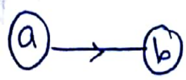
10/11/18

Graph Theory

*) Let, $G = (V, E)$ be a graph
here, V be a finite set called as vertex set.
E be another finite set called as Edge set.

*) (a,b) denotes a directed path/edge from vertex 'a' to vertex 'b'.



*) $\{a,b\}$ denotes a non-directed path/edge in between the vertices 'a' & 'b'.

*) If all the edges contained in a graph are directed then the graph is said to be a directed graph or digraph.

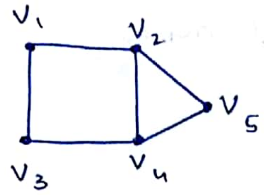
*) If all the edges contained in a graph are non directed then the graph is said to be nondirected graph.

*) If some of edges are directed and the remaining edges are non directed then the graph is said to be mixed graph.

*) The total no. of vertices contained in a graph is its order,
i.e. order of $G = |V|$

*) The total no. of edges contained in a graph is its ~~site~~ size,
ie, ~~site~~ size of $G = |E|$

Ex. (1)

G: 

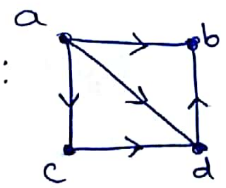
Here, $V = \{v_1, v_2, v_3, v_4, v_5\}$

$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_5\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}\}$

The graph is nondirected

order of G = 5

size of G = 6

(2) G: 

Here $V = \{a, b, c, d\}$

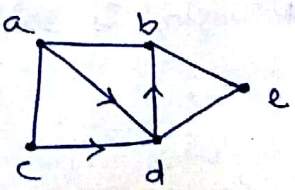
$E = \{(a, b), (a, c), (a, d), (c, d), (d, b)\}$

The graph is directed / digraph

order = 4

size = 5

(3)

G : 

It is mixed graph

$V = \{a, b, c, d, e\} \Rightarrow$ order = 5

$E = \{\{a, b\}, \{a, c\}, (a, d) (c, d), (d, b), (e, d), \{b, e\}\}$

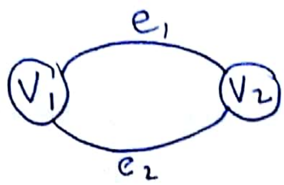
size = 7

*) Let $G = (V, E)$ be a non directed graph.

— $e = \{v_1, v_2\}$ be an edge.

Here, we can say that, v_1 & v_2 are the end vertices of the edge 'e'.

— If there exists more than one edge in between two end vertices, then those edges are said to be parallel edges.

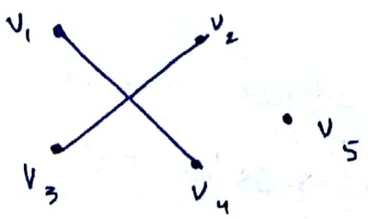


Here, e_1 & e_2 are parallel edges.

— If the end vertices of an edge are identical, then the edge is said to be a loop.

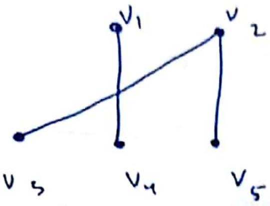
— If in a graph $G = (V, E)$ any one vertex is not the end vertex of any edge then that vertex is said to be & Isolated vertex.

G:



Here, v_5 is be the isolated vertex.

*) Two vertices which are joined by an edge are said to be adjacent or neighbours.



V_1 & V_4 are adjacent
V_2 & V_3 " "
V_2 & V_5 " "

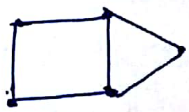
Here,

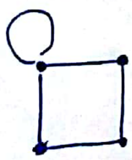
V_1 & V_2 are not adjacent
V_2 & V_4 are not adjacent
...

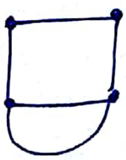
<u>Simple graph</u>

*) A graph G is said to be simple if it does not contain any loop or parallel edges, or in otherwords, any non directed graph without self loop and parallel edges is said to be a simple graph.

<u>Ex</u>.

G_1 :  Simple

G_2 :  Not simple
self loop exists.

G_3 :  Not simple
parallel edges.

Complete graph

A complete graph is a simple graph in which each pair of distinct vertices are to be joined by an edge

*) If order of a complete graph $= n$
then size " " $= \frac{n(n-1)}{2}$

*) The complete graph with order 'n' is to be denoted by K_n.

		order	size
K_1 :		1	$0 = \frac{1(1-1)}{2}$
K_2 :		2	$1 = \frac{2(2-1)}{2}$
K_3 :		3	$3 = \frac{3(3-1)}{2}$
K_4 :		4	$6 = \frac{4(4-1)}{2}$
$\vdots$			
K_{10}		10	$45 = \frac{10(10-1)}{2}$

Empty graph

*) If there exists any graph without any edge, then the graph is said to be empty graph or trivial graph.
that means, all the vertices are isolated.

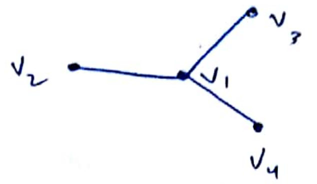
Ex

v_1, v_2, v_3 are isolated.

Bipartite graph

Let, $G = (V,E)$ be a graph.
This graph is said to be bipartite, if the vertex set is partitioned into two disjoint sets X and Y $\{$ie, $X \cup Y = V$, $X \cap Y = \phi \}$
in such a way that, one end of any edge lies in set X and other end of that edge lies in set Y.

Ex.



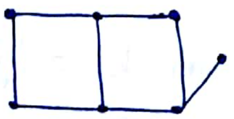
$V = \{v_1, v_2, v_3, v_4\}$

$X = \{v_1\}, \quad Y = \{v_2, v_3, v_4\}$

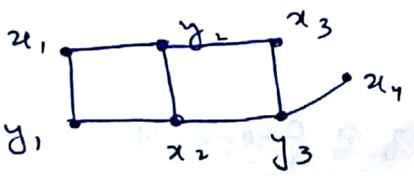
This is a bipartite graph.

Ex. Is the following graph bipartite?

①



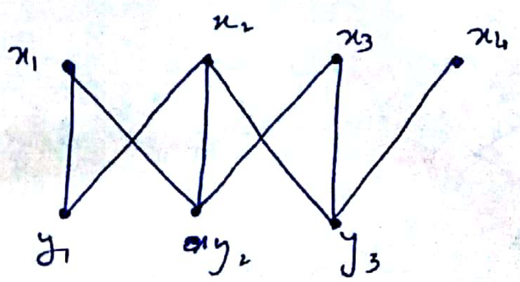
→ Naming of the vertices :



→ Let, $X = \{x_1, x_2, x_3, x_4\}$

$Y = \{y_1, y_2, y_3\}$ ~~so, it is bipartite~~

→



~~so $X = \{x_1, x_2, x_3, x_4\}$~~

~~$Y = \{$~~

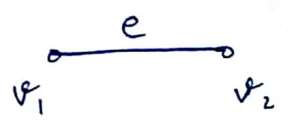
It is a bipartite set.

Degree of vertex

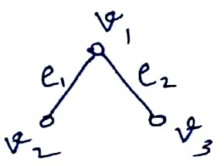
Let, $G = (V, E)$ be a graph

e be an edge of G

→ That edge is said to incendent with the vertex V if V is an end vertex of the edge e.



→ Two edges, e_1 & e_2 which are incident with a common vertex V are said to be adjacent.



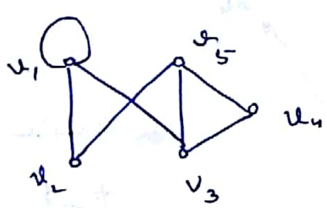
e_1 & e_2 are adjacent.

→ Let v be any vertex of a graph G.

It's degree is to be denoted by, $deg(v)$ and is defined as the no. of edges incident with that vertex v.

Ex.

G:



deg(self loop) = 2

$deg(v_1) = 4$

$deg(v_2) = 2$

$deg(v_3) = 3$

$deg(v_4) = 2$

$deg(v_5) = 3$

Order = 5

Size = 7.

sum of degree = 2 * size of graph.

Theorem 1 : Handshaking theorem

Let $G = (V, E)$ be a graph with n vertices such as $v_1, v_2, v_3, \ldots, v_n$,

~~tot~~ then $\boxed{\sum_{i=1}^{n} deg(v_i) = 2 * |E|}$

Proof :

Since, each edge has two end vertices which contributes ~~pble~~ precisely 2 to the sum of the degree of the vertices., ie, when the degrees of the vertices are to ~~be~~ be added, each edge is to be counted two times.

→ If the degree of a vertex is even, it is said to be an even vertex.

→ If the degree of a vertex is odd, it is said to be an odd vertex.

→ The no. of ~~edge~~ odd vertex should be even.

Theorem 2 :

In any graph G (nondirected), there is an even no. of odd vertices.

Given graph is $G = (V, E)$

Let us partition the vertex set V into two disjoint sets V_1 and V_2, ie, $V_1 \cup V_2 = V$ & $V_1 \cap V_2 = \phi$

where, V_1 be the set of odd vertices

V_2 " " " even vertices.

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By principal of inclusion exclusion,

$$|V| = |V_1| + |V_2|$$

By handshaking theorem,

$$\sum_{i=1}^{n} \deg(V_i) = 2|E|$$

$$\Rightarrow \sum_{V_i \in V_1} \deg(V_1) + \sum_{V_i \in V_2} \deg(V_2) = \text{even}$$

$$\Rightarrow \sum_{V_i \in V_1} \deg(V_1) + \text{even} = \text{even}$$

$$\Rightarrow \sum_{V_i \in V_1} \deg(V_1) = \text{even}$$

Since the terms over summation in left hand side are odd nos.

as we know, if odd nos. will be added for even times, then the result will be even.

Thus, there are <u>even no. of odd vertices</u>.

Degree Sequence

If $v_1, v_2, v_3 \dots v_n$ are n no. of vertices of a graph G with degrees $d_1, d_2, d_3 \dots d_n$.

If we will write those degrees either in increasing order or decreasing order, then ~~it will be said~~ $\langle d_1, d_2, d_3 \dots d_n \rangle$ will be said an as <u>degree sequence</u>.

$$d_1 \le d_2 \le d_3 \le d_4 \le \dots \le d_n$$

or,

$$d_1 \ge d_2 \ge d_3 \ge \dots \ge d_n$$

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Graphic or Graphical

Let, $d = \langle d_1, d_2, d_3 \ldots d_n \rangle$ be a given degree sequence. It is said to be graphic or graphical if there exists a simple non directed graph with that given degree sequence.

Ex. Is the following degree sequence graphic?

$$2, 2, 3, 3, 5, 8, 8$$

→ It is not graphic. As, here we have $3 = $ (odd) no. of od vertices.

Havel- Hakimi Theorem

~~Let~~ Let, us write the given degree sequence in decending order.

The degree sequence

$$S, d_1, d_2, d_3 \ldots d_S, t_1, t_2 \ldots t_n$$

is graphic, iff

$$d_1 - 1, d_2 - 1, d_3 - 1 \ldots d_S - 1, t_1, t_2, t_n$$

is graphic.

Ex. ① ⑥ 6, 6, 6, 4, 3, 3, 0

$\equiv$ ⑤, 5, 5, 3, 2, 2, 0

$\equiv$ ④, 4, 2, 1, 1, 0

$\equiv$ ③, 1, 0, 0, 0

$\equiv$ 0, -1, -1, 0

It is not graphical as any vertex cannot have degree negative.

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② 6, 5, 5, 4, 3, 3, 2, 2, 2

≡ 4, 4, 3, 2, 2, 1, 2, 2

≡ 4, 4, 3, 2, 2, 2, 2, 1

≡ 3, 2, 1, 1, 2, 2, 1

~~≡ 1, 1, 1, 2, 2, 2, 3~~ ≡ 3, 2, 2, 2, 1, 1, 1

~~≡ 0, 1, 2, 2, 2, 3~~ ≡ 1, 1, 1, 1, 1, 1

It is graphic.

This is a regular graph.

Regular graph

If the degree of each vertex is same then the graph is regular.

Matrix representation on nondirected graph

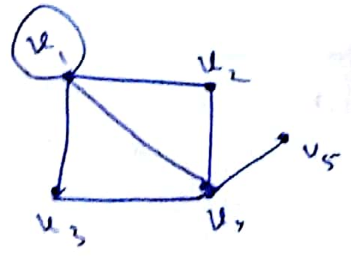
i] Adjacency matrix

*) Let $G = (V, E)$ be a nondirected graph, with n-vertices $v_1, v_2, v_3, v_4 \dots v_n$.

Then its adjacency matrix is denoted by $A(G)$, where
- $A(G)$ be a symmetric $n \times n$ matrix.
- $A(G)$ be the correspondence ~~to~~ from vertex to vertex.
- $A(G) = (a_{ij})_{n \times m}$

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise.} \end{cases}$$

Ex. G :



$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

12 Incidence matrix

Let $G = (V, E)$ be a non directed graph with the n-vertices $v_1, v_2, v_3 \ldots v_n$

and m edges, $e_1, e_2, e_3 \ldots e_m$

Then its incidence matrix is to be denoted by $I(G)$, where,

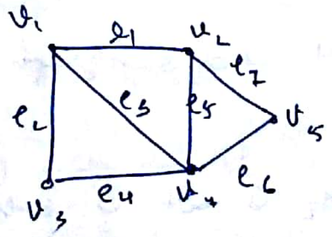
— $I(G)$ be a matrix of order $n \times m$

— $I(G)$ be the matrix with correspondence from vertex to edge

— $I(G) = (a_{ij})_{n \times m}$

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the end vertex of } e_j \\ 0, & \text{otherwise.} \end{cases}$$

Ex. G :



$$I(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

*) In some cases, if there are more than one edges, put 2, 3 ... in place of 1. (Generally not used)

Subgraph

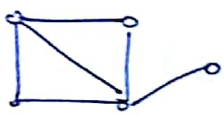
- Let G be a $G = (V, E)$ be a graph with the vertex set $V(G)$ and edge set $E(G)$.
- Let, H be another graph with the vertex set $V(H)$ & edge set $E(H)$.
- If $V(H) \subseteq V(G)$ & $E(H) \subset E(G)$ then we will say

  H is the subgraph of the graph G

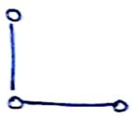
  &

  G is the supergraph of graph H.

Ex. G :

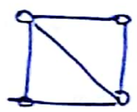


H_1 :



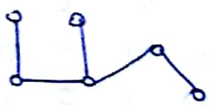
H_1 is a subgraph of G.

H_2 :



H_2 is not a subgraph of G as $E(H_2) \not\subset E(G)$

H_3 :

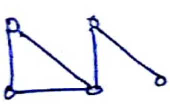


H_3 is not a subgraph of G as, $V(H_3) \not\subset V(G)$

*) The subgraph H of the graph G is said to be spanning subgraph iff,

$$V(H) = V(G)$$

G :



H :

Here H is the spanning subgraph of G.

Walk

A walk in a graph G is a finite sequence of vertices and edges alternatively.

$$w = v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$$

- It is the $v_0 - v_k$ walk.
  where, the vertex v_0 is the origin and v_k is the terminus of the walk.
- It need not be required that v_0 ~~and~~ and v_k are distinct.
- The no. of edges contained in the walk is the length of the walk.
- A walk without any edge is said to be a trivial walk.
- If the ~~exact vertex~~ origin & Terminus vertex of a walk are same, then the walk is closed, otherwise, the walk is open.
- If the edges contained in the walk are distinct, then the walk is said to be a trail.
- If the vertices contained in the walk are distinct, then the walk is said to be a path
- Two vertices u & v in a graph G are connected, if there exists a path from vertex u to vertex v.
- A non trivial closed trail in a graph is said to be a cycle if its origin and internal vertices are distinct.
- If a graph contains no cycle, then it is said to be an acyclic graph.
- Since a loop is a cycle of length 1 and a pair of parallel edges produces a cycle of length 2, so, any acyclic graph

must be simple.

Tree

Any connected acyclic graph is said to be a tree.

Spanning tree

A spanning tree of a graph G is a spanning subgraph of the graph G which is also a tree.

Weighted graph

If all the edges of a nondirected graph are assigned by some positive values, then that is said to be weighted graph.

Ex