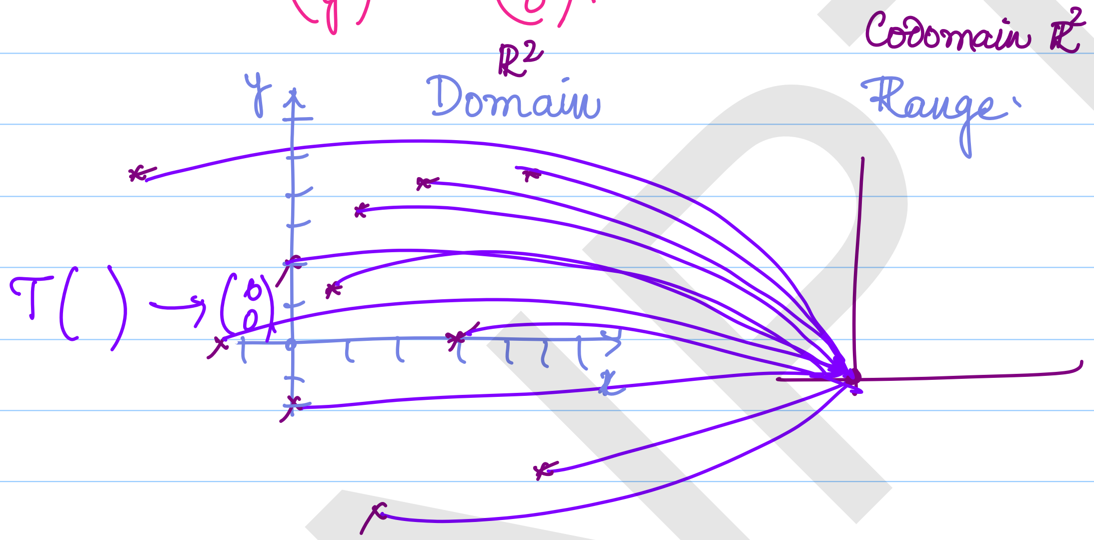
Obtaining the Standard matrix with a linear transformation T.

Ex 1: $T : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \qquad e_1 \ \& \ e_2$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= T\left[x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = xT\begin{pmatrix} 1 \\ 0 \end{pmatrix} + yT\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$M = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Ex2: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$;

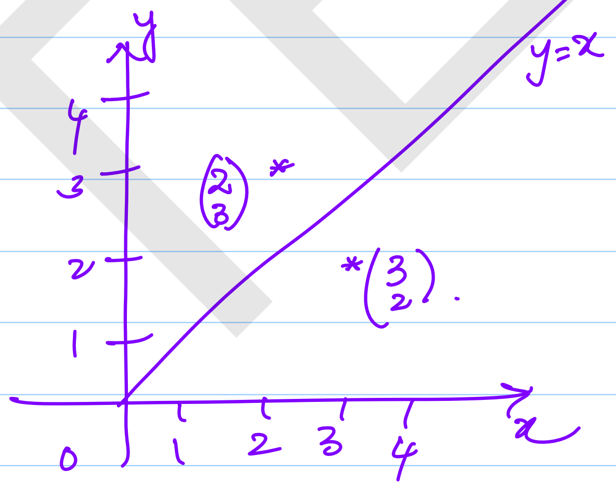
$T\begin{pmatrix}x\\y\end{pmatrix} \longrightarrow \begin{pmatrix}y\\x\end{pmatrix} \Rightarrow$ Reflection about $y=x$

$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ & $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$\therefore \quad M = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

For ex in obtaining RREF of a matrix we may want to swap rows

$M : \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \Rightarrow R_2 \longleftrightarrow R_1$



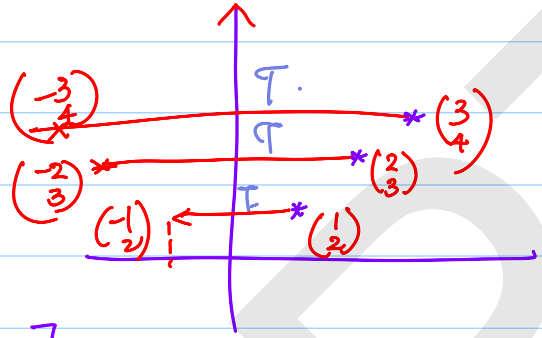
Re

$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Ex: 3 $\quad T: \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} -x \\ y \end{pmatrix}$. $\Rightarrow$ Reflection about y''.

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

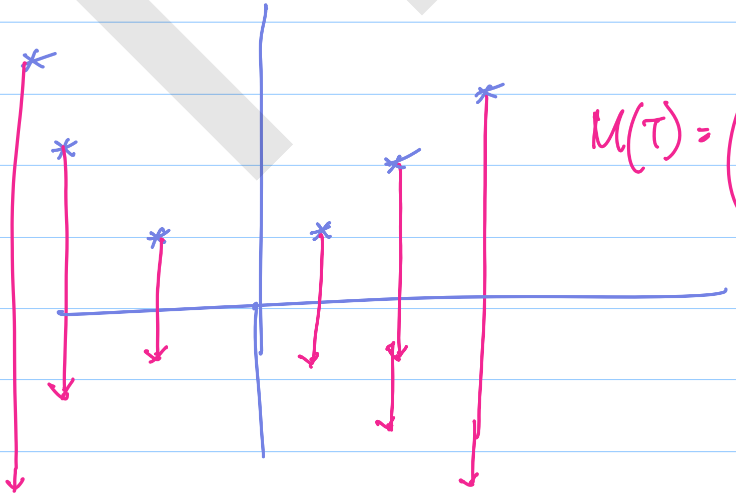
$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$\therefore \quad M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Ex: 4 $\quad T: \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ -y \end{pmatrix}$ $\quad$ Reflection about x''

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



$M(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$.

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ x+y \\ y \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+0=1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Ex: Let $V = \mathbb{R}^2$ & $W = \mathbb{R}^3$.

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ a_{31}x + a_{32}y \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11}(1) + a_{12}(0) \\ a_{21}(1) + a_{22}(0) \\ a_{31}(1) + a_{32}(0) \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a_{11}(0)+a_{21}(1)\\ a_{21}(0)+a_{22}(1)\\ a_{31}(0)+a_{32}(1)\end{pmatrix} = \begin{pmatrix}a_{21}\\ a_{22}\\ a_{23}\end{pmatrix}$$

$$M(T) = \begin{pmatrix}T(e_1) & & T(e_2)\\ & & \end{pmatrix}$$

$$= \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{pmatrix}$$

Does T map entire $\mathbb{R}^2$ to entire $\mathbb{R}^3$?

No

Ex: $V = \mathbb{R}^3 \longrightarrow W = \mathbb{R}^2$.

$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

$$T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1+x_2\\ x_3\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\\0\end{pmatrix},\ T\begin{pmatrix}0\\1\\0\end{pmatrix} \ \& \ T\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_3 \end{pmatrix}$$

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 \end{bmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Does T map entire $\mathbb{R}^3$ to entire $\mathbb{R}^2$? In this example

Yes.

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Recall:

There are 2 vector Subspaces associated with any linear transformation T namely

i) Kernel (T) = $\mathrm{Ker}(T)$

If $T: V \to W$

$$\{\underline{x} \in V : T(x) = 0 \in W\}.$$

Kernel of T is a subspace of V.

If $M(T)$ or M is the matrix associated with T, then the kernel of T corresponds to the soln to

$$Mx = 0 \qquad \text{where } M^{m\times n}$$

Kernel (T) = Soln to the Homogeneous Sys. of eqns $Mx = 0$.

$$\{x \in \mathbb{R}^n ; Mx = 0^{m\times 1}\}$$

$V = \mathbb{R}^n \qquad W = \mathbb{R}^m.$

2) Range of T:

Set of vectors in W that are obtained by transforming the vectors in V by the linear transformation T.

$$\{ w \in W : T(v) = w, \text{ where } v \in V \}.$$

Suppose M is the matrix representatⁿ of T, then the range of T is given by the set of vectors

$$\{ b \in \mathbb{R}^m : b = A\underline{\underline{x}}, \text{ for every } x \in \mathbb{R}^n \}$$

Range of T: Column Space of the matrix M.

Suppose U, V & W are three vector spaces and T_1 and T_2 be linear transformations such that

$$T_1 : U \to V$$
$$T_2 : V \to W$$

then

$T = T_2 \circ T_1$ then T is a linear transformation from U to W.

Let $u, v \in \underline{U}$.

$$T(u+v) = T_2(T_1(\underline{u}+\underline{v}))$$
$$= T_2(T_1(u) + T_1(v))$$

$$T_1(u) = v_1 \in V \quad \& \quad T_1(v) = v_2 \in V$$

$$= T_2(T_1(u) + T_1(v))$$
$$= T_2(v_1 + v_2)$$
$$= T_2(v_1) + T_2(v_2)$$
$$T(u+v) = T_2(T_1(u+v)) = T_2(T_1(u)) + T_2(T_1(v))$$

$$\underline{T(cu)} = T_2(T_1(cu)) = T_2(c\, T_1(u))$$

$$= \underline{c\, T_2(T_1(u))}$$

$\therefore$ $T = T_2 \circ T_1$ is a linear transformation.

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2(1)+0 \\ 1+2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2(0)+1 \\ 0+2(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

Under the transformation T which is represented by the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

let us look at the effect of T.

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$T\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$T\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

Given this T which maps every vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} 2x+y \\ x+2y \end{pmatrix}$, do we have vectors in $\mathbb{R}^2$ s.t

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} = k\begin{pmatrix} x \\ y \end{pmatrix}$$

Consider $f(x) = e^{ax}$.

$$\frac{df}{dx} = a e^{ax} = \frac{d(e^{ax})}{dx} =$$

$$\frac{d^2f}{dx^2} = a^2 e^{ax} = \frac{d}{dx}\left(ae^{ax}\right)$$