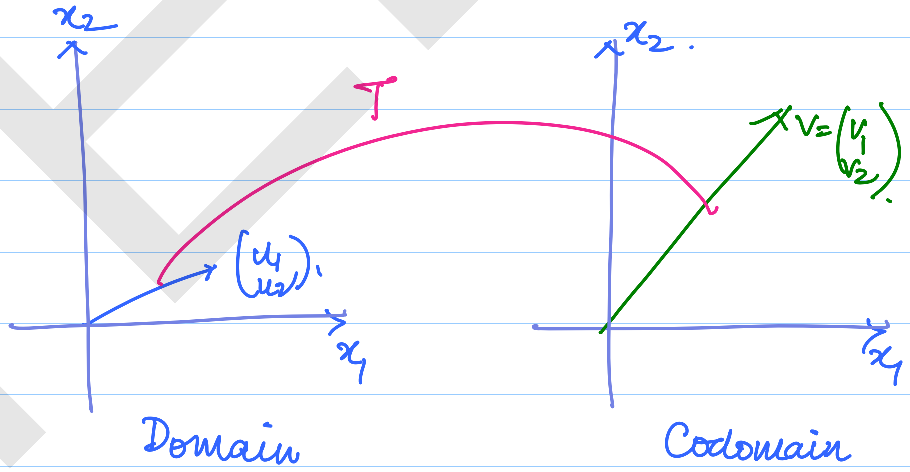
Let T be a linear map from V to V.

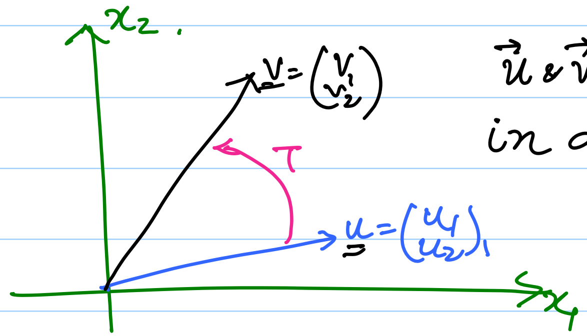
$$T : V \longmapsto V$$

$$T : \mathbb{R}^2 \longmapsto \mathbb{R}^2$$

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) \longrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

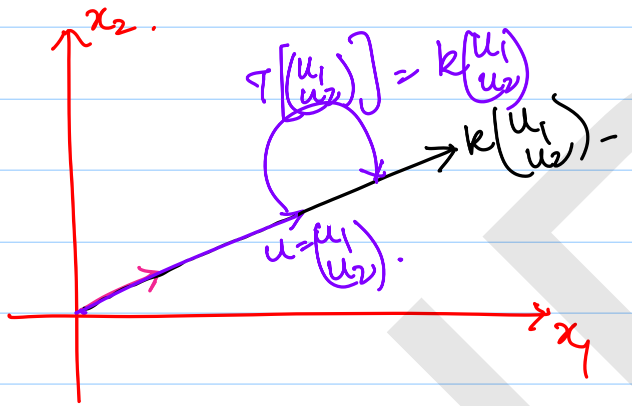




$\vec{u}$ & $\vec{v}$ are in diff. directions.

Consider a linear transformation T such that

$$T\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right] \longrightarrow k\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$$



This implies the direction of the output vector of T is the same as the input vector direction.

Let $\underset{\sim}{u}$ be the input vector to T. The result of the transformation T is scalar multiple of u.

$\Rightarrow$ The vector u is "INVARIANT" to the linear transformation T.

$\Rightarrow$ No direction change/no reorientn.

If $T(u) = ku$ where T is a linear transformation and k is a scalar,

we say/call the vector u as the eigenvector of T and the scalar k is called the eigenvalue

For any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ we see that

$$T(0) = 0.$$

The eigenvector is a non-zero vector.

How do we identify or obtain the eigenvectors for a l.t T from $\mathbb{R}^n \to \mathbb{R}^n$?

Suppose $A^{n\times n}$ is the matrix representation of T, then

$$T(\vec{u}) = k\vec{u}$$ can be expressed as

$$A\vec{u} = k\vec{u}$$

$A\vec{u} = k\vec{u}$ — k: Scalar.

$A\vec{u} = k(I\vec{u})$ — I: Identity matrix of order n.

$(A-kI)\vec{u} = \vec{0} \rightarrow$ Homogeneous Sys of equations.

$\Rightarrow$ Since $\vec{u}$ is a non-zero vector we are looking for a solution to the homogeneous sys. of eqns $(A-kI)\vec{u} = \vec{0}$

we infer that the matrix $(A-kI)$ is a singular matrix $\Rightarrow$ Matrix $(A-kI)$ is non-invertible

$$\Rightarrow \boxed{\det(A-kI) = 0.}$$

Characteristic eqn of A.

The roots of the char. eqn are precisely the scalars that scale the vector $\vec{u}$ upon the action of T on $\vec{u}$.

The roots of the characteristic eqns of A are called the eigenvalues of A.

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix}$$

$$M(T) = A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$A\vec{u} = \lambda\vec{u}$

$A\vec{u} = \lambda\vec{u}$ $\quad \lambda$: Scalar.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \lambda\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow (A - \lambda I)\vec{u} = \vec{0}$$

$$= \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{u} = \vec{0}$$

$$= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 4 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda-3)(\lambda-1) = 0$$

$\lambda_1 = 3$ & $\lambda_2 = 1$.

$\lambda_1 = 3$ $\quad A\vec{u} = 3\vec{u}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 3\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} -u_1 + u_2 &= 0 \\ u_1 - u_2 &= 0 \end{aligned}\right\} \Rightarrow u_1 = u_2.$$

$$\therefore \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix}$$

$$\vec{u} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\therefore$ eigenvector associated with $\lambda_1 = 3$ is $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

$Au \overset{?}{=} 3u$

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{A\vec{u} = 3u}$$

$\lambda_2 = 1.$

$$Av = \lambda_2 v$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 1\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$(A - \lambda_2 I)\vec{v} = \vec{0}$$

$$= \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad v_1 + v_2 = 0 \quad \Rightarrow \quad v_1 = -v_2.$$

$$\therefore \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} s \\ -s \end{pmatrix} = \left\{ s\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

s: ~~arbitrary~~ real.

$$Av = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow Av = \lambda_2 v.$$

$\therefore$ $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the eigenvector associated with $\lambda_2 = 1$.

$$T\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}$$

$$\Rightarrow T\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = \begin{bmatrix} 2+1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\underline{\underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}$$

$$T\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \begin{bmatrix} 2-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\underline{\underline{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}}$$

The invariant direction for the above T are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 3t \\ 3t \end{bmatrix} = 3\begin{bmatrix} t \\ t \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} s \\ -s \end{bmatrix} = \begin{bmatrix} s \\ -s \end{bmatrix} = 1\begin{bmatrix} s \\ -s \end{bmatrix}$$

$\left\{3\begin{pmatrix} t \\ t \end{pmatrix}\right\} \to$ Line passing thro' the origin & $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to$ SUBSPACE

$\left\{1\begin{pmatrix} s \\ -s \end{pmatrix}\right\} \to$ line passing thro the origin & $\begin{pmatrix} 1 \\ -1 \end{pmatrix} \to$ Subspace.

The invariant subspace of a linear transformation $T: V \mapsto V$ (for ex: $T: \mathbb{R}^n \longmapsto \mathbb{R}^n$)

is the set of vectors $\vec{u}$ s.t

$$\boxed{T(\vec{u}) = \lambda \vec{u}}$$

where λ is a scalar

For $T: \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix}$ the invariant subspaces are

$$\left\{ t\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ t \in \mathbb{R} \right\} \ \& \ \left\{ s\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ s \in \mathbb{R} \right\}.$$

Diagonalizⁿ of A & Matrix powers.

Examples of applications of eigenval & eigenvectors.