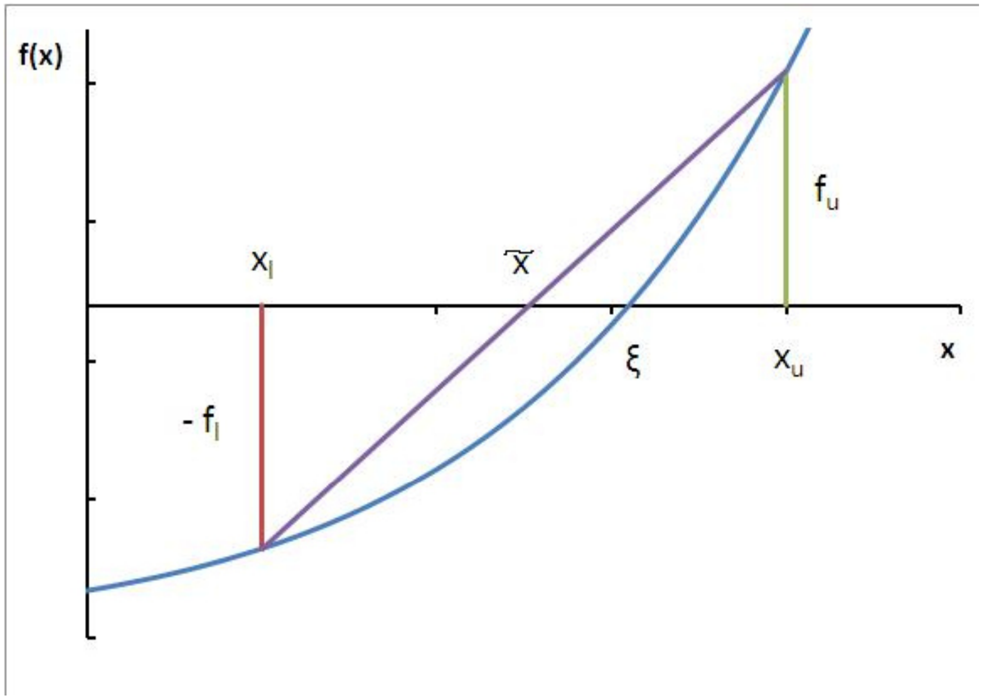
Linear Interpolation



$$\frac{\tilde{x}-x_l}{-f_l}=\frac{x_u-\tilde{x}}{f_u} \Rightarrow \tilde{x}=x_l-f_l\,\frac{x_u-x_l}{f_u-f_l}=x_u-f_u\,\frac{x_u-x_l}{f_u-f_l}$$

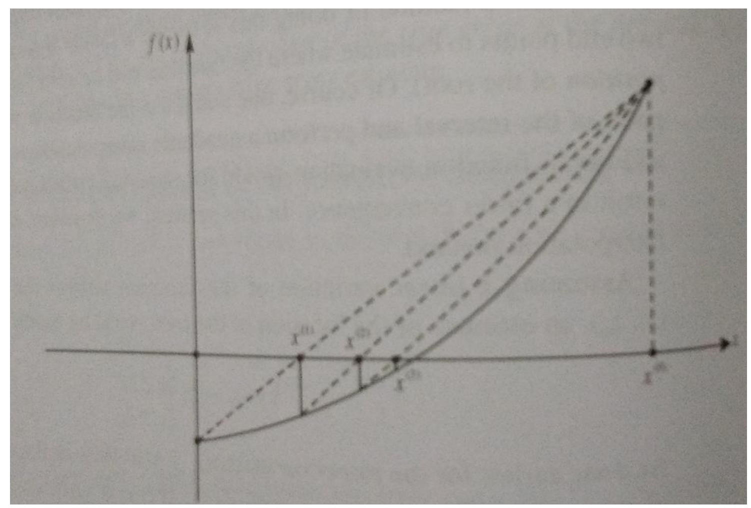
Linear Interpolation: Error Analysis

•Linear interpolation between two points, x_l and x_u,

$$f(x) = f(x_u) + (x - x_u)\frac{f_u - f_l}{x_u - x_l} + (x - x_u)(x - x_l)\frac{f''(\zeta)}{2};\, x, \zeta \in (x_l, x_u)$$

• If we assume the function to be uniformly concave/convex, one end of the interval, x_u(or x_l), remains fixed, say, $x^{(0)}$, and

$$x^{(i+1)} = x^{(0)} - f_0 \frac{x^{(0)} - x^{(i)}}{f_0 - f_i}$$



Linear Interpolation: Error Analysis

From the interpolation equation, applied at $x=\xi$:

$$f(\xi)=0=f_0+\left(\xi-x^{(0)}\right)\frac{f_0-f_i}{x^{(0)}-x^{(i)}}+\left(\xi-x^{(0)}\right)\left(\xi-x^{(i)}\right)\frac{f''(\zeta_1)}{2};\quad \zeta_1\in\left(x^{(i)},x^{(0)}\right)$$

$$\Rightarrow f_0\,\frac{x^{(0)}-x^{(i)}}{f_0-f_i}=-e^{(0)}-e^{(0)}e^{(i)}\,\frac{f''(\zeta_1)}{2f'(\zeta_2)};\qquad \zeta_1,\zeta_2\in\left(x^{(i)},x^{(0)}\right)$$

And, from the iteration equation

$$\xi-x^{(i+1)}=\xi-x^{(0)}+f_0\,\frac{x^{(0)}-x^{(i)}}{f_0-f_i}\;\Rightarrow e^{(i+1)}=-e^{(0)}e^{(i)}\,\frac{f''(\zeta_1)}{2f'(\zeta_2)}$$

In the limit, the iterations converge to the root, and

$$\lim_{i\to\infty}\left|\frac{e^{(i+1)}}{e^{(i)}}\right|=\left|e^{(0)}\,\frac{f''(\zeta_3)}{2f'(\zeta_4)}\right|;\quad \zeta_3,\zeta_4\in\left(\xi,x^{(0)}\right)$$

Linear Interpolation: Error Analysis

- Linearly convergent
- **Asymptotic error constant** is not constant (was ½ for bisection), but depends on the nature of the function

$$C = \left| e^{(0)} \frac{f''(\zeta_3)}{2f'(\zeta_4)} \right|$$

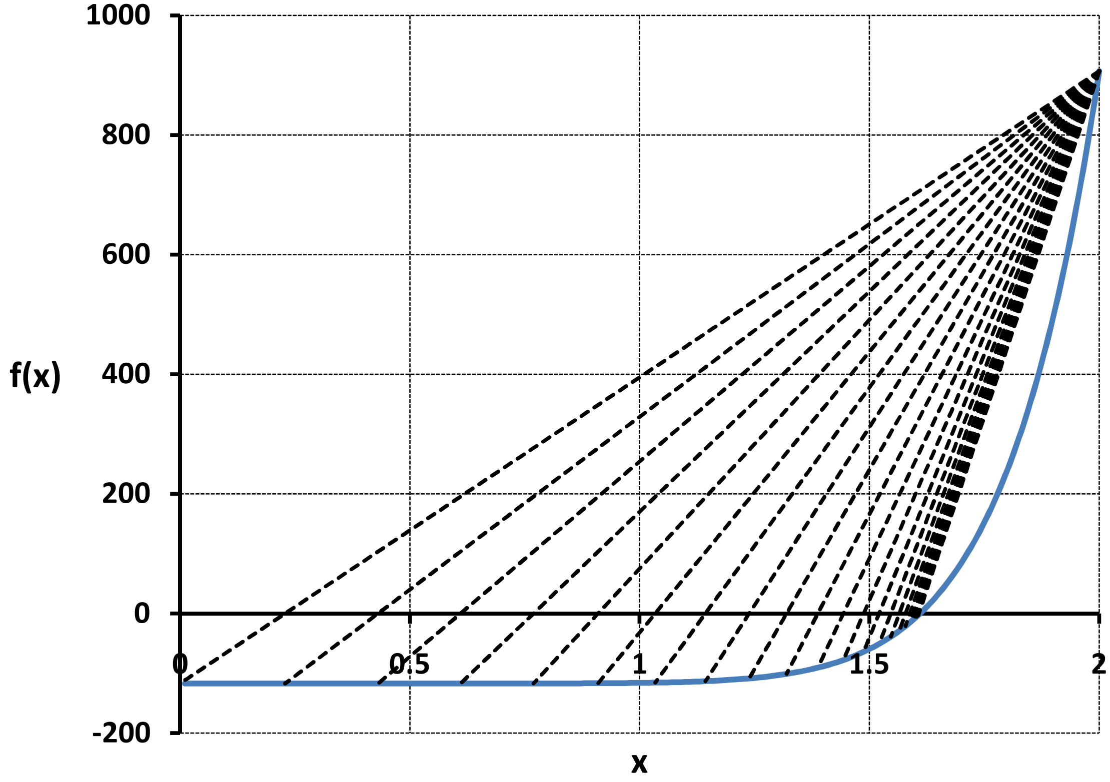
For the function used in the example,

$$f(x) = x^3 - 1.2500000x^2 - 1.5625250x + 1.9530938$$

The iterations show one end fixed at −2, the root is −1.25; $e^{(0)}$=0.75. |f”/2f’| varies from 0.5 to 0.8 over (−2, −1.25), indicating C between 0.35 and 0.6.

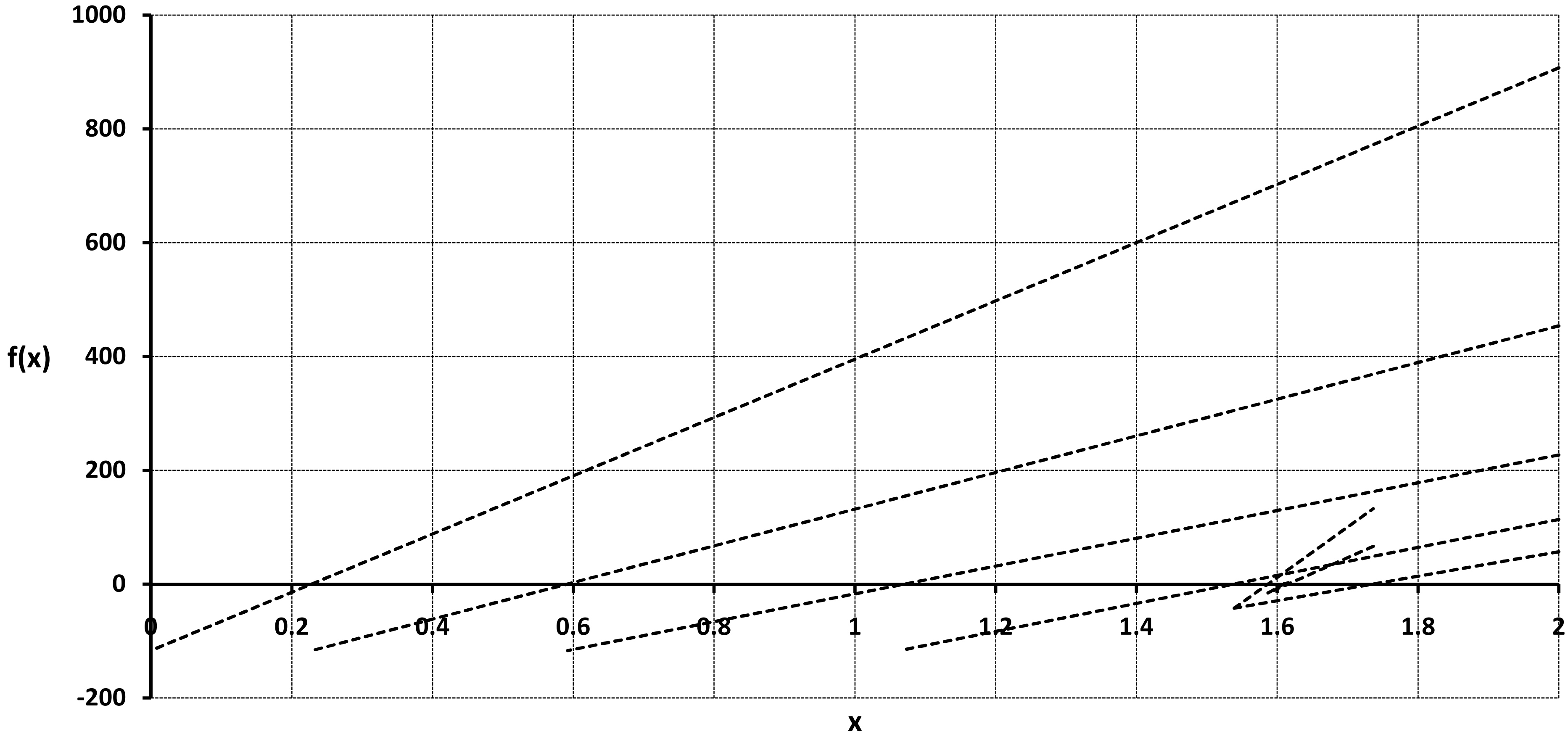
Linear Interpolation: Modification

- Called Modified False Position or Modified Regula Falsi Method
- To avoid slow convergence (E.g. : $x^{10} - 117.02 = 0$)



Modified Regula Falsi

- If one end is fixed for a few iterations (generally 2-3), reduce the function value to some fraction (usually half).



Roots of Nonlinear Equations: Open methods

- Because of linear convergence, bracketing methods are generally used only as starting methods
- Once we are close to the root, we may switch to “open methods” which generally converge faster
- However, since these do not bracket the root, convergence is not guaranteed

Different open methods are:

Fixed-point, Newton-Raphson, Secant, Muller ...

Fixed-point Method

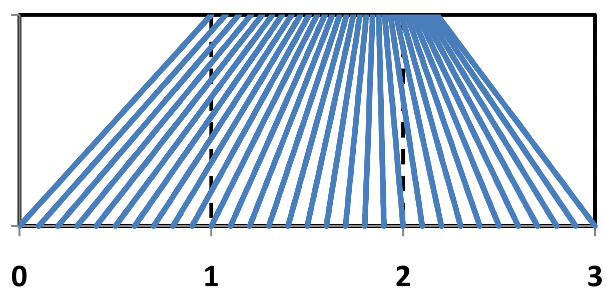
$$f(x) = 0 \text{ can always be written as } x = \phi(x)$$

$$\text{E.g., } f(x) = x^2 - 2 = 0 \text{ could be written as } x = \frac{2}{x} \text{ or } x = \frac{x}{2} + \frac{1}{x}$$

$$\text{If nothing else, } x = x + f(x)$$

$\phi(x)$ may be thought of as a mapping which maps each point on the x-axis to a different point, except for some points which are mapped on themselves [i.e., $\phi(x)=x$]. These are called the fixed-points of the function $\phi(x)$.

- We start with an initial guess of the root as $x^{(0)}$, and then obtain the next guess as $x^{(1)}= \phi(x^{(0)})$. Continue till $x^{(i+1)} \approx x^{(i)}$



Fixed-point Method: Example

- Flow depth in an open channel (T.V.=1.86)

$$f(x) = x^{2.5} - 2x - 1 = 0$$

- We could write, $\phi_1(\mathrm{x}) = (1 + 2\mathrm{x})^{0.4}$ or $\phi_2(x) = \left(x^{2.5} - 1\right)/2$

- Iterations shown below:

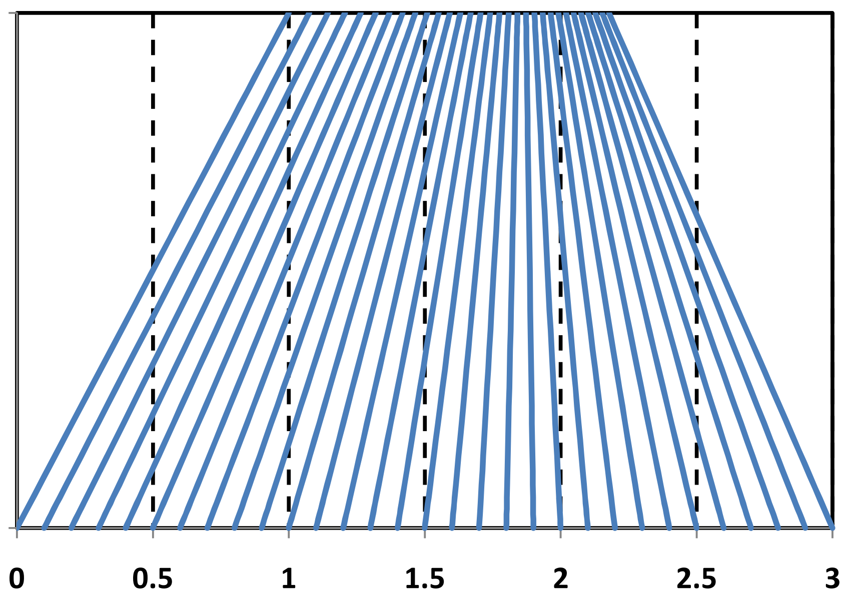
x	$\mathrm{Phi}_1(\mathrm{x})$
2.000000	1.903654
1.903654	1.873962
1.873962	1.864668
1.864668	1.861744
1.861744	1.860823
1.860823	1.860533
1.860533	1.860442

x	$\mathrm{Phi}_2(\mathrm{x})$
2.000	2.328
2.328	3.636
3.636	12.108
12.108	254.587
254.587	517084.882

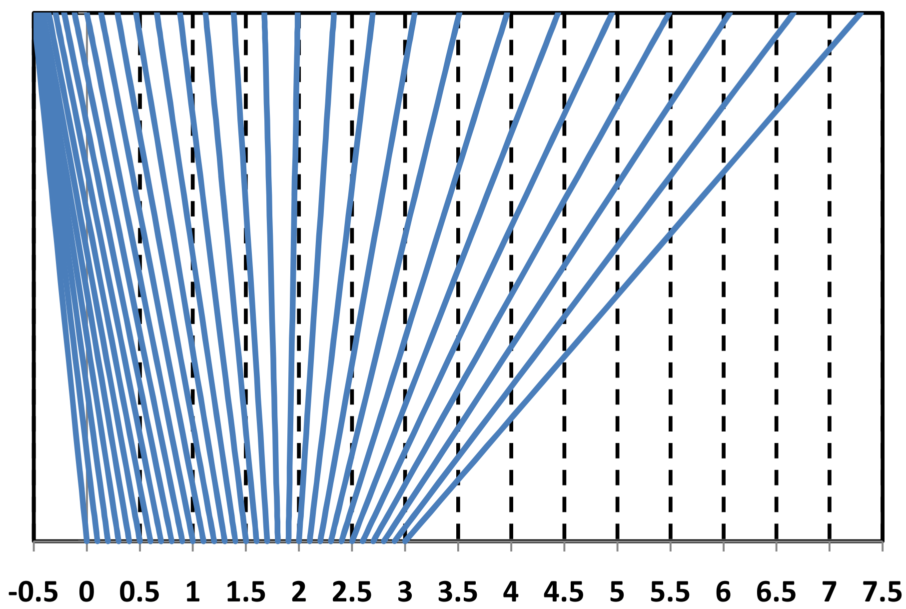
Fixed-point Method: Example

- The mappings are shown below:

$\phi_1(x) = (1+2x)^{0.4}$



$\phi_2(x) = (x^{2.5}-1)/2$



Fixed-point Method: Error Analysis

- True Error at iteration (i+1)

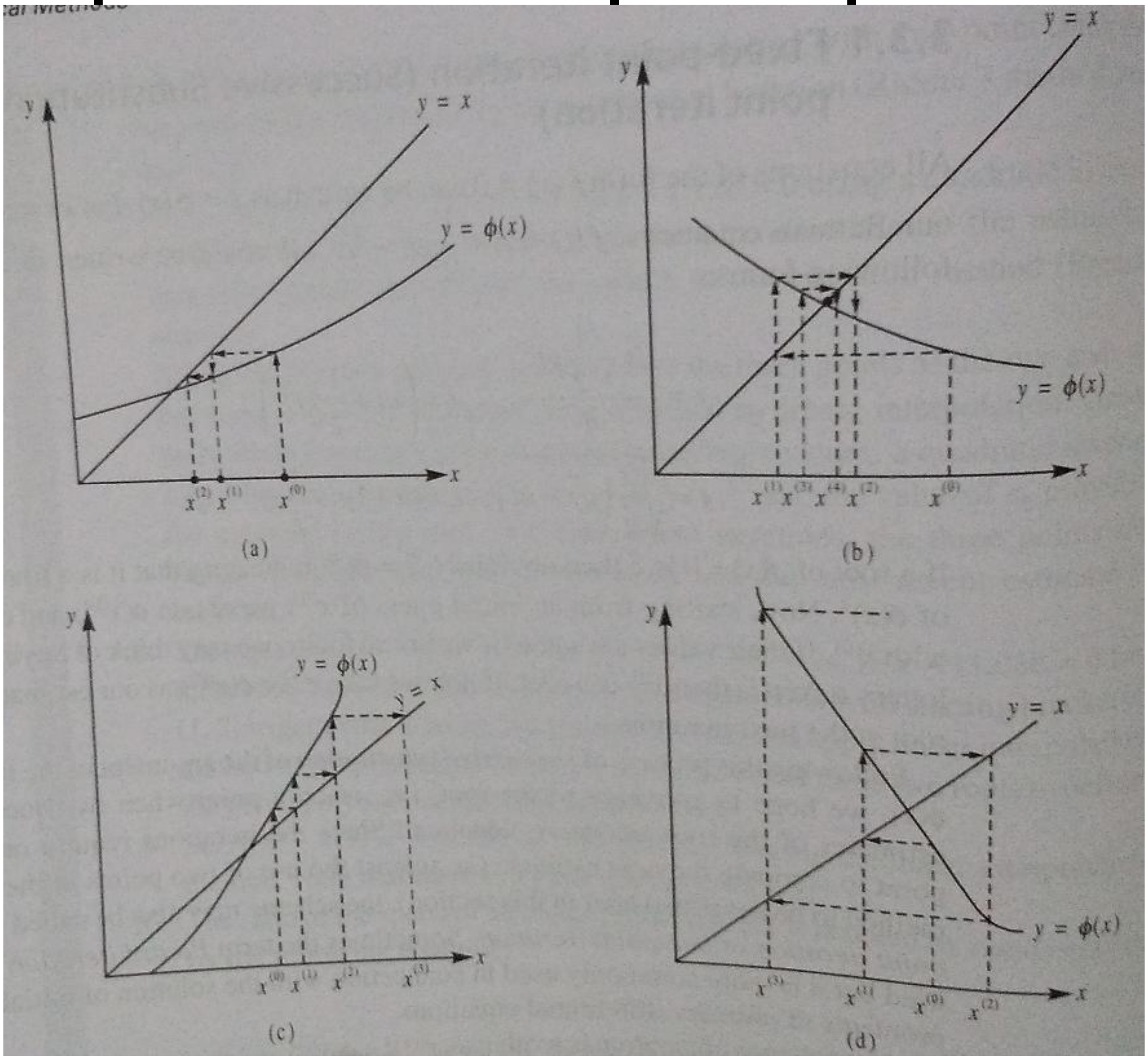
$$e^{(i+1)} = \xi - x^{(i+1)} = \phi(\xi) - \phi(x^{(i)})$$

- Using Mean Value Theorem, $e^{(i+1)} = \left(\xi - x^{(i)}\right)\phi'(\zeta)$ where $\zeta \in \left(x^{(i)}, \xi\right)$

- Linearly convergent with the Error Constant $C = \phi'(\zeta)$

- For convergence, *C* should be less than 1

Fixed-point Method: Graphical representation

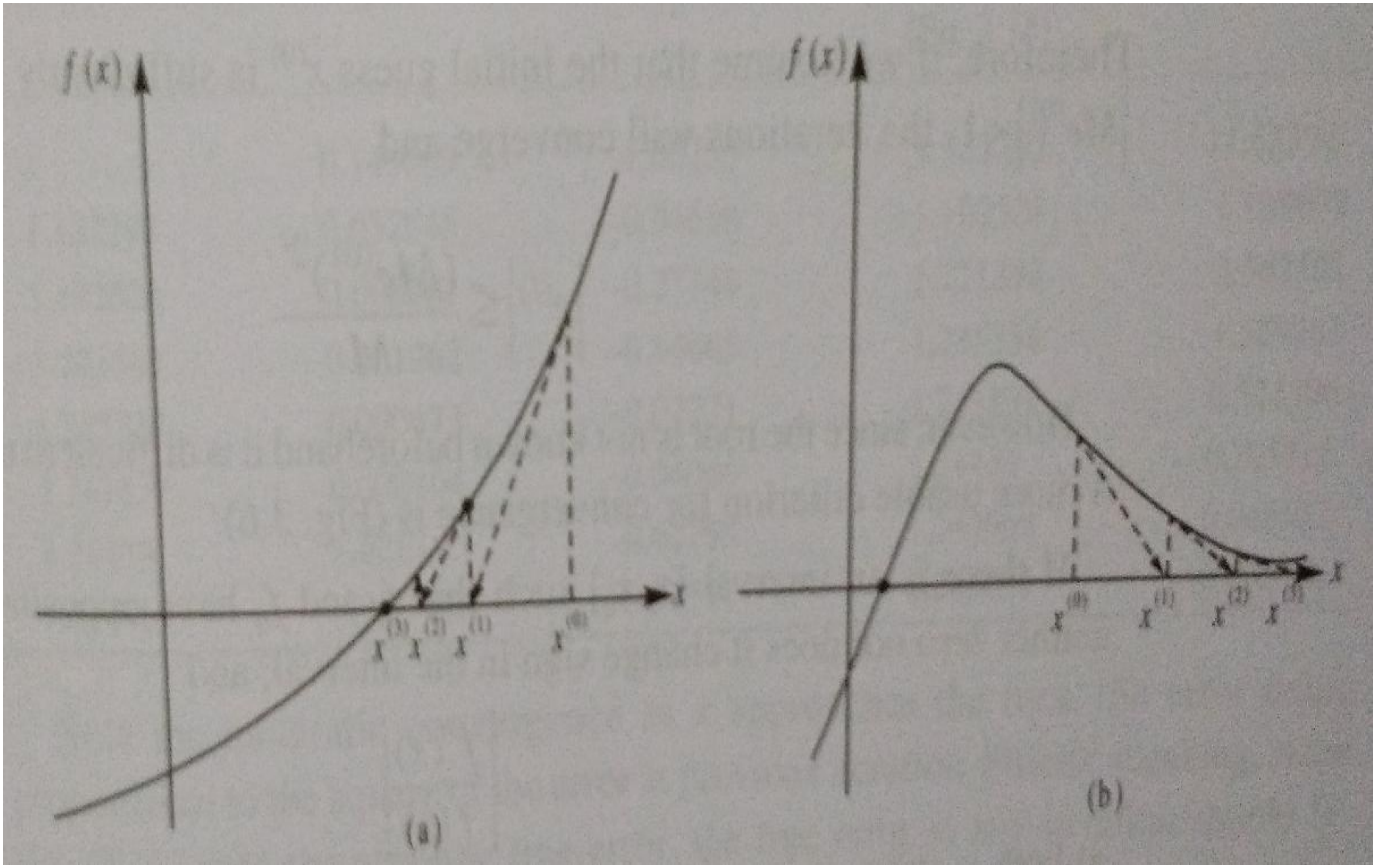


Roots of Nonlinear Equations: Newton Method

- Fixed point method has linear convergence, hence a better method is desirable
- Newton (or Newton-Raphson) method is one such method, which has a second-order convergence
- In linear interpolation, function is approximated by a linear function, with the slope estimated from the difference of the function values at the two points.
- In Newton method also, the function is approximated by a linear function, but the slope is obtained analytically

$$f(x) \approx f(x^{(i)}) + (x - x^{(i)}) f'(x^{(i)})$$

Newton Method: Graphical Representation



Newton Method: Algorithm and Error Analysis

- The straight-line approximation is

$$f(x) \approx f(x^{(i)}) + (x - x^{(i)}) f'(x^{(i)})$$

- It will intersect the x-axis at $x^{(i+1)}$, hence

$$0 = f(x^{(i)}) + (x^{(i+1)} - x^{(i)}) f'(x^{(i)}) \Rightarrow x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})}$$

- To write the error, use Taylor's series

$$f(x) = f(x^{(i)}) + (x - x^{(i)}) f'(x^{(i)}) + \frac{\left(x - x^{(i)}\right)^2}{2} f''(\zeta) \qquad \zeta \in \left(x, x^{(i)}\right)$$

- As *x* approaches the root,

$$f(\xi) = 0 = f(x^{(i)}) + (\xi - x^{(i)}) f'(x^{(i)}) + \frac{\left(\xi - x^{(i)}\right)^2}{2} f''(\zeta)$$

Newton Method: Error Analysis

$$x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{f'(x^{(i)})}$$

$$f(\xi) = 0 = f(x^{(i)}) + (\xi - x^{(i)}) f'(x^{(i)}) + \frac{\left(\xi - x^{(i)}\right)^2}{2} f''(\zeta)$$

From these equations,

$$e^{(i+1)} = \xi - x^{(i+1)} = \xi - x^{(i)} + \frac{f(x^{(i)})}{f'(x^{(i)})} = -\frac{f''(\zeta)}{2f'(x^{(i)})}\left(\xi - x^{(i)}\right)^2$$

$$\lim_{i\to\infty}\left|\frac{e^{(i+1)}}{\left[e^{(i)}\right]^2}\right| = \left|\frac{f''(\xi)}{2f'(\xi)}\right|$$

Second-order convergence.