R1 : R1$

$R2 : R2$

$R3 : R3 + R2$

$2x + y + z = 5$

$0x - 8y - 2z = -12$

$0x + 0y + z = 2$

Backward Substitution:

$z = 2$

$y = 1$

$x = 1$

Augmented Matrix (M)

A	x	b	M
$m \times n$	$n \times 1$	$m \times 1$	$m \times (n+1)$

$M \equiv [A \mid b]$

$$M = \begin{bmatrix} 2 & 1 & 1 \mid 5 \\ 4 & -6 & 0 \mid -2 \\ -2 & 7 & 2 \mid 9 \end{bmatrix}$$

Where $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

Echelon Matrix:

$$M = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Week 2 Session 2

Outline

Row Canonical Form: Gauss Jordan Elimination

Elementary Matrix Operations

LU Decomposition: LDU

Vector Spaces

Echelon Matrix

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 0 & 2 & 1 & 4 & -1 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Augmented Matrix

$Ax = b, M = [A|b]$

Row Canonical Form (Row-reduced Echelon Form)

1. Echelon Form
2. All non zero leading elements must be equal to 1
3. All the other values above and below a leading element must be 0

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$M = [A|b]$

Gauss-Jordan Elimination

$Ax = b$

$M = [A|b]$ - Augmented matrix

Reduce M to its row canonical form

$M' = [A'|b']$ (i.e., $A'x = b'$)

Example:

$2x_1 + x_2 + x_3 = 5$

$4x_1 - 6x_2 = -2$

$-2x_1 + 7x_2 + 2x_3 = 9$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$M \equiv [A|b] = \begin{bmatrix} 2 & 1 & 1|5 \\ 4 & -6 & 0|-2 \\ -2 & 7 & 2|9 \end{bmatrix}$$

$R1 : R1$

$R2 : R2 - 2R1$

$R3 : R3 + R1$

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & 8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 : R3 + R2$

$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & 8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ which is the Echelon Form

$R1 : R1 - R3$

$R2 : R2 + 2R3$

$R3 : R3$

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & -8 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R1 : R1$

$R2 : -1/8R2$

$R3 : R3$

$$\begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R1 : R1 - R2$

$R2 : R2$

$R3 : R3$

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R1 : 1/2R1$

$R2 : R2$

$R3 : R3$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$, which is in row canonical form

$x_1 = 1, x_2 = 2, x_3 = 2$

Linear combination of orthogonal vectors

Let $u_1, u_2, \ldots, u_n \in \mathbb{R}^n$ are mutually orthogonal

For any vector $v \in \mathbb{R}$

$v = u_1x_1 + \ldots + u_nx_n$

where $x_i = \frac{v \cdot u_1}{||u_i||^2}$ and $u_i \neq \mathbf{0}$ for $1 \leq i \leq n$

$$A = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots \\ u_1 & u_2 & \ldots & u_n \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

$Ax = v$ what is x ?

Proof:

$u_i \cdot u_j = \begin{cases} 0, & \text{if } i \neq j \\ ||u_i||^2, & \text{if } i = j \end{cases}$ Equation 1

$Ax = v$

$\sum_{j=1}^{n} x_j u_j = v$ Equation 2

$v \cdot u_i = \sum_{j=1}^{n} x_j u_j \cdot u_i$

$= \sum_{j=1}^{n} x_j (u_j \cdot u_i)$

$= (u_i \cdot u_i)x_i + \sum_{j=1, j \neq i}^{n} x_j (u_i \cdot u_j)$

$= ||u_i||^2 x_i$

Therefore, $v \cdot u_i = ||u_i||^2 x_i$ means that $x_i = \frac{v \cdot u_1}{||u_i||^2}$

$v = \sum_{j=1}^{n} x_i u_i = \sum_{j=1}^{n} \frac{v \cdot u_i}{||u_j||^2} u_i$

Inverse Matrix

Using Gauss Jordan Elimination for A^{-1}

If A ($n \times n$) is invertible, $\exists A^{-1}$ such that $AA^{-1} = I$

$AA^{-1} = I$

say $B = A^{-1}$

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \dots & \dots & \dots & \dots \\ b_1 & b_2 & \dots & b_n \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$Ab_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}$$

$M = [A|I]$ Row canonical $\rightarrow$ $[I|A^{-1}]$

Example 1:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$ Find A^{-1}

$$M \equiv \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 = R3 + R2$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$R1 : R1 - R3$

$R2 : R2$

$R3 : R3 + R2$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$R1 : R1 - R3$

$R2 : R2 + 2R3$

$R3 : R3$

$$\begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$R1 : R1$

$R2 = -1/8R2$

$R3 = R3$

$$\begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$R1 : R1 - R2$

$R2 : R2$

$R3 = R3$

$$\begin{bmatrix} 2 & 0 & 0 & 3/2 & -5/8 & -3/4 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$R1 : 1/2R1$

$R2 : R2$

$R3 : R3$

$$\begin{bmatrix} 1 & 0 & 0 & 3/4 & -5/16 & -3/8 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

where $A^{-1} = \begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix}$

Check:

$$AA^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

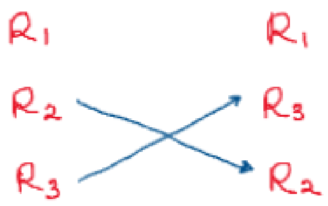
Elementary Matrix Operations

$eA \equiv EA$

where e is the elementary row operation, E is the elementary matrix operation

$e_n ... e_1 A = E_n ... E_1 A$

1. Row Swap $R_i \leftrightarrow R_j$



$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} \dots b_1 : \\ \dots b_2 : \\ \dots b_3 : \end{bmatrix}$$

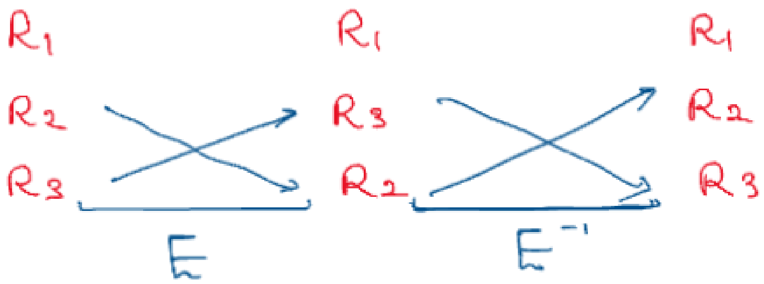
Let $E = I$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$EA \equiv B$ where $B = [b_{ij}]$

$\sum_{k=1}^{n} e_{ik} a_{kj} = b_{ij}$

where $e_{ik} = [e_{i1}, e_{i2}, \dots, e_{in}]$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Scalar Multiplication of a row

$R_i : kR_i$

$EA = B$

$R1 : R1$

$R2 : kR2$

$R3 : R3$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/k & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Row addition with a scalar multiple of another row

Operation	E	E^{-1}
$R1$	$R1$	$R1$
$R2$	$R2 + kR3$	$R2 + kR3 - kR3$
$R3$	$R3$	$R3$

This is an operation of E and E^{-1}

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix}$

LU decomposition

$A = LU \equiv LDU$

where A is in dimension $n \times n$, L is the lower triangular, U is the upper triangular, D is the diagonal matrix

A is a nonsingular matrix that can be reduced into triangular from U only row-addition operations

Example:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$e_n ... e_1 A = U = E_n ... E_1 A$

$E_n ... E_1 A = U$

$(E_n ... E_1)^{-1} = E_1^{-1} E_2^{-1} ... E_n^{-1}$

$(E_n ... E_1)^{-1}(E_n ... E_1)A = E_1^{-1} E_2^{-1} ... E_n^{-1} U$

$LHS : A = LU$

$RHS = LU$

$R1 : R1$

$R2 : R2 - 2R1$

$R3 : R3 + R1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Operations	E_1	E_1^{-1}
$R1$	$R1$	$R1$
$R2$	$R2 - 2R1 (+2R1)$	$R2$
$R3$	$R3 + R1\ (-R1)$	$R3$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 : R3 + R2$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$(E_2E1)A = U$

$A = (E_1^{-1}E_2^{-1})U$ and $E_1^{-1}E_2^{-1} = L$

Operations	E_1	E_1^{-1}
$R1$	$R1$	$R1$
$R2$	$R2$	$R2$
$R3$	$R3 + R2\,(-R2)$	$R3$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$L = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

Check:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Week 3 Session 1

Outlines

LU Decomposition: LDU

Vector Spaces: Fields, Span, Subspaces

Linear Independence: Invertibility

Uniqueness Theorem

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ 2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$A = LU$

$A = LDU$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix}$$

Vector Spaces

Field:

A field F is a collection of elements such that for binary operations: $+, \times$

We have the following: $\forall a, b, c \in F$

1. $a + b = b + a$; $a \cdot b = b \cdot a$
2. $a + (b + c) = (a + b) + c$; $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
3. $\exists 0 \in F : a + 0 = a$

   $\exists 1 \in F : a \cdot 1 = a$
4. $\exists a' \in F$: $a + a' = 0$

5. $a \times \frac{1}{a} = 1$ if $a \neq 0$
6. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Example:

$\mathbb{R}, \mathbb{Q}, \mathbb{C}$ - field

$\mathbb{Z}$ not a field ($5^{th} \frac{1}{a} \notin \mathbb{Z}$)

A vector V over field F is a collection of elements $\{\alpha, \beta, \gamma, \ldots\}$ (typically called vectors) and collection of elements $\{a, b, c, \ldots\} \in F$ ca;;ed scalars such that:

- Commutative group for $(V, +)$
    1. $\alpha + \beta \in V$
    2. $\alpha + \beta = \beta + \alpha$
    3. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
    4. $\forall \alpha, \exists \alpha' \in V : \alpha + \alpha' = \mathbf{0}$
    5. $\exists \mathbf{0} \in V : \forall \alpha \in V, 0 + \alpha = \alpha$
- Properties for combination of $+$ and $\times$
    1. $a\alpha \in V$
    2. $a(b\alpha) = (ab)\alpha$
    3. $a(\alpha + \beta) = a\alpha + a\beta$
    4. $(a + b)\alpha = a\alpha + b\alpha$
    5. $\exists 1 \in F : 1\alpha = \alpha$

1

K is field, K^n

$\alpha, \beta \in K^n$

$\alpha = \begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix}, a_1 \in K$

2

Polynomial Space: $P(t)$

$p(t) \in P(t)$

$p(t) = a_0 + a_1 t^1 + a_2 t^2 + \ldots + a_s t^s$

where $s \in \{1, 2, 3, \ldots\}$

3

Matrix over a field: $K_{m \times n}$

$A \in K_{m \times n}$
$A \equiv [a_{ij}]$ where $a_{ij} \in K$

Linear Combination:

Let $\alpha_1, \alpha_2, \ldots \alpha_n \in V$ where is a vector space over field V

w is a linear combination of the α_i's if:

$w = a_1\alpha_1 + \ldots + a_n\alpha_n$

where $a_1, a_2, \ldots, a_n \in F$

Alternatively:

$Ax = b$

$\begin{bmatrix} \ldots \\ \alpha_1 \\ \ldots \end{bmatrix} x_1 + \begin{bmatrix} \ldots \\ \alpha_2 \\ \ldots \end{bmatrix} x_2 + \ldots + \begin{bmatrix} \ldots \\ \alpha_n \\ \ldots \end{bmatrix} x_n = w$

Linear Span

Let $S = \{\alpha_1, \ldots \alpha_n\} \subset V$ for a vector space V over field F

S spans V means that $\forall w \in V, \exists a_1, \ldots, a_n \in F$ such that:

$w = a_1\alpha_1 + \ldots + a_n\alpha_n$

Subspace

u is a subspace of vector space V over field F, if

1. $u \subset V$ (u is a subset of V)
2. u is a vector space over F

Thm:

Let Vbe a vector space over field F and u is a subset of v ($u \subset V$), If:

1. $\mathbf{0} \in u$
2. $\forall \alpha, \beta \in u, \forall a, b \in F, a\alpha + b\beta \in u$

Then u is a space of V

Thm:

Let V be a vector space over field F. If u is a subspace of V, and w is a subspace of u, then w is a subspace of V

Thm:

Intersection of any number of subspaces of a vector V over field F is a subspace of V

Proof:

$u_1, u_2, \ldots$ are subspaces of V
u_1 is a subspace of V

u_2 is a subspace of $$

$\ldots$

If $\bigcap_{i=1}^{n} u_i$ a subspace of V?

Yes.

Example:

$$w = \begin{bmatrix}1\\2\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

where $\alpha_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\alpha_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\alpha_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ ✅

where $\alpha_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\alpha_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ ❎

$\mathbb{R}^2 \equiv \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

$\{\mathbf{0}\}$ subspace of $\mathbb{R}^2$ ✅

$ax + by = 1$ ❎

$ax + by = 0$ subspace of $\mathbb{R}^2$

Thm:

Let $S = \{\alpha_1, \ldots \alpha_n\} \subset V$ where V is a vector space over F and $L(s)$ be the set of all linear combinations of S with respect to F. Then $L(s)$ is a subspace of V.

1. Vector space V over field F
2. $S = \{\alpha_1, \ldots \alpha_n\} \subset V$
3. $L(s) = \{w : w = \sum_{i=1}^{n} a_i\alpha_1, a_i \in F, \alpha_i \in \S\}$

$\implies L(s)$ (span of S) is a subspace of V

Proof:

1. Show that $L(s) \subset V$

   $v \in L(s) \implies v \in V$

   Assume that $v \in L(s)$

   $v = \sum_{i=1}^{n} a_i\alpha_i$ - Def of $L(s)$

   $\alpha_1 \in S \implies \alpha_i \in V$ - because $S \subset V$

   $v = \sum_{i=1}^{n} a_i\alpha_i \in V$ - V is a vector space

   $L(s) \subset V$
2. Show that $\mathbf{0} \in L(s)$

   $\mathbf{0} = 0\alpha_1 + 0\alpha_2 + \ldots + 0\alpha_n = \sum_{i=1}^{n} 0\alpha_i \in L(s)$ - Def of S
3. Show that for $v, w \in L(s)$ and $c, d \in F$, $cv + dw \in L(s)$

   $cv + dw = c\sum_{i=1}^{n} a_i\alpha_i + d\sum_{i=1}^{n} b_i\alpha_i$ where $v = \sum_{i=1}^{n} a_i\alpha_i$ and $w = \sum_{i=1}^{n} b_i\alpha_i$

   $= \sum_{i=1}^{n} ca_i\alpha_i + \sum_{i=1}^{n} db_i\alpha_i$

   $= \sum_{i=1}^{n}(ca_i + db_i)\alpha_i$ where $ca_1 + db_i \in F$

   Therefore, $cv + dw \in L(s)$

$L(s)$ is a subspace of V

Linear Independence

Let v be a vector space over field F

$S = \{\alpha_1, \dots \alpha_n\} \subset v$

s is a linearly dependent set if there exist a_i's in F such that:

$a_1\alpha_1 + a_2\alpha_2, \dots, a_n\alpha_n = \mathbf{0}$

and at least one of the a_i's is non-zero

Linearly Independent:

s is linearly independent means that:

$a_1\alpha_1 + a_2\alpha_2, \dots, a_n\alpha_n = \mathbf{0}$ only holds when:

$a_1 = a_2 = \dots = a_n = 0$

$Ax = \mathbf{0}$ - Homogenous System

$$A = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix}, x = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}, b = \mathbf{0}$$

Note:

Let $S = \{\alpha_1, \dots \alpha_n\} \subset v$,then:

1. If $\mathbf{0} \in s$, then s is a linearly dependent set
2. If $s = \{\alpha_1\}$, then s is linearly dependent if and only if $\alpha_1 = 0$

Row Equivalence

A , B are in dimension of $m \times n$
A is row equivalent to B if fB can be obtained from a sequence of elementary row operations of A

Example

A row operations $\implies$ A' (Echelon Form) row operations $\implies$ A'' (Row Canonical Form)

Say A in dimension of $n \times n$

Echelon Form

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$, number of pivots (1, 2, 3, 1) $= n$

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$, number of pivots (1, 2, 2) $< n$ Linearly dependent, 0 row (R4)

Row Canonical Form

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$R = \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & y & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I$$

$I^{-1} = I$

$$A \in \mathbb{R}^{n \times n} = \begin{cases} A \sim (Row\ Equivalent)\ I \\ A \nsim I \end{cases}$$

$$B = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(zero row)}, B = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$BB^{-1} \neq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

There is no β_4 such that $i_{44} = 1$

So B^{-1} does not exist

Week 3 Session 2

Outlines

Uniqueness Theorem

Basis and Dimension: Dimension Theorem

Subspaces of a matrix

$A \in \mathbb{R}^{n \times n}$

1. Linearly independent rows $\leftrightarrow$ $A \sim I$
2. Linearly dependent rows $\leftrightarrow$ $A \sim B$ such that B^{-1} does not exist

Thm:

Let A be a square matrix, the following statement are equivalent:

1. A is invertible
2. A is row equivalent to I
3. A is a product of elementary matrices

Let P and Q be logical statements

If P then Q ($P \implies Q$)

1. Assume P is TRUE, and then show it logically implies that Q us TRUE
2. Proof by contradiction: $\sim Q \implies \sim P$

If and only if (Equivalence)

P if and only if ($P \leftrightarrow Q$)

- $P \implies Q$
- $Q \implies P$

Proof: $a \implies b, b \implies c, c \implies a$

$a \implies b, b \implies c, c \implies a$

Then, $a \leftrightarrow b$

- $a \implies b$

  A is invertible $\implies$ A is row equivalent to I

  $P \implies Q$

  $\sim Q$: If A if not row equivalent to I, then $A \sim B$ such that B^{-1} does not exist

  so, $B = E_n \dots E_1 A$

  $(E_n \dots E_1)^{-1} B = (E_n \dots E_1)^{-1} E_n \dots E_1 A = A$

  Due to $(A_1 A_2)^{-1} = A_2^{-1} A_1^{-1}$

  $A^{-1} = B^{-1}(E_n \dots E_1)$

  So, A is not invertible because B^{-1} does not exist

- $b \implies c$

  If A is row equivalent to I then A is a product of elementary matrices

  $P \implies Q$

  $E_n \dots E_1 A = I$

  $(E_n \dots E_1)^{-1}(E_n \dots E_1)A = (E_n \dots E_1)^{-1} I$

  so, $A = (E_n \dots E_1)^{-1} = E_1^{-1} \dots E_n^{-1}$

  For an elementary matrix E_i, E_i^{-1} is also an elementary matrix

- $c \implies a$

  If A is a product of elementary matrices then A is invertible

  $A = (E_1 \dots E_n)$

  $A^{-1} = (E_1 \dots E_n)^{-1} = E_n^{-1} \dots E_1^{-1}$ because E_1^{-1} exists

Therefore, $a \implies b \,, b \implies c \,, c \implies a$

Thm:

Let v be a vector space over F and $S = \{\alpha_1, \dots, \alpha_n\} \subset V$. Suppose S is a linearly independent set, then for every $w \in V$ there exist at most one representation as a linear combination of vectors in S.

Sketch:

If $S = \{\alpha_1, \dots, \alpha_n\} \subset V$ (linearly independent set), then $\forall w \in V$, there exist at least one representation: $w = \sum_{i=1}^n a_i \alpha_i$

$P \implies Q$

Proof:

$\sim Q$: Assume that $\exists w \in V$, we have two possible representations

$w = \sum_{i=1}^n a_i \alpha_i$ and $w = \sum_{i=1}^n b_i \alpha_i$, $\exists k : a_k \neq b_k, 1 \leq k \leq n$

So, $\mathbf{0} = w - w = \sum_{i=1}^n a_i \alpha_i - \sum_{i=1}^n b_i \alpha_i$

$= \sum_{i=1}^n (a_i - b_i)\alpha_i$

$(\mathbf{0}) = \sum_{i=1, i \neq k}^n (a_i - b_i)\alpha_i + (a_k - b_k)\alpha_k$, where $(a_k - b_k) \neq 0$

Therefore, S is linearly dependent set

$S = \{\alpha_1, \dots, \alpha_n\} \subset V$ (vector space over field F)

If S spans V then $\forall w \in v$, $\exists a_i$'s $\in F : w = \sum_{i=1}^n a_i \alpha_i$

$P \implies Q$

Thm:

Let $S = \{\alpha_1, \dots, \alpha_n\} \subset V$ where V is a vector space over field F

If:

1. S is linearly independent (number of representations ≤ 1)
2. S spans V (number of representations ≥ 1)

then every vector $w \in V$ has a unique representation as a linear combination of vectors in S

Properties

$S = \{\alpha_1, \dots, \alpha_n\} \subset V$ (vector space over field F)

1. If S is linearly dependent, then any larger set of vectors containing S is linearly dependent
2. If S is linearly independent, then any subset of S is linearly dependent

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Basis and Dimension

Vector space V over field F
Basis of V is a set of vectors $S \in V$ such that:

1. S span V
2. S is a linearly independent set

Dimension of V is the number of vectors in the basis of V

Example:

$V \equiv \mathbb{R}^3$

$V = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1, x_2, x_3 \in \mathbb{R}$

Basis: $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$

$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$ ✔️

$Dim(\mathbb{R}^3) = 3$

$P_n(t) =$ Polynomial of order $\leq n$

$p(t) \in P_n(t)$

$p(t) = c_0 + c_1 t^1 + \dots + c_n t^n$

Basis$= \{1, t, t^2, \dots t^n\}$

$p(t) = \sum_{i=0}^{n} c_0 t^n$

$Dim(P_n(t)) = n + 1$

Thm: Dimension Theorem

All basis of a vector space have the same number of vectors

Proof:

If $T = \{\alpha_1, \ldots \alpha_n\}$ (a basis) and $S = \{\beta_1, \ldots, \beta_m\}$ (a basis) then $n = m$

$P \implies Q$

Proof by contradiction:

$\sim Q : n \neq m \rightarrow (n < m)$ or $(n > m)$

Let $(n < m)$ - (Without Loss Of Generality)

$T = \{\alpha_1, \ldots \alpha_n\}$

$S = \{\beta_1, \ldots, \beta_n, \beta_{n+1}, \ldots \beta_m\}$

$A = \{\alpha_1, \ldots \alpha_n\}$, $B = \{\beta_1, \ldots, \beta_n\}$

$$B = \begin{bmatrix} \cdots & \beta_1 & \cdots \\ \cdots & \beta_2 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \beta_n & \cdots \end{bmatrix} \in \mathbb{R}^{n \times p}, C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}, A = \begin{bmatrix} \cdots & \alpha_1 & \cdots \\ \cdots & \alpha_2 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \alpha_n & \cdots \end{bmatrix} \in \mathbb{R}^{n \times p}$$

$B = CA$

$$\begin{bmatrix} \cdots & \beta_1 & \cdots \\ \cdots & \beta_2 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \beta_n & \cdots \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ c_{1n} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} \cdots & \alpha_1 & \cdots \\ \cdots & \alpha_2 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \alpha_n & \cdots \end{bmatrix}$$

$\beta_{lj} = \sum_{k=1}^{n} c_{lk} \alpha_{kj}$ - Matrix Multiplication

$\beta_l = \sum_{k=1}^{n} c_{lk} \alpha_k$

Lemma 1:

If A and B have linearly independent rows then C is invertible

$P \implies Q$ or $\sim Q \implies \sim P$

Note: C is invertible $\leftrightarrow$ C has linearly independent rows

$\sim Q$: C has linearly dependent rows

$c_l = \sum_{i=1, i \neq l} a_i c_i, c_{lk} = \sum_{i=1, i \neq l}^{n} a_i c_{lk}$

$\beta_l = \sum_{k=1}^{n} c_{lk} \alpha_k$

$= \sum_{k=1}^{n} \sum_{i=1, i \neq l}^{n} a_i c_{lk} \alpha_k$

$= \sum_{i=1, i \neq l}^{n} a_i \sum_{k=1}^{n} c_{lk} \alpha_k$

$= \sum_{i=1, i \neq l}^{n} a_i \beta_i$

So for $B = CA$ with invertible C then,

$A = C^{-1} B$

$C^{-1} = D \equiv [d_{ij}]$

$\alpha_{ij} = \sum_{k=1}^{n} d_{ik} \beta_{kj}$

$\alpha_i = \sum_{k=1}^{n} d_{ik} \beta_k$

$T = \{\alpha_1, \ldots \alpha_n\}$ is a basis of V and $\beta_{m+1} \in V$

$\beta_{n+1} = \sum_{i=1}^{n} e_i \alpha_i$ for some e_i's $\in F$

$\beta_{n+1} = \sum_{i=1}^{n} e_i(\sum_{k=1}^{n} d_{ik}\beta_k)$

$= \sum_{i=1}^{n}\sum_{k=1}^{n} e_i d_{ik}\beta_k$

$= \sum_{i=1}^{n}(\sum_{k=1}^{n} e_i d_{ik})\beta_k$

S is linearly dependent set

so S is a basis

Fundamental subspace of a matrix

$A \in \mathbb{R}^{m\times n}$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

$Ax = b$ where $x \in \mathbb{R}^{n\times 1}, b \in \mathbb{R}^{m\times 1}$

$T : \mathbb{R}^{n\times 1}$ (Domain) $\rightarrow \mathbb{R}^{m\times 1}$ (Co-domain)

$A^T y = d$ where $A^T \in \mathbb{R}^{n\times m}, y \in \mathbb{R}^{m\times 1}, d \in \mathbb{R}^{n\times 1}$

$T : \mathbb{R}^{m\times 1}$ (Domain) $\rightarrow \mathbb{R}^{n\times 1}$ (Co-domain)

1. Column Space: $C(A)$
   $C(A) = \{b \in \mathbb{R}^{m\times 1} : Ax = b, x \in \mathbb{R}^{n\times 1}\}$
2. Row Space: $C(A^T)$
   $C(A^T) = \{d \in \mathbb{R}^{n\times 1} : A^T y = d, y \in \mathbb{R}^{m\times 1}\}$
3. Null Space: $n(A)$
   $n(A) = \{x \in \mathbb{R}^{n\times 1} : Ax = \mathbf{0}\}$
4. Left Null Space: $n(A^T)$
   $n(A) = \{y \in \mathbb{R}^{m\times 1} : A^T y = \mathbf{0}\}$

Subspaces	Dimension
Domain	$n \equiv$ order
$C(A)$	$r \equiv$ rank
$n(A)$	$\zeta \equiv$ nullity

Fact: $n = r + \zeta$

$r \equiv$ rank

$r =$ number of pivots$= Dim(C(A)) = Dim(C(A^T))$

$\zeta =$ number of free variables$= Dim(n(A))$

Example:

$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}$ Find $n(A)$ and its dimension.

$Ax = \mathbf{0}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

$R1 : R1$

$R2 : R2 - 5R1$

$R3 : R3 - 2R1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 : R3 - R2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$r =$ number of pivots $= 2$$

$\zeta =$ number of free variables $= 0$

$x = 0, y = 0$

$n(A) = \{\mathbf{0}\}$

$Dim(n(A)) = \zeta = 0$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

$R1 : R1$

$R2 : R2 - 5R1$

$R3 : R3 - 2R1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

$R1 : R1$

$R2 : R2$

$R3 : R3 - R2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} r = 2, \zeta = 1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Let $x_3 = z, z \in \mathbb{R}$

$4x_3 + 4x_3 = 0 \implies x_2 = -Z$

$x_1 + x_3 = 0 \implies x_1 = -z$

$z\begin{bmatrix}-1\\-1\\1\end{bmatrix}, z \in \mathbb{R}$

$n(A) = span\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\}$

$Dim(C(A)) = 2, Dim(n(A)) = 1$

Thm:

Interchanging the rows of a matrix leaves its rank unchanged.

Thm:

If $Ax = 0$ and $Bx = 0$ have the same solution, then A and B have the same column rank

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Week 4 Session 1

Outlines

Dimension Theorem

Existence and Uniqueness

Inner Product Space

$Ax = b$ where $A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^{n\times 1} = y \in \mathbb{R}^{m\times 1}$, $m \neq n$

Principal Component Analysis (Dimension Reduction)

$X \in \mathbb{R}^{n\times p}$ where n is the number of sample, p is the number of features, $p \gg 1$

$A \in \mathbb{R}^{p\times s}$ where s is a very small dimension

$X \to \bar{X}$ mean$= 0 \implies K_{xx}$ where it is $p \times p \to$ Eigenvalues

$$E \in \mathbb{R}^{p\times p} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots. \\ e_1 & \cdots & e_s & \cdots & e_p \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\lambda_1) & \cdots & (\lambda_s) & \cdots & (\lambda_p) \end{bmatrix}$$

$\lambda_1 \gg \lambda_2 \gg \ldots \gg \lambda_p$

XE where X is $n \times p$, and E is $p \times p$

$X\bar{E} = \hat{X}$ where X is $n \times p$, $\bar{E}$ is $p \times s$, $\hat{X}$ is $n \times s$

Thm:

If $Ax = 0$ and $Bx = 0$ have the same solution, then A and B have the same column rank.

Proof:

$P \implies Q$

Let s be the column rank of A

Let t be the column rank of B

where $t \neq s$ so, $(t > s)/(s > t)$

Let $t > s$ (WLOG)

$B \in \mathbb{R}^{m \times n} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots \\ \beta_1 & \dots & \beta_s & \dots & \beta_t & \dots & \beta_n \\ \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots \end{bmatrix}$ so that $Bx = 0$

where column $1 \dots t$ are linearly independent, $t+1 \dots n$ are linearly dependent

$A \in \mathbb{R}^{m \times n} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots \\ \alpha_1 & \dots & \alpha_s & \dots & \alpha_t & \dots & \alpha_n \\ \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots \end{bmatrix}$ so that $Ax = 0$

Therefore $\exists d_i's \neq 0 : \sum_{i=1}^{\sigma} d_i \alpha_i = \mathbf{0}$ where $t > s$

$\sum_{j=1}^{\sigma} d_j \alpha_j + \sum_{j=t+1}^{n} 0 \alpha_j = \mathbf{0}$

$x_1 = d_1, x_2 = d_2, \dots x_t = d_t$, this is the solution to $Ax = 0$

$x_{t+1} = \dots = x_n = 0$

$\exists d_i's \neq 0 : \sum_{j=1}^{t} d_i \beta_j + \sum_{j=t+1}^{n} 0 \beta_j = \mathbf{0}$

$\exists d_i's \neq 0 : \sum_{j=1}^{t} d_i \beta_j = \mathbf{0}$

$\{\beta_1, \dots \beta_t\}$ is linearly dependent

Contradiction

Thm:

Elementary row operations preserve column rank

$Ax = b$ elementary operations $\implies A'x = b'$

$Ax = 0 \implies A'x = 0$

Thm:

Rank Theorem

Dimension of column space equals the dimension of row space.

$Ax = b$ where $A \in \mathbb{R}^{m \times n}$

Proof:

Let c be the column rank of A

Let r be the row rank of A

$c \leq r$ or $r \leq c$

Case 1: $c \leq r$

$A = \begin{bmatrix} \dots & \alpha_1 & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_r & \dots \\ \dots & \alpha_{r+1} & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_m & \dots \end{bmatrix}$

where $B \in \mathbb{R}^{r \times n} = \begin{bmatrix} \dots & \alpha_1 & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_r & \dots \end{bmatrix}$ is linearly independent rows where $D \in \mathbb{R}^{(m-r)\times n} = \begin{bmatrix} \dots & \alpha_{r+1} & \dots \\ \dots & \dots & \dots \\ \dots & \alpha_m & \dots \end{bmatrix}$ is linearly dependent rows

$\forall_j : r+1 \leq j \leq m, \exists t'_{ji}s :$

$\alpha_j = \sum_{i=1}^{r} t_{ji}\alpha_i \ , \ T \equiv [t_{ji}]$

$D = TB$

$(m-r) \times n = (m-r) \times r(t \times n)$

$A = \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} B \\ TB \end{bmatrix}$

If $Ax = \mathbf{0} \implies \begin{bmatrix} B \\ TB \end{bmatrix} x = \begin{bmatrix} Bx \\ TBx \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$

If $Bx = \mathbf{0} \implies Ax = \begin{bmatrix} B \\ TB \end{bmatrix} x = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$

Case 2: $r \leq c$

$A^T y = x$

$A^T \in \mathbb{R}^{n\times m}, y \in \mathbb{R}^{m\times 1}, x \in \mathbb{R}^{n\times 1}$

The column rank of A^T is r

The row rank of A^T is c

$$A^T = \begin{bmatrix} \dots & \beta_1 & \dots \\ \dots & \dots & \dots \\ \dots & \beta_c & \dots \\ \dots & \beta_{c+1} & \dots \\ \dots & \dots & \dots \\ \dots & \beta_n & \dots \end{bmatrix}$$

where $E \in \mathbb{R}^{c\times m} = \begin{bmatrix} \dots & \beta_1 & \dots \\ \dots & \dots & \dots \\ \dots & \beta_c & \dots \end{bmatrix}$ is linearly independent rows

where $F \in \mathbb{R}^{(n-c)\times m} = \begin{bmatrix} \dots & \beta_{c+1} & \dots \\ \dots & \dots & \dots \\ \dots & \beta_n & \dots \end{bmatrix}$ is linearly dependent rows

$\forall_j : c+1 \leq j \leq n, \exists r'_{ji}s :$

$\beta_j = \sum_{i=1}^{c} r_{ji}\beta_i \ , \ R \equiv [r_{ji}]$

$F = RE$

$A^T = \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} E \\ RE \end{bmatrix}$

$A^T y = \mathbf{0}$ if and only if $Ey = 0$

The column rank of $E = r$

$Ey = x$ where $E \in \mathbb{R}^{c\times m}, y \in \mathbb{R}^{m\times 1}, x \in \mathbb{R}^{c\times 1}$

$r \leq c$

Therefore $r = c$

$Bx = y$ where $B \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^{n\times 1}, y \in \mathbb{R}^{m\times 1}$

x is the domain and y is the co-domain

$Bx \equiv$ column smace or range space

column space $\subset$ co-domain

Counting Theorem

$A \in \mathbb{R}^{m \times n}$

Dimension of column space + Dimension of null sapce = n = number of columns

Proof:

$A \in \mathbb{R}^{m \times n}$

$A \implies R_r$ (row reduced Echelon)

$Ax = 0$ and $R_r x = 0$

Number of pivots in $R_r =$ column rank (A)

$R_r x = 0$ and $Ax = 0$

Dim of null sapce for $R_r = n - r$ where n is the number of columns and r is the number of pivots

Because A and R_r are row equivalent, then

$Ax = 0$ if and only if $R_r x = \mathbf{0}$

Dimension of null space of $A = n - r$

$n - r + r = n$

Dim($n(A)$)+Dim($C(A)$)=number of columns

Thm:

Fundamental Theorem: $A \in \mathbb{R}^{m \times n}$

1. The row space of A and nullsape of A are orthogonal complements in $\mathbb{R}^{n \times 1}$
2. The column space of A and left null sapce of A are orthoginal complements in $\mathbb{R}^{m \times 1}$

Let v be a vector space

u be a subspace of v

w be a subsapce of v

u and w are orthogonal complements means that $\forall \alpha \in u$ and $\forall \beta \in w$, $\alpha \perp \beta$

$\alpha \cdot \beta = 0$

Proof:

$Ax = y$ where $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n \times 1}, y \in \mathbb{R}^{m \times 1}$

1. Row Space: $C(A^T) = \{x \in \mathbb{R}^{n \times 1} : A^T y = x, y \in \mathbb{R}^{m \times 1}\}$
   Null Space: $n(A) = \{x \in \mathbb{R}^{n \times 1} : Ax = \mathbf{0}\}$
   Assume $\alpha \in C(A^T)$ and $\beta = n(A)$
   $\alpha \cdot \beta = \alpha^T \beta = (A^T y)^T x$
   $= y^T Ax$ where $Ax = 0$
   $= 0$
2. Column Space: $C(A) = \{y \in \mathbb{R}^{m \times 1} : Ax = y, x \in \mathbb{R}^{n \times 1}\}$
   Left null space $n(A^T) = \{y \in \mathbb{R}^{m \times 1} : A^T y = 0\}$
   Assume $\alpha \in C(A)$ and $\beta \in n(A^T)$

$$\begin{aligned}\alpha^T\beta &= (Ax)^Ty \\ &= x^TA^Ty \text{ where } A^Ty=0 \\ &= 0\end{aligned}$$

Summary:

$A \in \mathbb{R}^{m\times n}$

column rank =r

dimension of null space= $n-r$

row rank = r

dimension of left null space =$m-r$

$Ax=b$

$m \equiv$ number of equations

$n \equiv$ number of unknowns

$M=[Ab]$

Rank(M)   Rank(A)

Existence and Uniqueness

Thm:

Let $Ax=b$ be a system with n-unknowns m equations and augmented matrix $M=[Ab]$

1. The system has at least one solution if and only if $rank(M)=rank(A)$

$$M' = \begin{bmatrix}1&2&3&4\\0&3&1&2\\0&0&2&1\\0&0&0&0\end{bmatrix} A' = \begin{bmatrix}1&2&3\\0&3&1\\0&0&2\\0&0&0\end{bmatrix}$$

$n=3$

2. The system has a unique solution if and only if $rank(M)=n=rank(A)$

Inner Product Space

Vector space V over a field F

Real Inner Product space:

Let V be a vector space over field $\mathbb{R}$

$<\alpha,\beta>$ assign a real number for $\alpha,\beta\in V$

Then $<\alpha,\beta>$ is an inner product if:

[I_1] Linearity: $<\alpha, a\beta+b\gamma> = a<\alpha,\beta> + b<\alpha,\gamma>$, $\forall \alpha,\beta,\gamma\in V$ and $a,b\in\mathbb{R}$

[I_2] Symmetry: $<\alpha,\beta> = <\beta,\alpha>$, $\forall\alpha,\beta\in V$

[I_3] Positive Definite: $<\alpha,\alpha>\geq 0$ and $<\alpha,\alpha>=0$ if and only if $\alpha=\mathbf{0}$

Examples:

1. Euclidean $\mathbb{R}^n$

   $<u,v> = u\cdot v = \sum_{i=1}^n u_iv_i$

2. Function space $c[a,b]$ and polynomial space $P_n(t)$

$c[a, b]$ - vector space of all continuous functions on the closed interval $[a, b]$

$< f, g >= \int_a^b f(x)g(x)dx$

3. Matrix space $M = \mathbb{R}^{m\times n}$

   M - vector space of all real $m \times n$ matrices

   $< A, B >= Tr(B^T A)$

Week 4 Session 2

Outlines

Orthogonality and Inner Products

Gram-Schmidt Process

Inner Product

$< \alpha, \beta >$

Complex Inner Product Space

Vector V over field $\mathbb{C}$

$< \alpha, \beta >= \sum_{i=1}^n a_i b_i^*$ where $\alpha = \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix}$, $\beta = \begin{bmatrix} b_1 \\ \dots \\ b_n \end{bmatrix}$

$< \alpha, \beta >$ must satisfy the following properties:

$\forall \alpha, \beta, \gamma \in V; \forall a, b \in \mathbb{C}$

$[I_1]$: Linearity

$< \alpha, a\beta + b\gamma >= a^* < \alpha, \beta > +b^* < \alpha, \gamma >$

$[I_2]$: Conjugate Symmetry

$< \alpha, \beta >=< \beta, \alpha >^*$

$[I_3]$: Positive Definite:

$< \alpha, \alpha >\geq 0$ and $< \alpha, \alpha >= 0$ if and only if $\alpha = \mathbf{0}$

Normed Vector Spaces

Let $V = \{\alpha, \beta, \gamma, \dots\}$ be a vector space over a field F. A norm $|| \cdot ||$ of V is a function from the elements of v (vectors in V) into the non-negative real number such that:

$[N_1]$: $||\alpha|| \geq 0$, $\forall \alpha \in V$ and $||\alpha|| = 0$ if an only if $\alpha = \mathbf{0}$

$[N_2]$: $||k\alpha|| = |k|||\alpha||$, $\forall \alpha \in V$ and $\forall k \in F$

$[N_3]$: $||\alpha + \beta|| \leq ||\alpha|| + ||\beta||$, $\forall \alpha, \beta \in V$ (triangle inequality)

Example:

1. $v = \mathbb{R}^n, \alpha \in V, \alpha = [a_1, \dots a_n]$

   $||\alpha|| = \sqrt{(a_1)^2 + \dots (a_n)^2}$ - Euclidean Norm
2. $v = \mathbb{C}^n$ Complex field

Metric Space

Vector space V over F

$M(\alpha, \beta)$ - metric

Properties of a matric:

$[M_1] : M(\alpha, \beta) \geq 0$ and $M(\alpha, \beta) = 0$ if and only if $\alpha = \beta$

$[M_2] : M(\alpha, \beta) = M(\beta, \alpha)$

$[M_3] : M(\alpha, \gamma) \leq M(\alpha, \beta) + M(\beta, \gamma)$

Norm

l^p - norm : $\sqrt[p]{\sum_{i=1}^{n} |x_i|^p} = ||x||_p$

l^p - distance : $||x - y||_p$

Volume of an Euclidean ball of radians γ

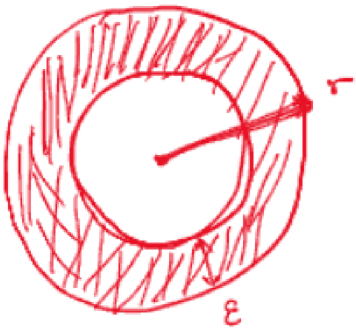
l^2 - norm $r = \sqrt[2]{\sum_{i=1}^{n} x_i^2} = n = 2$

$r = \sqrt{x^2 + y^2}$

Given Conditions:

$V_n(r) = c_n r^n \, ; \, c_n = \frac{2\pi}{n} c_{n-2}$

n	V_n
$c_1 = 2$	$2r$
$c_2 = \pi$	πr^2
$c_3 = \frac{2\pi}{3} c_1 = \frac{4\pi}{3}$	$\frac{4\pi}{3} r^3$
$c_4 = \frac{2\pi}{4} c_2 = \frac{\pi^2}{2}$	$\frac{\pi^2}{2} r^4$



$0 < \epsilon < r$

Volume shell - Entire Volume

$\frac{c_n r^n - c_n (r-\epsilon)^n}{c_n r^n}$

$= \frac{r^n - (r-\epsilon)^n}{r^n}$

$= 1 - (1 - \frac{\epsilon}{r})^n$

$0 < \epsilon < r \implies 0 < \frac{\epsilon}{r} < 1$

$1 > 1 - \frac{\epsilon}{r} > 0$

$lim_{n \to \infty} 1 - (1 - \frac{\epsilon}{r})^n = 1$

Orthogonality

Vector space V over field F

$\alpha, \beta \in V$
$\alpha \perp \beta$ if and only if $< \alpha, \beta >= 0$

Def: Let $S = \{\alpha_1, \ldots \alpha_n\} \subset V$ is mutually orthogonal if and only if

$\alpha_i \cdot \alpha_j = 0$ for $i \neq j$

Mutually Orthonormal

A vector is normal if and only if its norm $|| \cdot ||$ is equal to 1

Def: Let $S = \{\beta_1, \ldots \beta_n\} \subset V$ is mutually orthonormal if and only if

$\alpha_i \cdot \alpha_j = \begin{cases} 0, \text{if } i \neq j \\ 1, \text{if } i = j \end{cases}$

$S = \{\alpha_1, \ldots \alpha_n\}$ which is mutually orthogonal $\implies T = \{\frac{\alpha_1}{||\alpha_1||}, \ldots, \frac{\alpha_n}{||\alpha_n||}\}$ which is mutually orthonormal

---

S is linearly independent $\not\Longrightarrow$ S is mutually orthogonal

S is mutually orthogonal $\implies$ S is linearly independent

Example.

$S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$v = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$

where $\alpha_1 \cdot \alpha_2 = 0, \alpha_1 \cdot \alpha_3 = 0, \alpha_2 \cdot \alpha_3 = 0$

$c_1 = \frac{v \cdot \alpha_1}{\alpha_1 \cdot \alpha_1} = \frac{3}{1}$

$c_2 = \frac{v \cdot \alpha_2}{\alpha_2 \cdot \alpha_2} = \frac{7}{2}$

$c_3 = \frac{v \cdot \alpha_3}{\alpha_3 \cdot \alpha_3} = \frac{3}{2}$

Therefore: $v = 3\alpha_1 + \frac{7}{2}\alpha_2 + \frac{3}{2}\alpha_3$

Thm:

If $S = \{\alpha_1, \ldots \alpha_n\}$ is in a vector space V and S is mutually orthogonal (with $\alpha_i \neq 0$), then S is linearly independent

Proof:

$c_1\alpha_1 + \ldots c_n\alpha_n = \mathbf{0}$

$(c_1\alpha_1 + \ldots c_n\alpha_n) \cdot \alpha_i = \mathbf{0} \cdot \alpha_i = \mathbf{0}$

$\sum_{j=1}^{n} c_j(\alpha_j \cdot \alpha_i) = 0$

$\sum_{j=1, j \neq i}^{n} c_j(\alpha_j \cdot \alpha_i) + c_i(\alpha_i \cdot \alpha_i) = 0$

$\sum_{j=1, j \neq i}^{n} c_j(\alpha_j \cdot \alpha_i) = 0$ because S is mutually orthogonal

$c_i(\alpha_i \cdot \alpha_i) = 0$

$c_i = \frac{0}{\alpha_i \cdot \alpha_i}$ where $\alpha_i \neq 0$

Therefore, $c_1 = c_2 = \ldots = c_n$ is the only solution

Therefore, S is linearly independent

$S = \{\alpha_1, \alpha_2, \alpha_3\}$

where $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

It is linearly independent but not mutually orthogonal

where $\alpha_1 \cdot \alpha_2 = 1, \alpha_1 \cdot \alpha_3 = 2, \alpha_2 \cdot \alpha_3 = 1$

Thm:

If $S = \{\alpha_1, \ldots \alpha_n\}$ is in a vector space V , S is a basis of V and S is mutually orthogonal, then $\forall \beta \in V, \exists a_i's$ such that

$a_1\alpha_1 + \ldots + a_n\alpha_n = \beta$

$a_i = \frac{\beta \cdot \alpha_i}{\alpha_i \cdot \alpha_i}$

Proof:

S is a basis for V

$\forall \beta \in V$

$a_1\alpha_1 + \ldots + a_n\alpha_n = \beta$

$(a_1\alpha_1 + \ldots + a_n\alpha_n)\alpha_i = \beta \cdot \alpha_i$

$\sum_{j=1}^{n} a_j(\alpha_j \cdot \alpha_i) = \beta\alpha_i$

$\sum_{j=1, j \neq i}^{n} a_j(\alpha_j \cdot \alpha_i) + a_i(\alpha_i \cdot \alpha_i) = \beta \cdot \alpha_i$

$a_i(\alpha_i \cdot \alpha_i) = \beta \cdot \alpha_i$

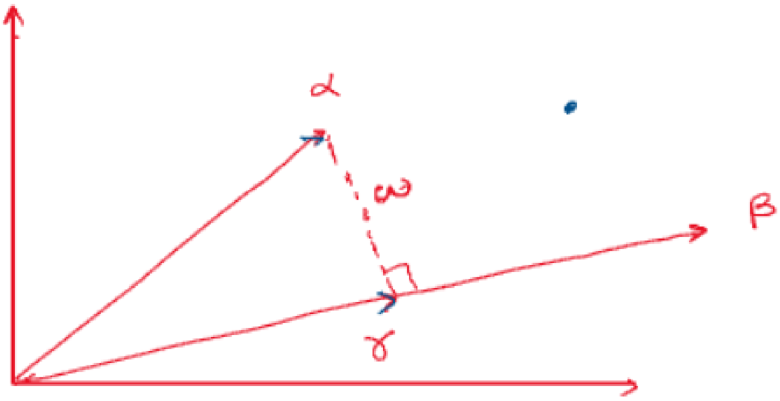
Therefore, $a_i = \frac{\beta \cdot \alpha_i}{\alpha_i \cdot \alpha_i}$

$S = \{\alpha_1, \ldots \alpha_n\}$ is a orthogonal basis of V

1. Basis
2. Mutually orthogonal

Projection

Projection of α onto β



$\gamma = Proj_\beta(\alpha) = c\beta$

$\omega = \alpha - \gamma$ and $\omega \perp \beta$

$(\alpha - \gamma) \perp \beta$

$(\alpha - c\beta) \perp \beta$

$(\alpha - c\beta) \cdot \beta = 0$

$\alpha \cdot \beta - c\beta \cdot \beta = 0$

$c = \frac{\alpha \cdot \beta}{\beta \cdot \beta}$

$Proj_\beta(\alpha) = c\beta = \frac{\alpha \cdot \beta}{\beta \cdot \beta}\beta$

$Orth_\beta(\alpha) \equiv \alpha - \gamma = \alpha - \frac{\alpha \cdot \beta}{\beta \cdot \beta}\beta$

$Proj_\beta(\alpha) + Orth_\beta(\alpha) = \alpha$

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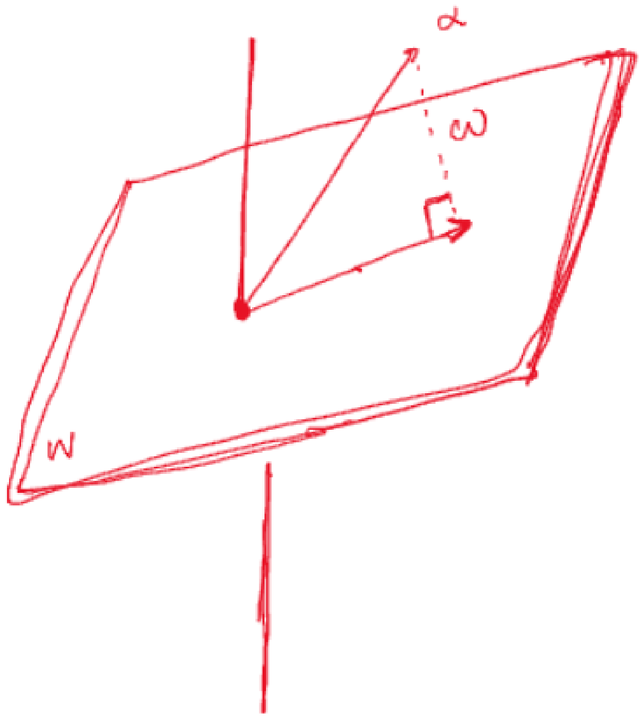
w is a subspace of V

Projection of α onto ω

$S = \{\beta_1, \ldots \beta_m\}$ is an orthogonal basis for ω



$\gamma = Proj_w(\alpha) = c_1\beta_1 + \ldots + c_m\beta_m$

1. $\gamma \cdot \beta_i = (c_1\beta_1 + \ldots + c_m\beta_m) \cdot \beta_i$
   $= c_i(\beta_i \cdot \beta_i) + \sum_{j=1, j\neq i}^{m} c_j(\beta_j \cdot \beta_i)$
2. $\omega = \alpha - \gamma : \omega \perp \beta_i$
   $(\alpha - \gamma) \perp \beta_i$
   $(\alpha - \gamma) \cdot \beta_i = 0$
   $\alpha \cdot \beta_i = \gamma \cdot \beta_i$

Then $\gamma \cdot \beta = \alpha \cdot \beta_i = c_i(\beta_i \cdot \beta_i)$

$c_i = \frac{\alpha \cdot \beta_i}{\beta_i \cdot \beta_i}$

$\gamma = \frac{\alpha \cdot \beta_1}{\beta_1 \cdot \beta_1} + \ldots + \frac{\alpha \cdot \beta_m}{\beta_m \cdot \beta_m} = Proj_\omega(\alpha)$

$S = \{\alpha_1, \ldots \alpha_n\}$ which is linearly independent "Gram Schmidt" $\implies T = \{\beta_1, \ldots \beta_n\}$ which is mutually orthogonal

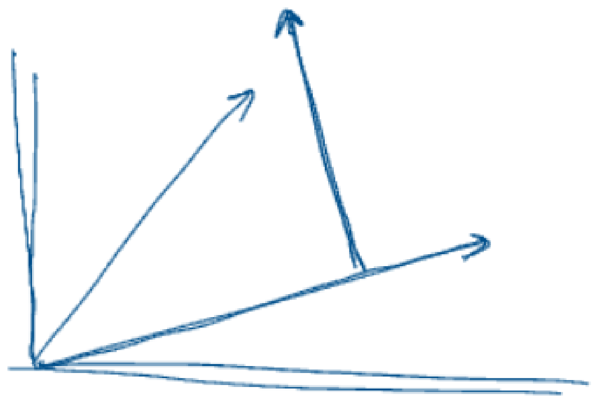
$L(S) = L(T)$

Def:

If V is a vector space and S is a subspace of V, then

$\omega = \{\alpha + \beta : \alpha \in S, \beta \in S^\perp\} = V = S \oplus S^\perp$

where $S^\perp$ is the orthogonal complement of S



Gram-Schmidt Process

Given $S = \{\alpha_1, \ldots, \alpha_n\}$ where S is linearly independent.

Find $T = \{\beta_1, \ldots, \beta_n\}$ where S is mutually orthogonal and $L(s) = L(\tau)$

$\beta_1 = \alpha_1$

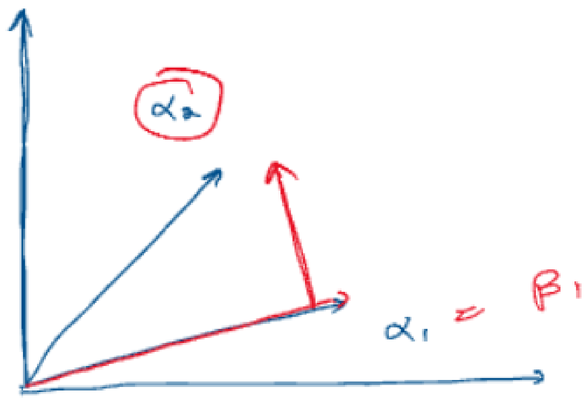
$V_1 = span\{\alpha_1\} = span\{\beta_1\}$

$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1$ where $Proj_{v_1}(\alpha_2) = \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1$

$V_2 = span(\{\alpha_1, \alpha_2\}) = span(\{\beta_1, \beta_2\})$

$\beta_3 = \alpha_3 - [\frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_2]$ where $Proj_{v_2}(\alpha_3) = \frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2}\beta_1$

$\beta_k = \alpha_k - [\frac{\alpha_k \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 + \ldots + \frac{\alpha_k \cdot \beta_{(k-1)}}{\beta_{(k-1)} \cdot \beta_{(k-1)}}\beta_{(k-1)}]$



$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1$

Ex. $\alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}$

$\beta_1 = \alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

$\beta_3 = \alpha_3 - [\frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 + \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2] = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$\beta_4 = \alpha_4 - [\frac{\alpha_4 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 + \frac{\alpha_4 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2 + \frac{\alpha_4 \cdot \beta_3}{\beta_3 \cdot \beta_3} \beta_3] = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} - \frac{4}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-3}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

Let V be a vector where

$V = \{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \text{where } a \in \mathbb{R} \}$

then $\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \}$ is a basis, and the dimension is 1

Week 4 Session 1 (Messed)

Outlines

- Review
- Dimension

  Rank Theorem

  Counting Theorem

  Fundamental Theorem

Existence and Uniqueness
- Inner Product, Orthogonality

Review:

Theorem 1:

Interchanging rows of a matrix leaves its row rank unchanged.

Theorem 2:

If $Ax = 0$ and $Bx = 0$ have the same solution then A and B have the same column rank

[Proof: Lecture 6]

Theorem 3:

Elementary row operation does not change the column rank.

Reason: Elementary row operation preserves solution, then apply Theorem 2

$Ax = b$ where $A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^{n\times 1}, b \in \mathbb{R}^{n\times 1}$

Theorem 4: Rank Theorem:

Dimension of column space = Dimension of row space

Proof:

column rank = Dimension of column space

row rank = Dimension of row space

Let $r =$ row rank of A

$c =$ column rank of A

Claim 1: $c \leq r$

Proof:

$A = \begin{bmatrix} \dots & a_1 & \dots \\ \dots & \dots & \dots \\ \dots & a_r & \dots \\ \dots & a_{r+1} & \dots \\ \dots & \dots & \dots \\ \dots & a_m & \dots \end{bmatrix}$, where $\begin{bmatrix} \dots & a_1 & \dots \\ \dots & \dots & \dots \\ \dots & a_r & \dots \end{bmatrix}$ is linearly independent rows, $\begin{bmatrix} \dots & a_{r+1} & \dots \\ \dots & \dots & \dots \\ \dots & a_m & \dots \end{bmatrix}$ is linearly dependent rows

Let $B = \begin{bmatrix} \dots & a_1 & \dots \\ \dots & \dots & \dots \\ \dots & a_r & \dots \end{bmatrix}$, where $B \in \mathbb{R}^{r\times n}$, $D = \begin{bmatrix} \dots & a_{r+1} & \dots \\ \dots & \dots & \dots \\ \dots & a_m & \dots \end{bmatrix}$, where $D \in \mathbb{R}^{(m-r)\times n}$

Note: \$$\forall j : r+1 \leq j \leq m, \exists t_{ji}$\$' s such that

$a_j = \sum_{i=1}^{r} t_{ji} a_i$ - Linearly dependent rows

Let $T = [t_{ji}]$

$D = TB$

$A = \begin{bmatrix} B \\ D \end{bmatrix} = \begin{bmatrix} B \\ TB \end{bmatrix}$

So, $Ax = \mathbf{0}$ means $\begin{bmatrix} B \\ TB \end{bmatrix} x = \begin{bmatrix} Bx \\ TBx \end{bmatrix} = \mathbf{0}$

$Ax = 0$ if and only if $Bx = 0$

The column rank of $A = c$

so the column rank of $B = c$

Remember:

$Bx = d \in \mathbb{R}^{r \times 1}$

so the column space of $B \subset R^{r \times 1}$

$Dim(C(B)) \leq D(\mathbb{R}^{r \times 1})$, where $C(B)$ is ye column space

$c \leq r$

Claim 2: $r \leq c$

Proof:

Definition of transpose

$c =$ row rank of A^T

$r =$ column rank of A^T

$A = \begin{bmatrix} \cdots & \alpha_1 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \alpha_c & \cdots \\ \cdots & \alpha_{c+1} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \alpha_n & \cdots \end{bmatrix}$, where $\begin{bmatrix} \cdots & \alpha_1 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \alpha_c & \cdots \end{bmatrix}$ is linearly independent rows, $\begin{bmatrix} \cdots & \alpha_{c+1} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & a_n & \cdots \end{bmatrix}$ is linearly dependent rows

Let $E = \begin{bmatrix} \cdots & \alpha_1 & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \alpha_c & \cdots \end{bmatrix}$, where $B \in \mathbb{R}^{c \times m}$, $F = \begin{bmatrix} \cdots & \alpha_{r+1} & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \alpha_n & \cdots \end{bmatrix}$, where $D \in \mathbb{R}^{(n-c) \times m}$

Note: \$$\forall i : c+1 \leq i \leq n, \exists r_{ij}$\$' s such that

$\alpha_i = \sum_{j=1}^{c} r_{ij} \alpha_j$

Let $R = [r_{ij}]$

then $F = RE$

$A^T = \begin{bmatrix} E \\ F \end{bmatrix} = \begin{bmatrix} E \\ RE \end{bmatrix}$

$A^T y = \mathbf{0}$ means $\begin{bmatrix} E \\ RE \end{bmatrix} y = \begin{bmatrix} Ey \\ REy \end{bmatrix} = \mathbf{0}$

$A^T y = \mathbf{0}$ if and only if $Ey = 0$

The column rank of $A^T = r$

So the column rank of $E = r$

Remember:

$Ex = f \in \mathbb{R}^{c \times 1}$

So the column space of $E \subset \mathbb{R}^{c \times 1}$

$Dim(C(E)) \leq Dim(\mathbb{R}^{c \times 1})$, where $C(E)$ is the column space of E

Then $r \leq c$

Therefore $c = r$

Theorem 5: Counting Theorem

Dimension of column space + Dimension of null space =number of columns

Reason: Let R_r be the row-reduced echelon form of A

- Row space of A=row space of R_r

  Because rows of R_r are linear combinations of rows of A and vice versa
- Column Space of $A =$ column space of R_r

  Because same solution for $Ax = 0$ and $R_r x = 0$
- Null space of $A =$ null space of R_r

  Because elementary row operation preserves solution

From R_r

n is the number of variables

r is the number of pivot variables

$n - r$ is the number of free variables

$n = r + (n - r)$

number of columns=$Dim(C(R_r)) + Dim(n(R_r))$

$= Dim(C(A)) + Dim(n(A))$

Similarly,

Dimension of row space + Dimension of left null space = Dimension of rows

---

Theorem 6: Fundamental Theorem

1. The row space and null space of A are orthogonal complements in $\mathbb{R}^n$
2. The column space and left null space of A are orthogonal complements in $\mathbb{R}^m$

Proof:

Definition: Let V be a vector space. Let U be a subspace of V and W be a subspace of V. u and w are orthogonal complements in V means that $\forall u \in U$ and $\forall w \in W$, $u \perp w$ ($u \cdot w = 0$)

1. Row space: $C(A^T) = \{A^T y : y \in \mathbb{R}^{m \times 1}\}$

   Null space: $n(A) = \{x \in \mathbb{R}^{n \times 1} : Ax = \mathbf{0}\}$

   Let $A^T y_1 \in C(A^T)$ and $x_0 \in n(A)$

   $(x_0)^T A^T y_1 = ((x_0)^T A^T y_1)^T$ where it is a representation of transpose of scalar

   $= y_1 A x_0$ where $Ax_0 = \mathbf{0}$

   $= \mathbf{0}$
2. Column space: $C(A) = \{Ax : x \in \mathbb{R}^{n \times 1}\}$

   Left null space: $n(A^T) = \{z \in \mathbb{R}^{m \times 1} : A^T z = \mathbf{0}\}$

   Let $Ax_1 \in C(A)$ and $z_0 \in n(A^T)$

   $(z_0)^T A x_1 = ((z_0)^T A x_1)^T$

$= (x_1)^T A^T z_0$ where $A^T z_0 = \mathbf{0}$
$= \mathbf{0}$

Summary

for $A \in \mathbb{R}^{m \times n}$

Column rank = Row rank = Rank = r

Dimension of the null space = $n - r$

Dimension of the left null space = $m - r$

Theorem 1: Existence and Uniqueness

Let $Ax = b$ be a system with n unknowns with augmented matrix $M = [A|b]$ then:

Existence

The system has at least one solution if and only if

rank(A)=rank(M)

Uniqueness

The system has a unique solution if and only if

rank(A)=rank(M)=n

Proof:

1. A has no solution if and only if there exist a degenerate row $[0, 0, \ldots, 0|b]$ in the echelon form of M
2. rank(A)=n if and only if no free variable

Inner product and orthogonality

Real inner product space:

Let V be a vector space over $\mathbb{R}$. Suppose that $\forall \alpha, \beta \in V < \alpha, \beta >$ assigns a real number. Then $< \alpha, \beta >$ is an inner product on V if

[I_1] Linearity: $< \alpha, a\beta + b\gamma >= a < \alpha, \beta > +b < \alpha, \gamma >$, $\forall \alpha, \beta, \gamma \in V$ and $a, b \in \mathbb{R}$

[I_2] Symmetry: $< \alpha, \beta >=< \beta, \alpha >$, $\forall \alpha, \beta \in V$

[I_3] Positive Definite: $< \alpha, \alpha >\geq 0$ and $< \alpha, \alpha >= 0$ if and only if $\alpha = \mathbf{0}$

Examples:

1. Euclidean $\mathbb{R}^n$

   $< u, v >= u \cdot v = \sum_{i=1}^{n} u_i v_i$
2. Function space $c[a, b]$ and polynomial space $P_n(t)$

   $c[a, b]$ - vector space of all continuous functions on the closed interval $[a, b]$

   $< f, g >= \int_a^b f(x)g(x)dx$
3. Matrix space $M = \mathbb{R}^{m \times n}$

   M - vector space of all real $m \times n$ matrices

   $< A, B >= Tr(B^T A)$

Complex Inner product Space

Vector space V: $\alpha, \beta, \gamma \in V$

The field is $\mathbb{C}$: $a, b \in \mathbb{C}$

$< u, v >$ must satisfy the following:

$[I_1]$: Linearity

$< \alpha, a\beta + b\gamma >= a^* < \alpha, \beta > +b^* < \alpha, \gamma >$

$[I_2]$: Conjugate Symmetry

$< \alpha, \beta >=< \beta, \alpha >^*$

$[I_3]$: Positive Definite:

$< \alpha, \alpha >\geq 0$ and $< \alpha, \alpha >= 0$ if and only if $\alpha = \mathbf{0}$

Normed Vector Spaces

Let $V = \{\alpha, \beta, \gamma, \dots\}$ be a vector space over a field F. A norm $|| \cdot ||$ of V is a function from the elements of v (vectors in V) into the non-negative real number such that:

$[N_1]$: $||\alpha|| \geq 0$, $\forall \alpha \in V$ and $||\alpha|| = 0$ if an only if $\alpha = \mathbf{0}$

$[N_2]$: $||k\alpha|| = |k|||\alpha||$, $\forall \alpha \in V$ and $\forall k \in F$

$[N_3]$: $||\alpha + \beta|| \leq ||\alpha|| + ||\beta||$, $\forall \alpha, \beta \in V$ (triangle inequality)

Example:

1. $v = \mathbb{R}^n, \alpha \in V, \alpha = [a_1, \dots a_n]$
   $||\alpha|| = \sqrt{(a_1)^2 + \dots (a_n)^2}$ - Euclidean Norm
2. $v = \mathbb{C}^n$ Complex field
   $\forall \alpha \in V$, $||\alpha|| = \sqrt{(a_1)^2 + \dots (a_n)^2}$

Definition: A metric $M(\alpha, \beta)$ on pairs of elements $\alpha, \beta \in V$ satisfies the following:

$[M_1]$: $M(\alpha, \beta) = 0$ if and only if $\alpha = \beta$

$[M_2]$: $M(\alpha, \beta) = M(\beta, \alpha)$

$[M_3]$: $M(\alpha, \beta) + M(\beta, \gamma) \geq M(\alpha, \gamma), \forall \alpha, \beta, \gamma \in V$

---

l^p - distance

$l^p(x, y) = \sqrt[p]{\sum_{i=1}^{n} |x_i - y_i|^p}, 1 \leq p \leq \infty$

$P = 1$

$l^1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ - Absolute

Let $x, y \in B^n = \{0, 1\}^n$

consider $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Mid Term 1 Review

Sample 1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Find A^{-1} if the inverse exists otherwise give sufficent reason.

Using Gaussian Jordan-Elimination

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 : R_1$

$R_2 : R_2 - 3R_1$

$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -8 & -3 & 1 & 0 \\ 0 & -3 & -3 & -2 & 0 & 1 \end{bmatrix}$$

$R_3 : R_3 - 3/4R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -8 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1/12 & -1/4 & 1/3 \end{bmatrix}$$

$R_1 : R_1 - 3R_3$

$R_2 : R_2 + 8R_3$

$$\begin{bmatrix} 1 & 2 & 0 & 3/4 & 3/4 & -1 \\ 0 & -4 & 0 & -7/3 & -1 & 8/3 \\ 0 & 0 & 1 & 1/12 & -1/4 & 1/3 \end{bmatrix}$$

$R_2 : -1/4R_2$

$$\begin{bmatrix} 1 & 2 & 0 & 3/4 & 3/4 & -1 \\ 0 & 1 & 0 & 7/12 & 1/4 & -2/3 \\ 0 & 0 & 1 & 1/12 & -1/4 & 1/3 \end{bmatrix}$$

$R_1 : R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & -5/12 & 1/4 & 1/3 \\ 0 & 1 & 0 & 7/12 & 1/4 & -2/3 \\ 0 & 0 & 1 & 1/12 & -1/4 & 1/3 \end{bmatrix}$$

Sample 2

Find the left null space of matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -2 & -1 & 1 \\ -1 & -8 & -10 & -11 \end{bmatrix}$$

Find the dimension of $C(A)$ and $C(A^T)$

$$N(A^T) = \{y \in \mathbb{R}^{3\times 1}, A^T y = 0\}$$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & -8 \\ 3 & -1 & -10 \\ 4 & 1 & -11 \end{bmatrix}$$

$$R_2 : R_2 - 2R_1$$

$$R_3 : R_3 - 3R_1$$

$$R_4 : R_4 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -6 & -6 \\ 0 & -7 & -7 \\ 0 & -7 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-6y_2 - 6y_3 = 0$$

$$y_2 = -y_3$$

$$y_1 + 2y_2 - y_3 = 0$$

$$y_1 = 3y_3$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3y_3 \\ -y_3 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Basis $(n(A^T)) = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

Null space of A $(Ax = 0, x \in \mathbb{R}^{4\times 1})$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -2 & -1 & 1 \\ -1 & -8 & -10 & -11 \end{bmatrix}$$

$$R_2 : R_2 - 2R_1$$

$$R_3 : R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -7 & -7 \\ 0 & -6 & -7 & -7 \end{bmatrix}$$

$$R_3 : R_3 - R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \text{dimension is 2}$$

According to the counting theorem

$D(C(A)) + D(N(A)) =$ number of columns =4

$D(N(A)) = 2$

$D(C(A^T)) + D(N(A^T)) =$ number of rows =3

$D(N(A^T)) = 2$

Sample 3

Use the Gram-Schmidt procedure to construct a set of orthonormal set of $\{[-1, 1, 0], [-1, 0, 1], [0, 1, 1]\}$

Express $v = [2 \quad 3 \quad 5]$ as linear combination of such orthonormal vectors.

$$\alpha_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \alpha_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\beta_1 = \alpha_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$||\beta_1||^2 = 2$

$\beta_2 = \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$||\beta_2||^2 = 3/2$

$\beta_3 = \alpha_3 - [\frac{\alpha_3 \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 + \frac{\alpha_3 \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2]$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - -\frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1/2}{3/2} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$||\beta_3||^2 = 12/9$

The orthonormal basis are:

$$\gamma_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\gamma_2 = \sqrt{2/3} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \sqrt{1/6} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\gamma_3 = \sqrt{9/12}\begin{bmatrix}2/3\\2/3\\2/3\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$v = \begin{bmatrix}2 & 3 & 5\end{bmatrix}$

$v = (v \cdot \gamma_1)\gamma_1 + (v \cdot \gamma_2)\gamma 2 + (v \cdot \gamma_3)\gamma_3$

$v \cdot \gamma_1 = \frac{1}{\sqrt{2}}$

$v \cdot \gamma_2 = \frac{5}{\sqrt{6}}$

$v \cdot \gamma_3 = \frac{10}{\sqrt{3}}$

$v = \frac{1}{\sqrt{2}}\gamma_1 + \frac{5}{\sqrt{6}}\gamma 2 + \frac{10}{\sqrt{3}}\gamma_3$

$$= 1/2\begin{bmatrix}-1\\1\\0\end{bmatrix} + 5/6\begin{bmatrix}-1\\-1\\2\end{bmatrix} + 10/3\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Week 5 Session 1 (Only 1)

Outlines

Linear Transformation (mappings)

Groups

Symmetric Group

Determinants

A is $n \times n$

Linear mapping

$Ax = b$
Non empty sets A and B

$f : A \to B$

f assigns a unique to $a \in A$ in B

$A \equiv \text{Domain of } f, B \equiv \text{Codomain}$

$A' \subset A, f(A') = \{f(a) : a \in A'\}$

$B' \subset B, f^{-1}(B') = \{a \in A : f(a) = b, b \in B'\}$

Matrix Mapping

Let $A \in K^{m \times n}$ (field K)

F_A is the transformation determined by $$A

$F_A : K^m \to K^n$

For $\alpha \in K^m$, $F_A(\alpha) = A\alpha$

Composition of Mappings:

$f : A \to B$ and $g : B \to C$

$g \cdot f : A \to C$ or $g \circ f : A \to C$

$(g \circ f)(\alpha) = g(f(\alpha)) \qquad \alpha \in A$

Let $f : A \to B$

1. f is injective (one to one) if

   $f(\alpha) = f(\alpha')$

   $\implies \alpha = \alpha'$

2. f is surjective (onto) if

   $\forall \beta \in B, \exists \alpha \in A : f(\alpha) = \beta$

3. f is bijective (one to one correspondence) means that

   f is injective and surjective

Identify mapping:
$f : A \to A$

$\mathbb{1}_A : \mathbb{1}_A(\alpha) = \alpha$

Inverse mapping:
$f : A \to B$ and $g : B \to A$, $g = f^{-1}$ if

$f \circ g = \mathbb{1}_B$ and $g \circ f = \mathbb{1}_A$

Linear Mapping

Let v and u are vector spaces over field K and

$F : v \to u$ then F is a linear mapping if :

1. For any $\alpha, \beta \in v$, $F(\alpha + \beta) = F(\alpha) + F(\beta)$
2. For any $k \in K, \alpha \in v$, $F(k\alpha) = kF(\alpha)$

Note:

$F : v \to u$ is a linear mapping if

For any $a, b \in k$ and $\alpha, \beta \in v$

$F(a\alpha + b\beta) = aF(\alpha) + bF(\beta)$

Example

Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a projection onto the xy plane, where $F(x, y, z) = (x, y)$. Is F a linear mapping?

Let $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

$F(\alpha + \beta) = F(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (x_1 + x_2, y_1 + y_2)$

$= (x_1, y_1) + (x_2, y_2)$

=$F(\alpha) + F(\beta)$

$F(k\alpha) = F(kx_1, ky_1, kz_1)$

$= (kx_1, ky_1)$

$= k(x_1, y_1)$

$= kF(\alpha)$

Example

$G : \mathbb{R}^2 \to \mathbb{R}^2$

$G(x, y) = (x + 1, y + 2)$. Is G a linear mapping?

Let $\alpha = (x_1, y_1), \beta = (x_2, y_2)$

$G(\alpha) = (x_1 + 1, y_1 + 2)$

$G(\beta) = (x_2 + 1, y_2 + 2)$

$G(\alpha + \beta) = G(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2 + 1, y_1 + y_2 + 2)$

$G(\alpha + \beta) - G(\alpha) = (x_2, y_2) \neq G(\beta)$

G is not a linear mapping

Example

$J : v \implies \mathbb{R}$

$J(f(t)) = \int_0^1 f(t)dt$

$J(af(t) + bg(t)) = \int_0^1 af(t) + bg(t)dt$

$= \int_0^1 af(t)dt + \int_0^1 bg(t)dt$

$= a\int_0^1 f(t)dt + b\int_0^1 g(t)dt$

$= aJ(f(t)) + bJ(g(t))$

Thm: Let v and u be vector space over field K and $S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ be a basis of v, then there exists a linear mapping $F : V \to U$ such that any $\beta_1, \beta_2, \dots \beta_n \in U$ is a unique representation with respect to F such that $F(\alpha_i) = \beta_i$

$F : V \to U$ where $V \to S = \{\alpha_1, \alpha_2, \dots \alpha_n\}, U \to \{\beta_1, \beta_2, \dots \beta_n\}$

Proof:

1. Define F
2. F is a linear mapping
3. F is unique

Claim 1:

$\gamma \in V$

$\gamma = a_1\alpha_1 + \dots a_n\alpha_n$ - a_1 is unique

$F(\gamma) = a_1\beta_1 + \dots + a_1\beta_n$

$F(\alpha_1) = F(1\alpha_1 + 0\alpha_2 + \dots + 0\alpha_n) = 1\beta_1$

$F(\alpha v) = 1\beta_i$

Claim 2: F is a linear mapping

Let $v, w \in V$

$v = a_1\alpha_1 + \dots a_n\alpha_n$

$w = b_1\alpha_1 + \dots b_n\alpha_n$

$F(v) = \sum_{j=1}^{n} a_j\beta_j$

$F(w) = \sum_{j=1}^{n} b_j\beta_j$

$F(v+w) = F((a_1+b_1)\alpha_1 + ... + (a_n+b_n)\alpha_n)$

$= \sum_{j=1}^{n}(a_j+b_j)\beta_j$

$= \sum_{j=1}^{n} \alpha_j\beta_j + \sum_{j=1}^{n} b_j\beta_j$

$= F(v) + F(w)$

$F(kv) = F(k(a_1\alpha_1 + ... a_n\alpha_n))$

$= F(ka_1\alpha_1 + ... ka_n\alpha_n)$

$= \sum_{j=1}^{n} ka_j\beta_j$

$= k\sum_{j=1}^{n} a_j\beta_j$

$= kF(v)$

F is a linear mapping

Claim 3:

$G : v \to u$ is a linear mapping and $G(\alpha_i) = \beta_i$

$G(a_1\alpha_1 + ... a_n\alpha_n) = \sum_{j=1}^{n} G(a_j\alpha_j)$ - G is a linear mapping

$= \sum_{j=1}^{n} G(a_j\alpha_j)$
$= \sum_{j=1}^{n} a_jG(\alpha_j)$

$= \sum_{j=1}^{n} a_j\beta_j$

$= F(v)$

=$F(a_1\alpha_1 + ... a_n\alpha_n)$

Isomorphism

Definition

Two vector space v and s over field K are isomorphic if there exist $F : v \to u$ such that

1. F is bijective
2. F is a linear mapping

Example

vector space v and $s = \{\alpha_1, ... \alpha_n\}$ is a basis of v

$Proj_s\alpha$ for all α in v is an isomorphism between v and K^n

---

Kernel and image of a linear mapping

$F : v \to u$

Kernel of F : $Ker(F)$

$Ker(F) = \{\alpha \in v : F(\alpha) = 0\}$

Image of F : $Im(F)$

$Im(F) = \{\beta \in u : \exists\alpha \in v, F(\alpha) = \beta\}$

Nullity of F : $Dim(Ker(F))$

Rank of F : $Dim(Im(F))$

$Dim(v) = Dim(Ker(F)) + Dim(Im(F))$

Note:

$Ker(F)$ is a subspace of v

$Im(F)$ is a subspace of u

Thm:
Suppose $\{\alpha_1, \ldots, \alpha_n\}$ spans V and $F : v \to u$ is a linear mapping, then $F(\alpha_1), \ldots F(\alpha_n)$ spans the image of F ($Im(F)$)

Proof:

If $\gamma \in v$ then $\gamma = a_1\alpha_1 + \ldots + a_n\alpha_n$ - $a_i's$

$F(\gamma) = F(a_1\alpha_1 + \ldots + a_n\alpha_n)$

$= \sum_{j=1}^{n} F(a_j\alpha_j)$

$= \sum_{j=1}^{n} a_j F(\alpha_j)$

$= a_1F(\alpha_1) + a_2F(\alpha_2) + \ldots + a_nF(\alpha_n)$

Singularity

$F : v \to u$

$Ker(F) = \{\mathbf{0}\}$

If $\exists \alpha \in v : \alpha \neq \mathbf{0} \text{ and } F(\alpha) = 0, \text{then } F$ is singular

---

$A \in n \times n$

$Det(A) = \sum_{\sigma \in S_n} sgn(\sigma) a_1\sigma^1 a_2\sigma^2 \ldots a_n\sigma^n$ where $sgn(\sigma)$ is the parity

Group

A collection of objects and a binary operation such that:

1. Closure: $\forall a, b \in G, a * b \in G$
2. Associativity: $\forall a, b, c \in G, a * (b * c) = (a * b) * c$
3. Identity: $\exists e \in G : \forall a \in G, a * e = e * a = a$
4. Inverse: $\forall a \in G, \exists a' \in G : a * a' = a' * a = e$

Semi-group operation $*$: A collection of objects and a binary such that it satisfies 1, 2, 3

Abelian Group (commutative group):

$(G, *)$ is an Abelian group if $(G, *)$ is a group and $\forall a, b \in G, a * b = b * a$

Ring

A collection of items and two operations (usually addition $+$ and multiplication $\times$) such that

1. $(R, +)$ is a commutative group
2. $(R, \times)$
   1. $\forall a, b \in R, a \times b \in R$
   2. $\forall a, b, c \in R, a \times (b + c) = (a \times b) + (a \times c)$
   3. $\forall a, b, c \in R, (a + b) \times c = (a \times c) + (b \times c)$
   4. $\forall a, b, c \in R, a \times (b \times c) = (a \times b) \times c$

5. $\exists \circ \in R : 0 \times a = 0 \; \forall a \in R$

Field

A field F is a ring where $F' =$ all the elements in F without the 0-element and $(F', \times)$ is commutative group

---

Symmetric Group S_x on x

Group G : $|G|$ is the order of G = number of elements in G

$S_x = \{\sigma : x \rightarrow x \text{ such that } \sigma \text{ is bijective}\}$

$\sigma - \text{shuffles } x$

σ is permutation of $x \in X$

$|S_n| = n! = n(n-1)\ldots 2 \cdot 1$

σ, τ

$\sigma \circ \tau$ or $\tau \circ \sigma$

$\sigma \circ \tau \in S_x$, $\tau \circ \sigma \in S_x$

Example

Let $X = \{1, 2, 3\}$

$|S_x| = 3! = 6$

$S_x = \{e, \sigma_1, \sigma_2, \tau_1, \tau_2, \tau_3\}$

$e = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{bmatrix}$

$\sigma_1 = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{bmatrix}$

$\sigma_2 = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{bmatrix}$

$\tau_1 = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{bmatrix}$

$\tau_2 = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{bmatrix}$

$\tau_3 = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{bmatrix}$

Is $\alpha_1 \circ \tau_1 = \tau_1 \circ \alpha_1$

$\alpha_1 \circ \tau_1 = \tau_1 = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \end{bmatrix} = \tau_2$

$$\tau_1 \circ \sigma_1 = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{bmatrix} = \tau_3$$

sign or parity $sgn(\sigma)$

$x = \{1, 2, 3\}, \sigma \in S_x$

$\sigma = \{\sigma^1, \sigma^2, \sigma^3\}$

Example: $\sigma = \sigma^1, \sigma_1^1 = 2, \sigma_1^2 = 1, \sigma_1^3 = 3$

Even and odd parity

$\sigma_1 = \{2, 1, 3\}$

number of i and $k : i < k$ but $\sigma^i > \sigma^k$

$\tau_2 = \{3, 2, 1\}$
$sgn(\alpha) = \prod_{i<k} \frac{\sigma^k - \sigma^i}{k-i}$

For S_3: $sgn(\sigma) = \frac{\sigma^3 - \sigma^2}{3-2} \frac{\sigma^2 - \sigma^1}{2-1} \frac{\sigma^3 - \sigma^1}{3-1}$

$= \frac{1-2}{3-2} \frac{2-3}{2-1} \frac{1-3}{3-1}$

$= -1$

Week 6 Session 1

Outline

Determinants

Eigenvalues and Eigenvectors

$f : \mathbb{R}^n \to \mathbb{R}^n$

$x_{t+1} = f(x_t)$

Fixed point: $f(\hat{x}) = \hat{x}$

$A\hat{x} = \hat{x}$

$A \in n \times n$

$Det(A) = \sum_{\sigma \in S_n} sgn(\sigma) a_1 \sigma^1 \ldots a_n \sigma^n$

$sgn(\sigma) = \prod_{i>k} \frac{\sigma^i - \sigma^k}{i-k}$

$$sgn(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

$sgn(\sigma) = (-1)^{N(\sigma)}$

$N(\sigma) \equiv$ number of (i, k) such that $(i > k)$ but $\sigma^i < \sigma^k$

$$\sigma = \begin{bmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{bmatrix}$$

$i = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$, $k = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$N(\sigma) = 2$

$sgn(\sigma) = (-1)^2 = 1$

Facts:

1. Let $g(x_1, \dots x_n) = \prod_{i>k}(x_i - x_k)$
2. Let $\sigma(g) = \prod_{i>k}(x_{\sigma^i} - x_{\sigma^k})$

   $\sigma(g) = \begin{cases} +g & \text{if } \sigma \text{ is even} \\ -g & \text{if } \sigma \text{ is odd} \end{cases}$

   $\sigma(g) = sgn(\sigma)g$
3. Let $\sigma, \tau \in S_n$, $sgn(\sigma \circ \tau) = sgn(\sigma)sgn(\tau)$

   $sgn(\sigma \circ \tau)g = (\sigma \circ \tau)g$

   $= \sigma(\tau(g))$

   $= sgn(\sigma)\tau(g)$

   $= sgn(\sigma)sgn(\tau)g$

   $sgn(\sigma \circ \tau) = sgn(\sigma)sgn(\tau)$
4. $sgn(\sigma) = sgn(\sigma^{-1})$

   $\sigma \circ \sigma^{-1} = \epsilon$

   $sgn(\sigma \circ \sigma^{-1}) = sgn(\sigma)sgn(\sigma^{-1}) = sgn(\epsilon) = 1$

   check the two cases of $sgn(\sigma) \in \{+1, -1\}$
5. Let $\sigma = j_1 j_2 \dots j_n$ for scalar a_{ij} and $a_{j1^1} a_{j2^2} \dots a_{jn^n} = a_{1k_1} a_{2k_2} \dots a_{nk_n}$

   $\sigma(k_i) = i$

   Let us assume that $\tau = k_1 k_2 \dots k_n$

   $\tau(j_i) = i$

   $\tau(j_i) = \tau(\sigma(i)) = i$

   $(\tau \circ \sigma)i = i$

   $\tau = \sigma^{-1}$

Thm: If σ^i and σ^j are interchanged in $\sigma = (\sigma^1, \dots \sigma^n)$ to given $\hat{\sigma}$, then $sgn(\hat{\sigma}) = -sgn(\sigma)$

Proof: $\prod_{i>k} \frac{\sigma^i - \sigma^j}{i-j}$

Example

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$Det(A) = \sum_{\sigma \in S_2} sgn(\sigma) a_{1\sigma^1} a_{2\sigma^2}$

$S_2 = \{\epsilon, \sigma\}$

$\epsilon = \begin{bmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 1 & 2 \end{bmatrix}$

$\sigma = \begin{bmatrix} 1 & 2 \\ \downarrow & \downarrow \\ 2 & 1 \end{bmatrix}$

$sgn(\epsilon) = +1$

$sgn(\sigma) = -1$

$Det(A) = (+1)a_{11}a_{22} + (-1)a_{12}a_{21}$

$= a_{11}a_{22} - a_{12}a_{21}$

Properties of Determinants

1. $Det(A) = Det(A^T)$

   $A = [a_{ij}]$

   $A^T = B = [b_{ij}]$ where $b_{ij} = a_{ij}$

   $Det(A) = \sum_{\sigma \in S_n} sgn(\sigma) a_{a\sigma^1} \ldots a_{n\sigma^n}$

   $Det(A^T) = \sum_{\sigma \in S_n} sgn(\sigma) b_{1\sigma^1} \ldots b_{n\sigma^n}$

   $= \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma^1 1} \ldots a_{\sigma^n n}$ - $A^T = B$

   $= \sum_{\sigma \in S_n, \tau = \sigma^{-1}} sgn(\sigma) a_{1\tau^1} \ldots a_{n\tau^n}$ - Fact 5

   $= \sum_{\tau \in S_n} sgn(\tau) a_{a\tau^1} \ldots a_{n\tau^n}$ - Fact 4

   $= Det(A)$

2. If A is a square matrix and two rows (or columns) are interchanged to form B, then

   $Det(A) = -Det(B)$

3. If A is a square matrix with a zero row (or zero column), then

   $Det(A) = 0$

   $\sum_{\sigma \in S_n} sgn(\sigma) a_{1\sigma^1} \ldots a_{n\sigma^n}$

4. If A has two identical rows (or two identical columns), then the determinant

   $Det(A) = 0$

   $A = \begin{bmatrix} \ldots & R_i & \ldots \\ \ldots & R_j & \ldots \end{bmatrix} R_i = R_j$

   $A' = \begin{bmatrix} \ldots & R_i & \ldots \\ \ldots & R_j & \ldots \end{bmatrix}$

   $Det(A) = -Det(A)$

   $Det(A) = -Det(A') = -Det(A) \implies Det(A) = 0$

5. If scaling a row (or a column) by k transforms a square A to B, then

   $Det(B) = kDet(A)$

   $A = [\ldots \quad R_i \quad \ldots]$

   $\sum_{\sigma \in S_n} sgn(\sigma) a_{1\sigma^1} \ldots a_{n\sigma^n}$

   $B = [\ldots \quad kR_i \quad \ldots]$

   $\sum_{\sigma \in S_n} sgn(\sigma) b_{1\sigma^1} \ldots b_{n\sigma^n}$

   $= \sum_{\sigma \in S_n} sgn(\sigma) k(a_{1\sigma^1} \ldots a_{n\sigma^n})$

   $Det(B) = kDet(A)$

6. $R_i : R_i + kR_j$

   If adding a scalar multiple of a row (or a column) to another transforms a square matrix A to B, then

   $Det(B) = Det(A)$

   $Det(B) = \sum_{\sigma \in S_n} sgn(\sigma) a_{1\sigma^1} \ldots a_{i\sigma^i} + ka_{j\sigma^i} \ldots a_{n\sigma^n}$

   $= \sum_{\sigma \in S_n} sgn(n) a_{1\sigma^1} \ldots a_{i\sigma^i} \ldots a_{n\sigma^n} + k \sum_{\tau \in S_n} sgn(\tau)(a_{1\tau^1 \ldots a_{j\tau^i}} \ldots a_{n\tau^n})$

   where $a_{1\tau^1 \ldots a_{j\tau^i}} \ldots a_{n\tau^n} = 0$

   $= Det(A)$

7. If E is an elementary matrix and A is a square matrix, then

   $Det(EA) = Det(E)Det(A)$

8. $Det(AB) = Det(A)Det(B)$

9. If A is a diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

then $Det(A) = \prod_{i=1}^{n} a_{ii}$

10. If A is a triangular matrix, then

    $Det(A) = \prod_{i=1}^{n} a_{ii}$

11. $Det(A^{-1}) = (Det(A))^{-1}$ if $Det(A) \neq 0$

Thm:

The following statements are equivalent

1. A is invertible - $M = [A|I] \sim [I|A^{-1}]$
2. $Ax = 0$ has only the zero solution
3. $Det(A) \neq 0$

$A = E_n E_{n-1} \dots E_1 I$

$Det(A) = Det(E_n) \dots Det(E_1) Det(I) \neq 0$

$Det(E_n) \neq 0$, $Det(E_1) \neq 0$, $Det(I) = 1$

Block matrix

$$M = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

A_{11} is $r \times r$

A_{22} is $s \times s$

$M_1 = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}$, $M_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$

$M = M_1 M_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

If $B = \begin{bmatrix} B_{11} & 0 & \dots & 0 \\ B_{21} & B_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ B_{n1} & B_{n2} & \dots & B_{nn} \end{bmatrix}$

$Det(B) = \prod_{i=1}^{n} Det(B_{ii})$

$Det(M) = Det(M_1) Det(M_2)$

$= 1 \times Det(A_{11}) Det(A_{22} - A_{21}A_{11}^{-1}A_{12})$

$= Det(A_{11}) Det(A_{22} - A_{21}A_{11}^{-1}A_{12})$

Determinants and volume

Let $u_1, \dots u_n \in \mathbb{R}^n$

$$V^T = \begin{bmatrix} \dots & \dots & \dots & \dots \\ v_1 & v_2 & \dots & v_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$V = \begin{bmatrix} \dots & v_1 & \dots \\ \dots & v_2 & \dots \\ \dots & \dots & \dots \\ \dots & v_n & \dots \end{bmatrix}$$

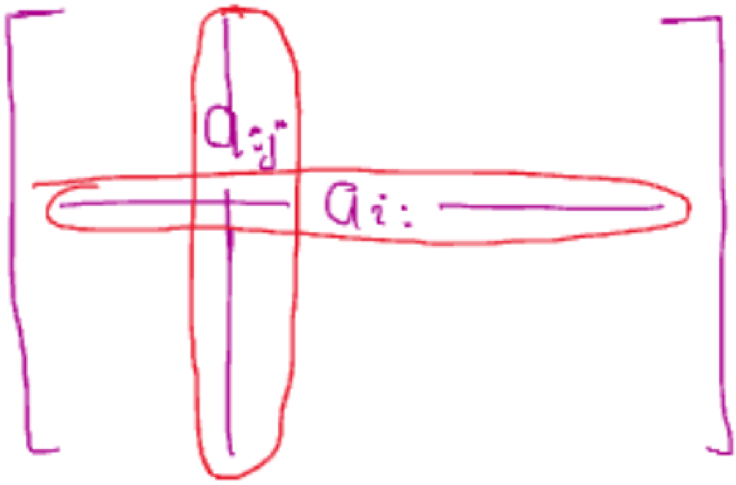
volume enclosed by $v_1, ... v_n$ is $|Det(v)|$

Cofactors and minors

$A \equiv [a_{ij}]$

$M \equiv$ Delete row i and column j from A

$M_{ij} \in (n-1) \times (n-1)$



say $A = \begin{bmatrix} 4 & 5 & 7 \\ -2 & 1 & 0 \\ 3 & 8 & 7 \end{bmatrix}$

then $M_{12} = \begin{bmatrix} -2 & 0 \\ 3 & 7 \end{bmatrix}$

Minor: $m_{ij} = Det(M_{ij})$

Cofactor: $c_{ij} = (-1)^{i+j} m_{ij}$

$Det(A) = \sum_{j=1}^{n} a_{ij} c_{ij} = \sum_{i=1}^{n} a_{ij} c_{ij}$ - Laplace Expansion

Adjoint: $A = [a_{ij}]$

$\tilde{A} = [c_{ij}]$

$Adj(A) = \widetilde{A}^T$

$A^{-1} = \frac{Adj(A)}{Det(A)}$

Eigenvalues and Eigenvectors

Fixed point:

$x_{t+1} = f(x_t)$

Fixed point $\hat{x}$ is such that $f(\hat{x}) = \hat{x}$

Example

$y = f(x) = x^2$

$f : \mathbb{R} \to \mathbb{R}$

$x = x^2$

$\hat{x} \in \{0, 1\}$

$y = f(x) = x^3$

$f : \mathbb{R} \to \mathbb{R}$

$\hat{x} \in \{-1, 0, 1\}$

$Df = \frac{df(x)}{dx} = f(x)$

$f(x) = e^x$ is the eigen function

$\frac{d}{dx} e^{cx} = ce^{cx}$ where c is the eigen value

Week 6 Session 2

Outline

Eigenvalues and Eigenvectors

Linearly Independent Eigenvectors (LIE)

Eigenvectors generalizes fixed points

$f : \mathbb{R}^n \to \mathbb{R}^n$

$\hat{x}$ is a fixed point for f is $f(\hat{x}) = \hat{x}$

Example

$y = f(x) = x^2; \hat{x} \in \{0, 1\}$

$y = f(x) = x^3; \hat{x} \in \{-1, 0, 1\}$

$D : f \to g$

$Df = \frac{df(x)}{dx}$

$\hat{f} = e^x$ because $\frac{d\hat{f}}{dx} = e^x = \hat{f}$

$\frac{d}{dx} e^{cx} = ce^{cx}$ where c is the eigen value and e^{cx} is the eigenfunction

---

Matrix

$A \in \mathbb{C}^{n \times n}$

$A : \mathbb{C}^n \to \mathbb{C}^n$

$\hat{x}$ is a fixed point for A if $A\hat{x} = \hat{x}$

Definition:

Suppose $x \neq 0$ and $\lambda \in \mathbb{C}$ if $Ax = \lambda x$ for $A \in C^{n \times n}$ then

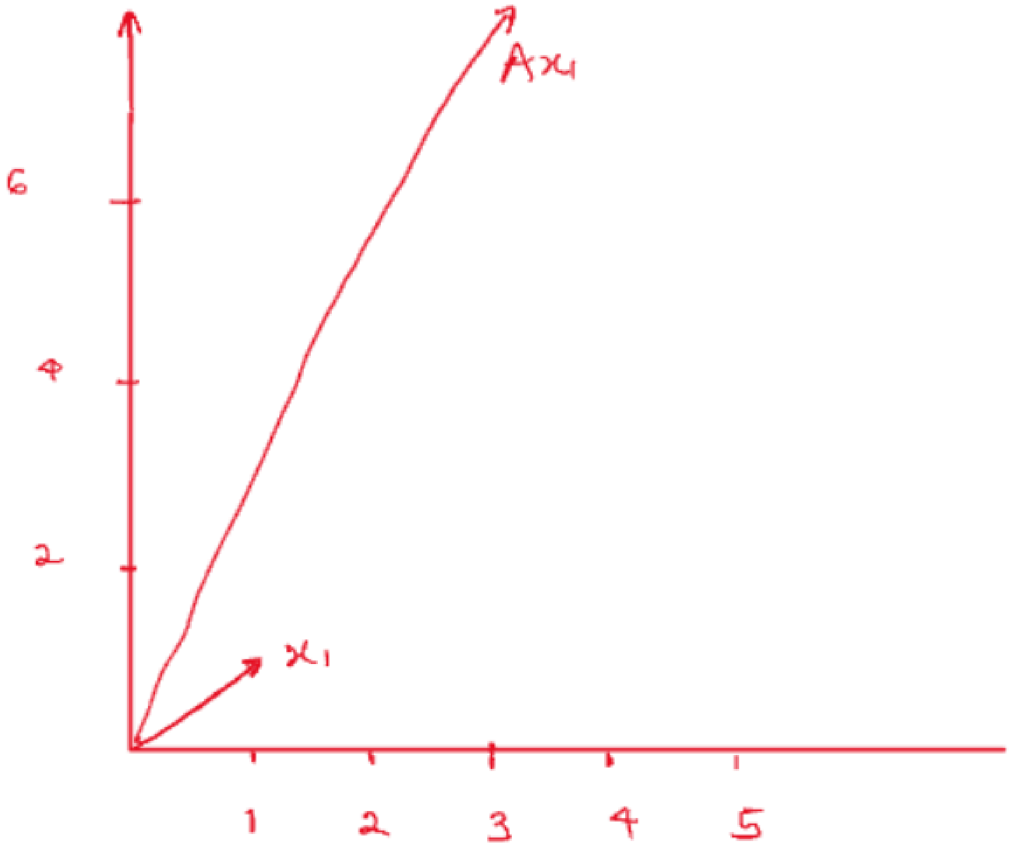
λ is the eigenvalue and x is the corresponding eigenvector

$Ax = \lambda x$

Example

$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Ax_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$



$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $Ax_2 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Eigenvectors are not unique

$Ax = \lambda x$

$A(cx) = \lambda(cx)$ - $c \in \mathbb{C}$

$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

$Ax_2 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Recall: The following statements are equivalent for a square matrix A

1. A is invertible
2. $Ax = 0$ has the zero-vector as the only solution
3. $Det(A) \neq 0$

Thm: $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $Det(A - \lambda I) = 0$

Proof: $Ax = \lambda x$ and $x \neq 0$ (Definition of eigenvector and eigenvalue)

$Ax = \lambda I x$

$Ax - \lambda I x = \mathbf{0}$

$(A - \lambda I)x = \mathbf{0}$ and $x \neq 0$

$Det(A - \lambda I) = 0$

Characteristic Polynomial

$P_A(\lambda) = Det(A - \lambda I)$

$P_A(\lambda) = 0$

Finding eigenvalues and eigenvectors

1. Define the characteristic polynomial $P_A(\lambda)$
2. Solve $P_A(\lambda) = 0$
3. For each λ, solve for x in $(A - \lambda I)x = 0$

Example

$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

Find its eigenvalues and eigenvectors

Step 1: $P_A(\lambda)$

$P_A(\lambda) = Det(A - \lambda I)$

$= \begin{bmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{bmatrix}$

$= (2-\lambda)(4-\lambda) - 3$

$= \lambda^2 - 6\lambda + 5$

Step 2: Solve $P_A(\lambda) = 0$

$\lambda_1 = 1, \lambda_2 = 5$

Step 3:

For $\lambda = 1$

$(A - \lambda I)x = \mathbf{0}$

$\begin{bmatrix} 2-1 & 1 \\ 3 & 4-1 \end{bmatrix} x = \mathbf{0}$

$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} x = \mathbf{0}$

$x = \begin{bmatrix} z \\ -z \end{bmatrix}$ for $z \in \mathbb{C}, z \neq 0$

$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the corresponding eigenvector for $\lambda = 1$

For $\lambda = 5$

$(A - \lambda I)x = \mathbf{0}$

$\begin{bmatrix} 2-5 & 1 \\ 3 & 4-5 \end{bmatrix} x = \mathbf{0}$

$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} x = \mathbf{0}$

$x = \begin{bmatrix} z \\ 3z \end{bmatrix}$ for $z \in \mathbb{C}, z \neq 0$

$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is the corresponding eigenvector for $\lambda = 5$

Eigenspace

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda_1 = 1, \lambda_2 = 1$

$Ax = \lambda x$

$Ix = 1x$

$P_A(\lambda) = Det(A - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda)$

$P_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = (1 - \lambda)(5 - \lambda) = \lambda^2 - 6\lambda + 5$

$Det(A - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda)$

$Det(A - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda)^{m_j}$ where $k \leq n$

$\sum_{j=1}^{k} m_j = n$

Let A be a diagonal matrix

$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & 0 & \dots & 0 \\ 0 & a_{22} - \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{nn} - \lambda \end{bmatrix}$

$Det(A - \lambda I) = \prod_{i=1}^{n}(a_{ii} - \lambda) = P_A(\lambda)$ where $a_{ii} = \lambda_i$

Let A be a triangular matrix

$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

$Det(A - \lambda I) = \prod_{i=1}^{n}(a_{ii} - \lambda)$ where $a_{ii} = \lambda_i$

Thm: $Det(A) = \prod_{i=1}^{n} \lambda_i$

Proof: $Det(A - I\lambda) = \prod_{i=1}^{n}(\lambda_i - \lambda)$ where $\lambda = 0$

$Det(A) = \prod_{i=1}^{n} \lambda_i$

LIE (Linearly independent eigenvectors)

Distinct Eigenvalues $\implies$ LIE (Linearly independent eigenvectors)

LIE (Linearly independent eigenvectors) $\not\Longrightarrow$ Distinct Eigenvalues

Thm: If $A \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$ Then A has n linearly independent eigenvectors $\alpha_1, \alpha_2, \dots \alpha_n$

Proof: Mathematical Induction

Basis step:

Show that A with a distinct eigenvalue has linearly independent eigenvector set

$Ax = \lambda x$ where $A, x \in 1 \times 1$

$cx = \lambda x, x \neq 0$

$\{x\}$ is linearly independent

Induction step:

Induction hypothesis: Assume that $\lambda_1, \lambda_2, \ldots \lambda_n$ are unique (distinct) and $\alpha_1, \alpha_2, \ldots \alpha_n$ are linearly independent

If $\lambda_1, \lambda_2, \ldots \lambda_{n+1}$ are distinct eigenvalues then $\alpha_1, \alpha_2, \ldots \alpha_{n+1}$ are linearly independent for $A\alpha_i = \lambda_i \alpha_i$

$P \to Q$

Proof: By contradiction ($\sim Q \to \sim P$)

$\sim Q : \exists c_j's$ such that $\alpha_{n+1} = \sum_{j=1}^{n} c_j \alpha_j$

$A\alpha_{n+1} = \lambda_{n+1}\alpha_{n+1}$

$A\alpha_{n+1} = A\sum_{j=1}^{n} c_j \alpha_j = \sum_{j=1}^{n} c_j A\alpha_j = \sum_{j=1}^{n} c_j \lambda_j \alpha_j \qquad Eq.\,3$

$A\alpha_{n+1} = \lambda_{n+1}\alpha_{n+1} = \lambda_{n+1}\sum_{j=1}^{n} c_j \alpha_j = \sum_{j=1}^{n} c_j \lambda_{n+1} \alpha_j \qquad Eq.\,4$

$Eq.\,3 - Eq.\,4$

$\sum_{j=1}^{n} c_j(\lambda_j - \lambda_{n+1})\alpha_j = 0$ where $\alpha_j \neq 0$

$\exists j : \lambda_j - \lambda_{n+1} = 0$

$\lambda_j = \lambda_{n+1}$

Contradiction $\sim P$

Thm: Projection matrix $p : p^2 = p$ and $p = p^T$, then $A_i = 0$ or $A_i = 1$

Proof: $px = \lambda x$

$ppx = p\lambda x = \lambda px = \lambda\lambda x = \lambda^2 x$

$ppx = px = \lambda x$

$\lambda^2 x = \lambda x$

$\lambda \in \{0, 1\}$

Thm: A and A^T have the same eigenvalues

Proof: $Det(A^T - \lambda I) = Det(A^T - \lambda I^T)$

$= Det((A - \lambda I)^T)$

$= Det(A - \lambda I)$ where $Det(A) = Det(A^T)$

$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2-\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$

Thm: $Tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$

Proof: $Det(A - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda) = P_A(\lambda)$

$(x + a_1)(x + a_2)\ldots(x + a_n) = x^n + x^{n-1}(a_1 + a_2 + \ldots + a_n) + x^{n-2}(a_1a_2 + a_1a_3 + \ldots + a_{n-1}a_n) + \ldots + (a_1a_2\ldots a_n)$

RHS: $\prod_{i=1}^{n}(\lambda_i - \lambda) = (-\lambda)^n + (-\lambda)^{n-1}(\lambda_1 + \lambda_2 + \ldots + \lambda_n) + (-\lambda)^{n-2}(\lambda_1\lambda_2 + \ldots \lambda_{n-1}\lambda_n)$

LHS: $Det(A - \lambda I)$

$$C = A - \lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix}$$

$Det(A - \lambda I) = \sum_{\sigma \in S_n} sgn(\sigma) c_1 \sigma^1 \dots c_n \sigma^n$

If look at the term of $(-\lambda)^{n-1}$

$(a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$

$(-\lambda)^n + (-\lambda)^{n-1}(a_{11} + \dots + a_{nn})$

$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{b} a_{ii}$

coefficient of $(-\lambda)^{n-1}$

Week 7 Session 1

Outline

Similar matrices

Diagonalizable matrices

Power and exponential of matrices

Stability of differential equation

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$T \equiv Tr(A) = a + d$

$D \equiv Det(A) = ad - bc$

$P_A(\lambda) = Det(A - \lambda I) = Det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$= (a - \lambda)(d - \lambda) - bc$

$= \lambda^2 - (a + d)\lambda + ad - bc$
$= \lambda^2 - T\lambda + D$

Let $P_A(\lambda) = 0$

$\lambda^2 - T\lambda + D = 0$

$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

Thm:

A and A^T have the same eigenvalues

Proof:

$P_{A^T}(\lambda) = Det(A^T - \lambda I) = Det(A^T - \lambda I^T)$

$= Det((A - \lambda I)^T)$

$= Det(A - \lambda I)$

$= P_A(\lambda)$

Similar matrices

A is similar to B ($A \sim B$)

Definition: A is similar to B is there exists an invertible matrix T such that

$AT = TB$

$T^{-1}AT = B$

$A = TBT^{-1}$

Thm:

If $A \sim B$, then A and B have the same eigenvalues

Proof:

Assume that $A \sim B$

$B = T^{-1}AT$

$P_B(\lambda) = Det(B - \lambda I) = Det(T^{-1}AT - \lambda I)$

$= Det(T^{-1}AT - \lambda T^{-1}T)$

$= Det(T^{-1}(AT - \lambda T))$

$= Det(T^{-1}(A - \lambda I)T)$

$= Det(T^{-1})Det(A - \lambda I)Det(T)$

$= (Det(T))^{-1}Det(A - \lambda I)Det(T)$

$= Det(A - \lambda I)$

$= P_A(\lambda)$

Example

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda_1 = \lambda_2 = 1$

$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\lambda_1 = \lambda_2 = 1$

$T^{-1}AT = T^{-1}IT$

$= T^{-1}T = I \neq B$

Let $x' = Ax$ be the initial coordinate system

$T\gamma = AT\beta$

$\gamma = T^{-1}AT\beta$ be the new coordinate system

$T = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$ where $\alpha_1, \alpha_2, \dots \alpha_n$ is basis of the new coordinate system

$x' = Ax$

$\exists c_i's : x = c_1\alpha_1 + c_2\alpha_2 + \ldots c_n\alpha_n$

$x = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_n \end{bmatrix} = T\beta$ where $T = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix}$ $\beta = \begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_n \end{bmatrix}$

$x' = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_n \end{bmatrix} = T\gamma$ where $\gamma = \begin{bmatrix} d_1 \\ d_2 \\ \ldots \\ d_n \end{bmatrix}$

Note:

T is invertible because $\alpha_1, \ldots \alpha_n$ are linearly independent

$\beta = T^{-1}x$, $\gamma = T^{-1}x'$

$\gamma = T^{-1}x'$

$\gamma = T^{-1}Ax$ - $x' = Ax$

$\gamma = T^{-1}AT\beta$ - $x = T\beta$

Recall:

A is similar to B if $\exists$ invertible T such that

$AT = TB$

$A = TBT^{-1}$

$B = T^{-1}AT$

$x' = Ax$

$\gamma = T^{-1}AT\beta$ where $B = T^{-1}AT$

T : Transforms β to the initial coordinate

A : Linear transformation in the initial coordinate

T^{-1} : Transforms back to the new coordinate

Note:

1. $A : A \sim A$
2. $A \sim B \implies B \sim A$
   $B = T^{-1}AT$
   Let $S = T^{-1}, S^{-1} = T$
   $TBT^{-1} = TT^{-1}ATT^{-1}$
   $A = TBT^{-1}$
   $A = S^{-1}BS$
3. If $A \sim B$ and $B \sim C$, then $A \sim C$

Diagonalizable matrices

Definition: A is diagonalizable if there exists a diagonal matrix Λ such that A is similar to Λ

$A \sim \Lambda$

$A = T\Lambda T^{-1}$ or $\Lambda = T^{-1}AT$

Thm:

A is diagonalizable if and only if A has linearly independent eigenvectors (LIE)

P if and only if Q must satisfies:

Claim 1: $P \to Q$

Claim 2: $Q \to P$

Claim 1: A is diagonalizable $\implies$ A has linearly independent eigenvectors

Proof:

$AT = T\Lambda$ - Assumption T is invertible

$$A \begin{bmatrix} \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} \text{ where } \Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ A\alpha_1 & A\alpha_2 & \dots & A\alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \alpha_1\lambda_1 & \alpha_2\lambda_2 & \dots & \alpha_n\lambda_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$A\alpha_i = \lambda_i \alpha_i$ for $i = \{1, 2, \dots, n\}$

α_i is an eigenvector

$\alpha_1, \dots \alpha_n$ are linear independent - T is invertible

A has linearly independent eigenvectors

Claim 2: A has linearly independent eigenvectors $\implies$ A is diagonalizable

Proof:

Let $\alpha_1, \dots \alpha_n$ be the eigenvectors of A and $T = \begin{bmatrix} \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$

$$AT = A \begin{bmatrix} \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$= \begin{bmatrix} \dots & \dots & \dots & \dots \\ A\alpha_1 & A\alpha_2 & \dots & A\alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$= \begin{bmatrix} \dots & \dots & \dots & \dots \\ \lambda_1\alpha_1 & \lambda_1\alpha_2 & \dots & \lambda_n\alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$= \begin{bmatrix} \dots & \dots & \dots & \dots \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

$= T\Lambda$

T is invertible

$\alpha_1, \dots \alpha_n$ are linear independent

A is diagonalizable if and only if A has linearly independent eigenvectors

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Example

$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

$AE = E\Lambda$

$A = E\Lambda E^{-1}$

$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$

$T = 2 + 4 = 6$

$D = 8 - 3 = 5$

$\lambda_1 = 5, \lambda_2 = 1$

For λ_1 :

$(A - \lambda I)x = \mathbf{0}$

$\begin{bmatrix} 2-5 & 1 \\ 3 & 4-5 \end{bmatrix} x = \mathbf{0}$

$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} x = \mathbf{0}$

$x = \begin{bmatrix} 1z \\ 3z \end{bmatrix}$

$\alpha_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

For λ_2 :

$(A - \lambda I)x = \mathbf{0}$

$\begin{bmatrix} 2-1 & 1 \\ 3 & 4-1 \end{bmatrix} x = \mathbf{0}$

$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} x = \mathbf{0}$

$x = \begin{bmatrix} z \\ -z \end{bmatrix}$

$\alpha_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

$E = [\alpha_1 \quad \alpha_2] = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

$A = E\Lambda E^{-1}$

$E^{-1} = \dfrac{1}{(1)(-1) - (1)(3)} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix}$

Inverse of a matrix:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ where $ad \neq bc$

$\frac{df}{dx} = f, f = e^x$

$\frac{de^{cx}}{dx} = ce^{cx}$

Power of matrices

$A^k = AAAAA\ldots A$, k in total

Assume that A is diagonalizable

$A^k = AAA\ldots A$

$= E\Lambda E^{-1}E\Lambda E^{-1}\ldots E\Lambda E^{-1}$ where $E^{-1}E = I$

$= E\Lambda^k E^{-1}$

$$= E\begin{bmatrix} \lambda_1^k & 0 & \ldots & 0 \\ 0 & \lambda_2^k & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \lambda_n^k \end{bmatrix}E^{-1}$$

Thm:

If A has eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ then A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$ for $k = 1, 2, \ldots$

Proof:

Basos step

Induction step

Thm:

If A is invertible and A has eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ then A^{-1} has eigenvalues $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$

Proof:

$A\alpha = \lambda\alpha$

$A^{-1}A\alpha = A^{-1}\lambda\alpha$

$I\alpha = \lambda A^{-1}\alpha$

$A^{-1}\alpha = \frac{1}{\lambda}\alpha = \lambda^{-1}\alpha$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$

Assume that A is disgonalizable

$A = E\Lambda E^{-1}$

$e^A = \sum_{n=0}^{\infty} \frac{(E\Lambda E^{-1})^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{E\Lambda^n E^{-1}}{n!}$

$= E(\sum_{n=0}^{\infty} \frac{\Lambda^n}{n!})E^{-1}$

$$= E\begin{bmatrix} \sum_{n=0}^{\infty} \frac{\lambda_1^n}{n!} & 0 & \ldots & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{\lambda_2^n}{n!} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \sum_{n=0}^{\infty} \frac{\lambda_1^n}{n!} \end{bmatrix}E^{-1}$$

$$= E\begin{bmatrix} e^{\lambda_1} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & e^{\lambda_n} \end{bmatrix}E^{-1}$$

Thm:

If $AB = BA$, then $e^{A+B} = e^A e^B$

Proof:

Binomial Theorem: $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$

$(x+y)^2 = (x+y)(x+y) = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$

$e^A e^B = (\sum_{i=0}^{\infty} \frac{A^i}{i!})(\sum_{j=0}^{\infty} \frac{B^j}{j!})$

$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{A^i B^j}{i!j!}$

Let $l = i + j$

$= \sum_{l=0}^{\infty} \sum_{j=0}^{l} \frac{A^{l-j} B^j}{(l-j)!j!}$

$= \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^{l} \frac{l!}{j!(l-j)!} A^{l-j} B^j$ - $\binom{n}{j} = \frac{n!}{j!(n-j)!}$

$= \sum_{l=0}^{\infty} \frac{1}{l!} \binom{l}{j} A^{l-j} B^j$

$= \sum_{l=0}^{\infty} \frac{1}{l!} (A+B)^l$ - $AB = BA$

$= e^{A+B}$

Example

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$\lambda_1 = -1, \lambda = -3$

$T = -4, D = 3$

$$\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$e^{At} = Ee^{\Lambda t}E^{-1}$

$$E = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$E^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\lim_{t \to \infty} e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lim_{t\to\infty} e^{-t} & 0 \\ 0 & \lim_{t\to\infty} e^{-3t} \end{bmatrix} \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\lim_{t\to\infty} e^{-t} = 0$

$\lim_{t\to\infty} e^{-3t} = 0$

$\frac{du}{dt} = u \implies u = e^t$

$\frac{du}{dt} = au \implies u = e^{at}$

$\frac{du}{dt} = Au \implies u = e^{At}$