$T \to A$ is the matrix representation of the linear transformation T.

$Au = \lambda u$ for some real λ.

$(A - \lambda I)\vec{u} = \vec{0}$

To find λ, Solve $\det(A - \lambda I) = 0$.

$\Downarrow$

Characteristic eqn of A.

Suppose $A^{n\times n}$ matrix is such that the eigenvalues are

Case 1: distinct

Case 2: Repeated

Case 3: Complex.

$A^{n\times n}$: $\det(A - \lambda I) = 0$

$\Rightarrow$ Degree n-polynomial.

$\Rightarrow$ n roots of the polynomials

1. Are the n-roots distinct?
2. Are there repeated roots among the n-roots

③ Are the roots complex?

Ex: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$\Rightarrow \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) = 0$

$\Rightarrow \lambda_1 = 3 \;\&\; \lambda_2 = 1$

eigenvalues are distinct.

$\lambda_1 = 3 \Rightarrow ev_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow$

$\lambda_2 = 1 \Rightarrow ev_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \rightarrow$

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Let T be a linear transform^n from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let A be the matrix representation of T.

A: 2×2 real matrix.

Let λ_1 and λ_2 be the distinct eigenvalues of A & let $\vec{u}$ & $\vec{v}$ be their Corresponding eigenvectors.

(λ_1, u) & (λ_2, v) are (eV) pairs.

Claim: $\vec{u}$ & $\vec{v}$ are linearly indep.

$$\Rightarrow \alpha_1 u + \alpha_2 v = \vec{0} \xrightarrow{(1)} \Rightarrow \alpha_1 = 0, \alpha_2 = 0.$$

$$T(\alpha_1 u + \alpha_2 v) = T(\vec{0}) = \vec{0}$$

$$= \alpha_1 T(u) + \alpha_2 T(v) = \vec{0}$$

$$= \alpha_1 \lambda_1 u + \alpha_2 \lambda_2 v = \vec{0} \rightarrow (2)$$

Multiply eqn 1 by λ_1

$$(\lambda_1 \alpha_1 u + \lambda_1 \alpha_2 v) = \lambda_1 \vec{0} = \vec{0}$$

$$\lambda_1 \alpha_1 \cancel{u} + \lambda_1 \alpha_2 \vec{v} = \vec{0} \longrightarrow ③$$

Eqn② $\cancel{\lambda_1} \alpha_1 u + \lambda_2 \alpha_2 \vec{v} = 0$

③ − ②

$$\alpha_2 (\lambda_1 - \lambda_2) \vec{v} = \vec{0} \longrightarrow (*)$$

Note λ_1 & λ_2 are distinct $\Rightarrow \lambda_1 - \lambda_2 \neq 0$

$\vec{v}$: Non zero vector.

(*) Can become zero vector on RHS is by having $\alpha_2 = 0$.

$$\alpha_1 u + \alpha_2 v = \vec{0}$$

$$\Rightarrow \alpha_1 \vec{u} = \vec{0}$$

$$\therefore \alpha_1 = 0$$

$\alpha_1 = 0,\ \alpha_2 = 0 \Rightarrow \alpha_1 u + \alpha_2 v = \vec{0}$ is possible only when $\alpha_1 = \alpha_2 = 0$

$\Rightarrow$ u & v are linearly indep.

eigenvectors associated with distinct eigenvalues are linearly indep.

(2) eigenvalues are repeated.

$T: A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow (\lambda-1)^2 = 0$

$\Rightarrow \underline{\underline{\lambda = 1}}$ twice.

The eigen vector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Defn: Algebraic Multiplicity (AM)

The number of times an eigenval. is repeated is called the algebraic multiplicity

Defn: Geometric Multiplicity (GM)

The number of linearly indep eigenvectors associated with a particular eigenvalue.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\lambda_1 = 3$, $ev_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\lambda_2 = 1$ $ev_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

AM of $\lambda_1 = 3 = 1$, GM of $\lambda_1 = 3 = 1$
AM of $\lambda_2 = 1 = 1$ GM of $\lambda_2 = 1 = 1$

Ex2: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = 1$ repeated twice
$\hookrightarrow$ ev: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

AM of $\lambda_1 = 1$ is 2
GM of $\lambda_1 = 1$ is 1

Note: GM $\leq$ AM. for each λ.

If GM of a specific eigenvalue is < AM, we say that the corresponding eigenvalue is deficient.

Ex 3: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow$ Rot^n by 90° Counterclockwise

$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 + 1 = 0$
$\Rightarrow \lambda = \pm i \rightarrow$ Complex

Diagonalizn of a matrix A.