Introduction: Engineering problems and computational methods

- 20 cm x 30 cm rectangular plate
- Initially at temperature $f(x,y)$. Then, insulated at x=0, zero temperature at y=0, and convective heat transfer at x=20 and at y=30
- Find temperature variation with time at the center

Experimental: Advantages: No equations, Arbitrary Shape

Issues: Size, Material, Time, *f(x,y)*

Analytical : $\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$ $T(x,y,t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{ij} e^{-\alpha\left(\beta_i^2+\gamma_j^2\right)t} \cos(\beta_i x)\sin(\gamma_j y)$

Adv: Fast, Adaptable, Accurate; Issues: Ideal conditions

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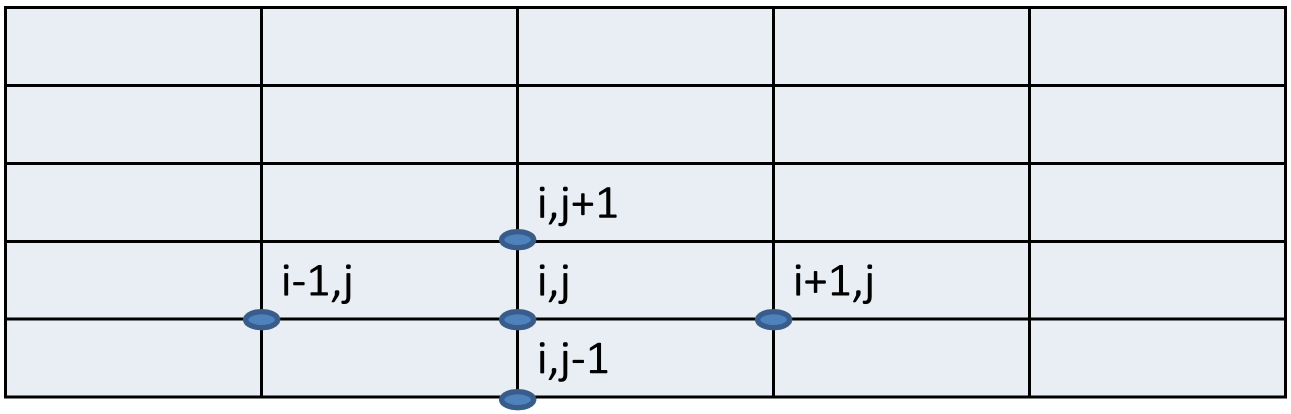
Numerical Method: Reduce to algebraic equations

E.g.,



$$\frac{\partial T}{\partial t} = \frac{T_{i,j}^{t+\Delta t} - T_{i,j}^{t}}{\Delta t}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\dfrac{T_{i+1,j} - T_{i,j}}{\Delta x} - \dfrac{T_{i,j} - T_{i-1,j}}{\Delta x}}{\Delta x} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}$$

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Results in a set of linear (or nonlinear, if α is a function of temperature) equations

Adv: Arbitrary shape, initial and boundary conditions; Usually fast.

Issues: Convergence, Accuracy, Efficiency

Error Analysis: Round-off and Truncation errors
Backward and Forward error analysis

All computational results have errors: Computers have finite storage length and computations cannot be carried out infinitely (We do not consider Model error and Data error)

Round-off Error: Due to finite storage length. E.g., with 8 significant digits: $\sqrt{2}$ is stored as 1.4142136 (actually as 0.14142136×10^1), an error of about 3×10^{-6} percent.

Truncation Error: Due to finite steps in computation. E.g., exp(2) is computed using the infinite series, truncated after a finite number of terms. True value is 7.389056098930650. 1 term – 1; 2 terms–3, 3–5, 4–6.333, 5-7, 6-7.356,…,15-7.3890561.

Error Analysis: Round-off and Truncation errors
Backward and Forward error analysis

Round-off Error: Considering decimal storage, real numbers are stored as mantissa (m) x 10 $^{\text{power (p)}}$ with $0.1 \leq m < 1$. If mantissa rounded-off at the t^{th} digit, maximum error = 0.5×10^{-t}. Maximum relative error = 5×10^{-t} called the round-off unit, *u*.

Definitions:

True Error (e) = True Value – Approximate Value

Approximate Error (ε) = Current Approximation – Previous Approximation

(e.g., x=x/2+1/x, x=1,1.5, 1.4166667,1.4142157,1.4142136)

True Relative Error (e_r) = (True Value – Approximate Value) / True Value

Approximate Relative Error (ε_r) = (Current Approximation – Previous Approximation) / Current Approximation

Error Propagation

y is a function of n independent variables, $x_1, x_2, \ldots, x_n$, i.e. vector x

y=f(x)

Due to error in the variables ($\Delta x_i = x_i - \widetilde{x}_i$), there is error in *y*

$$\widetilde{y} = f\left(\widetilde{x}\right)$$

Error: $\Delta y = y - \widetilde{y} = f\left(x\right) - f\left(\widetilde{x}\right) = \sum_{i=1}^{n} \Delta x_i \left.\frac{\partial f}{\partial x_i}\right|_{\widetilde{x}_i}$

(Neglecting higher order terms in the Taylor's series)

Error Propagation

$$g = \frac{2h}{t^2}$$

Height measurement accurate to a mm, time to 100th of sec. Error in *g*, if h=100.000 m and t=4.52 s?

$$\frac{\partial g}{\partial h} = \frac{2}{t^2}; \frac{\partial g}{\partial t} = -\frac{4h}{t^3}$$

$$\Delta g \cong \frac{2}{\tilde{t}^{\,2}} \Delta h - \frac{4\tilde{h}}{\tilde{t}^{\,3}} \Delta t$$

$$= 0.0978933 \Delta h - 4.33156 \Delta t$$

Max value 0.0434 (Relative error is more meaningful, roughly 0.4%.**)**

Error Propagation and Condition Number of a problem

Condition number: Ratio of the relative change in the function, *f(x),* to the relative change in the variable for a small perturbation Δx in the variable x.

$$C_P = \frac{\left|\dfrac{f(x+\Delta x) - f(x)}{f(x)}\right|}{\left|\dfrac{\Delta x}{x}\right|} \quad \text{or, as } \Delta x \to 0, \frac{x f'(x)}{f(x)}$$

Well-Conditioned (<1) and ***Ill-conditioned (>1)***

E.g.: Hyperbola $xy - y + 1 = 0$. Measure x, compute $y = f(x) = 1/(1-x)$.

$C_P = |x/(1-x)|$ **Ill-conditioned for x>0.5** (x=0.8, y=5; x=0.808,y=5.2083)

Similarly, $y = \sqrt{1+x} - 1$ for small x ;

$C_P = \left|\dfrac{x}{2\left(1+x-\sqrt{1+x}\right)}\right|$ close to but <1. (x=0.01, y=.004988; x=0.0101,y=.005037)

Error Analysis: Forward and Backward error analysis

Forward Error: requires the T.V.

(*Recall that **True Error (e) = True Value − Approximate Value***)

E.g.: √2 = 1.414213562373095. With 4 digit accuracy ($u=5\times10^{-4}$), approx. value = 1.414. True error, e = 2.134×10^{-4}. True relative error = 0.015%.

Backward Error: determines the change necessary in the input variable (or data) to explain the final error in the output (or result) obtained through errors introduced at various stages of calculation.

E.g.: $1.414^2=1.999396$. Error=6.04×10^{-4}, Relative error = 0.03%.

$y=\sqrt{1+x}-1$ with x=0.001616 gives y = √1.002 − 1 = 0.0009995, which is exact for x=0.002 (error=0.000384, 24%)

Condition Number of an algorithm

- **Condition number**: Exact input data, *x;* Corresponding function value *y;* Algorithm *A* operating on *x* produces y_A on a machine with precision unit *u*. Input data, x_A, which would produce *y*, if exact computations are done. *Condition number of the algorithm*

$$C_A = \frac{\left|\frac{x - x_A}{x}\right|}{u}$$

Well-Conditioned (<1) and ***Ill-conditioned (>1)***

$$y = \sqrt{1+x} - 1$$ for x =0.001616, we get x_A=0.002.

$$C_A = \frac{\left|\frac{0.001616 - 0.002}{0.001616}\right|}{5\times10^{-4}} = 475$$

$$y = \sqrt{1+x} - 1 = \frac{x}{\sqrt{1+x}+1}$$

x=0.001616, y=0.0008076,

x_A=0.00161585, C_A=0.18

Error Analysis: Vector Norms

What if the computed result is a vector?

$$[A]\{x\} = \{b\}$$

Error vector

$$\{\varepsilon\} = \{x\} - \{\widetilde{x}\}$$

Or, relative error vector

$$\{\varepsilon_r\} = \left\{\frac{x - \widetilde{x}}{x}\right\}$$

Magnitude: In terms of the "Norm" of the vector

Error Analysis: Vector Norms

Some properties of the vector norms are (α is a scalar):

- $\|x\| = 0$, only if x is a null vector, otherwise $\|x\| > 0$
- $\|\alpha x\| = |\alpha| \|x\|$
- $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$

Magnitude: The L_p norm of an n-dimensional vector, $\{x\}$, *i.e.*, $(x_1, x_2, \ldots, x_n)^T$, *is given by*

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + |x_3|^p + \ldots + |x_n|^p\right)^{1/p} \qquad p \geq 1$$

Common Vector Norms

- p=2, Euclidean norm: (length) $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$
- p=1, (total distance) $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
- p=∞, Maximum norm $\|x\|_\infty = \max_i |x_i|$