Linear Transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \boxed{A} \longrightarrow \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Consider $x \in \mathbb{R}^n$ & let

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$b \in \mathbb{R}^m$

$$\Rightarrow b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Let T be a transformation.

A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector in $\mathbb{R}^n$, a vector

$T(\vec{x})$ in $\mathbb{R}^m$

$x \in \mathbb{R}^n \longmapsto T(x) \in \mathbb{R}^m$

$\mathbb{R}^n$: Domain of T.

$\mathbb{R}^m$: Codomain of T.

$T(\vec{x})$, for $\vec{x} \in \mathbb{R}^n$, is a vector in $\mathbb{R}^m$ & we call $\vec{b} = T(\vec{x})$ as the image of $\vec{x}$ under T.

Set of all images $\{T(x) \mid x \in \mathbb{R}^n\}$ is called the Range of T.

$\Rightarrow$ $b = Ax$.

$\{b : b = Ax$ for $b \in \mathbb{R}^m$ & $x \in \mathbb{R}^n\}$.

$\rightarrow$ Range of A.

$Ax = b$.

$\Rightarrow$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m.$$

$$\Rightarrow \underbrace{x_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{\text{Linear Combination of cols of } A.} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

A is the matrix that represents T

Range of a transformation T, is the set of vectors that can be obtained as linear combinations of the column vectors of the matrix A.

$\Rightarrow$ Set of all possible vectors obtained as the linear combinations of cols. of A is called the COLUMN SPACE of the matrix A. Also called as the range of A.

Range of T is the Same as Col Sp of the matrix A that captures T is a subset of $\mathbb{R}^m$.

$T(x) \in \mathbb{R}^m$

$T(x) = \text{Col Sp}(A) = \text{Range of } T$

is a vector Subspace of $\mathbb{R}^m$.

$\mathbb{R}^n \qquad\qquad\qquad \mathbb{R}^m.$

$x \xrightarrow{T} T(x).$

$\{T(x)\}$

$\Downarrow$

Vector Subsp. of $\mathbb{R}^m$.

Let T be a transform^n from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$.

$$x \in \mathbb{R}^n \longrightarrow 0^m.$$

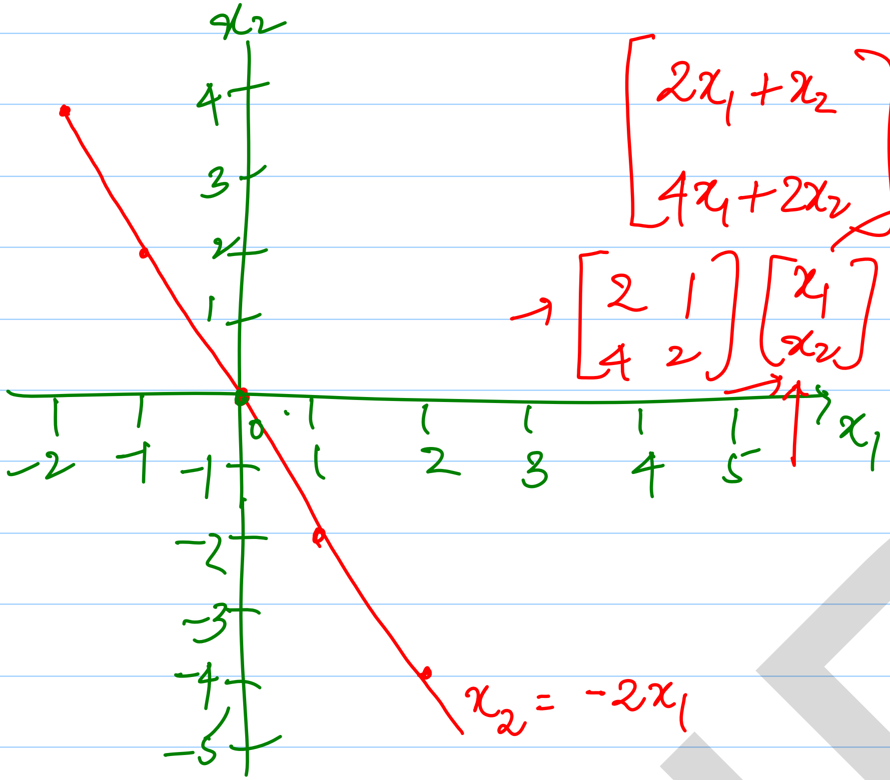
Ex: $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ A.

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 4x_1 + 2x_2 \end{bmatrix}$$

$$\begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix} \Rightarrow \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$x = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x_1 + x_2 \\ 4x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x_1 + x_2 \\ 4x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



$$\begin{bmatrix} 2x_1 + x_2 \\ 4x_1 + 2x_2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The set of all vectors $x \in \mathbb{R}^n$ such that, under the transform^n T, they get mapped to the zero vector in $\mathbb{R}^m$

$\Rightarrow$ Set of $\vec{x} \in \mathbb{R}^n$ s.t $T(\vec{x}) = \vec{0}$

$\Rightarrow$ Image of those vectors in $\mathbb{R}^n$, under T is the $\vec{0}$.

$\rightarrow$ KERNEL of T.

$$A\,x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}^{m\times 1}$$

$A\,x = 0 \rightarrow$ Soln to the Homogeneous Sys. of linear eqns

$A\underset{\sim}{x} = 0$.

Null Space of A: $\{x \in \mathbb{R}^n : Ax = \vec{0}\}$.

Null Space of A = Kernel of the Transf. T represented by the matrix A.

Let W_1 and W_2 be two Vector Spaces. A linear transformⁿ is a mapping from W_1 to W_2, denoted by $T: W_1 \longrightarrow W_2$, that obeys the following rules:

(i) $T(u_1+u_2) = T(u_1)+T(u_2)$.

Transformⁿ of the Sum = Sum of the transformations.

(ii) $T(\alpha u_1) = \alpha T(u_1)$ α is a scalar.

Transformⁿ of Scalar product is the scalar product of the transformⁿ

Suppose $B = \{v_1, v_2 \dots v_k\}$ is a basis for a vector space W and let x be any vector in W.

Let $x = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k$ for some scalars $\alpha_1 \dots \alpha_k$

Let T be a linear transformⁿ

$$T(\vec{x}) = T(\alpha_1 v_1 + \dots + \alpha_k v_k)$$

$$= T(\alpha_1 v_1) + T(\alpha_2 v_2) + \dots + T(\alpha_k v_k)$$

$$= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_k T(v_k)$$