EE503 Probability for Electrical and Computer Engineers

Week 1 Session 1

Outcome space / Sample space

$\Omega = \{set\ of\ all\ possible\ outcomes\ of\ a\ random\ experiment\}$

1. Flip 1 coin:

   $\Omega = \{H, T\}$

2. Flip 2 coins:

   $\Omega = \{HH, TT, HT, TH\}$

3. Number of emails in the inbox from 10:30 am to 12:30 pm:

   $\Omega = \{0, 1, 2, 3, \ldots\}$

4. Amplitude of the received signal at the radar:

   $\Omega = \{0, \infty\}$

Events:

Flip 2 coins:

$\Omega = \{HH, TT, HT, TH\}$

$A = \{HH, TT\}$

Event A: a subset of Ω

If the observed outcome belongs to event A, then event A has occured.

Radar:

$\Omega = \{0, \infty\}$

$A = \{0, 1\}$

$B = \{\pi\}$

Event Space: Collection of events.

1. Flip 1 coin:

   $\Omega = \{H, T\}$

Event Space: $\{H\}$, $\{T\}$, Ω, ϕ [All possible subsets of Ω]

Power set of Ω: 2^Ω

2. Flip 2 coins:

   $\Omega = \{HH, TT, HT, TH\}$

Event space 1: ϕ, Ω, $\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$, $\{HH, TT\}$, $\{HT, TH\}$, $\{HH, HT\}$, $\{HT, TT\}$ ...

[Power set of Ω]

For a set with n elements, number of possible subsets is 2^n.

Event Space 2: $\Omega = \{HH, TT, HT, TH\}$

$\{HH, TT\}$, $\{HT, TH\}$, Ω, $\phi \leftarrow$ Another possible event space for the experiment of flipping

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Requirement of an Event Space

1. Ω is in the event space (sure event)
2. If A is in the event space, A^c is in the event space
3. If A and B are in the event space, then $A \bigcup B$ and $A \bigcap B$ are also in the event space.

Deduction 1:

ϕ is always in event space

Deduction 2:

If $A_1, A_2, \ldots A_n$ in the event space, then:

$\bigcap_{i=1}^n A_i$ and $\bigcup_{i=1}^n A_i$ are in the event space.

Probability Law P

For each event A in the event space, $P(A)$ is a real number that describes our belief/ likelihood of event A.

Axioms of Probability

1. $P(\Omega) = 1$
2. For any event A, $0 \leq P(A) \leq 1$
3. Additivity Axiom

   (a) If A and B are 2 disjoint (i.e., mutually exclusive $\leftarrow A \bigcap B = \phi$) events, then:

   $P(A \bigcup B) = P(A) + P(B)$

   (b) If $A_1, A_2, \ldots A_n$ are pairwise disjoint events (i.e., $A_k \bigcap A_l = \phi$ for all $k \neq l$), then:

   $P(A_1 \bigcup A_2 \ldots \bigcup A_n) = \sum_{i=1}^n P(A_i)$

$Example\ 1$

$\Omega = \{H, T\}$

Event space=Power set of Ω

$P(\{H\}) = 1/2$

What is the value of $P(T)$

$P(\Omega) = 1$

$P(\{H, T\}) = 1$

$\{H, T\} = \{H\} \bigcup \{T\}$

$P(\{H\} \bigcup \{T\}) = 1 \leftarrow$ Additivity axiom

$P(\{H\}) + P(\{T\}) = 1$

$P(\{T\}) = 1 - P(\{H\})$

$P(\{T\}) = 1 - 1/2 = 1/2$

$Example\ 2$

Throw a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Event space=Power set of Ω

Probability law: For any event A, $P(A) = \frac{|A|}{6}$

$Notation : |A| = number\ of\ elements\ in\ A = cardinality\ of\ A$

$P(\{6\}) = 1/6$

$Prob\ of\ getting\ an\ even\ number :$

$P(\{2, 4, 6\}) = 3/6$

$P(\phi) = 0$

$Example\ 3$

Throw a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Event space=Power set of Ω

$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = 5/12$

$P(\{6\}) = 1/3$

$P(\{3, 4, 5\}) = P(\{3\} \bigcup \{4\} \bigcup \{5\})$

$= P(\{3\}) + P(\{4\}) + P(\{5\})$

$= 6/15$

$Example\ 4$

$\Omega = \{0, \infty\}$

Event space consist of all possible sub-interval of $\{0, \infty\}$ as well as their compliments, unions and intersections.

e.g., (a, b), $[a, b]$, $(a, b]$, $[a, \infty)$

$Borel\ event\ space\ or\ Borel\ sigma\ algebra$

Probability law: For any interval A

$P(A) = \int_A e^{-\omega} d\omega$

$P((1,2)) = \int_1^2 e^{-\omega} d\omega$

$P([2,\infty)) = \int_2^\infty e^{-\omega} d\omega$

Probability that the outcome is less than 1 or greater than 5 ?

$P([0,1]) \bigcup (5,\infty)) = P([0,1]) + P((5,\infty))$

$= \int_0^1 e^{-\omega} d\omega + \int_5^\infty e^{-\omega} d\omega$

$Example\ 5$

$\Omega = \{1, 2, 3, 4, \dots\}$

$\mathcal{F} = Power\ set\ of\ \Omega$

i.e., Event space $\mathcal{F}$ [sigma-algebra]

$P(\{k\}) = \frac{1}{2^k}$, where $k = 1, 2, 3, \dots$

Verify $P(\Omega) = 1$

Event space

If $A_1, A_2, \dots, A_n$ are in the event space, then:

$\bigcap_{i=1}^n A_i$ and $\bigcup_{i=1}^n A_i$ are in the event space.

If $A_1, A_2, A_3, \dots$ are in the event space, then:

$\bigcap_{i=1}^\infty A_i$ and $\bigcup_{i=1}^\infty A_i$ are also in the event space.

Probability Axioms

- Additivity axiom

  $A_1, A_2, \dots, A_n$ are pairwise disjoint events

  $P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$
- Countable additivity axiom

$A_1, A_2, A_3, ...$ are pairwise disjoint events

$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

$P(\Omega) = P(\{1, 2, 3, 4...\})$

$\{1, 2, 3, ...\} = \{1\} \bigcup \{2\} \bigcup ...$

$P(\{1\} \bigcup \{2\} \bigcup ...) = P(\{1\}) + P(\{2\}) + ...$

Countable additivity axiom: $P(\{k\}) = \frac{1}{2^k}$

$= \frac{1}{2} + \frac{1}{2^2} + ...$

$= \frac{1/2}{1-1/2}$

Note $\leftarrow$ Geometric series

$a + ar + ar^2 + ...$ where $r < 1$, then

$sum = \frac{a}{1-r}$

Probability that the outcome is an even number:

$P(\{2, 4, 6, 8...\}) = P(\{2\} \bigcup \{4\} \bigcup ...)$

$= P(\{2\}) + P(\{4\}) + ...$

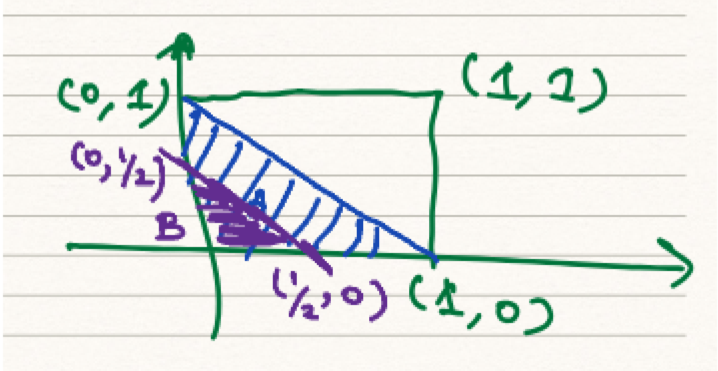
$= \frac{1}{2^2} + \frac{1}{2^4} + ...$

$= \frac{1/4}{1-1/4}$

$= 1/3$

$Example\ 6$

$\Omega = \{(x, y) : 0 \leq x, y \leq 1\}$



$P(A) = Area\ of\ A$

$P(\Omega) = Area\ of\ \Omega = 1$

$P(A) = 1/2$

B is the event that the sum of x and y coordinate is less than or equal to $1/2$

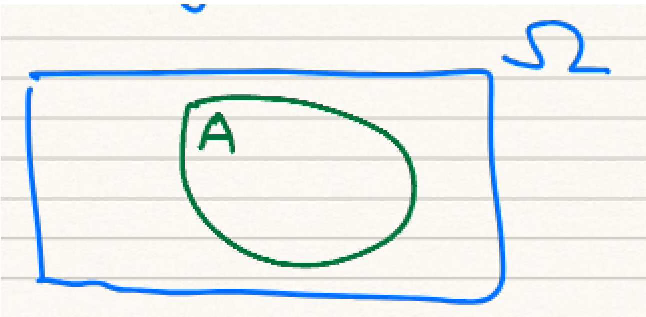
$B = (x, y) \in \Omega : x + y \leq 1/2$

$P(B) = Area\ of \in B = 1/2 \times 1/2 \times 1/2 = 1/8$

Week 1 Session 2

Random Experiment & Probability Model

- Outcome space / Sample Space Ω
- An event is a subset of Ω
- If the realized outcome of experiment lies in A, we say event A has occurred.



Event Space / Sigma algebra $\mathcal{F}$

Properties of $\mathcal{F}$:

1. Ω is in $\mathcal{F}$
2. If A is in $\mathcal{F}$, then A^c is in $\mathcal{F}$
3. (a) If $A_1, A_2, \ldots A_n$ are in $\mathcal{F}$, then:

   $\bigcup_{i=1}^{n} A_i$ is in $\mathcal{F}$ and $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{F}$

   (b) If $A_1, A_2, A_3, \ldots$ is an infinite sequence of events that are in $\mathcal{F}$, then:

   $\bigcup_{i=1}^{\infty} A_i$ is in $\mathcal{F}$ and $\bigcap_{i=1}^{\infty} A_i$ is in $\mathcal{F}$

Probability Law

For each event A in $\mathcal{F}$, $P(A)$ is a real number.

Probability Axioms

1. $P(\Omega) = 1$
2. $0 \leq P(A) \leq 1$
3. (a) If $A_1, A_2, \ldots A_n$ are pairwise disjoint events, then:

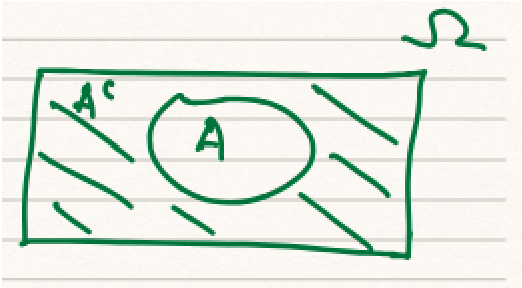
   $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

   (b) If $A_1, A_2, A_3, \ldots$ is an infinite sequence of pairwise disjoint events, then:

   $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Deduction from Axioms

1. $P(A) + P(A^c) = 1$



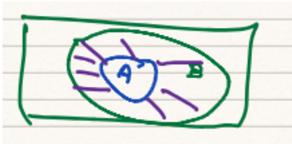
Proof:

$1 = P(\Omega)$

$= P(A \bigcup A^c)$

$= P(A) + P(A^c)$

2. If $A \subset B$, then $P(A) \leq P(B)$



Proof:

$B = A \bigcup C$

$P(B) = P(A \bigcup C)$

$= P(A) + P(C)$

$\rightarrow P(B) \geq P(A)$

$C = B \bigcap A^c$

3. Union Formula

   For any 2 events A and B

   $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$

$A \bigcap B$

$A \bigcap B^c$

$B \bigcap A^c$

Proof:

$P(A) = P(A \bigcap B^c) + P(A \bigcap B)$ ①

$P(B) = P(B \bigcap A^c) + P(A \bigcap B)$ ②

$P(A \bigcup B) = P(A \bigcap B^c) + P(A \bigcap B) + P(B \bigcap A^c)$

$= P(A) + P(B \bigcap A^c)$ ① $is\ applied$

$= P(A) + P(B) - P(A \bigcap B)$ ② $is\ applied$

4. $P(\phi) = 0$

Proof:

$P(\phi) + P(\phi^c) = 1$

$P(\phi) + P(\Omega) = 1$

$P(\phi) + 1 = 1$

$P(\phi) = 0$

Exercise 1.

A_1, A_2, A_3

$P(A_1) = a_1$, $P(A_2) = a_2$, $P(A_3) = a_3$

$P(A_1 \bigcap A_2) = b_1$, $P(A_2 \bigcap A_3) = b_2$, $P(A_3 \bigcap A_1) = b_3$

$P(A_1 \bigcap A_2 \bigcap A_3) = c$

What is the value of $P(A_1 \bigcup A_2 \bigcup A_3)$?

$B = A_2 \bigcup A2$

$P(A_1 \bigcup B) = P(A_1) + P(B) - P(A_1 \bigcap B)$

$= P(A_1) + P(A_2 \bigcup A_3) - P(A_1 \bigcap (A_2 \bigcap A_3))$

Exercise, show that:

$A_1 \bigcap (A_2 \bigcup A_3) = (A_1 \bigcap A_2) \bigcup (A_1 \bigcap A_3)$

Union Bound

Theorem:

$A_1, A_2, \dots A_n$ are n events $(n \geq 2)$

$P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

Proof: Induction argument

$n = 2$

$P(A_1 \bigcup A_2) = P(A_1) + P(A_2) - P(A_1 \bigcap A_2) \leq P(A_1) + P(A_2)$

Assume that the theorem is true for $n = k$

i.e., $P(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} P(A_i)$

Then in the $k + 1$ case, where

$A_1, A_2, \dots A_k, A_{k+1}$

$P(\bigcup_{i=1}^{k+1} A_i) = P((\bigcup_{i=1}^{k} \bigcup A_{k+1}) \leq P(\bigcup_{i=1}^{k} A_i + P(A_{k+1}))$

$P(\bigcup_{i=1}^{k+1} A_i) \leq \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$

$P(\bigcup_{i=1}^{k+1} A_i) = \sum_{i=1}^{k+1} P(A_i)$

Cardinality of sets

Finite sets

$\Omega = \{1, 2, 3, 4, 5, 6\}$

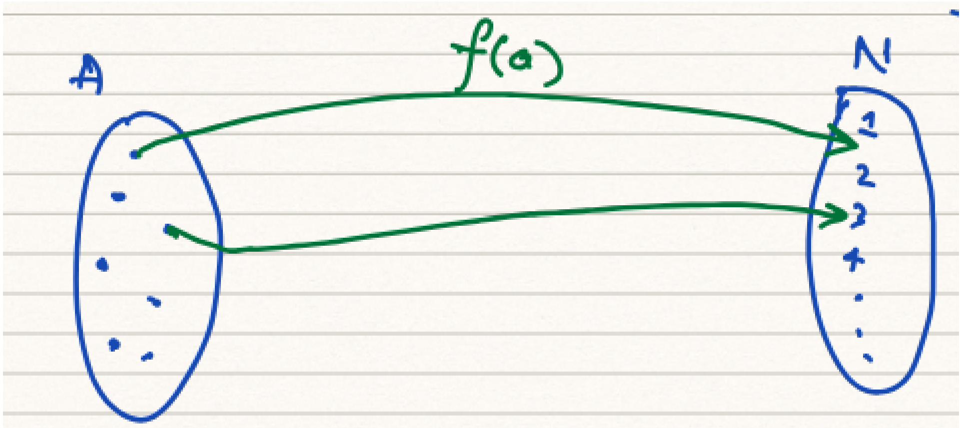
$\Omega = \{a, b, c, \dots z\}$

Infinite sets

Countably infinite sets

$N = \{1, 2, 3, \dots\}$

A set A that is "as large" as N is called a countably infinite set.



Formally, A is countably infinite if we can find a function f from A to N, such that

(i) f is a one-to-one function

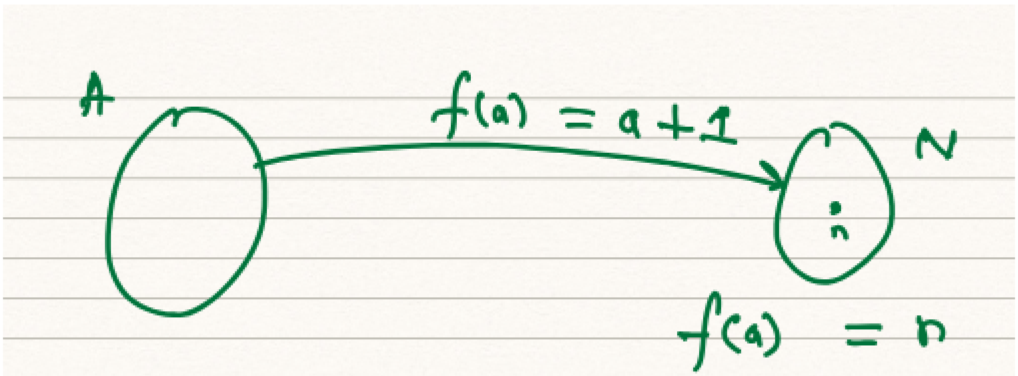
i.e., if $a \neq b$, then $f(a) \neq f(b)$

(ii) for every positive integer n, there is an $a \subset A$ such that $f(a) = n$

$Example 1$

$A = \{0, 1, 2, 3, \dots\}$

$N = \{1, 2, 3, \dots\}$



$f(a) = a + 1 = n$

$a = n - 1$

Therefore, A is countably infinite

$Example 2$

$B = \{2, 4, 6, 8, \dots\}$

$N = \{1, 2, 3, \dots\}$

$f(b) = b/2 = n$

$b = 2n$

$Example 3$

$C = \{2, 4, 8, 16, 32, \dots\}$

$f(c) = \log_2 c = n$

$c = 2^n$

$Example 4$

$\{-1, -2, -3, \dots\}$

$\{\dots, -1, 0, 1, 2, \dots\}$

are countably infinite sets

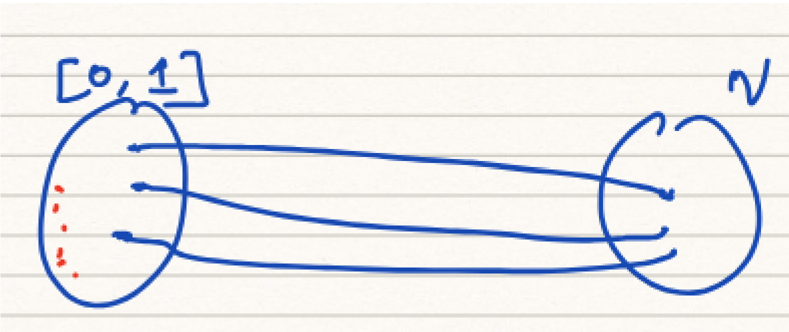
Uncountably infinite sets

Much larger sets of positive integers

e.g., $[0, 1]$, $[0, \infty]$, $(-\infty, \infty)$

$[0, 1]$

There is no way of finding a one-to-one association(correspondence) between $[0, 1]$ and N



$Example5$

$A_1 \subset A_2$

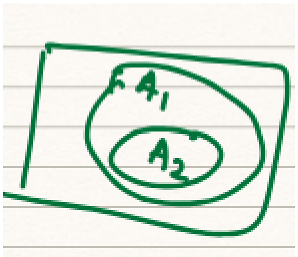
$P(A_1 \bigcup A_2) = P(A_2)$

$A_1 \subset A_2 \subset \ldots \subset A_k$

$P(\bigcup_{i=1}^{k} A_i) = P(A_k)$

$A_1 \subset A_2 \subset ... \subset A_k \subset A_{k+1}... $

$P(\bigcup_{i=1}^{\infty} A_i) = \lim_{k \to \infty} P(A_k)$



$A_1 \supset A_2$

$P(A_1 \supset A_2) = P(A_2)$

$A_1 \supset A_2 \supset \ldots \supset A_k$

$P(\bigcap_{i=1}^{k} A_i) = P(A_k)$

$A_1 \supset A_2 \supset \ldots \supset A_k \supset A_{k+1} \ldots$

$P(\bigcap_{i=1}^{\infty} A_i) = \lim_{k \to \infty} P(A_k)$

$Example6$

$\Omega = [0, 1]$

$P(interval) = length\ of\ interval$

$A_1 = [0, 1]$

$A_2 = [0, 1/2]$

$A_3 = [0, 1/3]$

$\ldots$

$A_k = [0, 1/k]$

$A_1 \supset A_2 \supset A_3 \supset \dots$

$P(A_1 \bigcap A_2 \bigcap A_3) = P(A_3) = 1/3$

$P(\bigcap_{i=1}^{\infty} A_i) = \lim_{k \to \infty} P(A_k) = \lim_{k \to \infty} 1/k = 0$

$\bigcap_{i=1}^{\infty} A_i = \{0\} = [0, 0]$

$P([0, 0]) = 0$

Finite outcome space

$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$

$\mathcal{F} = power\ set\ of\ \Omega$

$P(\omega_1) = p_1, P(\omega_2) = p_2, \dots P(\omega_n) = p_n$

$P(\{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_1\} \bigcup \{\omega_2\} \bigcup \{\omega_3\})$

$= P\{\omega_1\} + P\{\omega_2\} + P\{\omega_3\}$

$= p_1 + p_2 + p3$

$p_1 + p_2 + \dots + p_n = 1$

$1 = P(\Omega) = P(\{\omega_1, \dots, \omega_n\})$

Special case:

$\Omega = \{\omega_1, \dots, \omega_n\}$

$\mathcal{F} = Power\ set$

$P(\omega_i) = p$ for $i = 1, 2, \dots n$ Equally likely outcomes

$1 = p + p + \dots + p$

$1 = np$

$p = 1/n$

$P(\{\omega_2, \omega_4, \omega_6\}) = P(\{\omega_2\}) + P(\{\omega_4\}) + P(\{\omega_6\})$

$= 3/n$

$A = \{\omega_{k1}, \dots, \omega_{km}\}$

$P(A) = m/n$

$P(A) = \frac{|A|}{n}$

If Ω is countably infinite,

$\Omega = \{\omega_1, \omega_2, \dots\}$

we will usually work with power set on our event space

$Example 6$

$\Omega = [0, 1]$

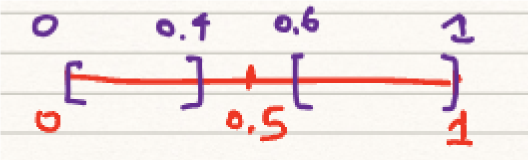
Borel Sigmal algebra: all sub-interval, union, intersections, compliments

$P(sub - interval\, of\, [0,1]) = length\, of\, sub - interval$

$0 \leq a \leq b \leq 1$

$P([a, b]) = b - a$

$A = \{\omega \in [0, 1] : |\omega - 0.5| \geq 0.1\}$



$P(A) = 0.8$

$A = [0, 0.4] \bigcup [0.6, 1]$

$P(A) = 0.4 + 0.4 = 0.8$

Exercise

$B = \{\omega \in [0, 1] : (\omega - 1/2)^2 \geq 1/4\}$

$P(B) = ?$

$P([0, 1]) = 1$

$P([0, 1]) = P(\bigcup_{0 \leq \omega \leq 1} \omega)$

$= \sum_{0 \leq \omega \leq 1} P(\omega)$

$\neq \sum_{0 \leq \omega \leq 1} P([\omega, \omega])$

$\neq 0$

Conditional Probability

Roll a fair die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = Power\ set$

Equally likely outcomes

$P(k) = 1/6, 1 \leq k \leq 6$

$B = \{2, 4, 6\}$

$A = \{2\}$

Given that B has occurred, the new probability for event $A = 1/3$

$C = \{1, 2, 3\}$

Given that B has occurred, what is the revised probability for event C?

$1/3$

$New\ prob\ of\ A = \frac{|A \bigcap B|}{|B|}$

$New\ prob\ of\ C = \frac{|C \bigcap B|}{|B|}$

Definition:

If A and B are 2 events, and $P(B) > 0$

$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$

$P(B|B) = 1$

$P(B^c|B) = 0$

$P(\Omega|B) = \frac{P(\Omega \bigcap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$