the rules for each step in the argument were given. However, formal proofs of useful theorems can be extremely long and hard to follow. In practice, the proofs of theorems designed for human consumption are almost always **informal proofs**, where more than one rule of inference may be used in each step, where steps may be skipped, where the axioms being assumed and the rules of inference used are not explicitly stated. Informal proofs can often explain to humans why theorems are true, while computers are perfectly happy producing formal proofs using automated reasoning systems. When you read proofs, you will often find the words “obviously” or “clearly.” These words indicate that steps have been omitted, however we will try to avoid using these words and try not to omit too many steps. There are many proof techniques that can be used to prove a wide variety of theorems. For example: direct proof, proof by contraposition, vacuous and trivial proofs, proofs by contradiction, proofs of equivalence, counterexamples, mathematical induction. A major goal of this section is to provide a thorough understanding of mathematical induction. In this section, we will describe how mathematical induction can be used and why it is a valid proof technique. It is extremely important to note that mathematical induction can be used only to prove results obtained in some other way. It is not a tool for discovering formulae or theorems.

Many mathematical statements assert that a property is true for all positive integers. Proofs using mathematical induction have two parts. First, we show that the statement holds for the positive integer 1. Second, we show that if the statement holds for a positive integer then it must also hold for the next larger integer. Mathematical induction is based on the rule of inference that tells us that if $P(1)$ and $\forall k\,(P(k) \rightarrow P(k+1))$ are true for the domain of positive integers, then $\forall n\, P(n)$ is true. Mathematical induction can be used to prove a tremendous variety of results. Understanding how to read and construct proofs by mathematical induction is a key goal of learning discrete mathematics.

In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. A proof by mathematical induction has two parts, a **basis step**, where we show that $P(1)$ is true, and an **inductive step**, where we show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.

To complete the inductive step of a proof using the principle of mathematical induction, we assume that $P(k)$ is true for an arbitrary positive integer k and show that under this assumption, $P(k + 1)$ must also be true. The assumption that $P(k)$ is true is called the **inductive hypothesis**. Once we complete both steps in a proof by mathematical induction, we have shown that $P(n)$ is true for all positive integers, that is, we have shown that $\forall n\, P(n)$ is true where the quantification is over the set of positive integers. In the inductive step, we show that $\forall k\, (P(k) \rightarrow P(k + 1))$ is true, where again, the domain is the set of positive integers. Expressed as a rule of inference, this proof technique can be stated as

$$\big(P(1) \wedge \forall k\, (P(k) \rightarrow P(k + 1))\big) \rightarrow \forall n\, P(n),$$

when the domain is the set of positive integers. Because mathematical induction is such an important technique, it is worthwhile to explain in detail the steps of a proof using this technique. The first thing we do to prove that $P(n)$ is true for all positive integers n is to show that $P(1)$ is true. This amounts to showing that the particular statement obtained when n is replaced by 1 in $P(n)$ is true. Then we must show that $P(k) \rightarrow P(k + 1)$ is true for every positive integer k. To prove that this conditional statement is true for every positive integer k, we need to show that $P(k + 1)$ cannot be false when $P(k)$ is true. This can be accomplished by assuming that $P(k)$ is true and showing that under this hypothesis $P(k + 1)$ must also be true.

Example: Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}.$$

Solution: Let $P(n)$ be the proposition that the sum of the first n positive integers, is $\frac{n(n+1)}{2}$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k + 1)$ is true for $k = 1, 2, 3, \ldots$.

Basis step: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$.

(The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $\frac{n(n+1)}{2}$.)

Inductive step: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k + 1)}{2}.$$

Under this assumption, it must be shown that $P(k + 1)$ is true, namely, that

$$1 + 2 + \cdots + k + (k + 1) = \frac{(k + 1)[(k + 1) + 1]}{2} = \frac{(k + 1)(k + 2)}{2}$$

is also true. When we add $k + 1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1 + 2 + \cdots + k + (k + 1) &= \frac{k(k + 1)}{2} + (k + 1) \\ &= \frac{k(k + 1) + 2(k + 1)}{2} \\ &= \frac{(k + 1)(k + 2)}{2}. \end{aligned}$$

This last equation shows that $P(k + 1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}$$

for all positive integers n.

Example: Conjecture a formula for the sum of the first *n* positive odd integers. Then prove your conjecture using mathematical induction.

Example: Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$

Example: Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever *n* is a positive integer.

Example: Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer *n*.

Example: If $a_n = 2^n + 3^n$ show that $a_n = 5a_{n-1} - 6a_{n-2}$, for $n \geq 2$ ($a_0 = 1$ and $a_1 = 5$).

Example: The **harmonic numbers** $H_j, j = 1, 2, 3, \ldots$, are

defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{j}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2}$$

whenever n is a nonnegative integer.

Strong Induction: There is another form of mathematical induction, called strong induction, which can often be used when we cannot easily prove a result using mathematical induction. The basis step of a proof by strong induction is the same as a proof of the same result using mathematical induction. That is, in a strong induction proof that $P(n)$ is true for all positive integers n, the basis step shows that $P(1)$ is true. However, the inductive steps in these two proof methods are different. In a proof by mathematical induction, the inductive step shows that if the inductive hypothesis $P(k)$ is true, then $P(k + 1)$ is also true. In a proof by strong induction, the inductive step shows that if $P(j)$ is true for all positive integers j not exceeding k, then $P(k + 1)$ is true. That is, for the inductive hypothesis we assume that $P(j)$ is true for $j = 1, 2, \ldots, k$. Thus **t**o prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis step: We verify that the proposition $P(1)$ is true.

Inductive step: We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k + 1)$ is true for all positive integers k.

Example: Given that $d_1 = 1, d_2 = 2, d_3 = 3$, $d_{n+3} = d_{n+2} + d_{n+1} + d_n$ for all positive integer n. Show by Strong Induction method that $d_n < 2^n$.

Solution: Given $d_1 = 1, d_2 = 2, d_3 = 3$, and for all positive integer n,

$$d_{n+3} = d_{n+2} + d_{n+1} + d_n.$$

To prove that, $d_n < 2^n$, for all $n \in \mathbb{N}$ by method of strong induction.

Step 1. To prove $d_n < 2^n$ for all $n \in \mathbb{N}$, let $P(n)$: $d_n < 2^n$.

Step 2. (Basis Step) Given that $d_1 = 1, d_2 = 2, d_3 = 3$.

Also $d_1 = 1 < 2^1, d_2 = 2 < 2^2, d_3 = 3 < 2^3$.

Therefore $P(1)$, $P(2)$ and $P(3)$ are true.

Step 3. (Inductive Step) Assume that $P(1), P(2), \cdots, P(k)$ are true for some $k \in \mathbb{N}$. We have to show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true. i.e $P(k+1)$ is true.

Now, $P(k+1): d_{k+1} < 2^{k+1}$.

Thus, we have to show $d_{k+1} < 2^{k+1}$.

As $P(k)$, $P(k-1)$ and $P(k-2)$ are true,

$$d_k < 2^k, d_{k-1} < 2^{k-1}, d_{k-2} < 2^{k-2}.$$

Given that

$$d_{n+3} = d_{n+2} + d_{n+1} + d_n.$$

So

$$\begin{aligned} d_{k+1} &= d_k + d_{k-1} + d_{k-2} \\ &< 2^k + 2^{k-1} + 2^{k-2} \\ &= 2^k\left(1 + \frac{1}{2} + \frac{1}{4}\right) \\ &< 2^k \times 2 \\ &= 2^{k+1} \end{aligned}$$

i.e. $d_{k+1} < 2^{k+1}$.

i.e. $P(k+1)$ is true.

Thus, the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true.

Step 4. (Conclusion) As we have already shown $P(0)$, $P(1)$ and $P(2)$ are true, and the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true, by method of strong induction $P(n)$ is true for all $n \in \mathbb{N}$. That is $d_n < 2^n$ for all $n \in \mathbb{N}$.

Example: Given a_0=2,$a_1 = 7$, and for all $n \geq 2$,$a_n = 5a_{n-1} - 6a_{n-2}$. Prove that, $a_n = 3^{n+1} - 2^n$ for all $n \in \mathbb{N}$ by method of strong induction.

Solution: Given a_0=2,$a_1 = 7$, and for all $n \geq 2$,

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

To prove that, $a_n = 3^{n+1} - 2^n$, for all $n \in \mathbb{N}$ by method of strong induction.

Step 1. *To prove* $a_n = 3^{n+1} - 2^n$ for all $n \in \mathbb{N}$, let $P(n)$: $a_n = 3^{n+1} - 2^n$.

Step 2.(Basis Step) Given that a_0=2,$a_1 = 7$. Also $a_0 = 3^{0+1} - 2^0 = 2$ and $a_1 = 3^{1+1} - 2^1 = 7$.

Now for all $n \geq 2$, $a_n = 5a_{n-1} - 6a_{n-2}$.

So, $a_2 = 5a_1 - 6a_0 = 5 \times 7 - 6 \times 2 = 23$. Also $a_2 = 3^{2+1} - 2^2 = 23$.

Therefore $P(0)$, $P(1)$ and $P(2)$ are true.

Step 3. (Inductive Step) Assume that $P(1), P(2), \cdots, P(k)$ are true for some $k \in \mathbb{N}$. We have to show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true. i.e $P(k+1)$ is true.

Now, $P(k+1)$: $a_{k+1} = 3^{k+1+1} - 2^{k+1} = 3^{k+2} - 2^{k+1}$.

Thus, we have to show $a_{k+1} = 3^{k+2} - 2^{k+1}$.

Given that

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

Therefore, $a_{k+1} = 5a_k - 6a_{k-1}$.

As $P(k)$ and $P(k-1)$ are true, $a_k = 3^{k+1} - 2^k$ and $a_{k-1} = 3^k - 2^{k-1}$.

Thus,

$$\begin{aligned}
a_{k+1} &= 5a_k - 6a_{k-1} \\
&= 5(3^{k+1} - 2^k) - 6(3^k - 2^{k-1}) \\
&= 15 \times 3^k - 10 \times 2^{k-1} - 6 \times 3^k + 6 \times 2^{k-1} \\
&= (15-6)3^k - (10-6)2^{k-1} \\
&= 9 \times 3^k - 4 \times 2^{k-1} \\
&= 3^{k+2} - 2^{k+1}
\end{aligned}$$

i.e.

$$a_{k+1} = 3^{k+2} - 2^{k+1}.$$

i.e $P(k+1)$ is true.

Thus, the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true.

Step 4. (Conclusion) As we have already shown $P(0)$, $P(1)$ and $P(2)$ are true, and the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true, by method of strong induction $P(n)$ is true for all $n \in \mathbb{N}$. That is $a_n = 3^{n+1} - 2^n$ for all $n \in \mathbb{N}$.

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