Linear Transformations.

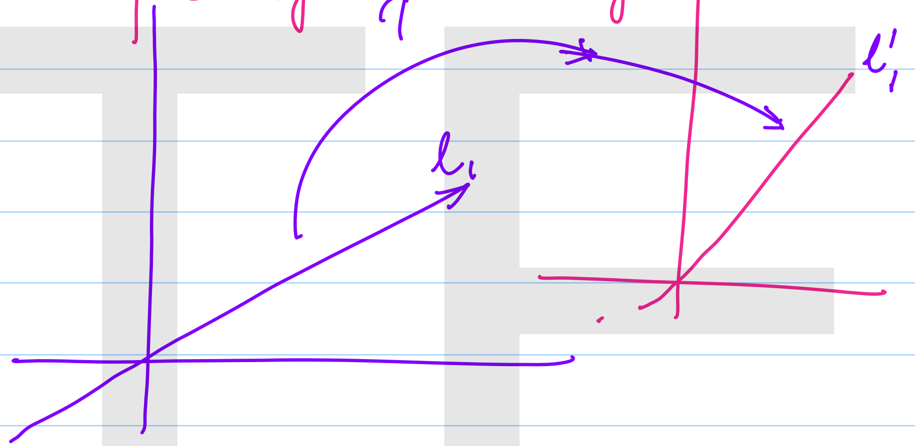
A linear transformation or a linear map T from a vector space V to a vector space W is a map that satisfies the following properties:

(i) $T(0) = 0$

(ii) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for $\vec{u}, \vec{v} \in V$.

(iii) $T(\alpha \vec{u}) = \alpha T(\vec{u})$ for $u \in V$

A linear map T always sends/maps lines passing thro' the origin to lines passing thro' the origin (or) onto the origin itself.



Ex: Let T be a linear transformation represented by the matrix A, where

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Let $x \in \mathbb{R}^2$,

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 2x+y \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

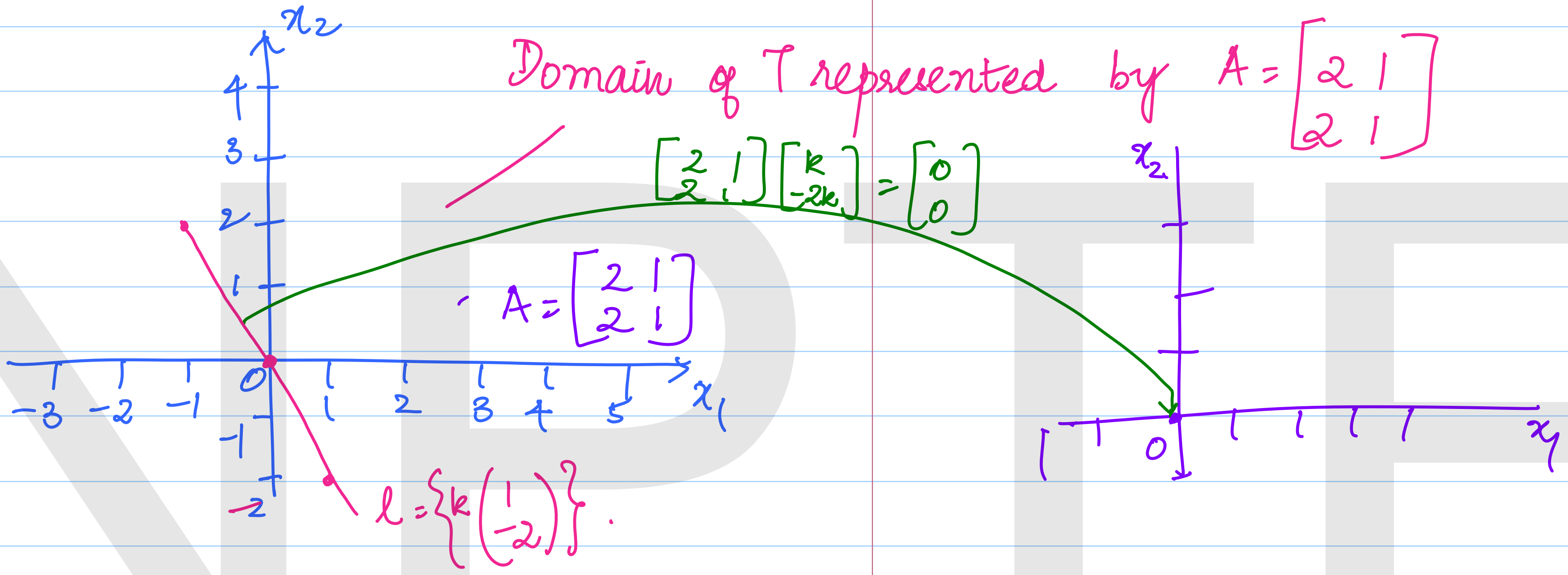
$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+-2 \\ 2+-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2+2 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} k \\ -2k \end{bmatrix} = \begin{bmatrix} 2k-2k \\ 2k-2k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

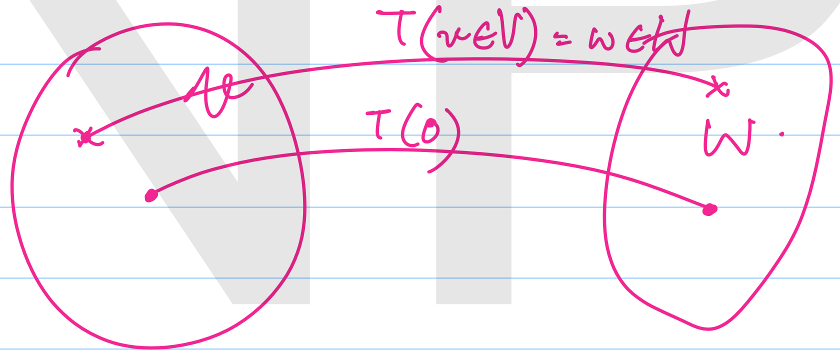
$$\left\{ \begin{pmatrix} k \\ -2k \end{pmatrix} \right\} = \left\{ k \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \Rightarrow$$

Domain of T represented by $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$



Given a transformation, how do we get the matrix representation of the transformation?

Let $T : V \to W$.



Any linear transformⁿ $T : V \to W$ is completely determined by what it does to the basis vectors of V.

Any l.T $T : V \to W$ is completely determined by its action on a basis of V.

Let V and W be two vector Spaces Such that a l.t $T: V \to W$

Let $\{v_1, v_2 \ldots v_k\}$ be a basis for V. Let w_i for $i = 1 \ldots n$ be any set of vectors in W.

Then there is a unique linear map $T: V \to W$ s.t $T(v_i) = w_i$

Proof: Let v be a vector in V

$\therefore v$ has a basis expansion in V.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k \quad \alpha_i$'s Scalars

Define

$T: V \to W$

$T(v) = \sum \alpha_i w_i$.

$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

$v_i = 0v_1 + 0v_2 + \ldots + 1v_i + 0v_{i+1} \ldots + 0v_k$

$T(v_i) = 0T(v_1) + 0T(v_2) + \ldots + 1T(v_i) + 0T(v_{i+1}) + \ldots + 0T(v_k)$

$= \boxed{T(v_i) \to w_i}$

Claim: T: is a linear map.

Let $u, v \in V$

$$u = \sum_{i=1}^{k} \alpha_i v_i \qquad \text{Let } v = \sum_{i=1}^{k} \beta_i v_i .$$

$$u + v = \sum_{i=1}^{k} \alpha_i v_i + \sum_{i=1}^{k} \beta_i v_i$$

$$\Rightarrow \sum_{i=1}^{k} (\alpha_i + \beta_i) v_i$$

$$T(u+v) = T\left(\sum_{i=1}^{k} (\alpha_i + \beta_i) v_i\right) = \sum_{i=1}^{k} (\alpha_i + \beta_i) w_i$$

$$= \sum_{i=1}^{k} \alpha_i w_i + \sum_{i=1}^{k} \beta_i w_i$$

$$= T(u) + T(v).$$

Let c be a real scalar. Then

$$cv = c \sum_{i=1}^{k} \alpha_i v_i$$

$$= \sum_{i=1}^{k} (c\alpha_i) v_i$$

$$T(cv) = \sum_{i=1}^{k} (c\alpha_i) w_i = c \underbrace{\sum_{i=1}^{k} \alpha_i w_i}_{} = c\,T(v)$$

$T(cv) = c\,T(v)$: T is a linear map.

Uniqueness of the linear map – Prove

Q: How do we get the matrix representation of any linear transformation?

Example: Let $V = \mathbb{R}^2$ and let W also be $\mathbb{R}^2$. Let T be a linear transformation s.t

$$T: \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

Obtain the matrix associated with this transformation.

$x \in \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = T\left[x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$

$$= xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$T:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x+y\\x-y\end{pmatrix}$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{pmatrix}1+0\\1-0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{pmatrix}0+1\\0-1\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$$

$$T\begin{pmatrix}x\\y\end{pmatrix}=x\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)+y\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$=x\begin{pmatrix}1\\1\end{pmatrix}+y\begin{pmatrix}1\\-1\end{pmatrix}$$

$$=\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$M(T)=\begin{bmatrix}1&1\\1&-1\end{bmatrix}$$

Where $T:\begin{pmatrix}x\\y\end{pmatrix}\longrightarrow\begin{pmatrix}x+y\\x-y\end{pmatrix}$