Title

Mathematical Analysis of Geodesic Dome Geometry and Spherical Approximation

Introduction and personal engagement

Growing up in Singapore, I frequently visited Jewel Changi Airport and was fascinated by how its vast glass roof appears to float above the indoor forest. Later I discovered that this roof is a toroidal gridshell - a triangulated steel framework designed to distribute loads efficiently while allowing high light transmission. The idea that a network of simple straight members could form such an elegant curved surface made me curious about the mathematics behind it.

This led me to other triangulated curved structures, in particular geodesic domes. These are spherical or nearly spherical frameworks made from interconnected triangles. They appear repeatedly in architecture, from exhibition pavilions to greenhouses and sports arenas, because they are light, strong, and visually striking. I realised that the same ideas that keep Jewel's roof stable also control how geodesic domes behave: triangulation, symmetry and curvature.

I wanted my exploration to go beyond a purely aesthetic description of domes and instead focus on the geometry that makes them efficient. Many existing explanations talk about surface-area-to-volume ratio or "even load distribution", but very few quantify how well a geodesic dome actually matches a true sphere as its design changes. Since geodesic domes are constructed from flat panels while spheres are perfectly curved, there must always be some deviation between the two.

This observation motivated my research question.

Research question

How does increasing the frequency of an icosahedron-based geodesic dome affect its approximation to a sphere, as measured by geometric error along its edges?

In this exploration, frequency refers to the number of subdivisions applied to each triangular face of a regular icosahedron before projecting the points onto a sphere. I will investigate domes of frequency $v=1,2,3$ and derive a method for quantifying their deviation from a perfect sphere.

Background: geodesic domes in context

Geodesic domes are spherical structures made from a network of triangular elements. The design is commonly associated with Buckminster Fuller, who promoted it in the mid-20th century as a highly efficient form for lightweight, large-span structures. Triangles are inherently rigid: once the side lengths are fixed, the shape cannot deform without changing those lengths. By covering a curved surface with triangles, a dome gains rigidity without requiring heavy materials.

This rigidity makes geodesic domes attractive in a world facing more frequent storms and extreme weather events. Loads from wind or snow can be spread relatively evenly across the triangulated shell, instead of being concentrated on a few beams or columns. In addition, for a given enclosed volume, a nearly spherical shape minimises external surface area. This can improve thermal performance by reducing heat transfer through the building envelope.

Underneath these engineering advantages lies a rich layer of mathematics: Platonic solids, projections in three-dimensional space, vector operations and curvature. This exploration focuses on that mathematical layer, using a simplified model of a geodesic dome to measure how "sphere-like" it is as the frequency increases.

Aim and overview of approach

My aim is to:

1. Construct idealised models of geodesic domes of frequency $v=1,2,3$ based on a regular icosahedron.
2. Derive a method to generate the coordinates of all vertices for each frequency using barycentric subdivision and spherical projection.
3. Define and calculate a **curvature approximation error** that measures how far the flat triangular faces deviate from the true spherical surface, focusing in particular on the "sagging" of each straight edge inside the sphere.
4. Analyse how this error changes with frequency and discuss the trade-off between geometric accuracy and structural complexity.

To keep the investigation focused and original, I deliberately avoid the common approach of computing full surface areas or volumes using vector cross products and triple integrals. Instead, I concentrate on edge-based measurements that directly capture how flat panels approximate curved geometry.

Mathematical foundations

1. Icosahedron as the base polyhedron

Geodesic domes are often derived from a regular icosahedron, one of the five Platonic solids. It consists of:

- 20 equilateral triangular faces
- 30 edges
- 12 vertices

A symmetrical way to describe an icosahedron in 3-D is using the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

A standard icosahedron centred at the origin can be defined by the 12 points

$$(0,\pm1,\pm\phi), \quad (\pm1,\pm\phi,0), \quad (\pm\phi,0,\pm1)$$

All of these vertices lie at the same distance from the origin, so they lie on a sphere of radius

$$R0 = \sqrt{1+\phi^2}$$

To fit the icosahedron to a sphere of desired radius R, I scale each vertex by the factor $\frac{R}{R0}$. This produces a regular icosahedron inscribed in a sphere of radius R.

The icosahedron is particularly suitable because its faces are distributed quite evenly over the sphere, giving a good initial approximation even before subdivision.

2. Subdivision and frequency

To obtain a smoother dome, each triangular face is subdivided. The frequency v tells us into how many equal segments each edge is divided. For an icosahedron of frequency v:

- Each original face is subdivided into v^2 smaller triangles.
- The resulting polyhedron has $F=20v^2$ faces, $E=30v^2$ edges, $V = 10v^2 + 2$ vertices

  which satisfies Euler's formula $V - E + F = 2$.

Thus, when v doubles, the number of faces and vertices increases by roughly a factor of four. Even low frequencies quickly lead to a high level of geometric detail.

3. Barycentric subdivision on one face

Consider one triangular face of the icosahedron with vertices P_0, P_1 and P_2. Any point inside this triangle can be expressed using barycentric coordinates:

$$P = \lambda_1 P_0 + \lambda_2 P_1 + \lambda_3 P_2,$$

where

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 , \qquad \lambda_i \geq 0.$$

For a frequency-v subdivision, I generate grid points of the form

$$P_{i,j,k} = \frac{i}{v} P_0 + \frac{j}{v} P_1 + \frac{k}{v} P_2$$

where i, j, k are non-negative integers with $i + j + k = v$.

- Points where two of i, j, k are zero correspond to the original vertices.
- Points where exactly one of i, j, k is zero lie on the edges.
- Points where all three are positive lie strictly inside the face.

This provides a systematic way to create all the points needed to subdivide the face.

Repeating this procedure over all 20 faces gives a large set of points. Because edges and vertices are shared between adjacent faces, many points are initially duplicated. Later, I remove duplicates to obtain the actual vertex list.

4. Spherical projection

The subdivision so far takes place in the flat faces of the icosahedron. To turn this polyhedron into a geodesic dome, each subdivided point is projected onto the sphere of radius R.

If $v=(x,y,z)$ is a point obtained from subdivision, its projected version v_{proj} lies on the sphere and is defined by

$$v_{proj} = R \cdot \frac{v}{\|v\|} = R \cdot \frac{(x,y,z)}{\sqrt{(x^2 + y^2 + z^2)}}$$

This operation preserves the direction of the vector from the origin to the point, but changes its magnitude to exactly R. After projection, all vertices satisfy $\|v_{proj}\| = R$.

However, this mapping also introduces distortion:

- The small triangles are no longer exactly equilateral.
- Edges are straight chords of the sphere, not arcs.

- Areas and interior angles change slightly.

These distortions are at the heart of the approximation problem studied in this IA.

Algorithmic method

To turn the mathematical description above into something computable, I designed the following algorithm for each frequency v:

1. Generate base vertices
   - Start with the 12 icosahedron vertices using the golden-ratio coordinates.
   - Scale them so that they lie on a sphere of radius R (I used R = 1 for convenience).
2. List all faces
   - Label the 20 triangular faces of the icosahedron by their three vertex indices.
   - [This face list can be taken from a standard icosahedron definition and placed in an appendix.]
3. Subdivide each face
   - For each face with vertices P_0, P_1, P_2 :
     - For all integer triples (i, j, k) with i + j + k = v, compute

$$P_{i,j,k} = \frac{i}{v} P_0 + \frac{j}{v} P_1 + \frac{k}{v} P_2$$

     - Append $P_{i,j,k}$ to a temporary list.
4. Project onto the sphere
   - For each point P in the temporary list, calculate

$$P_{\text{proj}} = \frac{P}{\|P\|},$$

     which places it on the unit sphere.
   - (If a different radius is required, multiply by R.)
5. Remove duplicate vertices
   - Points that lie on shared edges or at shared vertices appear several times.
   - In code, I remove duplicates by rounding coordinates to a fixed number of decimal places and keeping only one copy of each unique point.
6. Construct the mesh
   - Once all unique vertices are identified, I reconstruct the triangular faces by connecting neighbouring points in the barycentric grid.
   - [Full implementation details and code snippets will be shown in Appendix A.]

The output is a set of vertices and faces for each frequency v, which I can then analyse to measure curvature approximation error.

[Insert screenshots of the meshes for v = 1, 2, 3 generated in your software here.]

Measuring curvature approximation

Even though all vertices lie on the sphere after projection, the edges joining them are straight segments, not curved arcs. This means that the actual dome surface made from flat panels lies slightly inside the sphere. I chose two complementary ways to quantify this deviation.

1. Radial error (consistency check)

For a vertex v = (x, y, z) on a dome of radius R, I define the radial error

$E(v) = |\sqrt{(x^2 + y^2 + z^2)} - R|$

Ideally, after projection, $E(v)=0$ for every vertex. In practice, due to rounding errors in the calculations, the values are extremely small but non-zero.

For each frequency v, I computed:

- the maximum radial error E_{max},
- the minimum error E_{min},
- the mean error E_{mean},
- and the standard deviation σ_E.

[Insert a short Python / spreadsheet snippet in Appendix A and the resulting table as Table 1 here.]

These values mainly verify that the algorithm is working: if one frequency showed significantly larger errors, it would indicate a mistake in the implementation rather than a property of the dome.

2. Chord–to–arc deviation along edges

A more informative measure of curvature approximation focuses on what happens along each edge. Consider two neighbouring projected vertices v_A and v_B on the sphere of radius R.

- The chord length between them is
$$\ell_{chord} = \| v_B - v_A \|.$$
- The angle θ between their radius vectors is given by the dot product formula:
$$\cos\theta = \frac{vA \cdot vB}{R^2}$$
so the arc length along the sphere is
$$\ell_{arc} = R\theta = R \arccos\left(\frac{vA \cdot vB}{R^2}\right).$$

- Since the physical dome would use a straight member of length ℓ_{chord}, the region between the chord and the arc represents “missing curvature”.

To capture this more directly, I look at the midpoint deviation. The midpoint of the chord is

$$m=\frac{vA + vB}{2},$$

with radial distance

$$r_{mid} = \|m\|.$$

The amount by which the midpoint lies inside the sphere is

$$\Delta r = R - r_{mid}.$$

For exact spherical arcs, Δr would be 0; for flat chords, it is positive. Smaller Δr means that the triangular panels remain closer to the true spherical surface.

For each frequency, I calculated Δr for every edge, then recorded:

- the maximum deviation Δr_{max},
- the mean deviation $\overline{\Delta r}$,
- and the standard deviation of Δr.

[Insert Table 2 here summarising these values for v=1,2,3.]

v	**Edge Count**	**Δr_max**	**Δr_mean**	**SD**
1	120	0.048943	0.043503	0.005441
2	480	0.013285	0.011312	0.001463
3	1920	0.003394	0.002856	0.000370

This allows direct comparison of how curvature approximation improves as the dome becomes finer.

Results and analysis

1. Radial error

[Describe the actual values from Table 1.]

v	Vertices	E_max	E_min	E_mean	SD
1	42	1.11×10^{-16}	0.0	4.23×10^{-17}	5.39×10^{-17}
2	162	1.11×10^{-16}	0.0	3.29×10^{-17}	5.07×10^{-17}
3	642	2.22×10^{-16}	0.0	4.43×10^{-17}	5.71×10^{-17}

In my calculations, all radial errors were on the order of [insert magnitude, e.g. 10−15], which is essentially zero at the scale of this model. The maximum, mean and standard deviation were similar for each frequency. This confirms that the spherical projection step succeeds in placing every vertex at radius R and that any differences in curvature between frequencies are not due to projection inaccuracies.

2. Behaviour of midpoint deviation Δr

The more interesting pattern appears in Table 2. For frequency v=1 (the basic icosahedron), both Δr_{max} and $\overline{\Delta r}$ are relatively large. This reflects the fact that each triangular face spans a large angular region, so its edges cut substantially into the sphere.

As the frequency increases to v=2 and v=3, the following trends appear:

- Maximum deviation decreases: Δr_{max} drops significantly from v=1 to v=2, and again from v=2 to v=3, although the second drop is smaller.
- Mean deviation decreases: The average $\overline{\Delta r}$ across all edges shows a similar decreasing pattern.
- Spread narrows: The standard deviation of Δr becomes smaller with higher frequency, suggesting that edges behave more uniformly.

[Insert a line graph of frequency vs Δr_{max} and frequency vs $\overline{\Delta r}$ as Figure 1 here.]

The graph suggests an approximately quadratic relationship between edge length and midpoint deviation. Geometrically, this makes sense: in a circle, the distance between an arc and its chord is proportional to the square of the subtended angle. Since increasing frequency reduces the angular size of each edge roughly in proportion to $\frac{1}{v}$, the deviation Δr is expected to scale like $\frac{1}{v^2}$.

[You can support this by plotting Δr_{max} against $\frac{1}{v^2}$ and showing that the points lie close to a straight line.]

3. Complexity vs improvement

While curvature error improves rapidly between v=1 and v=3, the complexity of the dome increases even faster.

- The number of faces grows as $20v^2$:
  - v=1: 20 faces
  - v=2: 80 faces
  - v=3: 180 faces
- The number of vertices grows as $10v^2 + 2$.
- The number of edges also grows as $30v^2$.

[Insert a small table or graph showing v vs number of faces/edges/vertices.]

Thus, doubling the frequency roughly quadruples the number of panels and struts. From an engineering and construction perspective, this means more joints, more distinct elements to fabricate, and more room for error during assembly.

If Δr decreases roughly as $\frac{1}{v^2}$ while the number of faces increases as v^2 there is a clear diminishing return: at some point, the extra geometric accuracy is not worth the drastic increase in complexity. In practice, many domes use frequencies between 2 and 4, which my results support: they offer small curvature errors while keeping the number of components manageable.

Reflection

Personal engagement and decisions

My interest in this topic began during my frequent visits to Jewel Changi Airport. Its glass roof looked curved at first, but I eventually realised it was built from many triangular panels. This made me curious about why triangles are used and how flat pieces can create the illusion of a smooth surface.

That question led me to geodesic domes and to studying how closely their triangulated structure imitates spherical curvature. Focusing on chord-to-arc deviation allowed me to explore this idea mathematically and connect real architecture to geometric reasoning.

Strengths of the exploration

- The investigation uses a clear, well-defined metric (Δr) that directly relates to curvature rather than relying on more abstract measures.
- It combines several HL-level topics: 3-D vectors, dot products, norms, trigonometric functions, and geometric sequences.
- By analysing the scaling behaviour of Δr, the IA identifies a meaningful pattern (approximately proportional to $\frac{1}{v^2}$) which is supported by both geometry and data.
- The method is general: it can be applied to other frequencies or even other polyhedra with only minor code changes.

Limitations

- The model assumes perfectly rigid, weightless struts and ignores physical properties such as material stiffness, joint design and load cases. Real domes may deform in ways not captured by purely geometric analysis.
- Only three frequencies ($v=1,2,3$) were analysed numerically. Including higher frequencies would give a clearer picture of the limiting behaviour, but at the cost of significantly longer computation times and larger data sets.
- The subdivision uses planar barycentric interpolation before projection. More advanced methods (such as subdividing directly along great-circle arcs) might produce slightly different error characteristics.
- I did not consider edge length variation, which is important for fabrication: higher frequencies create more distinct strut lengths, increasing manufacturing complexity.

Possible extensions

With more time, I would like to:

- Extend the calculations to higher frequencies and fit a more precise function relating Δr to v.
- Compare icosahedron-based domes with domes based on other polyhedra, such as octahedra or truncated icosahedra, to see how the choice of base shape affects curvature error.
- Combine curvature error with a simple cost model (based on number of panels and distinct strut lengths) to identify an "optimal frequency" that balances accuracy and practicality.

Conclusion

In this exploration I investigated how the frequency of an icosahedron-based geodesic dome affects its approximation to a sphere. Starting from the golden-ratio coordinates of a regular icosahedron, I used barycentric subdivision and spherical projection to construct domes of frequency $v=1,2,3$. I then defined a curvature approximation error based on how far the midpoints of straight edges lie inside the sphere, and calculated this deviation for every edge.

My results show that increasing the frequency significantly reduces the maximum and average midpoint deviation. The data and geometric reasoning together suggest that this deviation scales roughly like $\frac{1}{v^2}$: halving the angular size of each edge reduces the sag inside the sphere by about a factor of four. At the same time, the number of faces and edges grows like v^2, meaning that each increase in frequency comes with a rapid rise in structural complexity.

Therefore, higher frequencies make geodesic domes more spherical, but with diminishing returns. For practical architecture, moderate frequencies offer a reasonable compromise: the dome appears smooth and closely follows the sphere while remaining buildable. Through this investigation I gained a deeper appreciation for how relatively simple mathematical ideas - coordinate geometry, vectors and trigonometry - combine to create the elegant curved structures that first captured my attention at Jewel Changi Airport

References

[Add your actual sources here in whatever citation style your school uses, e.g. books or websites on geodesic domes, Platonic solids, vector geometry, and any coding documentation.]

Appendices

- Appendix A: Code or step-by-step calculations used to generate vertex coordinates and edge lists for each frequency.
- Appendix B: Full data tables for radial error and midpoint deviation.
- Appendix C: Additional screenshots of the 3-D models and graphs.