Deep Learning

03 MLP: Representation Power of an MLP

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So far

1. Any Boolean function of n inputs can be exactly represented with one hidden layer!

Universal Approximation (for real functions)

1. We can represent any continuous function $(f : \mathcal{R}^m \rightarrow \mathcal{R}^n)$ to any desired approximation $(|g(x) - f(x)| < \epsilon)$ with a linear combination of sigmoid neurons

Universal Approximation (for real functions)

1. We can represent any continuous function $(f : \mathcal{R}^m \rightarrow \mathcal{R}^n)$ to any desired approximation $(|g(x) - f(x)| < \epsilon)$ with a linear combination of sigmoid neurons
2. In other words, neural networks with a single hidden layer can be used to approximate any continuous function to any desired precision

Universal Approximation

Math. Control Signals Systems (1989) 2: 303–314

Mathematics of Control, Signals, and Systems

Approximation by Superpositions of a Sigmoidal Function*

G. Cybenko†

Neural Networks, Vol. 4, pp. 251–257, 1991
Printed in the USA.

0893-6080/91 $3.00 + .00

ORIGINAL CONTRIBUTION

Approximation Capabilities of Multilayer Feedforward Networks

KURT HORNIK

Technische Universität Wien, Vienna, Austria

Universal Approximation

1. Let's look at the visual proof!

Universality with 1-input and 1-output

- Two hidden units and one output unit

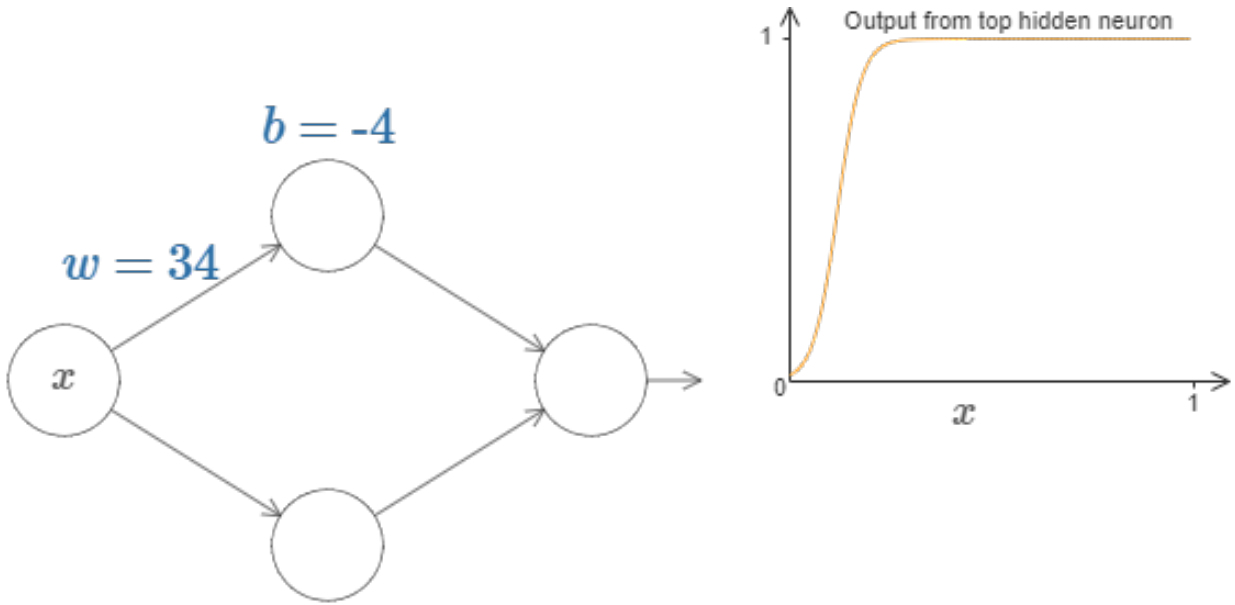
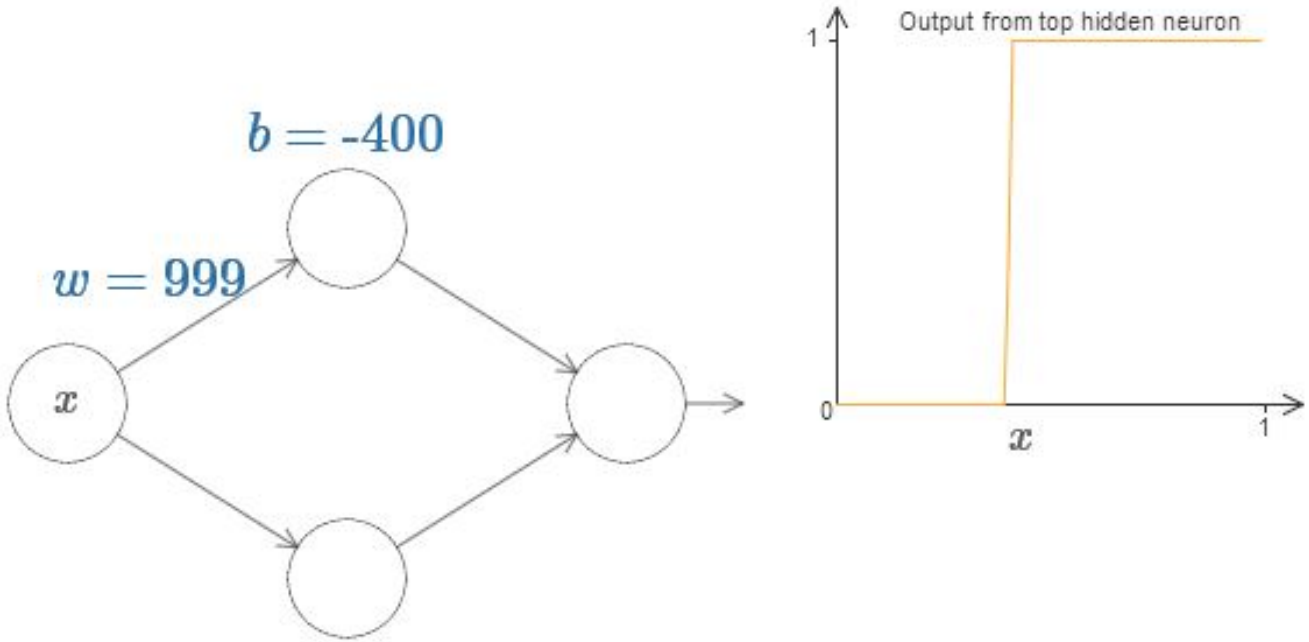


Figure from Michael Nielsen's NNDL textbook

Universality with 1-input and 1-output

- Sigmoid neurons can closely approximate a step function!



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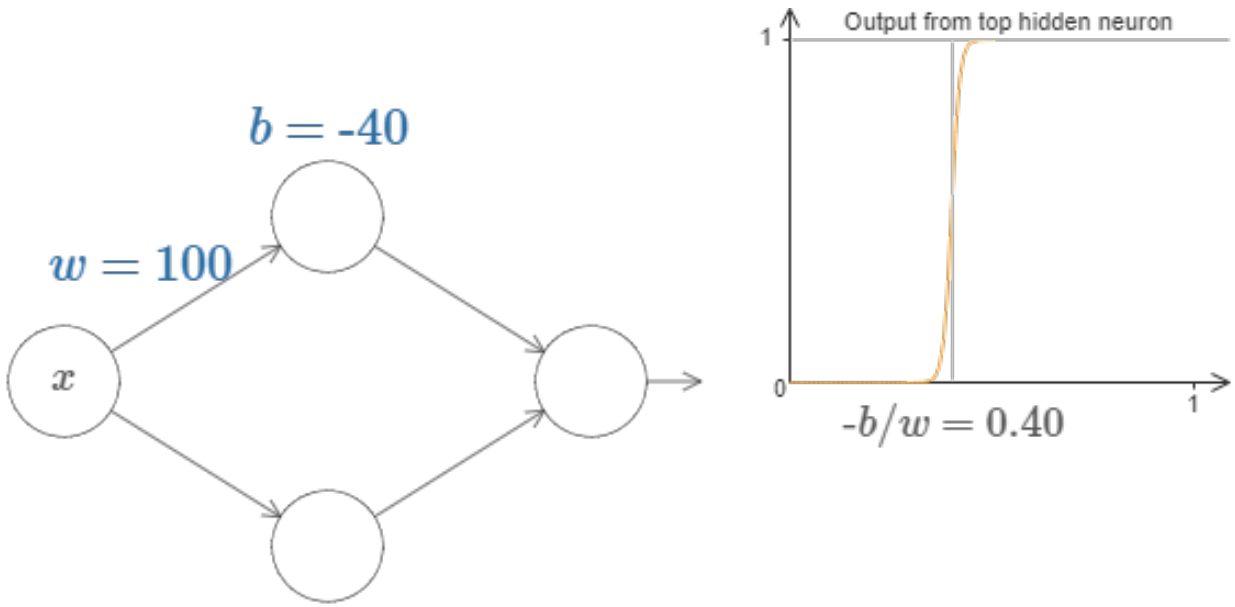
Universality with 1-input and 1-output

- Let's simplify the neuron representation with a single parameter (s)



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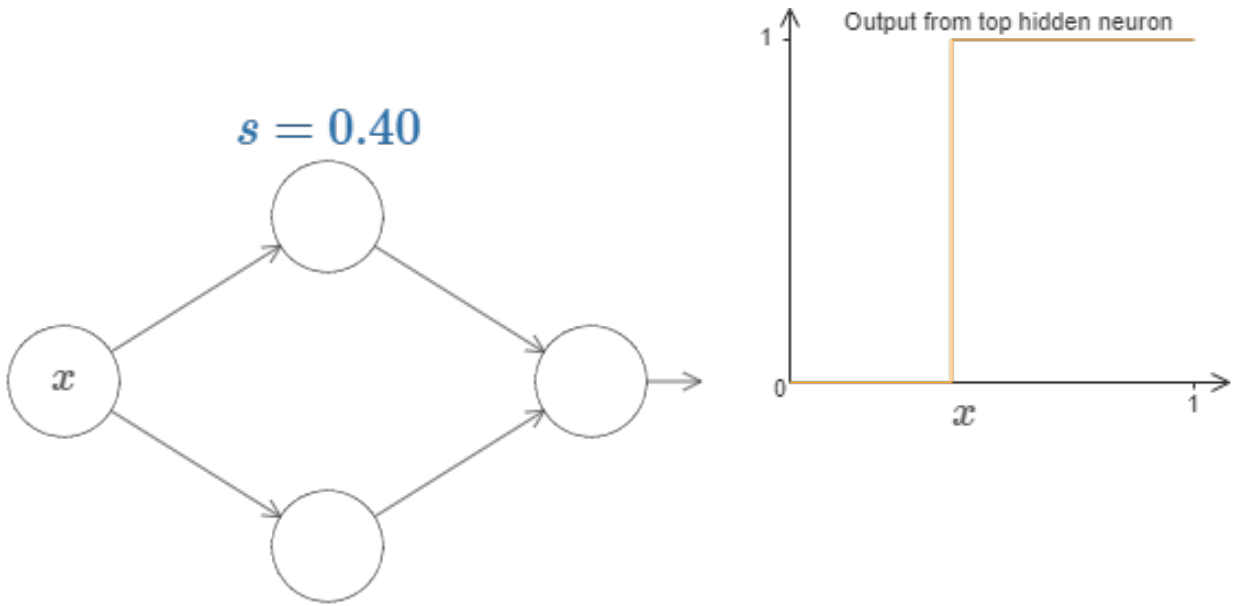
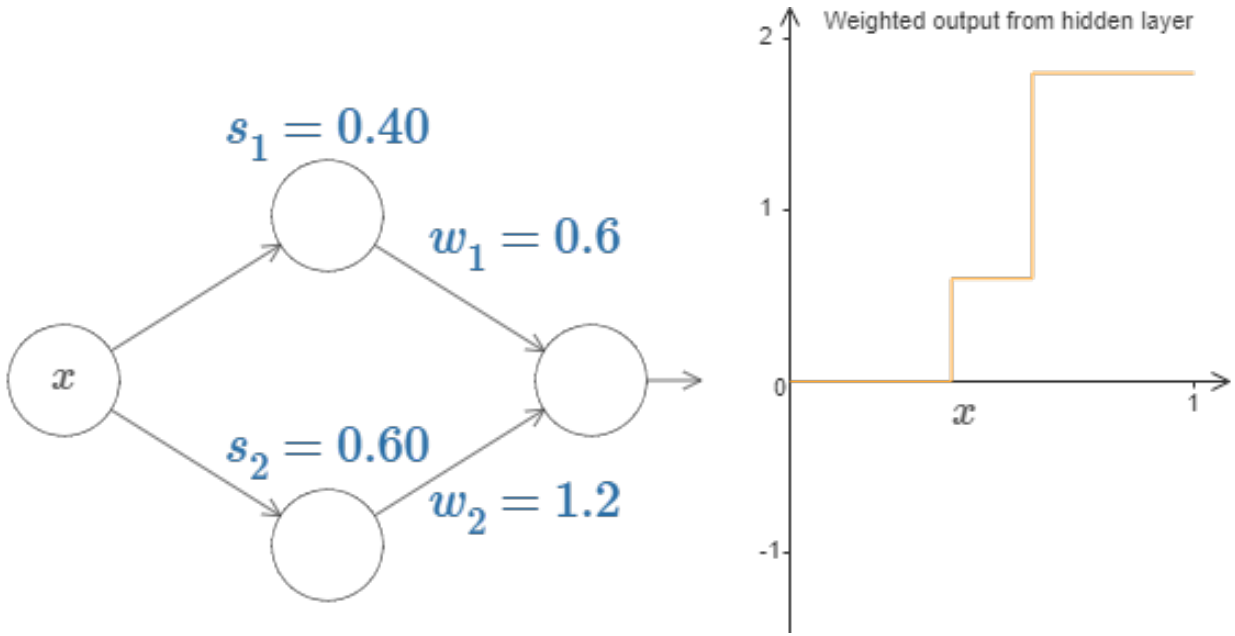


Figure from Michael Nielsen's NNDL textbook

Universality with 1-input and 1-output

- Weighted output of hidden neurons



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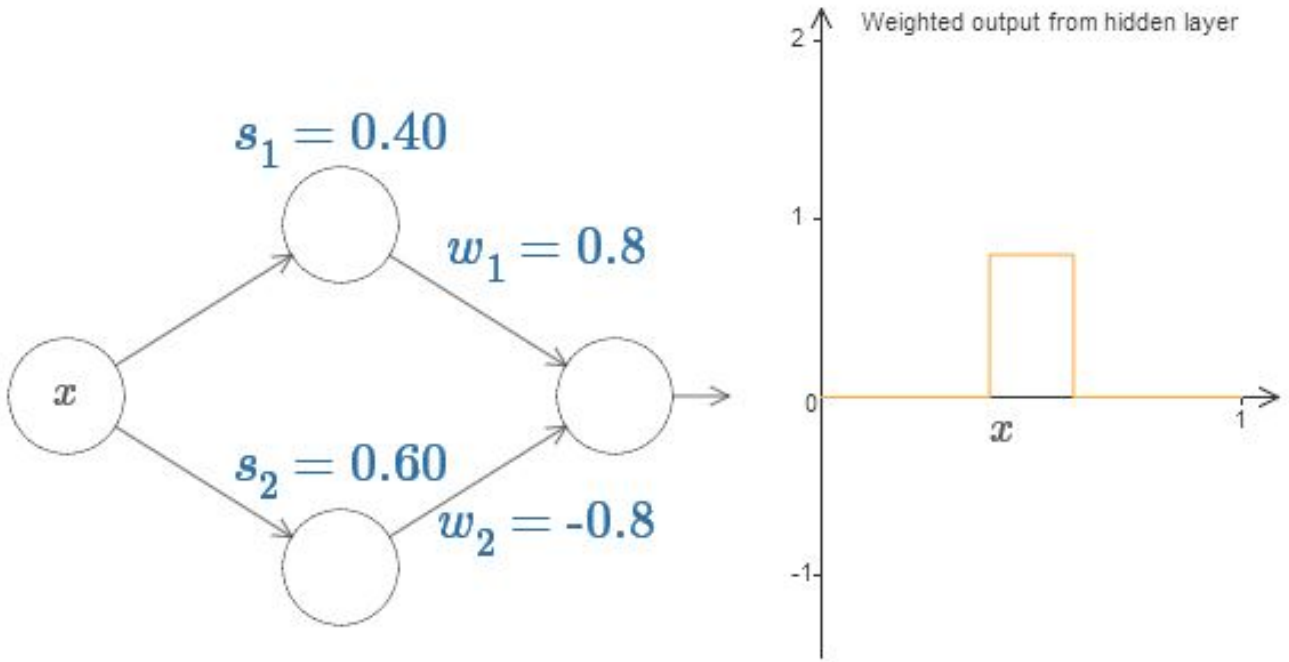
Universality with 1-input and 1-output

- Can output a pulse/tower of desired width and height!



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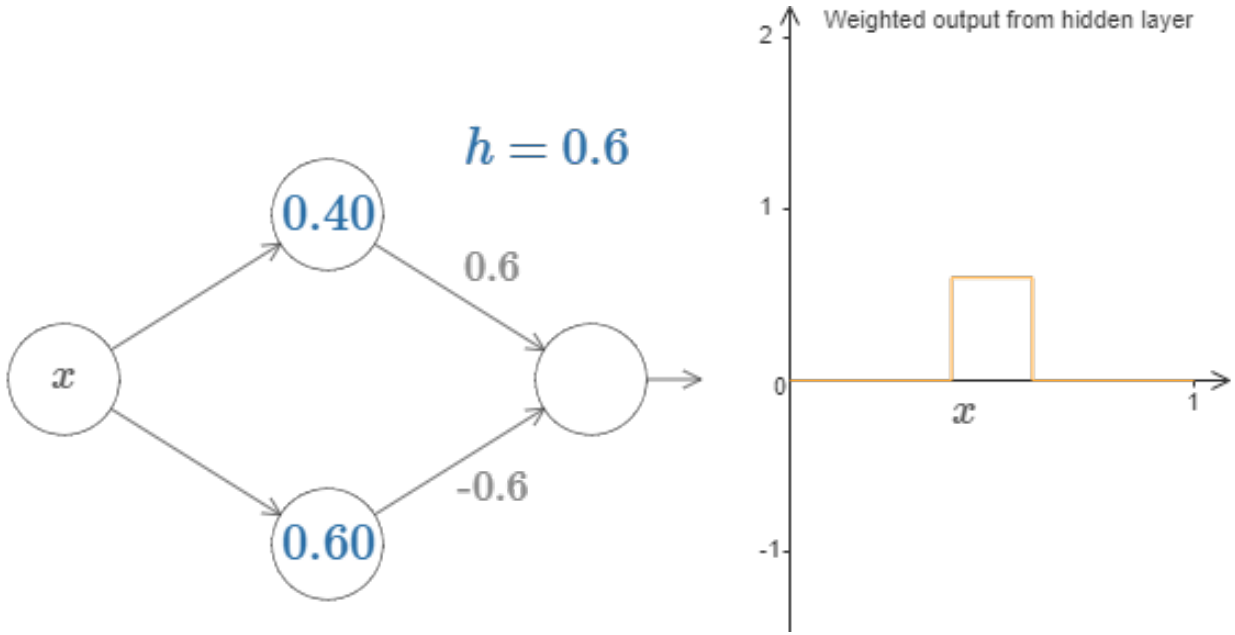

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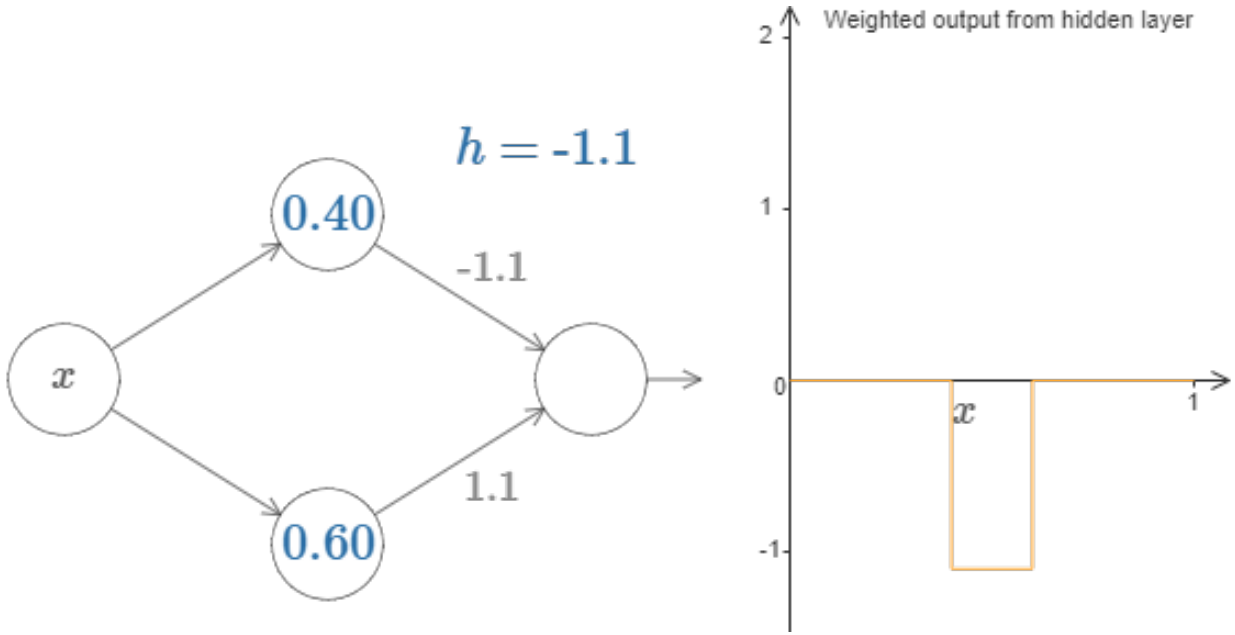

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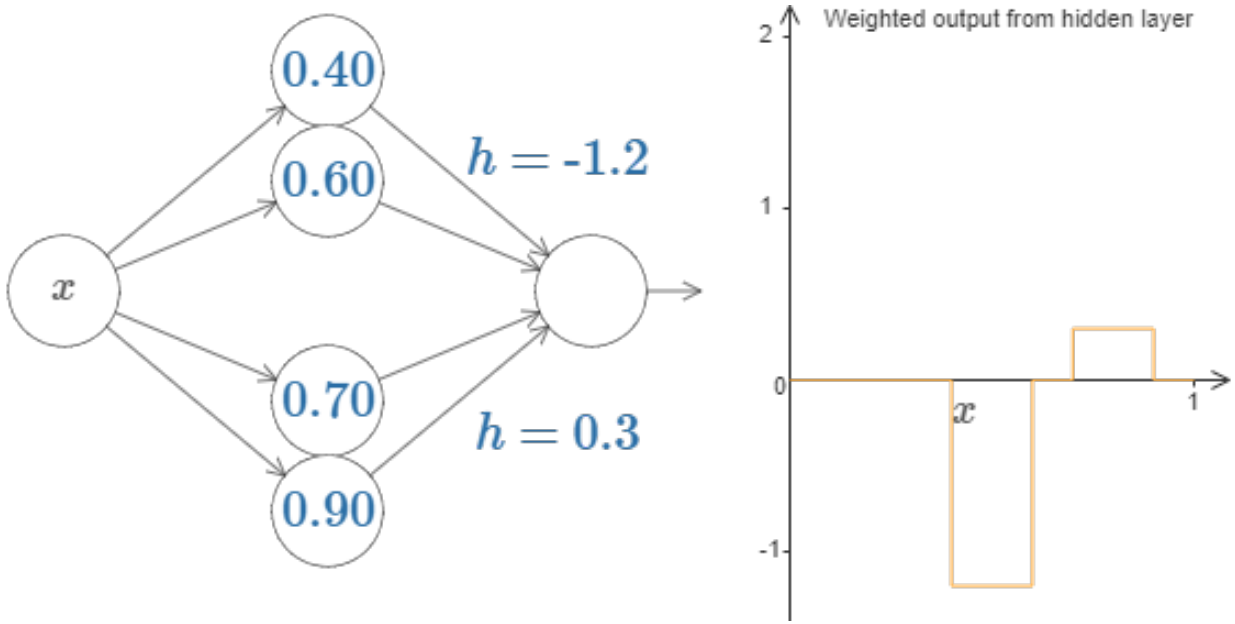
Universality with 1-input and 1-output

- With more neurons in the hidden layer, more towers!



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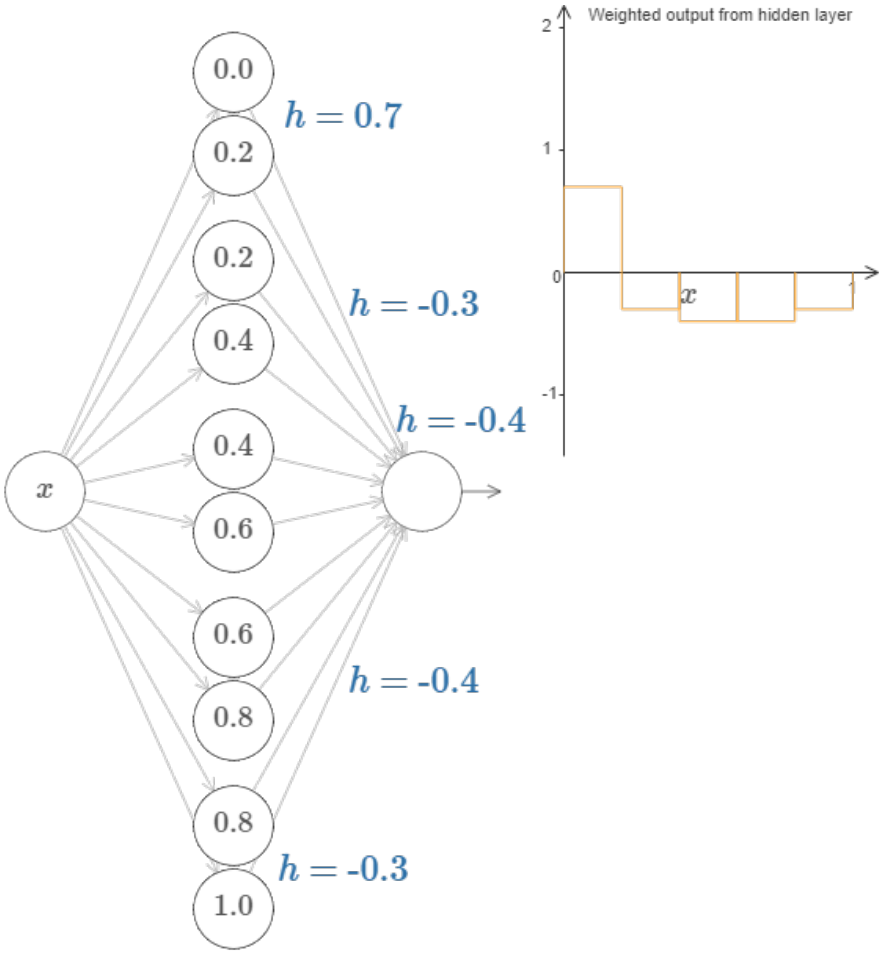

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Universality with 1-input and 1-output

1. Note that we computed only the weighted sum of the hidden outputs

Universality with 1-input and 1-output

1. Note that we computed only the weighted sum of the hidden outputs
2. It's not the output of our MLP

Universality with 1-input and 1-output

- For approximating $f(x)$, the input to the output neuron has to be $\sigma^{-1}(f(x))$ (note that the bias is zero)

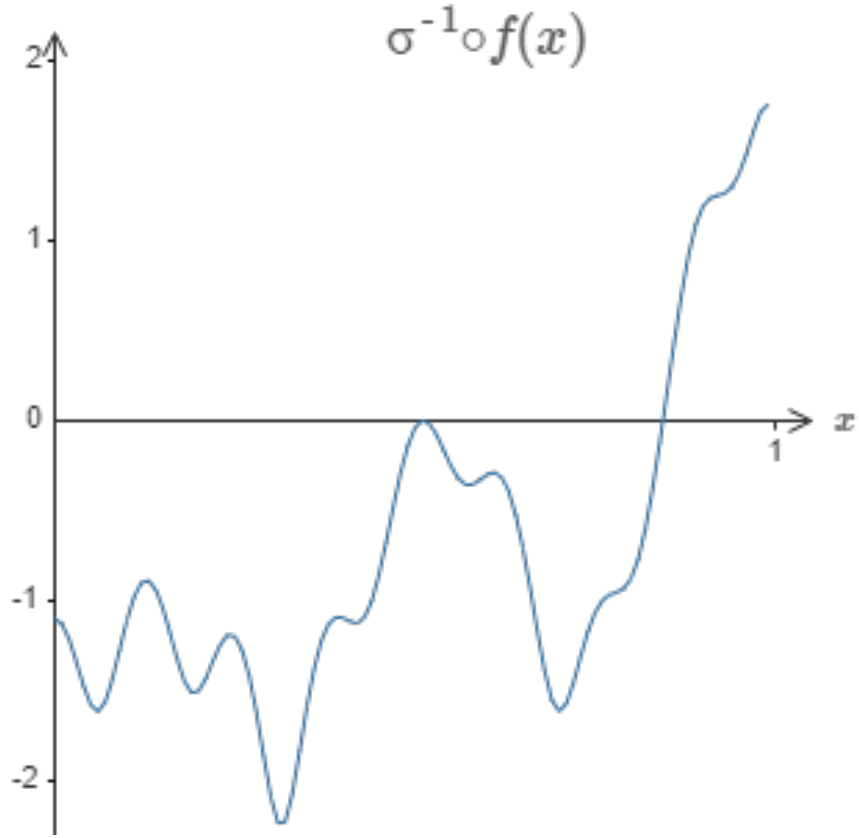


Figure from Michael Nielsen's NNDL textbook

Universality with 1-input and 1-output

- Manipulating the width and height of the towers $\rightarrow$ a better approximation of the function

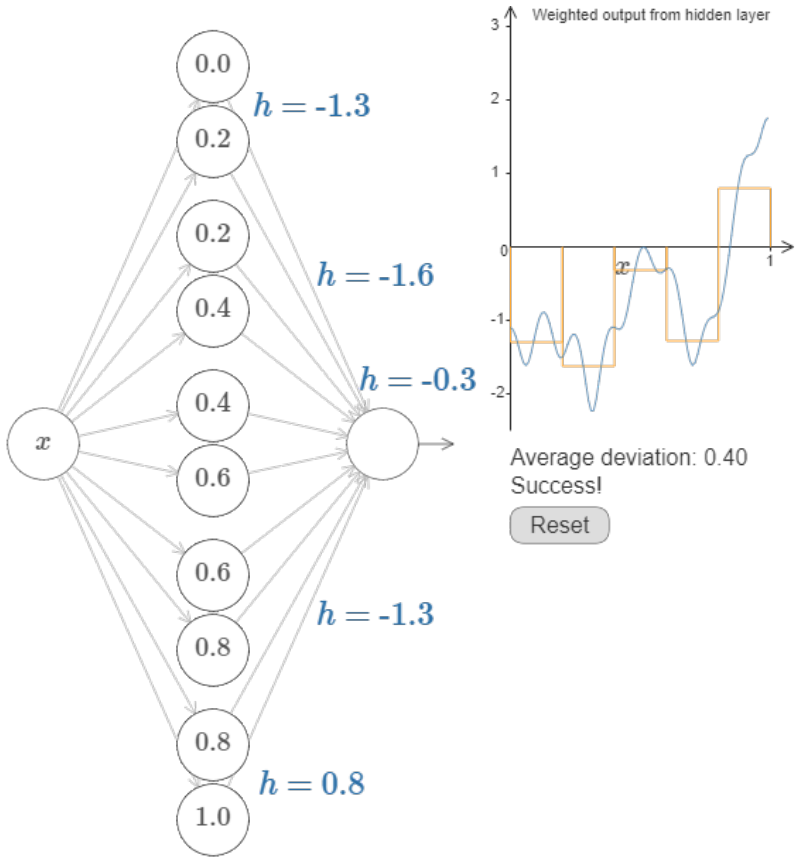


Figure from Michael Nielsen's NNDL textbook

Universality with multiple inputs

- Let's consider two input variables

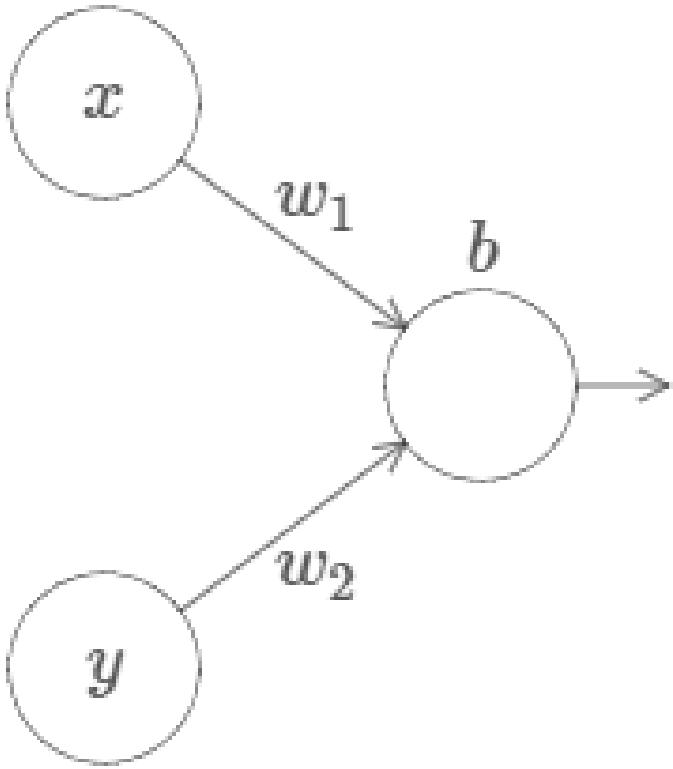
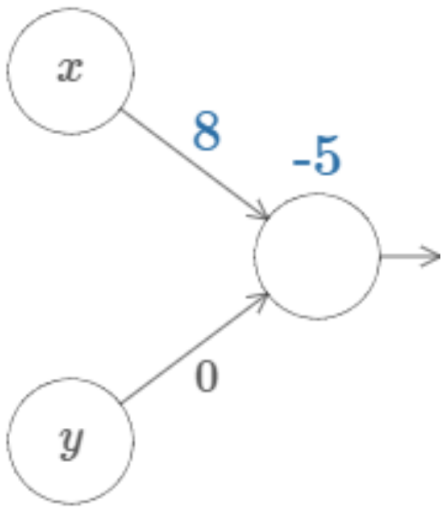
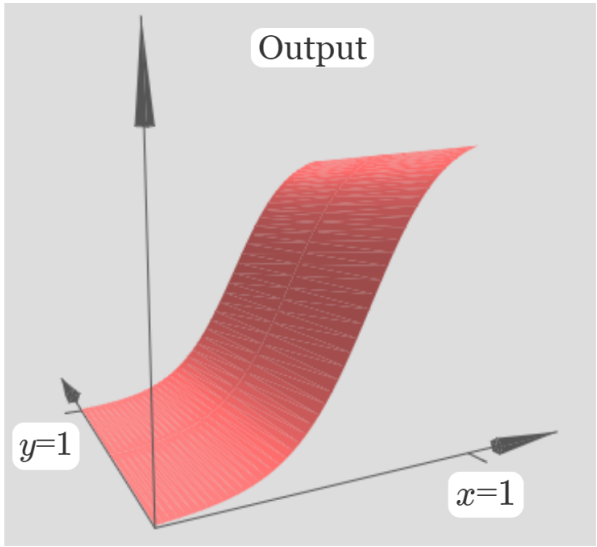


Figure from Michael Nielsen's NNDL textbook

Universality with multiple inputs

- Let's set $w_2 = 0$





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Universality with multiple inputs

- As seen earlier, let's approximate the step function

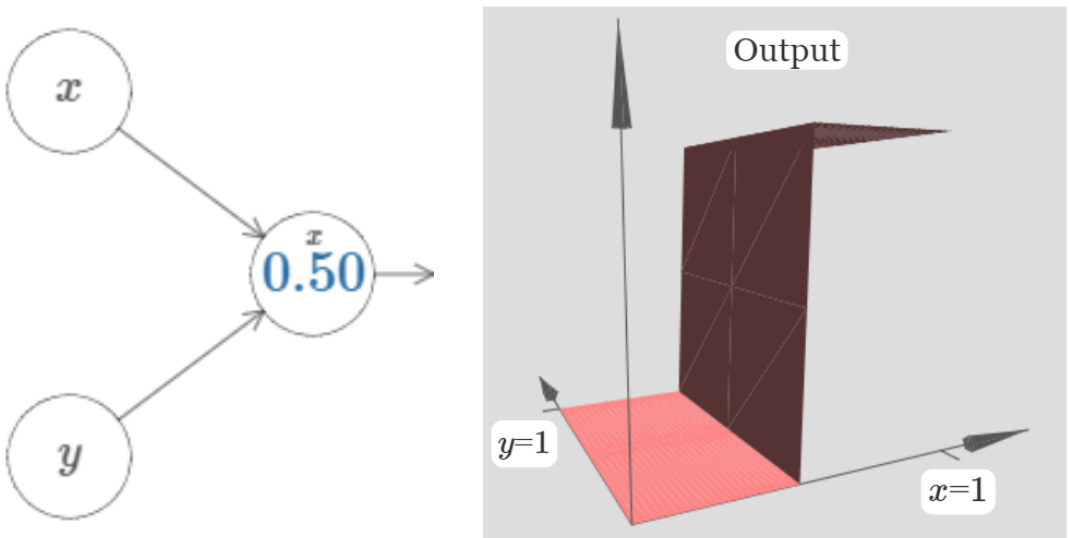
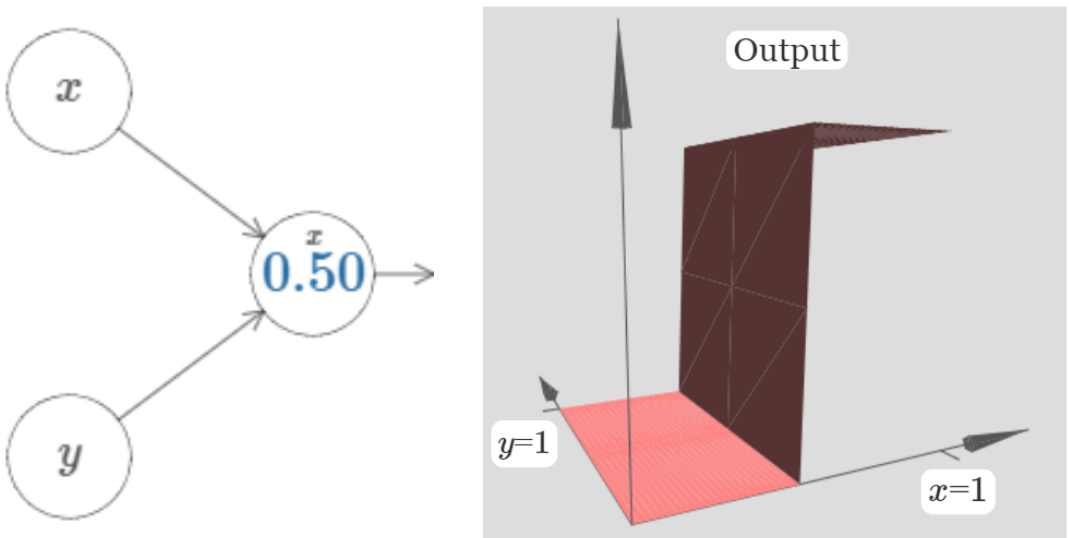


Figure from Michael Nielsen's NNDL textbook

Universality with multiple inputs

- As seen earlier, let's approximate the step function
- Use a single parameter $s = -b/w$ to represent



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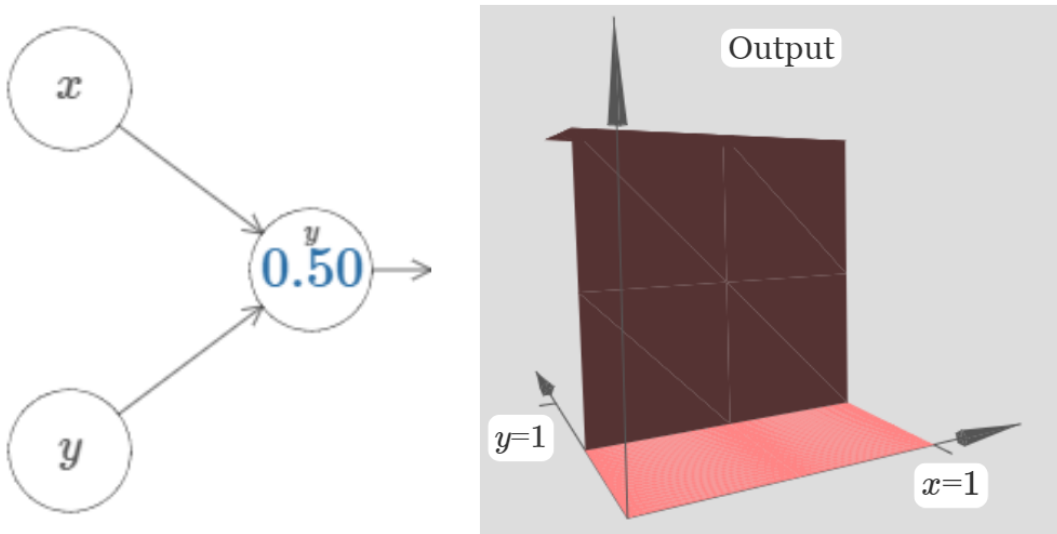
Universality with multiple inputs

- The step function in the y direction



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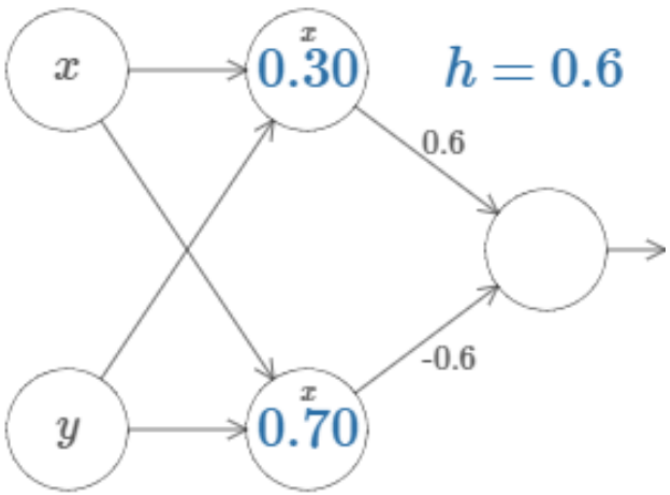
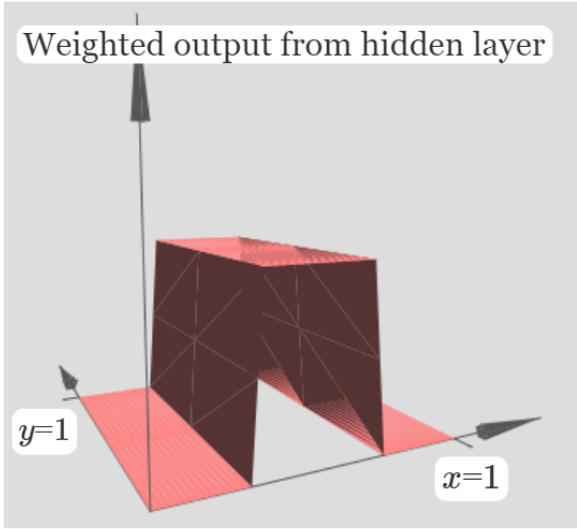
Universality with multiple inputs

- Towards the tower in 3D





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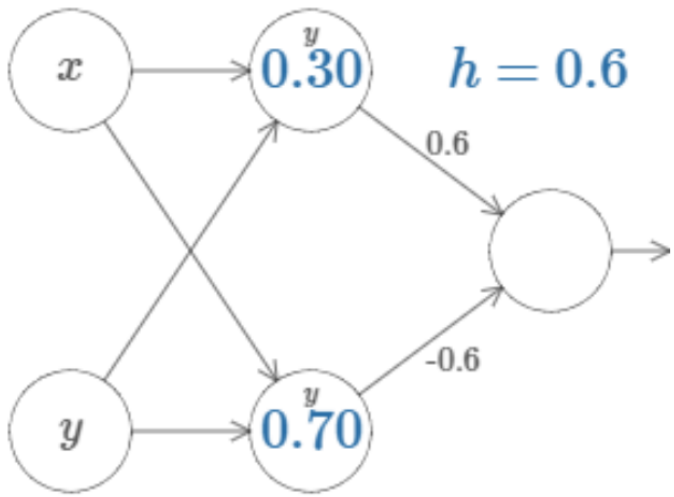

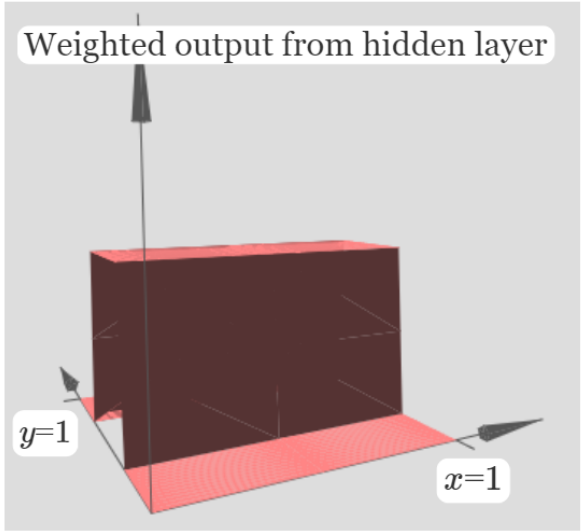

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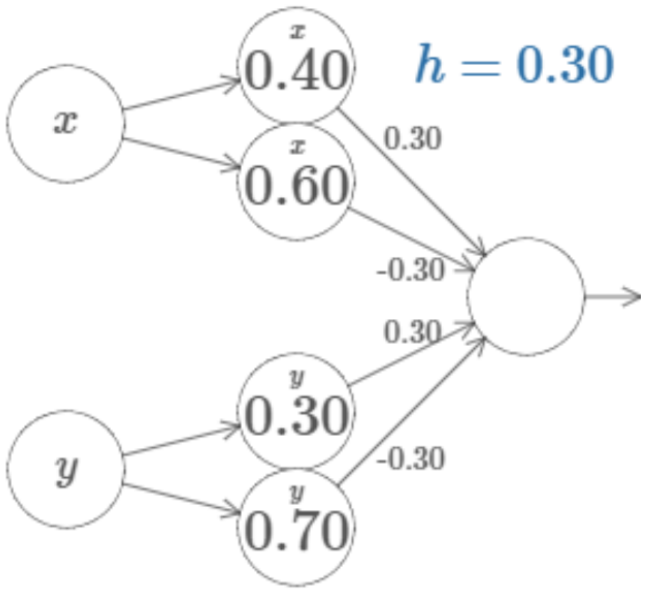

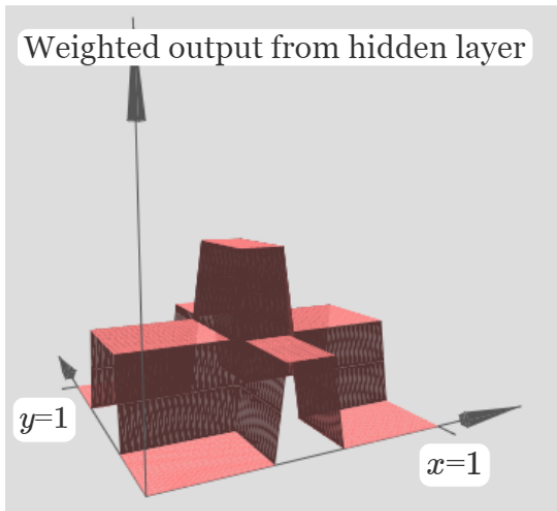

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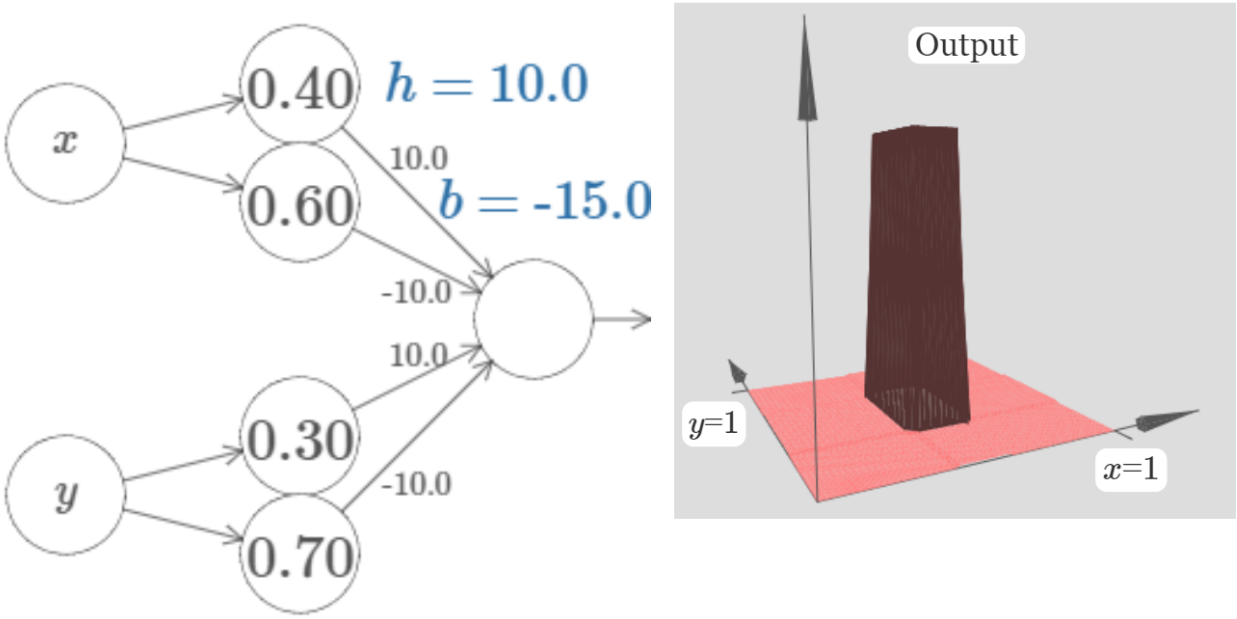

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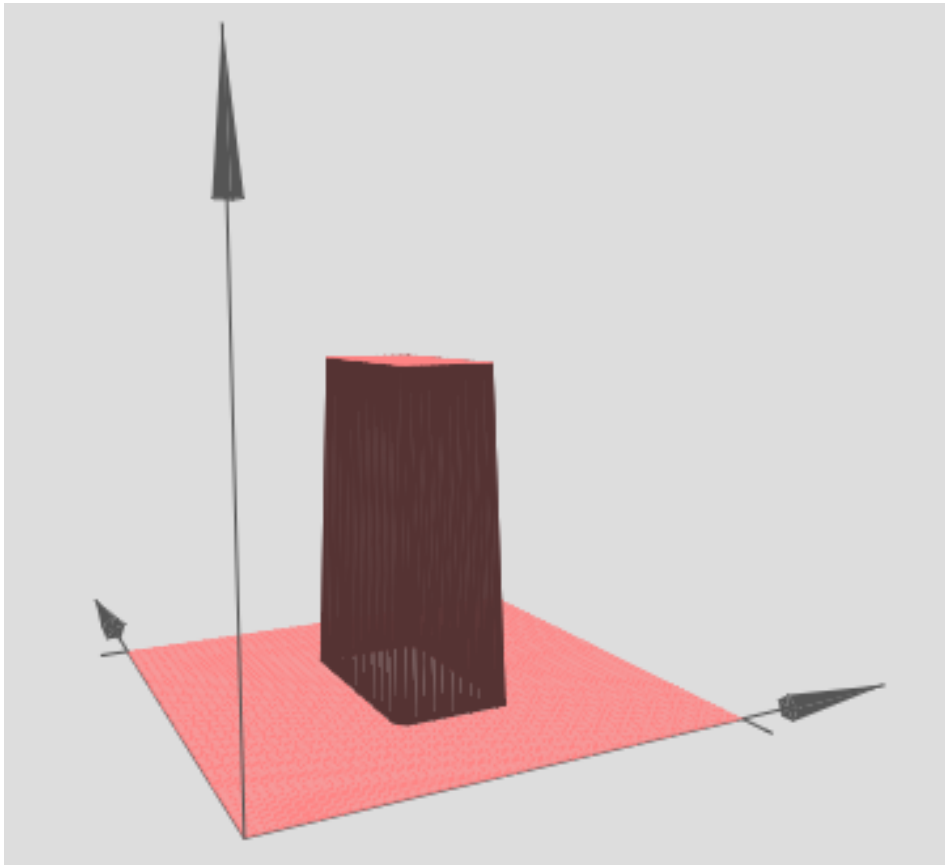

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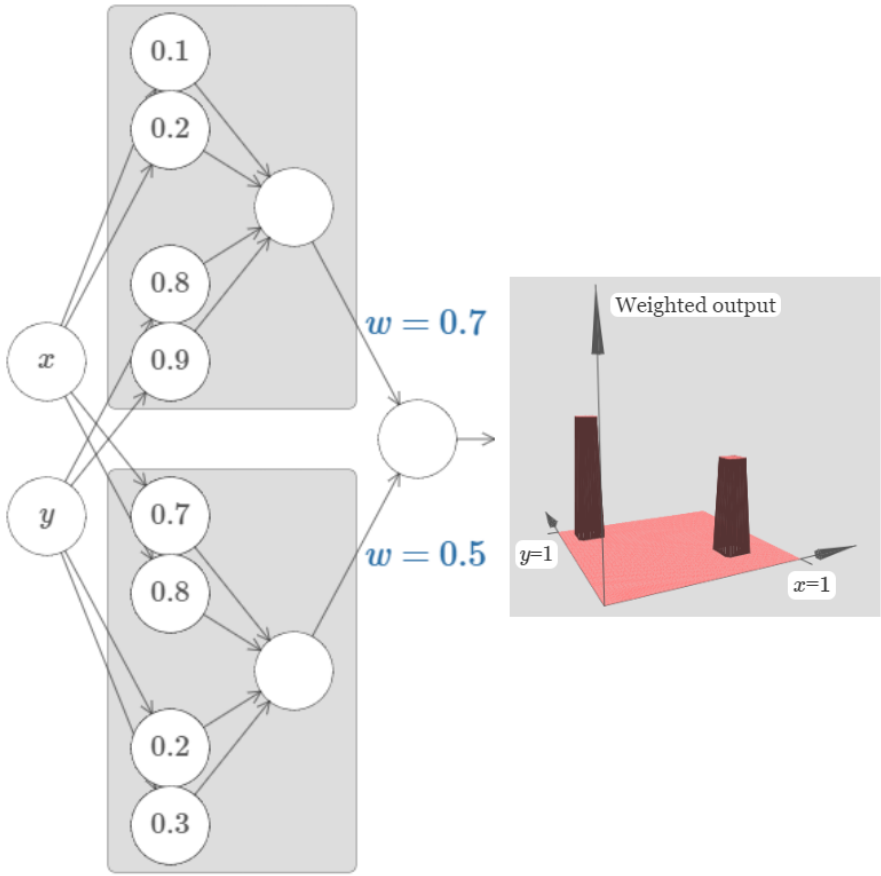
Universality with multiple inputs

- Several of the towers can approximate arbitrary functions



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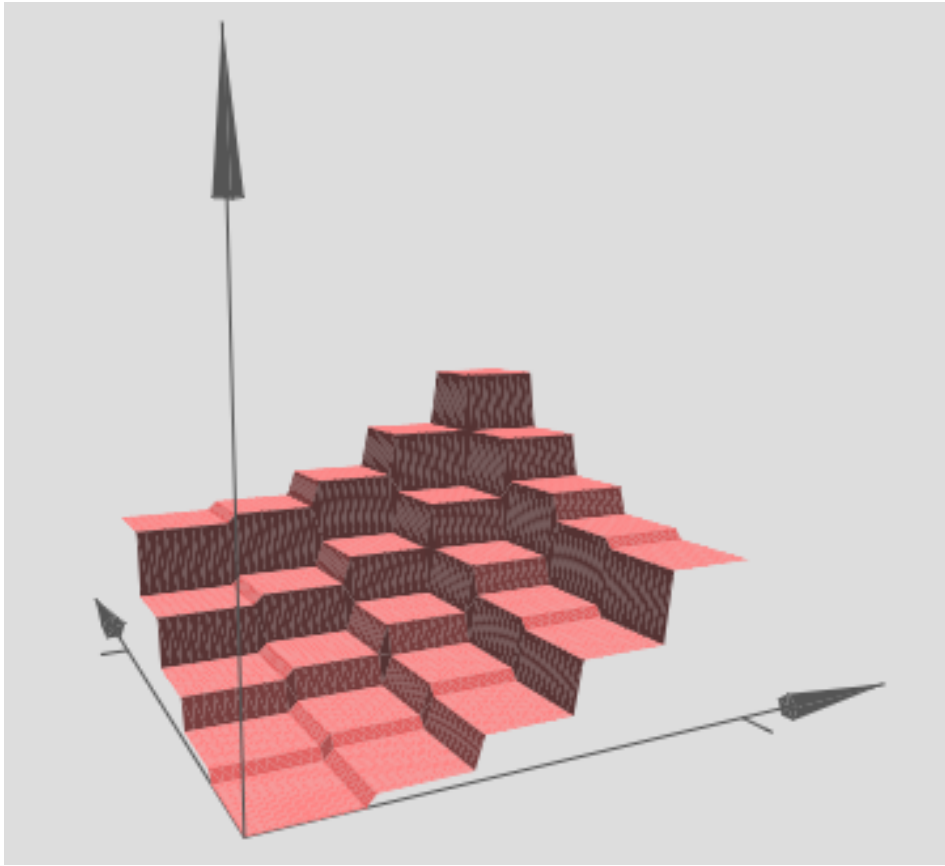

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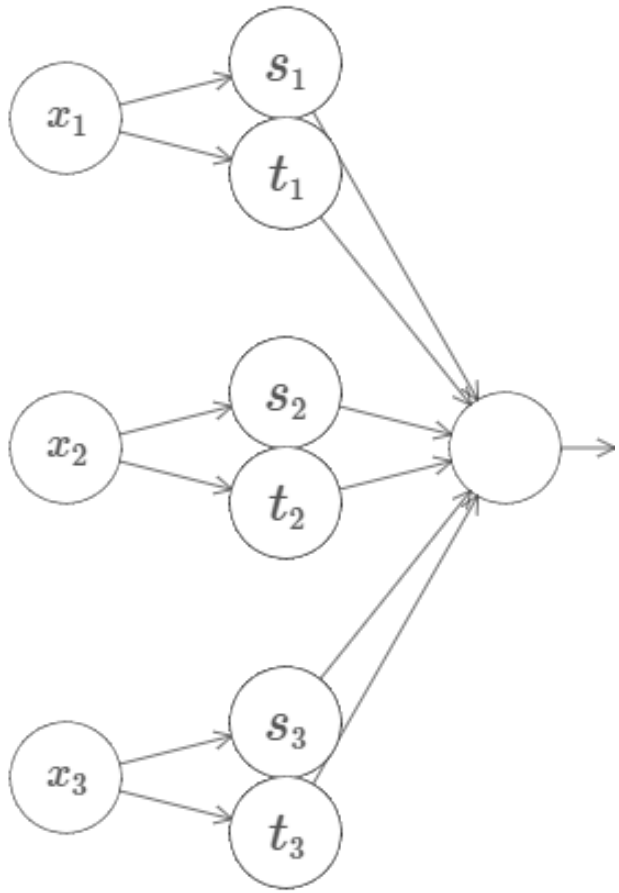
Universality with multiple inputs

- Three input variables



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Universality for vector-valued functions

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Universality for vector-valued functions

- $f(x) : \mathcal{R}^m \rightarrow \mathcal{R}^n$
- Can be regarded as n separate real-valued functions $f^1(x_1, \ldots, x_m), \ldots, f^n(x_1, \ldots, x_m)$
- Create a network approximating each function f^i and put them all together

Universal Approximation: Original form

Theorem 0.1 (UAT, [Cyb89, Hor91]). *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a non-constant, bounded, and continuous function. Let I_m denote the m-dimensional unit hypercube $[0,1]^m$. The space of real-valued continuous functions on I_m is denoted by $C(I_m)$. Then, given any $\varepsilon > 0$ and any function $f \in C(I_m)$, there exist an integer N, real constants $v_i, b_i \in \mathbb{R}$ and real vectors $w_i \in \mathbb{R}^m$ for $i = 1, \ldots, N$, such that we may define:*

$$F(\boldsymbol{x}) = \sum_{i=1}^{N} v_i \sigma\left(\boldsymbol{w}_i^T \boldsymbol{x} + b_i\right) = \boldsymbol{v}^\intercal \sigma\left(\boldsymbol{W}^\intercal \boldsymbol{x} + \boldsymbol{b}\right)$$

as an approximate realization of the function f; that is,

$$|F(\boldsymbol{x}) - f(\boldsymbol{x})| < \varepsilon$$

for all $\boldsymbol{x} \in I_m$.

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- Target function may lie on a space other than the hypercube (has to be bounded)
- Discontinuous targets can be approximated arbitrarily well
- σ can be as general as any nonpolynomial function ($\sigma(z)$ well-defined and different $z \to \infty$ and $z \to -\infty$; at least one side bounded)

Universal Approximation

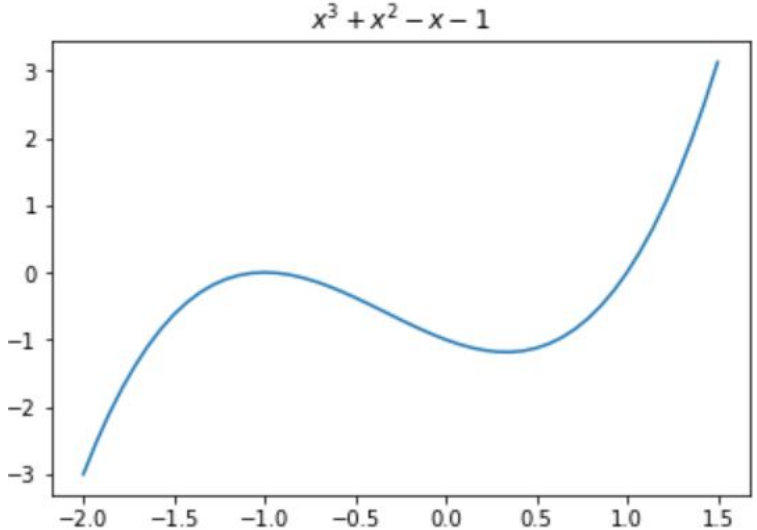
- Note that our visual proof had a network with two hidden layers

Universal Approximation

- Note that our visual proof had a network with two hidden layers
- One can show that a single hidden layer can do this

Universal Approximation using ReLU functions

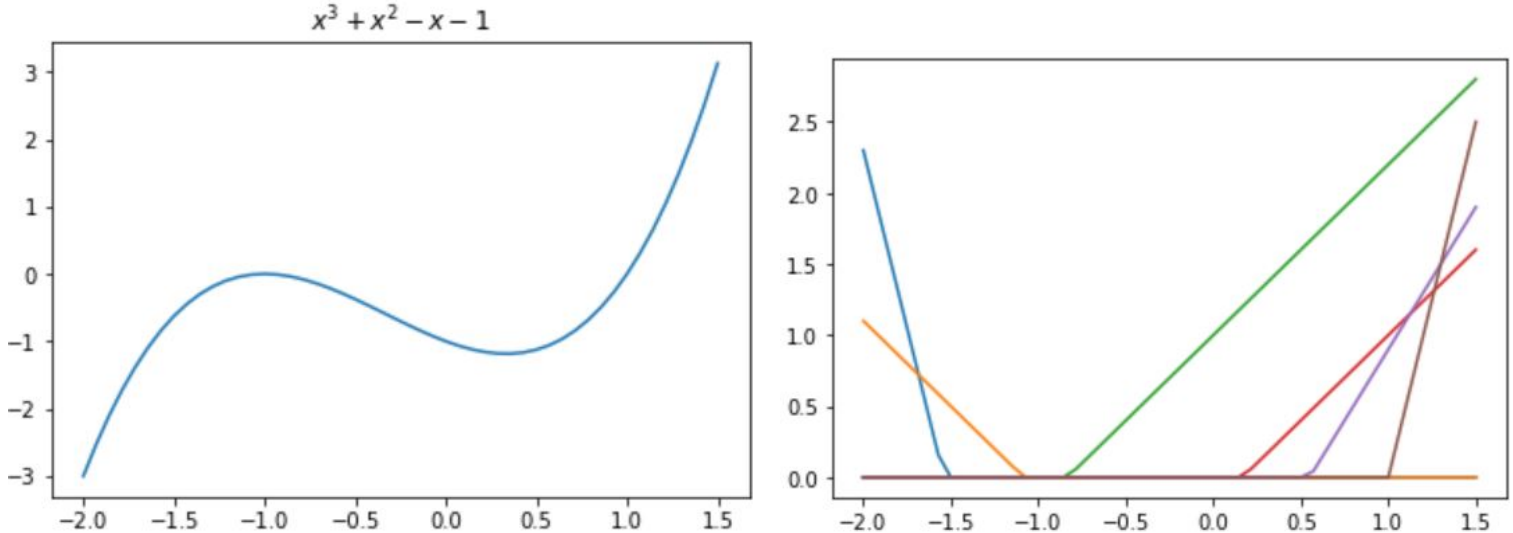
1. Let's approximate the following function using a bunch of ReLUs:



Example credits: Brendan Fortuner, and https://towardsdatascience.com/

Universal Approximation using ReLU functions

1. $n_1 = ReLU(-5x - 7.7), n_2 = ReLU(-1.2x - 1.3), n_3 = ReLU(1.2x + 1), n_4 = ReLU(1.2x - 0.2), n_5 = ReLU(2x - 1.1), n_6 = ReLU(5x - 5)$



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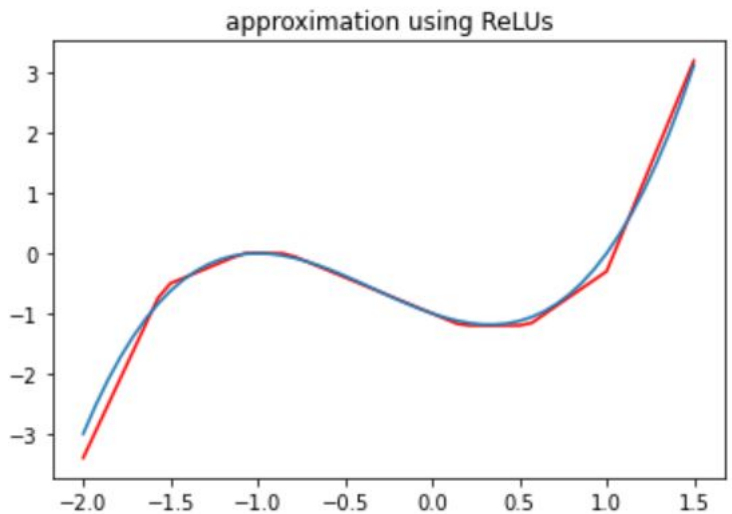
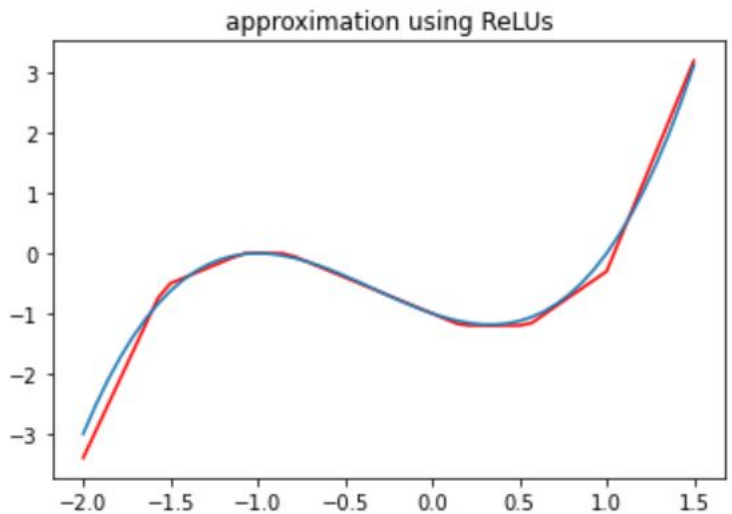
Universal Approximation using ReLU functions

1. Appropriate combination of these ReLUs:
   $-n_1 - n_2 - n_3 + n_4 + n_5 + n_6$



Universal Approximation using ReLU functions

1. Appropriate combination of these ReLUs:
   $-n_1 - n_2 - n_3 + n_4 + n_5 + n_6$
2. Note that this also holds in case of other activation functions with mild assumptions.



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If one hidden layer is good enough, why Deep learning?

1. May require an infeasible size for the hidden layer
2. May not generalize well
3. Doesn't enable the hierarchical learning