EE503 Probability for Electrical and Computer Engineers

Week 1 Session 1

Outcome space / Sample space

$\Omega = \{set\ of\ all\ possible\ outcomes\ of\ a\ random\ experiment\}$

1. Flip 1 coin:

   $\Omega = \{H, T\}$

2. Flip 2 coins:

   $\Omega = \{HH, TT, HT, TH\}$

3. Number of emails in the inbox from 10:30 am to 12:30 pm:

   $\Omega = \{0, 1, 2, 3, \ldots\}$

4. Amplitude of the received signal at the radar:

   $\Omega = \{0, \infty\}$

Events:

Flip 2 coins:

$\Omega = \{HH, TT, HT, TH\}$

$A = \{HH, TT\}$

Event A: a subset of Ω

If the observed outcome belongs to event A, then event A has occured.

Radar:

$\Omega = \{0, \infty\}$

$A = \{0, 1\}$

$B = \{\pi\}$

Event Space: Collection of events.

1. Flip 1 coin:

   $\Omega = \{H, T\}$

Event Space: $\{H\}$, $\{T\}$, Ω, ϕ [All possible subsets of Ω]

Power set of Ω: 2^{Ω}

2. Flip 2 coins:

   $\Omega = \{HH, TT, HT, TH\}$

Event space 1: ϕ, Ω, $\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$, $\{HH, TT\}$, $\{HT, TH\}$, $\{HH, HT\}$, $\{HT, TT\}$ ...

[Power set of Ω]

For a set with n elements, number of possible subsets is 2^n.

Event Space 2: $\Omega = \{HH, TT, HT, TH\}$

$\{HH, TT\}$, $\{HT, TH\}$, Ω, $\phi \leftarrow$ Another possible event space for the experiment of flipping

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Requirement of an Event Space

1. Ω is in the event space (sure event)
2. If A is in the event space, A^c is in the event space
3. If A and B are in the event space, then $A \bigcup B$ and $A \bigcap B$ are also in the event space.

Deduction 1:

ϕ is always in event space

Deduction 2:

If $A_1, A_2, \ldots A_n$ in the event space, then:

$\bigcap_{i=1}^{n} A_i$ and $\bigcup_{i=1}^{n} A_i$ are in the event space.

Probability Law P

For each event A in the event space, $P(A)$ is a real number that describes our belief/ likelihood of event A.

Axioms of Probability

1. $P(\Omega) = 1$
2. For any event A, $0 \leq P(A) \leq 1$
3. Additivity Axiom

   (a) If A and B are 2 disjoint (i.e., mutually exclusive $\leftarrow A \bigcap B = \phi$) events, then:

   $P(A \bigcup B) = P(A) + P(B)$

   (b) If $A_1, A_2, \ldots A_n$ are pairwise disjoint events (i.e., $A_k \bigcap A_l = \phi$ for all $k \neq l$), then:

   $P(A_1 \bigcup A_2 \ldots \bigcup A_n) = \sum_{i=1}^{n} P(A_i)$

$Example\ 1$

$\Omega = \{H, T\}$

Event space=Power set of Ω

$P(\{H\}) = 1/2$

What is the value of $P(T)$

$P(\Omega) = 1$

$P(\{H, T\}) = 1$

$\{H, T\} = \{H\} \bigcup \{T\}$

$P(\{H\} \bigcup \{T\}) = 1 \leftarrow$ Additivity axiom

$P(\{H\}) + P(\{T\}) = 1$

$P(\{T\}) = 1 - P(\{H\})$

$P(\{T\}) = 1 - 1/2 = 1/2$

$Example\ 2$

Throw a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Event space=Power set of Ω

Probability law: For any event A, $P(A) = \frac{|A|}{6}$

$Notation : |A| = number\ of\ elements\ in\ A = cardinality\ of\ A$

$P(\{6\}) = 1/6$

$Prob\ of\ getting\ an\ even\ number :$

$P(\{2, 4, 6\}) = 3/6$

$P(\phi) = 0$

$Example\ 3$

Throw a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Event space=Power set of Ω

$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = 5/12$

$P(\{6\}) = 1/3$

$P(\{3, 4, 5\}) = P(\{3\} \bigcup \{4\} \bigcup \{5\})$

$= P(\{3\}) + P(\{4\}) + P(\{5\})$

$= 6/15$

$Example\ 4$

$\Omega = \{0, \infty\}$

Event space consist of all possible sub-interval of $\{0, \infty\}$ as well as their compliments, unions and intersections.

e.g., (a, b), $[a, b]$, $(a, b]$, $[a, \infty)$

$Borel\ event\ space\ or\ Borel\ sigma\ algebra$

Probability law: For any interval A

$P(A) = \int_A e^{-\omega} d\omega$

$P((1, 2)) = \int_1^2 e^{-\omega} d\omega$

$P([2, \infty)) = \int_2^\infty e^{-\omega} d\omega$

Probability that the outcome is less than 1 or greater than 5 ?

$P([0, 1]) \bigcup (5, \infty)) = P([0, 1]) + P((5, \infty))$

$= \int_0^1 e^{-\omega} d\omega + \int_5^\infty e^{-\omega} d\omega$

$Example\ 5$

$\Omega = \{1, 2, 3, 4, \ldots\}$

$\mathcal{F} = Power\ set\ of\ \Omega$

i.e., Event space $\mathcal{F}$ [sigma-algebra]

$P(\{k\}) = \frac{1}{2^k}$, where $k = 1, 2, 3, \ldots$

Verify $P(\Omega) = 1$

Event space

If $A_1, A_2, \ldots, A_n$ are in the event space, then:

$\bigcap_{i=1}^n A_i$ and $\bigcup_{i=1}^n A_i$ are in the event space.

If $A_1, A_2, A_3, \ldots$ are in the event space, then:

$\bigcap_{i=1}^\infty A_i$ and $\bigcup_{i=1}^\infty A_i$ are also in the event space.

Probability Axioms

- Additivity axiom

  $A_1, A_2, \ldots, A_n$ are pairwise disjoint events

  $P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$
- Countable additivity axiom

$A_1, A_2, A_3, \ldots$ are pairwise disjoint events

$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

$P(\Omega) = P(\{1, 2, 3, 4 \ldots\})$

$\{1, 2, 3, \ldots\} = \{1\} \bigcup \{2\} \bigcup \ldots$

$P(\{1\} \bigcup \{2\} \bigcup \ldots) = P(\{1\}) + P(\{2\}) + \ldots$

Countable additivity axiom: $P(\{k\}) = \frac{1}{2^k}$

$= \frac{1}{2} + \frac{1}{2^2} + \ldots$

$= \frac{1/2}{1-1/2}$

Note $\leftarrow$ Geometric series

$a + ar + ar^2 + \ldots$ where $r < 1$, then

$sum = \frac{a}{1-r}$

Probability that the outcome is an even number:

$P(\{2, 4, 6, 8 \ldots\}) = P(\{2\} \bigcup \{4\} \bigcup \ldots)$

$= P(\{2\}) + P(\{4\}) + \ldots$

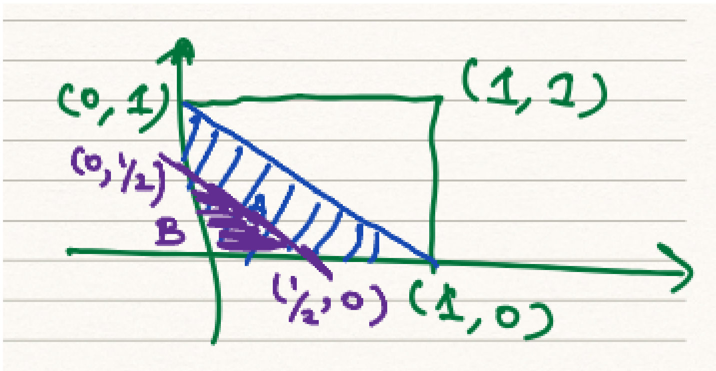
$= \frac{1}{2^2} + \frac{1}{2^4} + \ldots$

$= \frac{1/4}{1-1/4}$

$= 1/3$

$Example\ 6$

$\Omega = \{(x, y) : 0 \leq x, y \leq 1\}$



$P(A) = Area\ of\ A$

$P(\Omega) = Area\ of\ \Omega = 1$

$P(A) = 1/2$

B is the event that the sum of x and y coordinate is less than or equal to $1/2$

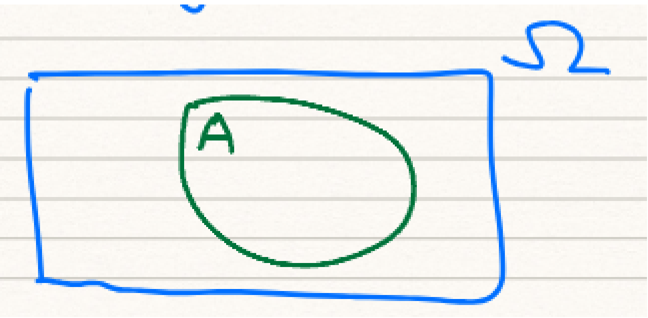
$B = (x, y) \in \Omega : x + y \leq 1/2$

$P(B) = Area\ of \in B = 1/2 \times 1/2 \times 1/2 = 1/8$

Week 1 Session 2

Random Experiment & Probability Model

- Outcome space / Sample Space Ω
- An event is a subset of Ω
- If the realized outcome of experiment lies in A, we say event A has occurred.



Event Space / Sigma algebra $\mathcal{F}$

Properties of $\mathcal{F}$:

1. Ω is in $\mathcal{F}$
2. If A is in $\mathcal{F}$, then A^c is in $\mathcal{F}$
3. (a) If $A_1, A_2, \ldots A_n$ are in $\mathcal{F}$, then:

   $\bigcup_{i=1}^{n} A_i$ is in $\mathcal{F}$ and $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{F}$

   (b) If $A_1, A_2, A_3, \ldots$ is an infinite sequence of events that are in $\mathcal{F}$, then:

   $\bigcup_{i=1}^{\infty} A_i$ is in $\mathcal{F}$ and $\bigcap_{i=1}^{\infty} A_i$ is in $\mathcal{F}$

Probability Law

For each event A in $\mathcal{F}$, $P(A)$ is a real number.

Probability Axioms

1. $P(\Omega) = 1$
2. $0 \leq P(A) \leq 1$
3. (a) If $A_1, A_2, \ldots A_n$ are pairwise disjoint events, then:

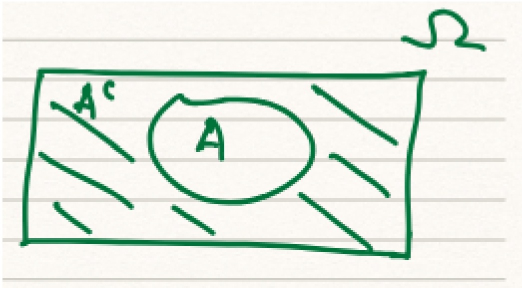
   $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

   (b) If $A_1, A_2, A_3, \ldots$ is an infinite sequence of pairwise disjoint events, then:

   $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Deduction from Axioms

1. $P(A) + P(A^c) = 1$



Proof:

$1 = P(\Omega)$

$= P(A \bigcup A^c)$

$= P(A) + P(A^c)$

2. If $A \subset B$, then $P(A) \leq P(B)$

Proof:

$B = A \bigcup C$

$P(B) = P(A \bigcup C)$

$= P(A) + P(C)$

$\rightarrow P(B) \geq P(A)$

$C = B \bigcap A^c$

3. Union Formula

For any 2 events A and B

$P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$

$A \bigcap B$

$A \bigcap B^c$

$B \bigcap A^c$

Proof:

$P(A) = P(A \bigcap B^c) + P(A \bigcap B)$ ①

$P(B) = P(B \bigcap A^c) + P(A \bigcap B)$ ②

$P(A \bigcup B) = P(A \bigcap B^c) + P(A \bigcap B) + P(B \bigcap A^c)$

$= P(A) + P(B \bigcap A^c)$ ① $is\ applied$

$= P(A) + P(B) - P(A \bigcap B)$ ② $is\ applied$

4. $P(\phi) = 0$

Proof:

$P(\phi) + P(\phi^c) = 1$

$P(\phi) + P(\Omega) = 1$

$P(\phi) + 1 = 1$

$P(\phi) = 0$

Exercise 1.

A_1, A_2, A_3

$P(A_1) = a_1$, $P(A_2) = a_2$, $P(A_3) = a_3$

$P(A_1 \bigcap A_2) = b_1$, $P(A_2 \bigcap A_3) = b_2$, $P(A_3 \bigcap A_1) = b_3$

$P(A_1 \bigcap A_2 \bigcap A_3) = c$

What is the value of $P(A_1 \bigcup A_2 \bigcup A_3)$?

$B = A_2 \bigcup A2$

$P(A_1 \bigcup B) = P(A_1) + P(B) - P(A_1 \bigcap B)$

$= P(A_1) + P(A_2 \bigcup A_3) - P(A_1 \bigcap (A_2 \bigcap A_3))$

Exercise, show that:

$A_1 \bigcap (A_2 \bigcup A_3) = (A_1 \bigcap A_2) \bigcup (A_1 \bigcap A_3)$

Union Bound

Theorem:

$A_1, A_2, \ldots A_n$ are n events $(n \geq 2)$

$P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

Proof: Induction argument

$n = 2$

$P(A_1 \bigcup A_2) = P(A_1) + P(A_2) - P(A_1 \bigcap A_2) \leq P(A_1) + P(A_2)$

Assume that the theorem is true for $n = k$

i.e., $P(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} P(A_i)$

Then in the $k+1$ case, where

$A_1, A_2, \ldots A_k, A_{k+1}$

$P(\bigcup_{i=1}^{k+1} A_i) = P((\bigcup_{i=1}^{k} \bigcup A_{k+1}) \leq P(\bigcup_{i=1}^{k} A_i + P(A_{k+1}))$

$P(\bigcup_{i=1}^{k+1} A_i) \leq \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$

$P(\bigcup_{i=1}^{k+1} A_i) = \sum_{i=1}^{k+1} P(A_i)$

Cardinality of sets

Finite sets

$\Omega = \{1, 2, 3, 4, 5, 6\}$

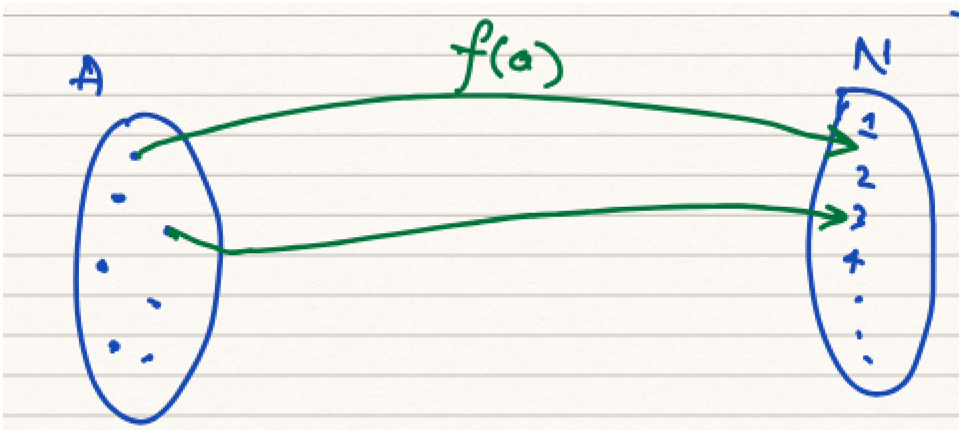
$\Omega = \{a, b, c, \ldots z\}$

Infinite sets

Countably infinite sets

$N = \{1, 2, 3, \ldots\}$

A set A that is "as large" as N is called a countably infinite set.



Formally, A is countably infinite if we can find a function f from A to N, such that

(i) f is a one-to-one function

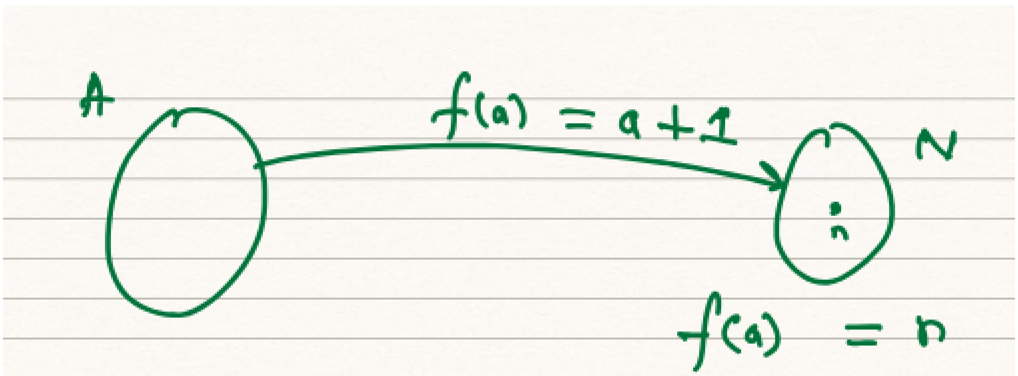
i.e., if $a \neq b$, then $f(a) \neq f(b)$

(ii) for every positive integer n, there is an $a \subset A$ such that $f(a) = n$

$Example 1$

$A = \{0, 1, 2, 3, \ldots\}$

$N = \{1, 2, 3, \ldots\}$



$f(a) = a + 1 = n$

$a = n - 1$

Therefore, A is countably infinite

$Example 2$

$B = \{2, 4, 6, 8, \ldots\}$

$N = \{1, 2, 3, \ldots\}$

$f(b) = b/2 = n$

$b = 2n$

$Example 3$

$C = \{2, 4, 8, 16, 32, \ldots\}$

$f(c) = \log_2 c = n$

$c = 2^n$

$Example 4$

$\{-1, -2, -3, \ldots\}$

$\{\ldots, -1, 0, 1, 2, \ldots\}$

are countably infinite sets

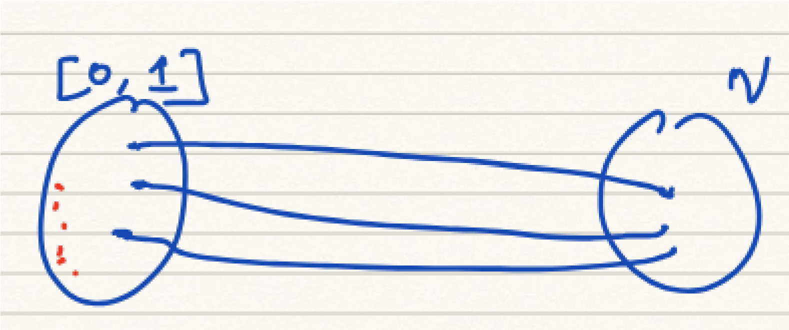
Uncountably infinite sets

Much larger sets of positive integers

e.g., $[0, 1]$, $[0, \infty]$, $(-\infty, \infty)$

$[0, 1]$

There is no way of finding a one-to-one association(correspondence) between $[0, 1]$ and N



$Example 5$

$A_1 \subset A_2$

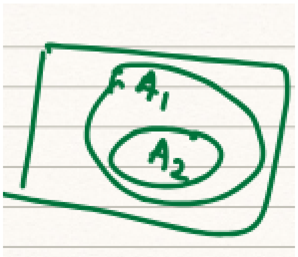
$P(A_1 \bigcup A_2) = P(A_2)$

$A_1 \subset A_2 \subset \ldots \subset A_k$

$P(\bigcup_{i=1}^{k} A_i) = P(A_k)$

$A_1 \subset A_2 \subset \ldots \subset A_k \subset A_{k+1} \ldots$

$P(\bigcup_{i=1}^{\infty} A_i) = \lim_{k \to \infty} P(A_k)$



$A_1 \supset A_2$

$P(A_1 \supset A_2) = P(A_2)$

$A_1 \supset A_2 \supset \ldots \supset A_k$

$P(\bigcap_{i=1}^{k} A_i) = P(A_k)$

$A_1 \supset A_2 \supset \ldots \supset A_k \supset A_{k+1} \ldots$

$P(\bigcap_{i=1}^{\infty} A_i) = \lim_{k \to \infty} P(A_k)$

$Example 6$

$\Omega = [0, 1]$

$P(interval) = length\, of\, interval$

$A_1 = [0, 1]$

$A_2 = [0, 1/2]$

$A_3 = [0, 1/3]$

$\ldots$

$A_k = [0, 1/k]$

$A_1 \supset A_2 \supset A_3 \supset \ldots$

$P(A_1 \bigcap A_2 \bigcap A_3) = P(A_3) = 1/3$

$P(\bigcap_{i=1}^{\infty} A_i) = \lim_{k \to \infty} P(A_k) = \lim_{k \to \infty} 1/k = 0$

$\bigcap_{i=1}^{\infty} A_i = \{0\} = [0, 0]$

$P([0, 0]) = 0$

Finite outcome space

$\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$

$\mathcal{F} = power\ set\ of\ \Omega$

$P(\omega_1) = p_1, P(\omega_2) = p_2, \ldots P(\omega_n) = p_n$

$P(\{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_1\} \bigcup \{\omega_2\} \bigcup \{\omega_3\})$

$= P\{\omega_1\} + P\{\omega_2\} + P\{\omega_3\}$

$= p_1 + p_2 + p3$

$p_1 + p_2 + \ldots + p_n = 1$

$1 = P(\Omega) = P(\{\omega_1, \ldots, \omega_n\})$

Special case:

$\Omega = \{\omega_1, \ldots, \omega_n\}$

$\mathcal{F} = Power\ set$

$P(\omega_i) = p$ for $i = 1, 2, \ldots n$ Equally likely outcomes

$1 = p + p + \ldots + p$

$1 = np$

$p = 1/n$

$P(\{\omega_2, \omega_4, \omega_6\}) = P(\{\omega_2\}) + P(\{\omega_4\}) + P(\{\omega_6\})$

$= 3/n$

$A = \{\omega_{k1}, \ldots, \omega_{km}\}$

$P(A) = m/n$

$P(A) = \frac{|A|}{n}$

If Ω is countably infinite,

$\Omega = \{\omega_1, \omega_2, \ldots\}$

we will usually work with power set on our event space

$Example6$

$\Omega = [0, 1]$

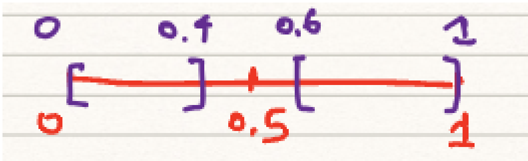
Borel Sigma algebra: all sub-interval, union, intersections, compliments

$P(sub - interval\ of\ [0, 1]) = length\ of\ sub - interval$

$0 \leq a \leq b \leq 1$

$P([a, b]) = b - a$

$A = \{\omega \in [0, 1] : |\omega - 0.5| \geq 0.1\}$



$P(A) = 0.8$

$A = [0, 0.4] \bigcup [0.6, 1]$

$P(A) = 0.4 + 0.4 = 0.8$

Exercise

$B = \{\omega \in [0, 1] : (\omega - 1/2)^2 \geq 1/4\}$

$P(B) = ?$

$P([0, 1]) = 1$

$P([0, 1]) = P(\bigcup_{0 \leq \omega \leq 1} \omega)$

$= \sum_{0 \leq \omega \leq 1} P(\omega)$

$\neq \sum_{0 \leq \omega \leq 1} P([\omega, \omega])$

$\neq 0$

Conditional Probability

Roll a fair die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$\mathcal{F} = Power\ set$

Equally likely outcomes

$P(k) = 1/6, 1 \leq k \leq 6$

$B = \{2, 4, 6\}$

$A = \{2\}$

Given that B has occurred, the new probability for event $A = 1/3$

$C = \{1, 2, 3\}$

Given that B has occurred, what is the revised probability for event C?

$1/3$

$New\ prob\ of\ A = \frac{|A \bigcap B|}{|B|}$

$New\ prob\ of\ C = \frac{|C \bigcap B|}{|B|}$

Definition:

If A and B are 2 events, and $P(B) > 0$

$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$

$P(B|B) = 1$

$P(B^c|B) = 0$

$P(\Omega|B) = \frac{P(\Omega \bigcap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Week 2 Session 1

Deductions from Axioms

1. $P(A) + P(A^c) = 1$
2. $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$
3. If $A \subset B$, then $P(A) \leq P(B)$

Finite outcome space

$\Omega = \{\omega_1, \ldots \omega_n\}$

Typically, $\mathcal{F}$ $=$ Power set of ω

$P(\{\omega_1)\}) = p_1, \ldots, P(\{\omega_n\}) = p_n$

Special case: Finite ω with equally likely outcome

$P(\{\omega_i\}) = \frac{1}{n}$, $n = |\Omega|$

$P(A) = \frac{|A|}{n}$

Conditional Probabilities

Definition: If A and B are 2 events, and $P(B) > 0$

$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$

Example

4 sided die, roll it twice

$\Omega = \{(x, y) : x, y \in \{1, 2, 3, 4\}\}$

$= \{(1, 1), (1, 2), (1, 3), \ldots (4, 4,)\}$

Equally likely outcome $P(\{x, y\}) = \frac{1}{16}$

$E = \{\text{Both number are less than } 3\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

$F = \{\text{Both numbers are } 1\} = \{(1, 1)\}$

$P(F|E) = \frac{P(F \bigcap E)}{P(E)}$

$P(E) = \frac{4}{16}$

$P(F \bigcap E) = \frac{1}{16}$

$P(F|E) = \frac{1}{4}$

$P(E|F) = \frac{P(E \bigcap F)}{P(F)}$

$P(E|F) = \frac{1}{16} / \frac{1}{16} = 1$

$B = \{\text{minimum of 2 number is } 2\} = \{(2, 2), (2, 3), (3, 2), (2, 4), (4, 2)\}$

$A = \{\text{maximum of 2 number is } 3\} = \{(3, 3), (3, 1), (1, 3), (2, 3), (3, 2)\}$

$P(A|B) = \frac{P(A \bigcap B)}{P(B))} = \frac{2/16}{5/16} = 2/5$

$C = \{\text{maximum of 2 number is } 1\}$

$P(C|B) = \frac{P(C \bigcap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0$

Probability Axioms

1. $P(\Omega) = 1$
2. $0 \leq P(A) \leq 1$
3. Additivity axiom

Fix event B with $P(B) > 0$. Consider conditional probability of various event given B. These new probability will satisfy probability axioms.

1. $P(\Omega|B) = \frac{P(\Omega \bigcap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$
2. $0 \leq P(A|B) \leq 1$

   $0 \leq \frac{P(A \bigcap B)}{P(B)} \leq 1$
3. If A_1 and A_2 are disjoint events

   $P(A_1 \bigcup A_2|B) = \frac{P((A_1 \bigcup A_2) \bigcap B)}{P(B)}$

$= \frac{P((A_1 \bigcap B) \bigcup (A_2 \bigcap B))}{P(B)}$

$= \frac{P(A_1 \bigcap B) + P(A_2 \bigcap B)}{P(B)}$

$= P(A_1|B) + P(A_2|B)$

Deduction from Axioms

1. $P(A|B) + P(A^c|B) = 1$
2. $P(A \bigcup C|B) = P(A|B) + P(C|B) - P(A \bigcap C|B)$
3. If $A \subset C$. then $P(A|B) \leq P(C|B)$

Chain Rule / Multiplication Rule

If $P(B) > 0$, then $P(A|B) = \frac{P(A \bigcap B)}{P(B)}$

$P(A \bigcap B) = P(A|B)P(B)$ - Chain Rule

If $P(A) > 0$, then $P(B|A) = \frac{P(B \bigcap A)}{P(A)}$

$P(B \bigcap A) = P(B|A)P(A)$

$P(B) > 0, P(B^c) > 0$

$P(A|B)$, $P(A|B^c)$

$P(A \bigcap B) = P(A|B)P(B)$

$P(A \bigcap B^c) = P(A|B^c)P(B^c)$

Law of total probability

$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

Partition

Partition of Ω

$B_1, \ldots, B_n$ forms a partition of Ω

if $B_i \bigcap B_j = \phi$ for $i \neq j$ (pairwise disjoint) and $\bigcup_{i=1}^n B_i = \Omega$

$P(B_i)$ for $i = 1, \ldots, n$

$P(A|B_i)$ for $i = 1, \ldots, n$

$P(A) = P(A \bigcap \Omega)$

$= P(A \bigcap (\bigcup_{i=1}^n B_i))$

$= P((A \bigcap B_1) \bigcup \ldots (A \bigcap B_n))$

$= \sum_{i=1}^n P(A \bigcap B_i)$

Law of total probability

$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Example

Say a sender and receiver with transmission route 1 and 2

$P(R1) = 3/4, P(D|R1) = 1/3, P(R2) = 1/4, P(D|R2) = 2/3$ where D is the packet get dropped

$P(D) = P(D \bigcap R1) + P(D \bigcap R2)$

$= P(D|R1)P(R1) + P(D|R2)P(R2)$

$= 1/3 \cdot 3/4 + 2/3 \cdot 1/4 = 5/12$

$P(\text{Packet not getting dropped}|\text{R1}) = P(D^c|R1)$

$= 1 - P(D|R1)$

$= 2/3$

$P(R1|D) = \frac{P(R1 \bigcap D)}{P(D)}$

$= \frac{P(D|R1)P(R1)}{5/12}$

Bayes' Rule

$P(A) > 0, P(B) > 0$

$P(B|A) = \frac{P(B \bigcap A)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$

$P(R1|D^c) = \frac{P(D^C|R1)P(R1)}{P(D^c)}$

$= \frac{(1-P(D|R1)) \cdot 3/4}{1-P(D)} \neq P(R1)$

This it because $P(R1|D^c)$ is the posterior probability and $P(R1)$ is the prior probability.

Law of total probability + Bayes' Rule

Partition of $\Omega : B_1, \dots B_n$

$B_i \bigcap B_j = \phi$

$\bigcup_{i=1}^{n} B_i = \Omega$

Given $P(B_i), P(A|B_i)$

$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)}$

Bayes' Rule:

$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$

Example

Binary communication channel

$P(S0) = 1 - \alpha$, $P(S1) = \alpha$

$P(R0|S0) = 1 - q$, $P(R1|S0) = q$

$P(R0|S0) = p$, $P(R1|S1) = 1 - p$

$P(R1) = P(R1 \bigcap S1) + P(R1 \bigcap S0)$

$= P(R1|S1)P(S1) + P(R1|S0)P(S0)$

$= (1-p)\alpha + q(1-\alpha)$

$P(S1|R1) = \frac{P(R1|S1)P(S1)}{P(R1)}$

$= \frac{(1-p)\alpha}{(1-p)\alpha + q(1-\alpha)}$

Chain Rule

$P(A \bigcap B) = P(A|B)P(B)$

$P(A \bigcap B \bigcap C) = P(A|(B \bigcap C))P(B \bigcap C)$

$= P(A|(B \bigcap C))P(B|C)P(C)$

$P(A_1 \bigcap ... \bigcap A_n) = P(A_1|A_2 \bigcap ... \bigcap A_n)P(A_2|A_3 ... \bigcap A_n) ... P(A_{n-1}|A_n)P(A_n)$

Example

Two urns

Urn 1: 5 red balls and 5 green balls

Urn 2: 2 red balls and 4 green balls

Randomly pick one ball from the selected urn and remove it

Randomly pick 2^{nd} ball from the same urn

$U1 = \{\text{Urn 1 is selected}\}$, $P(U1) = 2/3$

$U2 = \{\text{Urn 2 is selected}\}$, $P(U2) = 1/3$

$R1 = \{\text{first ball is red}\}$

$R2 = \{\text{second ball is red}\}$

$P(R1 \bigcap R2) = P(R1 \bigcap R2 \bigcap U1) + P(R1 \bigcap R2 \bigcap U2)$

$P(R2 \bigcap R1 \bigcap U1) = P(R2|R1 \bigcap U1)P(R1|U1)P(U1)$

$= 4/9 \cdot 5/10 \cdot 2/3$

$P(R2 \bigcap R1 \bigcap U2) = P(R2|R1 \bigcap U2)P(R1|U2)P(U2)$

$= 1/5 \cdot 2/6 \cdot 1/3$

$P(U1|R1 \bigcap R2) = \frac{P(R1 \bigcap R2|U1)P(U1)}{P(R1 \bigcap R2)}$

$= \frac{4/9 \cdot 5/10 \cdot 2/3}{P(R1 \bigcap R2)}$

Week 2 Session 2

$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$ if $P(B) > 0$

Chain Rule:

$P(A \bigcap B) = P(A|B)P(B)$

$P(A \bigcap B \bigcap C) = P(A|B \bigcap C)P(B|C)P(C)$

Law of total probability:

If $B_1, \ldots B_n$ is a partition of Ω
Then $P(A) = \sum_{i=1}^{n} P(A \bigcap B_i)$

$= \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Can be extended to a countable partition, i.e., $B_1, B_2, \ldots$

that are pairwise disjoint and $\bigcup_{i=1}^{\infty} B_i = \Omega$

Then, $P(A) = \sum_{i=1}^{\infty} P(A \bigcap B_i) = \sum_{i=1}^{\infty} P(A|B_i)P(B_i)$

Bayes' Rule

$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

$P(A \bigcap B) = P(A|B)P(B)$

Statement: $P(A \bigcap B|C) = P(A|B \bigcap C)P(B|C)$

LHS: $P(A \bigcap B|C) = \frac{P(A \bigcap B \bigcap C)}{P(C)}$

RHS: $P(A|B \bigcap C)P(B|C) = \frac{P(A \bigcap B \bigcap C)}{P(B \bigcap C)} \frac{P(B \bigcap C)}{P(C)}$

Two urns example

$P(R2 \bigcap R1|U1)P(U1)$

$= P(R2|R1 \bigcap U1)P(R1|U1)P(U1)$

$P(A \bigcap B) \sim \frac{\text{number of}(A \bigcap B)}{n}$ relative frequency

$P(B) \sim \frac{\text{number of} B}{n}$

$P(A|B) \sim \frac{\text{number of }(A \bigcap B)}{\text{number of } B}$

Independent Events

Definition: Two events A and B are independent if $P(A \bigcap B) = P(A)P(B)$

Let A and B be independent events and $P(B) > 0$
$P(A|B) = \frac{P(A \bigcap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$

$P(B^c) > 0$

$P(A|B^c) = \frac{P(A \bigcap B^c)}{P(B^c)} = \frac{P(A)-P(A \bigcap B)}{1-P(B)} = \frac{P(A)-P(A)P(B)}{1-P(B)} = P(A)$

Independence $\neq$ Disjoint events

$C \bigcap D = \phi$, $P(C), P(D) > 0$

<u>Are C and D independent?</u>

$P(C \bigcap D) = P(\phi) = 0$

$P(C)P(D) > 0$

$P(C \bigcap D) \neq P(C)P(D)$

C and D are not independent

$P(C|D) = 0$

Lemma 1

If A and B are independent events, then so are the following events:

(a) A and B^c

$P(A \bigcap B^c) = P(A)P(B^c)$

(b) A^c and B are independent

(c) A^c and B^c are independent

Proof:

(a) $P(A \bigcap B^c) = P(A) - P(A \bigcap B)$

$= P(A) - P(A)P(B)$

$= P(A)(1 - P(B))$

$= P(A)P(B^c)$

(b) A^c and B

(c) A^c and B^c

Using property: $(A \bigcup B)^c = A^c \bigcap B^c$

$P(A^c \bigcap B^c) = 1 - P(A \bigcup B)$

$= 1 - P(A) - P(B) + P(A \bigcap B)$

$= 1 - P(A) - P(B) - P(A)P(B)$

$= (1 - P(A))(1 - P(B))$

$= P(A^c)P(B^c)$

Example

4 sided die, $\Omega = \{1, 2, 3, 4\}$

Equally likely outcomes

$A = \{1, 4\}$, $B = \{2, 4\}$

$P(A) = 1/2, P(B) = 1/2$

$P(A \bigcap B) = 1/4$

Example

$\Omega = \{1, 2, 3, 4\}$

$P(\{1\}) = P(\{2\}) = P(\{3\}) = p$

$A = \{1, 4\}$, $B = \{2, 4\}$

$P(A) = 1 - 2p$

$P(B) = 1 - 2p$

$P(A \bigcap B) = P(\{4\}) = 1 - 3p$

A and B are independent if $1 - 3p = (1 - 2p)^2$

Independence of 3 events

Def: A, B, C are independent (mutually independent) if

1. $P(A \bigcap B \bigcap C) = P(A)P(B)P(C)$
2. $P(A \bigcap B) = P(A)P(B)$
3. $P(B \bigcap C) = P(B)P(C)$
4. $P(C \bigcap A) = P(C)P(A)$

$P(A \bigcap B | C) = \frac{P(A \bigcap B \bigcap C)}{P(C)} = \frac{P(A)P(B)P(C)}{P(C)} = P(A)P(B) = P(A \bigcap B)$

$P(A|C) = P(A)$

$P(A \bigcup B | C) = \frac{P((A \bigcup B) \bigcap C)}{P(C)}$

$= \frac{P((A \bigcap C) \bigcup (B \bigcap C))}{P(C)}$

$= \frac{P(A \bigcap C) + P(B \bigcap C) - P(A \bigcap B \bigcap C)}{P(C)}$

$= \frac{P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)}{P(C)}$

$= P(A) + P(B) - P(A)P(B)$

$= P(A \bigcup B)$

$P(A \bigcup C) = P(A|C) + P(B|C) - P(A \bigcap B|C)$

$= P(A) + P(B) - P(A \bigcap B)$

$= P(A \bigcup B)$

Pairwise independence

A, B, C are pairwise independent if

$P(A \bigcap B) = P(A)P(B)$

$P(B \bigcap C) = P(B)P(C)$
$P(C \bigcap A) = P(C)P(A)$

Example

4 sided die equally likely outcomes $\Omega = \{1, 2, 3, 4\}$

$A = \{1, 4\}$, $B = \{2, 4\}$, $C = \{3, 4\}$

$P(A \bigcap B) = 1/4$, $P(A) = 1/2$, $P(B) = 1/2$

A, B, C are pairwise independent

$P(A \bigcap B \bigcap C) = 1/4$

$P(A)P(B)P(C) = 1/8$

$P(A \bigcap B \bigcap C) \neq P(A)P(B)P(C)$ meaning A, B, C are not independent

$P(A \bigcap B|C) = \frac{P(A \bigcap B \bigcap C)}{P(C)} = \frac{1/4}{1/2} = 1/2$

Lemma 2

A, B, C are independent then

(a) A^c, B, C are independent

Proof:

$P(A^c \bigcap B) = P(A^c)P(B)$

$P(A^c \bigcap C) = P(A^c)P(C)$

$P(B \bigcap C) = P(B)P(C)$

$P(A^c \bigcap B \bigcap C) = P(B \bigcap C) - P(B \bigcap C \bigcap A)$

$= P(B)P(C) - P(B)P(C)P(A)$

$= (1 - P(A))P(B)P(C)$

$= P(A^c)P(B)P(C)$

(b) A^c, B^c, C are independent

$E = A^c$

From Part (a) E, B, C are independent

$\implies E, B^c, C$ are independent

$\implies A^c, B^c, C$ are independent

(c) A^c, B^c, C^c are independent

$E = A^c, F = B^c$

From Part (b)

$\implies E, F, C$ are independent

using Part (a)

$\implies E, F, C^c$ are independent

$\implies A^c, B^c, C^c$ are independent re

Example

3 bits are transmitted over a noisy channel

For each bit, the probability of correct reception is λ

$P(C_i) = \lambda\,, P(E_i) = 1 - \lambda$

The error events for the 3 bits are mutually independent

$E_i = \{\text{bit } i \text{ incorrectly received}\}$

E_1, E_2, E_3 are independent

$C_i = \{\text{bit } i \text{ correctly received}\}$

C_1, C_2, C_3 are independent

C_1, C_2, E_3 are independent

Example

Find the probability that the number of correctly received bits is 2

$S = \{\text{Number of correct bits is } 2\}$

$(C_1 \bigcap C_2 \bigcap E_3) \bigcup (C_1 \bigcap E_2 \bigcap C_3) \bigcup (E_1 \bigcap C_2 \bigcap C_3)$

say $F_1 = C_1 \bigcap C_2 \bigcap E_3, F_2 = C_1 \bigcap E_2 \bigcap C_3, F_3 = E_1 \bigcap C_2 \bigcap C_3$

$S = F_1 \bigcup F_2 \bigcup F_3$

$P(S) = P(F_1) + P(F_2) + P(F_3)$

because F_1, F_2, F_3 are pairwise disjoint

$P(C_1 \bigcap C_2 \bigcap E_3) = \lambda\lambda(1-\lambda)$

$P(S) = 3\lambda^2(1-\lambda)$

Probability that all bits are correctly received:

$= P(C_1 \bigcap C_2 \bigcap C_3) = \lambda^3$

Probability that at least 2 bits are correctly received:

$= \lambda^3 + 3\lambda^2(1-\lambda)$

Finite Ω and equally likely outcomes

$P(\{\omega\}) = \frac{1}{|\Omega|}$, $|\Omega|$ is the cardinality of Ω

$P(A) = \sum_{\omega \in A} P(\{\omega\}) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$

Example

Antenna array consists of n antennas

Arrange n antennas in a straight line

m out of the n antennas are defective

All arrangement of n antennas are equally likely

The array will not work if 2 defective antennas are next to each other

Probability of the array that does not work?

$n = 4, m = 2$

$A = \{DDNN, NNDD, NDDN, NDND, DNDN, DNND\}$

$P(\text{Array not work}) = \frac{|A|}{|\Omega|} = \frac{3}{6}$

$n = 12, m = 4$

Systematic / Efficient way of counting number of elements in a set without listing all elements $\rightarrow$ Combinatorics

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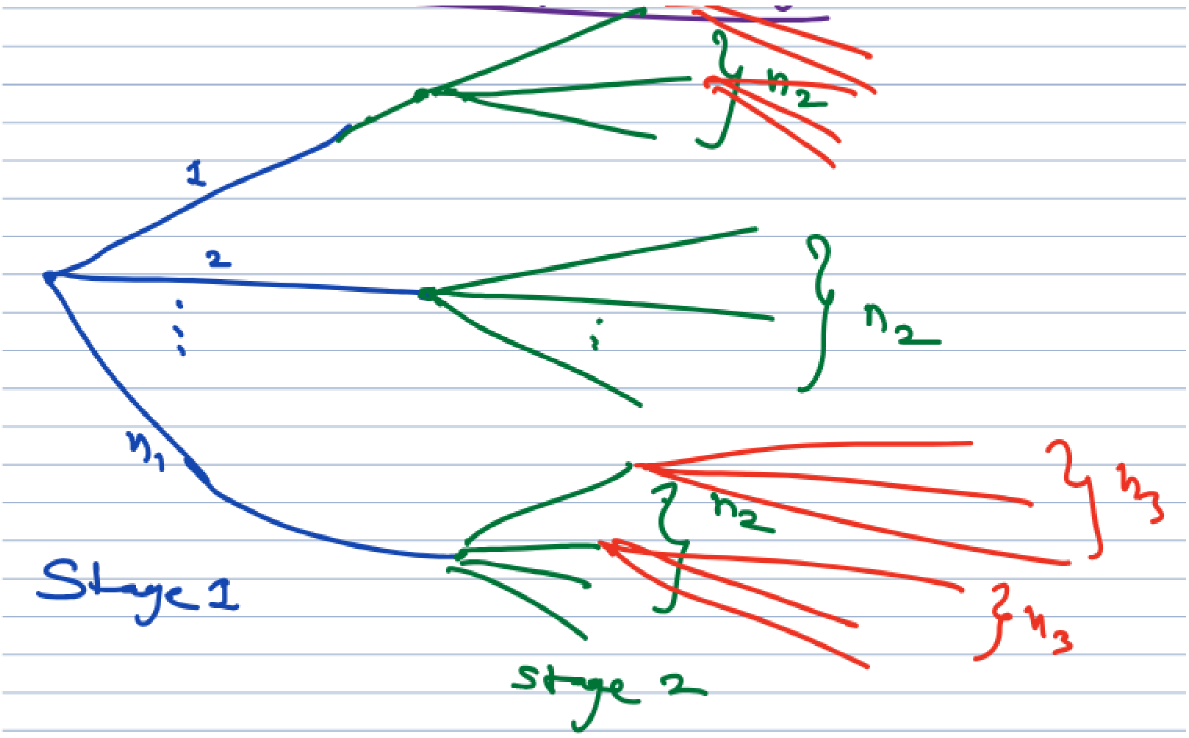
Basis principle of counting



$n_1 n_2$ ways of doing this procedure

Example

4 digit passcode

10^4

Example

7 character license plate where first 3 characters are letter other 4 are digits

$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

Repetition of letter or digits is not allowed

$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

Week 3 Session 1

Independent events

Definition: Two events A and B are independent if $P(A \bigcap B) = P(A)P(B)$

Independence $\implies P(A|B) = P(A|B^c) = P(A)$

Independence of 3 events

Def: A, B, C are independent (mutually independent) if

1. $P(A \bigcap B \bigcap C) = P(A)P(B)P(C)$
2. $P(A \bigcap B) = P(A)P(B)$

3. $P(B \bigcap C) = P(B)P(C)$
4. $P(C \bigcap A) = P(C)P(A)$

Independence of multiple events

Definition: $A_1, \ldots A_n$ are (mutually) independent if

$P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$

for every non-empty $I \subset \{1, 2, \ldots, n\}$

Definition: $A_1, A_2, \ldots$ are (mutually) independent if

$P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$

for every non-empty and finite $I \subset \{1, 2, \ldots\}$

Finite Ω and equally like outcomes

$P(A) = \frac{|A|}{|\Omega|}$

Basic principle of Counting

Total number of ways of doing 2-stage procedure is $n_1 n_2$

Permutations

Definition: Any arrangement of elements of S in a sequence

$S = \{I_1, \ldots I_n\}$

How many permutation of S are possible?

Imagine have n slots,

$n \cdot (n-1) \ldots \cdot 1 = n!$

Example

4 digit passcode using all of these digits 2,4,6,8

number of passcodes=number of permutations=$4!$

$S = \{I_1, \ldots I_n\}$

Sampling an ordered k-tuple with repetitions allowed

$k = 2$, $(I_1 \, I_2) \neq (I_2 \, I_1)$, $(I_1 \, I_1) \neq (I_1 \, I_1)$

How many ordered pairs are possible?

n^2

Ordered k-tuple

n^k ordered k-tuple

Example

Flipping a coin k times. How many sequence of H, T are possible?

k-tuple: 2^k

Sampling ordered k-tuple without repetitions $1 \leq k \leq n$

$S = \{I_1, \dots I_n\}$

Within k slots, then $n \cdot (n-1) \cdot \dots \cdot (n-k+1)$

Number of ordered k-tuple without repitition

$n \cdot (n-1) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}$

Example

n people in a department.

Need to form a committee consisting of a chair, a vice chair and a secretary

Same person cannot serve in more than 1 role

How many such committees are possible?

$n(n-1)(n-2) = \frac{n!}{(n-3)!}$

Example: Birthday problem

k people , $1 \leq k \leq 365$

All were born in non-leap year

a) How many birthday sequences are possible?

$(d_1, \dots d_k)$ where every slot has 365 possibilities

365^k

b) How many birthday sequences are possible without repitition?

$\frac{365!}{(365-k)!} = 365 \cdot 364 \cdot \dots \cdot (365-k+1)$

c) Assume that all birthday sequence are equally likely?

Probability that the group of k-people have distinct birthdays

$\frac{|A|}{|\Omega|} = \frac{\frac{365!}{(365-k)!}}{365^k}$

d) Probability that at least 2 people in this k-people group have the same birthday?

$P(B) = 1 - P(\text{everyone has a different birthday})$

$= 1 - \frac{\frac{365!}{(365-k)!}}{365^k}$

Selecting a subset with k elements, $1 \leq k \leq n$

- Order is not important
- No repetitions
- A subset of k elements is called a "combination" with k-elements

Say $S = \{I_1, I_2, I_3\}$ has a subset of size 2

$\{I1, I2\}, \{I1, I3\}, \{I3, I2\}$ all 3 possibilities

With n elements in S

number of k element subsets $n\mathrm{C}_k$

$n\mathrm{C}_k = \frac{n!}{(n-k)!k!} = \binom{n}{k}$ where $\frac{n!}{(n-k)!k!}$ is the binomial coefficient

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number of subset of 0 elements=1

$k = 0$

$\binom{n}{0} = \frac{n!}{n!0!} = 1$

---

for $k \geq n+1$

$n\mathrm{C}_k = 0$

Why is the number of subsets of S with k-elements=$\binom{n}{k}$

Task: Select an order k-tuple from S without repititions

$n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!k!}$

Indirect method for task

Stage1:

Pick a subset of k elements (without considering order)

$n\mathrm{C}_k$

Stage2:

Pick a permutation for the k-element subset chosen in stage1

$k!$

$n\mathrm{C}_k k! = \frac{n!}{(n-k)!}$

$n\mathrm{C}_k = \frac{n!}{(n-k)!k!}$

Example

a) 20 people in an organization need a committee of 3 people

number of possible committees$= \binom{20}{3} = \frac{20!}{3!17!}$

b) 12 people - 5 women and 7 men

Need to form a committee with 2 women and 3 men

number of possible committees$= \binom{5}{2}\binom{7}{3}$

c) 7 people

Committee of 3 people

Two of 7 people - Person 1 and Person 2 refuse to serve together

number of possible committees$= \binom{7}{3} - \binom{5}{1}$

Method 2: $S = P_1, \ldots P_7$

Committees with P_1 but not $P_2 = \binom{5}{2}$

Committees with P_2 but not $P_1 = \binom{5}{2}$

Committees with neither P_1 or $P_2 = \binom{5}{3}$

$\binom{5}{2} + \binom{5}{2} + \binom{5}{3} = 30$

Example

50 items: 10 of items are defective, 40 are functional

Randomly pick 10 items

Probability that exactly 5 of selected items are defective

$\frac{\binom{10}{5}\binom{40}{5}}{\binom{50}{10}}$

Example

32 bit binary number. How many such numbers have exactly 5 zeros

$\binom{32}{5}$

A computer randomly generate a 32 bit binary number.

Probability that the number generated has exactly 5 zeros.

$\frac{\binom{32}{5}}{2^{32}}$

Example

n antennas, m defective , $n-m$ functional, $1 \leq m \leq n$

n antennas are arranged in a row

a) How many ways can I arrange m defective (0) and $(n-m)$ functional (1) antennas?

say $n=4, m=2$

$\binom{4}{2} = \frac{4!}{2!2!} = 6$

n - bit binary number with m 0 and $n-m$ 1

$\binom{n}{m}$

b) How many arrangement where no 2 defective antennas are adjacent to each other?

n bits where no 2 zeros are adjacent

m zeros and $n-m$ ones

say $m=2$ then $n-2$ ones simply

$\binom{n-1}{2}$

${}^{n-m+1}\mathrm{C}_m$ number of valid arrangement

If $m \leq n-m+1$

${}^{n-m+1}\mathrm{C}_m = \binom{n-m+1}{m}$

If $m > n-m+1$

${}^{n-m+1}\mathrm{C}_m = 0$

c) Antenna system works as long as no two defective antennas are next to each other

$P(\text{Antenna system works}) = \frac{{}^{n-m+1}\mathrm{C}_m}{\binom{n}{m}}$

Example

$S = (I_1, \dots I_n)$

Divide S into groups: Group 1 with k_1 items, Group with k_2 items

where $k_1 + k_2 = n$, $1 \leq k_1 \leq n$, $1 \leq k_2 \leq n$

How many such division of S are possible?

$\binom{n}{k_1}\binom{n-k_1}{k_2} = \binom{n}{k_1}\binom{k_2}{k_2} = \binom{n}{k_1}$

Problem:

Divide S into r groups

Group 1: k_1 items

...

Group r: k_r items

where $k_1 + \ldots + k_r = n$

How many such divisions are possible?

$\binom{n}{k_1}\binom{n-k_1}{k_2}\ldots\binom{n-k_1-k_2-\ldots-k_{r-1}}{k_r}$

$= \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{k_2!(n-k_1-k_2)!}\ldots$

$= \frac{n!}{k_1!k_2!\ldots k_r!}$

Multinomial coefficient

Example

6 sided die. Roll it 12 times.

How man sequence with 10 ones, 1 twos and 1 threes?

12 positions

Divide 12 positions into 3 groups

G1- 10

G2- 1

G3- 1

$\frac{12!}{10!1!1!}$

Exercise: How many outcomes where each number appears exactly twice

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Week 3 Session 2

Combinatorial Problems

$S = I_1, \ldots I_n$

1. Number of ordered k-tuple with repetitions allowed

   n^k

2. Number of ordered k-tuples without repetition $1 \leq k \leq n$

   $\frac{n!}{(n-k)!}$

3. Number of possible subsets of size k, $1 \leq k \leq n$

   $\binom{n}{k}$

4. Partition S into r groups of sizes $k_1, k_2, \ldots k_r$ respectively $(k_1 + k_2 + \ldots + k_r = n)$

   Number of possible partitions

   $\frac{n!}{k_1!\ldots k_r!} = \binom{n}{k_1, k_2, \ldots k_r}$

Example: k identical items (k \$1 bills)

Divide among two people

$P1$	$P2$
0	k
1	$k-1$
2	$k-2$
$\ldots$	$\ldots$
$k-1$	1
k	0

Each outcome compounds to $k+1$ bit number with exactly one 0

Number of such number$=\binom{k+1}{1}$

Divide among 3 people

Each outcome compounds to a $k+2$ bit number with exactly two 0

$\binom{k+2}{2}$

Divide k identical item among n people

$\binom{n-1+k}{n-1}$ $=$ number of possible division of k identical items among n people

$\binom{n-1+k}{n-1} = \binom{n-1+k}{k}$

Example

Consider the equation

$x_1 + x_2 + \ldots + x_n = k$

Non-negative integer $x_1, \ldots, x_n$ that satisfy the equation above.

<u>How many such solutions are possible?</u>

$\binom{n-1+k}{n-1}$

Example

Unordered sampling with replacement

$S = \{a, b, c\}$

Sample this set k times with repitition allowed

Outcome is triplet (x_a, x_b, x_c)

$x_a + x_b + x_c = k$ where x_a is the number of times a was picked, x_b is the number of times b was picked, x_c is the number of times c was picked

Number of possible outcomes$=\binom{k+2}{2}$

Ordered sampling with replacement

$S = \{a, b, c\}$

Sample this set k times with repitition allowed

Outcome is the sequence of items selected

E.g., $aabb \ldots b$

$S = \{a, b, c\}$

Sampling without repetition, 3 times

Number of ordered triplets$= 3 \times 2 \times 1$

Picking a subset of size 3

Number of subsets$= \binom{3}{3} = 1$

$S = \{a, b, c\}$

Sampling with repetition, ordered k-tuple

Number of order k-tuple$= 3^k$

Number of such tuples with 1a, 1b, $k-2\ \ c$

$\frac{k!}{1!1!(k-2)!}$

Example

Roll a 6-sided die n times

Outcome is the sequence of number obtained

Number of possible outcomes$= 6^n$

Number of possible outcomes with k_1 ones, k_2 twos, ... k_6 sixes

$\binom{n}{k_1, k_2, \ldots, k_6}$

Number of possible outcomes with x even numbers and $n-x$ odd numbers

---

Sequential experiment with independent sub-experiments

Toss a coin n times $(n=3)$

$\Omega = \{HHH, TTT, \ldots\}$

$P(\{HTH\}) = P(\{\text{Heads on 1st toss}\}\{\text{Tails on 2nd toss}\}\{\text{Heads on 3rd toss}\})$

Assume Independence

$= P(\text{Heads on 1st toss})P(\text{Tails on 2nd toss})P(\text{Heads on 3rd toss})$

$= p(1-p)p$

$P(\text{Heads on } k^{th} \text{ toss}) = p$

$P(\text{Tails on } k^{th} \text{ toss}) = 1 - p$

Example

Roll a 4-sided die 3 times.

Rolls are independent. The die is fair.

$P((\text{Even number of 1st roll}) \bigcap (\text{Even number of 2nd roll}) \bigcap (\text{4 on 3rd roll}))$

$= P((\text{Even number of 1st roll}) P(\text{Even number of 2nd roll}) P(\text{4 on 3rd roll})$

$= 1/2 \cdot 1/2 \cdot 1/4$

Suppose we have a sequential experiment with n independent sub-experiments

This means that if we consider any n events $A_1, \ldots, A_n$

where A_i occurrence depends only on the outcomes of ith sub-experiments

Then, $A_1, \ldots, A_n$ are independent

Example

Coin toss n times

Each toss is H with probability p and T with probability $(1-p)$

Coin tosses are independent

$P(HHTT\ldots T) = p \cdot p \cdot (1-p) \ldots \cdot (1-p)$

$= p^2(1-p)^{n-2}$

$P(THHT\ldots T) = p^2(1-p)^{n-2}$

$P(\text{a sqeunce of length } n \text{ with } k\ H, (n-k)\ T) = p^k(1-p)^{n-k}$

Probability that we get k heads and $n-k$ tails

$= \sum_{\text{sequence with } k\ H, (n-k)\ T)} P(\text{sequence})$

$= \sum_{\text{sequence with } k\ H, (n-k)\ T)} p^k(1-p)^{n-k}$

$= \text{sequence with k H, (n-k) T}\ p^k(1-p)^{n-k}$

$= \binom{n}{k} p^k(1-p)^{n-k}$

$P(\{\text{Getting } k \text{ heads in } n \text{independent coin tosses}\}) = \binom{n}{k} p^k(1-p)^{n-k}$

$k = 0, 1, 2, \ldots n$

Binomial probability law

Example

n trials of a new drug

Each trial is a success with probability or a failure with probability $1-p$

Trials are independent

$P(\{\text{Get exactly } k \text{ success in } n \text{ trials}\}) = \binom{n}{k} p^k (1-p)^{n-k}$

Example

100 bits are transmitted over a noisy channel. The transmission of each bit are independent.

Each bit is received correctly with probability λ or incorrectly with probability $1-\lambda$

$P(\{\text{Get } k \text{ bits correctly}) = \binom{100}{k} \lambda^k (1-\lambda)^{100-k}$

Example

A coin is tossed infinitely many times. Coin tosses are independent

For each toss, $H \to p, T \to 1-p \, , 0 \leq p \leq 1$

Probability that the first H appears on the m^{th} toss

$m=1$	p
$m=2$	$(1-p)\cdot p$

Probability that first H on m^{th} toss

$= P(\{TT \ldots TH\}) = (1-p)^{m-1} p$ where $m = 1, 2, 3, \ldots$

Exercise: Probability that it takes more than m tosses to get the first H

$P(\{\text{first } H \text{ on } m+1 \text{ toss}\} \bigcup \{\text{first } H \text{ on } m+2 \text{ toss}\} \bigcup \{\ldots\})$

Additivity since disjoint events

$= \sum_{k=m+1}^{\infty} P(\text{first } H \text{ on } k^{th} \text{toss})$

$= \sum_{k=m+1}^{\infty} P(TT \ldots H)$

$= \sum_{k=m+1}^{\infty} (1-p)^{k-1} p$

$= (1-p)^m p + (1-p)^{m+1} p + \ldots$

$= \frac{(1-p)^m p}{1-(1-p)} = (1-p)^m$

$P(\{TTT \ldots T\}) = (1-p)^m$

Example

Roll a 6-sided die 7 times. Rolls are independent

Once each roll, probability of getting k, $(k = 1, 2, \ldots, 6)$ is p_k

$p_1 + p_2 + \ldots + p_6 = 1$

Probability of getting this sequence (1123456)$= p_1 \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6$

Probability of getting (2311564) $= p_1 \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot p_6$

Probability of getting 2 ones, 1 two, 1 three, 1 four, 1 five, 1 six?

$= \sum_{\text{sequence with 2 ones, 1 twos, ...}} P(\text{sequence})$

$= \{\text{Number of sequence with 2 ones, 1 twos, ...}\} \cdot p_1^2 p_2 \ldots p_6$

$= \frac{7!}{2!1!\ldots 1!} p_1^2 p_2 \ldots p_6$

Sequential experiment with n independent sub-experiments

On each sub-experiment, there are m possible outcomes with probability $p_1, p_2, \ldots p_m$

$\sum_{i=1}^{m} p_i = 1$

Probability(Outcome 1 happens k_1 times, Outcome 2 happen k_2 times, ... Outcome m happens k_m times)

$k_1 + k_2 + \ldots k_m = n$

$= \{\text{Number of valid sequences}\} p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$

$= \binom{n}{k_1, k_2, \ldots k_m} p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$

Multinomial probability Law

Random Variable

6-sided die

Roll it 3 times

Number of sequence with 1 even 2 odds

Pick 1 position for even number: $\binom{3}{1}$

Pick an even number to put in the selected position: 3

Pick 2 odd number to put in the other 2 position: 3^2

$\binom{3}{1} \cdot 3 \cdot 3^2 = 3^4$

6-sided die

n rolls, k even numbers, $n - k$ odd numbers

Pick k positions for the evens: $\binom{n}{k}$

Pick k even numbers: 3^k

Pick $(n-k)$ odds: 3^{n-k}

$\sum_{k=0}^{n}\binom{n}{k}3^n = \sum_{k=1}^{n}\text{Number of sequence with }\ k\text{ events} = \text{Total number of sequence}$

$3^n\sum_{k=0}^{n}\binom{n}{k} = 6^n$

$3^n2^n = 6^n$

Week 4 Session 1

Suppose we have a sequential experiment with n independent sub-experiments

Consider event $A_1, A_2, \ldots A_n$

where $A_i's$ occurrence depends only on outcome of i^{th} sub-experiment

Then $A_1, A_2, \ldots A_n$ are independent

Example

Coin toss n times

Each toss is H with probability p and T with probability $(1-p)$

Coin tosses are independent

$P(\text{Getting } k \text{ heads in } n \text{ independent coin tosses}) = p^k(1-p)^{n-k}$

$k = \{0, 1, 2, \ldots n\}$

Binomial probability law

$A = \{\text{first } k \text{ out of } n \text{ coin tosses are } H\}$

$B = \{\text{There were }\} kH \text{ in } n \text{ coin tosses}$

$P(A|B) = \frac{P(A\bigcap B)}{P(B)} = \frac{p^k(1-p)^{n-k}}{\binom{n}{k}\ p^k(1-p)^{n-k}} = \frac{1}{\binom{n}{k}}$

$A\bigcap B = HH\ldots HT\ldots T$ where there are $kH, (n-k)T$

$\sum_{k=0}^{n}\binom{n}{k}\ p^k(1-p)^{n-k} = 1$

Probabilistic way:

$\sum_{k=0}^{n} P(\{\text{Getting } kH \text{ in tosses}\})$

$= P(\{\text{Getting 0H}\}\bigcap\{1H\}\ldots\bigcap nH)$

$= P(\Omega) = 1$

Algebraic way:

Binomial expansion:

$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}$

where $a = p, b = 1 - p$

$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1$

say $p = 1/2$

$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^n} = 1$

$\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Random Variables

Probability model $\Omega, \mathcal{F}, p$

Definition: A random variable (RV) is a function from Ω to real number $\mathbb{R}$

$X : \Omega \to \mathbb{R}$

Example

Coin toss $n = 2$ times with independent tosses

$X(\omega) = \{\text{Number of } \ H \text{ in } \omega\}$

Example

$H : \Omega \to \mathbb{R}$

$H(\omega) = \{\text{Height of selected student}\}$

Remark

1. X is a function

   Fix ω, then $X(\omega)$ is a fixed real number
2. X may be a many-to-one function

   It is possible to have $X(\omega_1) = X(\omega_2)$ for some $\omega_1 \neq \omega_2$

Events in terms of RVs

$X : \omega \to \mathbb{R}$

$A = \{\omega \in \Omega : X(\omega) = 2\}$ is an event

$B = \{\omega \in \Omega : 0 \leq X(\omega) \leq 2\}$

$C = \{\omega \in \Omega : sin(X(\omega)) > 0\}$

$P(\{\omega \in \Omega : X(\omega) = 2\})$

Notation: $P(X = 2)$

$P(\{\omega \in \Omega : X(\omega) \leq 2\})$

Notation: $P(X \leq 2)$

$P(\{\omega \in \Omega : X(\omega) \in (a, b)\})$

Notation: $P(X \in (a, b)) = P(a < x < b)$

Example

n coin toss, independent toss

Probability of $H = p$

Probability of $T = 1 - p$

$X(\omega) = \text{Number of } H \text{ in } \omega$

$P(X = 0) = P(\{\omega \in \Omega : X(\omega) = 0\})$

$= P(\{TT \dots T\})$

$= (1 - p)^n$

$P(X \leq 1) = P(\{T \dots T, HT \dots T, TH \dots T, \dots\})$

$= (1 - p)^n + np(1 - p)^{n-1}$

$P(X \leq 1) = P(\{X = 0\} \bigcap \{X = 1\})$

$= P(X = 0) + P(X = 1)$

$= P(\text{Getting 0H}) + P(\text{Getting 1H})$

$= \binom{n}{0} p^k 0 (1 - p)^{n-0} + \binom{n}{1} p^1 (1 - p)^{n-1}$

$= (1 - p)^n + np(1 - p)^{n-1}$

Example

Roll a 4-sided die two times

Rolls are independent. Die is fair.

$\Omega = \{(1, 1), (1, 2), \dots (4, 4)\}$

$z = \text{Number on the 2nd roll}$

$P(z = 1) = P(\text{Get 1 on 2nd roll}) = 1/4$

$P(\{1, 1\} = P(\text{1 on first}) P(\text{1 on second}) = 1/4 \cdot 1/4 = 1/16$

$P(\{(1, 1), (1, 2), \dots (4, 4)\}) = 4 \cdot 1/16 = 1/4$

$X =$ minimum if the numbers on the 2 roll

$P(X = 4) = P(\{4, 4\}) = 1/16$

$P(X = 2) = P(\{2, 2\}, \{2, 3\}, \{2, 4\}, \{3, 2\}, \{4, 2\}) = 5/16$

Exercise: $P(X=3), P(X=1)$

Discrete RVs

Range of RV $X =$ all possible values $X(\omega)$ can take

Coin toss example $X = \text{number of } H$

Range of X: $S_x = \{0, 1, 2.., , n\}$

Definition: A RV X is discrete if S_x is a finite or a countably infinite set.

Example

Coin with probability $H = p, T = 1-p$

Independent coin tosses.

Keep tossing until a H appears

$N = \text{Number of tosses until an } H \text{ appears}$

$S_N = \{1, 2, 3, \ldots\}$

N is a discrete RV

$P(N=1) = p$

$P(N=k) = (1-p)^{k-1}p$ where $k \geq 2$

$S_x = \{x_1, x_2, \ldots x_n\}$

$S_x = \{x_1, x_2, x_3, \ldots\}$

Notations: X, Y, Z to indicate RV

$P(X=x)$

x, y, z indicates real number/ values that a RV may take

Probability Mass Function (PMF)

$S_x = \{x_1, x_2, \ldots x_n\}$

$P_X(x_i) = P(X = x_i) = P(\{\omega \in \Omega : X(\omega) = x_i\})$

1. $0 \leq P_X(x_i) \leq 1$
2. $S_x = \{x_1, x_2, \ldots x_n\}$

   $a \neq S_x$

   $P(X=a) = P(\{\omega \in \Omega : X(\omega) = a\}) = P(\phi) = 0$
3. $P(\{X = x_1\} \bigcap \{X = x_2\}) = 0$

   Disjoint Events
4. $P(\{X = x_1\} \bigcup \{X = x_2\} \bigcup \ldots \{X = x_n\}) = P(\Omega) = 1$

$\sum_{i=1}^{n} P(X = x_i)$

$\sum_{i=1}^{n} P_X(x_i)$

$\sum_{i=1}^{n} P_X(x_i) = 1$

PMF add up to 1

5. $\{X = x_1\}, \{X = x_2\}, \ldots, \{X = x_n\}$ form a partition of Ω
6. Let (a, b) be an interval in $\mathbb{R}$

   $P(X \in (a, b))$

   Law of total probability: $\sum_{i=1}^{n} P(X \in (a, b) \bigcap X = x_i)$

   $= \sum_{i:x_i \in (a,b)} P(X \in (a, b) \bigcap X = x_i) + \sum_{i:x_i \notin (a,b)} P(X \in (a, b) \bigcap X = x_i)$

   where $\sum_{i:x_i \notin (a,b)} P(X \in (a, b) \bigcap X = x_i) = 0$

   $= \sum_{i:x_i \in (a,b)} P(X = x_i)$

   $= \sum_{i:x_i \in (a,b)} P_X(x_i)$

   Therefore: $P(X \in (a, b)) = \sum_{i:x_i \in (a,b)} P_X(x_i)$

$P(X \in (1, 2) \bigcup X \in (4, 5))$

$= P(X \in (1, 2)) + P(X \in (4, 5))$

$= \sum_{x_i \in (1,2) P_X(x_i)} + \sum_{x_i \in (4,5) P_X(x_i)}$

$= \sum_{x_i \in (1,2) \bigcup (4,5)} P_X(x_i)$

For any arbitrary subset B pf the real line

$P(X \in B) = \sum_{i:x_i \in B} P_X(x_i)$

Example

Coin toss n times independent.

$X = \text{Number of } H \text{ that appear}$

$S_x = \{0, 1, \ldots, n\}$

Binomial RV:

$P_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}$ where $0 \leq k \leq n$

$X \sim \text{Binomial}(n, p)$

$X \sim \text{Binomial}(10, 1/2)$

$P_X(k) = \binom{10}{k} \frac{1}{2^k} \frac{1}{2^{n-k}}$ where $0 \leq k \leq 10$

$P(X \geq 9) = P_X(9) + P_X(10)$

$P(X < 0.8) = P_X(0)$

$P(|X - 5| \leq 1) = P_X(4) + P_X(5) + P_X(6)$

Example

If $X \geq n-1$, then I win \$100.

Otherwise if $X < n-1$, I lose \$10

$Y =$ my earnings, $y = g(x)$

<u>What is the PMF of y?</u>

$S_y = \{100, -10\}$

$P_Y(100) = P(y = 100) = P(X \geq n-1) = P_X(n-1) + P_X(n)$

$P_Y(-10) = 1 - P_Y(100) = 1 - P_X(n-1) - P_X(n)$

Example

Keep repeating independent coin tosses until H

$N =$ Number of coin tosses until H

<u>What is the PMF of N</u>

$S_N = \{1, 2, 3, \dots\}$

For $k \geq 1$

$P_N(k) = P(N = k) = (1-p)^{k-1}p$

Geometric RV

$X \sim \text{Geometric}(p)$

$X \sim \text{Binomial}(n, p)$

$P_X(k) = \binom{n}{k}p^k(1-p)^{n-k}$

Example

Roll a fair 4-sided die.

$X =$ Number that appears

$S_X = \{1, 2, 3, 4\}$

$P_X(k) = P(X = K) = 1/4$

Uniform RV

Uniform RV with range $= \{1, 2, \dots, m\}$ where $1 \leq k \leq m$

$P_X(k) = 1/m$

Uniform RV with range $= \{-m, -(m-1), \dots, m\}$ where $-m \leq k \leq m$

$P_X(k) = 1/(2m+1))$

Week 4 Session 2

Random Variable

$X : \Omega \to \mathbb{R}$

for each $\omega \in \Omega$, $X(\omega)$ is a real number

Events:

$P(a < X \leq b) = P(\{\omega \in \Omega : a < X(\omega) \leq b\})$

Discrete RV:

finite or countably infinite range S_X

$S_x = \{x_1, x_2, \dots x_n\}$

$S_x = \{x_1, x_2, x_3, \dots\}$

$S_X = \{\text{All possible values of } X(\omega) \text{ for all } \omega \in \Omega\}$

PMF of X:

$P(X \in B) = \sum_{i:x_i \in B} P_X(x_i)$

$P(a < X \leq b) = \sum_{i:a<x_i \leq b} P_X(x_i)$

$P(ax^3 + bx^2 + cx > 0) = \sum_{i:ax_i^3+bx_i^2+cx_i>0} P_X(x_i)$

Example

$X \sim \text{Binomial}(n, p)$

$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ where $0 \leq k \leq n$

$P(X \geq 1) = \sum_{k \geq 1} P_X(k)$

$= \sum_{k \geq 1} \binom{n}{k} p^k (1-p)^{n-k}$

or

$= 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - \binom{n}{0} p^0 k (1-p)^{n-0}$

$= 1 - (1-p)^n$

Bernoulli RV

$S_X = \{0, 1\}$

$P_X(1) = p, P_X(0) = 1 - p$

$X \sim \text{Bernoulli}(p)$

Example

$X : \Omega \to \mathbb{R}$

$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$

Indicator RV for event A, I_A

$P_X(1) = P(X = 1)$

$= P(\{\omega \in \Omega : X(\omega) = 1\})$

$= P(A)$

$P_X(0) = 1 - P_X(1) = 1 - P(A) = P(A^c)$

Poisson RV

$S_X = \{0, 1, 2, 3 \ldots\}$

$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k = 0, 1, 2 \ldots$

Here, λ is a fixed positve number

$X \sim \text{Poisson}(\lambda)$

$P_X(k) \geq 0$

$\sum_{k \geq 0} P_X(k) = \sum_{k \geq 0} \frac{e^{-\lambda}\lambda^k}{k!}$

$= e^{-\lambda} \sum_{k \geq 0} \frac{\lambda^k}{k!}$

$= e^{-\lambda} e^{\lambda} = 1$

$P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - \frac{e^{-\lambda}\lambda^0}{0!}$

$= 1 - e^{-\lambda}$

Example

Z is a discrete RV

$P(Z \leq 1) \leq P(Z \leq 2)$

where $P(Z \leq 1) = 0.1, P(Z \leq 2) = 0.2$

$P(Z \leq 2) = P(\{Z \leq 1\} \bigcup \{1 < Z \leq 2\})$

$= P(Z \leq 1) + P(1 < Z \leq 2)$

$P(1 < Z \leq 2) = 0.2$

Expected Value of RV

Definition: X is a discrete RV

$E[X] = \sum_{x \in S_X} x P_X(x)$

$S_X = \{x_1, x_2, \ldots x_n\}$

$E[X] = \sum_{x_i \in S_X} x_i P_X(x_i)$, finite value

$S_X = \{x_1, x_2, \ldots\}$

$E[X] = \sum_{i=1}^{\infty} x_i P_X(x_i)$

It is possible that this infinite series doesn't converge. In that case, $E[X]$ is undefined.

Example

$X \sim \text{Bernoulli}(p)$

$S_X = \{0, 1\}$

$P_X(1) = p, P_X(0) = 1 - p$

$E[X] = 0 \cdot (1 - p) + 1 \cdot p = p$

Example

$X \sim \text{Uniform with range } S_X = \{0, 1, \ldots, (M-1)\}$

$P_X(k) = 1/M, 0 \leq k \leq M - 1$

$E[X] = \sum_{k=0}^{M-1} k \cdot 1/M$

$= 1/M \cdot \sum_{k=0}^{M-1} k$

$= 1/M \cdot \frac{M(0+M-1)}{2}$

$= \frac{M-1}{2}$

In general of PMF is symmetric about same number c

$P_X(c + a) = P_X(c - a), \forall a \in \mathbb{R}$

Then $E[X] = c$

Example

$X \sim \text{Binomial}(3, 0.5)$

$P_X(k) \binom{3}{k} \frac{1}{2^k} (1/2)^{3-k}$, $k = 0, 1, 2, 3$

$E[X] = \sum_{k=0}^{3} k \cdot P_X(k)$

$= 1 \cdot \binom{3}{1} \frac{1}{2^3} + 2 \cdot \binom{3}{2} \frac{1}{2^3} + 3 \cdot \binom{3}{3} \frac{1}{2^3} = 1.5$

Result:

$X \sim \text{Binomial}(n, p)$

$E[X] = np$

Interpretation of $E[X]$

$S_X = \{1, 2, 4\}$

$P_X(1) = 1/4, P_X(2) = 1/2, P_X(4) = 1/4$

N independent repetitions of the underlying random experiment and record the value of X in each experiment

$2, 2, 4, 1, 2, 1, 2, 4, \ldots$

Average value$= \frac{\text{Add up all numbers}}{N}$

$= \frac{(\text{number of times } 1) \cdot 1 + (\text{number of times } 2) \cdot 2 + (\text{number of times } 4) \cdot 4}{N}$

$= \sum_{x \in S_X} x \cdot \frac{\text{number of times } x \text{ appears}}{N}$

As $N \to \infty$, $\text{number of times } x \text{ appears} \to P_X(x)$

Empirical average $\to \sum_{n \in S_X} x \cdot P_X(x) = E[X]$

$X \sim \text{Binomial}(n, p), E[X] = np$

$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \leq k \leq n$

Binomial expansion

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$E[x] = \sum_{k=0}^{n} k P_X(k)$

$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$

$= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$

$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k}$

$= \sum_{k=1}^{n} n \frac{(n-1)!}{(k-1)!((n-1)-(k-1)!)} p^k (1-p)^{n-k}$

say $j = k - 1$

$= n \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} p^{j+1} (1-p)^{(n-1)-j}$

$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j}$

where $\binom{n-1}{j} p^j (1-p)^{(n-1)-j}$ is a term from Binomial $(n-1, p)$

$= np(p + 1 - p)^{n-1}$

$= np$

$X \sim \text{Geometric}(p), p > 0$

$P_X(k) = (1-p)^{k-1} p$, $k = 1, 2, 3, \ldots$

say $q = 1 - p$

$P_X(k) = q^{k-1}p$

$E[X] = \sum_{k=1}^{\infty} kq^{k-1}p$

$= p\sum_{k=1}^{\infty} kq^{k-1}$

$= p\frac{1}{(1-q)^2}$

$= 1/p$

Mathematics used:

$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$, $0 < a < 1$

$\sum_{k=1}^{\infty} ka^{k-1} = \frac{1}{(1-a)^2}$, $0 < a < 1$

$\sum_{k=1}^{\infty} ka^k = \frac{a}{(1-a)^2}$, $0 < a < 1$

Example

$X \sim \text{Poisson}(\lambda)$

$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$, $k = 0, 1, 2, 3...$

$E[X] = \sum_{k\geq 0} k\frac{e^{-\lambda}\lambda^k}{k!}$

$= \sum_{k\geq 1} k\frac{e^{-\lambda}\lambda^k}{k!}$

$= \sum_{k\geq 1} \lambda\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}$

$= \lambda e^{-\lambda}\sum_{k\geq 1} \frac{\lambda^{k-1}}{(k-1)!}$

where $\sum_{k\geq 1} \frac{\lambda^{k-1}}{(k-1)!} = e^{\lambda}$

$= \lambda$

Example

$S_X = 3$ constant RV, degenerate RV

$P_X(3) = 1$

$E[X] = \sum_{x\in S_X} xP_X(x) = 3 \cdot 1 = 3$

$E[3] = 3$

In general $E[a] = a$ where $a \in \mathbb{R}$

Example

$S_X = \{1, 2, 3, \dots\}$

$P_X(k) = \frac{1}{ck^2}$ where $k = 1, 2, \dots$, c is a positive constant

$\sum_{k\geq 1} \frac{1}{ck^2} = 1$

$\frac{1}{c}\sum_{k\geq 1}\frac{1}{k^2}=1$

$\sum_{k\geq 1}\frac{1}{k^2}=c$

where $\sum_{k\geq 1}\frac{1}{k^2}\rightarrow\frac{\pi^2}{6}$

$E[X]=\sum_{k\geq 1}kP_X(k)$

$=\frac{1}{c}\sum_{k\geq 1}\frac{k}{k^2}$

$=\frac{1}{c}\sum_{k\geq 1}\frac{1}{k}$

where $\sum_{k\geq 1}\frac{1}{k}\rightarrow\infty$

$E[X]=\infty$

Remarks

If S_X is countably infinite and it included both positive and negative values

$E[X]=\sum_{x\in S_X}xP_X(x)$

$=\sum_{x\in S_X,x\geq 0}xP_X(x)+\sum_{x\in S_X,x<0}xP_X(x)$

If $\sum_{x\in S_X,x\geq 0}xP_X(x)\rightarrow\infty$

If $\sum_{x\in S_X,x<0}xP_X(x)\rightarrow-\infty$

$E[X]=\infty-\infty$ which is undefined

Properties of $E[X]$

$S_X=\{x_1,x_2,\ldots\}$

Suppose $x_i\geq a$ for $i=1,2..\,.$

$E[X]\geq a$

$E[X]=\sum_{x\in S_X}xP_X(x)=\sum_{x\in S_X}aP_X(x)=a\sum_{x\in S_X}P_X(x)=a$

Similarly if $x_i\leq b\forall x_i\in S_X$

$E[X]\leq b$

Function of a RV

$S_X=\{-3,-1,1,3\}$

$X\sim\text{uniform over the range }\ S_X$

$E[X]=0,Y=X^2$

$S_Y=\{1,9\}$

$P_Y(y)$

$P_Y(1)=P(Y=1)=P(X^2=1)=P_X(1)+P_X(-1)=1/2$

$P_Y(9)=1-P_Y(1)=1/2$

$E[Y] = 1 \cdot 1/2 + 9 \cdot 1/2 = 5$

Result

If $y = g(x)$, then

$E[Y] = \sum_{x \in S_X} g(x) P_X(x)$

$E[Y] = E[X^2] = 5 \neq (E[X])^2$

In general $E[g(x)] \neq g(E[X])$

Proof of result

$Y = g(x)$, $S_y = \{y_1, y_2, \dots y_m\}$

$E[Y] = \sum_{y \in S_y} y P_Y(y)$

$P_Y(y) = P(Y = y) = P(g(x) = y)$

$= \sum_{x_i : g(x_i) = y} P_X(x_i)$

So, $E[Y] = \sum_{y \in S_y} y (\sum_{x_i : g(x_i) = y} P_X(x_i))$

RHS:

$\sum_{x \in S_X} g(x) P_X(x)$

$= \sum_{x \in S_X : g(x) = y_1} g(x) P_X(x) + \sum_{x \in S_X : g(x) = y_2} g(x) P_X(x) + \dots \sum_{x \in S_X : g(x) = y_m} g(x) P_X(x)$

$= \sum_{x \in S_X : g(x) = y_1} y_1 P_X(x) + \sum_{x \in S_X : g(x) = y_2} y_2 P_X(x) + \dots \sum_{x \in S_X : g(x) = y_m} y_m P_X(x)$

$= y_1 \sum_{x \in S_X : g(x) = y_1} P_X(x) + y_2 \sum_{x \in S_X : g(x) = y_2} P_X(x) + \dots y_m \sum_{x \in S_X : g(x) = y_m} P_X(x)$

Week 5 Session 1

Expected value of RV

Definition: X is a discrete RV

$E[X] = \sum_{x \in S_X} x P_X(x)$

Function of a RV

If $y = g(x)$

Then $E[Y] = \sum_{x \in S_X} g(x) P_X(x)$

Why?

By definition of $E[Y]$

$E[Y] = \sum_{y \in S_Y} y P(Y = y)$ where $P(Y = y) = P(g(x)) = y)$

$= \sum_{y \in S_Y} y (\sum_{x : g(x) = y} P_X(x))$

$= y_1 \sum_{x : g(x) = y_1} P_X(x) + y_2 \sum_{x : g(x) = y_2} P_X(x) + \dots$

Moreover

$\sum_{x \in S_X} g(x)P_X(x)$ rearrange terms equals to the above

---

Example

X uniform $S_X = \{-3, -1, 1, 3\}$

$E[X] = 0$

$Y = g(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$

$E[Y] = \sum_{x \in S_X} g(x)P_X(x) = 1 \cdot 1/4 + 1 \cdot 1/4 + 0 \cdot 1/4 + 0 \cdot 1/4 = 1/2$

$g(E[X]) = 0 \neq E[g(x)]$

Linearity properties of Expectation

1. $E[aX + b] = aE[X] + b$

   $Y = aX + b$

   $E[Y] = \sum_{x \in S_X}(ax + b)P_X(x)$

   $= \sum_{x \in S_X} a_x P_X(x) + \sum_{x \in S_X} bP_X(x)$

   $= aE[X] + b$
2. $E[ag(x) + bh(x) + c]$

   $= aE[g(x)] + bE[h(x)] + c$

   $Z = ag(x) + bh(x) + c$

   $E[Z] = \sum_{x \in S_X}(ag(x) + bh(x) + c)P_X(x)$

   $= a\sum_x g(x)P_X(x) + b\sum h(x)P_X(x) + c\sum P_X(x)$

   $= aE[g(x)] + bE[h(x)] + c$

Example

$E[X] = 0$,mean of X, mean value of X, 1^{st} moment of X

$E[X^2] = 5$, 2^{nd} moment of X

$E[X^n]$, n^{th} moment of X

$Y = (2x + 10)^2$

$E[Y] = E[(2X + 10)^2]$

$= E[4X^2 + 40X + 100]$

$= 4E[X^2] + 40E[X] + 100$

$= 120$

Example

$X \sim \text{Binomial}(48, 1/3)$, $S_X = \{0, 1, \ldots, 48\}$

X is the number of voice packets that need to be transmitted.

Transmitter can only send 20 packets.

If X is more than 20, the excess packets are discarded.

Expected number of discarded packets

$Y = g(x) = \begin{cases} x - 20 & \text{if } x > 20 \\ 0 & \text{if } x \leq 20 \end{cases}$

$E[Y] = \sum_{x \in S_X} g(x) P_X(x)$

$= \sum_{k=0}^{20} 0 \cdot P_X(x) + \sum_{k=21}^{48} (k - 20) \cdot P_X(k)$

$= \sum_{k=21}^{48} (k - 20) \binom{48}{k} (\frac{1}{3})^k (\frac{2}{3})^{48-k}$

$= 0.182$

$E[X] = 0$

$Var(X) = E[(X - 0)^2]$

$= E[X^2] = (1)^2 \cdot 1/2 + (-1)^2 \cdot 1/2$

$= 1$

$E[Y] = 0$

$Var[Y] = E[(Y - 0)^2]$

$= E[Y^2] = 100$

Variance of a RV

$E[(X - m_X)^2]$

Variance of X : $Var(X)$ or σ_X^2

Standard Deviation

$\sigma_X = +\sqrt{Var(X)}$

Example

$X \sim \text{Bernoulli}(p)$

$S_X = \{0, 1\}$

$P_X(1) = p, P_X(0) = 1 - p$

$E[X] = 0 \cdot P_X(0) + 1 \cdot P_X(1) = p$

$Var[X] = E[(X-p)^2]$

$= (0-p)^2 P_X(0) + (1-p)^2 P_X(1)$

$= p^2(1-p) + (1-p)^2 p$

$= p(1-p)$

Example: Degenerate/ Constant RV

$S_X = \{c\}$

$E[X] = c \cdot 1 = c$

$Var(X) = E[(X-c)^2]$

$= \sum_{x \in S_X} (x-c)^2 P_X(x) = 0$

$Var(X) = E[(X-m_X)^2] = \sum_{x \in S_X} (x-m_X)^2 P_X(x)$

$Var(X) = E[(X-m_X)^2] = E[X^2 - 2m_X X + m_X^2]$

$= E[X^2] - 2m_X E[X] + m_X^2$

$= E[X^2] - 2m_X m_X + m_X^2$

$= E[X^2] - m_X^2$

$E[ag(x) + bh(x) + c] = aE[g(x)] + bE[h(x)] + c$ applied

$Var(X) = E[X^2] - m_X^2$

Example

$Y = X + c$

$E[Y] = E[X+c] = E[X] + c$ where $E[X] = m_X$

$Var(Y) = E[(Y-m_Y)^2]$ where $m_Y = m_X + c$

$= E[(x + c - (m_X + c))^2]$

$= E[(X-m_X)^2]$

$= \sigma_X^2$

If $Y = X + c$, then $Var(Y) = Var(X)$

Example

$Y = c \cdot X$

$E[Y] = E[cX] = cE[X] = cm_X$

$Var(Y) = E[(Y-m_Y)^2]$

$= E[(cX - cm_X)^2]$

$= E[c^2(X-m_X)^2]$

$= c^2E[(X - m_X)^2]$

$= c^2Var(X)$

If $Y = cX$, then $Var(Y) = c^2Var(X)$

If $Y = -X$, then $Var(Y) = Var(X)$

Example

X, m_X, σ_X^2

$Y = \frac{X - m_X}{\sigma_X}$

$E[Y] = E[\frac{1}{\sigma_X}(X - m_X)]$

$= \frac{1}{\sigma_X}E[(X - m_X)]$

$= \frac{1}{\sigma_X}(E[X] - m_X)$

$= 0$

$Var(Y) = E[(Y - 0)^2]$

$= E[Y^2]$

$= E[\frac{(X - m_X)^2}{\sigma_X^2}]$

$= E[\frac{1}{\sigma_X^2}(X - m_X)^2]$

$= \frac{1}{\sigma_X^2}E[(X - m_X)^2]$

$= \frac{\sigma_X^2}{\sigma_X^2} = 1$

$\frac{X - m_X}{\sigma_X}$ $\rightarrow$ Normalized form of X has mean 0 and variance 1

Conditional probability

Bayes' Rule

Total probability law

Let C be any event with $P(C) > 0$

$P(A|C) = \frac{P(A \bigcap C)}{P(C)}$

$A = \{X = x\} = \{\omega \in \Omega : X(\omega) = x\}$

$P(A|C) = P(X = x|C)$

$= \frac{P(\{X=x\} \bigcap C)}{P(C)}$

$P_X(x|C) := \frac{P(\{X=x\} \bigcap C)}{P(C)} = P(X = x|C)$

We can use the above definition for all $x \in S_X$

$S_X = \{x_1, x_2, \ldots, x_n\}$

$P_X(x_1|C), P_X(x_2|C), \ldots$ are conditional PMF of X given C

Results

1. $0 \leq P_X(x_i|C) \leq 1$
2. $\sum_{x \in S_X} P_X(x|C) = 1$
3. a) $P(x \in (a,b)|C) = \sum_{x \in (a,b)} P_X(x|C)$

   b) If B is any subset of $\mathbb{R}$

   $P(X \in B|C) = \sum_{x \in B} P_X(x|C)$

$S_X = \{x_1, \ldots, x_n\}$

$\sum_{i=1}^{n} P(X = x_i|C) = P(\Omega|C) = 1$

Example

X has uniform PMF with $S_X = \{1, 2, \ldots, L\}$ for $1 \leq k \leq L$

$P_X(k) = \frac{1}{L}$

$C = \{X > 1\}$

Conditional PMF of X given C

$P_X(1|C) = P(X = 1|X > 1) = \frac{P(X=1 \bigcap X>1)}{P(X>1)} = 0$

For $2 \leq k \leq L$

$P_X(k|C) = P(X = 2|X > 1) = \frac{P(X=2 \bigcap X>1)}{P(X>1)} = \frac{1/L}{1-1/L} = \frac{1}{L-1}$

Example

$X \sim \text{Geometric}(p)$

$S_X = \{1, 2, 3\ldots\}$

$P_X(k) = (1-p)^{k-1}p$ where $k \geq 1$

$C = \{X > 1\}$

$Y = X - 1$

Conditional PMF of Y given C

$P(Y = k|C) = P(Y = k|X > 1)$

$= P(X - 1 = k|X > 1)$

$= P(X = 1 + k|X > 1)$

$= \frac{P(X=1+k \bigcap X>1)}{P(X>1)}$

$= \begin{cases} \frac{0}{1-p} & k = 0 \\ \frac{P_X(1+k)}{1-p} & k = 1, 2, \ldots \end{cases}$

$P(Y = k|X > 1) = P(X = k)$

Exercise

$X \sim \text{Geometric}(p)$

$C = \{X > m\}$ where m is a positive integer

$Z = X - m$

Find the conditional PMF of Z given C

Chain Rule

$P(A \bigcap B) = P(A|B)P(B)$

Bayes' Rule

$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Law of total probability

$P(A) = \sum_{i=1}^{n} P(A \bigcap C_i)$

$= \sum_{i=1}^{n} P(A|C_i)P(C_i)$

Example

Factory that produce 2 types of devices

Type 1 is produced with probability α

Lifetime $\sim \text{Geometric}(r)$

Type 2 is produced with probability $1 - \alpha$
Lifetime $\sim \text{Geometric}(s)$

I purchase a device from this factory $X =$ lifetimes of purchased device

$S_X = \{1, 2, 3, \ldots\}$

PMF of X

$P_X(k) = P(X = k)$

$= P(X = k|\text{Type } 1)P(\text{Type } 1) + P(X = k|\text{Type } 2)P(\text{Type } 2)$

$= (1 - r)^{k-1} r\alpha + (1 - s)^{k-1} s(1 - \alpha)$

Verify that $P_X(k)$ adds up to 1 of summed over all $k \geq 1$

Suppose I observe that $X = m$

$P(\text{Type 1 device was purchased}|X = m) = \frac{P(X=m|\text{Type } 1)P(\text{Type } 1)}{P(X=m)}$

$= \frac{(1-r)^{m-1} r\alpha}{(1-r)^{m-1} r\alpha + (1-s)^{m-1} s(1-\alpha)}$

$P(\text{Type } 1|X > m) = \frac{P(X>m|\text{Type } 1)P(\text{Type } 1)}{P(X>m)}$

where $P(X > m|\text{Type } 1) = \sum_{k=m+1}^{\infty} (1 - r)^{k-1} r$

$P(X > m) = P(X > m|\text{Type } 1)\alpha + P(X > m|\text{Type } 2)(1 - \alpha)$

PMF	Expectation
$P_X(k)$	$E[X] = \sum_{x \in S_X} x P_X(x)$
Conditional PMF	Conditional Expectation
$P_X(x\|C)$	$E[X\|C] = \sum_{x \in S_X} x P_X(x\|C)$

$E[\text{lifetime}|\text{Type 1}] = \sum_{k=1}^{\infty} k P_X(k|\text{Type 1})$

$= \sum_{k=1}^{\infty} k(1-r)^{k-1} r$

Week 5 Session 2

1. $E[aX + b] = aE[X] + b$
2. $Var[X] = E[(X - m_X)^2] = E[X^2] - m_X^2$
3. Conditional PMF of RV X given event C

   $P_X(x|C)$ for $x \in S_X$

Using law of total probability with RVs

Suppose $C_1, C_2, \dots C_n$ is a partition of Ω, then

$P_X(x) = P(X = x)$

$= \sum_{i=1}^{n} P(\{X = x\} \bigcap C_i)$

$= \sum_{i=1}^{n} P(X = x|C_i) P(C_i)$

$= \sum_{i=1}^{n} P_X(x|C_i) P(C_i)$

Example

X is a discrete RV with range $S_X = \{x_1, x_2, \dots x_n\}$

For any real number c, the mean squared error is defined as follows

$f(c) = E[(X - c)^2]$

What choice of c given the minimum value of $f(c)$

$E[(X - c)^2] = \sum_{i=1}^{n} (x_i - c)^2 P_X(x_i)$

$f(c) = \sum_{i=1}^{n} (x_i - c)^2 P_X(x_i)$

$\frac{df(c)}{dc} = 0$

$\sum_{i=1}^{n} -2(x_i - c) P_X(x_i) = 0$

$\sum_{i=1}^{n} x_i P_X(x_i) = c \sum_{i=1}^{n} P_X(x_i)$

$c = m_X$

$\frac{d^2 f(c)}{dc} > 0$

$c = m_X$

$f(c) = f(m_X) = E[(X - m_X)^2] = Var(X)$

Conditional PMF given C

$P_X(x|C), x \in S_X$

Conditional expectation given C

$m_{X|c} = E[X|C] = \sum_{x \in S_X} xP_X(x|C)$

$E[g(X)|C] = \sum_{x \in S_X} g(x)P_X(x|C)$

Conditional variance given C

$E[(X - m_{X|C})^2|C] = \sum_{x \in S_X} (x - m_{X|C})^2 P_X(x|C)$

Law of total expectation

Theorem: Suppose $C_1, \ldots C_n$ is a partition of Ω. Then

$P_X(x) = \sum_{i=1}^{n} P_X(x|C_i)P(C_i)$

$E[X] = \sum_{i=1}^{n} E[X|C_i]P(C_i)$

Proof:

$E[X] = \sum_{x \in S_X} xP_X(x)$

$= \sum_{x \in S_X} x \sum_{i=1}^{n} P_X(x|C_i)P(C_I)$

$= \sum_{i=1}^{n} \sum_{x \in S_X} xP_X(x|C_i)P(C_i)$

$= \sum_{i=1}^{n} E[X|C_i]P(C_i)$

$Y = g(x)$

$E[Y] = \sum_{i=1}^{n} E[Y|C_i]P(C_i)$

$E[g(x)] = \sum_{i=1}^{n} E[g(x)|C_i]P(C_i)$

Example

C_1, C_2 form a partition of Ω

Given C_1. $X \sim \text{Geometric}(r)$

Given C_2. $X \sim \text{Geometric}(s)$

$Z \sim \text{Geometric}(p)$

$E[Z] = 1/p$

$E[Z^2] = \frac{1+p}{p^2}$

Find $Var(X)$

$Var(X) = E[X^2] - m_X^2$

$m_X = E[X] = E[X|C_1]P(C_1) + E[X|C_2]P(C_2)$

$= \frac{1}{r}P(C_1) + \frac{1}{s}P(C_2)$

$E[X^2] = E[X^2|C_1]P(C_1) + E[X^2|C_2]P(C_2)$

$= \frac{1+r}{r^2}P(C_1) + \frac{1+s}{s^2}P(C_2)$

$Var(X) = E[X^2] - (m_X)^2$

Example

n independent Bernoulli trials

Each trial may result in a success with probability p or a failure with probability $1-p$
$X =$ number of successes in n trials

$P_X(k) = \binom{n}{k}p^k(1-p)^{n-k}$ where $0 \leq k \leq n$

Suppose $X = 1$

What is the conditional probability that the first trial was successful?

$A_1 = \{\text{first trial is success}\}$

$P(A_1|X=1) = \frac{P(A_1 \bigcap X=1)}{P(X=1)}$

$= \frac{P(SFFFF...F)}{\binom{n}{1}p^1(1-p)^{n-1}}$

$= \frac{p^1(1-p)^{n-1}}{\binom{n}{1}p^1(1-p)^{n-1}}$

$= \frac{1}{n}$

$A_2 = \{\text{second trial is success}\}$

$P(A_2|X=1) = \frac{P(A_2 \bigcap X=1)}{P(X=1)}$

$= \frac{P(FSFFF...F)}{\binom{n}{1}p^1(1-p)^{n-1}}$

$= \frac{1}{n}$

Example

Given A_1, find the conditional PMF of X

$S_X = \{0, 1, \ldots, n\}$

$P_X(0|A_1) = P(X=0|A_1) = 0$

For $n \geq k \geq 1$

$P_X(k|A_1) = P(X=k|A_1) = \frac{P(X=k \bigcap A_1)}{P(A_1)}$

$P(A_1) = p$

$P(X=k \bigcap A_1) = P(A_1 \bigcap k-1 \text{ success trials } 2, 3, \ldots n)$

$= P(A_1)P(k-1 \text{ success trials } 2, 3, \ldots n)$

$= p\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}$

$P_X(k|A_1) = \binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}$ for $1 \le k \le n$

$P(X \ge n-1|A_1) = P_X(n-1|A_1) + P_X(n|A_1)$

$= \binom{n-1}{n-2}p^{n-2}(1-p)^1 + p^{n-1}$

$= (n-1)p^{n-2}(1-p) + p^{n-1}$

Conditional PMF of X given A_1^c

$S_X = \{0, 1, \ldots, n\}$

$P_X(n|A_1^c) = P(X = n|A_1^c) = 0$

For $0 \le k \le n-1$

$P_X(k|A_1^c) = P(X = k|A_1^c)$

$= \frac{P(X=k \bigcap A_1^c)}{P(A_1^c)}$

$= \frac{P(A_1^c \bigcap k \ \text{ success in trials 2,...,n})}{1-p}$

$= \frac{P(A_1^c)P(\text{success in trials 2,...,n})}{1-p}$

$= \binom{n-1}{k}p^k(1-p)^{n-1-k}$

$P_X(k|A_1), P_X(k|A_1^c)$ where $0 \le k \le n$

Verify

$P(X = k) = P(X = k|A_1)P(A_1) + P(X = k|A_1^c)P(A_1^c)$

$k = 0$

LHS: $P(X = 0) = (1-p)^n$

RHS: $P(X = 0|A_1)P(A_1) + P(X = 0|A_1^c)P(A_1^c)$

$= 0P(A_1) + (1-p)^{n-1}(1-p)$

$= (1-p)^n$

$P(A_1|X = 2) = \frac{P(X=2|A_1)P(A_1)}{P(X=2)}$

$P(A_1 \bigcap A_2|X = 2) = \frac{P(A_1 \bigcap A_2 \bigcap X=2)}{P(X=2)}$

$= \frac{pp(1-p)^{n-2}}{\binom{n}{2}p^2(1-p)^{n-2}}$

$= \frac{1}{\binom{n}{2}}$

If A, B are independent

$P(A \bigcap B) = P(A)P(B)$

If A, B are disjoint

$P(A \bigcup B|C) = P(A|C) + P(B|C)$

$P(A \bigcup B|C) = \frac{P(A \bigcup B \bigcap C)}{P(C)}$

$= \frac{P(A \bigcap C \bigcup B \bigcap C}{P(C)}$

$= \frac{P(A \bigcap C) + P(B \bigcap C)}{P(C)}$

$= P(A|C) + P(B|C)$

$Z \sim \text{Poisson}(\lambda)$

$P_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ for $k = 0, 1, 2\ldots$

$E[Z] = \lambda$

$Var[Z] = \lambda$

$E[Z^2] = \lambda + \lambda^2$

Example

Optical Communication

Receiver in an optical communication system. If message is being sent, then the receiver gets a random number of photons that has a Poisson PMF λ_1

If message is not being sent, then receiver gets a random number of photons with a Poisson PMF λ_0

$\lambda_0 < \lambda_1$

The prior probability that a message is being sent is p

a) Suppose the receiver get k photons. Conditional probability that a message was sent

$X =$ number of photons at the receiver

$M = \{\text{message is sent}\}$

$A = \{\text{message is absent}\}$, $A = M^c$

$P(M|X = k) = \frac{P(X=k|M)P(M)}{P(X=k)}$

Number$= \frac{\lambda_1^k e^{-\lambda_1}}{k!} p$

Denominator $P(X = k) = P(X = k|M)P(M) + P(X = k|A)P(A)$

$= \frac{\lambda_1^k e^{-\lambda_1}}{k!} p + = \frac{\lambda_0^k e^{-\lambda_0}}{k!} (1 - p)$

b) Receiver calculates $I = P(M|X = k)$ and $II = P(A|X = k)$

Declares message present if $I > II$

Declares message absent if $I < II$

Declare present if:

$$P(M|X=k) > P(A|X=k)$$

$$\frac{\frac{\lambda_1^k e^{-\lambda_1}}{k!}p}{P(X=k)} > \frac{\frac{\lambda_0^k e^{-\lambda_0}}{k!}(1-p)}{P(X=k)}$$

$$\frac{\lambda_1}{\lambda_0}^k > \frac{1-p}{p}e^{\lambda_1-\lambda_0}$$

$$k > \frac{log\frac{1-p}{p}+(\lambda_1-\lambda_0)}{log(\frac{\lambda_1}{\lambda_0})} = c$$

Receiver decision rule is

Declare present if Number of photons $> c$

Declare absent if Number of photons $\leq c$

c) Suppose that a message is present. What is the probability that the receiver makes a wrong declaration?

$$P(\text{Declaring Absent}|M) = P(X \leq c|M)$$

$$= \sum_{0 \leq k \leq c} e^{-\lambda_1}\frac{\lambda_1^k}{k!}$$

d) $P(\text{Declares present}|A) = P(X > c|A)$

$$= \sum_{k>c} e^{-\lambda_0}\frac{\lambda_0^k}{k!}$$

e) Probability that receiver makes a wrong declaration

$$W = \{\text{Weong declaration}\}$$

$$P(W) = P(W|M)P(M) + P(W|A)P(A)$$

$$= P(W|M)p + P(W|A)(1-p)$$

f) $M \rightarrow X \sim \text{Poisson}(\lambda_1)$

$$A \rightarrow X \sim \text{Poisson}(\lambda_0)$$

$$E[X] = E[X|M]P[M] + E[X|A]P[A]$$

$$= \lambda_1 p + \lambda_0(1-p)$$

g) $E[X^2] = E[X^2|M]P[M] + E[X^2|A]P[A]$

$$= (\lambda + \lambda^2)p + (\lambda_0 + \lambda_0^2)(1-p)$$

$$P(X=2), x \in S_X$$

Cumulative Distribution Function (CDF)

$$F_X(x) = P(X \leq x) \forall x \in \mathbb{R}$$