

IB Mathematics: Analysis and Approaches HL

Internal Assessment

Approximating Curved Surface Areas of 3D Shapes Using Triangular Meshes and Vector Cross Products

I) Introduction

In many real-world contexts, we can only work with digital or physical models of curved objects rather than with their exact continuous surfaces, which makes numerical approximation of surface area an essential tool.

Computer graphics and 3D modelling software, for example, display smooth objects such as spheres, cylinders and tori by rendering them as collections of small flat polygons, while engineers and designers use similar meshes in computer-aided design (CAD) and finite element analysis to estimate stresses and material usage on curved components.

All of these applications rely on subdividing curved surfaces into simpler elements whose areas are easier to compute and sum, so understanding how such subdivision behaves mathematically can clarify when these approximations are reliable and how fine a mesh is needed for a given accuracy.

In this investigation I adopt this subdivision approach explicitly by replacing each smooth surface with a mesh of small triangles, because triangles are the simplest polygons in three dimensions and their areas can be computed directly from vector cross products without needing more advanced surface integral techniques.

This choice allows me to connect familiar vector methods from the IB Mathematics AA HL syllabus to a practical numerical scheme and to compare how effectively the same triangulation strategy approximates three different curved geometries.

Aim

The aim of this investigation is to approximate the curved surface areas of three common three-dimensional shapes – a sphere, a cylinder, and a torus – using triangular mesh approximations and to compare the accuracy of these approximations at different levels of refinement.

A torus is a three-dimensional donut-shaped surface formed by rotating a circle of radius r around an axis in the same plane at distance R ; it has no singular points and can be smoothly parameterised using two angular variables, making it particularly suitable for triangular mesh approximations.

By subdividing each curved surface into a network of flat triangular facets and calculating their combined area using vector cross products, this investigation explores how closely straight-edged polygonal meshes can represent smooth curved surfaces.

The accuracy of these approximations is evaluated against known exact formulas for surface area, and the relationship between mesh refinement (measured by the number of triangles in the mesh) and error reduction is analysed.

Parameters and Limits

For consistency and comparability across shapes, the following dimensional parameters are fixed throughout this investigation:

- Sphere: $R_{\text{sphere}} = 6$
- Cylinder: $r = 4$, $h = 9$
- Torus: Major radius $R_{\text{torus}} = 8$, Minor radius $r = 2$

In each case, the curved surface is represented by a two-parameter vector-valued function $\mathbf{r}(\mathbf{u}, \mathbf{v})$ that assigns each ordered pair (\mathbf{u}, \mathbf{v}) in a rectangular parameter domain to a unique point $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in three-dimensional space on the surface.

For the sphere, the parameters are two angular variables; for the cylinder, an angular variable and a linear height coordinate; and for the torus, two angular variables. This ensures that every point on each surface corresponds uniquely to a single pair of parameter values within the specified ranges.

Three levels of mesh refinement are applied to each shape:

- Coarse mesh: Low grid resolution
- Medium mesh: Moderate grid resolution
- Fine mesh: High grid resolution

For each shape, the parameter domain is subdivided into a regular $\mathbf{m} \times \mathbf{n}$ grid, where \mathbf{m} denotes the number of equal divisions in the first parameter and \mathbf{n} denotes the number of equal divisions in the second parameter.

Each rectangular grid cell is then divided into two triangles using a fixed diagonal, meaning that the total number of triangles depends directly on the values of \mathbf{m} and \mathbf{n} . The specific values

of \mathbf{m} and \mathbf{n} corresponding to the three levels of mesh refinement are summarised in Table 2, and these parameters determine both the structure of the mesh and the total number of triangles used in the subsequent calculations.

II) Mathematical Background

Vector Cross Product

The cross product of two vectors in three-dimensional space produces a third vector perpendicular to both. Given two vectors \mathbf{v}_1 and \mathbf{v}_2 , their cross product $\mathbf{v}_1 \times \mathbf{v}_2$ has magnitude equal to the area of the parallelogram spanned by the two vectors (Stewart 805).

For vectors $\mathbf{v}_1 = (a_1, b_1, c_1)$ and $\mathbf{v}_2 = (a_2, b_2, c_2)$, the cross product is computed as:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{pmatrix} b_1c_2 - c_1b_2 \\ c_1a_2 - a_1c_2 \\ a_1b_2 - b_1a_2 \end{pmatrix}$$

The magnitude of this vector is:

$$\|\mathbf{v}_1 \times \mathbf{v}_2\| = \sqrt{(b_1c_2 - c_1b_2)^2 + (c_1a_2 - a_1c_2)^2 + (a_1b_2 - b_1a_2)^2}$$

Triangle Area Justification:

Consider a triangle with vertices P_1 , P_2 , and P_3 in three-dimensional space.

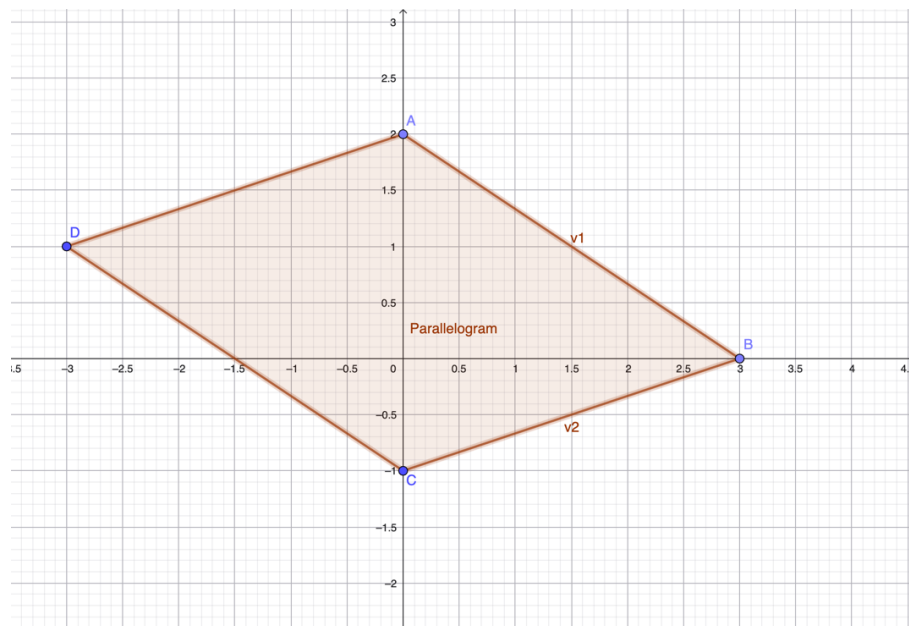
Two edge vectors can be defined as:

$$\mathbf{v}_1 = P_2 - P_1$$

$$\mathbf{v}_2 = P_3 - P_1$$

The area of the triangle is exactly half the area of the parallelogram formed by \mathbf{v}_1 and \mathbf{v}_2 :

$$A_{\text{parallelogram}} = \|\mathbf{v}_1 \times \mathbf{v}_2\|$$



$$A_{\text{triangle}} = \frac{1}{2} \|v_1 \times v_2\|$$

This formula is geometrically justified because the cross product magnitude gives the parallelogram area, and a triangle occupies half that area (Larson and Edwards 782).

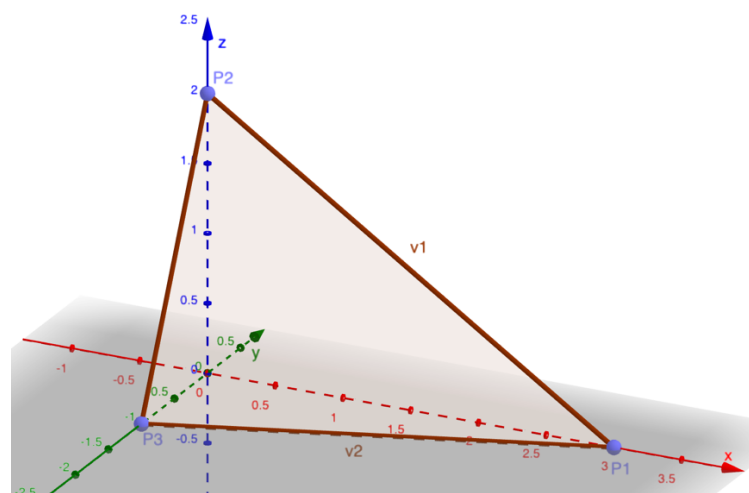


Figure 1: Triangle Area via Cross Product

Exact Surface Area Formulas

The exact surface areas for the three shapes under investigation are given by well-known geometric formulas (Clapham and Nicholson 512):

Sphere:

$$S = 4\pi R^2$$

For $R = 6$:

$$S = 4\pi(6)^2 = 144\pi \approx 452.389 \text{ (3 decimal places)}$$

Cylinder (curved surface area):

$$S = 2\pi rh$$

For $r = 4$, $h = 9$:

$$S = 2\pi(4)(9) = 72\pi \approx 226.195 \text{ (3 decimal places)}$$

Torus:

$$S = 4\pi^2 Rr$$

For $R = 8$, $r = 2$:

$$S = 4\pi^2(8)(2) = 64\pi^2 \approx 631.655$$

These exact values serve as benchmarks against which the triangular mesh approximations are compared.

Subdivision Concept

The fundamental approach is to approximate a smooth curved surface by covering it with a mesh of

flat triangular facets. Each triangle lies in a plane tangent or nearly tangent to the surface at that region. As the mesh becomes finer, containing more triangles of a smaller size, the approximation converges toward the true surface area. This links the vector cross-product formula for the area of a single triangle to a global approximation method for curved surfaces, where the total approximated area is the sum of the areas of all triangles in the mesh.

The construction process involves:

1. Parameterisation: Express the surface using two parameters (u, v) that map to three-dimensional coordinates (x, y, z).
2. Grid Generation: Divide the parameter space into a regular rectangular grid.
3. Vertex Placement: Compute the 3D coordinates of grid vertices using the parameterization.
4. Triangulation: Split each rectangular grid cell into two triangles by adding a diagonal.
5. Area Calculation: Use the vector cross product formula to compute the area of each triangle.
6. Summation: Add all triangular areas to obtain the total approximated surface area.

Refinement Levels

Three levels of mesh refinement are defined by the number of divisions along each parameter:

Table 1: Parameter definitions

Shape	Parameter 1	Range	Parameter 2	Range	Description
Sphere	θ	$0 \leq \theta \leq 2\pi$	ϕ	$0 \leq \phi \leq \pi$	θ : Rotation around vertical axis ϕ : Angle from positive z-axis
Cylinder	θ	$0 \leq \theta \leq 2\pi$	y	$0 \leq y \leq h$ (h: height of cylinder)	θ : Rotation around axis y : Vertical height from base of cylinder

Torus	θ	$0 \leq \theta \leq 2\pi$	ϕ	$0 \leq \phi \leq 2\pi$	θ : Rotation around central axis ϕ : Rotation around tube
-------	----------	---------------------------	--------	-------------------------	--

Table 2: Mesh refinement values

Shape	Coarse mesh (divisions)	Medium mesh (divisions)	Fine mesh (divisions)
Sphere	12 x 6	24 x 12	48 x 24
Cylinder	12 x 6	24 x 12	48 x 24
Torus	12 x 12	24 x 24	48 x 48

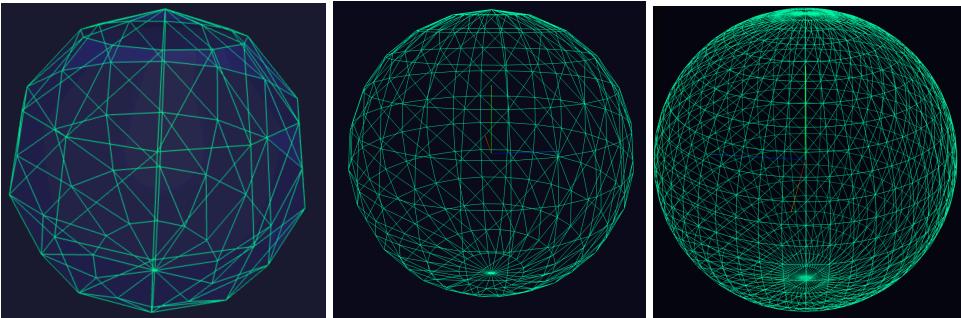


Figure 2: Spheres (coarse → medium → fine)

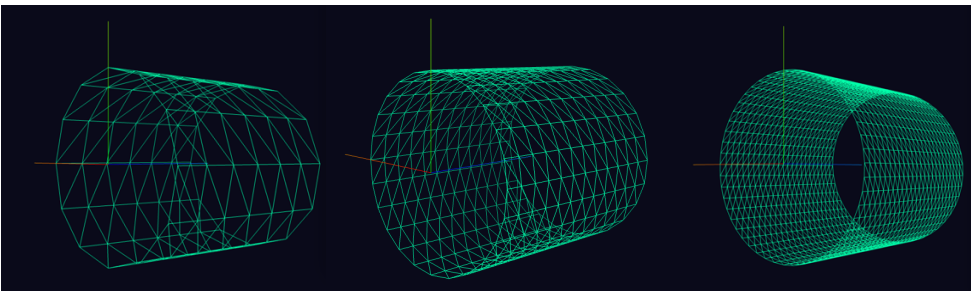


Figure 3: Cylinders (coarse → medium → fine)

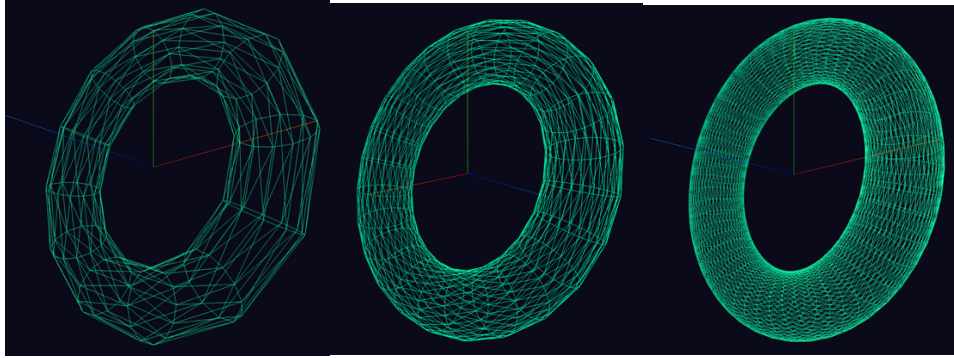


Figure 4: Torus (coarse \rightarrow medium \rightarrow fine)

Sphere:

m = divisions in θ (longitude), n = divisions in ϕ (latitude)

Cylinder:

m = divisions in θ (around the axis), n = divisions in y (height)

Torus:

m = divisions in θ (around the central axis), n = divisions in ϕ (around the tube)

For the sphere, the parameter grid forms m segments around θ . However, at $\phi = 0$ and $\phi = \pi$, the grid collapses to single points (the poles of the sphere), so the usable “bands” between poles are $n - 1$. Each band has m cells, and each cell is split into two triangles:

$$N_{\Delta, \text{sphere}} = 2m(n - 1)$$

Considering the cylinder, the grid forms m subdivisions around the circumference and n subdivisions along the height, giving mn rectangular cells. Each cell is split into two triangles:

$$N_{\Delta, \text{cylinder}} = 2mn$$

The surface of a torus is periodic in both parameters, so the rectangular parameter grid wraps around in two directions. This produces mn rectangular cells, each split into two triangles:

$$N_{\Delta, \text{torus}} = 2mn$$

Calculation of total triangles:**Sphere:**

Coarse mesh: $N = 2m(n-1) = 2(12)(6-1) = 2(12)(5) = 120$

Medium mesh: $N = 2(24)(12-1) = 2(24)(11) = 528$

Fine mesh: $N = 2(48)(24-1) = 2(48)(23) = 2208$

Cylinder:

Coarse mesh: $N = 2mn = 2(12)(6) = 144$

Medium mesh: $N = 2(24)(12) = 576$

Fine mesh: $N = 2(48)(24) = 2304$

Torus:

Coarse mesh: $N = 2mn = 2(12)(12) = 288$

Medium mesh: $N = 2(24)(24) = 1152$

Fine mesh: $N = 2(48)(48) = 4608$

Explanation of shape appearances:

Parameterisation is the process of representing a curved surface using two variables that uniquely determine the three-dimensional coordinates of each point on the surface, allowing the surface to be systematically divided into a grid for approximation and calculation.

Sphere:

The sphere is parameterised using two angular parameters, longitude $\theta \in [0, 2\pi]$ and latitude $\phi \in [0, \pi]$, which map each point on the surface to three-dimensional Cartesian coordinates. The

parameter space is divided into a rectangular $m \times n$ grid, where m represents divisions in the longitudinal direction and n represents divisions in the latitudinal direction. Each rectangular grid cell between adjacent latitude bands is subdivided into two triangles using a consistent diagonal. Special consideration is required at the north and south poles, where lines of longitude converge to a single point. These regions are handled using triangular fan structures, ensuring that no degenerate rectangular cells are formed. Figure 2 illustrates how increasing values of m and n produce progressively finer triangular approximations of the spherical surface.

Cylinder:

The curved lateral surface of the cylinder is parameterised using an angular coordinate $\theta \in [0, 2\pi]$ around the circular cross-section and a linear height coordinate $y \in [0, h]$. Together, these parameters map each point on the surface to three-dimensional Cartesian coordinates. The parameter domain is divided into a rectangular $m \times n$ grid, where m denotes the number of divisions in the angular direction and n denotes the number of divisions along the height. Each rectangular grid cell between adjacent height levels is subdivided into two triangles using a consistent diagonal, producing a uniform triangular mesh over the entire lateral surface. Because the cylinder has no singular points on its curved surface, the same triangulation rule applies throughout, and the mesh wraps seamlessly when θ reaches 2π . Figure 3 shows how increasing values of m and n generate progressively finer triangular approximations of the cylindrical surface.

Torus:

The surface of the torus is parameterised using two angular variables, $\theta \in [0, 2\pi]$ and $\phi \in [0, 2\pi]$, where θ represents rotation about the central axis of the torus and ϕ represents rotation around the circular cross-section of the tube. The parameter domain forms a rectangular grid with m divisions in the θ -direction and n divisions in the ϕ -direction. Each rectangular grid cell in the (θ, ϕ) parameter space is subdivided into two triangles using a consistent diagonal, producing a uniform triangular mesh over the entire toroidal surface. Because the torus is periodic in both parameters, the grid wraps seamlessly in both directions, and no boundaries or singular points require special treatment. Figure 4 illustrates how the mesh becomes progressively denser as the grid resolution is refined.

IV) Modelling & Calculations

Before modelling each individual shape, it is useful to describe the common structure of the meshes used. Each mesh is generated from a rectangular grid in the surface's parameter space: I define m as the number of equal divisions in the first parameter and n as the number of equal divisions in the second parameter. A mesh described as $m \times n$ therefore has m segments in the first parameter direction and n segments in the second, and each rectangular parameter cell is split into two triangles using a consistent diagonal. This construction determines both the total number of triangles and the approximate shape and distribution of the triangular facets for all three surfaces.

4.1 Sphere

Parameterisation of the Sphere

The spherical surface is represented using the vector-valued function

$$\mathbf{r}(\theta, \phi) = (R \sin \phi \cos \theta, R \sin \phi \sin \theta, R \cos \phi),$$

where

$$\theta \in [0, 2\pi], \phi \in [0, \pi].$$

For every ordered pair (θ, ϕ) , the function $\mathbf{r}(\theta, \phi)$ outputs the Cartesian coordinates (x, y, z) of a point on the surface of a sphere with radius R . The parameter θ controls rotation around the vertical axis, while ϕ controls the vertical position on the sphere from the top pole to the bottom pole. By varying both parameters across their ranges, the entire curved surface of the sphere is generated.

For this investigation, the radius of the sphere is fixed as $R = 6$.

Vertex Generation

To construct a triangular mesh, the parameter ranges are divided into equal intervals. For a mesh defined by m divisions in θ and n divisions in ϕ , the step sizes are

$$\Delta\theta = \frac{2\pi}{m}, \Delta\phi = \frac{\pi}{n}.$$

Vertices are generated at parameter values

$$\theta_i = i\Delta\theta (i = 0, 1, \dots, m-1),$$

$$\phi_j = j\Delta\phi (j = 0, 1, \dots, n).$$

Each vertex on the surface is therefore given by

$$P_{i,j} = \mathbf{r}(\theta_i, \phi_j).$$

Since all longitude lines converge at $\phi = 0$ and $\phi = \pi$, these values correspond to the north and south poles of the sphere and require special handling during triangulation.

Triangulation Rule

Between adjacent latitude bands ϕ_j and ϕ_{j+1} , rectangular parameter cells are formed. Each rectangular cell is subdivided into two triangles using a consistent diagonal. At the poles, where rectangular cells collapse into triangular regions, triangular fan structures are used to connect each latitude ring to the pole without creating degenerate triangles. For the sphere, the parameter grid forms m segments in the θ direction. At $\phi = 0$ and $\phi = \pi$, all lines of longitude converge to a single point, corresponding to the poles of the sphere. As a result, there are $n - 1$ usable bands between the poles, with each band containing m rectangular cells. Since each rectangular cell is subdivided into two triangles, the total number of triangles in the spherical mesh is

$$N_{\Delta, sphere} = 2m(n-1)$$

Worked Example: One Triangle (Coarse Mesh)

For the coarse mesh, the values $m = 12, n = 6$ are used.

The corresponding step sizes are

$$\Delta\theta = \frac{2\pi}{12} = \frac{\pi}{6}, \Delta\phi = \frac{\pi}{6}.$$

Consider a triangle formed between the parameter points:

$$(\theta_0, \phi_1), (\theta_1, \phi_1), (\theta_0, \phi_2).$$

Substituting values:

$$\theta_0 = 0, \theta_1 = \frac{\pi}{6}, \phi_1 = \frac{\pi}{6}, \phi_2 = \frac{\pi}{3}.$$

The Cartesian coordinates of the vertices are:

$$P_1 = \mathbf{r}(0, \frac{\pi}{6}) = (6\sin \frac{\pi}{6}, 0, 6\cos \frac{\pi}{6}) = (3, 0, 3\sqrt{3}),$$

$$P_2 = \mathbf{r}(\frac{\pi}{6}, \frac{\pi}{6}) = (3\cos \frac{\pi}{6}, 3\sin \frac{\pi}{6}, 3\sqrt{3}) = (\frac{3\sqrt{3}}{2}, \frac{3}{2}, 3\sqrt{3}),$$

$$P_3 = \mathbf{r}(0, \frac{\pi}{3}) = (6\sin \frac{\pi}{3}, 0, 6\cos \frac{\pi}{3}) = (3\sqrt{3}, 0, 3).$$

Two edge vectors are then defined:

$$\mathbf{v}_1 = P_2 - P_1, \mathbf{v}_2 = P_3 - P_1.$$

The area of the triangle is calculated using the cross product formula:

$$A_{\Delta} = \frac{1}{2} \|\mathbf{v}_1 \times \mathbf{v}_2\|.$$

This method is applied identically to all triangles in the mesh.

Total Area Summation

The approximated surface area of the sphere for a given mesh level is obtained by summing the areas of all individual triangles:

$$S_{\text{approx}} = \sum A_{\Delta}.$$

This summation is performed computationally using Python with NumPy to ensure numerical accuracy. The exact surface area of the sphere is given by

$$S_{\text{exact}} = 4\pi R^2 = 144\pi.$$

Table 3a: Approximated Surface Area of the sphere

Mesh level	Triangles	Approximated area (units ²)	Exact area (units ²)
Coarse	120	427.006	144π
Medium	528	445.959	144π
Fine	2208	450.776	144π

Error Calculations:

The accuracy of the approximation is measured using absolute and percentage error:

$$E_{\text{abs}} = |S_{\text{approx}} - S_{\text{exact}}|,$$

$$E_{\text{rel}} = \frac{E_{\text{abs}}}{S_{\text{exact}}} \times 100\%$$

Table 3b: Approximate Error for the Sphere

Mesh level	Absolute error	Percentage error (%)
Coarse	25.383	5.61
Medium	6.431	1.42
Fine	1.613	0.36

4.2 Cylinder (Curved Surface Only)

Parameterisation of the Cylinder

The curved (lateral) surface of the cylinder is represented using the vector-valued function

$$\mathbf{r}(\theta, y) = (r \cos \theta, r \sin \theta, y),$$

where

$$\theta \in [0, 2\pi], y \in [0, h].$$

This expression is read as follows: for each ordered pair (θ, y) , the function $\mathbf{r}(\theta, y)$ outputs the Cartesian coordinates (x, y, z) of a point on the curved surface of the cylinder. The parameter θ controls the position around the circular cross-section, while the parameter y determines the vertical position along the height of the cylinder. Varying both parameters across their ranges generates the entire lateral surface, excluding the top and bottom circular faces.

For this investigation, the dimensions of the cylinder are fixed as $r = 4, h = 9$.

Vertex Generation

For a mesh defined by m divisions in θ and n divisions along the height, the parameter step sizes are

$$\Delta\theta = \frac{2\pi}{m}, \Delta y = \frac{h}{n}.$$

Vertices are generated at

$$\begin{aligned}\theta_i &= i\Delta\theta (i = 0, 1, \dots, m-1), \\ y_j &= j\Delta y (j = 0, 1, \dots, n).\end{aligned}$$

Each vertex on the curved surface is given by

$$P_{i,j} = \mathbf{r}(\theta_i, y_j).$$

Since the surface wraps continuously around the circumference and has no singular points, the grid structure remains uniform across the entire surface.

Triangulation Rule

Adjacent values of θ and y form rectangular parameter cells, each of which is subdivided into two triangles using a consistent diagonal. The surface is periodic in the θ -direction, meaning that the first and last angular divisions are joined to form a closed surface. For the cylinder, the parameter grid forms m subdivisions around the circumference and n subdivisions along the height, producing mn rectangular cells. Since each rectangular cell is subdivided into two triangles, the total number of triangles in the cylindrical mesh is

$$N_{\Delta, \text{cylinder}} = 2mn.$$

Worked Example: One Triangle (Coarse Mesh)

For the coarse mesh,

$$m = 12, n = 6,$$

giving

$$\Delta\theta = \frac{\pi}{6}, \Delta y = \frac{9}{6} = 1.5.$$

Consider the triangle formed by the parameter points

$$(\theta_0, y_0), (\theta_1, y_0), (\theta_0, y_1),$$

where

$$\theta_0 = 0, \theta_1 = \frac{\pi}{6}, y_0 = 0, y_1 = 1.5.$$

The Cartesian coordinates of the vertices are:

$$P_1 = (4, 0, 0),$$

$$P_2 = (4\cos \frac{\pi}{6}, 4\sin \frac{\pi}{6}, 0) = (2\sqrt{3}, 2, 0),$$

$$P_3 = (4, 0, 1.5).$$

Two edge vectors are defined as

$$\mathbf{v}_1 = P_2 - P_1, \mathbf{v}_2 = P_3 - P_1.$$

The area of the triangle is calculated using

$$A_{\Delta} = \frac{1}{2} \|\mathbf{v}_1 \times \mathbf{v}_2\|.$$

This procedure is applied identically to all triangles in the cylindrical mesh.

The area of each triangle is calculated using

$$A_{\Delta} = \frac{1}{2} \| \mathbf{v}_1 \times \mathbf{v}_2 \|.$$

This calculation is applied to every triangle in the cylindrical mesh using the corresponding edge vectors, and the resulting triangle areas are summed to obtain the total approximated curved surface area.

Total Surface Area Approximation

The approximated curved surface area of the cylinder is obtained by summing the areas of all triangular facets:

$$S_{\text{approx}} = \sum A_{\Delta}.$$

This summation is performed computationally using Python and NumPy.

The exact curved surface area of the cylinder is

$$S_{\text{exact}} = 2\pi rh = 72\pi.$$

Table 4a: Approximated Curved Surface Area of the cylinder

Mesh level	Number of triangles	Approximated area	Exact area
Coarse	144	223.620	72π
Medium	576	225.549	72π
Fine	2304	226.033	72π

Error Calculations

The approximation accuracy is measured using

$$E_{\text{abs}} = |S_{\text{approx}} - S_{\text{exact}}|,$$

$$E_{\text{rel}} = \frac{E_{\text{abs}}}{S_{\text{exact}}} \times 100\%.$$

Table 4b: Approximated error of the cylinder

Mesh level	Absolute error	Percentage error (%)
Coarse	2.575	1.14
Medium	0.645	0.29
Fine	0.161	0.07

As the mesh resolution increases, the percentage error decreases, indicating improved approximation of the curved surface.

4.3 Torus

Parameterisation of the Torus

The torus is represented using the vector-valued function

$$\mathbf{r}(\theta, \phi) = ((R + r \cos \phi) \cos \theta, (R + r \cos \phi) \sin \theta, r \sin \phi),$$

Where $\theta \in [0, 2\pi]$, $\phi \in [0, 2\pi]$.

This expression is read as follows: for each ordered pair (θ, ϕ) , the function $\mathbf{r}(\theta, \phi)$ produces the Cartesian coordinates of a point on the surface of a torus. The parameter θ controls rotation around the central axis of the torus, while ϕ controls rotation around the circular cross-section of the tube. Varying both parameters across their ranges generates the entire closed surface.

For this investigation, the radii are fixed as $R = 8, r = 2$.

Vertex Generation

For a mesh defined by m divisions in θ and n divisions in ϕ , the parameter step sizes are

$$\Delta\theta = \frac{2\pi}{m}, \Delta\phi = \frac{2\pi}{n}.$$

Vertices are generated at

$$\theta_i = i\Delta\theta (i = 0, 1, \dots, m-1),$$

$$\phi_j = j\Delta\phi (j = 0, 1, \dots, n-1).$$

Each vertex is given by

$$P_{i,j} = \mathbf{r}(\theta_i, \phi_j).$$

Due to periodicity in both parameters, the grid wraps seamlessly in both directions and contains no boundaries or singular points.

Triangulation Rule

Adjacent parameter values form rectangular cells in the (θ, ϕ) parameter space. Each cell is subdivided into two triangles using a consistent diagonal. Since the torus is periodic in both parameters, the first and last divisions are connected in each direction to form a closed mesh. For the torus, the surface is periodic in both parameters, so the rectangular parameter grid wraps around in two directions and produces mn rectangular cells.

Since each rectangular cell is subdivided into two triangles, the total number of triangles in the toroidal mesh is

$$N_{\Delta, \text{torus}} = 2mn.$$

Worked Example: One Triangle (Coarse Mesh)

For the coarse mesh,

$$m = 12, n = 12,$$

giving

$$\Delta\theta = \frac{\pi}{6}, \Delta\phi = \frac{\pi}{6}.$$

Consider the triangle formed by the parameter points

$$(\theta_0, \phi_0), (\theta_1, \phi_0), (\theta_0, \phi_1),$$

where

$$\theta_0 = 0, \theta_1 = \frac{\pi}{6}, \phi_0 = 0, \phi_1 = \frac{\pi}{6}.$$

The Cartesian coordinates of the vertices are:

$$P_1 = (10, 0, 0),$$

$$P_2 = (10\cos \frac{\pi}{6}, 10\sin \frac{\pi}{6}, 0) = (5\sqrt{3}, 5, 0),$$

$$P_3 = (8 + 2\cos \frac{\pi}{6}, 0, 2\sin \frac{\pi}{6}) = (8 + \sqrt{3}, 0, 1).$$

Two edge vectors are defined as

$$\mathbf{v}_1 = P_2 - P_1, \mathbf{v}_2 = P_3 - P_1.$$

The area of the triangle is calculated using

$$A_{\Delta} = \frac{1}{2} \|\mathbf{v}_1 \times \mathbf{v}_2\|.$$

Total Surface Area Approximation

The approximated surface area of the torus is obtained by summing the areas of all triangular facets:

$$S_{\text{approx}} = \sum A_{\Delta}.$$

This summation is performed computationally using Python and NumPy.

The exact surface area of the torus is

$$S_{\text{exact}} = 4\pi^2 Rr = 64\pi^2.$$

Table 5a: Approximated Curved Surface Area of the torus

Mesh level	Number of triangles	Approximated area	Exact area
Coarse	288	606.883	$64\pi^2$
Medium	1152	625.372	$64\pi^2$
Fine	4608	630.078	$64\pi^2$

Error Calculations

The approximation accuracy is measured using

$$E_{\text{abs}} = |S_{\text{approx}} - S_{\text{exact}}|,$$

$$E_{\text{rel}} = \frac{E_{\text{abs}}}{S_{\text{exact}}} \times 100\%.$$

Table 5b: Approximated error of the torus

Mesh level	Absolute error	Percentage error (%)
Coarse	24.772	3.92
Medium	6.283	0.99
Fine	1.576	0.25

As the mesh resolution increases, the percentage error decreases, demonstrating convergence of the triangular approximation.

V) Results

Summary Table: Approximated Surface Areas and Errors

Shape	Refinement	Triangles	Approximated Area (sq units)	Exact Area (sq units)	Absolute Error (sq units)	Percentage Error (%)
Sphere	Coarse	120	427.006	452.389	25.383	5.61
Sphere	Medium	528	445.959	452.389	6.431	1.42
Sphere	Fine	2208	450.776	452.389	1.613	0.36
Cylinder	Coarse	144	223.620	226.195	2.575	1.14
Cylinder	Medium	576	225.549	226.195	0.645	0.29
Cylinder	Fine	2304	226.033	226.195	0.161	0.07
Torus	Coarse	288	606.883	631.655	24.772	3.92
Torus	Medium	1152	625.372	631.655	6.283	0.99
Torus	Fine	4608	630.078	631.655	1.576	0.25

Graph: Number of Triangles vs. Percentage Error

Observations:

- **Sphere:** The percentage error decreases substantially from 5.61% for the coarse mesh to 0.36% for

the fine mesh. This indicates steady convergence of the triangular approximation as the number of surface facets increases.

- **Cylinder:** The cylinder exhibits the lowest percentage error at all refinement levels, decreasing from 1.14% to 0.07%. This suggests that the curved surface of a cylinder is approximated very efficiently by planar triangular facets.

- **Torus:** The torus shows a clear reduction in error from 3.92% to 0.25% as mesh resolution increases, demonstrating convergence despite its more complex surface geometry.

Comparative Analysis

Overall, all three shapes display a consistent trend of decreasing percentage error with increasing mesh refinement. However, the **rate of convergence differs between shapes**. The cylinder converges most rapidly, followed by the torus, while the sphere exhibits the slowest convergence. This suggests that the geometric properties of a surface influence how effectively it can be approximated using flat triangular elements.

VI) Reflection & Evaluation

Accuracy Improvement:

For all three shapes, increasing the number of triangular facets led to a clear reduction in percentage error, demonstrating improved accuracy with mesh refinement.

For the **sphere**, the percentage error decreased from 5.61% for the coarse mesh to 0.36% for the fine mesh. This represents a reduction of approximately 93.6% in error.

For the **cylinder**, the percentage error decreased from 1.14% to 0.07%, corresponding to a reduction of approximately 93.9%.

For the **torus**, the percentage error decreased from 3.92% to 0.25%, representing a reduction of approximately 93.6%.

Although the percentage reduction is similar across shapes, the **absolute error and rate of convergence** differ. The error reduction does not appear linear with respect to the number of triangles;

instead, the error decreases more rapidly as the mesh becomes finer. This behaviour is consistent with numerical approximation theory, where polygonal surface approximations typically converge non-linearly as mesh resolution increases (Press et al.). No explicit convergence order was calculated, as this investigation focuses on numerical comparison rather than formal error modelling.

Straight-Edge vs. Curved Surface Error:

A fundamental limitation of this method is that **triangular facets are flat**, while the surfaces being approximated are curved. Each triangle approximates a small portion of the surface using straight edges, which introduces error whenever the surface has curvature.

The **sphere** has positive curvature at every point, meaning that no region of the surface is locally flat. As a result, planar triangles consistently underestimate the true surface area unless the mesh is sufficiently refined.

The **cylinder** has curvature in only one direction and can be unrolled into a flat rectangle without distortion. This geometric property explains why the cylinder exhibits the lowest approximation error at all mesh levels.

The **torus** has varying curvature across its surface, with regions of higher and lower curvature. This leads to intermediate error behaviour between the sphere and the cylinder.

These differences in geometric structure directly influence how effectively each surface can be approximated using flat triangular elements.

Grid Limitations:

The rectangular parameter grids used in this investigation impose several limitations. First, equal spacing in parameter space does not always correspond to equal spacing on the actual surface, particularly for the sphere and torus. Second, the triangulation topology is fixed, meaning that all grid cells are subdivided in the same manner regardless of local curvature. Third, boundaries or singular regions may introduce distortion when mapped onto a flat grid.

These limitations could be addressed by using adaptive mesh refinement, where smaller triangles are placed in regions of higher curvature. However, such methods fall outside the scope of this investigation and the IB Mathematics AA HL syllabus.

Shape-Specific Issue: Sphere Pole Singularities:

For the sphere, the parameterisation introduces singular points at the north and south poles, where multiple grid lines converge. Near these regions, triangles become narrower and less uniform, which can reduce local approximation accuracy. This effect contributes to the relatively higher error observed for the sphere compared to the cylinder at equivalent mesh resolutions.

Shape Comparison

Among the three shapes, the **cylinder** was the easiest to approximate, exhibiting the smallest errors at all refinement levels. The **sphere** was the most difficult to approximate due to its constant curvature in all directions. The **torus** showed intermediate difficulty, as its curvature varies across the surface.

Ranking the shapes by approximation difficulty:

1. Cylinder (easiest)
2. Torus
3. Sphere (hardest)

This ranking aligns with the geometric properties of each surface and highlights the relationship between curvature and approximation accuracy.

Real-World Relevance

Triangular mesh approximation is widely used in real-world applications, including computer graphics, animation, computer-aided design (CAD), 3D printing, and finite element analysis. In these contexts, complex curved surfaces are often represented using polygonal meshes to enable efficient computation and rendering.

This investigation demonstrates that **mesh refinement significantly improves accuracy**, but also increases computational complexity. Understanding this trade-off is essential in practical applications, where engineers and designers must balance precision against performance. The results of this investigation illustrate why surfaces with simpler curvature require fewer facets for accurate representation, while highly curved surfaces demand finer meshes.

VII) Conclusion

The aim of this investigation was to approximate the curved surface areas of a sphere, a cylinder, and a torus using triangular mesh approximations and the vector cross product, and to evaluate how mesh refinement affects accuracy. This aim was successfully achieved by subdividing each surface into triangular facets, calculating individual triangle areas using vector cross products, and comparing the summed approximations to exact surface area formulas.

Across all three shapes, the results showed a clear relationship between mesh refinement and accuracy. As the number of triangular facets increased from coarse to fine meshes, the percentage error consistently decreased, demonstrating convergence of the approximation method. The cylinder produced the lowest errors at all refinement levels, while the sphere exhibited the highest, with the torus lying between the two.

These differences can be explained by geometric properties. The cylinder's surface is developable and can be unrolled into a flat shape, making it easier to approximate with planar triangles. In contrast, the sphere has curvature in all directions, increasing approximation difficulty. The torus, with varying curvature, showed intermediate behaviour.

The investigation also confirmed the mathematical validity of the vector cross product as a method for calculating surface area through triangulation. This approach is widely applicable in fields such as computer graphics, engineering, and 3D modelling.

Limitations include the use of uniform grids and fixed triangulation patterns. Further exploration could investigate adaptive meshes or alternative parameterisations to improve accuracy.

Works Cited

Appendix