Metric learning from lazy, opinionated crowds

i.e., from limited pairwise preference comparisons

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An opinionated member of society

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I prefer Blade Runner over Godzilla.

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For it is more similar to my favorite movie The Matrix.

Metric learning from preferences

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Suppose a lot of people on the internet tell us these sorts of pairwise movie rankings.

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Suppose a lot of people on the internet tell us these sorts of pairwise movie rankings.

- Can we learn a metric that captures the similarity of movies in general?

Background

Metric learning: raison d'être

Distance-based algorithms

- nearest neighbor methods
- margin-based classification
- information retrieval
- clustering
- etc.

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Automatically learn a good metric for these downstream tasks

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- esp. metrics aligning with human values, perception, and preferences.

The alignment problem

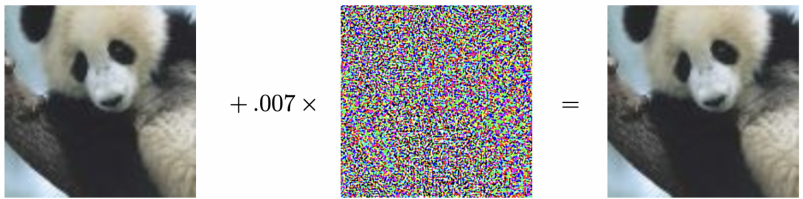

Figure 1: These two images are visually indistinguishable to a human, but very well-separated under the Euclidean distance (Goodfellow et al., 2014).

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- Assumption: a user has an *ideal item* A and prefers items more similar to A.

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- Assumption: a user has an *ideal item* A and prefers items more similar to A.
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Xu and Davenport (2020) and Canal et al. (2022)

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Our work: Let's just give up on trying to learn the ideal points. We ask:
Can we recover the metric using $m \ll d$ measurements per user?

Preliminaries

Formal setting

Representation space

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- We receive measurements from users of the form:

$$(x, x', y) \qquad \text{where} \quad y = \mathbf{1}\big\{\rho_M(u, x) < \rho_M(u, x')\big\}.$$

Mahalanobis distances

A *Mahalanobis distance* ρ_M on $\mathbb{R}^d$ is a metric of the form:

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- $M = A^\top A$ for some $A \in \mathbb{R}^{d \times d}$ since $M \succ 0$.
- Let $\Phi(x) = Ax$ be a new (linear) representation. Then:

$$\rho_M(x, x') = \|\Phi(x) - \Phi(x')\|_2.$$

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- **Continuous responses:** measurements of the form (x, x', ψ), where:

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- **Binary responses:** measurements of the form (x, x', y) where:

$$y = \mathbf{1}\{\psi < 0\}.$$

  - Later, we consider the setting where labels are binary and noisy.

A linear reparametrization (Canal et al., 2022)

Let $x, x' \in \mathbb{R}^d$ be two items. If a user has ideal point $u \in \mathbb{R}^d$, then:

$$\psi_M(x, x'; u) = \underbrace{\left\langle xx^\top - x'x'^\top, M \right\rangle}_{(1)} + \underbrace{\left\langle x - x', v \right\rangle}_{(2)}, \qquad \text{where } \underbrace{v = -2Mu}_{(3)}.$$

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Upshot: Reparametrize (M, u) to (M, v). Then, the following map is linear:

$$(M, v) \mapsto \psi_M(x, x'; u).$$

Design matrices

Let $\{(x_{i_0}, x_{i_1})\}_{i=1}^{m}$ be a set of item pairs.

- Define the linear map $D : \mathrm{Sym}(\mathbb{R}^d) \oplus \mathbb{R}^d \to \mathbb{R}^m$:

$$D_i(A, w) = \left\langle x_{i_0} x_{i_0}^\top - x'_{i_1} x'^\top_{i_1}, A \right\rangle + \left\langle x_{i_0} - x'_{i_1}, w \right\rangle.$$

- We call D the design matrix induced by the item pairs.

Metric learning from continuous responses, single user case

Suppose a user provides us with measurements $\left\{(x_{i_0}, x_{i_1}, \psi_i)\right\}_{i=1}^{m}$, where:

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- To recover the metric and ideal point, $m = \frac{d(d+1)}{2} + d$ measurements is necessary.

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Generalization to multiple users

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  - from $d + 1$ measurements per user if $K = \Omega(d^2)$.

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- Our regime: too few measurements per user to learn latent parameters.

An impossibility result

Setting for impossibility result

Setting.

- Let $\mathcal{X} \subset (\mathbb{R}^d, \rho_M)$ be a countable set of items.
- Let user $k \in \mathbb{N}$ have pseudo-ideal point v_k.
- We ask $m \leq d$ pairwise comparisons per user over items in $\mathcal{X}$.
  - Let $D^{(k)}$ be the design matrix for user k.

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$$D^{(k)}(M, v_k) = D^{(k)}(M', v'_k), \qquad \forall k \in \mathbb{N}.$$

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- That is, M' is consistent with observed data.
- Each user introduces enough degrees of freedom to account for all variation in data.
- Not only is recovery impossible, but we learn nothing at all about M.

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When (i) $\mathcal{X}$ has generic pairwise relations, (ii) . . . the impossibility result holds.

- We introduce a notion of genericity, slightly stronger than *general linear position*.
- Almost all finite sets are generic in this sense (w.r.t. Lebesgue measure on $\mathbb{R}^d$).

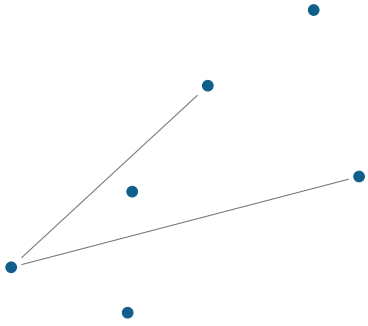
General linear position

Definition

A set $\mathcal{X} \subset \mathbb{R}^d$ is in general linear position if the following is linearly independent:

$$\{x_i - x_0 : i = 1, \ldots, n\},$$

for any distinct $x_0, x_1, \ldots, x_n \in \mathcal{X}$ and $n \leq d$.



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General linear position: alternate definition

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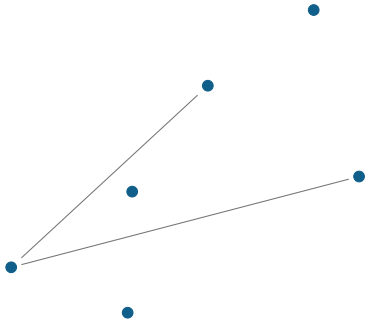
A set $\mathcal{X} \subset \mathbb{R}^d$ is in general linear position if for any star graph $G = (V \subset \mathcal{X}, E)$ with $|E| \leq d$, the following is linearly independent:

$$\{x - x' : (x, x') \in E\}.$$



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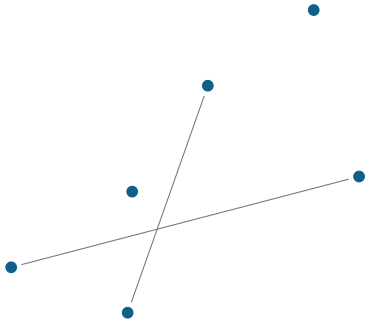
Generic pairwise relation

Definition

A set $\mathcal{X} \subset \mathbb{R}^d$ has generic pairwise relations if for any acyclic graph $G = (\mathcal{X}, E)$ with $|E| \leq d$, the following is linearly independent:

$$\{x - x' : (x, x') \in E\}.$$



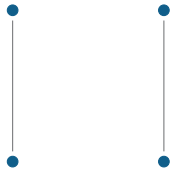
A set of points $\mathcal{X} \subset \mathbb{R}^d$.

Generic pairwise relations $\Longrightarrow$ general linear position

Proof.
A star graph with at most d edges is an acyclic graph with at most d edges. $\square$

General linear position $\not\Longrightarrow$ generic pairwise relations

- ✓ General linear position—no three points are colinear.
- × These points do not have generic pairwise relations.



General takeaway I

(Not) learning from crowd data

- Weaker feedback may make data easier/cheaper to collect
  - e.g. triplet $\rightarrow$ binary feedback (with latent comparator)

General takeaway I

(Not) learning from crowd data

- Weaker feedback may make data easier/cheaper to collect
  - e.g. triplet $\rightarrow$ binary feedback (with latent comparator)
- But we may need to pay for it elsewhere
  - e.g. new fundamental limits/regimes where data carries no information

Metric learning with subspace-cluster structure

Real data often exhibit additional structure

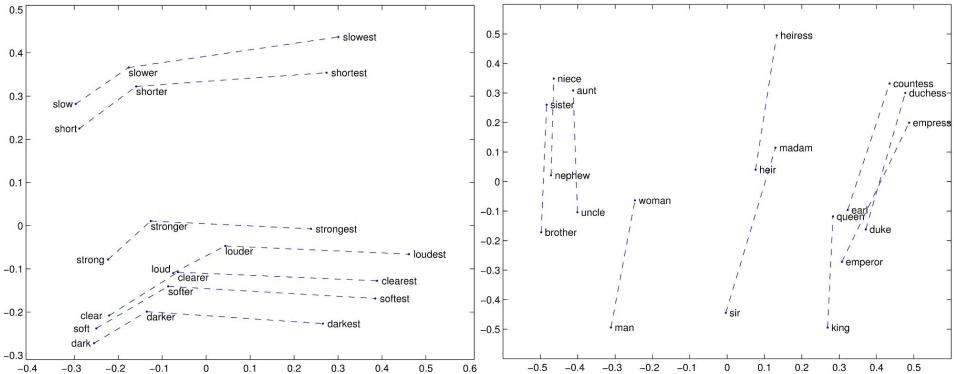


Figure 2: An example of data that approximately *does not have* generic pairwise relations (Pennington et al., 2014).

Subspace-clusterability assumption

Assumption:
There are low-dimensional subspaces of $\mathbb{R}^d$ that are 'rich' with items.

- That is, assume that $\mathcal{X}$ lies on a union of low-rank subspaces.

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- That is, assume that $\mathcal{X}$ lies on a union of low-rank subspaces.
- e.g. $\mathcal{X}$ is *sparsely encodable*, in the sense of dictionary learning.

Divide-and-conquer approach

A natural approach:

1. Learn the metric restricted to each of the item-rich subspaces.
2. Stitch the subspace metrics together.

Subspace Mahalanobis distances

Definition

Let $V \subset \mathbb{R}^d$ be a subspace. A metric on V is a subspace Mahalanobis distance if it is the subspace metric of a Mahalanobis distance ρ on $\mathbb{R}^d$,

$$\rho\big|_V(x, x') = \rho(x, x'), \qquad \forall x, x' \in V.$$

Why can we divide?

Simple case: both items $\mathcal{X}$ and user ideal point u belong to V.

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- It turns out for any $u \in \mathbb{R}^d$, there exists a phantom ideal point $\tilde{u}$ in V such that:

$$\psi_M(x, x'; u) = \psi_M(x, x'; \tilde{u}), \qquad \forall x, x' \in V\,.$$

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- We can no longer recover u, but we can learn $\rho\big|_V$.

Why can we recombine?

After dividing, we end up with a collection of subspace metric:

$$\rho\big|_{V_1}, \ldots, \rho\big|_{V_n}.$$

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After dividing, we end up with a collection of subspace metric:

$$\rho\big|_{V_1}, \ldots, \rho\big|_{V_n}.$$

Result: As long as the subspaces $V_1, \ldots, V_n$ *quadratically span* $\mathbb{R}^d$, there is a unique Mahalanobis distance on $\mathbb{R}^d$ generating the joint subspace metrics.

Geometric proof

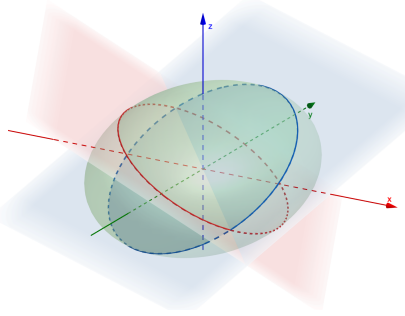

Figure 3: Unit spheres of Mahalanobis distances are ellipsoids in $\mathbb{R}^d$.

Geometric proof

For Mahalanobis distances:

- Metric learning is equivalent to recovering its unit ellipsoid $\mathcal{E}$.

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Fact from geometry:
We can reconstruct an ellipsoid given enough low-dimensional slices.

Quadratic spanning

Definition

*The subspaces $V_1, \ldots, V_n \subset \mathbb{R}^d$ **quadratically span** $\mathbb{R}^d$ if the (linear) span satisfies:*

$$\mathrm{Sym}(\mathbb{R}^d) = \mathrm{span}\left(\left\{xx^\top : x \in V_1 \cup \cdots \cup V_n\right\}\right).$$

Metric learning from lazy crowds (simple math setting)

We asked: Suppose we can obtain very few $m \ll d$ measurements per user. Though ideal points can no longer be learned, is metric learning still possible?

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Answer (continuous response model):

- In general, this is not possible.

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Answer (continuous response model):

- In general, this is not possible.
- If $\mathcal{X}$ is a union of r-dimensional subspaces ($r \ll d$), it is possible with:

number of users	d^2/r
measurements per user	$2r$

General takeaway II

Learning from crowd data

- Fundamental limit overcome using additional structural assumptions
  - e.g. generic pairwise relations $\rightarrow$ subspace-cluster structure

General takeaway II

Learning from crowd data

- Fundamental limit overcome using additional structural assumptions
  - e.g. generic pairwise relations $\rightarrow$ subspace-cluster structure
- These structural assumptions could be (approximately) realistic
  - we could even enforce the structure upsteam
  - e.g. generate representations via dictionary learning

Goals of the rest of the talk

Up to now:

- Fundamental limits of weak and per-user-budgeted crowdsourced data
- Paying for weak feedback if there is additional structure

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- Fundamental limits of weak and per-user-budgeted crowdsourced data
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Rest of the talk:

- High-level description of statistical/learning-theoretic techniques
- A commonly used model for analyzing preference feedback
- A fundamental open question: crowdsourced sensing with latent parameters

Metric learning from non-idealized data

Divide-and-conquer for idealized data

Divide step:
For each subspace $V_1, \ldots, V_n$, solve a system of linear equations:

$$\mathbf{D}_i(\hat{Q}_i, w_1, \ldots, w_K) = \Psi_i.$$

Recombine step:
Define $\Pi(M) = (Q_1, \ldots, Q_n)$ to be the linear map:

$$\Pi : \text{parameters of Mahalanobis distances} \mapsto \text{parameters of subspace metrics.}$$

Solve a system of linear equations:

$$\hat{M} = \Pi(\hat{Q}_1, \ldots, \hat{Q}_n).$$

From linear systems to regression

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Recombine step:
We need to show that we can recombine estimated subspace metrics.

- **Algorithm:** perform linear regression instead, and project onto the PSD cone.

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- Let $\mathcal{V}_n = \{V_1, \ldots, V_n\}$ be a collection of subspaces of $\mathbb{R}^d$.

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- Let $\hat{Q}_1, \ldots, \hat{Q}_n$ be independent estimators of the subspace metrics.
- Let $\hat{M}$ be the projected ordinary least squares solution (on the PSD cone):

$$\hat{M}_{\text{OLS}} = \operatorname*{arg\,min}_{A \in \text{Sym}(\mathbb{R}^d)} \sum_{i=1}^{n} \|\hat{Q}_i - \Pi_{V_i}(A)\|^2$$

$$\hat{M} = \operatorname*{arg\,min}_{A \succeq 0} \|\hat{M}_{\text{OLS}} - A\|_F^2.$$

Recombination recovery guarantee

Assumptions:

- The estimators have low-bias: $\| \mathbb{E}[\hat{Q}_i] - Q_i \| \leq \gamma$.
- The estimators have bounded spread: $\| \hat{Q}_i - \mathbb{E}[\hat{Q}_i] \| \leq \varepsilon$.

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Theorem

There is a constant $c > 0$ such that for any $p \in (0, 1]$, with probability at least $1 - p$,

$$\|\hat{M} - M\|_F \leq c \cdot \frac{1}{\sigma(\mathcal{V}_n)} \cdot \left(\gamma \sqrt{n} + \varepsilon d \sqrt{\log \frac{2d}{p}} \right),$$

where $\sigma(\mathcal{V})$ quantifies the 'quadratic spread' of subspaces $V_1, \ldots, V_n$ in $\mathrm{Sym}(\mathbb{R}^d)$.

Proof sketch

For simplicity, we just show bound for $\|\hat{M}_{\mathrm{OLS}} - M\|_F$.

1. Recall the linear map $\Pi(M) = (Q_1, \ldots, Q_n)$.

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$$\|\hat{M}_{\text{OLS}} - M\|_F^2 = \|\Pi^+(\hat{Q}_1 - Q, \ldots, \hat{Q}_n - Q_n)\|_F^2 \leq \sigma_{\max}^2(\Pi^+) \sum_{i=1}^{n} \|\hat{Q}_1 - Q_i\|^2.$$

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4. A more fine-grained bound by decomposition: $\hat{Q} - Q = \underbrace{\hat{Q} - \mathbb{E}[\hat{Q}]}_{\text{mean-zero r.v.}} + \underbrace{\mathbb{E}[\hat{Q}] - Q}_{\text{bias}}$.

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   - For independent mean-zero error terms, can apply Chernoff-style concentration.

Interpretation of the bound

Key quantities: $n =$ number of subspaces; $\gamma, \varepsilon =$ subspace recovery bias/accuracy

$$\|\hat{M} - M\|_F \leq c \cdot \frac{1}{\sigma(\mathcal{V})} \cdot \left(\gamma\sqrt{n} + \varepsilon d \sqrt{\log \frac{2d}{p}} \right)$$

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- $\sigma(\mathcal{V}_n)$ grows with the number of subspaces,

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As $n \to \infty$, the dominating term is possibly the bias term γ.

- e.g. if the estimators $\hat{Q}$ have a systematic constant biases $\gamma > 0$.

A noisy feedback model with recovery guarantee

Probabilistic model

Generalized linear model:

- Continuous response: (x, x', ψ)

$$\psi \equiv \psi_M(x, x'; u) = D_{x,x'}(M, v).$$

- Noisy response: (x, x', y)

$$\Pr\left[Y = y \,\middle|\, M, x, x', u\right] = f\big(y \cdot D_{x,x'}(M, v)\big),$$

where f is a (non-linear) *link function*.

- The link function is the first (and only) instance of a non-linearity in this work.
- When $f(z) = \frac{1}{1+\exp(-z)}$ is the sigmoid function, this leads to a logistic regression.

Setting for subspace metric recovery

Setting:

- Assume that user provide measurements (x, x', Y) where $Y \in \{-1, +1\}$,

$$\Pr\big[Y = y\big] = f\big(-y \cdot D_{x,x'}(M, v)\big),$$

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- We can perform maximum likelihood estimation:

$$(\hat{M}, \hat{v}_1, \ldots, \hat{v}_k) \leftarrow \operatorname*{arg\,max}_{(A, w_1, \ldots, w_K)} \sum_k \sum_{(x,x',Y)} \log f\big(-Y \cdot D_{x,x'}(M, v_k)\big).$$

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- Assume $\|M\|_\infty \leq 1$ and items and ideal points are contained in unit Euclidean ball.

Analysis via generalization

Theorem (Metric recovery, adapted from Canal et al. (2022))

Let $\mathcal{X}$ quadratically span $\mathbb{R}^d$. There exists designs $D^{(k)}$ asking for m responses from each of K users such that from that data, the maximum likelihood estimator $\hat{M}$ satisfies w.h.p.:

$$\|\hat{M} - M\|_F^2 = \mathcal{O}\left(\sqrt{\frac{d^2 + dK}{mK}}\right).$$

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- Proof uses standard techniques from generalization theory.
- The $d^2 + dK$ term comes from a metric entropy bound on:

$$\{(A, u_1, \ldots, u_K) : \|A\|_\infty \leq 1 \text{ and } \|u_k\| \leq 1, \forall k\}.$$

- When $K \gg d^2$, the dominating term is $\sqrt{d/m}$.

Open question

Weakness of analysis, weakness of naive ERM, or fundamental limit?

- The generalization approach actually shows:

$$\|\hat{M} - M\|_F^2 + \sum_{k=1}^{K} \|\hat{v}_k - v_k\|^2 = \mathcal{O}\left(\sqrt{\frac{d^2 + dK}{mK}}\right).$$

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- But, we only care about learning the parameters of M.
- This analysis does not seem to allow us to decouple estimating $\hat{M}$ and $\hat{v}_k$.
- Is the analysis loose? Is there a better algorithm? Is there a fundamental limit?

Implication for metric learning

Suppose K users provide m measurements on rank-r subspaces.

Subspace metric error:

$$\gamma + \varepsilon \leq \mathcal{O}\left(\sqrt{\frac{r^2 + rK}{mK}}\right).$$

Metric error after recombination:

$$\|\hat{M} - M\|_F \leq c \cdot \frac{1}{\sigma(\mathcal{V})} \cdot \left(\gamma\sqrt{n} + \varepsilon d\sqrt{\log\frac{2d}{p}}\right)$$

When $K \gg d$, then there are settings with: $\|\hat{M} - M\|_F = \mathcal{O}\left(\sqrt{\frac{r}{m}}\right)$.

Additional open problems

Further questions

Other structure:

- Low rank metrics; non-linear representations/kernel extension
- Learning with approximate subspace clusters
- Learning with structured user sets

Inducing structure:

- What are good representations for human/crowdsourced labeling?

Statistics:

- Other noise/preference models (e.g. Bradley-Terry model)
- Semi-parametric estimation
- Robust recovery

Acknowledgments

Collaborators

Zhi Wang
UC San Diego

Ramya Korlakai Vinayak
UW–Madison

Thank you!

See *https://geelon.github.io/* for preprint.

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