Chapter Four

BP: An Illustrating Example

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四川大学 计算机学院 章毅教授

Zhang Yi, *IEEE Fellow*
Autumn 2018

Outline

- Brief Review of Backpropagation Algorithm
- An Illustrating Example
- Experiments
- Assignment

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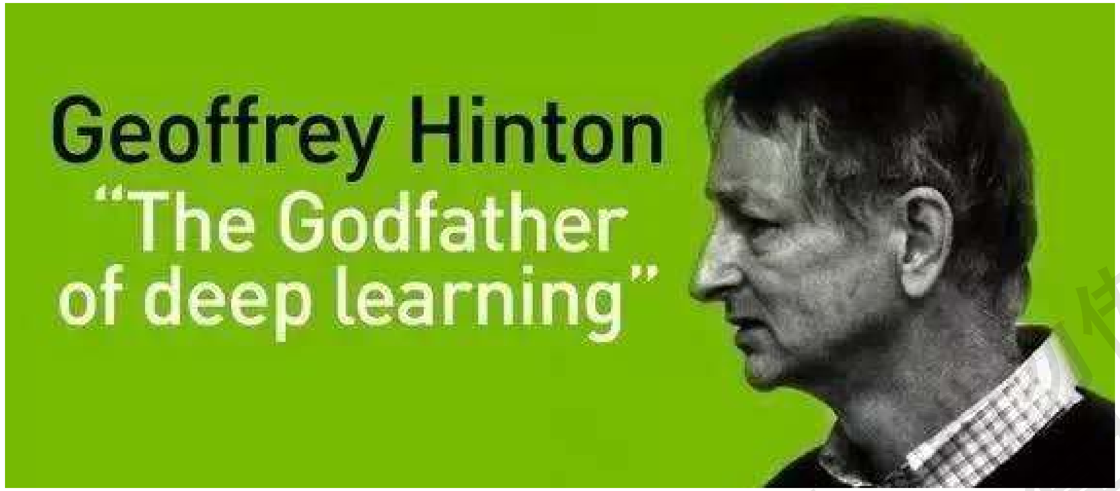
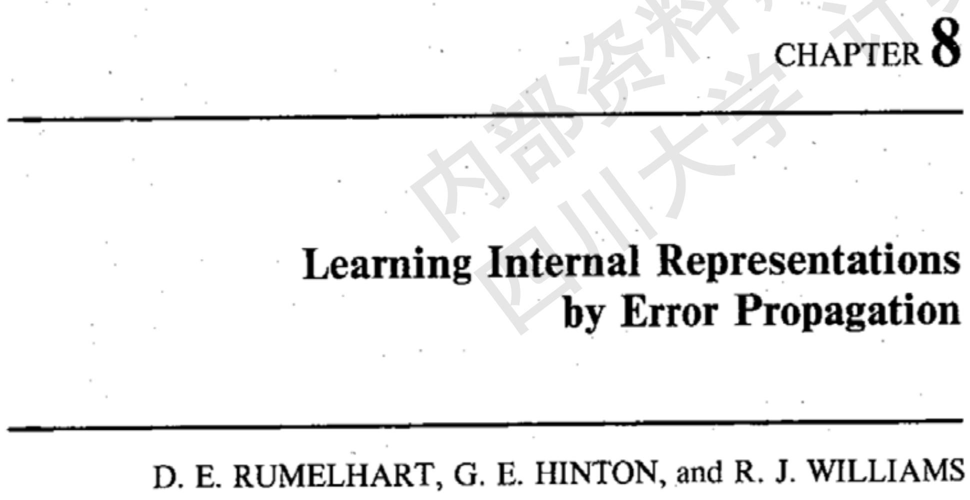
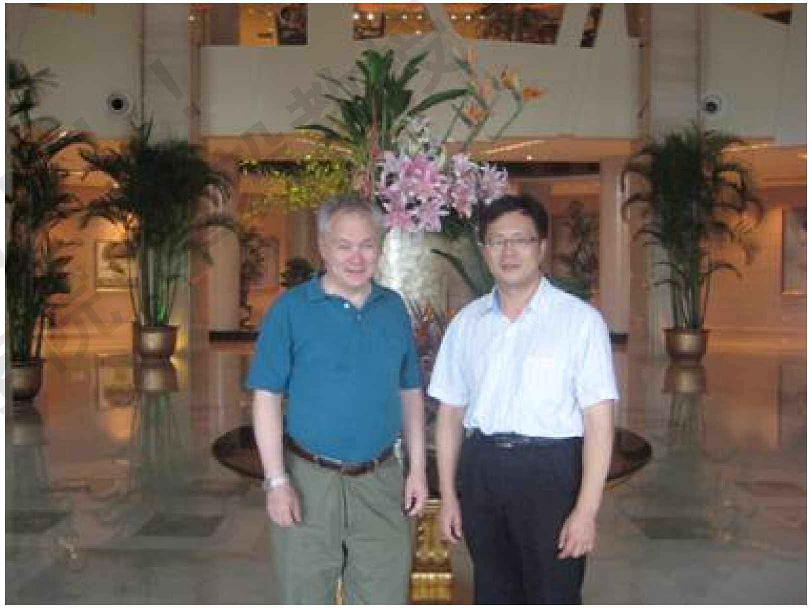
Brief History of BP




CHAPTER 8

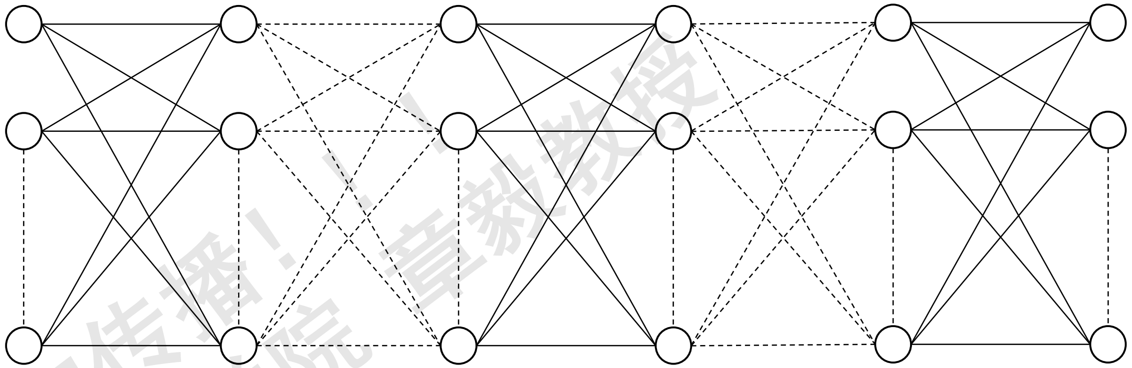
Learning Internal Representations by Error Propagation

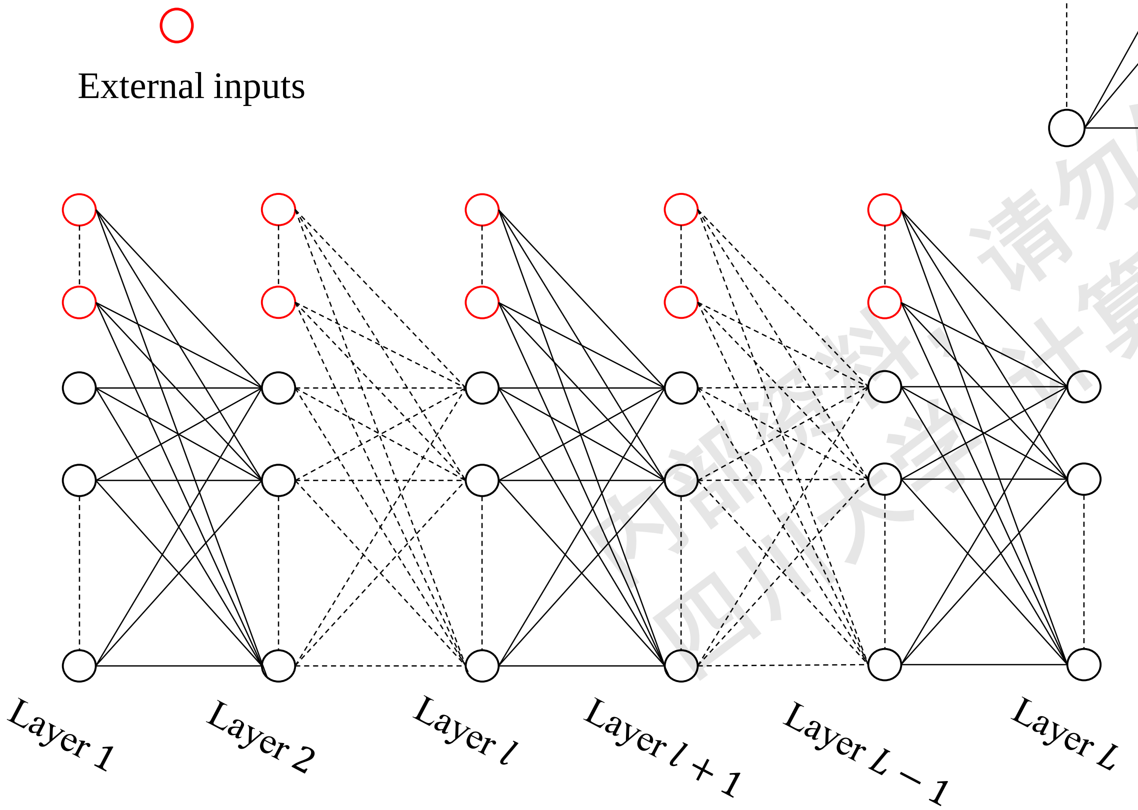
D. E. RUMELHART, G. E. HINTON, and R. J. WILLIAMS



Professor P. Werbos

Computational Model of Neural Networks





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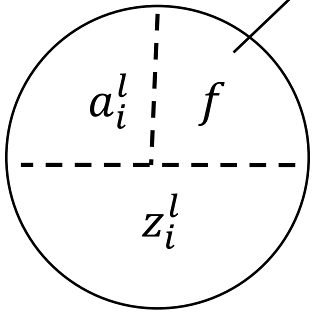
Local function defined on neuron



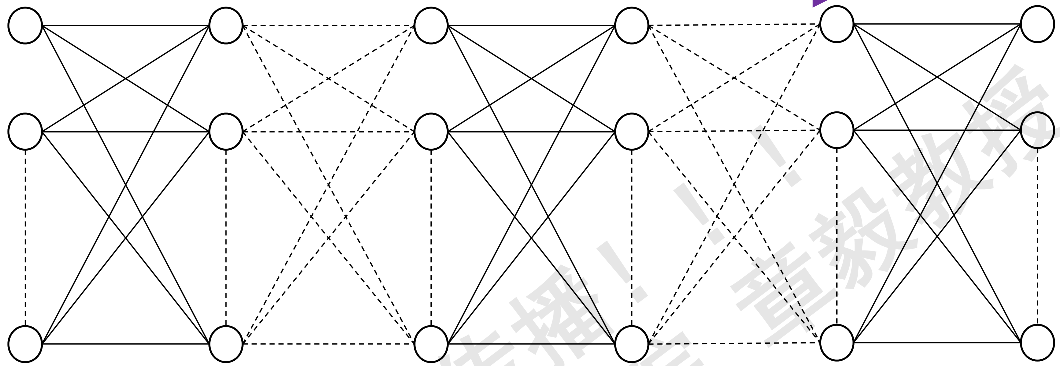
$$a_i^l = f(z_i^l)$$

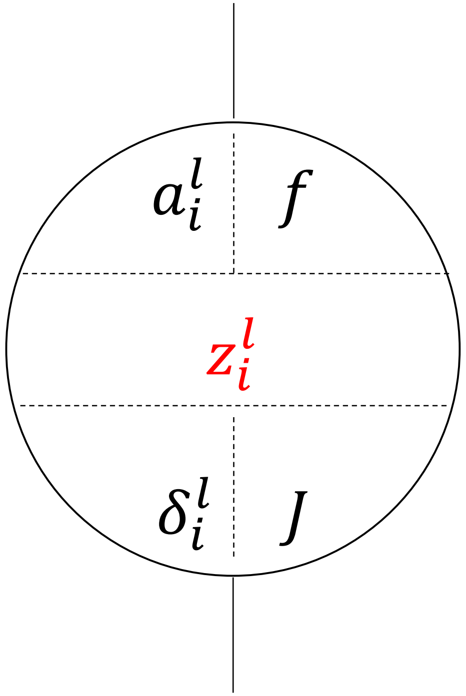
$$a^l = \begin{bmatrix} a_1^l \\ a_2^l \\ \vdots \\ a_{n_l}^l \end{bmatrix}$$

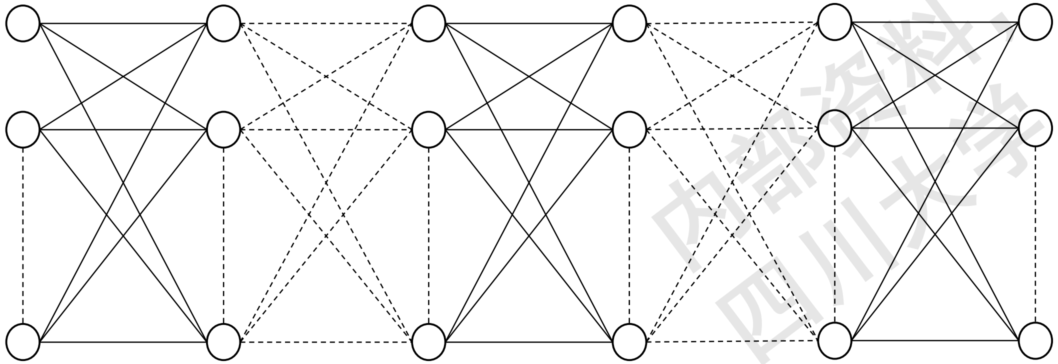
l layer i^{th} neuron

Local activation function f

Forward computing a^l





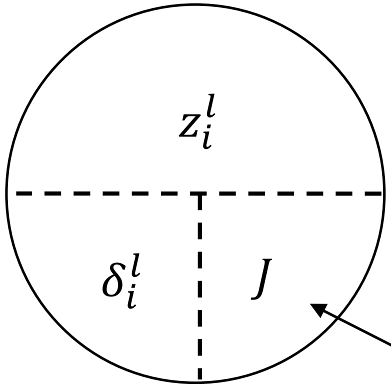


Backpropagation δ^l

Global cost function J

l layer i^{th} neuron

$$\delta_i^l = \frac{\partial J}{\partial z_i^l}$$



$$\delta^l = \begin{bmatrix} \delta_1^l \\ \delta_2^l \\ \vdots \\ \delta_{n_l}^l \end{bmatrix}$$

Global function defined on network

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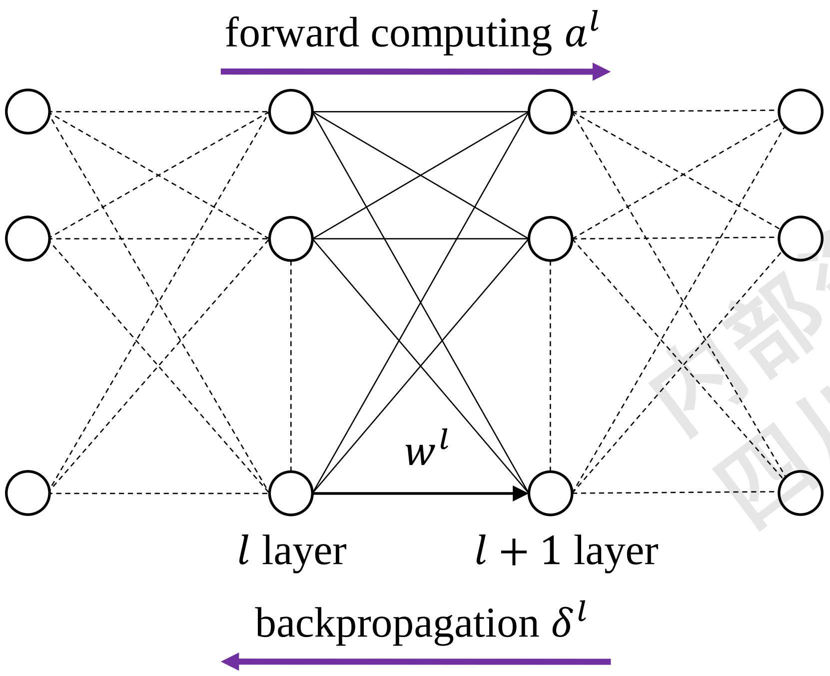
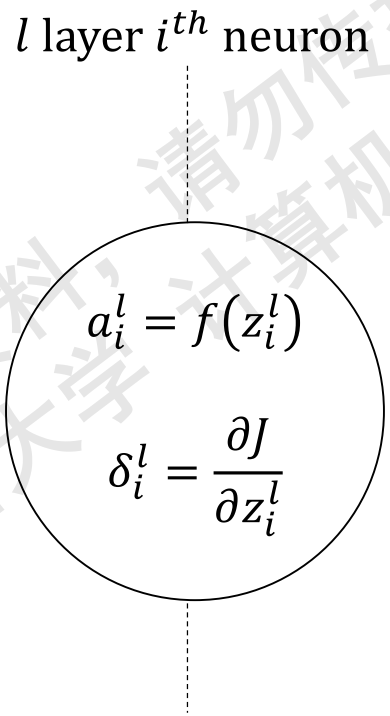
One Page to Understand BP

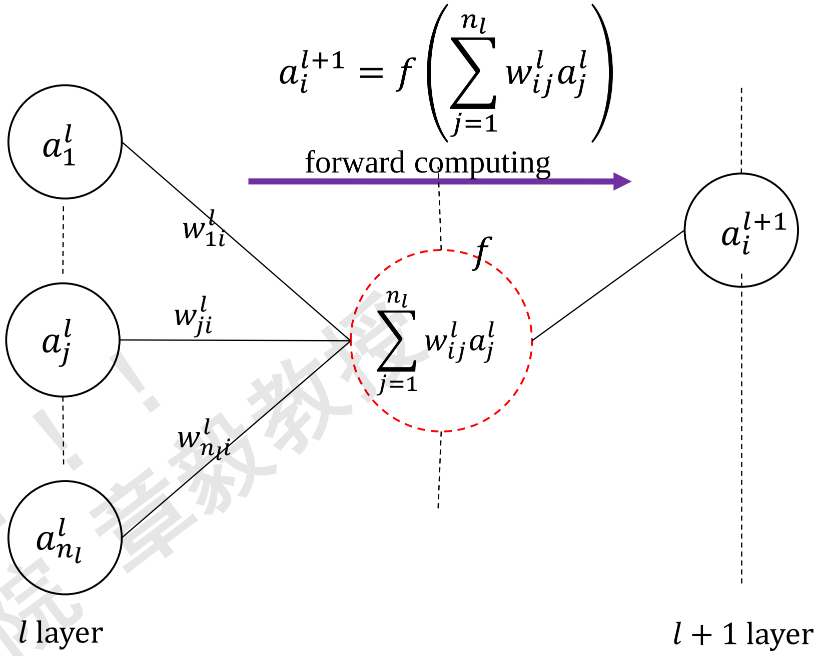
Cost function: $J(w^1, \cdots, w^L)$

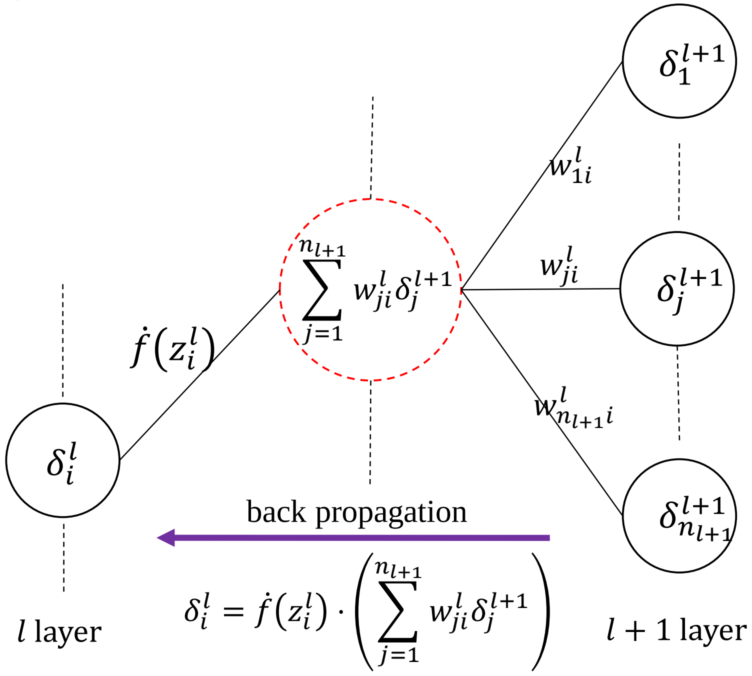
Updating rule: $w_{ji}^l \leftarrow w_{ji}^l - \alpha \cdot \frac{\partial J}{\partial w_{ji}^l}$

Relationship: $\frac{\partial J}{\partial w_{ji}^l} = \delta_j^{l+1} \cdot a_i^l$









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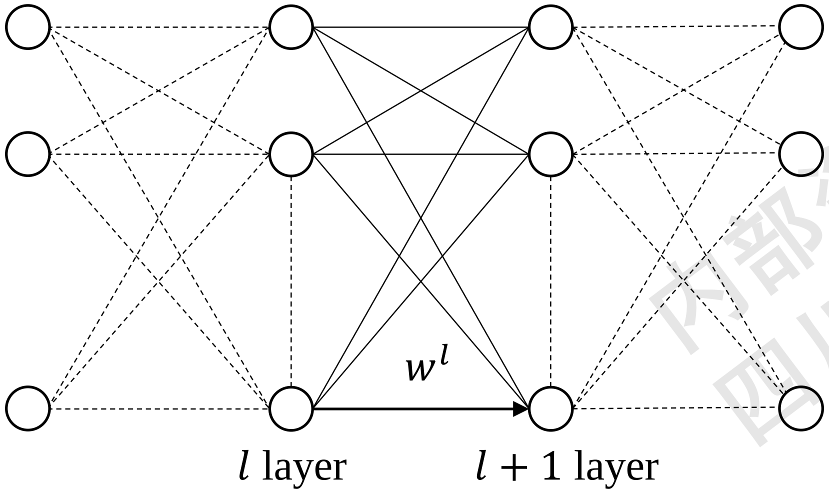
BP Functions

Cost function: $J(w^1, \cdots, w^L)$

Updating rule: $w_{ji}^l \leftarrow w_{ji}^l - \alpha \cdot \frac{\partial J}{\partial w_{ji}^l}$

Relationship: $\frac{\partial J}{\partial w_{ji}^l} = \delta_j^{l+1} \cdot a_i^l$

forward computing a^l



backpropagation δ^l

l layer i^{th} neuron

$$a_i^l = f(z_i^l)$$

$$\delta_i^l = \frac{\partial J}{\partial z_i^l}$$

%forward computing

function $fc(w^l, a^l)$

$for\ i = 1: n_{l+1}$

$$z_i^{l+1} = \sum_{j=1}^{n_l} w_{ij}^l a_j^l$$

$$a_i^{l+1} = f(z_i^{l+1})$$

end

%backpropagation

function $bc(w^l, \delta^{l+1})$

$for\ i = 1: n_l$

$$\delta_i^l = \dot{f}(z_i^l) \cdot \left(\sum_{j=1}^{n_{l+1}} w_{ji}^l \delta_j^{l+1} \right)$$

end

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The BP Algorithm

BP Algorithm:

Step 1. Input the training data set $D = \{(x, y)\}$

Step 2. Initial each w_{ij}^l, and choose a learning rate α.

Step 3. For each mini-batch sample $D_m \subseteq D$

$$a^1 \leftarrow x \in D_m;$$

for $l = 2 : L$

$$a^l \leftarrow fc(w^l, a^l);$$

end

$$\delta^L = \frac{\partial J}{\partial z^L};$$

for $l = L - 1 : 2$

$$\delta^l \leftarrow bc(w^l, \delta^{l+1});$$

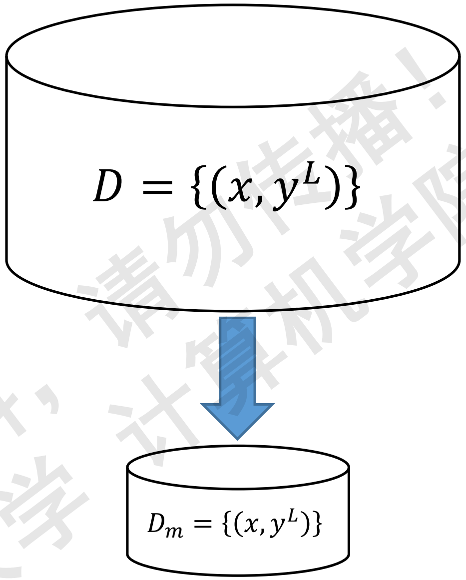
end

$$\frac{\partial J}{\partial w_{ji}^l} \leftarrow \frac{\partial J}{\partial w_{ji}^l} + \delta_j^{l+1} \cdot a_i^l;$$

Step 4. Updating

$$w_{ji}^l \leftarrow w_{ji}^l - \alpha \cdot \frac{\partial J}{\partial w_{ji}^l};$$

Step 5. Return to Step 3 until each w^lconverge.



function $fc(w^l, a^l)$

$for\ i = 1 : n_{l+1}$

$$z_i^{l+1} = \sum_{j=1}^{n_l} w_{ij}^l a_j^l$$

$$a_i^{l+1} = f(z_i^{l+1})$$

end

Relationship:

$$\frac{\partial J}{\partial w_{ji}^l} = \delta_j^{l+1} \cdot a_i^l$$

function $bc(w^l, \delta^{l+1})$

$for\ i = 1 : n_l$

$$\delta_i^l = \dot{f}(z_i^l) \cdot \left(\sum_{j=1}^{n_{l+1}} w_{ji}^l \delta_j^{l+1} \right)$$

end

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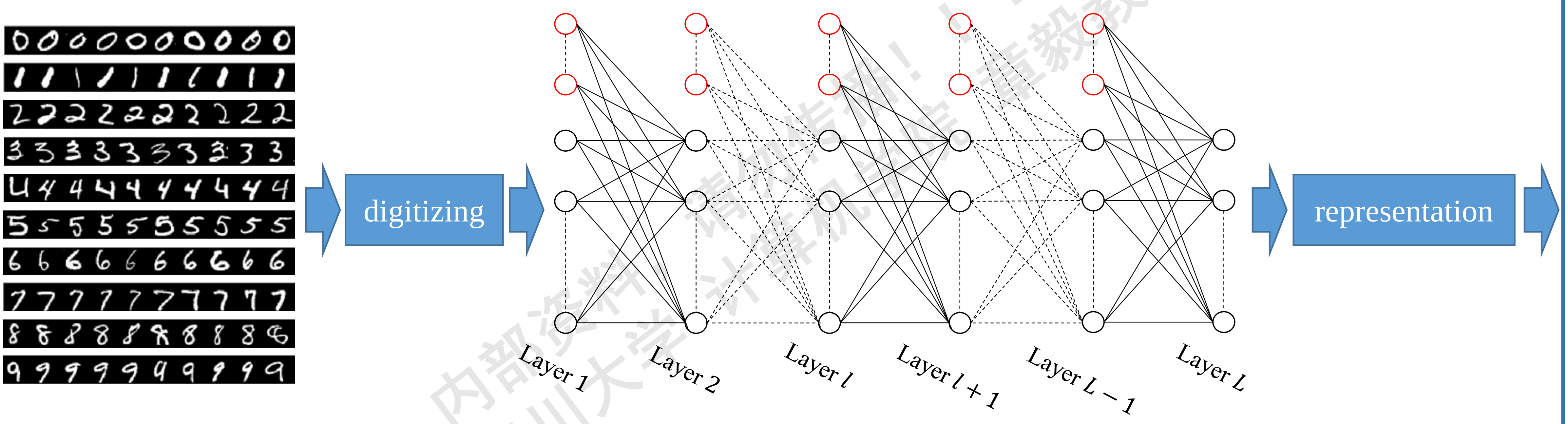
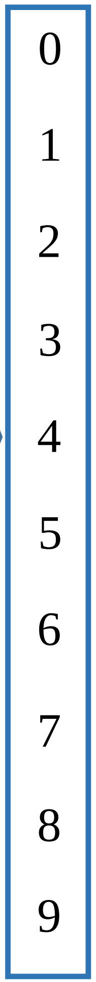
Handwritten digits recognition problem





Task:
Use Backpropagation algorithm to train a neural network to recognize handwritten digits.

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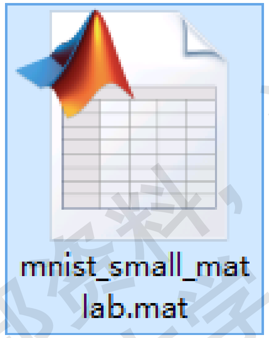
Step 1: Data Preparation

Dataset: **MNIST_small**

MNIST is a database of handwritten digits created by "re-mixing" the samples from MNIST's original datasets. It contains digits written by high school students and employees of the United States Census Bureau. The digits have been size-normalized and centered in 28×28 images.

MNIST_small dataset is a subset of MNIST containing 10000 training samples and 2000 testing samples.

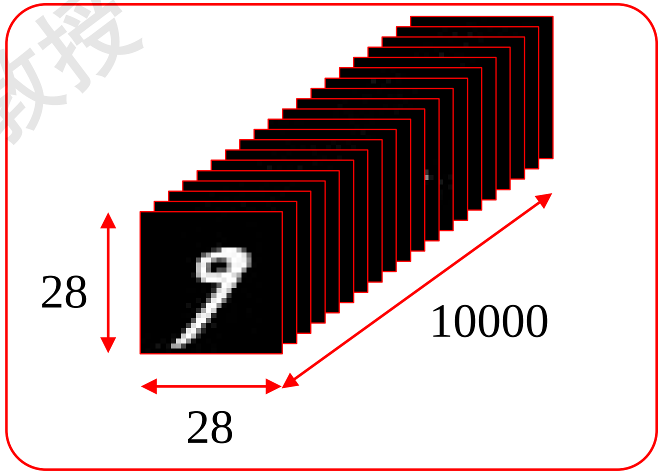
New York University

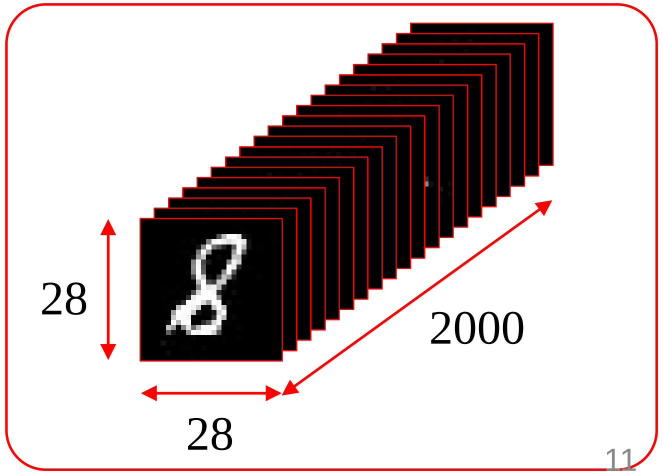


Training set
- Used for training network
- 10000 samples

Testing set
- Used for evaluating network performance
- 2000 samples

Data





Download link:
MNIST http://yann.lecun.com/exdb/mnist/
MNIST_small: https://github.com/kswersky/nnet/blob/master/mnist_small.mat

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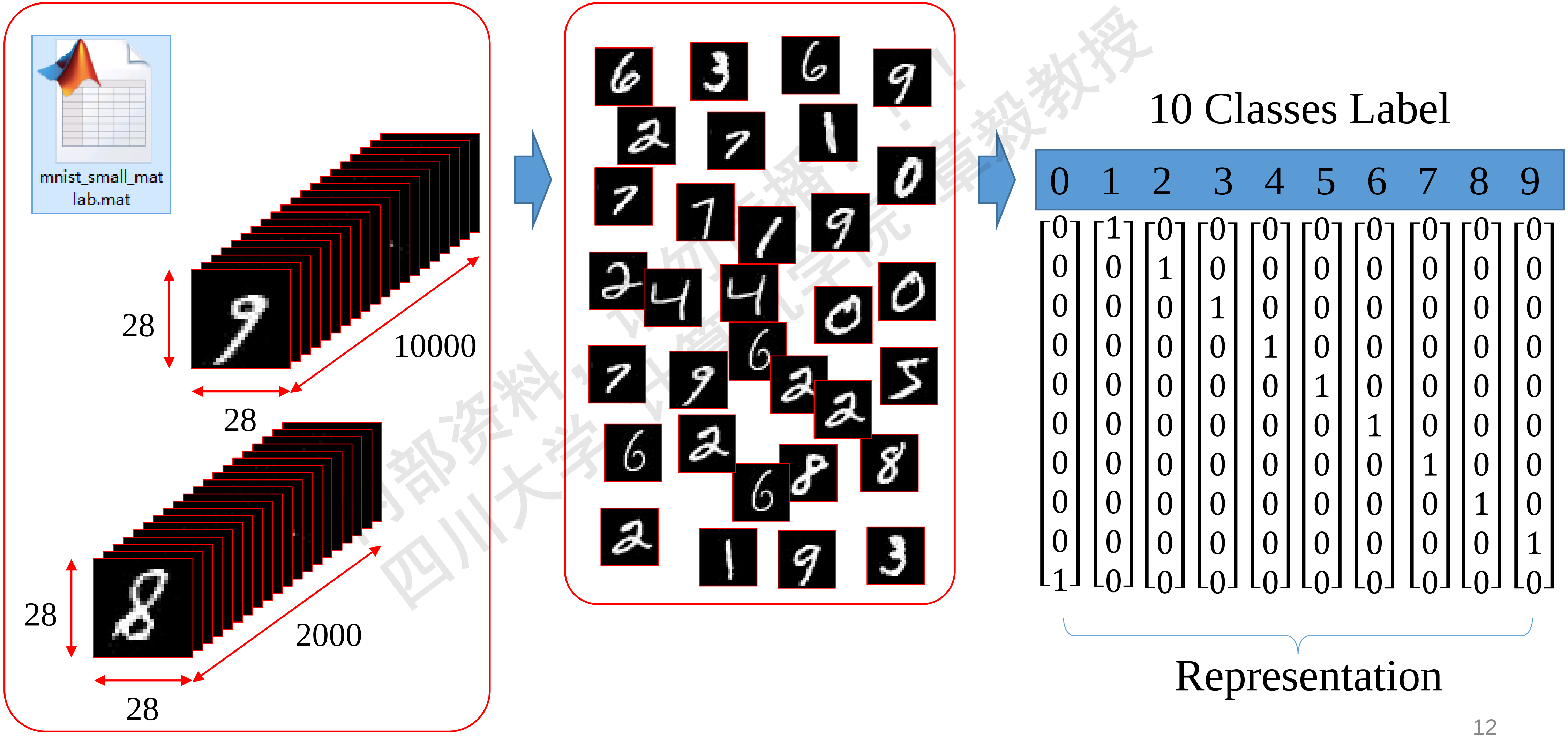
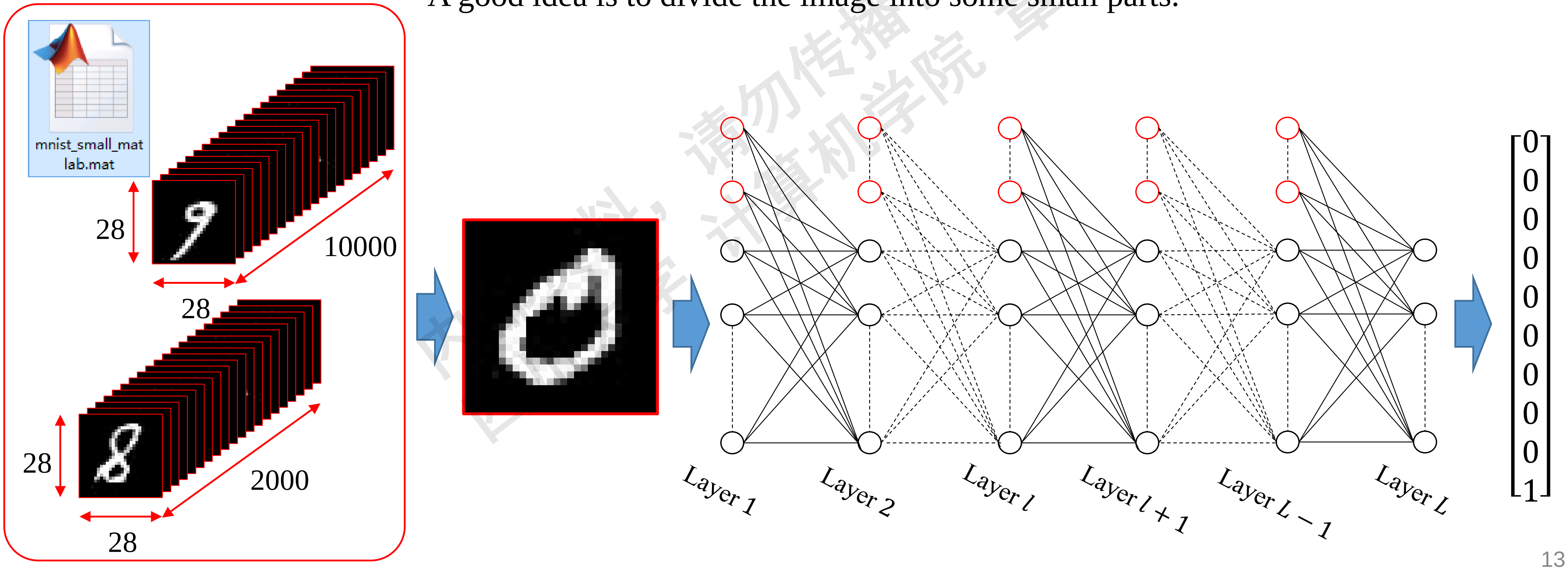
Step 1: Data Preparation



Step 1: Data Preparation

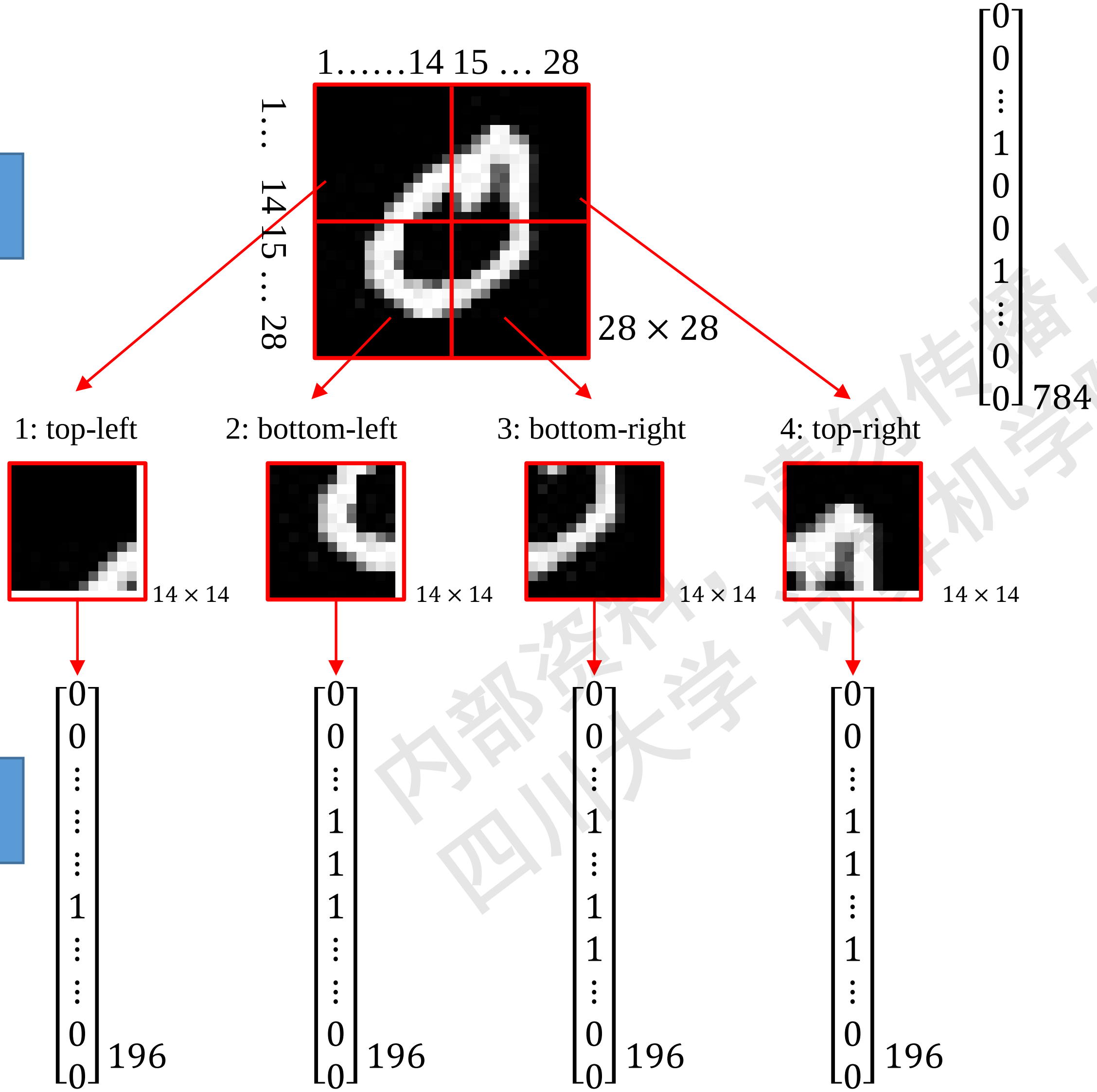
The input image is a $28 \times 28 = 784$ dimensional vector, relatively large in some situation.
A good idea is to divide the image into some small parts.



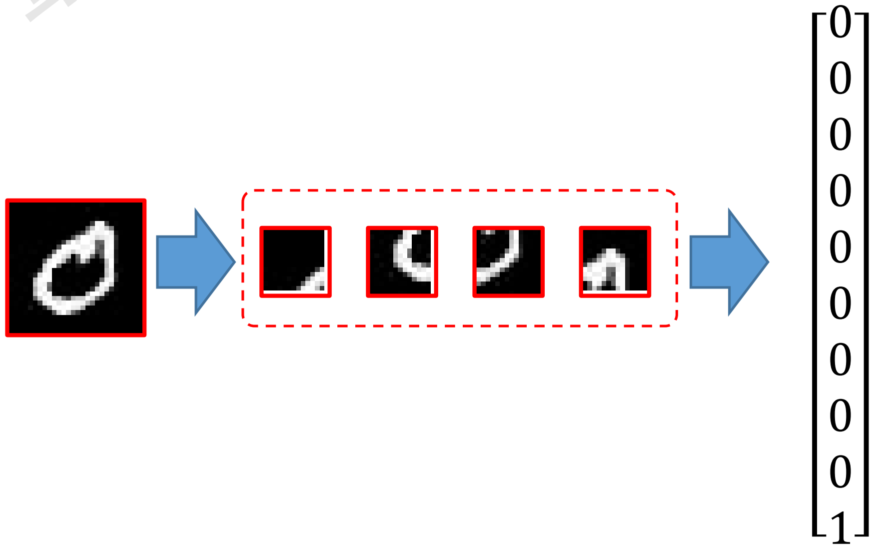
Step 1: Prepare Data Preparation

Dividing

digitizing



Representation



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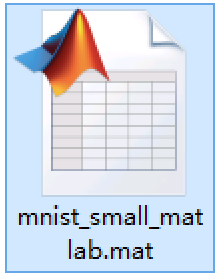
Step 1: Data Preparation



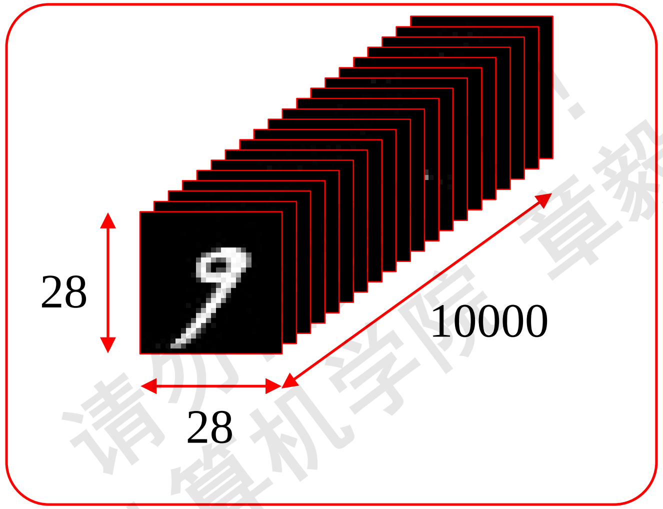
Training set

- ❑ Used for training network
- ❑ 10000 samples
- ❑ each sample contains four elements

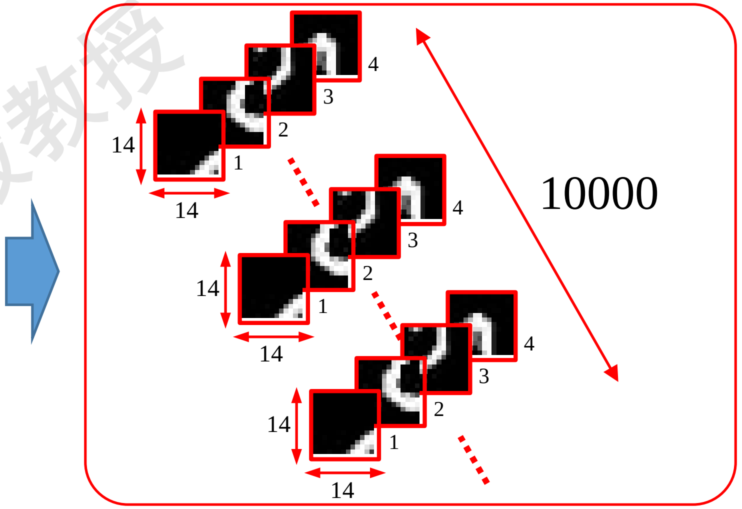
Testing set

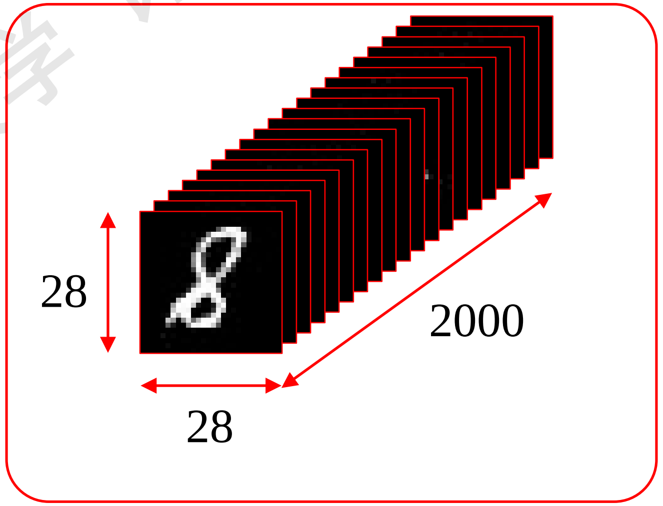
- ❑ Used for evaluating network performance
- ❑ 2000 samples
- ❑ each sample contains four elements

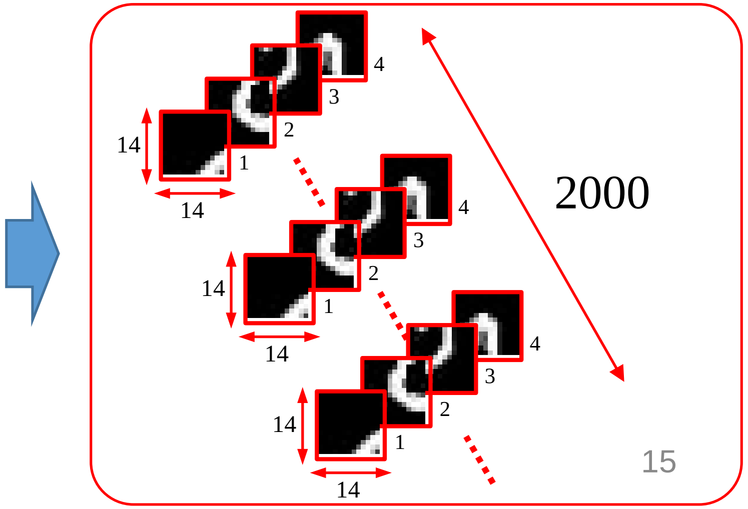
Training Data



Training Data







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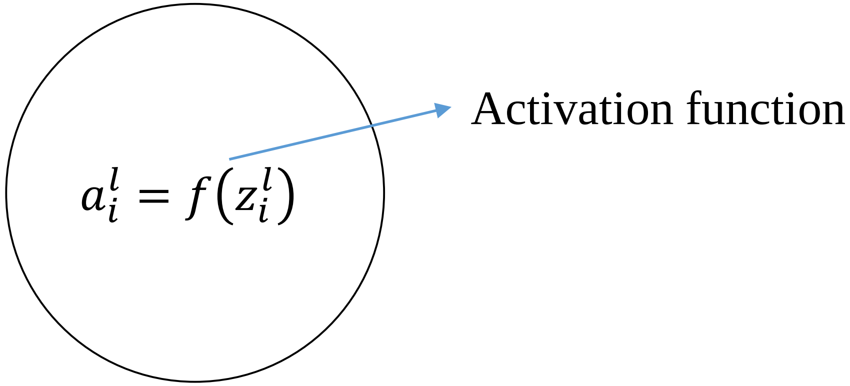
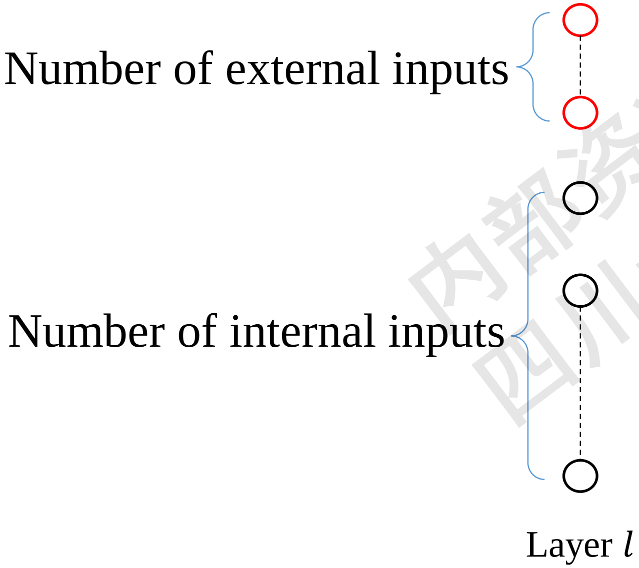
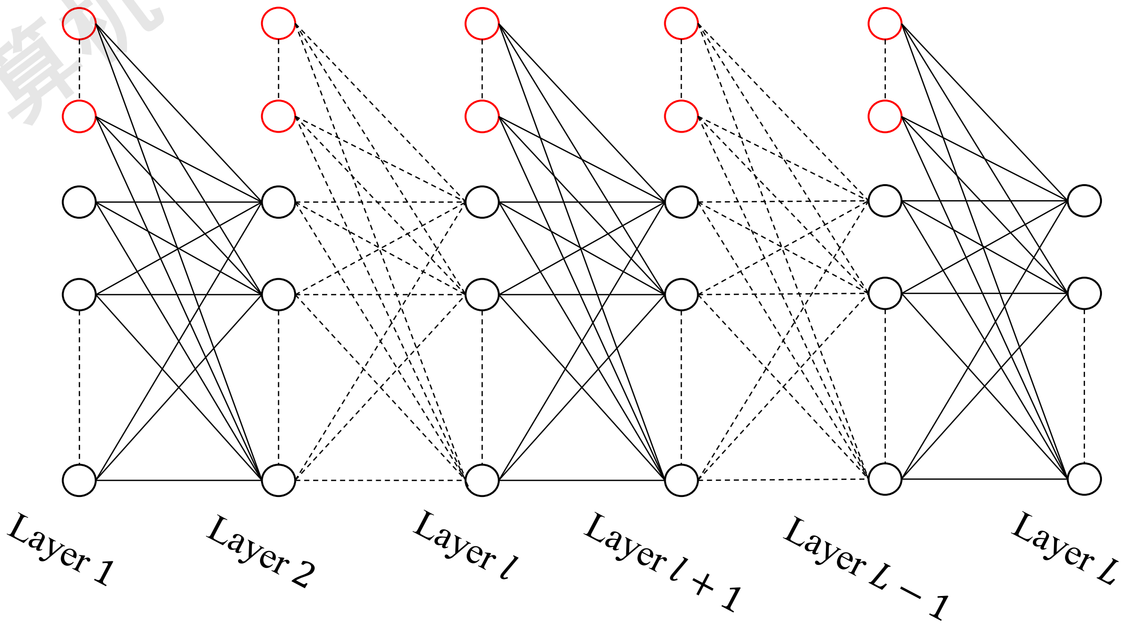
Step 2: Design Network Architecture



Network architecture design:

1. Number of layers
2. Number of neurons in each layer (external neurons and internal neurons)
3. Activation function





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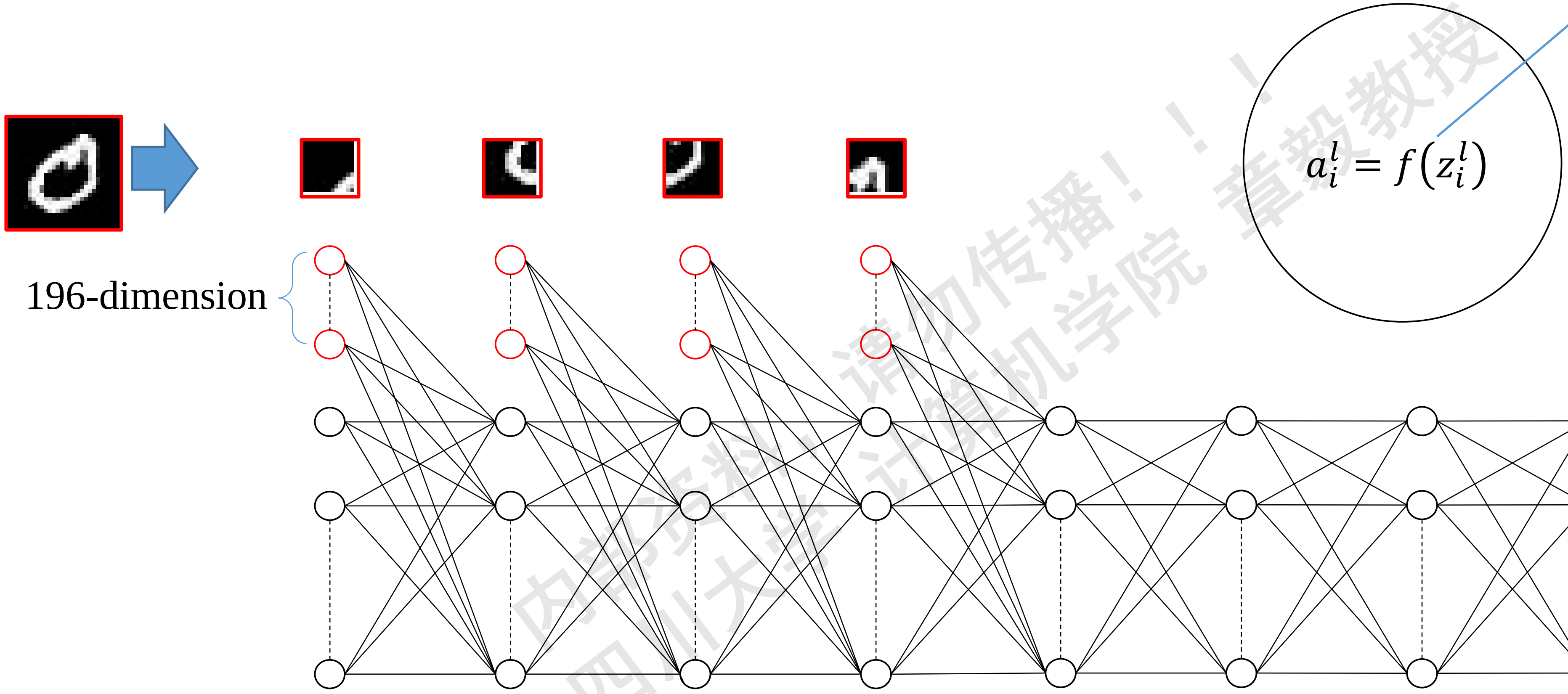
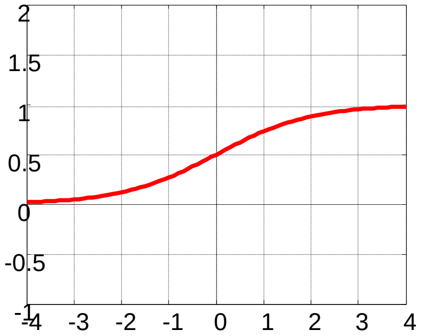
Step 2: Design Network Architecture



Layer 1 (196, 32) | Layer 2 (196, 32) | Layer 3 (196, 32) | Layer 4 (196, 32) | Layer 5 (0, 64) | Layer 6 (0, 64) | Layer 7 (0, 64) | Layer 8 (0, 10)

Sigmoid function

$$f(z) = \frac{1}{1+e^{-z}}$$



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Step 3: Initial Weights and Learning Rate

Initialize Weight Connections

Random initialization:

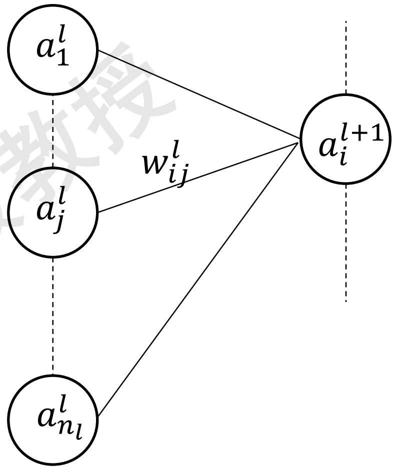
Method 1: Gaussian distribution: $w_{ij}^l \sim N(0,1)$

Method 2: Uniform distribution: $w_{ij}^l \sim U\left(-r^l, r^l\right)$

$$r^l = \sqrt{\frac{6}{p^l + q^{l+1}}}$$

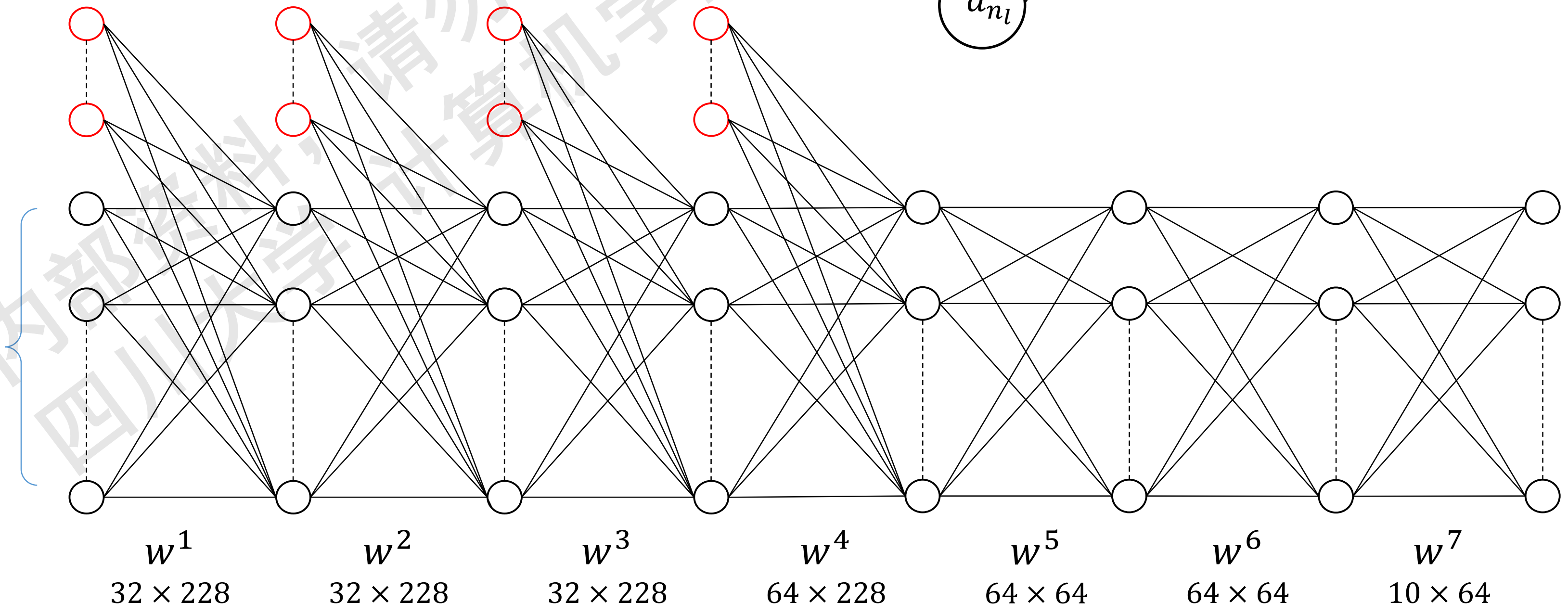
p^l: number of neurons in l layer

q^{l+1}: number of internal neurons in $l+1$ layer



Initialize Internal Representation of Layer 1:

$$a_i^1 = 0,$$
or
$$a_i^1 = 1,$$
…



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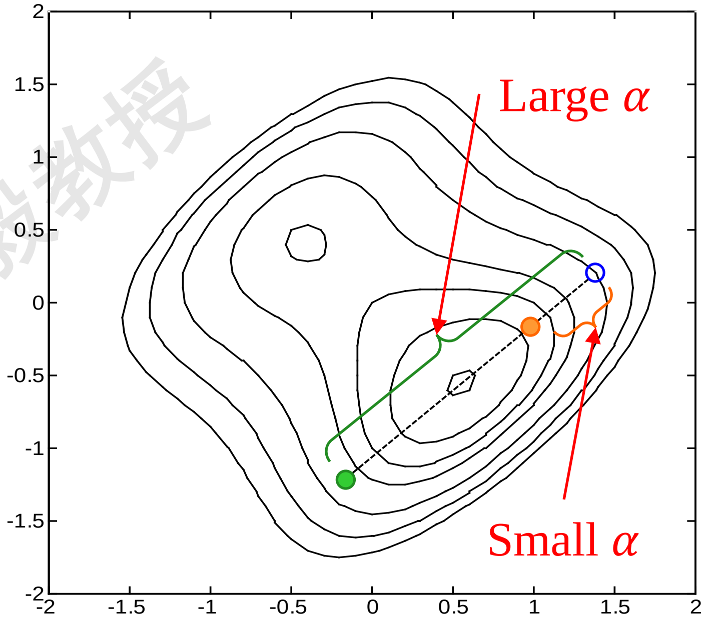
Step 3: Initialization and Learning Rate

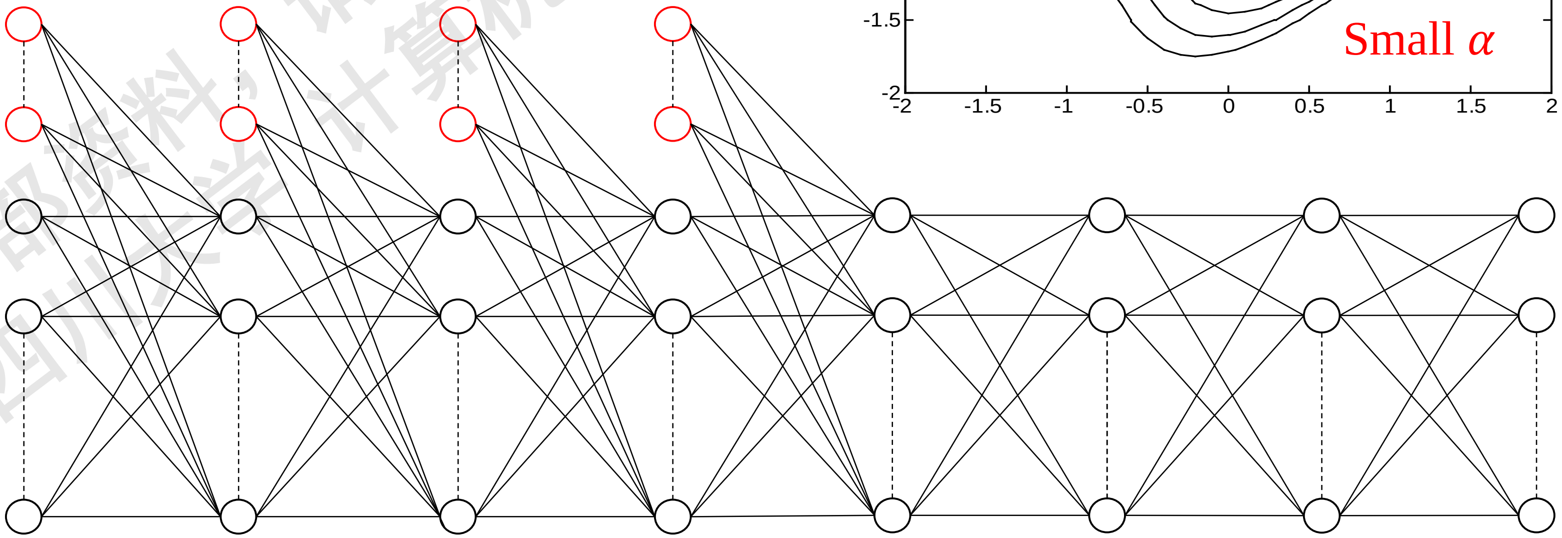
Learning rate:

- Small: slow learning, long learning time.
- Large: fast learning, possibly not converge to minima.



$$w_{ji}^{l} \leftarrow w_{ji}^{l} - \alpha \cdot \frac{\partial J}{\partial w_{ji}^{l}}$$

$$\alpha = \cdots, 0.5, 1, 2, 4, \cdots$$



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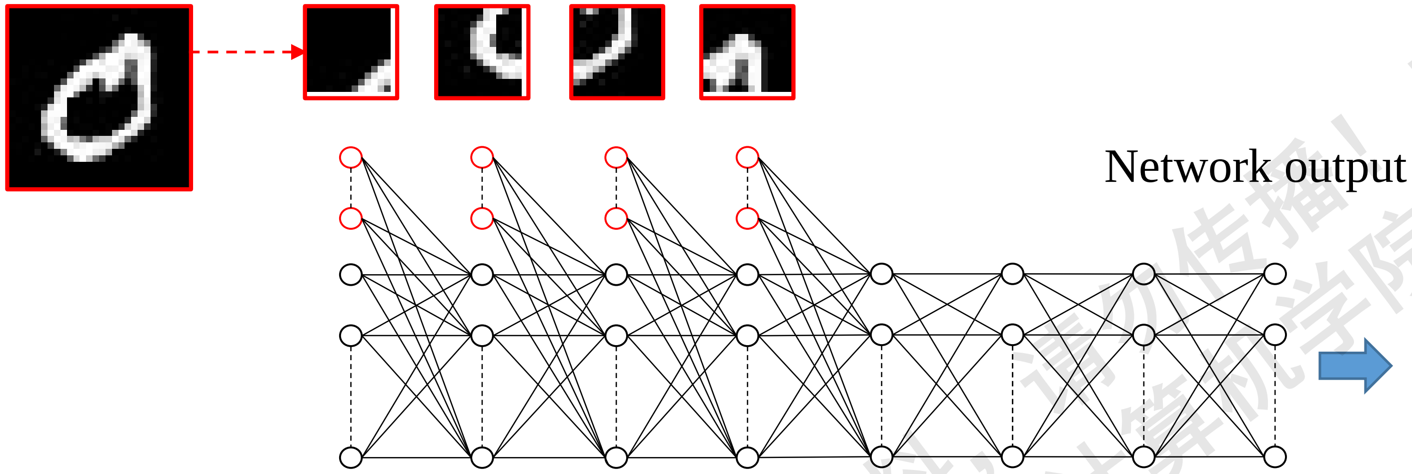
Step 4: Define Cost Function



Network output $\begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \\ 0 \\ 0 \\ 0.2 \\ 0 \\ 0 \\ 0.7 \end{bmatrix}$

Cost function

$$J = \frac{1}{2}\sum_{j=1}^{n_L} e_j^2$$

$$e_j = a_j^L - y_j^L \qquad e = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \\ 0 \\ 0 \\ 0.2 \\ 0 \\ 0 \\ -0.3 \end{bmatrix}$$

Target output $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Label

0 → representation → Target output

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Step 5: Define Evaluation Index

$$\text{Accuracy} = \frac{\text{number of correct prediction}}{\text{number of samples}}$$

An example

Tested data 9

Prediction 7 9 0 4 8 8 8 3 1

Correct prediction
7

Incorrect prediction
2

$$\text{Accuracy} = \frac{7}{9} = 77.78\%$$

Test on training set:

- Reflect the progress of training.
- Evaluate the ability of the model to fit given data.

Test on testing set:

- Evaluate the ability of the model to generalize the knowledge.

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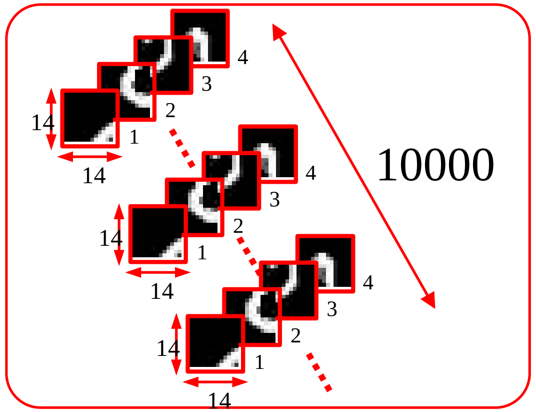
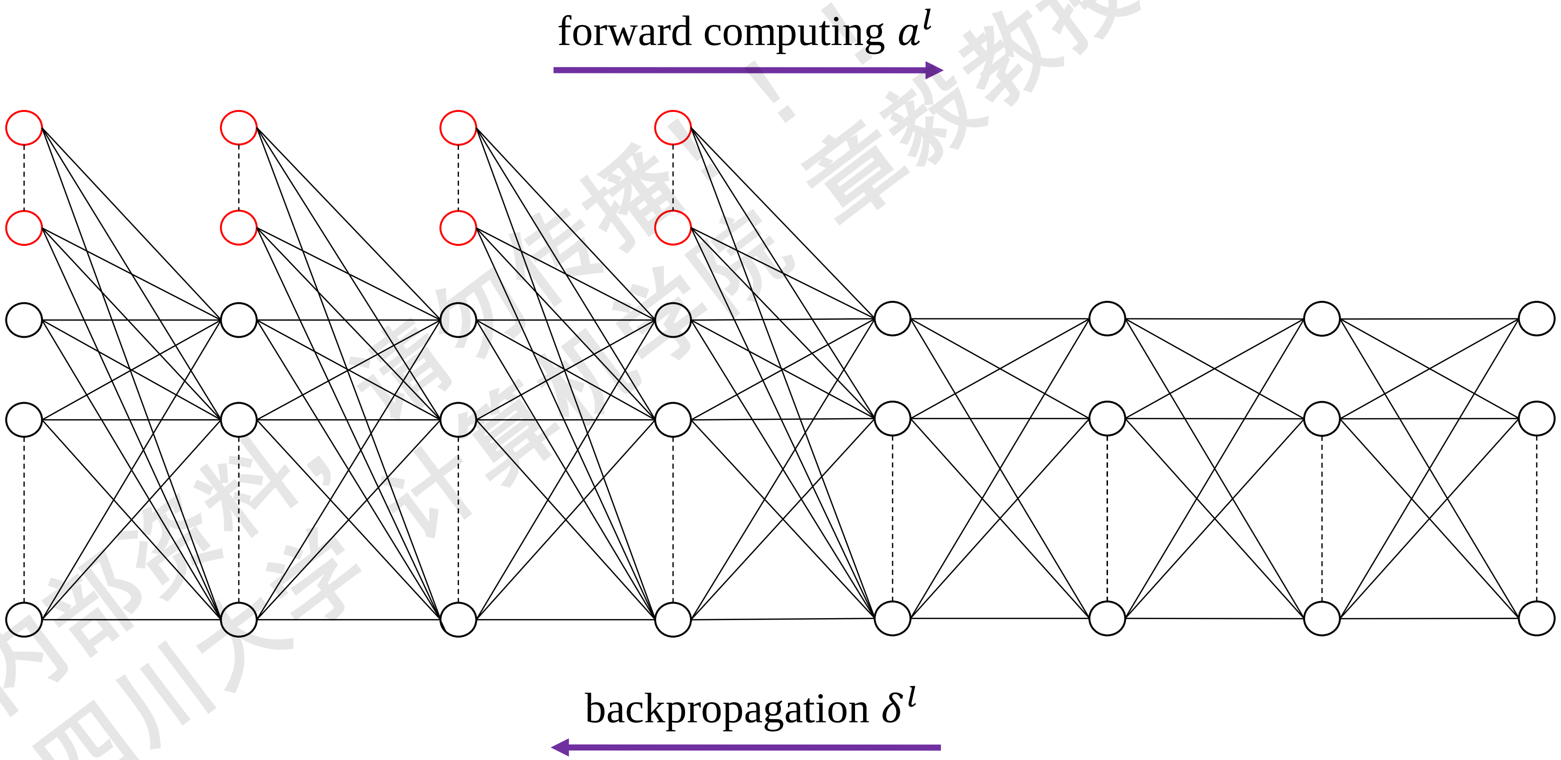
Step 6: Train the Network

Training Data





Updating weights

$$w_{ji}^l \leftarrow w_{ji}^l - \alpha \cdot \frac{\partial J}{\partial w_{ji}^l}$$

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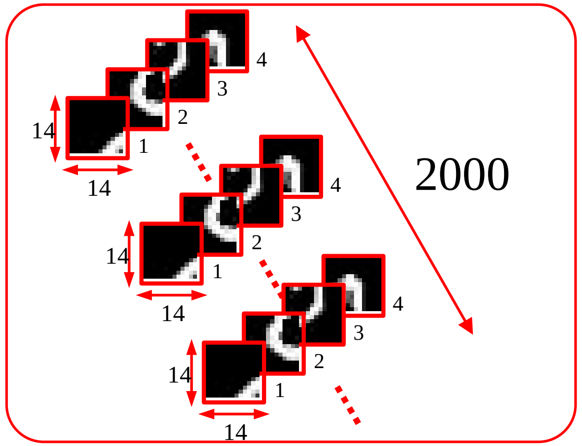
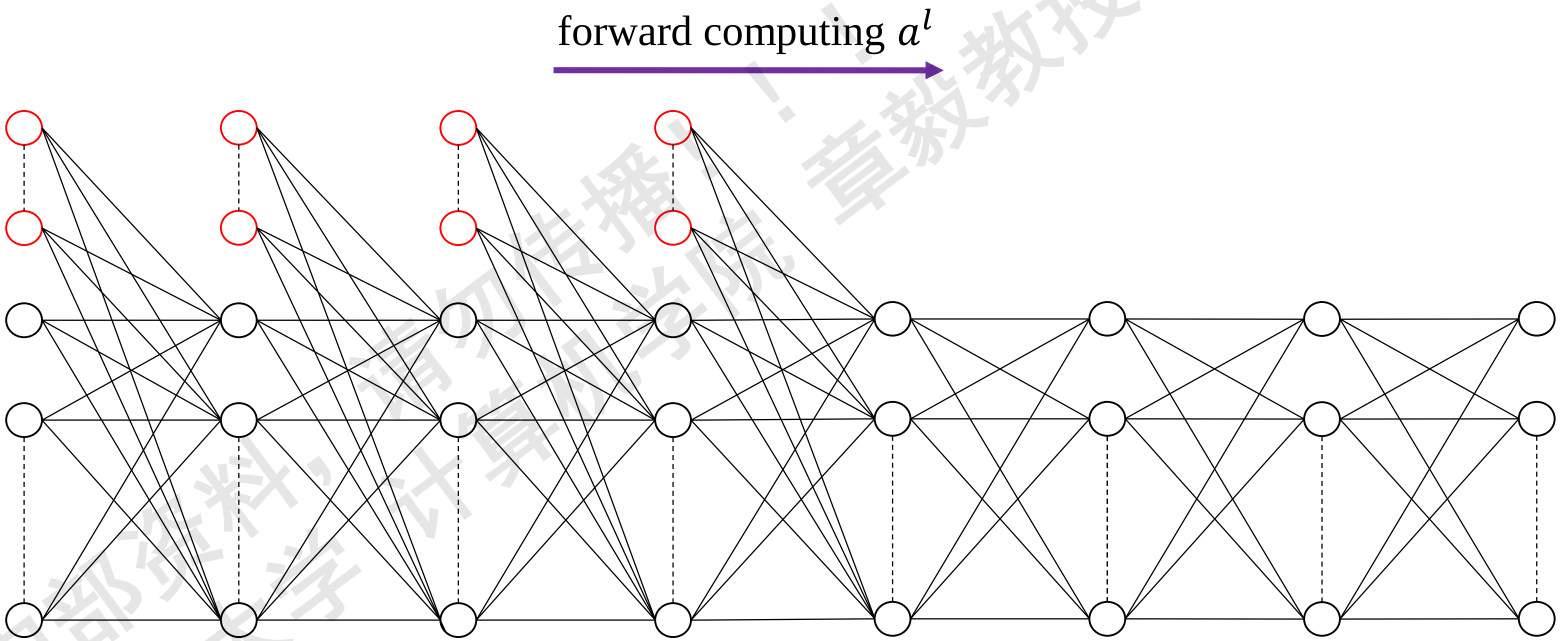
Step 7: Test the Network

Test Data





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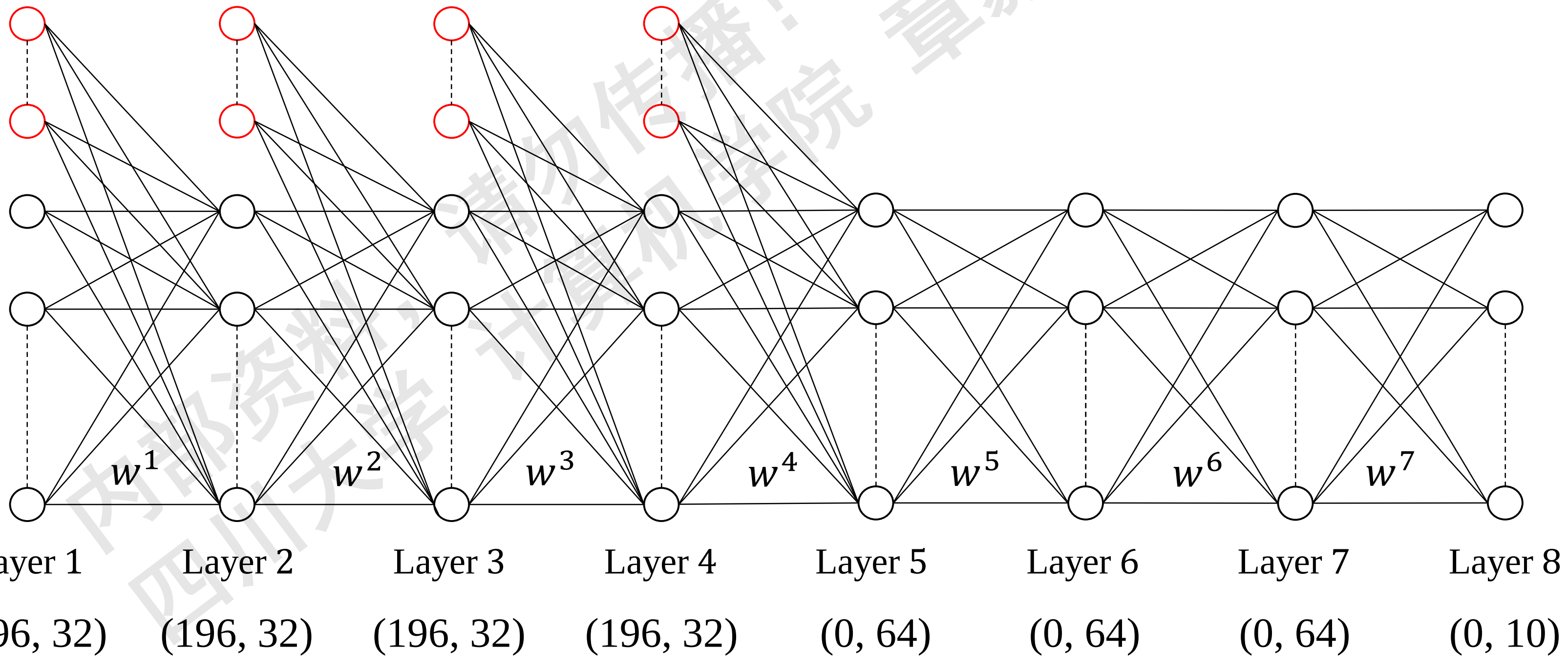
Calculating the Evaluation index

$$\text{Accuracy} = \frac{\text{number of correct prediction}}{2000}$$

Step 8: Store the Network Parameters

Learning rate α

Initialize value of Layer 1



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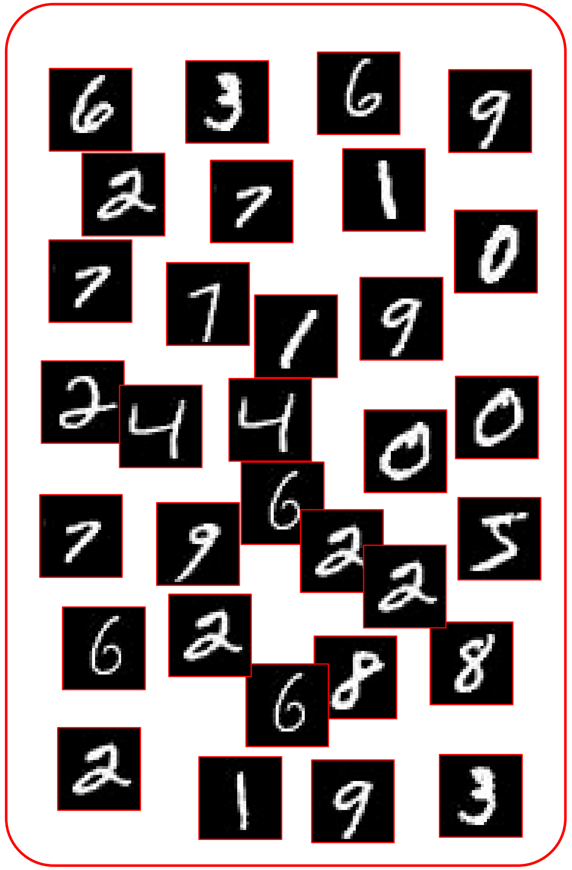
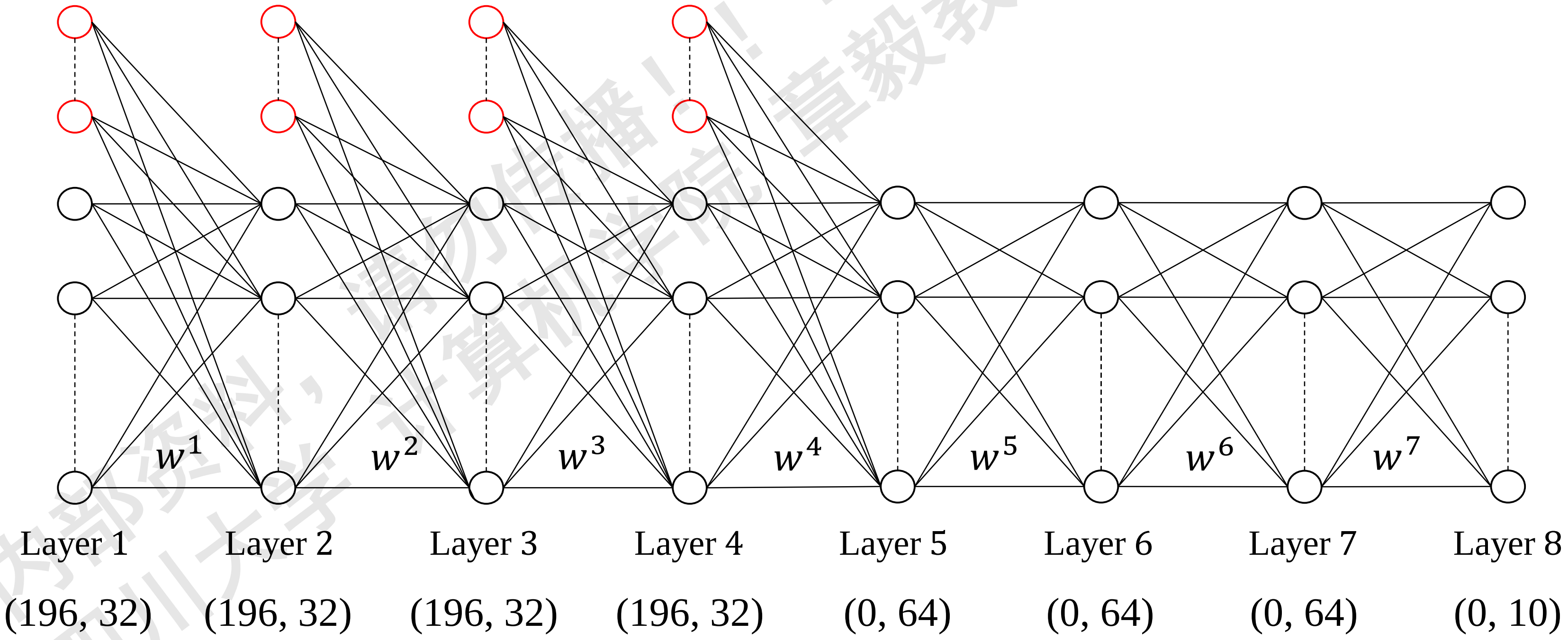
Step 9: Using Trained Network for Applications





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Outline

- Brief Review of Backpropagation Algorithm
- An Illustrating Example
- Experiments
- Assignment

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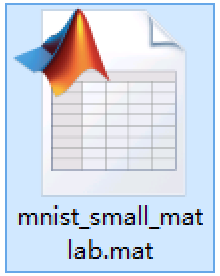
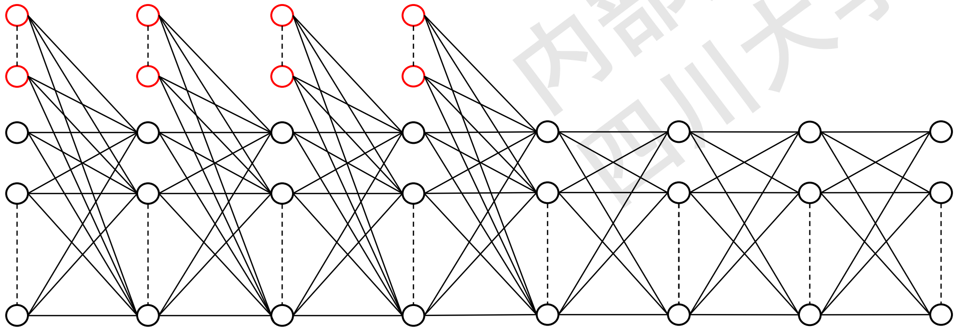
Experiments: Data Preparation

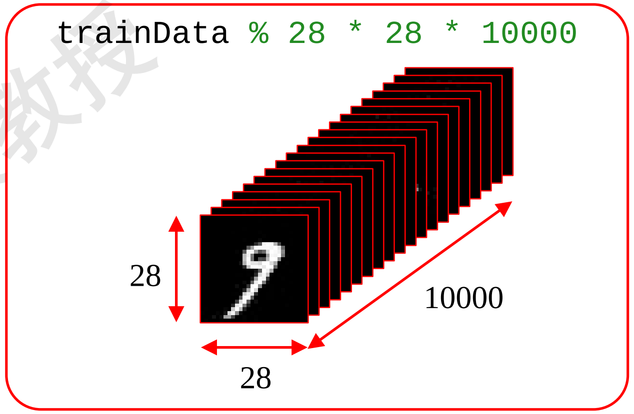


名称	值
trainData	28x28x10000 double
testData	28x28x2000 double
trainLabels	10x10000 double
testLabels	10x2000 double

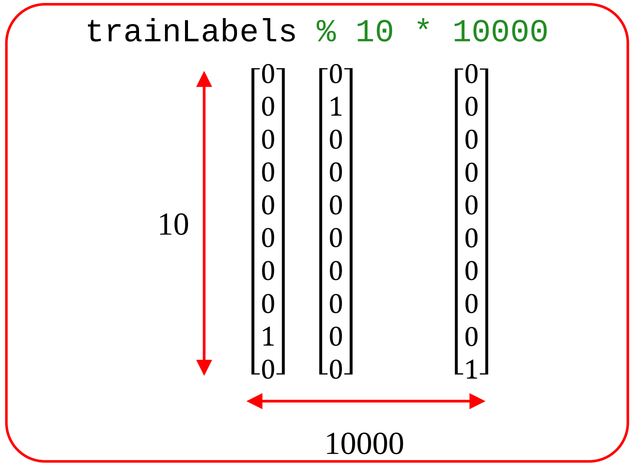
```
% prepare the data set
load mnist_small_matlab.mat
```



Training Data



Label



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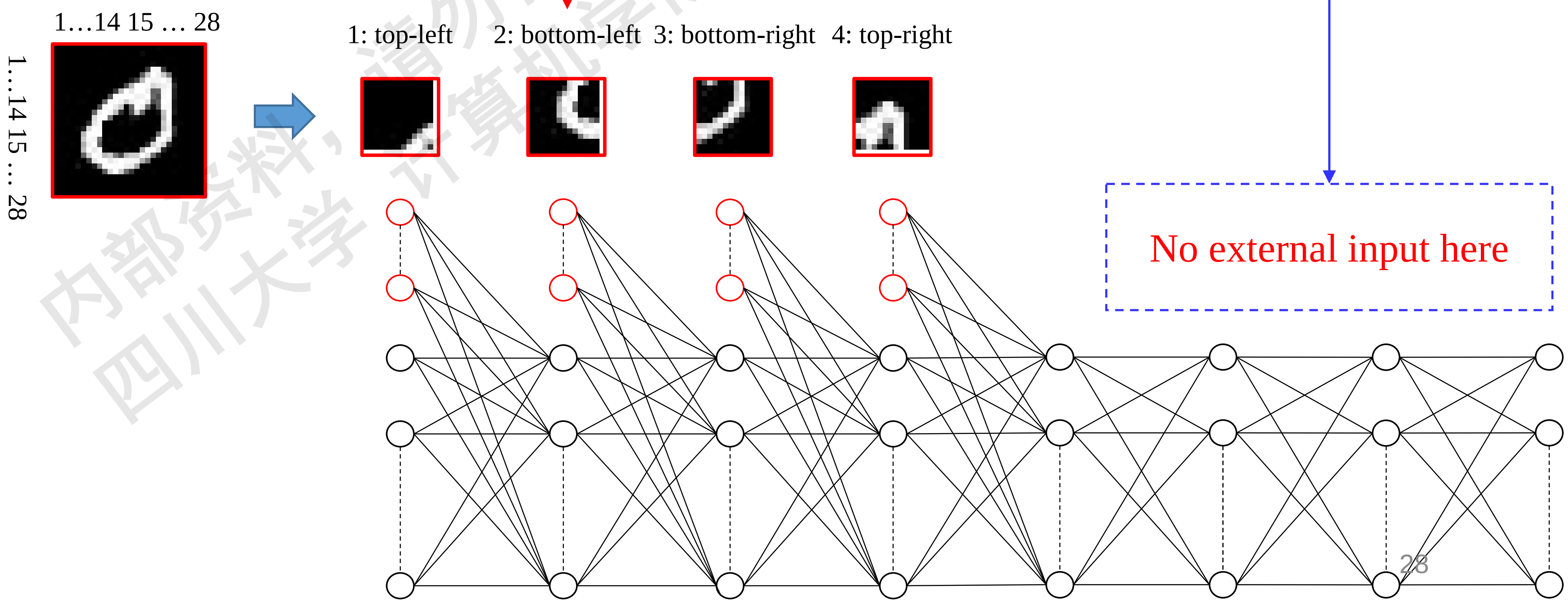
```
% prepare training data
train_size = 10000;
X_train{1} = reshape(trainData(1:14,1:14,:),[],train_size);% top-left
X_train{2} = reshape(trainData(15:28,1:14,:),[],train_size); % bottom-left
X_train{3} = reshape(trainData(15:28,15:28,:),[],train_size); % bottom-right
X_train{4} = reshape(trainData(1:14,15:28,:),[],train_size); % top-right
X_train{5} = zeros(0, train_size);
X_train{6} = zeros(0, train_size);
X_train{7} = zeros(0, train_size);
X_train{8} = zeros(0, train_size);
% prepare testing data
% ...
```



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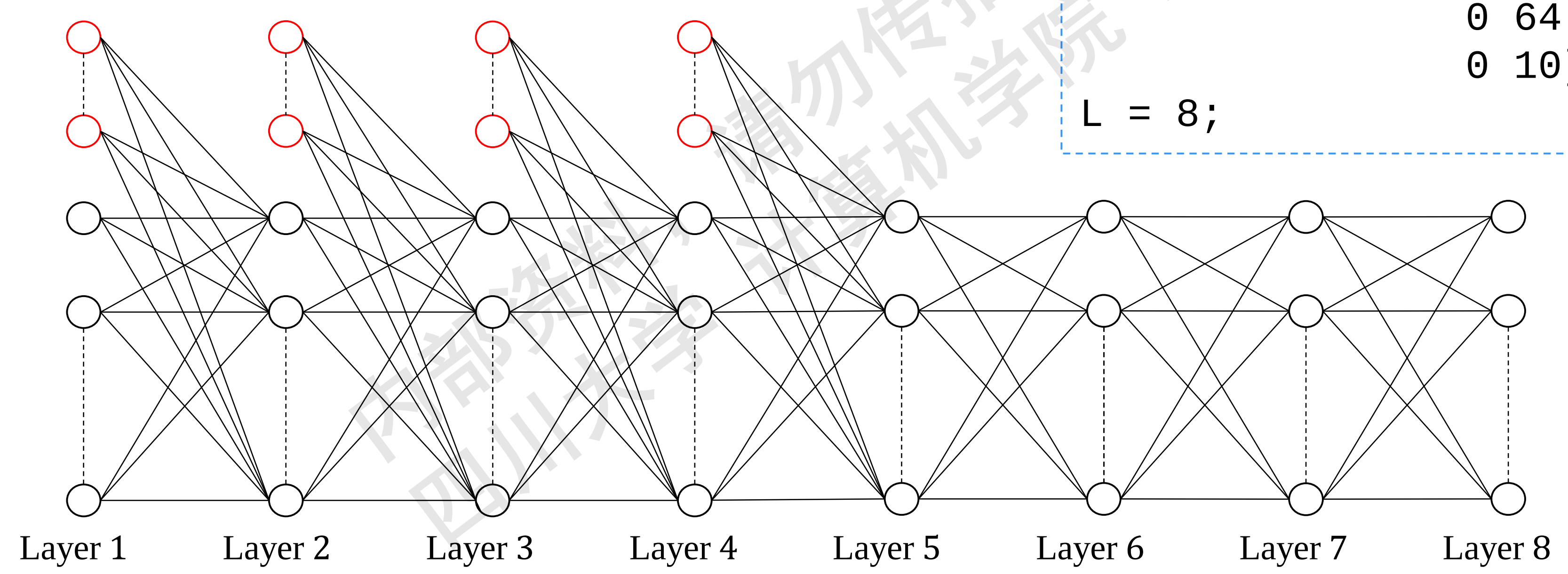
Experiments: Architecture

```
% define network architecture
% - 1st column: external neurons
% - 2nd column: internal neurons
layer_size = [196 32    % layer 1
              196 32    % layer 2
              196 32    % layer 3
              196 32    % layer 4
                0 64    % layer 5
                0 64    % layer 6
                0 64    % layer 7
                0 10];  % layer 8
L = 8;
```



Dimensions (196, 32) (196, 32) (196, 32) (196, 32) (0, 64) (0, 64) (0, 64) (0, 10)

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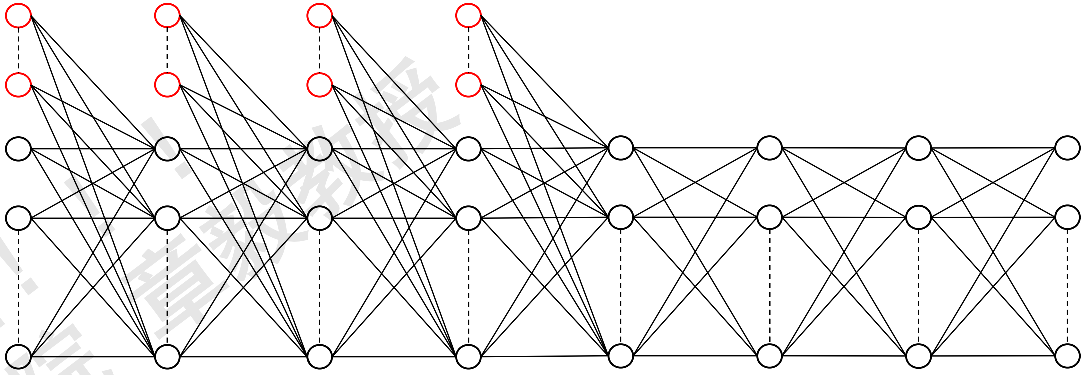
Experiments: Initialize Weights



Gaussian distribution: $w_{ij}^{l} \sim N(0,1)$

```
% initialize weights
for l = 1:L-1
    w{l} = randn(layer_size(l+1,2), sum(layer_size(l,:)));
end
```

Uniform distribution: $w_{ij}^{l} \sim U\left(-r^{l}, r^{l}\right)$

```
% initialize weights
for l = 1:L-1
    % a tricky, but effective, initialization
    w{l} = (rand(layer_size(l+1,2), sum(layer_size(l,:))) * 2 -1)
               * sqrt(6/(layer_size(l+1,2)+sum(layer_size(l,:))));
end
```

$$r^{l} = \sqrt{\frac{6}{p^{l} + q^{l+1}}}$$

p^{l}: number of neurons in l layer
q^{l+1}: number of internal neurons in $l+1$ layer

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Experiments: Run the Network

Learning rate

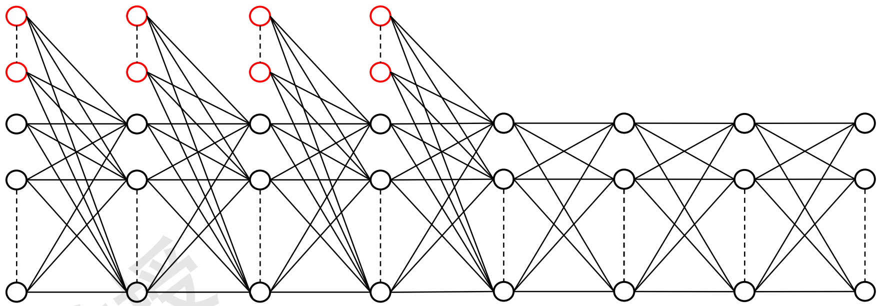
Number of iteration

Number of samples in a batch

```
% choose parameters
alpha = 1;
max_iter = 300;
mini_batch = 100;
```

```
% loop until converge
for iter = 1:max_iter
    % for each mini-batch
        % batch forward computation
        % batch backward computation
        % cumulate and update weight
end
```

Mini-batch BP implement



BP Algorithm:

Step 1. Input the training data set $D = \{(x, y)\}$

Step 2. Initial each w_{ij}^{l}, and choose a learning rate α.

Step 3. For each mini-batch sample $D_m \subset D$

$$a^1 \leftarrow samples\ in\ D_m$$

for $l = 2: L$

$$a^l \leftarrow fc(w^l, a^l);$$

end

$$\delta^L = \frac{\partial J}{\partial z^L};$$

for $l = L-1: 2$

$$\delta^l \leftarrow bc(w^l, \delta^{l+1});$$

end

$$\frac{\partial J}{\partial w_{ji}^l} \leftarrow \frac{\partial J}{\partial w_{ji}^l} + \delta_j^{l+1} \cdot a_i^l;$$

Step 4. Updating

$$w_{ji}^l \leftarrow w_{ji}^l - \alpha \cdot \frac{\partial J}{\partial w_{ji}^l};$$

Step 5. Return to Step 3 until each w^l converge.

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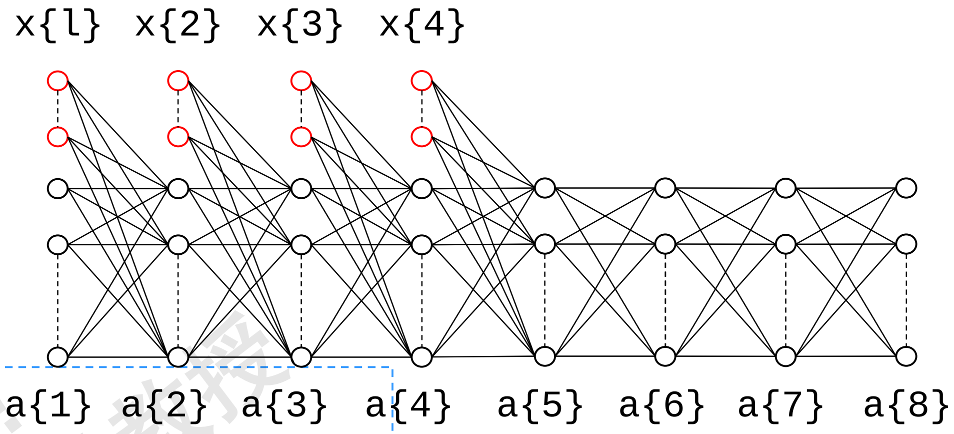
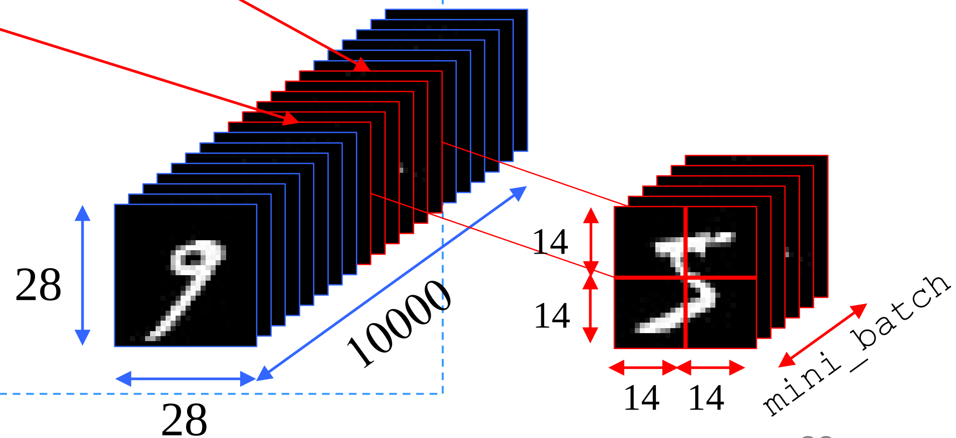
Experiments: mini-batch BP



shuffle index

```
ind = randperm(train_size);
% for each mini-batch
for k = 1:ceil(train_size/mini_batch)
    % prepare internal inputs
    a{1} = zeros(layer_size(1,2),mini_batch);
    % prepare external inputs
    for l=1:L
        x{l} = X_train{l}(:,ind((k-1)*mini_batch+1:min(k*mini_batch, train_size)));
    end
    % prepare labels
    y = double(trainLabels(:,ind((k-1)*mini_batch+1:min(k*mini_batch, train_size))));
    % forward computation
    % ...
    % cost function and error
    % ...
    % backward computation
    % ...
    % update weight
    % ...
end
```



training data

mini-batch

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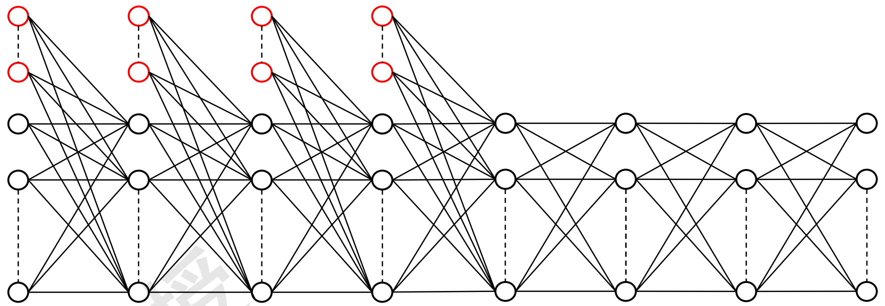
Experiments: BP



```
% forward computation
for l=1:L-1
    [a{l+1}, z{l+1}] = fc(w{l}, a{l}, x{l});
end

% Compute delta of last layer
delta{L} = (a{L} - y).* a{L} .*(1-a{L});

% backward computation
for l=L-1:-1:2
    delta{l} = bc(w{l}, z{l}, delta{l+1});
end

% update weight
for l=1:L-1
    gw = delta{l+1} * [x{l};a{l}]' / mini_batch;
    w{l} = w{l} - alpha * gw;
end
```

BP Algorithm:

Step 1. Input the training data set $D=\{(x,y)\}$

Step 2. Initial each w_{ij}^{l}, and choose a learning rate α.

Step 3. For each mini-batch sample $D_m \subset D$

$a^1 \leftarrow samples\ in\ D_m$

for $l = 2{:}L$

$a^l \leftarrow fc(w^l, a^l);$

end

$$\delta^L = \frac{\partial J}{\partial z^L};$$

for $l = L-1{:}2$

$\delta^l \leftarrow bc(w^l, \delta^{l+1});$

end

$$\frac{\partial J}{\partial w_{ji}^{l}} \leftarrow \frac{\partial J}{\partial w_{ji}^{l}} + \delta_j^{l+1} \cdot a_i^l;$$

Step 4. Updating

$$w_{ji}^{l} \leftarrow w_{ji}^{l} - \alpha \cdot \frac{\partial J}{\partial w_{ji}^{l}};$$

Step 5. Return to Step 3 until each w^l converge.

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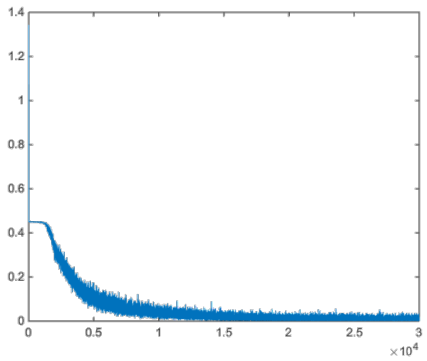
Experiments: Plotting

Cost function

$$J = \frac{1}{2}\sum_{j=1}^{n_L}\left(a_j^L - y_j^L\right)^2$$

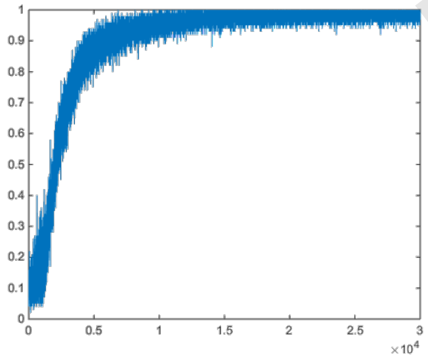



```
% cost function
J = [J 1/2/mini_batch*sum((a{L}(:)-y(:)).^2)];
figure
plot(J);
```

Accuracy

$$\text{Acc} = \frac{\textit{number of correct prediction}}{\textit{number of samples}}$$

Use max output as prediction




```
% accuary on training batch
[~,ind_train] = max(y);
[~,ind_pred] = max(a{L});
Acc= [Acc sum(ind_train == ind_pred) / mini_batch];
figure
plot(Acc);
```

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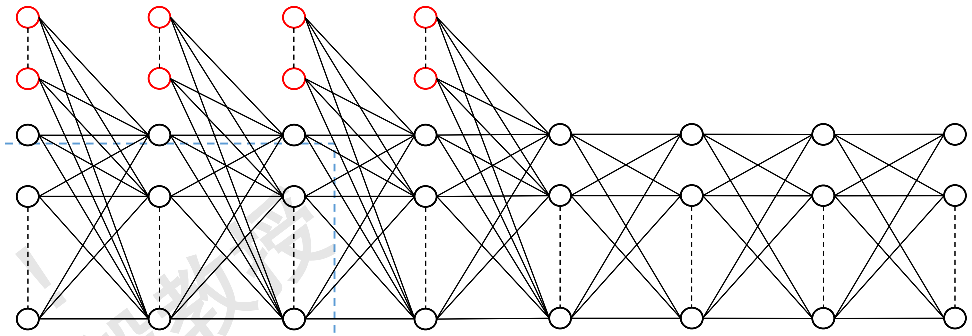
Experiments: Testing



```
% test on training set
a{1} = zeros(layer_size(1,2),train_size);
for l=1:L-1
    a{l+1} = fc(w{l}, a{l}, X_train{l});
end
[~,ind_test] = max(trainLabels);
[~,ind_pred] = max(a{L});
train_acc = sum(ind_test == ind_pred)/train_size;
fprintf('Accuracy on training dataset is %f%%\n', train_acc*100);
```

```
% test on testing set
a{1} = zeros(layer_size(1,2),test_size);
for l=1:L-1
    a{l+1} = fc(w{l}, a{l}, X_test{l});
end
[~,ind_test] = max(testLabels);
[~,ind_pred] = max(a{L});
test_acc = sum(ind_test == ind_pred)/test_size;
fprintf('Accuracy on testing dataset is %f%%\n', test_acc*100);
```

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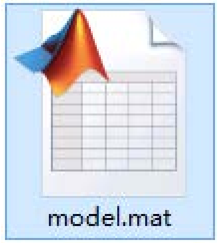
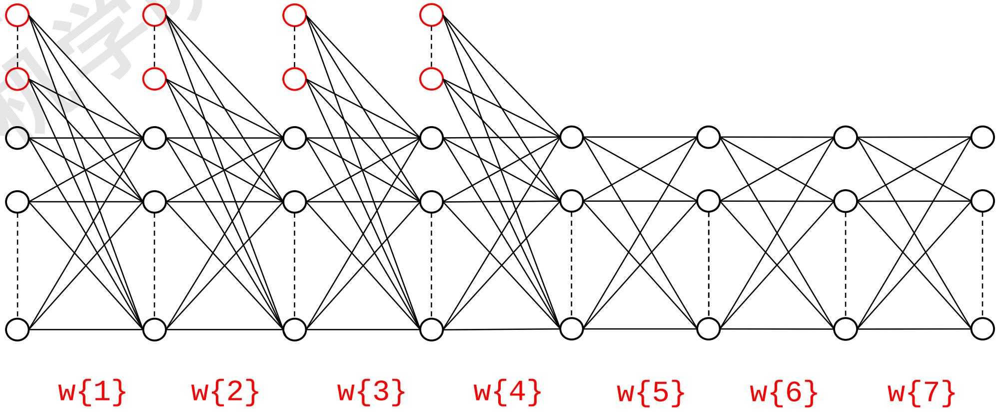
Experiments: Store the Network Parameters

```
% save model
save model.mat w layer_size
```



名称	值
layer_size	8x2 double
w	1x7 cell

This is very important!



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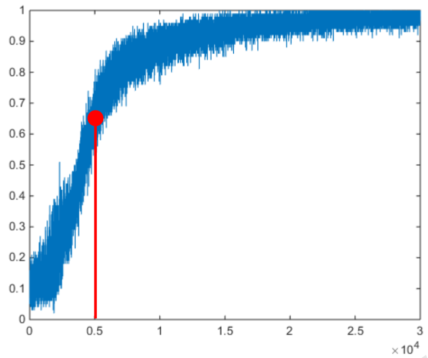
Results: Learning Rate

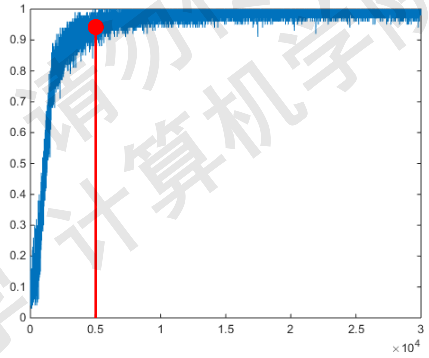
```
% learning rate
alpha = 0.5;
```



Accuracy

- Training=98.05%
- Testing=94.40%

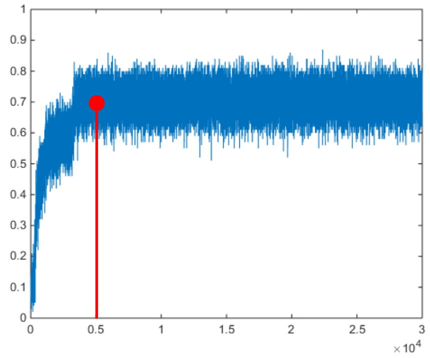
```
% learning rate
alpha = 2;
```



Accuracy

- Training=99.14%
- Testing=95.40%

```
% learning rate
alpha = 8;
```



Accuracy

- Training=71.02%
- Testing=69.25%

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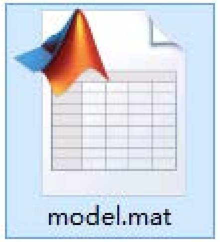
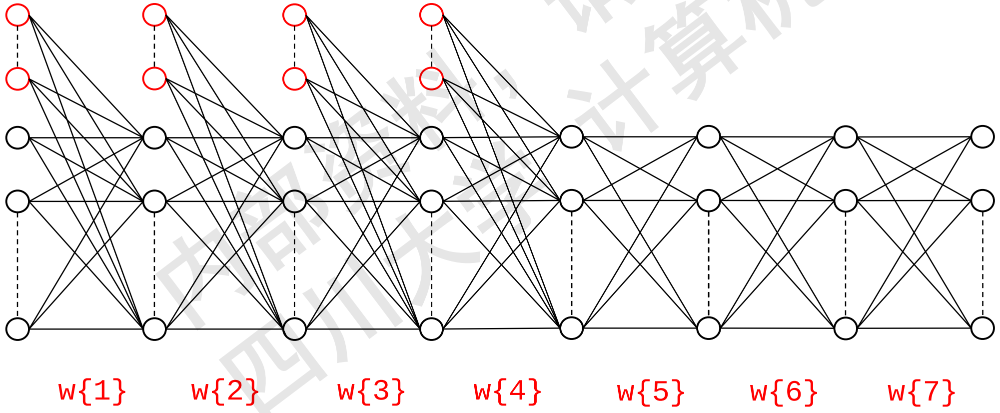
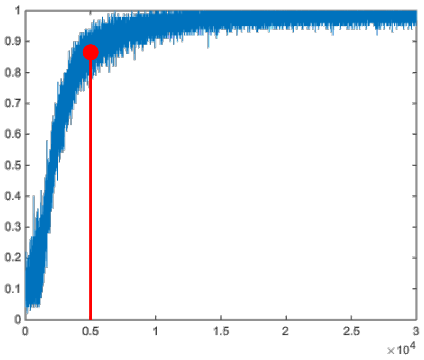
Results: Number of Layers



名称	值
layer_size	8x2 double
w	1x7 cell



8 layers



Accuracy
Training=98.65%
Testing=95.10%

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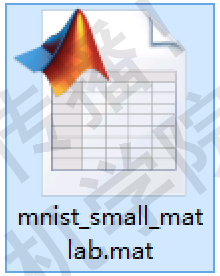
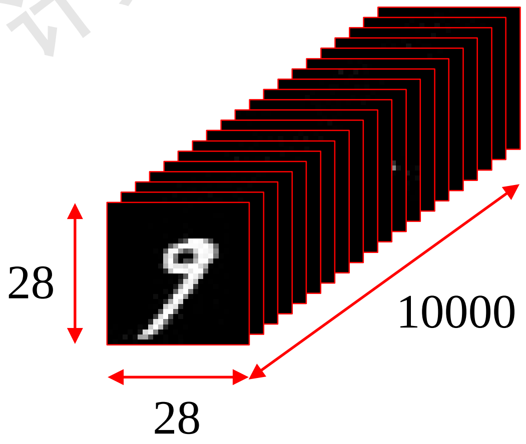
Assignment

Implement the handwritten digits recognition by MATLAB using only one layer of external input.





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Thanks

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