LEC: 4 WEEK 1

Let $u_1, u_2 \dots u_k$ be k n-component vectors and $\alpha_1, \alpha_2 \dots \alpha_k$ be scalars.

Look at the l.c that results in 0_n.

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k = 0_n$$

If the above l.c. is that the only way to get the 0_n is by making the scalars 0, then we say that $u_1 \dots u_k$ are linearly indep vectors.

For example

$$u_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$\alpha_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \alpha_1 = 0$ & $\alpha_2 = 0$.

u_1 & u_2 are l.i.

Some observations:

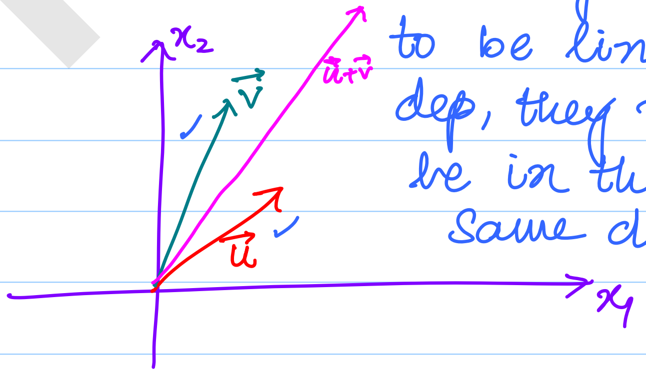
1) A linearly indep set cannot contain the $\underset{\sim}{0}$ vector.

2) A single vector is always linearly indep unless it is the zero vector.

3) Any subset of a linearly indep set of vectors is always linearly independent.

4) Any superset of a linearly dependent set of vectors is linearly dependent.

5) Two vectors are linearly indep if one is not a multiple of the other. $\Rightarrow$ For any 2 vectors to be linearly dep, they must be in the same direction



Span of a Set of vectors.

Let $u_1, u_2 \dots u_k$ be k vectors

Span := Set of all possible l.c. of $u_1, u_2 \dots u_k$.

Span is a vector space

Span of a Set of linearly indep. vectors.

Let $v_1, v_2 \dots v_n$ be l.i. Set of vectors.

Span : $\{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$
$\alpha_i \in \mathbb{R}.\}$
for $i = 1 \dots n$.

$\Downarrow$

Smallest Subspace that contains the set of linearly indep vectors.

For ex: $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ & $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$\left\{\alpha_1 v_1 + \alpha_2 v_2 \,,\ \alpha_1, \alpha_2 \in \mathbb{R}\right\}$$

$$\Rightarrow \left\{\underbrace{\alpha_1 \begin{pmatrix}2\\1\end{pmatrix} + \alpha_2}\begin{pmatrix}1\\2\end{pmatrix} \,,\ \alpha_1, \alpha_2 \in \mathbb{R}\right\}$$

(i) $\begin{pmatrix}0\\0\end{pmatrix} \in \operatorname{Span}\left(\begin{pmatrix}2\\1\end{pmatrix} \times \begin{pmatrix}1\\2\end{pmatrix}\right)$

(ii) $\operatorname{Span}\left(\begin{pmatrix}2\\1\end{pmatrix} \times \begin{pmatrix}1\\2\end{pmatrix}\right)$ is closed under VA.

(iii) $\operatorname{Span}\left(\begin{pmatrix}2\\1\end{pmatrix} \times \begin{pmatrix}1\\2\end{pmatrix}\right)$ is closed under SM.

$\operatorname{Span}\left(\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right)$ is a vector space.

BASIS:

A set of n linearly indep n-component vectors is called a basis for the vector sub space that contains these n-linearly indep n-comp. vectors.

Basis: Sampling set for a vector space.

Suppose $u_1, u_2 \dots u_k$ are linearly indep vectors, and let

$$x = \alpha_1 u_1 + \dots + \alpha_k u_k \longrightarrow (1)$$

Let x also have another representation in terms of $u_1, u_2 \dots u_k$

$$x = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_k u_k \text{ --- } (2)$$

$(1) - (2)$

$$\vec{x} - \vec{x} = \vec{0} = (\alpha_1 - \beta_1)u_1 + (\alpha_2 - \beta_2)u_2) + \dots$$
$$+ (\alpha_k - \beta_k)u_k$$

$$\vec{0} = (\alpha_1 - \beta_1)u_1 + (\alpha_2 - \beta_2)u_2 + \dots + (\alpha_k - \beta_k)u_k$$

Since $u_1, u_2 \dots u_k$ are l.i. $\vec{0}$ has only one rep, where the scalars $\alpha_1 - \beta_1, \alpha_2 - \beta_2 \dots \alpha_k - \beta_k$ are all 0

$\Rightarrow \alpha_1 = \beta_1, \quad \alpha_2 = \beta_2 \quad \ldots \quad \alpha_k = \beta_k.$

$\Rightarrow$ Any vector, when expressed as a l.c. of a l.i. Set of vectors has A UNIQUE SET OF SCALARS.

$\Rightarrow$ Any vector in a vector space has a unique representation in terms of the basis vectors.

For exp: $V = \mathbb{R}^2$,

$B_1 = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$v = \underline{\underline{\begin{pmatrix} 3 \\ 3 \end{pmatrix}}} = \underline{\underline{\alpha_1}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \underline{\underline{\alpha_2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow \underline{1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \underline{1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Consider the vector space $V = \mathbb{R}^n$ for any n.

There are infinitely many bases for $\mathbb{R}^n$. However all the basis will have exactly 'n' vectors.

The number of elements/vectors in any given basis for a vector space IS CALLED THE DIMENSION OF THE VECTOR SPACE.