Diagonalizⁿ of a matrix A.

$T: \mathbb{R}^2 \longmapsto \mathbb{R}^2$ represented by A.

A: 2×2 matrix

A: has 2 distinct eigenvalues λ_1 & λ_2 & each eigenvalue is associated with eigenvectors $\vec{u}$ & $\vec{v}$ resp.

Define a matrix P such that the columns of P are the eigenvectors of A.

$$P = \begin{bmatrix} u & v \end{bmatrix}$$

$$\underset{=}{A}\,\underset{=}{P} = A\begin{bmatrix} u & v \end{bmatrix}$$

$$= \begin{bmatrix} Au & Av \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 u & \lambda_2 v \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} u & v \end{bmatrix}}_{P} \overbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}^{D}$$

$AP = PD.$

$P = \begin{bmatrix} \underline{u} & \underline{v} \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

P is invertible.

$P = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$ det $P = u_1 v_2 - u_2 v_1 \neq 0.$

$\therefore P^{-1}$ exists.

$AP = PD.$

$(AP)P^{-1} = PDP^{-1}$

$\Rightarrow \boxed{A = PDP^{-1}} \Rightarrow$ eigendecomp of A.

$AP = PD$

$P^{-1}AP = \underbrace{P^{-1}P}_{I} D$

$= ID$

$\boxed{P^{-1}AP = D} \rightarrow$ Diagonalizn of A.

$$\underline{A = PDP^{-1}}$$

$$A^2 = A \cdot A = (PDP^{-1})(PDP^{-1})$$

$$= PD^2P^{-1}$$

$$A^3 = \overset{A}{(PDP^{-1})}\overset{A^2}{(PD^2P^{-1})}$$

$$= PD^3P^{-1}$$

$\vdots$

$$\boxed{A^k = PD^kP^{-1}}$$

If $\lambda_1, \lambda_2 \ldots \lambda_n$ are eigenvalues such that λ_i has algebraic multiplicity $\underline{\underline{>1}}$ and s.t

GM of λ_1 + GM(λ_2) + $\cdots$ + $\underline{GM(\lambda_i)}$ + $\cdots$ GM(λ_n) $\underline{\underline{<n}}$.

Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \lambda_1 = 1 \quad$ AM: 2

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ GM = 1.

$\lambda = 1$ is deficient. $\quad P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ P matrix is not invertible

$\Rightarrow$ A cannot be written as PDP^{-1} & D cannot be expressed as $P^{-1}AP$.

$\Rightarrow$ A is not diagonalizable.

If an eigenvalue of A, say λ, is such that the GM of λ < AM of λ, then A is not diagonalizable.

$A^{2\times 2}$, with repeated eigenvalues, the matrix A is not diagonalizable
$\Rightarrow$ The eigen val. λ is deficient.

Q: Suppose $A^{3\times 3}$ is such that
(i) it has one eigenvalue with $AM=3$ (OR)
(ii) it has 2 eigenval λ_1 & λ_2 with $AM(\lambda_1)=2$ & $AM(\lambda_2)=1$
(iii) it has 3 distinct eigenval

What can we say about diagonalizability of A in each of those cases?