# Nearest neighbor for realizable online classification

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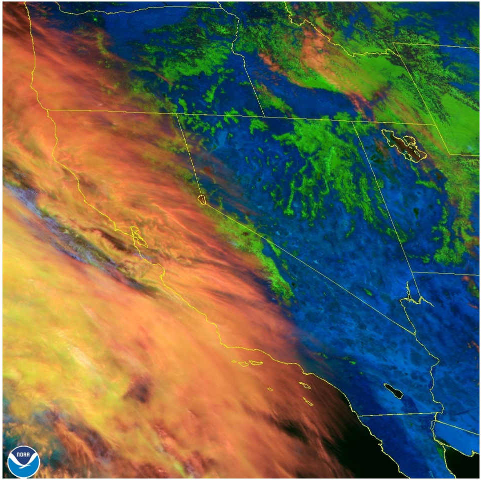
EnCORE Student Social — Mar 20, 2023

# Weather forecasting problem

**The Weather Channel's task**

Each day:

- receive atmospheric data



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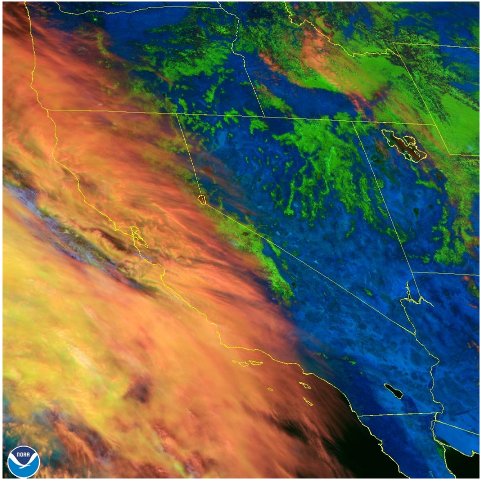

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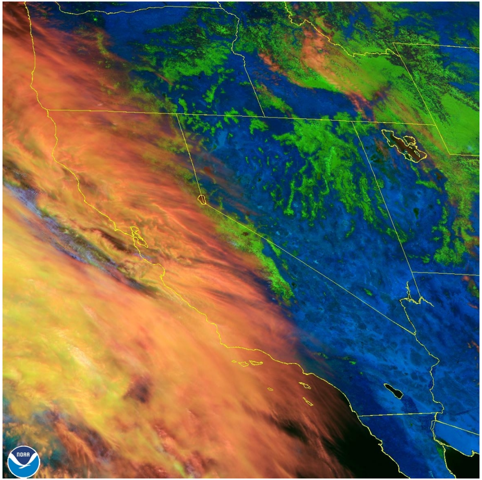

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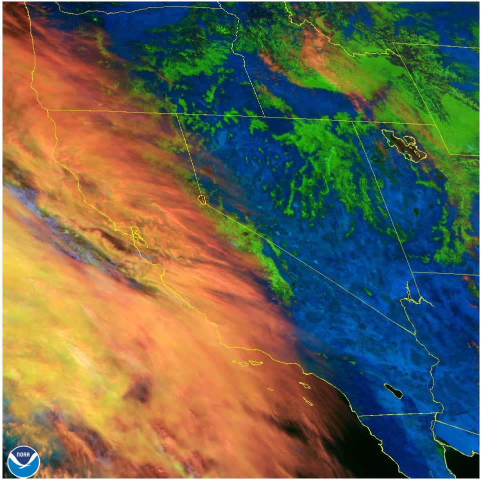

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**NEAREST NEIGHBOR ALGORITHM**

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- predict weather according to the most similar conditions in memory

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Let $(\mathcal{X}, \rho)$ be a metric space.

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- predict using corresponding label

$$\hat{y}(x) = y_{\mathrm{NN}(x)}$$

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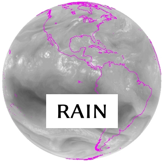

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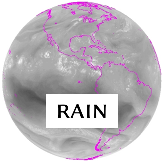


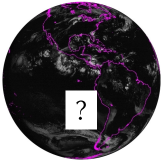

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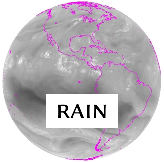


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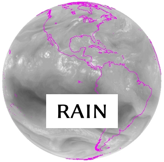


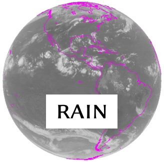


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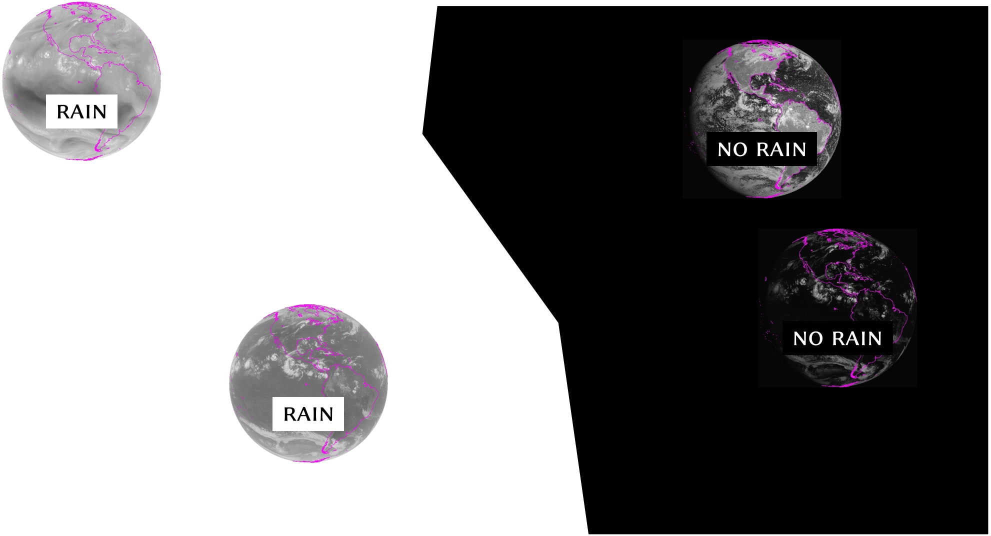

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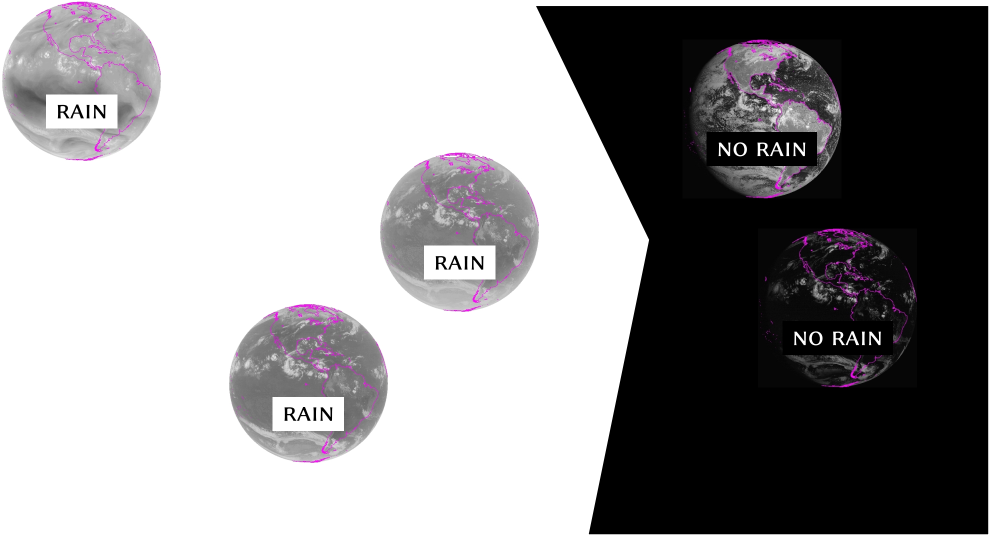

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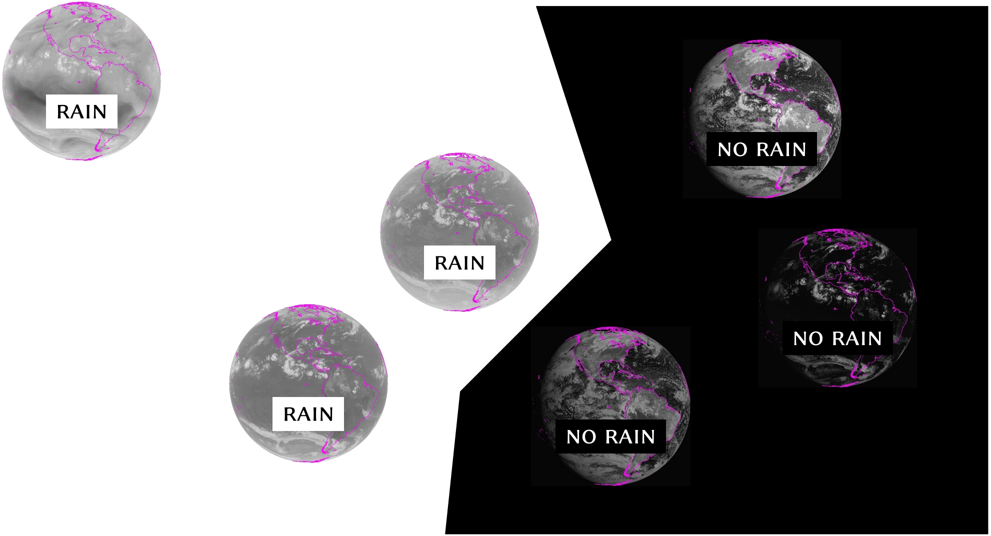

# Behavior of online nearest neighbor

**QUESTION**

When is the *nearest neighbor rule* a reasonable online prediction strategy?

# Online learning setting

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For $t = 1, 2, \ldots$

- receive instance $x_t$

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- receive instance $x_t$
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- observe true label $y_t$
- incur loss $\ell(x_t, y_t, \hat{y}_t)$

# Online learning setting

REALIZABILITY ASSUMPTION

The true labels are generated by some underlying function $f : \mathcal{X} \to \mathcal{Y}$,

$$y_t = f(x_t).$$

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Make fewer and fewer mistakes over time.

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$$\underbrace{\mathrm{er}_T := \frac{1}{T}\sum_{t=1}^{T} \ell\big(x_t, y_t, \hat{y}_t\big) \to 0}_{\text{achieve vanishing error rate}} .$$

# Connection to regret

In the usual goal in the online learning setting is to achieve sublinear regret:

$$\text{regret}_T := \sum_{t=1}^{T} \ell\big(x_t, y_t, \hat{y}_t\big) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell\big(x_t, y_t, h(x_t)\big).$$

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- In the realizable setting, if $\mathcal{H}$ is non-parametric (e.g. all nearest neighbor classifiers), no mistakes are made by any optimal $h \in \mathcal{H}$ on $(x_1, y_1), \ldots, (x_T, y_T)$.

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- Thus, sublinear regret is equivalent to vanishing error rate.

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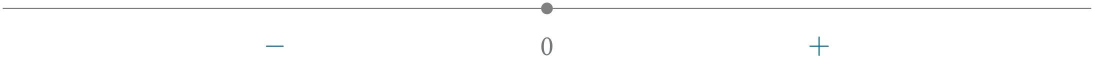
- The sequence of instances $x_t$ do not come i.i.d. from some distribution.
- In the worst-case, each $x_t$ is selected so that learner makes a mistake each time.

# Negative example: learning the sign function

**GOAL**

Learn the sign function $f(x) := \begin{cases} + & x \geq 0 \\ - & x < 0 \end{cases}$



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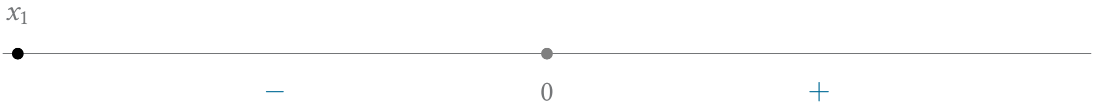


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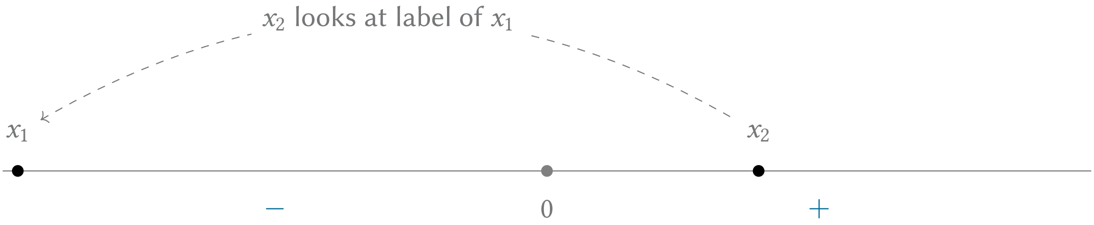


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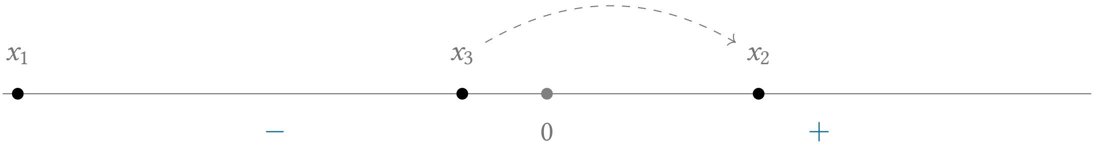


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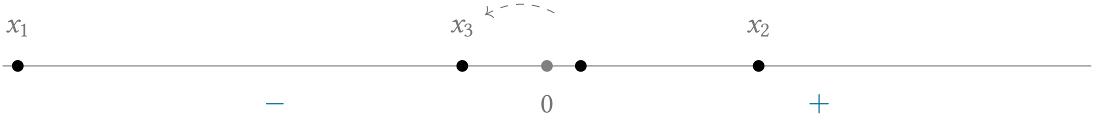


**EXAMPLE.** A worst-case sequence where the nearest neighbor rule errs every time.

- The sequence alternate signs and the nearest neighbor of $x_{t+1}$ is $x_t$ out of $x_1, \ldots, x_t$.

# Negative result

**SETTING**

Let $(\mathcal{X}, \rho)$ be a totally bounded metric space and $f : \mathcal{X} \to \{-, +\}$.

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## Proposition (Non-convergence in the worst-case)

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- **Proof idea:** can always find arbitrarily close pairs $(x, x')$ with opposite signs
  - can select sequence so that $x_{2t}$ is closest to $x_{2t-1}$, which has the opposite sign

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The worst-case adversary is too powerful—learning may not be possible in this setting.

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Under what *general conditions* is realizable online learning possible?

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- How much do we need to relax the worst-case adversary?

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  - Show that almost all problems are easy (the hard instances have measure zero).
    - Or, problems are easy with high probability/on average.

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The smoothed online setting is also studied by Rakhlin et al. (2011); Haghtalab et al. (2020).

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# Example: Gaussian perturbation model

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**GAUSSIAN-SMOOTHED ADVERSARY:**

- adversary selects $\overline{x}$
- test instance $x$ is a perturbed version $\overline{x} + \xi$ where $\xi \sim \mathcal{N}(0, \sigma^2 I)$, so:

$$\mu = \mathcal{N}(\bar{x}, \sigma^2 I).$$

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**$\sigma$-SMOOTHED ADVERSARY:**

- let $\nu$ be an underlying distribution over $\mathcal{X}$
- the adversary can select any distribution $\mu$ satisfying:

$$\mu(A) \leq \frac{1}{\sigma} \cdot \nu(A),$$

for all $A \subset \mathcal{X}$ measurable.

# Dominated adversary

In this work, we generalize both by the $\nu$-*dominated adversary*.

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## Definition (Dominated adversary)

*The measure $\nu$ uniformly dominates a family $\mathcal{M}$ of probability distributions on $\mathcal{X}$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that:*

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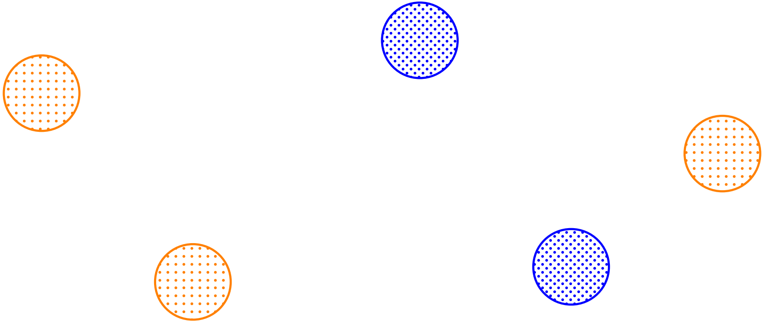
*for all $A \subset \mathcal{X}$ measurable and distribution $\mu \in \mathcal{M}$. We say that adversary is $\nu$-dominated if at all times $t$ it selects $\mu_t$ from a family of distributions uniformly dominated by $\nu$.*

# Example: learning labels for well-separated clusters

**SETTING**

Suppose that the instance space $\mathcal{X}$ consists of countably many well-separated clusters



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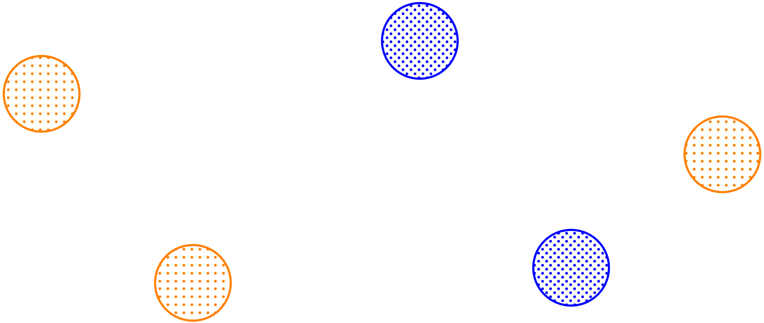

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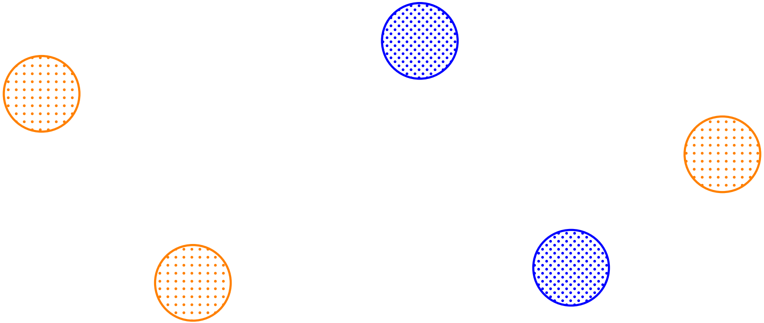

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**THE NEAREST NEIGHBOR LEARNER**

makes at most one mistake made per cluster

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Let $\nu$ be a finite measure on $\mathcal{X}$.

# Example: learning labels for well-separated clusters

**CONVERGENCE RESULT FOR WELL-SEPARATED CLUSTERS**

Let $\nu$ be a finite measure on $\mathcal{X}$.The nearest neighbor learner achieves vanishing error rate against any $\nu$-dominated adversary.

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- Nearest neighbor can only make finitely many mistakes on $\mathcal{X}_{\text{easy}}$.
  - These mistakes contribute nothing to the asymptotic mistake rate.
- The $\nu$-dominated adversary selects points from $\mathcal{X}_{\text{small}}$ at rate $\mu(\mathcal{X}_{\text{small}}) < \varepsilon$.
  - By the law of large number, at most an $\varepsilon$-fraction of $(x_t)_t$ comes from $\mathcal{X}_{\text{small}}$:

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}\{x_t \in \mathcal{X}_{\text{small}}\} = \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mu_t(\mathcal{X}_{\text{small}}) < \varepsilon \quad \text{a.s.}$$

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- Split $\mathcal{X}$ into two pieces $\mathcal{X}_{\text{easy}} \cup \mathcal{X}_{\text{small}}$, where:
  - $\mathcal{X}_{\text{easy}}$ is the union of a finite collection of clusters
  - $\mathcal{X}_{\text{small}}$ satisfies $\nu(\mathcal{X}_{\text{small}}) < \delta$.

- Nearest neighbor can only make finitely many mistakes on $\mathcal{X}_{\text{easy}}$.
  - These mistakes contribute nothing to the asymptotic mistake rate.

- The $\nu$-dominated adversary selects points from $\mathcal{X}_{\text{small}}$ at rate $\mu(\mathcal{X}_{\text{small}}) < \varepsilon$.
  - By the law of large number, at most an $\varepsilon$-fraction of $(x_t)_t$ comes from $\mathcal{X}_{\text{small}}$:

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}\{x_t \in \mathcal{X}_{\text{small}}\} = \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mu_t(\mathcal{X}_{\text{small}}) < \varepsilon \quad \text{a.s.}$$

Thus, the asymptotic mistake rate is upper bounded by any $\varepsilon > 0$ by selecting $\mathcal{X}_{\text{small}}$ sufficiently small.

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Thus, the asymptotic mistake rate is upper bounded by any $\varepsilon > 0$ by selecting $\mathcal{X}_{\text{small}}$ sufficiently small. The asymptotic mistake rate is zero by taking $\varepsilon \downarrow 0$. $\square$

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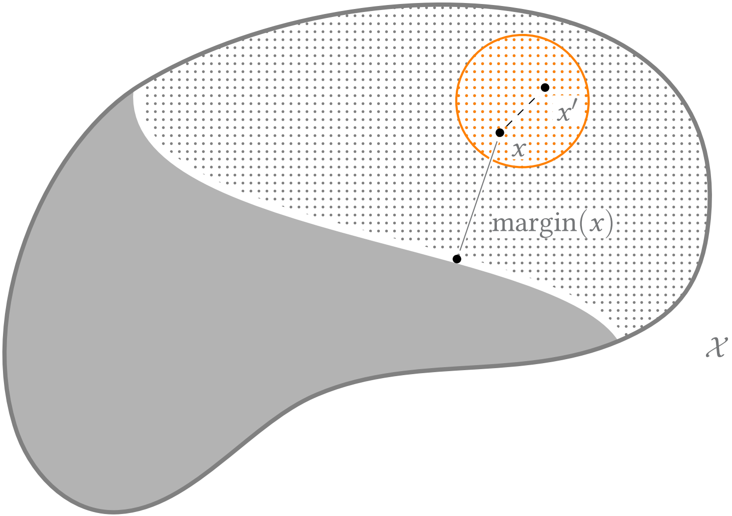
The nearest neighbor learner makes at most one mistake per mutually-labeling set.

- We introduce the device of mutually-labeling sets $U \subset \mathcal{X}$ satisfying the property:

$$\text{interpoint distances in } U < \text{distance to points with different labels.}$$

# Mutually-labeling set



# Generalizing argument

## Definition (Mutually-labeling set)

*A subset $U \subset \mathcal{X}$ is* ***mutually labeling*** *if for all $x, x' \in U$:*

$$\underbrace{\rho(x, x')}_{\textit{interpoint distances}} \quad < \quad \underbrace{\mathrm{margin}(x)}_{\textit{distance to decision boundary}}$$

*where* $\mathrm{margin}(x)$ *is the smallest distance between $x$ and points with different labels:*

$$\mathrm{margin}(x) = \inf\{\rho(x, \overline{x}) : f(x) \neq f(\overline{x})\}.$$

# Convergence result

**SETTING**

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## Theorem (Convergence of nearest neighbor)

*The nearest neighbor rule achieves vanishing mistake rate against a $\nu$-dominated adversary:*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}\{\hat{y}_t \neq y_t\} = 0 \quad \text{a.s.}$$

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Thus, the mistake rate converges to zero almost surely. $\square$

# Discussion about this work

**UPSHOT**

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**QUESTIONS**

- Does the $\nu$-dominated adversary balance between generality and tractability well?
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- What do meaningful rates of convergence look like?

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OPPORTUNITY: we might not live in the worst-case adversarial setting

- Is the $\nu$-dominated online learning setting realistic and tractable?
- If so, can we design and analyze algorithms specifically for this setting?
  - e.g. a minimax optimal algorithm might not be optimal in this setting

# Thank you

**ACKNOWLEDGEMENTS**

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