Learning when learning is reasonable

Thesis Proposal

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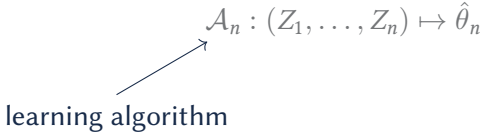
Outline of talk

1. Motivation
2. Online classification
3. Non-worst case sequences
4. Consistency of the nearest neighbor rule
5. Ergodic continuity of nearest neighbor processes
6. Looking forward

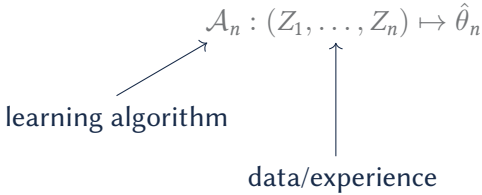
A working definition of learning

$$\mathcal{A}_n : (Z_1, \ldots, Z_n) \mapsto \hat{\theta}_n$$

A working definition of learning



A working definition of learning



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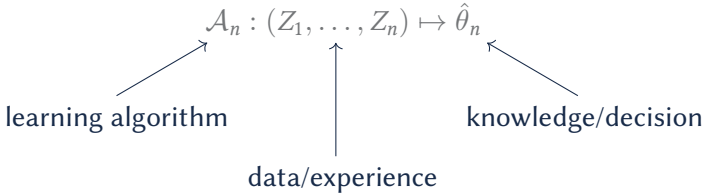

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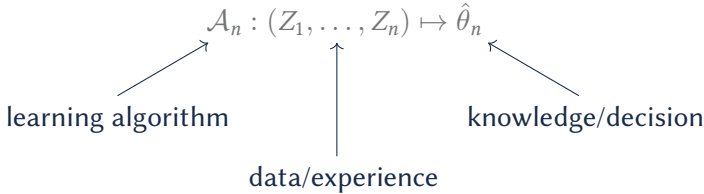

Learning: to derive better knowledge/decisions with more data/experience.

Question. What do past experiences have to do with the current task?

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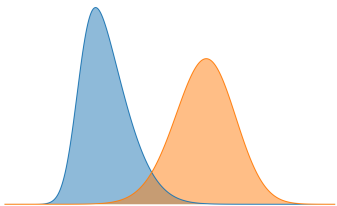
- Different assumptions about this lead to different models of learning.

Two standards of machine learning



Statistical (i.i.d.)

Online (worst-case)

Learning beyond I.I.D. and worst-case settings

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- We introduce general conditions for learning that admit simple analyses.

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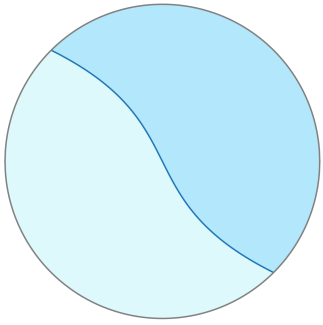
Contributions

- We introduce general conditions for learning that admit simple analyses.
- We show that the nearest neighbor rule is consistent under much broader conditions than previously known.

Online classification

The realizable online setting

Setup. Let $\mathcal{X}$ be an instance space and $\mathcal{Y}$ be a finite label space. Let $\eta : \mathcal{X} \to \mathcal{Y}$ be the target classifier.



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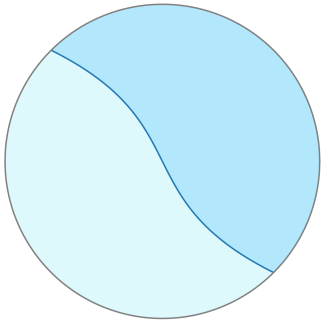
Online classification loop.
For $n = 1, 2, \ldots$

- A test instance X_n is generated.



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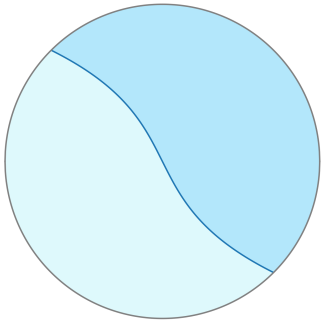

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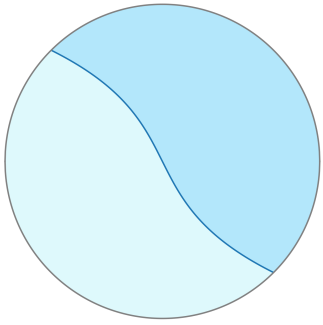
- The answer $Y_n = \eta(X_n)$ is revealed.



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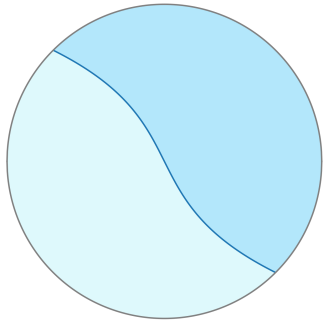

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- A test instance X_n is generated.
- The learner makes prediction $\hat{Y}_n$.
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Consistency of learner: $\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}\{\hat{Y}_n \neq Y_n\} = 0.$

The nearest neighbor rule (Fix and Hodges, 1951)

- Memorize all data points as they come.

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- Predict using the label of the most similar instance in memory.

Nearest neighbor process

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Definition

A nearest neighbor process is a sequence $\tilde{\mathbb{X}} = (\tilde{X}_n)_{n>0}$ satisfying

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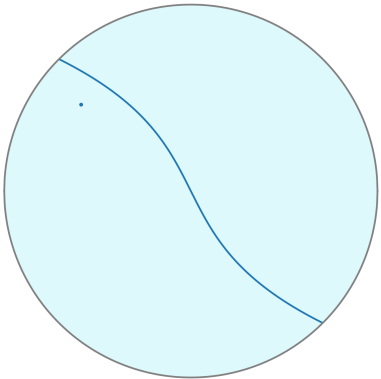
$$\tilde{X}_n = \underset{x \in \mathbb{X}_{<n}}{\arg\min}\ \rho(X_n, x).$$

- The nearest neighbor rule: $\hat{Y}_n = \eta(\tilde{X}_n)$.

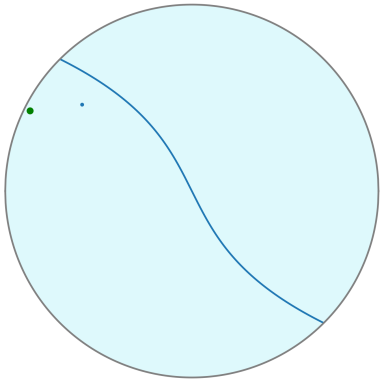
Behavior of the nearest neighbor rule in the i.i.d. setting.

I.I.D. sequence



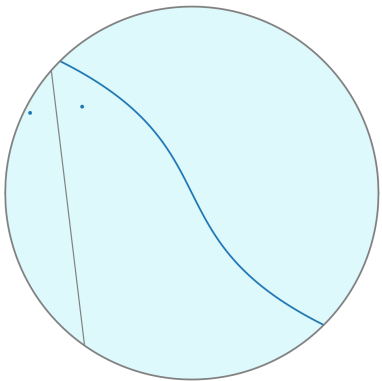
Time	0
Mistake counter	0

I.I.D. sequence



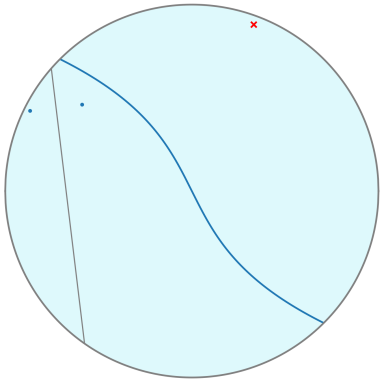
Time	1
Mistake counter	0

I.I.D. sequence



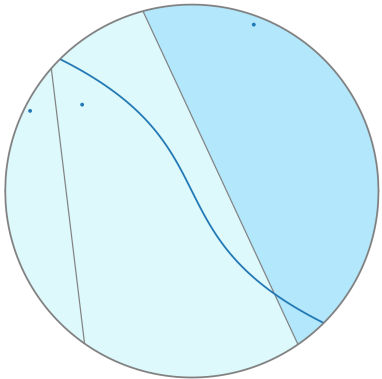
Time	1
Mistake counter	0

I.I.D. sequence



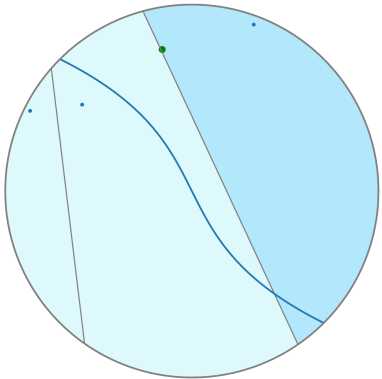
Time	2
Mistake counter	1

I.I.D. sequence



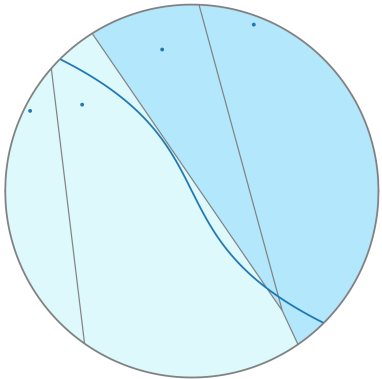
Time	2
Mistake counter	1

I.I.D. sequence



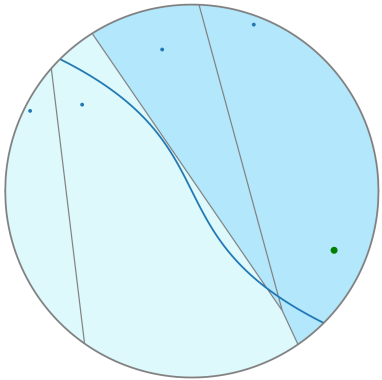
Time	3
Mistake counter	1

I.I.D. sequence



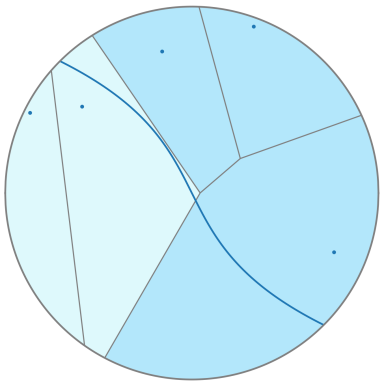
Time	3
Mistake counter	1

I.I.D. sequence



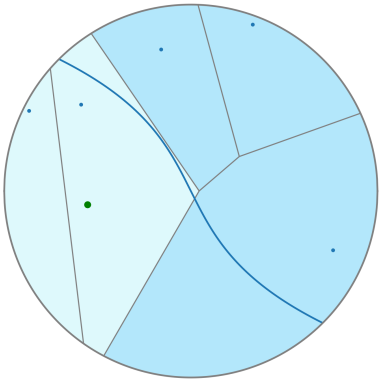
Time	4
Mistake counter	1

I.I.D. sequence



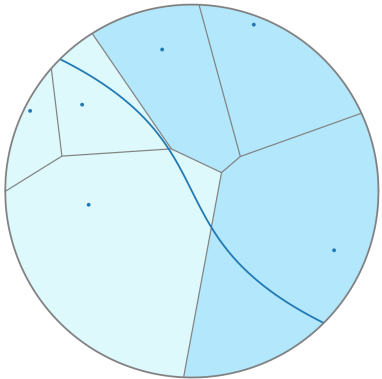
Time	4
Mistake counter	1

I.I.D. sequence



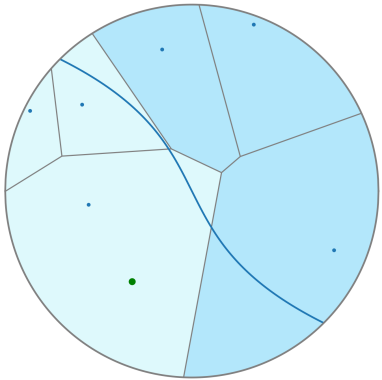
Time	5
Mistake counter	1

I.I.D. sequence



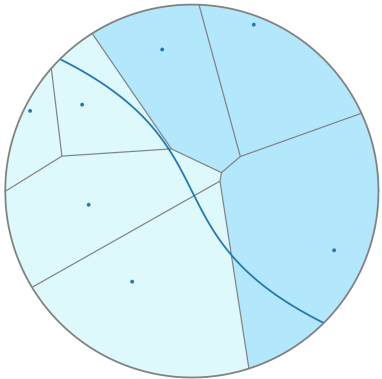
Time	5
Mistake counter	1

I.I.D. sequence



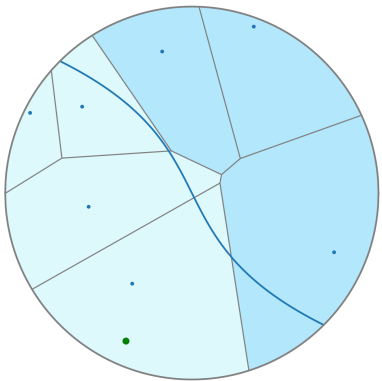
Time	6
Mistake counter	1

I.I.D. sequence



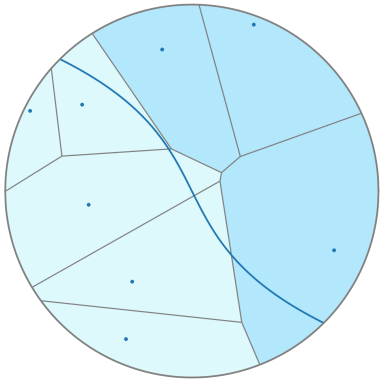
Time	6
Mistake counter	1

I.I.D. sequence



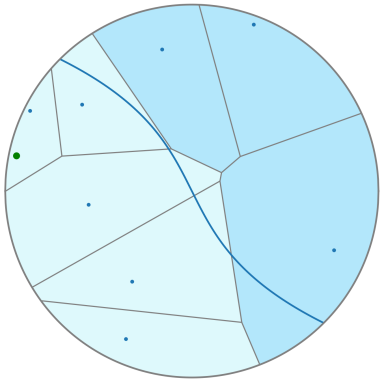
Time	7
Mistake counter	1

I.I.D. sequence



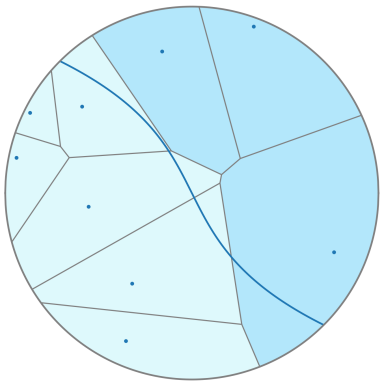
Time	7
Mistake counter	1

I.I.D. sequence



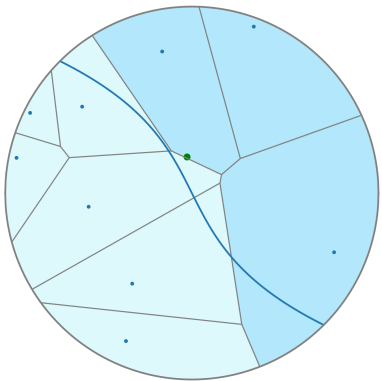
Time	8
Mistake counter	1

I.I.D. sequence



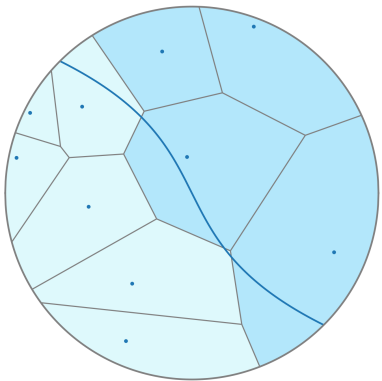
Time	8
Mistake counter	1

I.I.D. sequence



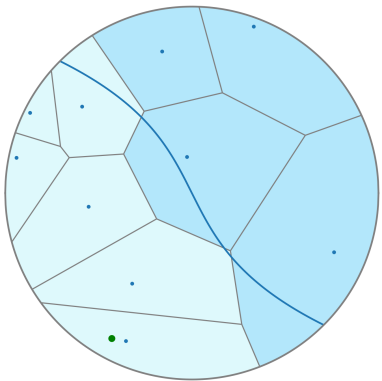
Time	9
Mistake counter	1

I.I.D. sequence



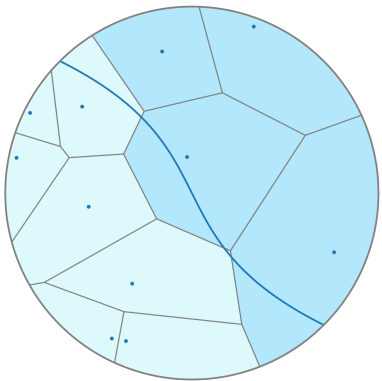
Time	9
Mistake counter	1

I.I.D. sequence



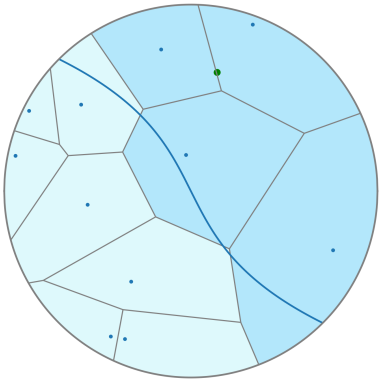
Time	10
Mistake counter	1

I.I.D. sequence



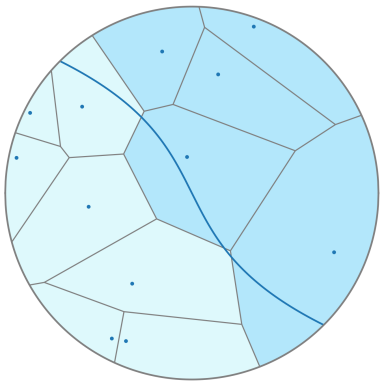
Time	10
Mistake counter	1

I.I.D. sequence



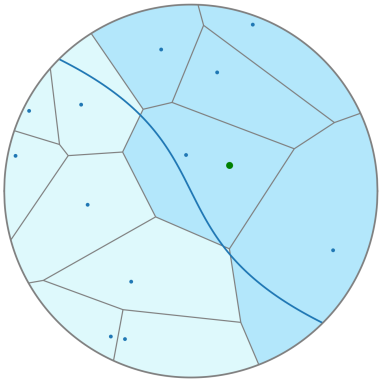
Time	11
Mistake counter	1

I.I.D. sequence



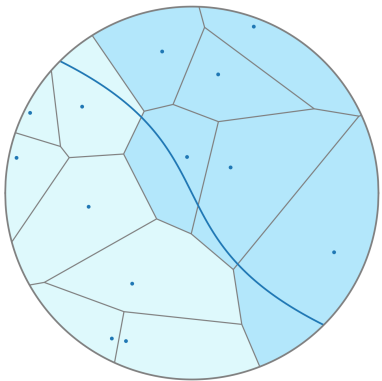
Time	11
Mistake counter	1

I.I.D. sequence



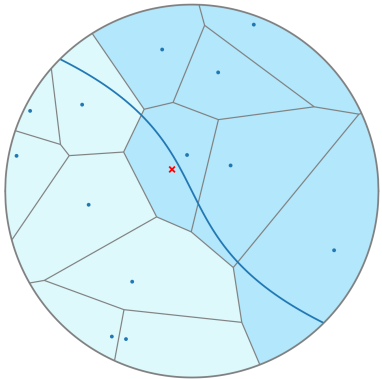
Time	12
Mistake counter	1

I.I.D. sequence



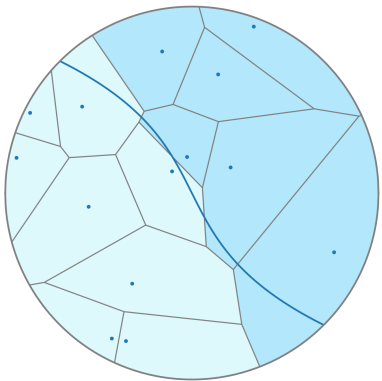
Time	12
Mistake counter	1

I.I.D. sequence



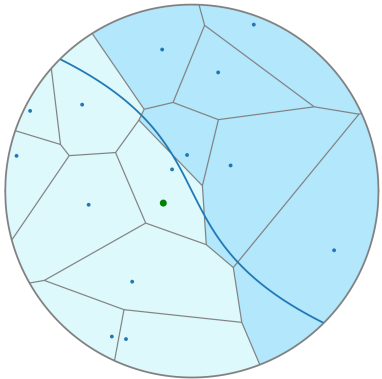
Time	13
Mistake counter	2

I.I.D. sequence



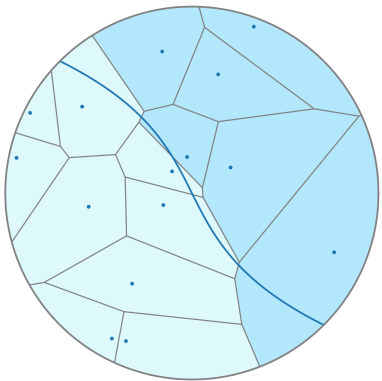
Time	13
Mistake counter	2

I.I.D. sequence



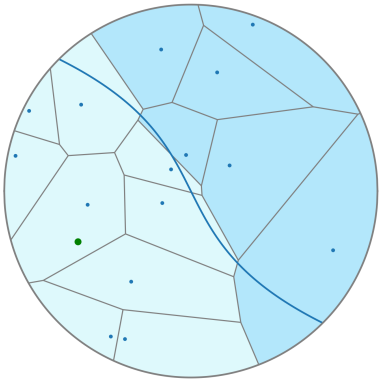
Time	14
Mistake counter	2

I.I.D. sequence



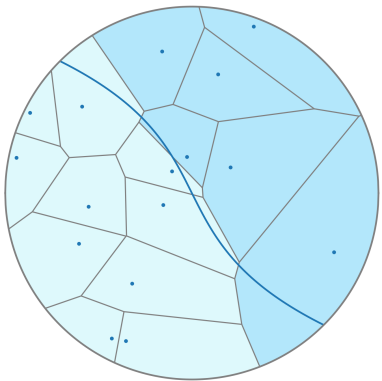
Time	14
Mistake counter	2

I.I.D. sequence



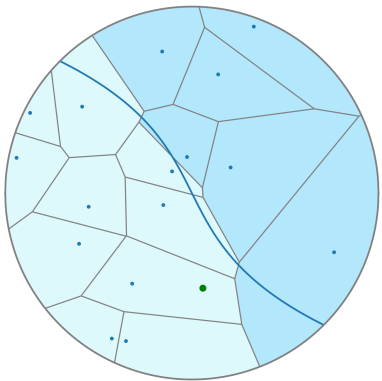
Time	15
Mistake counter	2

I.I.D. sequence



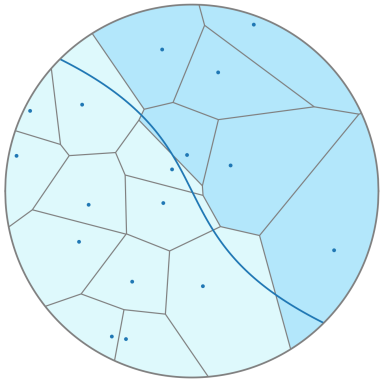
Time	15
Mistake counter	2

I.I.D. sequence



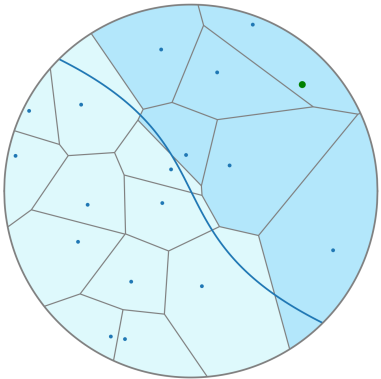
Time	16
Mistake counter	2

I.I.D. sequence



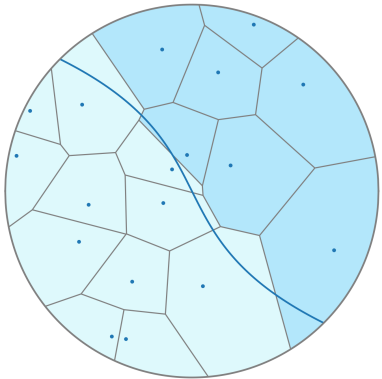
Time	16
Mistake counter	2

I.I.D. sequence



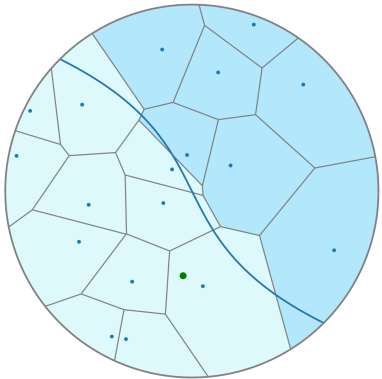
Time	17
Mistake counter	2

I.I.D. sequence



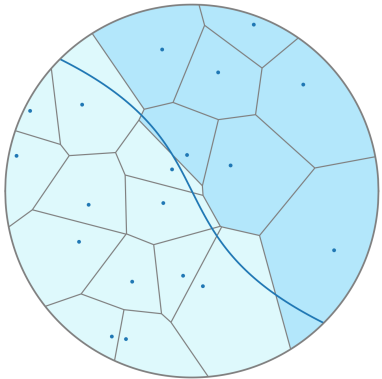
Time	17
Mistake counter	2

I.I.D. sequence



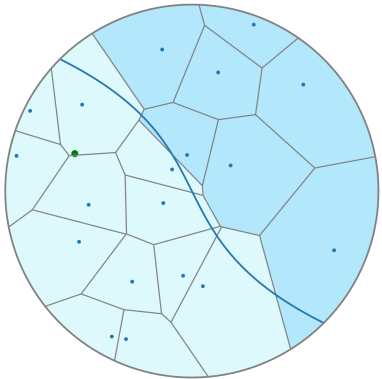
Time	18
Mistake counter	2

I.I.D. sequence



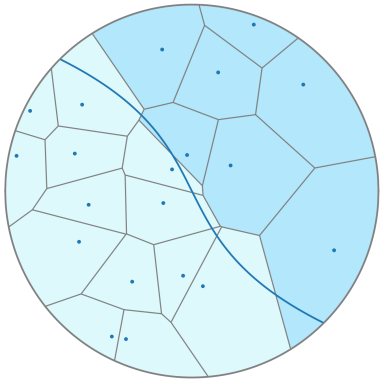
Time	18
Mistake counter	2

I.I.D. sequence



Time	19
Mistake counter	2

I.I.D. sequence

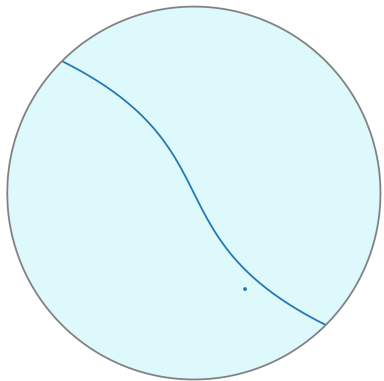


Time	19
Mistake counter	2

Behavior of the nearest neighbor rule in the worst-case setting.

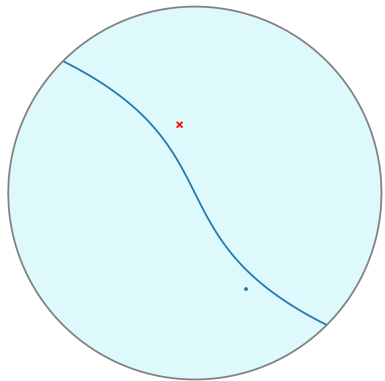
Worst-case sequence

Time	0
Mistake counter	0



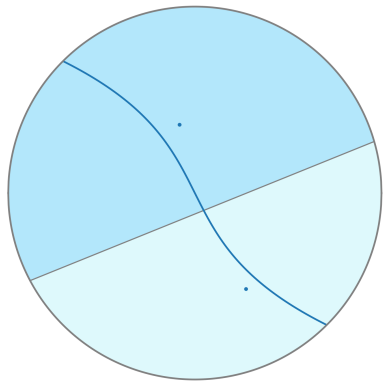
Worst-case sequence

Time	1
Mistake counter	1



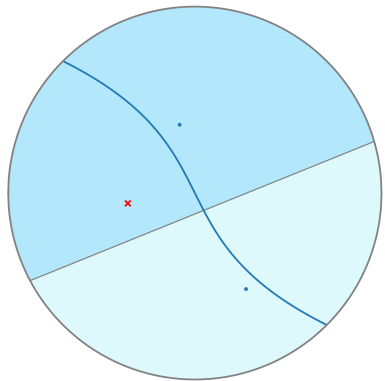
Worst-case sequence

Time	1
Mistake counter	1



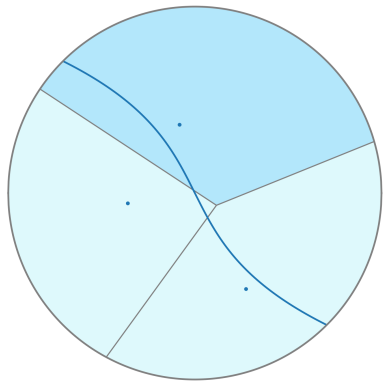
Worst-case sequence

Time	2
Mistake counter	2



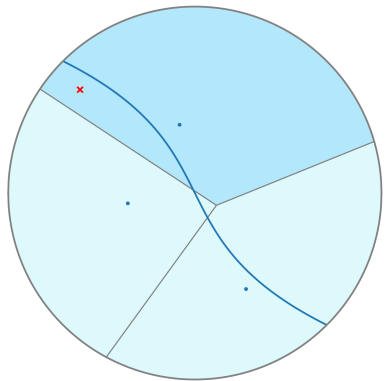
Worst-case sequence

Time	2
Mistake counter	2



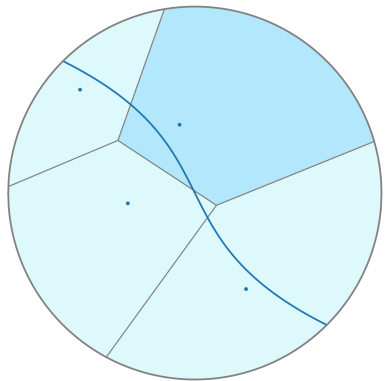
Worst-case sequence

Time	3
Mistake counter	3



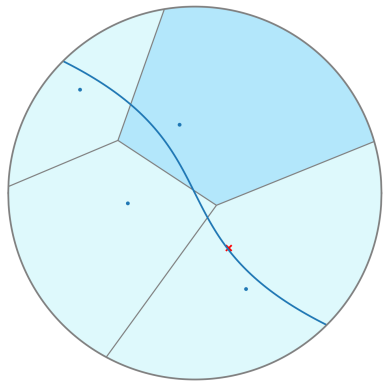
Worst-case sequence

Time	3
Mistake counter	3



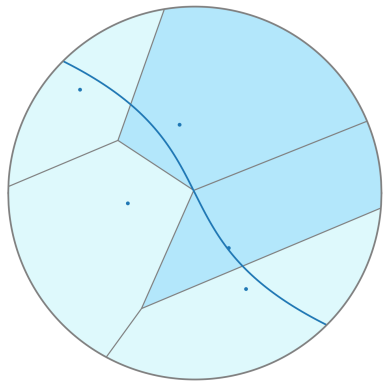
Worst-case sequence

Time	4
Mistake counter	4



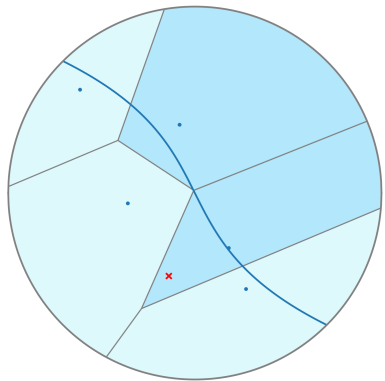
Worst-case sequence

Time	4
Mistake counter	4



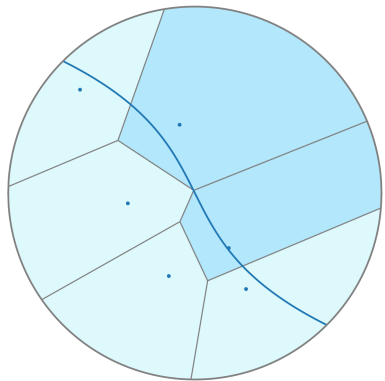
Worst-case sequence

Time	5
Mistake counter	5



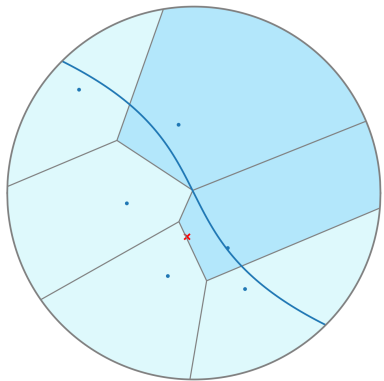
Worst-case sequence

Time	5
Mistake counter	5



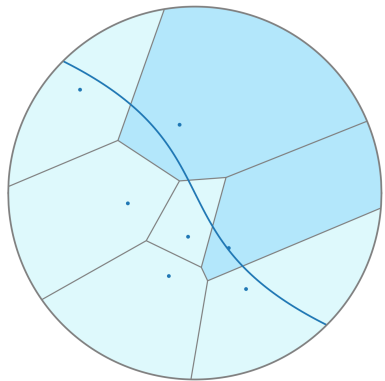
Worst-case sequence

Time	6
Mistake counter	6



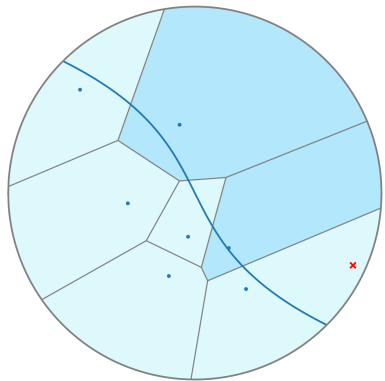
Worst-case sequence

Time	6
Mistake counter	6



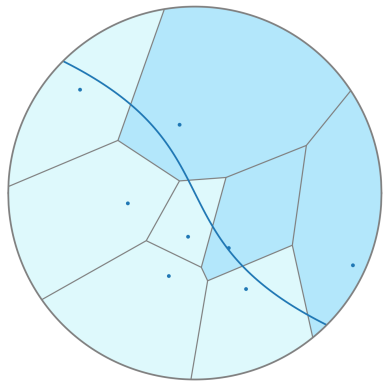
Worst-case sequence

Time	7
Mistake counter	7



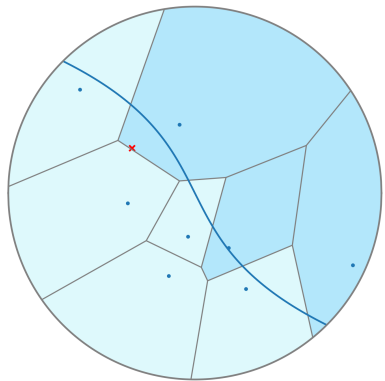
Worst-case sequence

Time	7
Mistake counter	7



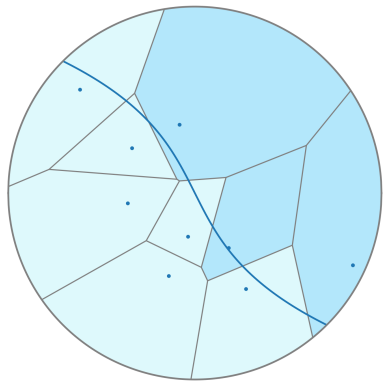
Worst-case sequence

Time	8
Mistake counter	8



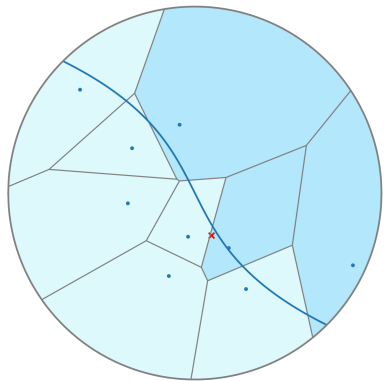
Worst-case sequence

Time	8
Mistake counter	8



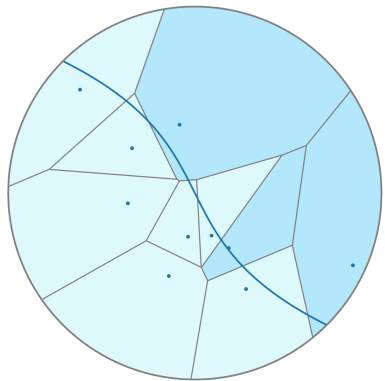
Worst-case sequence

Time	9
Mistake counter	9



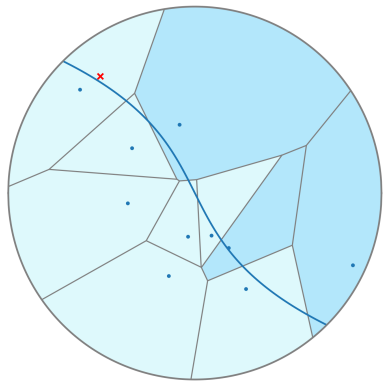
Worst-case sequence

Time	9
Mistake counter	9



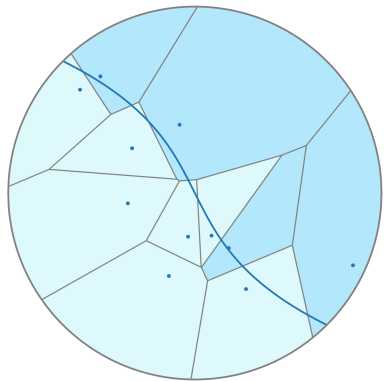
Worst-case sequence

Time	10
Mistake counter	10



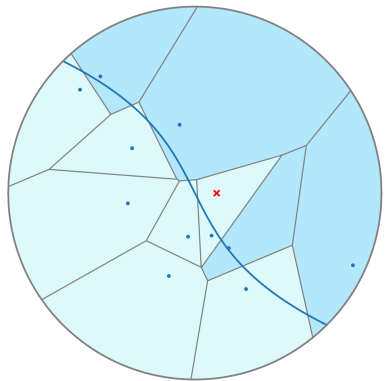
Worst-case sequence

Time	10
Mistake counter	10



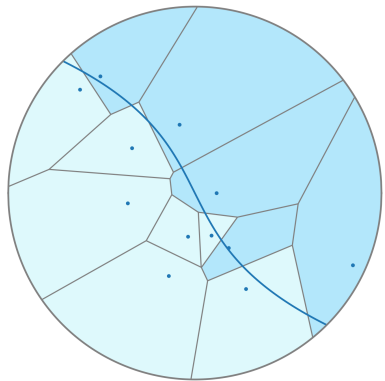
Worst-case sequence

Time	11
Mistake counter	11



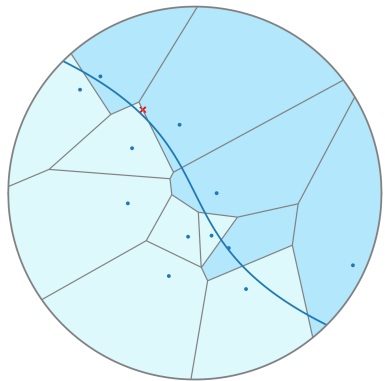
Worst-case sequence

Time	11
Mistake counter	11



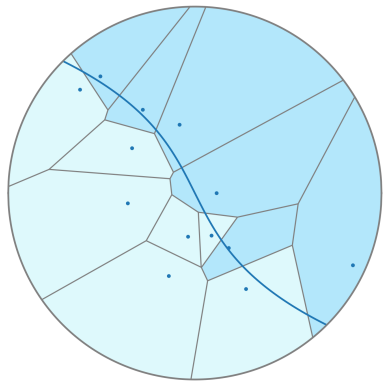
Worst-case sequence

Time	12
Mistake counter	12



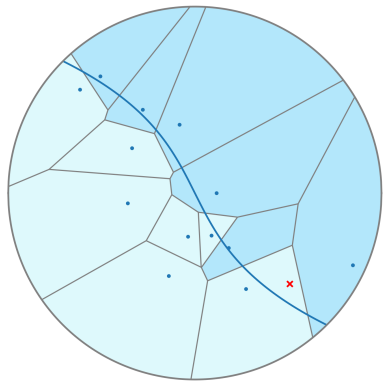
Worst-case sequence

Time	12
Mistake counter	12



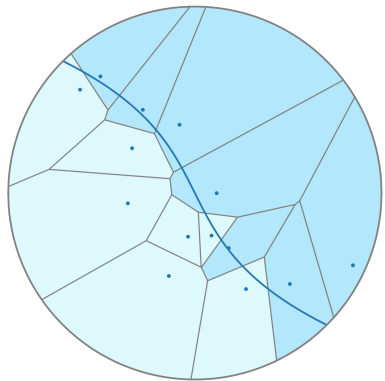
Worst-case sequence

Time	13
Mistake counter	13



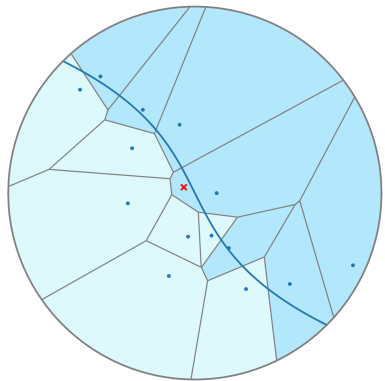
Worst-case sequence

Time	13
Mistake counter	13



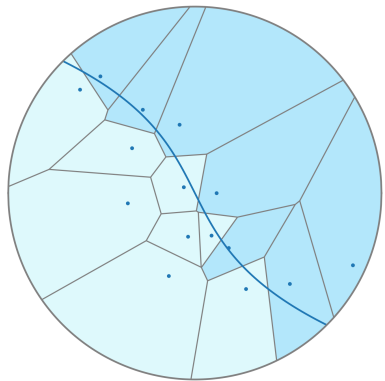
Worst-case sequence

Time	14
Mistake counter	14



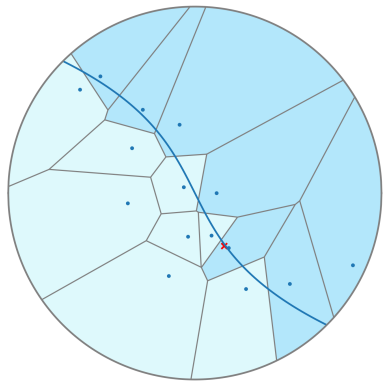
Worst-case sequence

Time	14
Mistake counter	14



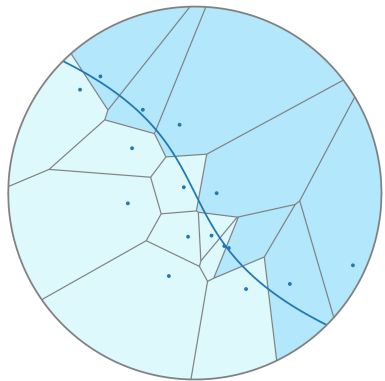
Worst-case sequence

Time	15
Mistake counter	15



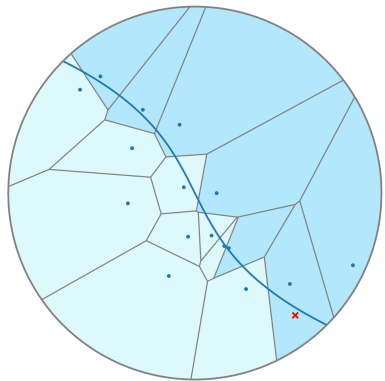
Worst-case sequence

Time	15
Mistake counter	15



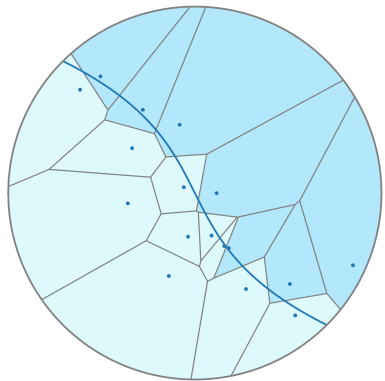
Worst-case sequence

Time	16
Mistake counter	16



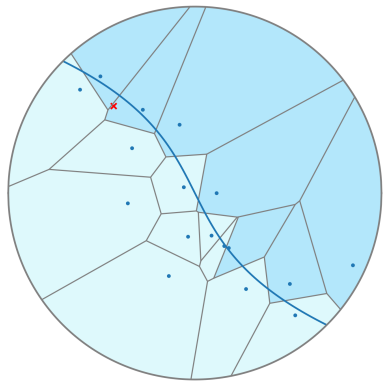
Worst-case sequence

Time	16
Mistake counter	16



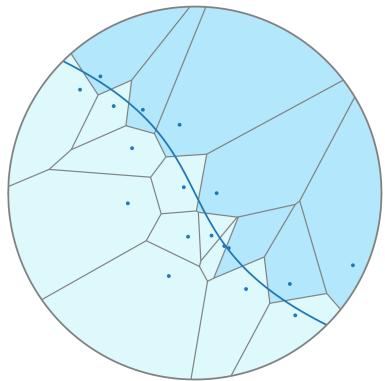
Worst-case sequence

Time	17
Mistake counter	17



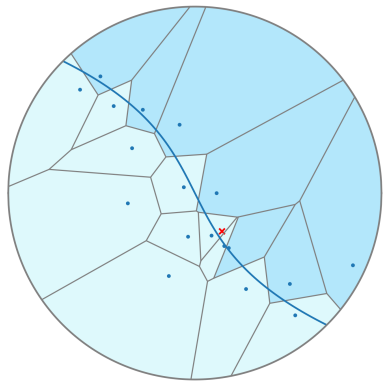
Worst-case sequence

Time	17
Mistake counter	17



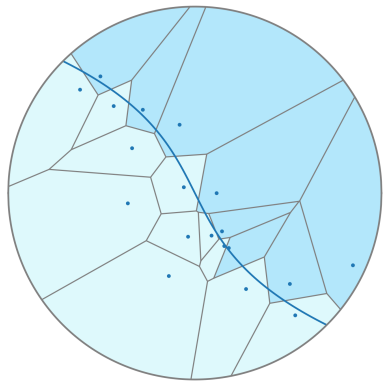
Worst-case sequence

Time	18
Mistake counter	18



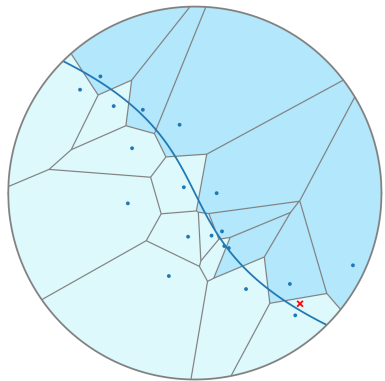
Worst-case sequence

Time	18
Mistake counter	18



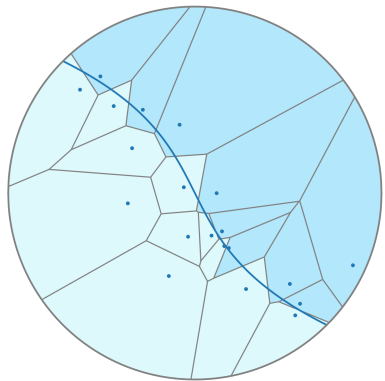
Worst-case sequence

Time	19
Mistake counter	19



Worst-case sequence

Time	19
Mistake counter	19



Question. When is the nearest neighbor rule consistent in the worst case?

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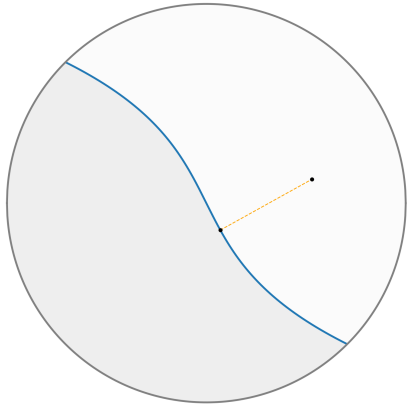
Answer. When different classes have positive separation.

Classification margin

Definition

The margin of x with respect to η is given by:

$$\text{margin}_\eta(x) = \inf_{\eta(x) \neq \eta(x')} \rho(x, x').$$



Worst-case guarantee for the nearest neighbor rule

Let $(\mathcal{X}, \rho)$ be a totally bounded metric space and $\eta : \mathcal{X} \to \mathcal{Y}$.

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Worst-case guarantee for the nearest neighbor rule

($\Longleftarrow$) By contradiction.

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- Asymptotic mistake rate is at least $1/2$.

A general class of non-worst case sequences

Excluding the bad cases

Mode of failure.
Provided the learner has not exactly recovered η, a worst-case adversary can always select a point from the mistake region, no matter how small.

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Question.
Can we rule out adversaries that can generate points with arbitrary precision?

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Setup. Let $\mathcal{X}$ be a measurable instance space and $\mathcal{Y}$ be a finite label space. Let $\eta : \mathcal{X} \to \mathcal{Y}$ be the target classifier.

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For $n = 1, 2, \ldots$

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- We can constrain μ_n with respect to ν (e.g. $\mu_n \ll \nu$).
- Intuitively, it should be hard to generate points from regions with small ν-mass.

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Intuition

If $A \subset \mathcal{X}$ has small ν-mass, then the generating process for $\mathbb{X}$ cannot select instances from A with large probability at any point in time.

Arguments via uniform absolute continuity

Proposition

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- *Uniform absolute continuity: it is hard to generate these bad points.*
- *This proposition: on average, we will not see these points very often.*

Consistency for functions with negligible boundaries

Observation.
The nearest neighbor rule works well away from the decision boundary.

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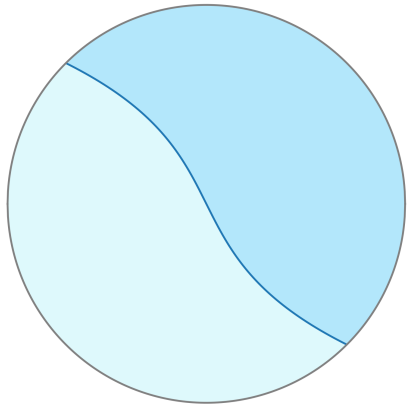
This section.
The nearest neighbor rule has no problem learning functions when boundary points have ν-measure zero and when $\mathbb{X}$ is uniformly dominated by ν.

Functions with negligible boundaries

Definition

Let $\eta : \mathcal{X} \to \mathcal{Y}$. The boundary of η is the set:

$$\partial_\eta \mathcal{X} = \{x \in \mathcal{X} : \text{margin}_\eta(x) = 0\}.$$



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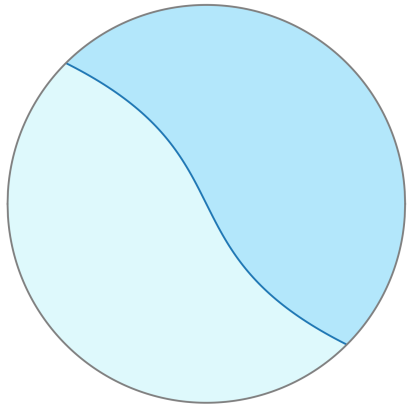
Denote functions with negligible boundaries:

$$\mathcal{F}_0 = \{\eta \text{ measurable} : \nu(\partial_\eta \mathcal{X}) = 0\}.$$



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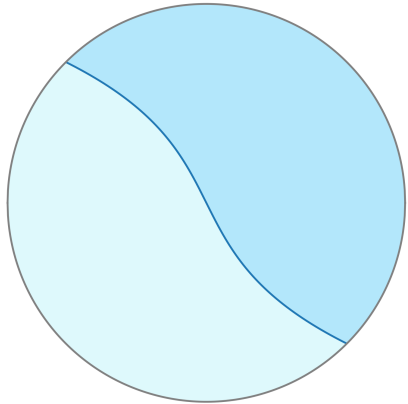
Example

Let $\eta : \mathbb{R}^d \to \{0, 1\}$ have a smooth decision boundary, which is a $(d-1)$-dimensional object in $\mathbb{R}^d$, and thus has zero Lebesgue measure.



Consistency for $\mathcal{F}_0$

Theorem

Let $(\mathcal{X}, \rho, \nu)$ be a space where ρ is a separable metric and ν is a finite Borel measure.

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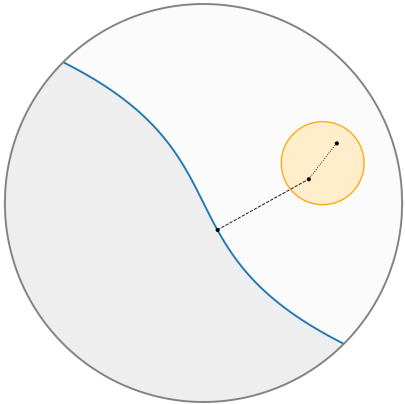
$$\underbrace{\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}\{\eta(X_n) \neq \eta(\tilde{X}_n)\} = 0}_{\text{the nearest neighbor rule is online consistent for } (\mathbb{X}, \eta).} \qquad \text{a.s.}$$

Mutually-labeling set

Definition

*A set $U \subset \mathcal{X}$ is **mutually-labeling** for η when:*

$$\operatorname{diam}(U) < \operatorname{margin}_\eta(x), \qquad \forall x \in U.$$



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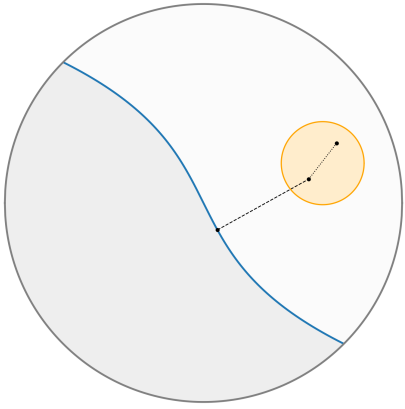
Proposition

When x is not a boundary point, the ball $B(x, r)$ is mutually labeling when:

$$r < \operatorname{margin}_\eta(x)/3.$$



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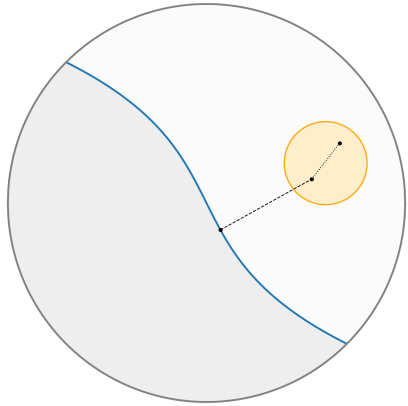
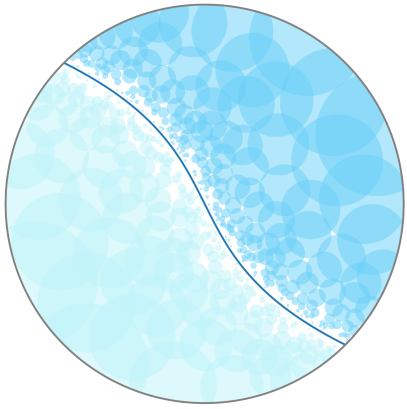
Proposition

Let U be a mutually-labeling set for η. The nearest neighbor rule makes at most one mistake on U,

$$\sum_{n=1}^{\infty} \mathbb{1}\{\eta(X_n) \neq \eta(\tilde{X}_n) \wedge X_n \in U\} \leq 1.$$



Proof by a δ-approximate mutually-labeling cover



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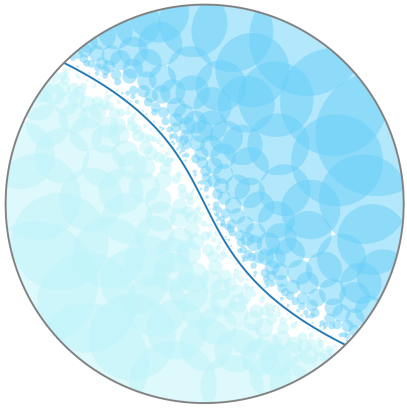

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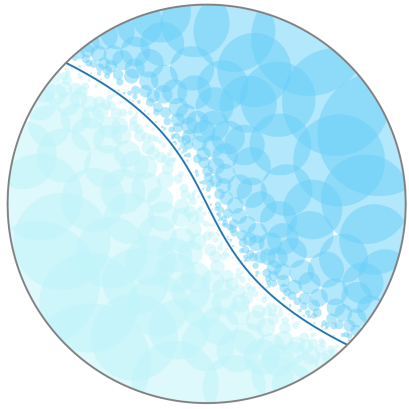

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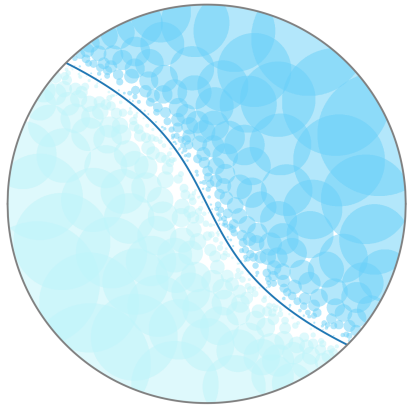

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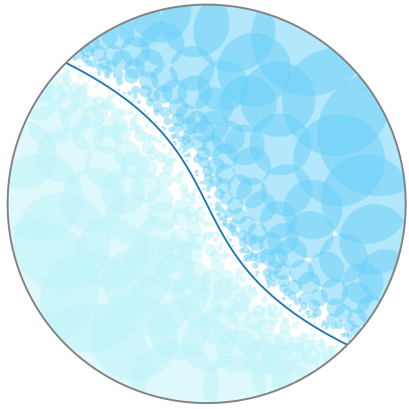

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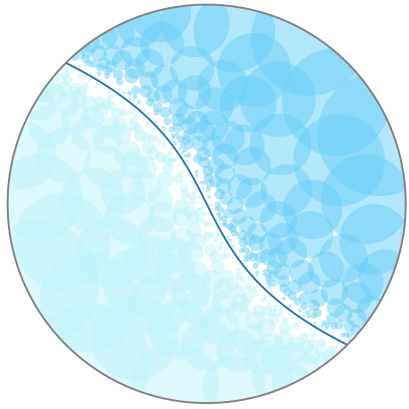

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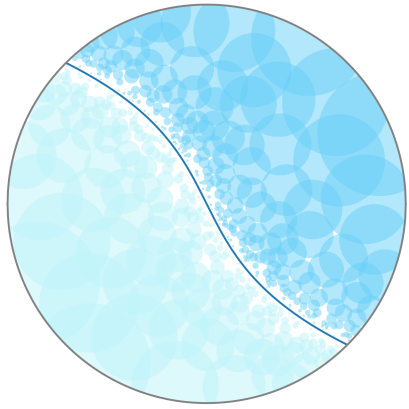

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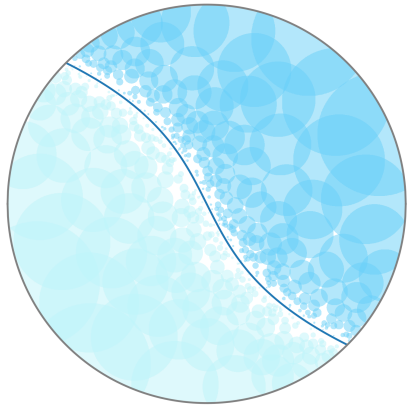

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Consistency on upper doubling spaces

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In Euclidean space, simple functions are dense in L^1.

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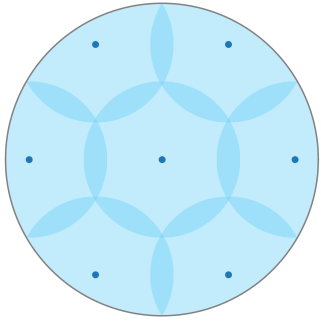
It turns out that $\mathcal{F}_0$ is dense in L^1 for nice metric measure spaces.

The nearest neighbor rule behaves similarly for similar functions $\eta \approx \eta'$ when $\mathbb{X}$ is uniformly dominated in a geometrically nice space.

Upper doubling spaces

Definition (Hytönen (2010))

A metric measure space $(\mathcal{X}, \rho, \nu)$ is upper doubling when:



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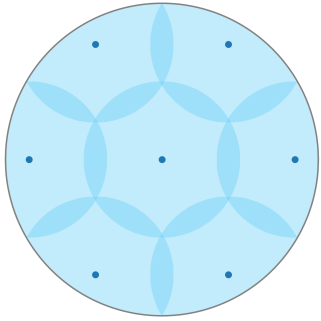

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A metric measure space $(\mathcal{X}, \rho, \nu)$ is upper doubling when:

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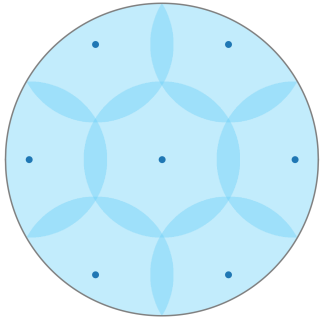
$$\mathcal{N}_{r/2}\big(B(x,r)\big) \leq 2^d.$$



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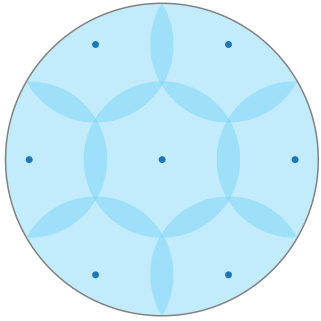

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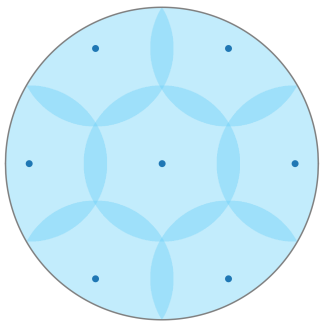
2. *The measure ν is upper doubling. That is, there exists a constant $c > 0$ where:*

$$\nu\big(B(x,r)\big) < cr^d.$$



$\mathcal{F}_0$ is dense in $L^1(\mathcal{X}; \nu)$

Let $\mathcal{X}$ be equipped with a doubling metric and a finite measure.

Question. Can we approximate η arbitrarily well by $\eta_0 \in \mathcal{F}_0$? Yes.

Lebesgue differentiation theorem. For almost every $x \in \mathcal{X}$, we can find arbitrarily small balls $B(x, r)$ such that η is nearly constant on $B(x, r)$.

Vitali covering theorem. There is a countable disjoint subcollection of such balls $B(x_1, r_1), B(x_2, r_2), \ldots$ that cover almost all of $\mathcal{X}$.

Construction. Define $\eta_0 = \sum_{i=1}^{\infty} \eta(x_i) \cdot \mathbb{1}_{B(x_i, r_i)}$, where $B(x_i, r_i) \cap \partial_{\eta_0} \mathcal{X} = \varnothing$.

Consistency for all measurable functions

Theorem

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Let $(\mathcal{X}, \rho, \nu)$ be upper doubling, where ρ is separable and ν is finite. Let η be measurable. Suppose that $\mathbb{X}$ is uniformly dominated by ν. Then:

$$\underbrace{\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}\{\eta(X_n) \neq \eta(\tilde{X}_n)\} = 0}_{\text{the nearest neighbor rule is online consistent for } (\mathbb{X}, \eta).} \qquad \text{a.s.}$$

Proof strategy

1. Approximate η by $\eta_0 \in \mathcal{F}_0$ such that:

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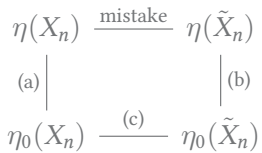

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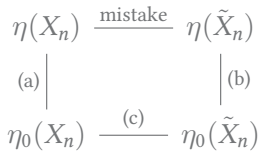

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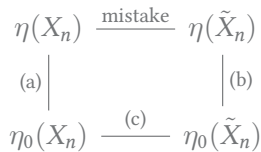

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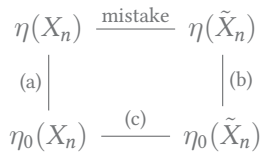

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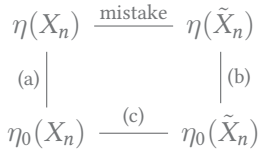

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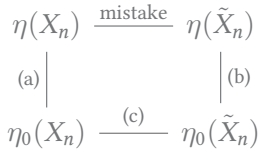

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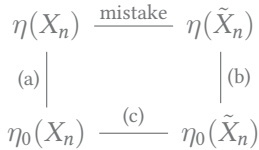

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$\tilde{\mathbb{X}}$ is ergodically dominated, so we can control this rate as well.

By consistency on $\mathcal{F}_0$, the rate for this term vanishes.

Ergodic continuity of nearest neighbor processes

Influence filtered through the nearest neighbor process

Question. For any $A \subset \mathcal{X}$, what is the long-term influence of points in $\mathbb{X} \cap A$ on $\tilde{\mathbb{X}}$?

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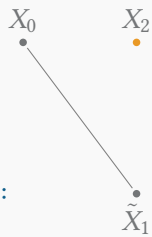

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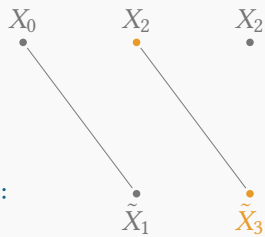

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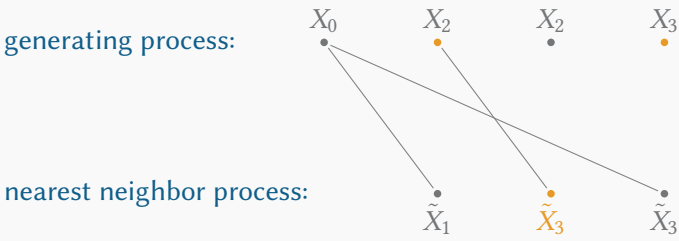

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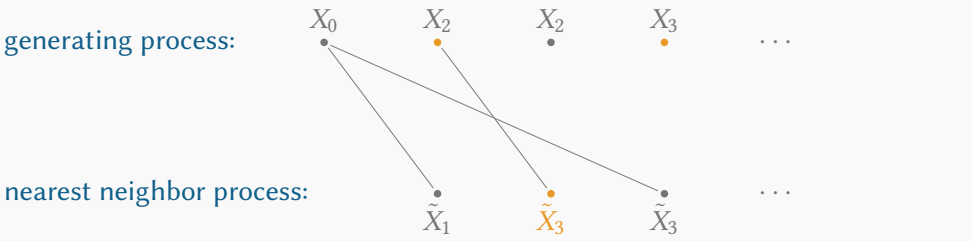

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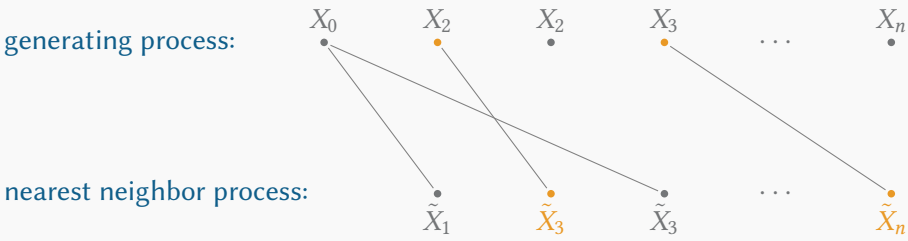

Two opposing dynamics

Goal. Upper bound the asymptotic rate that $\tilde{\mathbb{X}}$ hits A:

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Upshot. These two dynamics balance out when $\mathbb{X}$ is a uniformly dominated process in an upper doubling space.

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- $(\mathcal{X}, \rho)$ is any metric space and $U \subset \mathcal{X}$

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Lemma

The number of times that $\mathbb{X}$ *hits* U *while having nearest neighbor distance at least* r *is:*

$$\sum_{n=1}^{\infty} \mathbb{1}\{ X_n \in U \ \wedge \ \rho(X_n, \tilde{X}_n) \geq r \} \leq \mathcal{P}_r(U).$$

Warm-up: diminishing influence of a single point

Proposition

Let $\mathcal{X}$ be a bounded doubling space. Let $\mathbb{X}$ be any process.

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- The second term can be controlled when the space is *upper* doubling:

$$\nu(B_0) = 2^{-\Omega(k)}.$$

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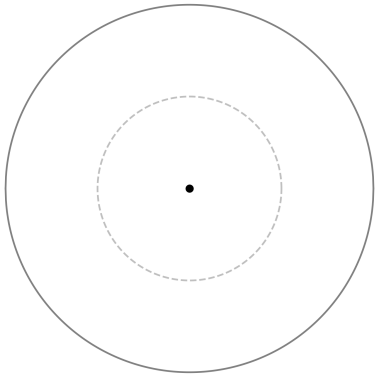
X_0 is the nearest neighbor:

1. at scale $[R/2, R)$ no more than 2^d times.



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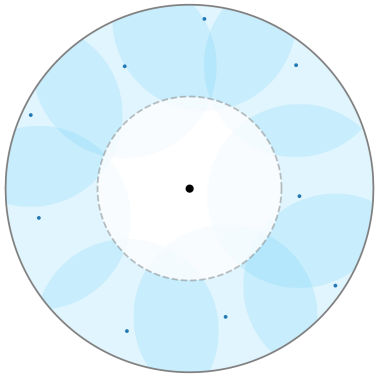

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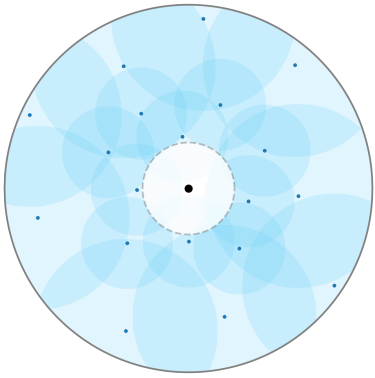

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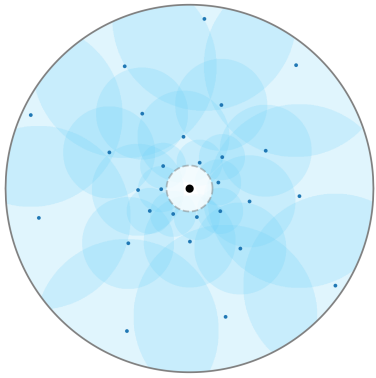

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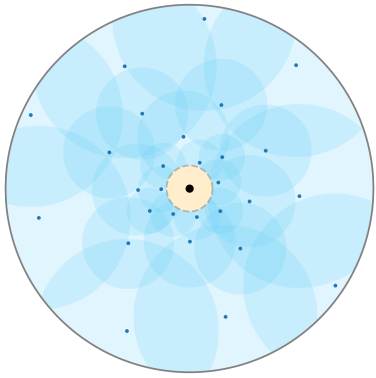
If $\mathcal{X}$ is upper doubling and $\mathbb{X}$ is uniformly dominated, then X_0 is the nearest neighbor:

k. at scale $< R \cdot 2^{-k}$ at a rate $2^{-\Omega(k)}$.

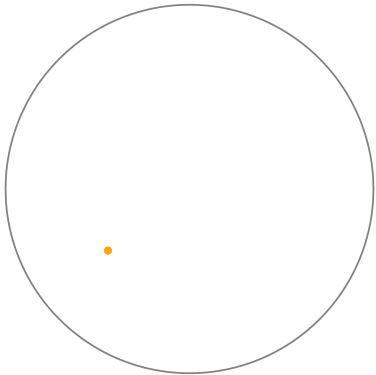


Bounding the long-term influence of a subsequence

Generating process:

X_0 X_1 X_2 X_3 $\cdots$ X_n

- At time $n = 1$, we observe $X_1 \in A$.



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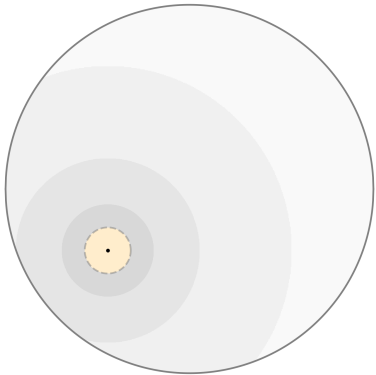

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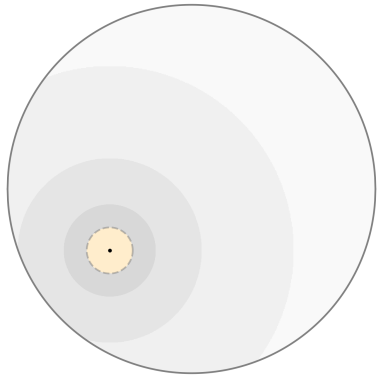
Packing Argument.

1. For each ring of outer radius r, the event:

$$\{X_n, \tilde{X}_n \in B(X_1, r) \wedge \rho(X_n, \tilde{X}_n) > r/2\},$$

happens no more than $O(2^d)$ times.



Bounding the long-term influence of a subsequence

Generating process:

X_0 X_1 X_2 X_3 $\dots$ X_n

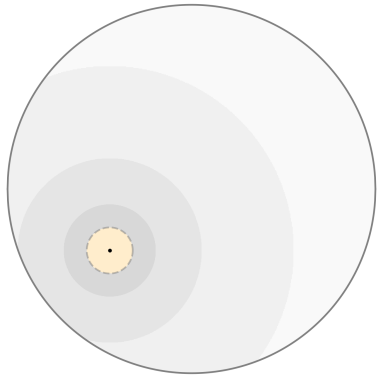
- At time $n = 1$, we observe $X_1 \in A$.
- We can pay upfront for part of $\{\tilde{X}_n = X_1\}$.

Packing Argument.

1. For each ring of outer radius r, the event:

$$\{X_n, \tilde{X}_n \in B(X_1, r) \wedge \rho(X_n, \tilde{X}_n) > r/2\},$$

   happens no more than $O(2^d)$ times.
2. Remainder contained in $\{X_n \in B(X_1, r/2)\}$.



Bounding the long-term influence of a subsequence

Generating process:

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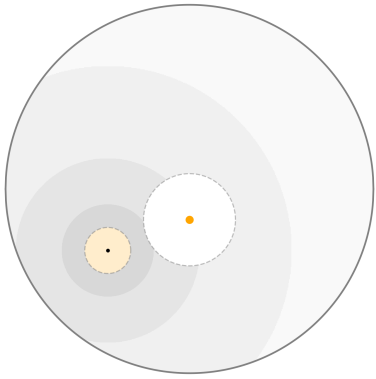
- At $n = 3$, we observe $X_3 \in A$.



Bounding the long-term influence of a subsequence

Generating process:

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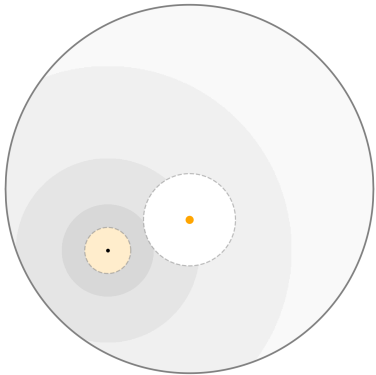

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- At $n = 3$, we observe $X_3 \in A$.
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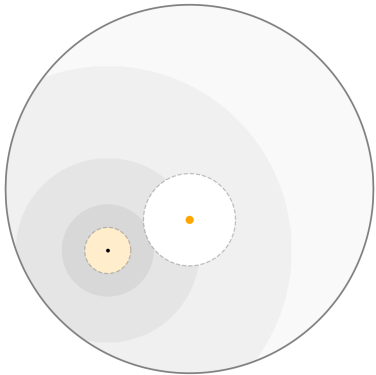

Bounding the long-term influence of a subsequence

Generating process:

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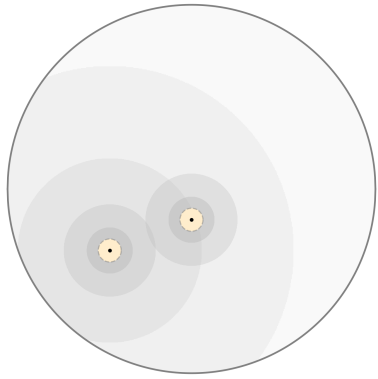
- At $n = 3$, we observe $X_3 \in A$.
- We need to account for the event $\{\tilde{X}_n = X_3\}$.
- Part of this event is already accounted for.
- Pay for more scales for both X_3 and parent(X_3).



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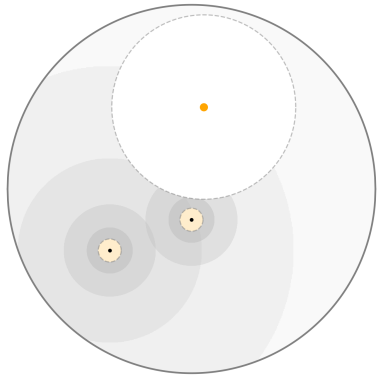
- Part of this event is already accounted for.
- Pay for more scales for both X_3 and $\text{parent}(X_3)$.
- Induct on each X_n that comes from A,



Bounding the long-term influence of a subsequence

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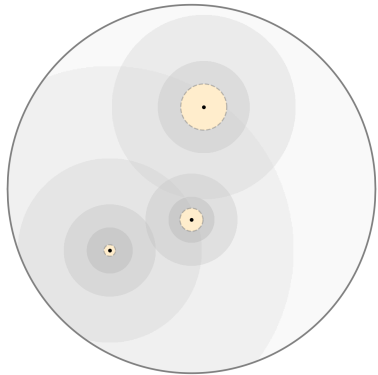
- Part of this event is already accounted for.
- Pay for more scales for both X_3 and $\text{parent}(X_3)$.
- Induct on each X_n that comes from A, ensuring:

$$\nu\big(\text{remainder region}\big) < \delta.$$



Bounding the long-term influence of a subsequence

By this argument, at every point in time N,

$$\sum_{n=1}^{N} \mathbb{1}\{\tilde{X}_n \in A\} \leq O\left(\left|\mathbb{X}_{<N} \cap A\right|\right) + \sum_{n=1}^{N} \mathbb{1}\{X_n \in R_n\},$$

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where R_n is the sequence of remainder regions with $\nu(R_n) < \delta$.

- The first term grows at rate depending on $O\left(\varepsilon(\delta) \log \frac{1}{\delta}\right)$.
- The second term grows at a rate bounded by $\varepsilon(\delta)$.

Ergodic continuity of the nearest neighbor process

Theorem

Let $(\mathcal{X}, \rho, \nu)$ be bounded and upper doubling.

Ergodic continuity of the nearest neighbor process

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Ergodic continuity of the nearest neighbor process

Theorem

Let $(\mathcal{X}, \rho, \nu)$ be bounded and upper doubling. Suppose $\mathbb{X}$ is uniformly dominated by ν at a rate of $\varepsilon(\delta)$. Then, $\tilde{\mathbb{X}}$ is ergodically dominated by ν at a slower rate of $O\big(\varepsilon(\delta)\log\frac{1}{\delta}\big)$.

Ergodic continuity

Definition

A stochastic process $\mathbb{X} = (X_n)_n$ *is* ***ergodically dominated*** *by* ν

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We say that $\mathbb{X}$ is ergodically continuous with respect to ν at rate $\varepsilon(\delta)$.

Intuition

If $A \subset \mathcal{X}$ has small ν-mass, then the generating process for $\mathbb{X}$ cannot have selected instances from A very often in retrospect.

Uniform absolute continuity vs. ergodic continuity

Setting. Let $A \subset \mathcal{X}$ have measure $\nu(A) < \delta$.

Uniform absolute continuity vs. ergodic continuity

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$$\underbrace{\sup_{n \in \mathbb{N}} \Pr\left(X_n \in A\right) < \varepsilon}_{\text{time-uniform guarantee}}$$

Uniform absolute continuity vs. ergodic continuity

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Uniform absolute continuity

$$\underbrace{\sup_{n \in \mathbb{N}} \Pr\left(X_n \in A\right) < \varepsilon}_{\text{time-uniform guarantee}} \implies$$

Uniform absolute continuity vs. ergodic continuity

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Uniform absolute continuity

$$\underbrace{\sup_{n \in \mathbb{N}} \Pr\left(X_n \in A\right) < \varepsilon}_{\text{time-uniform guarantee}}$$

$\Longrightarrow$

Ergodic continuity

$$\underbrace{\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}\{X_n \in A\} < \varepsilon}_{\text{time-averaged guarantee}}$$

Upshot

Suppose that $(\mathcal{X}, \rho, \nu)$ is a reasonable space, and test instances are generated by an adversary with time-uniform bounded precision.

Then, the nearest neighbor rule will eventually make no mistakes on average.

Looking forward

Immediate follow-up questions

1. Online consistency of the k_n-nearest neighbor rule for noisy labels
2. Nearest neighbor methods for adversarial labels
3. Learning with bounded or rate-limited memory
4. Extension to general distance spaces
5. Comparison of uniform absolute continuity to weaker conditions in literature

Some other ongoing work

- *Learning in repeated games with a shared signal*, with Yian Ma and Yixin Wang.
- *Optimization on the Pareto set*, with Yian Ma and Abhishek Roy.

Thank you!

Additional slides

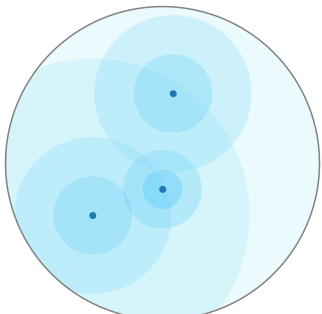
Cover tree data structure

Let $\mathcal{D}$ be a finite dataset in $(\mathcal{X}, \rho)$.

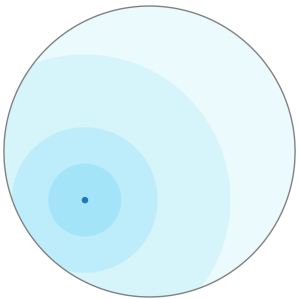
Definition (Beygelzimer et al. (2006))

A cover tree on $\mathcal{D}$ is a sequence $(\mathcal{C}_k)_k$ of coverings of $\mathcal{D}$:

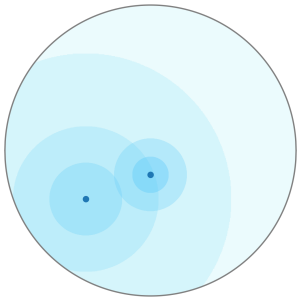
- *$\mathcal{C}_k$ is a subset of $\mathcal{D}$,*
- *$\mathcal{C}_k$ forms a 2^{-k}-net of $\mathcal{D}$,*
- *$\mathcal{C}_{k+1} \supset \mathcal{C}_k$.*



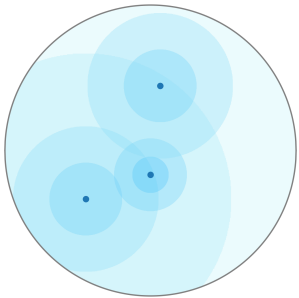
Sequential insertion for cover trees



$\text{root} \leftarrow X_1$



$\text{parent}(X_2) \leftarrow X_1$



$\text{parent}(X_3) \leftarrow X_1$

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