Digital signal processing

February 13, 2026

Contents

1 Conventions

Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic, integrable function. The Fourier coefficients of f are defined by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-ikx}\, \mathrm{d}x, \quad k \in \mathbb{Z}.$$

The Fourier series of f is given by

$$f(x) \ = \ \sum_{k\in\mathbb{Z}} \hat{f}(k)\, e^{ikx}$$

For 2π-periodic functions $f, g \colon \mathbb{R} \to \mathbb{C}$ we define the circular convolution by

$$(f * g)(x) \ = \ \int_0^{2\pi} f(y)\, g(x-y)\, \mathrm{d}y$$

The root of unity:

$$\omega_N = e^{\frac{2\pi i}{N}}$$

2 DFT

Lemma 2.1. *The following holds for roots of unity:*

2.1.1 $\omega_N^{kN} = 1 \; \forall k \in \mathbb{Z}$

2.1.2 N *even.* $\omega_{N/2} = \omega_N^2$

2.1.3 $\overline{\omega_N} = \omega_N^{-1}$

Proof.

1. $$\omega_N^N = \left(e^{\frac{2\pi i}{N}}\right)^N = e^{\frac{2\pi i N}{N}} = e^{2\pi i} = \cos(2\pi) + i\sin(2\pi) = 1$$

   Then:
   $$\omega_N^{kN} = \left(\omega_N^N\right)^k = 1^k = 1$$

2. $\omega_{N/2} = e^{\frac{2\pi i}{N/2}} = e^{2\cdot\frac{2\pi i}{N}} = \left(e^{\frac{2\pi i}{N}}\right)^2 = \omega_N^2$

3. $\overline{\omega_N} = \overline{e^{\frac{2\pi i}{N}}} = e^{-\frac{2\pi i}{N}} = \left(e^{\frac{2\pi i}{N}}\right)^{-1} = \omega_N^{-1}$

□

Suppose we have $f : \mathbb{R} \to \mathbb{C}$ which is 2π-periodic. Consider the values of this function at points $\frac{0\cdot 2\pi}{N}$, $\frac{1\cdot 2\pi}{N}$, ..., $\frac{(N-1)\cdot 2\pi}{N}$:

$$x_j := \frac{j \cdot 2\pi}{N}$$

$$y_j := f(x_j) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx_j} = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i \frac{kj}{N}} = \sum_{k=-\infty}^{\infty} \hat{f}(k) \omega_N^{kj}$$

By Lemma 2.1.1 we have that $\omega_N^{(k+nN)j} = \omega_N^{kj}\omega_N^{(nj)N} = \omega_N^{kj}$
By grouping the terms with $k_1 = k_2 \mod N$ and obtain the following:

$$y_j = \sum_{k=0}^{N-1} \omega_N^{kj} \left(\sum_{n=-\infty}^{\infty} \hat{f}(k+nN) \right) = \sum_{k=0}^{N-1} \omega_N^{kj} z_k \tag{1}$$

We define the matrix $F_{kj} = \frac{1}{\sqrt{N}}\omega_N^{kj}$. Then:

$$y_j = \sqrt{N}(Fz)_j \Rightarrow y = \sqrt{N}Fz \Rightarrow z = \frac{1}{\sqrt{N}}F^{-1}y \tag{2}$$

Later we use the vector z to indicate the output of the DFT, namely $F^{-1}y$

The crucial observation is that the oscillation of the form $e^{i(k+nN)x}$ for $n > 0$ and $k \in 0, ..., N-1$ cannot be detected using the given discretized points. This simply follows from the fact that $e^{i(k+nN)x_j} = e^{i(k+nN)\frac{j\cdot 2\pi}{N}} = e^{i\frac{kj\cdot 2\pi}{N}}e^{2\pi i\cdot nj} = e^{2\pi i\frac{kj}{N}} = e^{ikx_j}$ as computed previously. Even though the original signal might have an oscillation with rate above N, we are unable to detect it using the discrete values of the function f, therefore we may just assume each z_k corresponds to the coefficient $\hat{f}(k)$ (basically, contribution of the waves corresponding to the frequencies above N cannot be distinguished from the wave with the respective frequency in the range from 0 to $N-1$). In sound processing this doesn't pose any issue since human ear cannot hear any frequency above 20 khz, so with enough discretization we basically do not lose any perceivable data.

Theorem 2.1. *The matrix* $F_{kj} = \frac{1}{\sqrt{N}}\omega_N^{kj}$ *is unitary*

Proof.

$$(F\overline{F^T})_{jl} = \frac{1}{\sqrt{N}}\overline{\left(\frac{1}{\sqrt{N}}\right)}\sum_{k=0}^{N-1} F_{jk}(\overline{F^T})_{kl} = \frac{1}{N}\sum_{k=0}^{N-1} F_{jk}\overline{F_{lk}} =$$
$$\frac{1}{N}\sum_{k=0}^{N-1}\omega_N^{jk}\overline{\omega_N^{lk}} \underset{(a)}{=} \frac{1}{N}\sum_{k=0}^{N-1}\omega_N^{jk}\omega_N^{-lk} = \frac{1}{N}\sum_{k=0}^{N-1}\omega_N^{(j-l)k}$$

In (a) we used the identity $\overline{\omega_N^{lk}} = \omega_N^{-lk}$ from Lemma 2.1.3

1. $j \neq l$. Using the formula for sum of geometric series:

$$\sum_{k=0}^{N-1}\omega_N^{(j-l)k} = \frac{1-\left(\omega_N^{j-l}\right)^N}{1-\omega^{j-l}} = \frac{1-\left(\omega_N^N\right)^{j-l}}{1-\omega^{j-l}} \underset{(a)}{=} \frac{1-1^{j-l}}{1-\omega^{j-l}} = 0$$

In (a) we used the identity $\omega_N^N = 1$ from Lemma 2.1.1. The condition $j \neq l$ ensures that the denominator is non-zero, so the expression is well-defined.

2. $j = l$. Then:

$$\sum_{k=0}^{N-1} \omega_N^{(j-l)k} = \sum_{k=0}^{N-1} 1 = N \Rightarrow (F\overline{F^T})_{jl} = \frac{1}{N} N = 1$$

The matrix $FF^* = I$, so F unitary and $F^{-1} = \overline{F^T}$
The entries of F^{-1} are given by $F_{ij}^{-1} = (\overline{F_{ji}}) = \frac{1}{\sqrt{N}} \overline{(\omega_N^{ji})} = \frac{1}{\sqrt{N}} \omega_N^{-ij}$. $\square$

Using the result of Theorem 2.1 and equations (2), (1) we define the discrete Fourier transform:

$$z_j = \left(F^{-1} y\right)_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-kj} y_k \tag{DFT}$$

$$y_j = \left(F z\right)_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{kj} z_k \tag{IDFT}$$

Theorem 2.2 (Real signal conjugate symmetry)**.** *Let N be even. The following are equivalent:*

1. $y = \overline{y}, y \in \mathbb{C}^N$
2. $z_j = \overline{z_{N-j}} \quad \forall j \in \{1, \dots, N/2 - 1\}$ *and* $z_0, z_{N/2} \in \mathbb{R}$

Proof.
"(1) $\Rightarrow$ (2)" Using the formula for DFT

$$z_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-kj} y_k = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \overline{\omega_N^{kj}} y_k \underset{y=\overline{y}}{=} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \overline{\omega_N^{kj} y_k} =$$
$$\overline{\left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{kj} y_k \right)} \underset{\omega_N^{kN}=1}{=} \overline{\left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-kN} \omega_N^{kj} y_k \right)} =$$
$$\overline{\left(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-k(N-j)} y_k \right)} = \overline{z_{N-j}}$$

$$z_0 = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-k \cdot 0} y_k = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} y_k \in \mathbb{R}$$

Observe:

$$\omega_N^{-k\cdot N/2} = \left(e^{-2\pi i \frac{N/2}{N}}\right)^k = \left(e^{-\pi i}\right)^k = (-1)^k$$

$$z_{N/2} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{-k\cdot N/2} y_k = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (-1)^k y_k \in \mathbb{R}$$

"(2) $\Rightarrow$ (1)" Using the formula for IDFT

$$y_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega_N^{kj} z_k = \frac{1}{\sqrt{N}} \left(z_0 + (-1)^j z_{N/2} + \sum_{k=1}^{N/2-1} \omega_N^{kj} z_k + \sum_{l=N/2+1}^{N-1} \omega_N^{lj} z_l \right) \tag{3}$$

$$\sum_{k=1}^{N/2-1} \omega_N^{kj} z_k + \sum_{l=N/2+1}^{N-1} \omega_N^{lj} z_l \underset{l'=N-l}{=} \sum_{k=1}^{N/2-1} \omega_N^{kj} z_k + \sum_{l=1}^{N/2-1} \omega_N^{(N-l)j} z_{N-l} =$$

$$\sum_{k=1}^{N/2-1} \left(\omega_N^{kj} z_k + \omega_N^{(N-k)j} z_{N-k} \right) = \sum_{k=1}^{N/2-1} \left(\omega_N^{kj} z_k + \overline{\omega_N^{kj} z_k} \right) \in \mathbb{R}$$

By plugging this expression back to (3) we get:

$$(3) = \frac{1}{\sqrt{N}} \left(z_0 + (-1)^j z_{N/2} + \sum_{k=1}^{N/2-1} \left(\omega_N^{kj} z_k + \overline{\omega_N^{kj} z_k} \right) \right) \in \mathbb{R}$$

since all the summands and the factor in front are real. $\square$

3 FFT