Deep Learning

05 Backpropagation-1

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Recap

- Gradient of a scalar valued function $f(\mathbf{x})$: $\mathbf{x} \rightarrow \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_D}\right)$

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- Gradient of a vector valued function $\mathbf{f}(\mathbf{x})$ is called Jacobian:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

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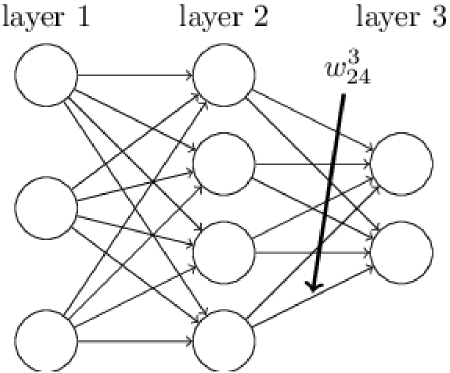

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4. Vector of activations (or, biases) at a layer l is denoted by a bold-faced $\mathbf{x}^l$ (or $\mathbf{b}^l$) and W^l is the matrix of weights into layer l

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4. σ is the activation function that applies element-wise

Gradient descent on MLP

- Loss is $\mathcal{L}(W, \mathbf{b}) = \sum_n l(f(x_n; W, \mathbf{b}), y_n) = \sum_n l(\mathbf{x}^L, y_n)$ (L is the number of layers in the MLP)

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- For applying Gradient descent, we need gradient of individual sample loss with respect to all the model parameters

$$l_n = l(f(x_n; W, \mathbf{b}), y_n)$$

$\frac{\partial l_n}{\partial W_{jk}^{(l)}}$ and $\frac{\partial l_n}{\partial \mathbf{b}_j^{(l)}}$ for all layers l

Forward pass operation

$$x^{(0)} = x \xrightarrow{W^{(1)},\mathbf{b}^{(1)}} s^{(1)} \xrightarrow{\sigma} x^{(1)} \xrightarrow{W^{(2)},\mathbf{b}^{(2)}} s^{(2)} \dots x^{(L-1)} \xrightarrow{W^{(L)},\mathbf{b}^{(L)}} s^{(L)} \xrightarrow{\sigma} x^{(L)} = f(x; W, \mathbf{b})$$

Formally, $x^{(0)} = x, f(x; W, \mathbf{b}) = x^{(L)}$

$$\forall l = 1, \dots, L \quad \begin{cases} s^{(l)} & = W^{(l)} x^{(l-1)} + \mathbf{b}^{(l)} \\ x^{(l)} & = \sigma\left(s^{(l)}\right) \end{cases}$$

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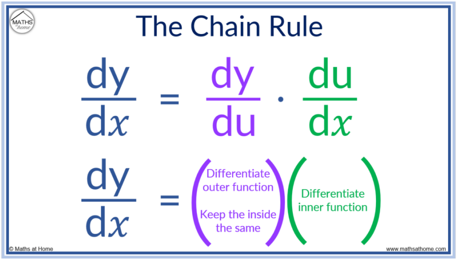
- $$\frac{\partial}{\partial x} f(g(x)) = \left.\frac{\partial f(a)}{\partial a}\right|_{a=g(x)} \cdot \frac{\partial g(x)}{\partial x}$$

Chain rule of differential calculus



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- $y = f(z) \rightarrow \Delta y = \frac{df}{dz} \Delta z = \frac{df}{dz} \frac{dg(x)}{dx} \Delta x = \frac{df}{dg(x)} \frac{dg(x)}{dx} \Delta x$

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भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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Chain rule of differential calculus

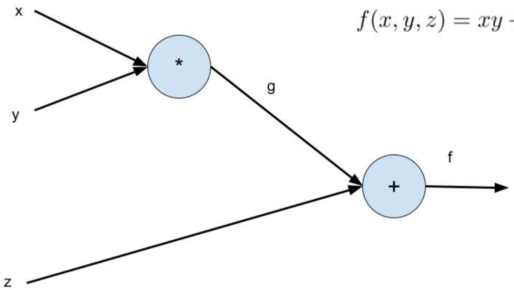
1. $f(x) = e^{sin(x^2)}$, let's find $\frac{\partial f}{\partial x}$

Chain rule of differential calculus

$$g = xy$$

$$f = g + z$$

$$f(x, y, z) = xy + z$$



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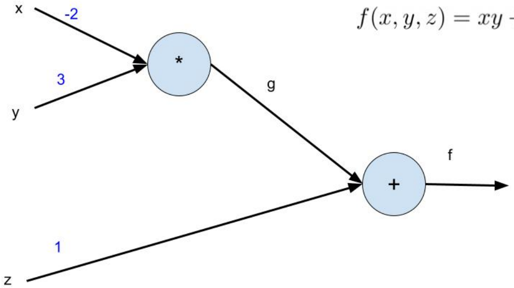

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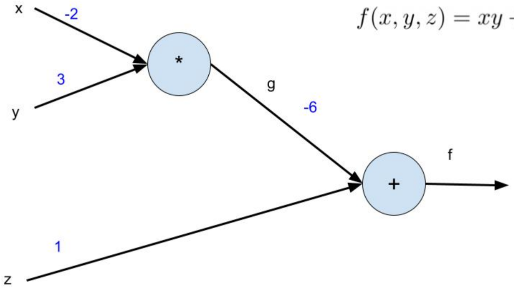

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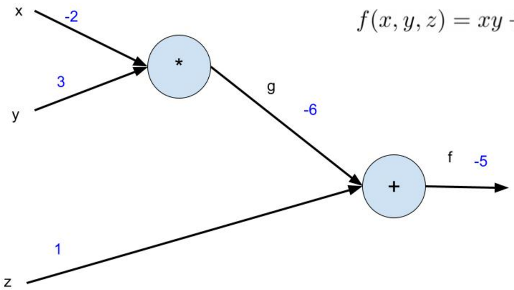

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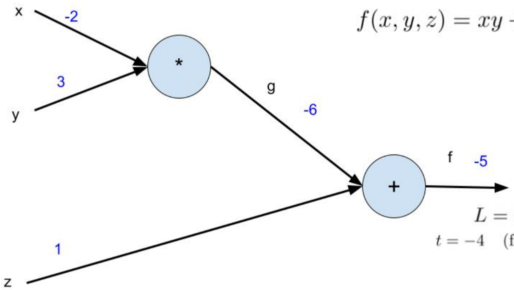

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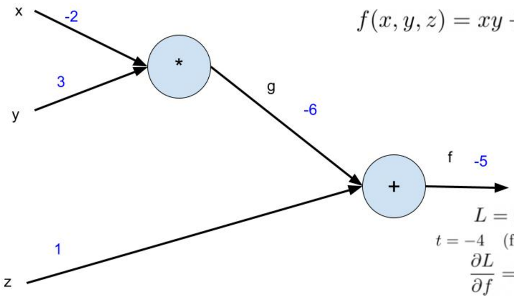

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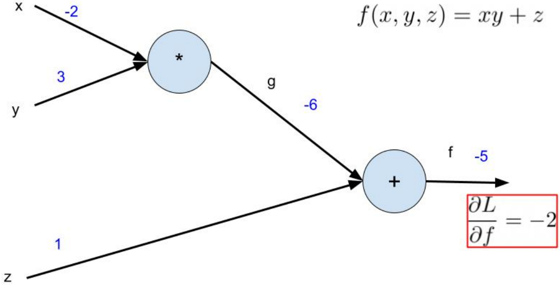

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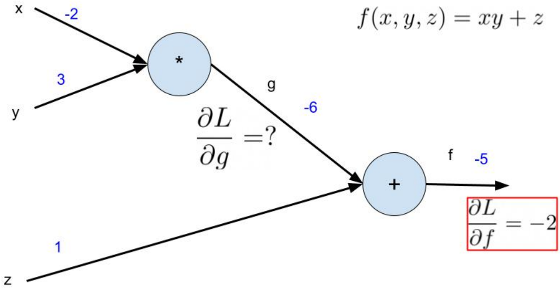

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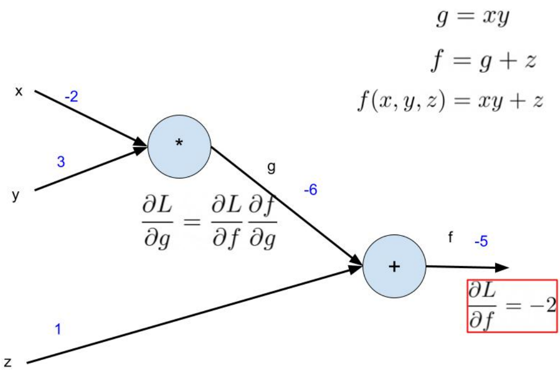

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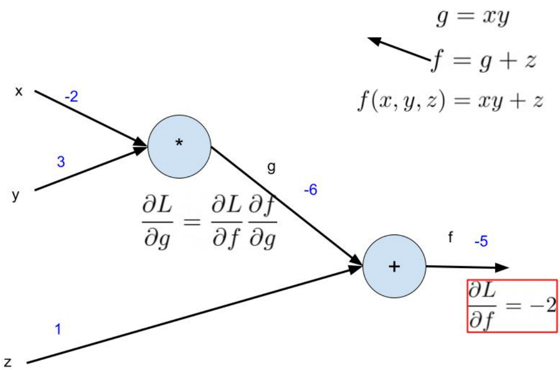

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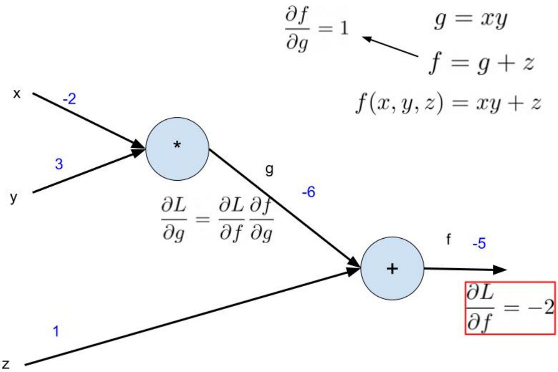

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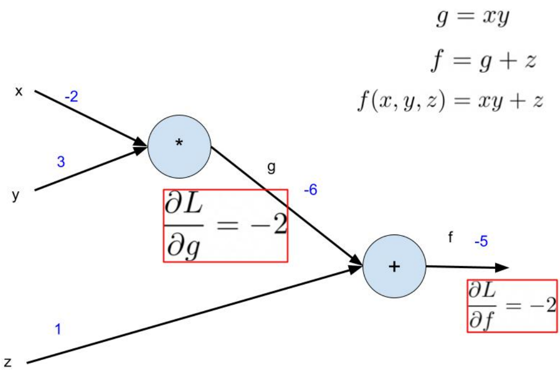

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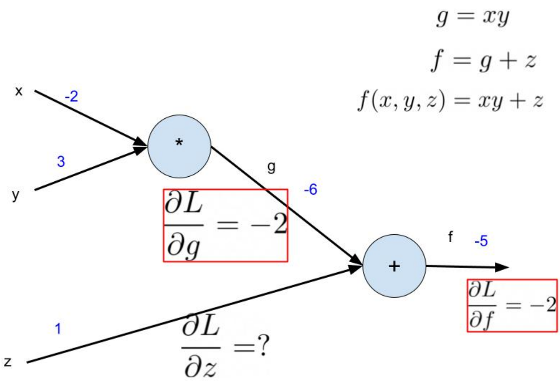

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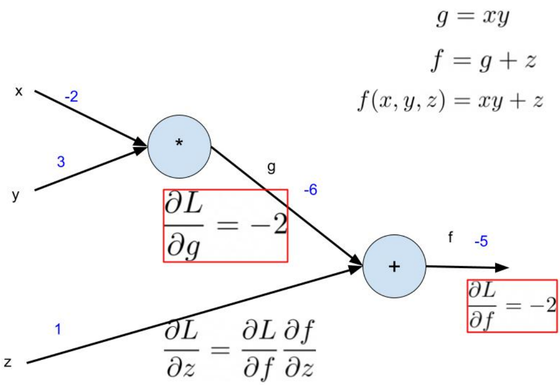

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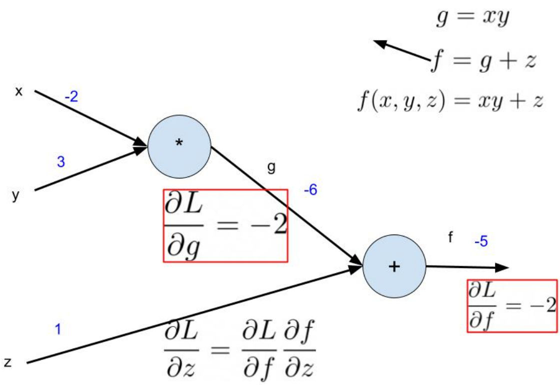

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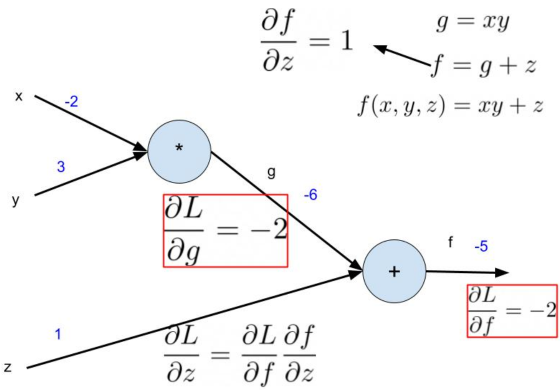

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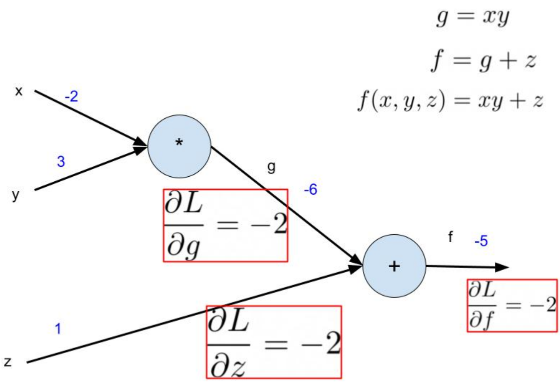

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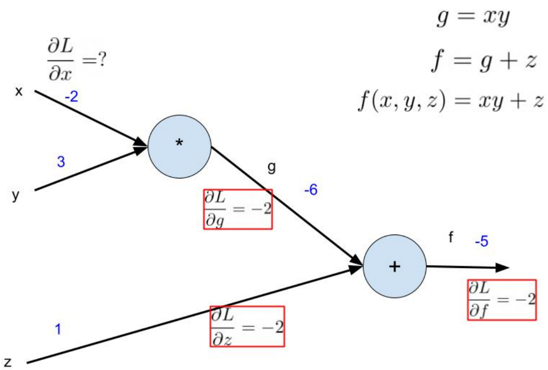

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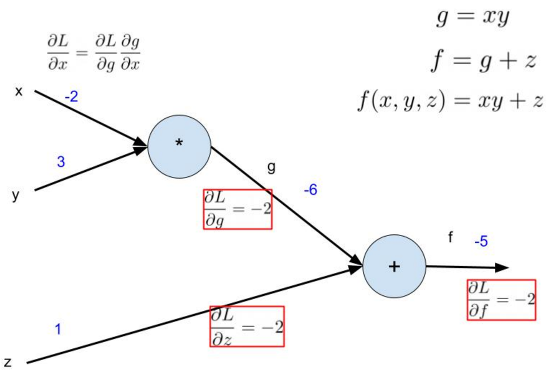

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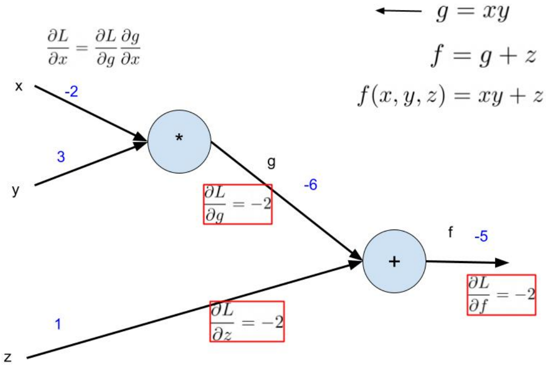

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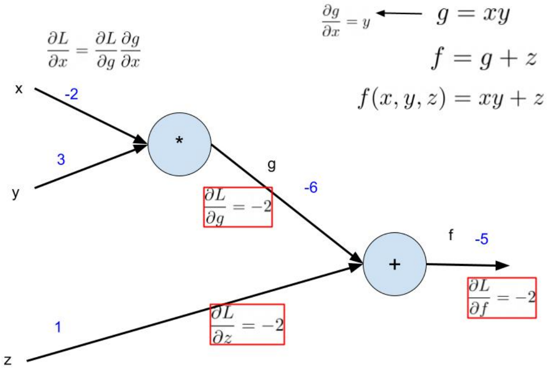

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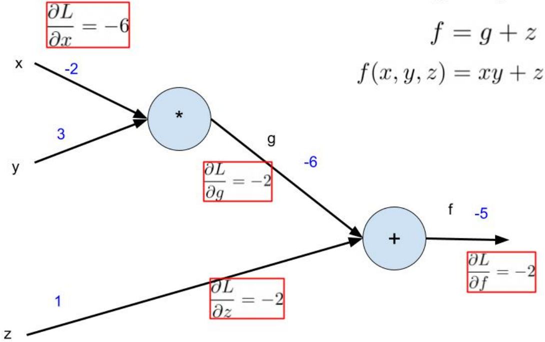

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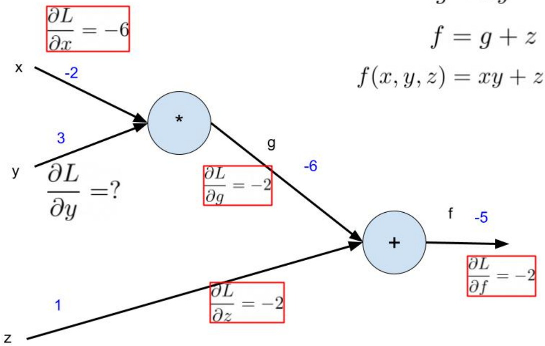

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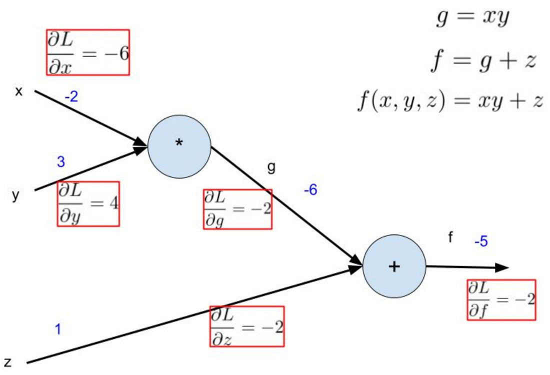
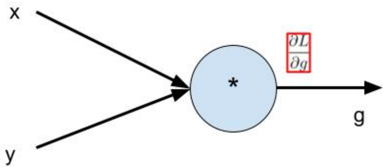


Gradient Flow



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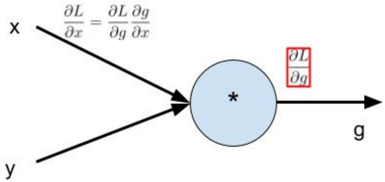

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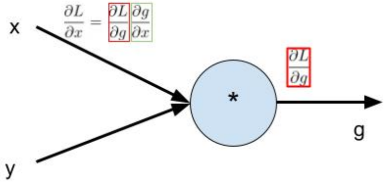

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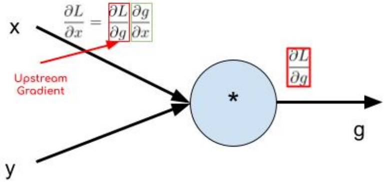

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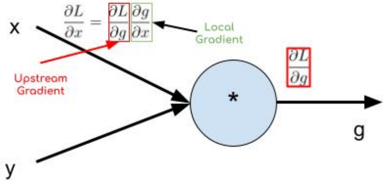

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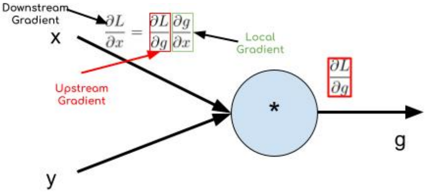

Chain rule of differential calculus for an MLP

- $$J_{f_N \circ f_{N-1} \circ \dots f_1(x)} = J_{f_N(f_{N-1}(\dots f_1(x)))} \cdot J_{f_{N-1}(f_{N-2}(\dots f_1(x)))} \cdots J_{f_2(f_1(x))} \cdot J_{f_1(x)}$$

  $J_{f(x)}$ is Jacobian of f computed at x.