భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

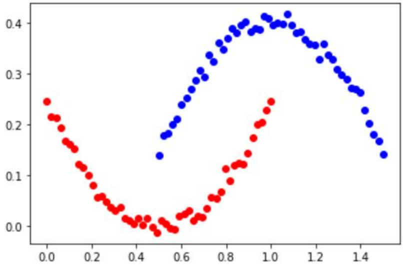
Deep Learning

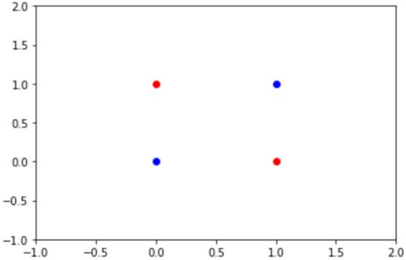
02 Network of Perceptrons

Dr. Konda Reddy Mopuri
Dept. of Artificial Intelligence
IIT Hyderabad
Jan-May 2025

Linear Classifiers: Shortcomings

- Lower capacity: data has to be linearly separable
- Some times no hyper-plane can separate the data (e.g. XOR data)





Pre-processing

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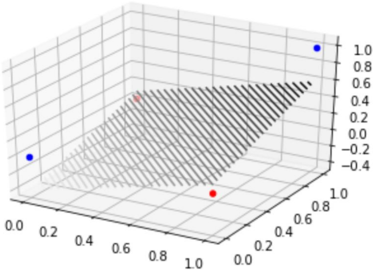
1. Sometimes, data specific pre-processing makes the data linearly separable
2. Consider the xor case
   $\phi(\mathbf{x}) = \phi(x_u, x_v) = (x_u, x_v, x_u x_v)$
3. Consider the perceptron in the new space $f(\mathbf{x}) = \sigma(\mathbf{w}^T \phi(\mathbf{x}) + b)$



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1. (Recap: Polynomial regression): increasing the degree $\rightarrow$ increase the model capacity
2. (Recap: Bias-Variance decomposition): to reduce the bias error, we increased the model capacity
3. Feature design (or pre-processing) may also be another way to reduce the capacity without affecting (or improving) the bias

Consider the XOR function

x_1	x_2	XOR
0	0	0
0	1	1
1	0	1
1	1	0

Consider the XOR function

- If we attempt to realize XOR function with a single perceptron

x_1	x_2	XOR	
0	0	0	$w_0 < 0$
0	1	1	$w_2 + w_0 \geq 0$
1	0	1	$w_1 + w_0 \geq 0$
1	1	0	$w_1 + w_2 + w_0 < 0$

Consider the XOR function

- If we attempt to realize XOR function with a single perceptron

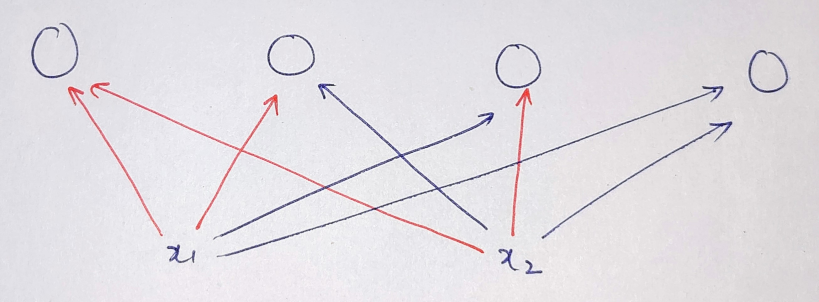
x_1	x_2	XOR		
0	0	0	$w_0 < 0$	
0	1	1	$w_2 + w_0 \geq 0$	$\Rightarrow \quad w_2 \geq -w_0$
1	0	1	$w_1 + w_0 \geq 0$	$\Rightarrow \quad w_1 \geq -w_0$
1	1	0	$w_1 + w_2 + w_0 < 0$	$\Rightarrow \quad w_1 + w_2 < -w_0$

Consider the XOR function

- Clearly, a single perception cannot represent the XOR function!

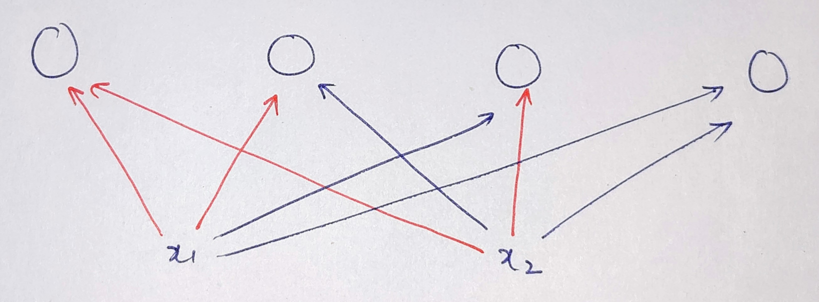
Let's see if multiple perceptions can do this

- Consider 4 perceptions



Let's see if multiple perceptions can do this

- Consider 4 perceptions
- $\color{red}{\rightarrow} = -1$ and $\color{blue}{\rightarrow} = +1$



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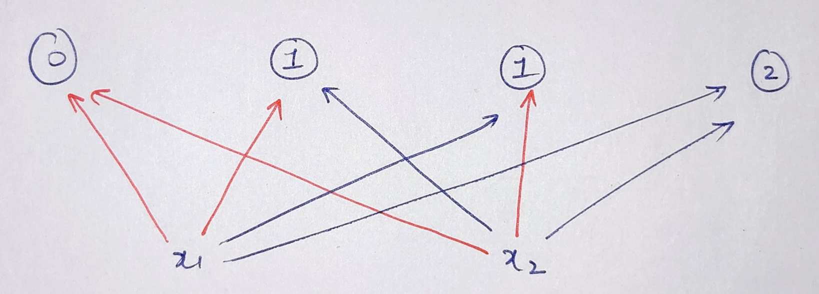
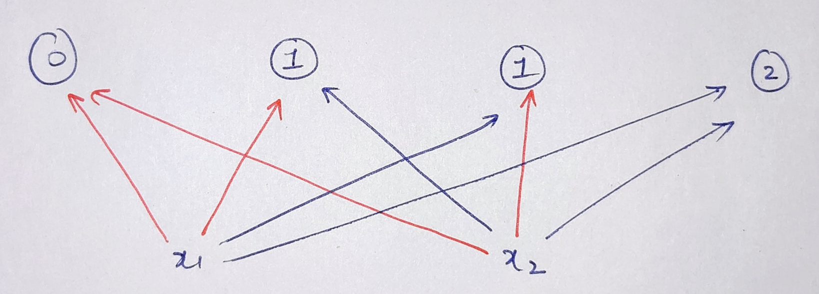
- Let's have these thresholds



Let's see if multiple perceptions can do this

- Let's have these thresholds
- Notice, each of them fire for exactly one specific input pattern



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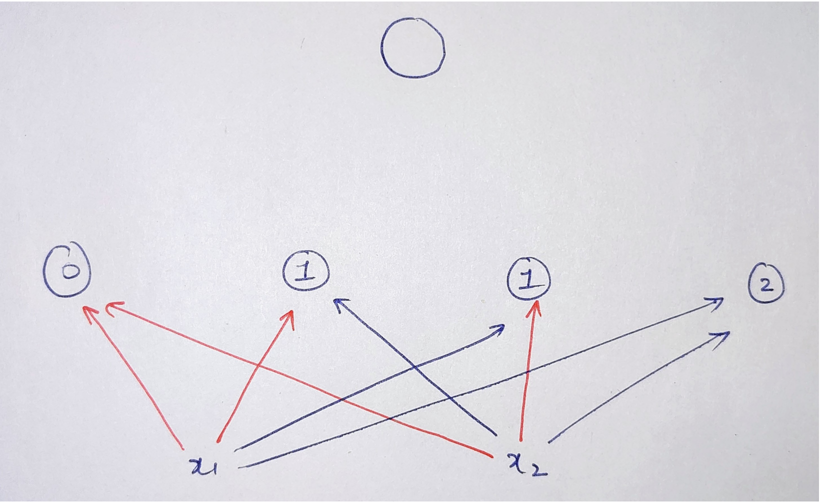
- Let's now add another perceptron



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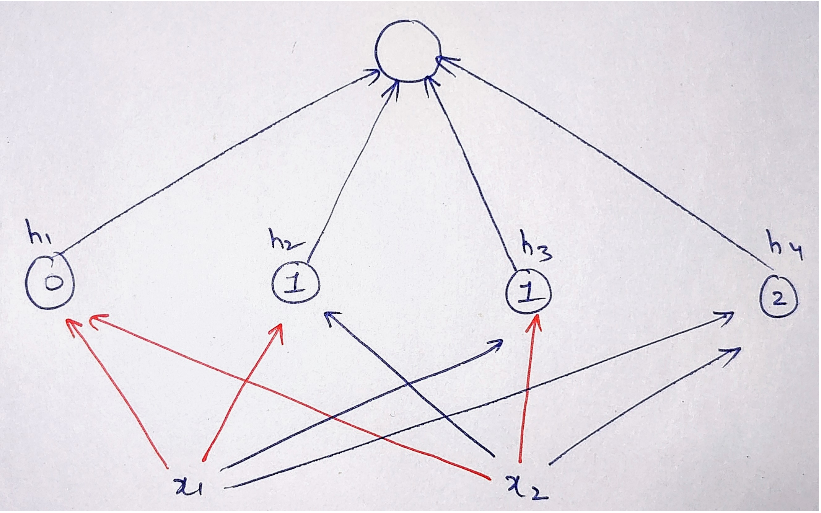
- Connect the previous (hidden) ones and call it the output perceptron



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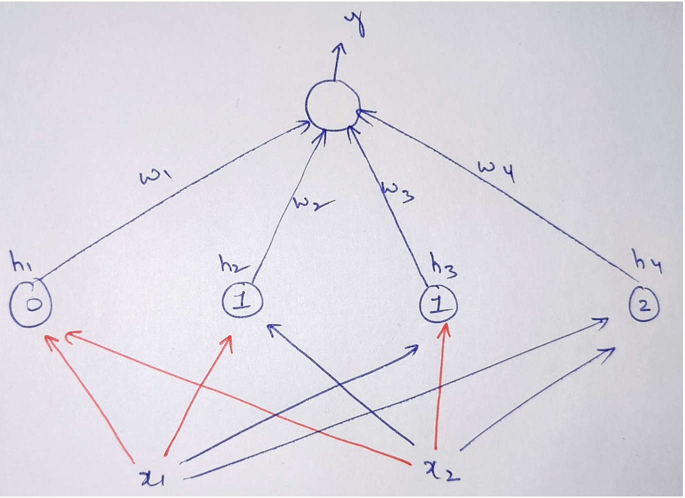
- See if we can find a set of weights (W_i) to represent the XOR function



Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0					
0	1					
1	0					
1	1					

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x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	
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x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	
0	1	0	1	0	0	
1	0	0	0	1	0	
1	1	0	0	0	1	

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x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	w_1
0	1	0	1	0	0	w_2
1	0	0	0	1	0	w_3
1	1	0	0	0	1	w_4

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y
0	0	1	0	0	0	$w_1 + w_0$
0	1	0	1	0	0	$w_2 + w_0$
1	0	0	0	1	0	$w_3 + w_0$
1	1	0	0	0	1	$w_4 + w_0$

Let's see if multiple perceptions can do this

x_1	x_2	h_1	h_2	h_3	h_4	y	XOR
0	0	1	0	0	0	$w_1 + w_0$	0
0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

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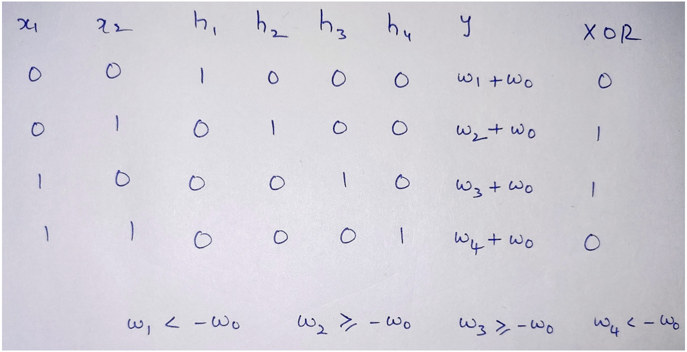

x_1	x_2	h_1	h_2	h_3	h_4	y	XOR
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0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

$w_1 < -w_0 \quad w_2 \geq -w_0 \quad w_3 \geq -w_0 \quad w_4 < -w_0$

Let's see if multiple perceptions can do this

- Clearly possible to find such weights $\rightarrow$ represent the XOR function!

x_1	x_2	h_1	h_2	h_3	h_4	y	XOR
0	0	1	0	0	0	$w_1 + w_0$	0
0	1	0	1	0	0	$w_2 + w_0$	1
1	0	0	0	1	0	$w_3 + w_0$	1
1	1	0	0	0	1	$w_4 + w_0$	0

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What about other 2-input Boolean functions?

- Possible to represent!

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- Possible to represent!
- Leads to finding a different set of non-contradicting weights

What if there are more inputs?

- Can do the same with 2^n perceptions in the hidden layer and 1 in the output layer!

What did we just find?

- Any Boolean function of n inputs can be exactly represented with 2^n perceptions in the hidden layer and 1 in the output layer!

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- Any Boolean function of n inputs can be exactly represented with 2^n perceptions in the hidden layer and 1 in the output layer!
- Note that $2^n + 1$ is a sufficient but not necessary
- Caveat: the size of the hidden layer grows exponentially!

Network of Perceptrons

- Generally referred to as MLP (Multi-Layered Network of Perceptrons)

Moving on from Boolean functions

- $y = f(x)$, where $x \in \mathcal{R}^n$ and $y \in \mathcal{R}$

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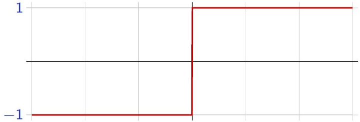
- $y = f(x)$, where $x \in \mathcal{R}^n$ and $y \in \mathcal{R}$
- Can MLPs represent such functions?

Threshold-ing is very harsh!

1. Perceptron's o/p is discontinuous!

$$\sigma(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ -1 & \text{else} \end{cases}$$



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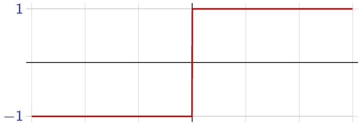

2. Think of inputs -0.0001 and 0

Enough of Boolean functions!

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1. Many real world problems have non-binary outputs
2. Perceptron only gives two outputs!
3. Sigmoid neuron

$$f(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{1} + \mathbf{e}^{-\mathbf{w}^{\mathbf{T}}\mathbf{x}}}$$