EE503 Probability for Electrical and Computer Engineers

Week 1 Session 1

Outcome space / Sample space

$\Omega = \{set\ of\ all\ possible\ outcomes\ of\ a\ random\ experiment\}$

1. Flip 1 coin:

   $\Omega = \{H, T\}$

2. Flip 2 coins:

   $\Omega = \{HH, TT, HT, TH\}$

3. Number of emails in the inbox from 10:30 am to 12:30 pm:

   $\Omega = \{0, 1, 2, 3, \ldots\}$

4. Amplitude of the received signal at the radar:

   $\Omega = \{0, \infty\}$

Events:

Flip 2 coins:

$\Omega = \{HH, TT, HT, TH\}$

$A = \{HH, TT\}$

Event A: a subset of Ω

If the observed outcome belongs to event A, then event A has occured.

Radar:

$\Omega = \{0, \infty\}$

$A = \{0, 1\}$

$B = \{\pi\}$

Event Space: Collection of events.

1. Flip 1 coin:

   $\Omega = \{H, T\}$

Event Space: $\{H\}$, $\{T\}$, Ω, ϕ [All possible subsets of Ω]

Power set of Ω: 2^Ω

2. Flip 2 coins:

   $\Omega = \{HH, TT, HT, TH\}$

Event space 1: ϕ, Ω, $\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$, $\{HH, TT\}$, $\{HT, TH\}$, $\{HH, HT\}$, $\{HT, TT\}$ ...

[Power set of Ω]

For a set with n elements, number of possible subsets is 2^n.

Event Space 2: $\Omega = \{HH, TT, HT, TH\}$

$\{HH, TT\}$, $\{HT, TH\}$, Ω, $\phi \leftarrow$ Another possible event space for the experiment of flipping

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Requirement of an Event Space

1. Ω is in the event space (sure event)
2. If A is in the event space, A^c is in the event space
3. If A and B are in the event space, then $A \bigcup B$ and $A \bigcap B$ are also in the event space.

Deduction 1:

ϕ is always in event space

Deduction 2:

If $A_1, A_2, ... A_n$ in the event space, then:

$\bigcap_{i=1}^{n} A_i$ and $\bigcup_{i=1}^{n} A_i$ are in the event space.

Probability Law P

For each event A in the event space, $P(A)$ is a real number that describes our belief/ likelihood of event A.

Axioms of Probability

1. $P(\Omega) = 1$
2. For any event A, $0 \leq P(A) \leq 1$
3. Additivity Axiom

   (a) If A and B are 2 disjoint (i.e., mutually exclusive $\leftarrow A \bigcap B = \phi$) events, then:

   $P(A \bigcup B) = P(A) + P(B)$

   (b) If $A_1, A_2, ... A_n$ are pairwise disjoint events (i.e., $A_k \bigcap A_l = \phi$ for all $k \neq l$), then:

   $P(A_1 \bigcup A_2 ... \bigcup A_n) = \sum_{i=1}^{n} P(A_i)$

$Example\ 1$

$\Omega = \{H, T\}$

Event space=Power set of Ω

$P(\{H\}) = 1/2$

What is the value of $P(T)$

$P(\Omega) = 1$

$P(\{H, T\}) = 1$

$\{H, T\} = \{H\} \bigcup \{T\}$

$P(\{H\} \bigcup \{T\}) = 1 \leftarrow$ Additivity axiom

$P(\{H\}) + P(\{T\}) = 1$

$P(\{T\}) = 1 - P(\{H\})$

$P(\{T\}) = 1 - 1/2 = 1/2$

$Example\ 2$

Throw a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Event space=Power set of Ω

Probability law: For any event A, $P(A) = \frac{|A|}{6}$

$Notation : |A| = number\ of\ elements\ in\ A = cardinality\ of\ A$

$P(\{6\}) = 1/6$

$Prob\ of\ getting\ an\ even\ number :$

$P(\{2, 4, 6\}) = 3/6$

$P(\phi) = 0$

$Example\ 3$

Throw a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Event space=Power set of Ω

$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = 5/12$

$P(\{6\}) = 1/3$

$P(\{3, 4, 5\}) = P(\{3\} \bigcup \{4\} \bigcup \{5\})$

$= P(\{3\}) + P(\{4\}) + P(\{5\})$

$= 6/15$

$Example\ 4$

$\Omega = \{0, \infty\}$

Event space consist of all possible sub-interval of $\{0, \infty\}$ as well as their compliments, unions and intersections.

e.g., (a, b), $[a, b]$, $(a, b]$, $[a, \infty)$

$Borel\ event\ space\ or\ Borel\ sigma\ algebra$

Probability law: For any interval A

$P(A) = \int_A e^{-\omega} d\omega$

$P((1, 2)) = \int_1^2 e^{-\omega} d\omega$

$P([2, \infty)) = \int_2^\infty e^{-\omega} d\omega$

Probability that the outcome is less than 1 or greater than 5 ?

$P([0, 1]) \bigcup (5, \infty)) = P([0, 1]) + P((5, \infty))$

$= \int_0^1 e^{-\omega} d\omega + \int_5^\infty e^{-\omega} d\omega$

$Example\ 5$

$\Omega = \{1, 2, 3, 4, \ldots\}$

$\mathcal{F} = Power\ set\ of\ \Omega$

i.e., Event space $\mathcal{F}$ [sigma-algebra]

$P(\{k\}) = \frac{1}{2^k}$, where $k = 1, 2, 3, \ldots$

Verify $P(\Omega) = 1$

Event space

If $A_1, A_2, \ldots, A_n$ are in the event space, then:

$\bigcap_{i=1}^n A_i$ and $\bigcup_{i=1}^n A_i$ are in the event space.

If $A_1, A_2, A_3, \ldots$ are in the event space, then:

$\bigcap_{i=1}^\infty A_i$ and $\bigcup_{i=1}^\infty A_i$ are also in the event space.

Probability Axioms

- Additivity axiom

  $A_1, A_2, \ldots, A_n$ are pairwise disjoint events

  $P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$
- Countable additivity axiom

$A_1, A_2, A_3, ...$ are pairwise disjoint events

$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

$P(\Omega) = P(\{1, 2, 3, 4...\})$

$\{1, 2, 3, ...\} = \{1\} \bigcup \{2\} \bigcup ...$

$P(\{1\} \bigcup \{2\} \bigcup ...) = P(\{1\}) + P(\{2\}) + ...$

Countable additivity axiom: $P(\{k\}) = \frac{1}{2^k}$

$= \frac{1}{2} + \frac{1}{2^2} + ...$

$= \frac{1/2}{1-1/2}$

Note $\leftarrow$ Geometric series

$a + ar + ar^2 + ...$ where $r < 1$, then

$sum = \frac{a}{1-r}$

Probability that the outcome is an even number:

$P(\{2, 4, 6, 8...\}) = P(\{2\} \bigcup \{4\} \bigcup ...)$

$= P(\{2\}) + P(\{4\}) + ...$

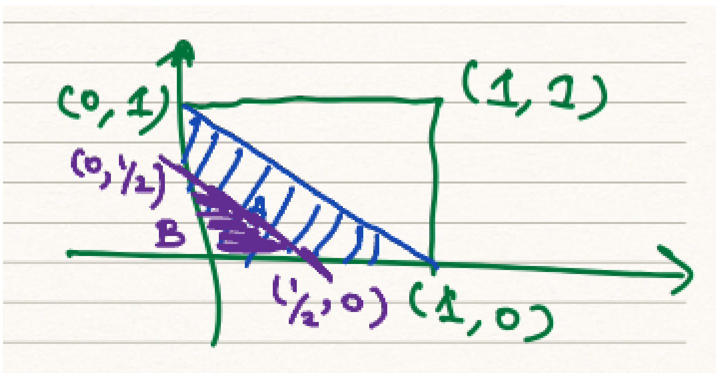
$= \frac{1}{2^2} + \frac{1}{2^4} + ...$

$= \frac{1/4}{1-1/4}$

$= 1/3$

$Example\ 6$

$\Omega = \{(x, y) : 0 \leq x, y \leq 1\}$



$P(A) = Area\ of\ A$

$P(\Omega) = Area\ of\ \Omega = 1$

$P(A) = 1/2$

B is the event that the sum of x and y coordinate is less than or equal to $1/2$

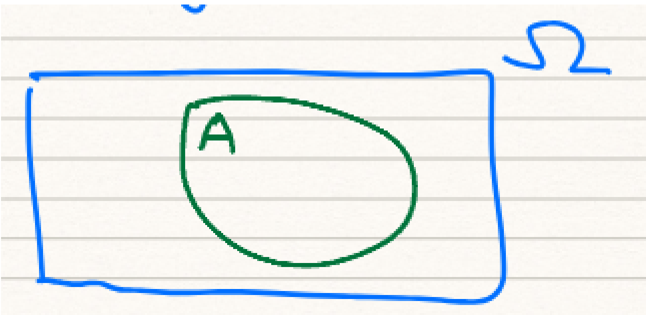
$B = (x, y) \in \Omega : x + y \leq 1/2$

$P(B) = Area\ of \in B = 1/2 \times 1/2 \times 1/2 = 1/8$

Week 1 Session 2

Random Experiment & Probability Model

- Outcome space / Sample Space Ω
- An event is a subset of Ω
- If the realized outcome of experiment lies in A, we say event A has occurred.



Event Space / Sigma algebra $\mathcal{F}$

Properties of $\mathcal{F}$:

1. Ω is in $\mathcal{F}$
2. If A is in $\mathcal{F}$, then A^c is in $\mathcal{F}$
3. (a) If $A_1, A_2, \ldots A_n$ are in $\mathcal{F}$, then:
   $\bigcup_{i=1}^{n} A_i$ is in $\mathcal{F}$ and $\bigcap_{i=1}^{n} A_i$ is in $\mathcal{F}$
   (b) If $A_1, A_2, A_3, \ldots$ is an infinite sequence of events that are in $\mathcal{F}$, then:
   $\bigcup_{i=1}^{\infty} A_i$ is in $\mathcal{F}$ and $\bigcap_{i=1}^{\infty} A_i$ is in $\mathcal{F}$

Probability Law

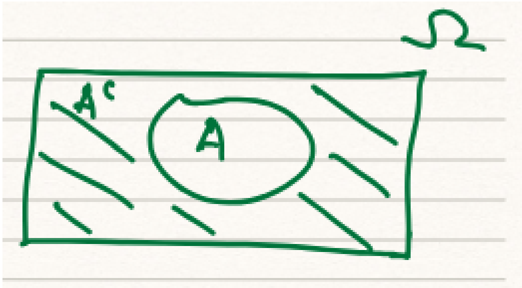
For each event A in $\mathcal{F}$, $P(A)$ is a real number.

Probability Axioms

1. $P(\Omega) = 1$
2. $0 \leq P(A) \leq 1$
3. (a) If $A_1, A_2, \ldots A_n$ are pairwise disjoint events, then:
   $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$
   (b) If $A_1, A_2, A_3, \ldots$ is an infinite sequence of pairwise disjoint events, then:
   $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

Deduction from Axioms

1. $P(A) + P(A^c) = 1$



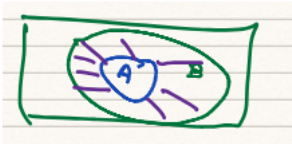
Proof:

$1 = P(\Omega)$

$= P(A \bigcup A^c)$

$= P(A) + P(A^c)$

2. If $A \subset B$, then $P(A) \leq P(B)$



Proof:

$B = A \bigcup C$

$P(B) = P(A \bigcup C)$

$= P(A) + P(C)$

$\rightarrow P(B) \geq P(A)$

$C = B \bigcap A^c$

3. Union Formula

   For any 2 events A and B

   $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$

$A \bigcap B$

$A \bigcap B^c$

$B \bigcap A^c$

Proof:

$P(A) = P(A \bigcap B^c) + P(A \bigcap B)$ ①

$P(B) = P(B \bigcap A^c) + P(A \bigcap B)$ ②

$P(A \bigcup B) = P(A \bigcap B^c) + P(A \bigcap B) + P(B \bigcap A^c)$

$= P(A) + P(B \bigcap A^c)$ ① $is\ applied$

$= P(A) + P(B) - P(A \bigcap B)$ ② $is\ applied$

4. $P(\phi) = 0$

Proof:

$P(\phi) + P(\phi^c) = 1$

$P(\phi) + P(\Omega) = 1$

$P(\phi) + 1 = 1$

$P(\phi) = 0$

Exercise 1.

A_1, A_2, A_3

$P(A_1) = a_1$, $P(A_2) = a_2$, $P(A_3) = a_3$

$P(A_1 \bigcap A_2) = b_1$, $P(A_2 \bigcap A_3) = b_2$, $P(A_3 \bigcap A_1) = b_3$

$P(A_1 \bigcap A_2 \bigcap A_3) = c$

What is the value of $P(A_1 \bigcup A_2 \bigcup A_3)$?

$B = A_2 \bigcup A2$

$P(A_1 \bigcup B) = P(A_1) + P(B) - P(A_1 \bigcap B)$

$= P(A_1) + P(A_2 \bigcup A_3) - P(A_1 \bigcap (A_2 \bigcap A_3))$

Exercise, show that:

$A_1 \bigcap (A_2 \bigcup A_3) = (A_1 \bigcap A_2) \bigcup (A_1 \bigcap A_3)$

Union Bound

Theorem:

$A_1, A_2, \ldots A_n$ are n events $(n \geq 2)$

$P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$

Proof: Induction argument

$n = 2$

$P(A_1 \bigcup A_2) = P(A_1) + P(A_2) - P(A_1 \bigcap A_2) \leq P(A_1) + P(A_2)$

Assume that the theorem is true for $n = k$

i.e., $P(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} P(A_i)$

Then in the $k+1$ case, where

$A_1, A_2, \ldots A_k, A_{k+1}$

$P(\bigcup_{i=1}^{k+1} A_i) = P((\bigcup_{i=1}^{k} \bigcup A_{k+1}) \leq P(\bigcup_{i=1}^{k} A_i + P(A_{k+1}))$

$P(\bigcup_{i=1}^{k+1} A_i) \leq \sum_{i=1}^{k} P(A_i) + P(A_{k+1})$

$P(\bigcup_{i=1}^{k+1} A_i) = \sum_{i=1}^{k+1} P(A_i)$

Cardinality of sets

Finite sets

$\Omega = \{1, 2, 3, 4, 5, 6\}$

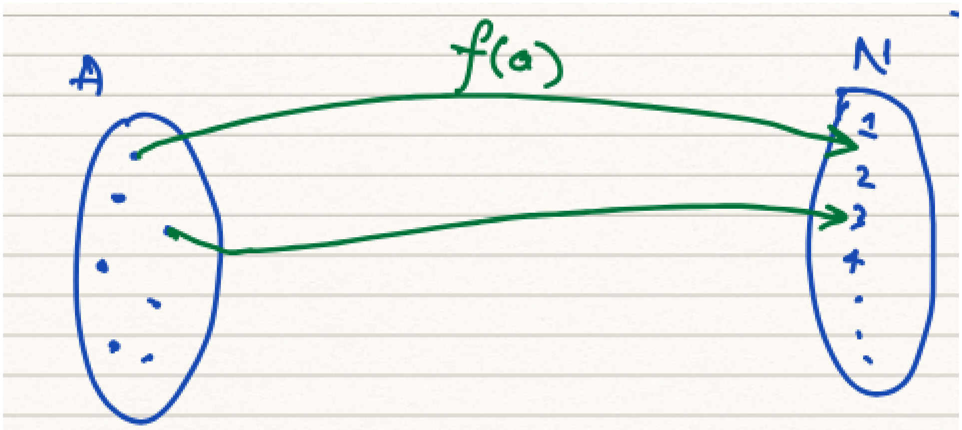
$\Omega = \{a, b, c, \dots z\}$

Infinite sets

Countably infinite sets

$N = \{1, 2, 3, \dots\}$

A set A that is "as large" as N is called a countably infinite set.



Formally, A is countably infinite if we can find a function f from A to N, such that

(i) f is a one-to-one function

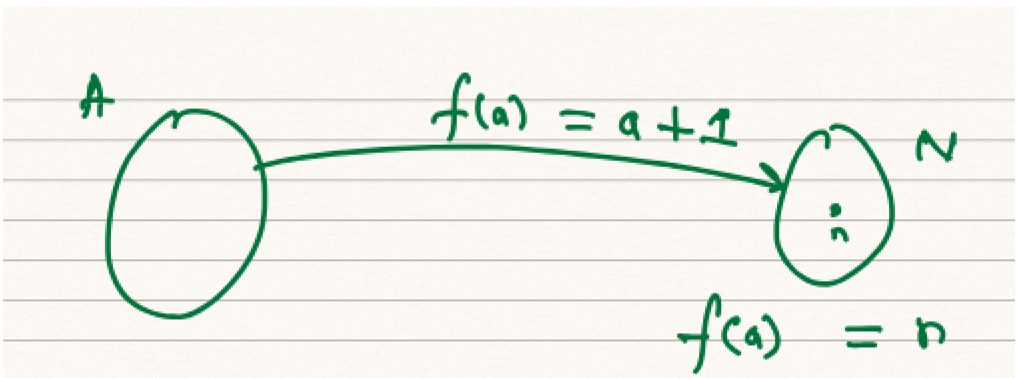
i.e., if $a \neq b$, then $f(a) \neq f(b)$

(ii) for every positive integer n, there is an $a \subset A$ such that $f(a) = n$

$Example 1$

$A = \{0, 1, 2, 3, \dots\}$

$N = \{1, 2, 3, \dots\}$



$f(a) = a + 1 = n$

$a = n - 1$

Therefore, A is countably infinite

$Example 2$

$B = \{2, 4, 6, 8, \dots\}$

$N = \{1, 2, 3, \dots\}$

$f(b) = b/2 = n$

$b = 2n$

$Example 3$

$C = \{2, 4, 8, 16, 32, \dots\}$

$f(c) = \log_2 c = n$

$c = 2^n$

$Example 4$

$\{-1, -2, -3, \dots\}$

$\{\dots, -1, 0, 1, 2, \dots\}$

are countably infinite sets

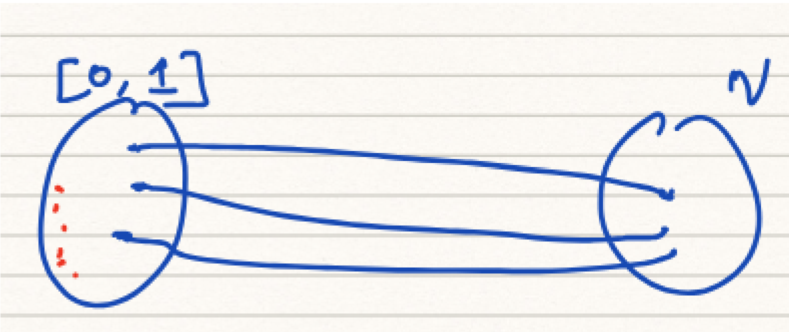
Uncountably infinite sets

Much larger sets of positive integers

e.g., $[0, 1]$, $[0, \infty]$, $(-\infty, \infty)$

$[0, 1]$

There is no way of finding a one-to-one association(correspondence) between $[0, 1]$ and N



$Example 5$

$A_1 \subset A_2$

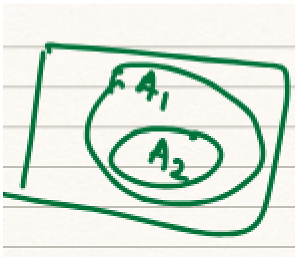
$P(A_1 \bigcup A_2) = P(A_2)$

$A_1 \subset A_2 \subset \ldots \subset A_k$

$P(\bigcup_{i=1}^{k} A_i) = P(A_k)$

$A_1 \subset A_2 \subset \ldots \subset A_k \subset A_{k+1} \ldots$

$P(\bigcup_{i=1}^{\infty} A_i) = \lim_{k \to \infty} P(A_k)$



$A_1 \supset A_2$

$P(A_1 \supset A_2) = P(A_2)$

$A_1 \supset A_2 \supset \ldots \supset A_k$

$P(\bigcap_{i=1}^{k} A_i) = P(A_k)$

$A_1 \supset A_2 \supset \ldots \supset A_k \supset A_{k+1} \ldots$

$P(\bigcap_{i=1}^{\infty} A_i) = \lim_{k \to \infty} P(A_k)$

$Example 6$

$\Omega = [0, 1]$

$P(interval) = length\, of\, interval$

$A_1 = [0, 1]$

$A_2 = [0, 1/2]$

$A_3 = [0, 1/3]$

$\ldots$

$A_k = [0, 1/k]$

$A_1 \supset A_2 \supset A_3 \supset \dots$

$P(A_1 \bigcap A_2 \bigcap A_3) = P(A_3) = 1/3$

$P(\bigcap_{i=1}^{\infty} A_i) = \lim_{k\to\infty} P(A_k) = \lim_{k\to\infty} 1/k = 0$

$\bigcap_{i=1}^{\infty} A_i = \{0\} = [0, 0]$

$P([0, 0]) = 0$

Finite outcome space

$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$

$\mathcal{F} = power\ set\ of\ \Omega$

$P(\omega_1) = p_1, P(\omega_2) = p_2, \dots P(\omega_n) = p_n$

$P(\{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_1\} \bigcup \{\omega_2\} \bigcup \{\omega_3\})$

$= P\{\omega_1\} + P\{\omega_2\} + P\{\omega_3\}$

$= p_1 + p_2 + p3$

$p_1 + p_2 + \dots + p_n = 1$

$1 = P(\Omega) = P(\{\omega_1, \dots, \omega_n\})$

Special case:

$\Omega = \{\omega_1, \dots, \omega_n\}$

$\mathcal{F} = Power\ set$

$P(\omega_i) = p$ for $i = 1, 2, \dots n$ Equally likely outcomes

$1 = p + p + \dots + p$

$1 = np$

$p = 1/n$

$P(\{\omega_2, \omega_4, \omega_6\}) = P(\{\omega_2\}) + P(\{\omega_4\}) + P(\{\omega_6\})$

$= 3/n$

$A = \{\omega_{k1}, \dots, \omega_{km}\}$

$P(A) = m/n$

$P(A) = \frac{|A|}{n}$

If Ω is countably infinite,

$\Omega = \{\omega_1, \omega_2, \dots\}$

we will usually work with power set on our event space

$Example 6$

$\Omega = [0,1]$

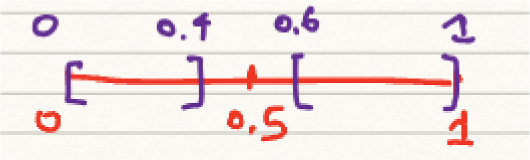
Borel Sigma algebra: all sub-interval, union, intersections, compliments

$P(sub - interval\ of\ [0,1]) = length\ of\ sub - interval$

$0 \leq a \leq b \leq 1$

$P([a,b]) = b - a$

$A = \{\omega \in [0,1] : |\omega - 0.5| \geq 0.1\}$



$P(A) = 0.8$

$A = [0, 0.4] \bigcup [0.6, 1]$

$P(A) = 0.4 + 0.4 = 0.8$

Exercise

$B = \{\omega \in [0,1] : (\omega - 1/2)^2 \geq 1/4\}$

$P(B) = ?$

$P([0,1]) = 1$

$P([0,1]) = P(\bigcup_{0 \leq \omega \leq 1} \omega)$

$= \sum_{0 \leq \omega \leq 1} P(\omega)$

$\neq \sum_{0 \leq \omega \leq 1} P([\omega, \omega])$

$\neq 0$

Conditional Probability

Roll a fair die

$\Omega = \{1,2,3,4,5,6\}$

$\mathcal{F} = Power\ set$

Equally likely outcomes

$P(k) = 1/6, 1 \leq k \leq 6$

$B = \{2,4,6\}$

$A = \{2\}$

Given that B has occurred, the new probability for event $A = 1/3$

$C = \{1, 2, 3\}$

Given that B has occurred, what is the revised probability for event C?

$1/3$

$New\ prob\ of\ A = \frac{|A \bigcap B|}{|B|}$

$New\ prob\ of\ C = \frac{|C \bigcap B|}{|B|}$

Definition:

If A and B are 2 events, and $P(B) > 0$

$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$

$P(B|B) = 1$

$P(B^c|B) = 0$

$P(\Omega|B) = \frac{P(\Omega \bigcap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

Week 2 Session 1

Deductions from Axioms

1. $P(A) + P(A^c) = 1$
2. $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$
3. If $A \subset B$, then $P(A) \leq P(B)$

Finite outcome space

$\Omega = \{\omega_1, \ldots \omega_n\}$

Typically, $\mathcal{F}$ = Power set of ω

$P(\{\omega_1)\}) = p_1, \ldots, P(\{\omega_n\}) = p_n$

Special case: Finite ω with equally likely outcome

$P(\{\omega_i\}) = \frac{1}{n}$, $n = |\Omega|$

$P(A) = \frac{|A|}{n}$

Conditional Probabilities

Definition: If A and B are 2 events, and $P(B) > 0$

$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$

Example

4 sided die, roll it twice

$\Omega = \{(x, y) : x, y \in \{1,2,3,4\}\}$

$= \{(1,1),(1,2),(1,3),\dots(4,4,)\}$

Equally likely outcome $P(\{x, y\}) = \frac{1}{16}$

$E = \{\text{Both number are less than } 3\} = \{(1,1),(1,2),(2,1),(2,2)\}$

$F = \{\text{Both numbers are } 1\} = \{(1,1)\}$

$P(F|E) = \frac{P(F \bigcap E)}{P(E)}$

$P(E) = \frac{4}{16}$

$P(F \bigcap E) = \frac{1}{16}$

$P(F|E) = \frac{1}{4}$

$P(E|F) = \frac{P(E \bigcap F)}{P(F)}$

$P(E|F) = \frac{1}{16} / \frac{1}{16} = 1$

$B = \{\text{minimum of 2 number is } 2\} = \{(2,2),(2,3),(3,2),(2,4),(4,2)\}$

$A = \{\text{maximum of 2 number is } 3\} = \{(3,3),(3,1),(1,3),(2,3),(3,2)\}$

$P(A|B) = \frac{P(A \bigcap B)}{P(B))} = \frac{2/16}{5/16} = 2/5$

$C = \{\text{maximum of 2 number is } 1\}$

$P(C|B) = \frac{P(C \bigcap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0$

Probability Axioms

1. $P(\Omega) = 1$
2. $0 \leq P(A) \leq 1$
3. Additivity axiom

Fix event B with $P(B) > 0$. Consider conditional probability of various event given B. These new probability will satisfy probability axioms.

1. $P(\Omega|B) = \frac{P(\Omega \bigcap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$
2. $0 \leq P(A|B) \leq 1$

   $0 \leq \frac{P(A \bigcap B)}{P(B)} \leq 1$
3. If A_1 and A_2 are disjoint events

   $P(A_1 \bigcup A_2|B) = \frac{P((A_1 \bigcup A_2) \bigcap B)}{P(B)}$

$= \frac{P((A_1 \bigcap B) \bigcup (A_2 \bigcap B))}{P(B)}$
$= \frac{P(A_1 \bigcap B) + P(A_2 \bigcap B)}{P(B)}$
$= P(A_1|B) + P(A_2|B)$

Deduction from Axioms

1. $P(A|B) + P(A^c|B) = 1$
2. $P(A \bigcup C|B) = P(A|B) + P(C|B) - P(A \bigcap C|B)$
3. If $A \subset C$. then $P(A|B) \leq P(C|B)$

Chain Rule / Multiplication Rule

If $P(B) > 0$, then $P(A|B) = \frac{P(A \bigcap B)}{P(B)}$

$P(A \bigcap B) = P(A|B)P(B)$ - Chain Rule

If $P(A) > 0$, then $P(B|A) = \frac{P(B \bigcap A)}{P(A)}$

$P(B \bigcap A) = P(B|A)P(A)$

$P(B) > 0, P(B^c) > 0$

$P(A|B)$, $P(A|B^c)$

$P(A \bigcap B) = P(A|B)P(B)$

$P(A \bigcap B^c) = P(A|B^c)P(B^c)$

<u>Law of total probability</u>

$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

Partition

Partition of Ω

$B_1, \ldots, B_n$ forms a partition of Ω

if $B_i \bigcap B_j = \phi$ for $i \neq j$ (pairwise disjoint) and $\bigcup_{i=1}^n B_i = \Omega$

$P(B_i)$ for $i = 1, \ldots, n$

$P(A|B_i)$ for $i = 1, \ldots, n$

$P(A) = P(A \bigcap \Omega)$

$= P(A \bigcap (\bigcup_{i=1}^n B_i))$

$= P((A \bigcap B_1) \bigcup \ldots (A \bigcap B_n))$

$= \sum_{i=1}^n P(A \bigcap B_i)$

Law of total probability

$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Example

Say a sender and receiver with transmission route 1 and 2

$P(R1) = 3/4, P(D|R1) = 1/3, P(R2) = 1/4, P(D|R2) = 2/3$ where D is the packet get dropped

$P(D) = P(D \bigcap R1) + P(D \bigcap R2)$

$= P(D|R1)P(R1) + P(D|R2)P(R2)$

$= 1/3 \cdot 3/4 + 2/3 \cdot 1/4 = 5/12$

$P(\text{Packet not getting dropped}|\text{R1}) = P(D^c|R1)$

$= 1 - P(D|R1)$

$= 2/3$

$P(R1|D) = \frac{P(R1 \bigcap D)}{P(D)}$

$= \frac{P(D|R1)P(R1)}{5/12}$

Bayes' Rule

$P(A) > 0, P(B) > 0$

$P(B|A) = \frac{P(B \bigcap A)}{P(A)} = \frac{P(A|B)P(B)}{P(A)}$

$P(R1|D^c) = \frac{P(D^C|R1)P(R1)}{P(D^c)}$

$= \frac{(1-P(D|R1)) \cdot 3/4}{1-P(D)} \neq P(R1)$

This it because $P(R1|D^c)$ is the posterior probability and $P(R1)$ is the prior probability.

Law of total probability + Bayes' Rule

Partition of $\Omega : B_1, \dots B_n$

$B_i \bigcap B_j = \phi$

$\bigcup_{i=1}^{n} B_i = \Omega$

Given $P(B_i), P(A|B_i)$

$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)}$

Bayes' Rule:

$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$

Example

Binary communication channel

$P(S0) = 1 - \alpha$, $P(S1) = \alpha$

$P(R0|S0) = 1 - q$, $P(R1|S0) = q$

$P(R0|S0) = p$, $P(R1|S1) = 1 - p$

$P(R1) = P(R1 \bigcap S1) + P(R1 \bigcap S0)$

$= P(R1|S1)P(S1) + P(R1|S0)P(S0)$

$= (1 - p)\alpha + q(1 - \alpha)$

$P(S1|R1) = \frac{P(R1|S1)P(S1)}{P(R1)}$

$= \frac{(1-p)\alpha}{(1-p)\alpha+q(1-\alpha)}$

Chain Rule

$P(A \bigcap B) = P(A|B)P(B)$

$P(A \bigcap B \bigcap C) = P(A|(B \bigcap C))P(B \bigcap C)$

$= P(A|(B \bigcap C))P(B|C)P(C)$

$P(A_1 \bigcap ... \bigcap A_n) = P(A_1|A_2 \bigcap ... \bigcap A_n)P(A_2|A_3 ... \bigcap A_n) ... P(A_{n-1}|A_n)P(A_n)$

Example

Two urns

Urn 1: 5 red balls and 5 green balls

Urn 2: 2 red balls and 4 green balls

Randomly pick one ball from the selected urn and remove it

Randomly pick 2^{nd} ball from the same urn

$U1 = \{\text{Urn 1 is selected}\}$, $P(U1) = 2/3$

$U2 = \{\text{Urn 2 is selected}\}$, $P(U2) = 1/3$

$R1 = \{\text{first ball is red}\}$

$R2 = \{\text{second ball is red}\}$

$P(R1 \bigcap R2) = P(R1 \bigcap R2 \bigcap U1) + P(R1 \bigcap R2 \bigcap U2)$

$P(R2 \bigcap R1 \bigcap U1) = P(R2|R1 \bigcap U1)P(R1|U1)P(U1)$

$= 4/9 \cdot 5/10 \cdot 2/3$

$P(R2 \bigcap R1 \bigcap U2) = P(R2|R1 \bigcap U2)P(R1|U2)P(U2)$

$= 1/5 \cdot 2/6 \cdot 1/3$

$P(U1|R1 \bigcap R2) = \frac{P(R1 \bigcap R2|U1)P(U1)}{P(R1 \bigcap R2)}$

$= \frac{4/9 \cdot 5/10 \cdot 2/3}{P(R1 \bigcap R2)}$

Week 2 Session 2

$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$ if $P(B) > 0$

Chain Rule:

$P(A \bigcap B) = P(A|B)P(B)$

$P(A \bigcap B \bigcap C) = P(A|B \bigcap C)P(B|C)P(C)$

Law of total probability:

If $B_1, \ldots B_n$ is a partition of Ω
Then $P(A) = \sum_{i=1}^{n} P(A \bigcap B_i)$

$= \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Can be extended to a countable partition, i.e., $B_1, B_2, \ldots$

that are pairwise disjoint and $\bigcup_{i=1}^{\infty} B_i = \Omega$

Then, $P(A) = \sum_{i=1}^{\infty} P(A \bigcap B_i) = \sum_{i=1}^{\infty} P(A|B_i)P(B_i)$

Bayes' Rule

$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

$P(A \bigcap B) = P(A|B)P(B)$

Statement: $P(A \bigcap B|C) = P(A|B \bigcap C)P(B|C)$

LHS: $P(A \bigcap B|C) = \frac{P(A \bigcap B \bigcap C)}{P(C)}$

RHS: $P(A|B \bigcap C)P(B|C) = \frac{P(A \bigcap B \bigcap C)}{P(B \bigcap C)} \frac{P(B \bigcap C)}{P(C)}$

Two urns example

$P(R2 \bigcap R1|U1)P(U1)$

$= P(R2|R1 \bigcap U1)P(R1|U1)P(U1)$

$P(A \bigcap B) \sim \frac{\text{number of}(A \bigcap B)}{n}$ relative frequency

$P(B) \sim \frac{\text{number of} B}{n}$

$P(A|B) \sim \frac{\text{number of } (A \bigcap B)}{\text{number of } B}$

Independent Events

Definition: Two events A and B are independent if $P(A \bigcap B) = P(A)P(B)$

Let A and B be independent events and $P(B) > 0$
$P(A|B) = \frac{P(A \bigcap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$

$P(B^c) > 0$

$P(A|B^c) = \frac{P(A \bigcap B^c)}{P(B^c)} = \frac{P(A)-P(A \bigcap B)}{1-P(B)} = \frac{P(A)-P(A)P(B)}{1-P(B)} = P(A)$

Independence $\neq$ Disjoint events

$C \bigcap D = \phi$, $P(C), P(D) > 0$

<u>Are C and D independent?</u>

$P(C \bigcap D) = P(\phi) = 0$

$P(C)P(D) > 0$

$P(C \bigcap D) \neq P(C)P(D)$

C and D are not independent

$P(C|D) = 0$

Lemma 1

If A and B are independent events, then so are the following events:

(a) A and B^c

$P(A \bigcap B^c) = P(A)P(B^c)$

(b) A^c and B are independent

(c) A^c and B^c are independent

Proof:

(a) $P(A \bigcap B^c) = P(A) - P(A \bigcap B)$

$= P(A) - P(A)P(B)$

$= P(A)(1 - P(B))$

$= P(A)P(B^c)$

(b) A^c and B

(c) A^c and B^c

Using property: $(A\bigcup B)^c = A^c \bigcap B^c$

$P(A^c \bigcap B^c) = 1 - P(A\bigcup B)$

$= 1 - P(A) - P(B) + P(A\bigcap B)$

$= 1 - P(A) - P(B) - P(A)P(B)$

$= (1 - P(A))(1 - P(B))$

$= P(A^c)P(B^c)$

Example

4 sided die, $\Omega = \{1, 2, 3, 4\}$

Equally likely outcomes

$A = \{1, 4\}$, $B = \{2, 4\}$

$P(A) = 1/2, P(B) = 1/2$

$P(A\bigcap B) = 1/4$

Example

$\Omega = \{1, 2, 3, 4\}$

$P(\{1\}) = P(\{2\}) = P(\{3\}) = p$

$A = \{1, 4\}$, $B = \{2, 4\}$

$P(A) = 1 - 2p$

$P(B) = 1 - 2p$

$P(A\bigcap B) = P(\{4\}) = 1 - 3p$

A and B are independent if $1 - 3p = (1 - 2p)^2$

Independence of 3 events

Def: A, B, C are independent (mutually independent) if

1. $P(A\bigcap B\bigcap C) = P(A)P(B)P(C)$
2. $P(A\bigcap B) = P(A)P(B)$
3. $P(B\bigcap C) = P(B)P(C)$
4. $P(C\bigcap A) = P(C)P(A)$

$P(A\bigcap B|C) = \frac{P(A\bigcap B\bigcap C)}{P(C)} = \frac{P(A)P(B)P(C)}{P(C)} = P(A)P(B) = P(A\bigcap B)$

$P(A|C) = P(A)$

$P(A\bigcup B|C) = \frac{P((A\bigcup B)\bigcap C)}{P(C)}$

$= \frac{P((A \bigcap C) \bigcup (B \bigcap C))}{P(C)}$

$= \frac{P(A \bigcap C) + P(B \bigcap C) - P(A \bigcap B \bigcap C)}{P(C)}$

$= \frac{P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)}{P(C)}$

$= P(A) + P(B) - P(A)P(B)$

$= P(A \bigcup B)$

$P(A \bigcup C) = P(A|C) + P(B|C) - P(A \bigcap B|C)$

$= P(A) + P(B) - P(A \bigcap B)$

$= P(A \bigcup B)$

Pairwise independence

A, B, C are pairwise independent if

$P(A \bigcap B) = P(A)P(B)$

$P(B \bigcap C) = P(B)P(C)$
$P(C \bigcap A) = P(C)P(A)$

Example

4 sided die equally likely outcomes $\Omega = \{1, 2, 3, 4\}$

$A = \{1, 4\}$, $B = \{2, 4\}$, $C = \{3, 4\}$

$P(A \bigcap B) = 1/4$, $P(A) = 1/2$, $P(B) = 1/2$

A, B, C are pairwise independent

$P(A \bigcap B \bigcap C) = 1/4$

$P(A)P(B)P(C) = 1/8$

$P(A \bigcap B \bigcap C) \neq P(A)P(B)P(C)$ meaning A, B, C are not independent

$P(A \bigcap B|C) = \frac{P(A \bigcap B \bigcap C)}{P(C)} = \frac{1/4}{1/2} = 1/2$

Lemma 2

A, B, C are independent then

(a) A^c, B, C are independent

Proof:

$P(A^c \bigcap B) = P(A^c)P(B)$

$P(A^c \bigcap C) = P(A^c)P(C)$

$P(B \bigcap C) = P(B)P(C)$

$P(A^c \bigcap B \bigcap C) = P(B \bigcap C) - P(B \bigcap C \bigcap A)$

$= P(B)P(C) - P(B)P(C)P(A)$

$= (1 - P(A))P(B)P(C)$

$= P(A^c)P(B)P(C)$

(b) A^c, B^c, C are independent

$E = A^c$

From Part (a) E, B, C are independent

$\implies E, B^c, C$ are independent

$\implies A^c, B^c, C$ are independent

(c) A^c, B^c, C^c are independent

$E = A^c, F = B^c$

From Part (b)

$\implies E, F, C$ are independent

using Part (a)

$\implies E, F, C^c$ are independent

$\implies A^c, B^c, C^c$ are independent re

Example

3 bits are transmitted over a noisy channel

For each bit, the probability of correct reception is λ

$P(C_i) = \lambda\,, P(E_i) = 1 - \lambda$

The error events for the 3 bits are mutually independent

$E_i = \{\text{bit } i \text{ incorrectly received}\}$

E_1, E_2, E_3 are independent

$C_i = \{\text{bit } i \text{ correctly received}\}$

C_1, C_2, C_3 are independent

C_1, C_2, E_3 are independent

Example

Find the probability that the number of correctly received bits is 2

$S = \{\text{Number of correct bits is } 2\}$

$(C_1 \bigcap C_2 \bigcap E_3) \bigcup (C_1 \bigcap E_2 \bigcap C_3) \bigcup (E_1 \bigcap C_2 \bigcap C_3)$

say $F_1 = C_1 \bigcap C_2 \bigcap E_3, F_2 = C_1 \bigcap E_2 \bigcap C_3, F_3 = E_1 \bigcap C_2 \bigcap C_3$

$S = F_1 \bigcup F_2 \bigcup F_3$

$P(S) = P(F_1) + P(F_2) + P(F_3)$

because F_1, F_2, F_3 are pairwise disjoint

$$P(C_1 \bigcap C_2 \bigcap E_3) = \lambda\lambda(1-\lambda)$$

$$P(S) = 3\lambda^2(1-\lambda)$$

Probability that all bits are correctly received:

$$= P(C_1 \bigcap C_2 \bigcap C_3) = \lambda^3$$

Probability that at least 2 bits are correctly received:

$$= \lambda^3 + 3\lambda^2(1-\lambda)$$

Finite Ω and equally likely outcomes

$P(\{\omega\}) = \frac{1}{|\Omega|}$, $|\Omega|$ is the cardinality of Ω

$$P(A) = \sum_{\omega \in A} P(\{\omega\}) = \sum_{\omega \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

Example

Antenna array consists of n antennas

Arrange n antennas in a straight line

m out of the n antennas are defective

All arrangement of n antennas are equally likely

The array will not work if 2 defective antennas are next to each other

Probability of the array that does not work?

$$n = 4, m = 2$$

$$A = \{DDNN, NNDD, NDDN, NDND, DNDN, DNND\}$$

$$P(\text{Array not work}) = \frac{|A|}{|\Omega|} = \frac{3}{6}$$

$$n = 12, m = 4$$

Systematic / Efficient way of counting number of elements in a set without listing all elements $\rightarrow$ Combinatorics

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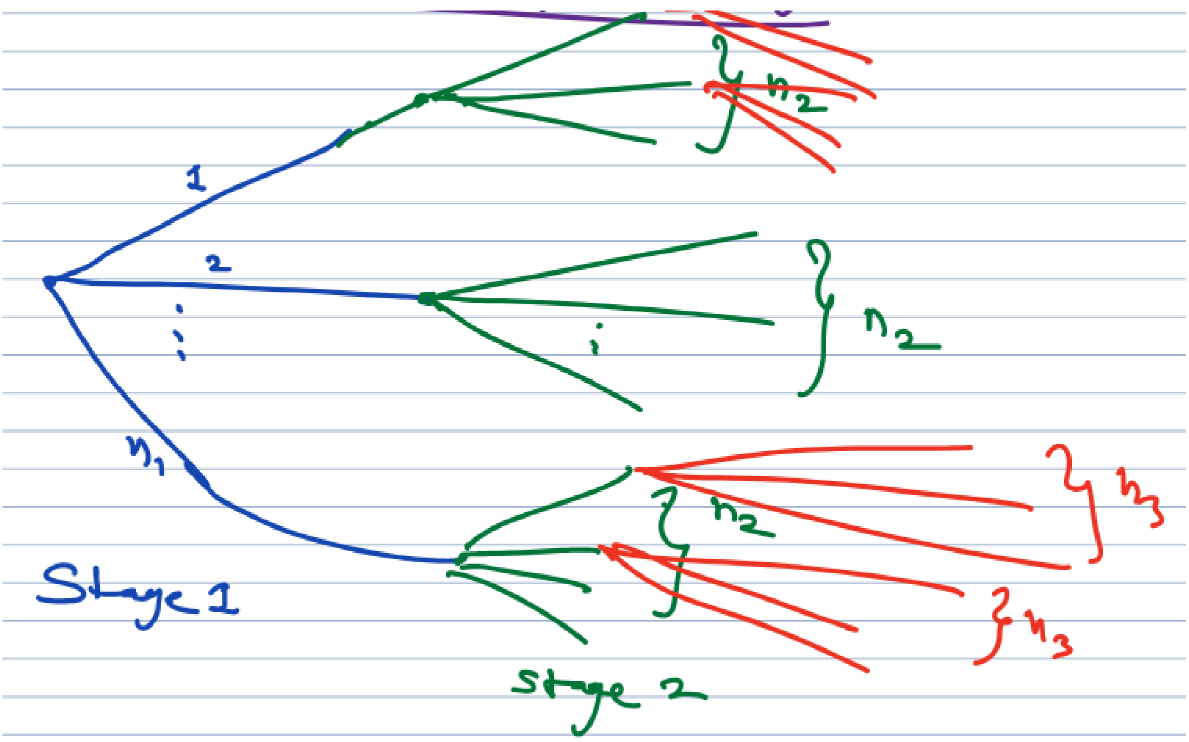
Basis principle of counting



$n_1 n_2$ ways of doing this procedure

Example

4 digit passcode

10^4

Example

7 character license plate where first 3 characters are letter other 4 are digits

$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

Repetition of letter or digits is not allowed

$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

Week 3 Session 1

Independent events

Definition: Two events A and B are independent if $P(A \bigcap B) = P(A)P(B)$

Independence $\implies P(A|B) = P(A|B^c) = P(A)$

Independence of 3 events

Def: A, B, C are independent (mutually independent) if

1. $P(A \bigcap B \bigcap C) = P(A)P(B)P(C)$
2. $P(A \bigcap B) = P(A)P(B)$

3. $P(B \bigcap C) = P(B)P(C)$
4. $P(C \bigcap A) = P(C)P(A)$

Independence of multiple events

Definition: $A_1, \ldots A_n$ are (mutually) independent if

$P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$

for every non-empty $I \subset \{1, 2, \ldots, n\}$

Definition: $A_1, A_2, \ldots$ are (mutually) independent if

$P(\bigcap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$

for every non-empty and finite $I \subset \{1, 2, \ldots\}$

Finite Ω and equally like outcomes

$P(A) = \frac{|A|}{|\Omega|}$

Basic principle of Counting

Total number of ways of doing 2-stage procedure is $n_1 n_2$

Permutations

Definition: Any arrangement of elements of S in a sequence

$S = \{I_1, \ldots I_n\}$

How many permutation of S are possible?

Imagine have n slots,

$n \cdot (n-1) \ldots \cdot 1 = n!$

Example

4 digit passcode using all of these digits 2,4,6,8

number of passcodes=number of permutations=$4!$

$S = \{I_1, \ldots I_n\}$

Sampling an ordered k-tuple with repetitions allowed

$k = 2$, $(I_1\ I_2) \neq (I_2\ I_1)$, $(I_1\ I_1) \neq (I_1\ I_1)$

How many ordered pairs are possible?

n^2

Ordered k-tuple

n^k ordered k-tuple

Example

Flipping a coin k times. How many sequence of H, T are possible?

k-tuple: 2^k

Sampling ordered k-tuple without repetitions $1 \leq k \leq n$

$S = \{I_1, \ldots I_n\}$

Within k slots, then $n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$

Number of ordered k-tuple without repitition

$n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$

Example

n people in a department.

Need to form a committee consisting of a chair, a vice chair and a secretary

Same person cannot serve in more than 1 role

How many such committees are possible?

$n(n-1)(n-2) = \frac{n!}{(n-3)!}$

Example: Birthday problem

k people , $1 \leq k \leq 365$

All were born in non-leap year

a) How many birthday sequences are possible?

$(d_1, \ldots d_k)$ where every slot has 365 possibilities

365^k

b) How many birthday sequences are possible without repitition?

$\frac{365!}{(365-k)!} = 365 \cdot 364 \cdot \ldots \cdot (365-k+1)$

c) Assume that all birthday sequence are equally likely?

Probability that the group of k-people have distinct birthdays

$\frac{|A|}{|\Omega|} = \frac{\frac{365!}{(365-k)!}}{365^k}$

d) Probability that at least 2 people in this k-people group have the same birthday?

$P(B) = 1 - P(\text{everyone has a different birthday})$

$= 1 - \frac{\frac{365!}{(365-k)!}}{365^k}$

Selecting a subset with k elements, $1 \leq k \leq n$

- Order is not important
- No repetitions
- A subset of k elements is called a "combination" with k-elements

Say $S = \{I_1, I_2, I_3\}$ has a subset of size 2

$\{I1, I2\}, \{I1, I3\}, \{I3, I2\}$ all 3 possibilities

With n elements in S

number of k element subsets $n\mathrm{C}_k$

$n\mathrm{C}_k = \frac{n!}{(n-k)!k!} = \binom{n}{k}$ where $\frac{n!}{(n-k)!k!}$ is the binomial coefficient

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number of subset of 0 elements=1

$k = 0$

$\binom{n}{0} = \frac{n!}{n!0!} = 1$

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for $k \geq n+1$

$n\mathrm{C}_k = 0$

Why is the number of subsets of S with k-elements=$\binom{n}{k}$

Task: Select an order k-tuple from S without repititions

$n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!k!}$

Indirect method for task

Stage1:

Pick a subset of k elements (without considering order)

$n\mathrm{C}_k$

Stage2:

Pick a permutation for the k-element subset chosen in stage1

$k!$

$n\mathrm{C}_k k! = \frac{n!}{(n-k)!}$

$n\mathrm{C}_k = \frac{n!}{(n-k)!k!}$

Example

a) 20 people in an organization need a committee of 3 people

number of possible committees$= \binom{20}{3} = \frac{20!}{3!17!}$

b) 12 people - 5 women and 7 men

Need to form a committee with 2 women and 3 men

number of possible committees$= \binom{5}{2}\binom{7}{3}$

c) 7 people

Committee of 3 people

Two of 7 people - Person 1 and Person 2 refuse to serve together

number of possible committees$= \binom{7}{3} - \binom{5}{1}$

Method 2: $S = P_1, \ldots P_7$

Committees with P_1 but not $P_2 = \binom{5}{2}$

Committees with P_2 but not $P_1 = \binom{5}{2}$

Committees with neither P_1 or $P_2 = \binom{5}{3}$

$\binom{5}{2} + \binom{5}{2} + \binom{5}{3} = 30$

Example

50 items: 10 of items are defective, 40 are functional

Randomly pick 10 items

Probability that exactly 5 of selected items are defective

$\frac{\binom{10}{5}\binom{40}{5}}{\binom{50}{10}}$

Example

32 bit binary number. How many such numbers have exactly 5 zeros

$\binom{32}{5}$
