Nearest neighbor for realizable online classification

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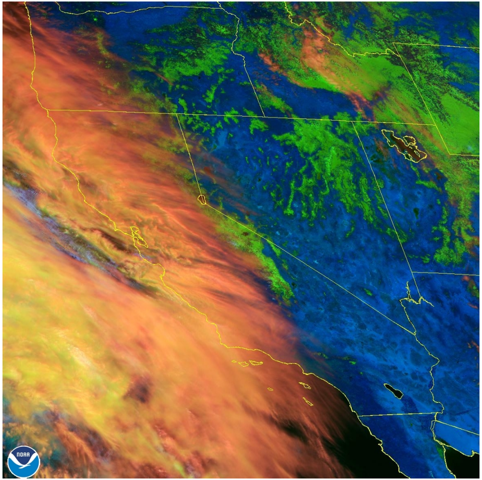
Weather forecasting problem

THE WEATHER CHANNEL'S TASK

Each day:

- receive atmospheric data



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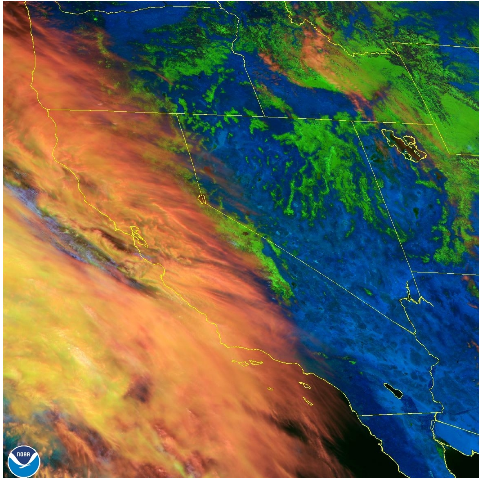

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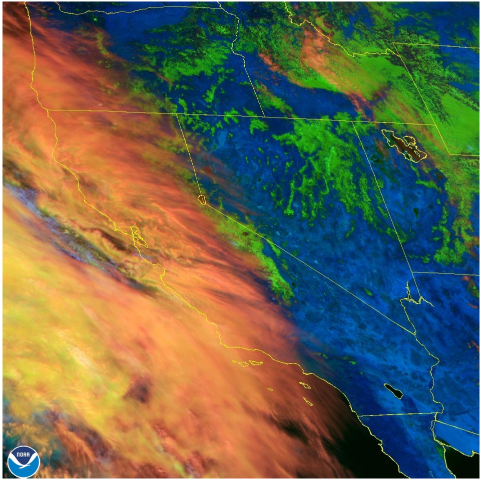

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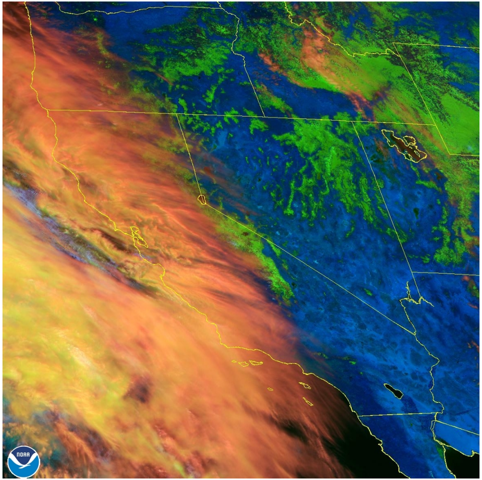

Nearest neighbor for weather prediction

NEAREST NEIGHBOR ALGORITHM

- remember all past conditions + weather outcomes

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- predict weather according to the most similar conditions in memory

The nearest neighbor rule

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Let $(\mathcal{X}, \rho)$ be a metric space.

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$$\hat{y}(x) = y_{\mathrm{NN}(x)}$$

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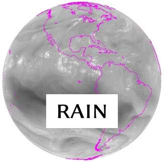

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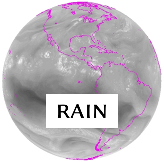

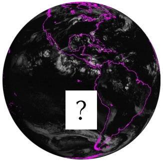

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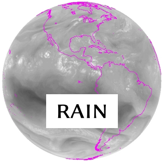

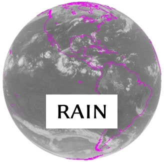

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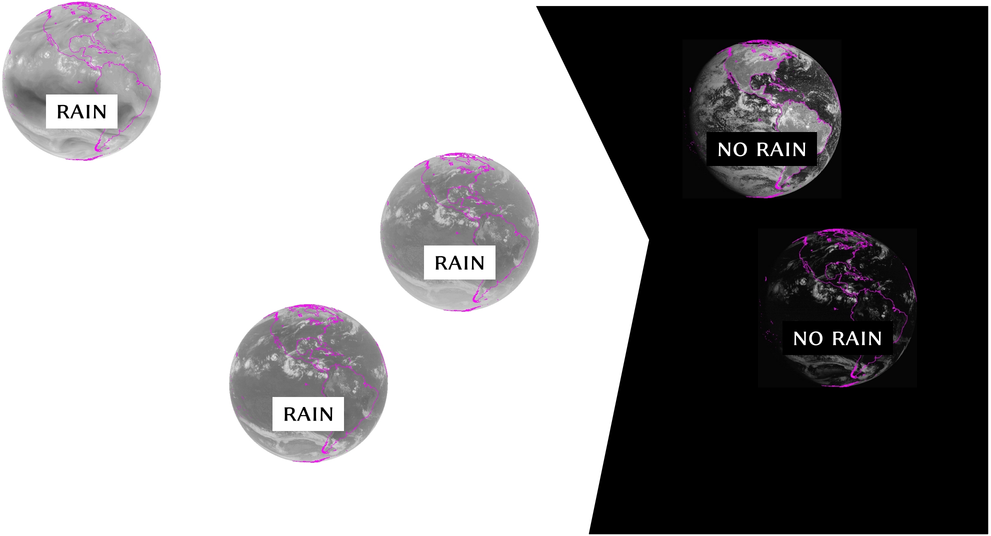

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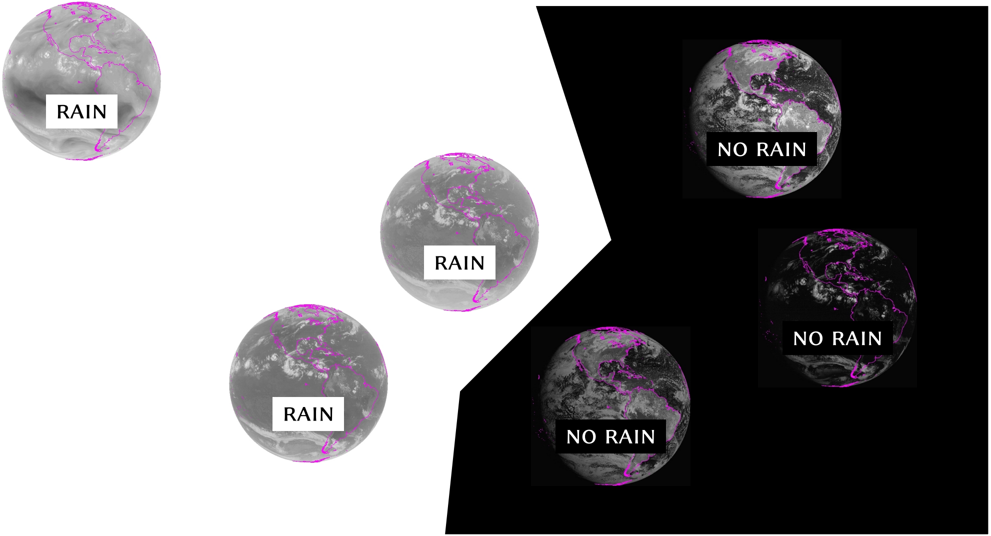

Behavior of online nearest neighbor

QUESTION

When is the *nearest neighbor rule* a reasonable online prediction strategy?

Online learning setting

ONLINE LEARNING LOOP

For $t = 1, 2, \ldots$

- receive instance x_t

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- incur loss $\ell(x_t, y_t, \hat{y}_t)$

Online learning setting

REALIZABILITY ASSUMPTION

The true labels are generated by some underlying function $f : \mathcal{X} \to \mathcal{Y}$,

$$y_t = f(x_t).$$

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GOAL

Make fewer and fewer mistakes over time.

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Make fewer and fewer mistakes over time. Formally:

$$\underbrace{\mathrm{er}_T := \frac{1}{T}\sum_{t=1}^{T} \ell\big(x_t, y_t, \hat{y}_t\big) \to 0}_{\text{achieve vanishing error rate}}\,.$$

Connection to regret

In the usual goal in the online learning setting is to achieve sublinear regret:

$$\mathrm{regret}_T := \sum_{t=1}^{T} \ell\big(x_t, y_t, \hat{y}_t\big) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell\big(x_t, y_t, h(x_t)\big).$$

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- In the realizable setting, if $\mathcal{H}$ is non-parametric (e.g. all nearest neighbor classifiers), no mistakes are made by any optimal $h \in \mathcal{H}$ on $(x_1, y_1), \ldots, (x_T, y_T)$.

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- Thus, sublinear regret is equivalent to vanishing error rate.

Difficulty of realizable online learning

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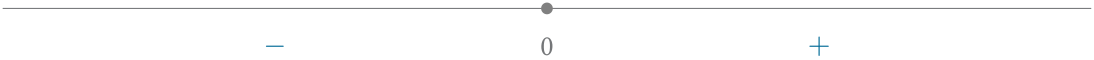
- In the worst-case, each x_t is selected so that learner makes a mistake each time.

Negative example: learning the sign function

GOAL

Learn the sign function $\quad f(x) := \begin{cases} + & x \geq 0 \\ - & x < 0 \end{cases}$



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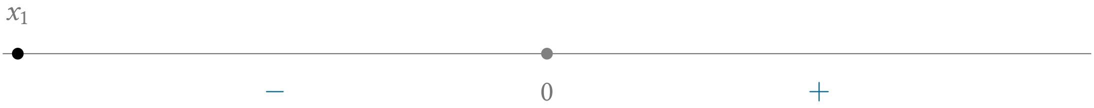

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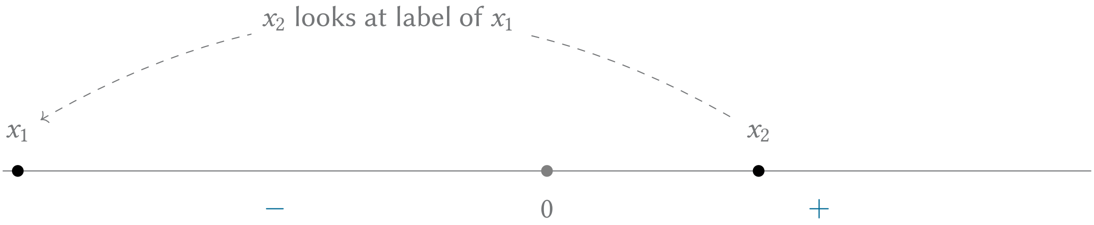

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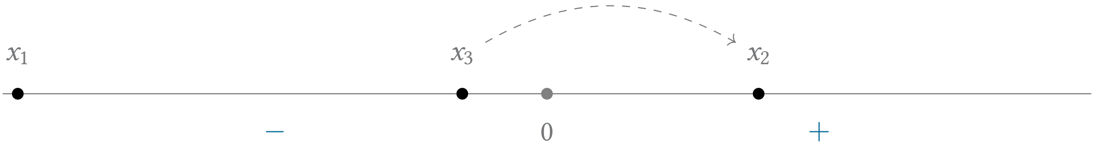

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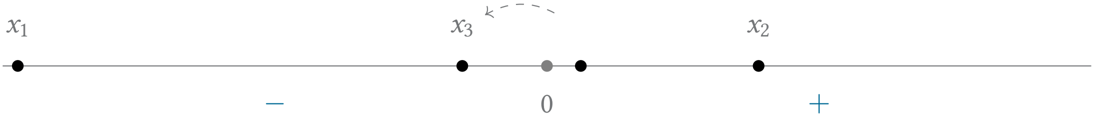

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- The sequence alternate signs and the nearest neighbor of x_{t+1} is x_t out of $x_1, \ldots, x_t$.

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SETTING

Let $(\mathcal{X}, \rho)$ be a totally bounded metric space and $f : \mathcal{X} \to \{-, +\}$.

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Proposition (Non-convergence in the worst-case)

There is a sequence of instances $(x_t)_t$ on which the nearest neighbor error rate is bounded away from zero if and only if there is no positive separation between classes:

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- Proof idea: can always find arbitrarily close pairs (x, x') with opposite signs
  - can select sequence so that x_{2t} is closest to x_{2t-1}, which has the opposite sign

Implications of negative result

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RESEARCH QUESTION

Under what *general conditions* is realizable online learning possible?

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Under what *general conditions* is realizable online learning possible?

- How much do we need to relax the worst-case adversary?

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  - Show that almost all problems are easy (the hard instances have measure zero).
    - Or, problems are easy with high probability/on average.

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The smoothed online setting is also studied by Rakhlin et al. (2011); Haghtalab et al. (2020).

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GAUSSIAN-SMOOTHED ADVERSARY:

- adversary selects $\overline{x}$
- test instance x is a perturbed version $\overline{x} + \xi$ where $\xi \sim \mathcal{N}(0, \sigma^2 I)$, so:

$$\mu = \mathcal{N}(\bar{x}, \sigma^2 I).$$

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σ-SMOOTHED ADVERSARY:

- let ν be an underlying distribution over $\mathcal{X}$
- the adversary can select any distribution μ satisfying:

$$\mu(A) \leq \frac{1}{\sigma} \cdot \nu(A),$$

for all $A \subset \mathcal{X}$ measurable.

Dominated adversary

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Definition (Dominated adversary)

The measure ν uniformly dominates a family $\mathcal{M}$ of probability distributions on $\mathcal{X}$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that:

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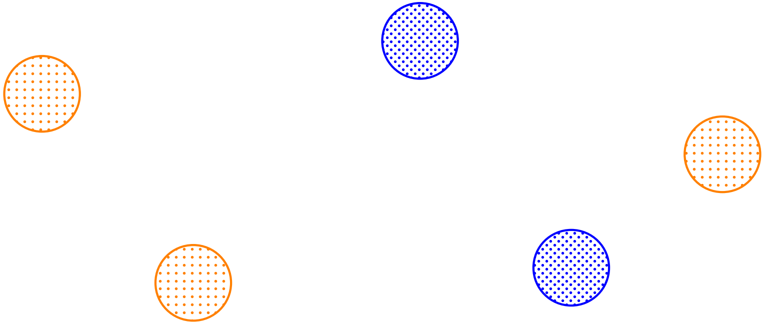
for all $A \subset \mathcal{X}$ measurable and distribution $\mu \in \mathcal{M}$. We say that adversary is ν-dominated if at all times t it selects μ_t from a family of distributions uniformly dominated by ν.

Example: learning labels for well-separated clusters

SETTING

Suppose that the instance space $\mathcal{X}$ consists of countably many well-separated clusters



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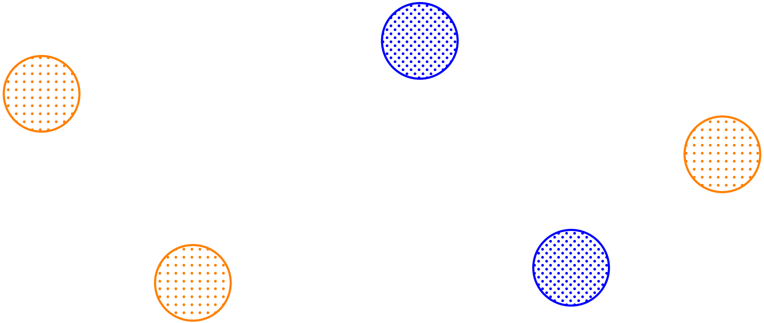

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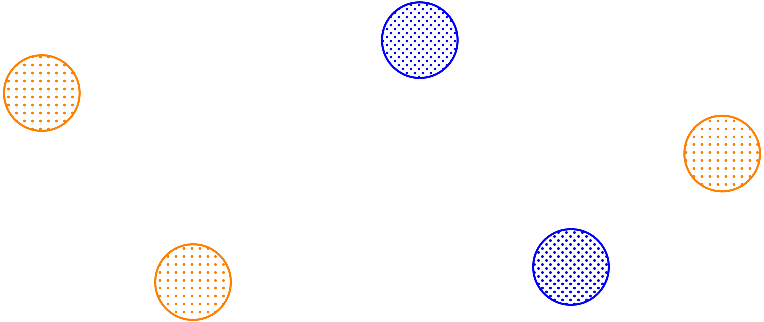

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THE NEAREST NEIGHBOR LEARNER

makes at most one mistake made per cluster

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CONVERGENCE RESULT FOR WELL-SEPARATED CLUSTERS

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CONVERGENCE RESULT FOR WELL-SEPARATED CLUSTERS

Let ν be a finite measure on $\mathcal{X}$. The nearest neighbor learner achieves vanishing error rate against any ν-dominated adversary.

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- The ν-dominated adversary selects points from $\mathcal{X}_{\text{small}}$ at rate $\mu(\mathcal{X}_{\text{small}}) < \varepsilon$.
  - By the law of large number, at most an ε-fraction of $(x_t)_t$ comes from $\mathcal{X}_{\text{small}}$:

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}\{x_t \in \mathcal{X}_{\text{small}}\} = \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mu_t(\mathcal{X}_{\text{small}}) < \varepsilon \quad \text{a.s.}$$

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Thus, the asymptotic mistake rate is upper bounded by any $\varepsilon > 0$ by selecting $\mathcal{X}_{\mathrm{small}}$ sufficiently small.

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  - $\mathcal{X}_{\text{small}}$ contains very little mass $\nu(\mathcal{X}_{\text{small}}) < \delta$.
- Nearest neighbor can only make finitely many mistakes on $\mathcal{X}_{\text{easy}}$.
  - These mistakes contribute nothing to the asymptotic mistake rate.
- The ν-dominated adversary selects points from $\mathcal{X}_{\text{small}}$ at rate $\mu(\mathcal{X}_{\text{small}}) < \varepsilon$.
  - By the law of large number, at most an ε-fraction of $(x_t)_t$ comes from $\mathcal{X}_{\text{small}}$:

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}\{x_t \in \mathcal{X}_{\text{small}}\} = \limsup_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \mu_t(\mathcal{X}_{\text{small}}) < \varepsilon \quad \text{a.s.}$$

Thus, the asymptotic mistake rate is upper bounded by any $\varepsilon > 0$ by selecting $\mathcal{X}_{\text{small}}$ sufficiently small. The asymptotic mistake rate is zero by taking $\varepsilon \downarrow 0$. $\square$

Generalizing the argument

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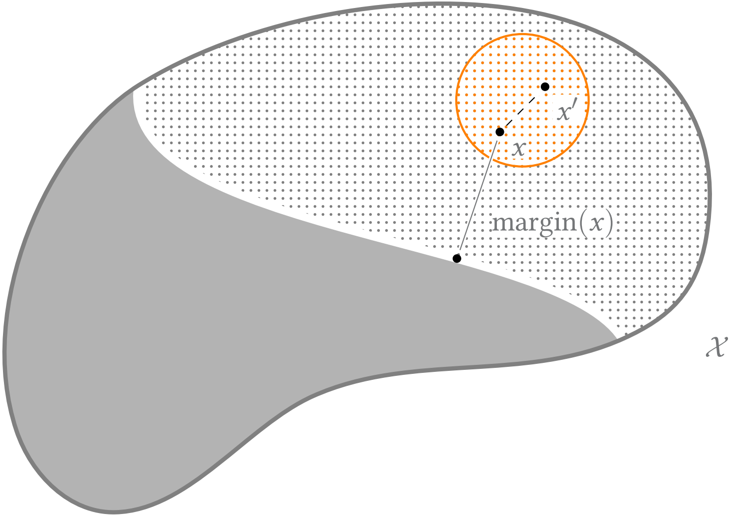
KEY PROPERTY USED

The nearest neighbor learner makes at most one mistake per mutually-labeling set.

- We introduce the device of mutually-labeling sets $U \subset \mathcal{X}$ satisfying the property:

$$\text{interpoint distances in } U < \text{distance to points with different labels.}$$

Mutually-labeling set



Generalizing argument

Definition (Mutually-labeling set)

*A subset $U \subset \mathcal{X}$ is **mutually labeling** if for all $x, x' \in U$:*

$$\underbrace{\rho(x, x')}_{\textit{interpoint distances}} \quad < \quad \underbrace{\mathrm{margin}(x)}_{\textit{distance to decision boundary}}$$

where $\mathrm{margin}(x)$ *is the smallest distance between x and points with different labels:*

$$\mathrm{margin}(x) = \inf\{\rho(x, \overline{x}) : f(x) \neq f(\overline{x})\}.$$

Convergence result

SETTING

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Theorem (Convergence of nearest neighbor)

The nearest neighbor rule achieves vanishing mistake rate against a ν-dominated adversary:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}\{\hat{y}_t \neq y_t\} = 0 \quad \text{a.s.}$$

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Proof sketch.

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Thus, the mistake rate converges to zero almost surely. $\square$

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- What do meaningful rates of convergence look like?

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- Is the ν-dominated online learning setting realistic and tractable?
- If so, can we design and analyze algorithms specifically for this setting?
  - e.g. a minimax optimal algorithm might not be optimal in this setting

Thank you

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