Online nearest neighbor classification

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Weather forecasting problem

THE WEATHER CHANNEL'S TASK

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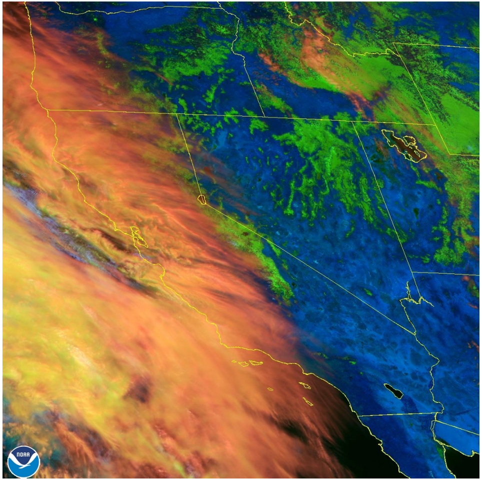
Each day:

- **receive** current atmospheric data



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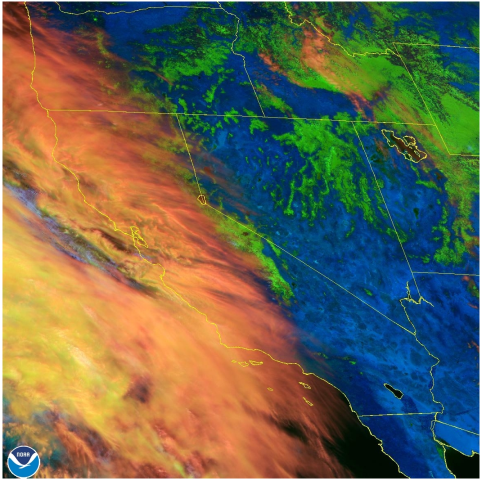

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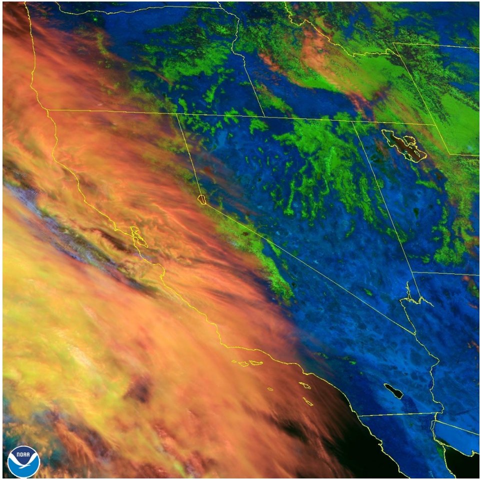

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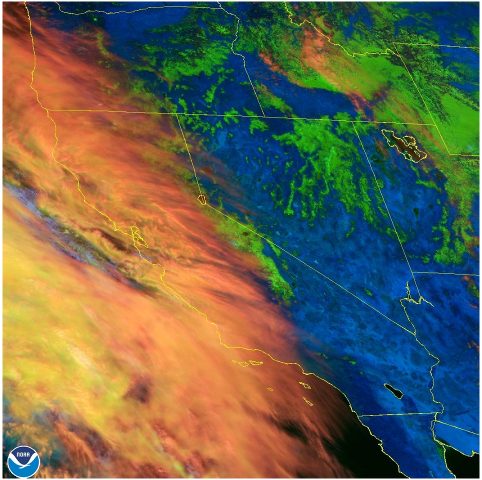

Nearest neighbor for weather prediction

NEAREST NEIGHBOR ALGORITHM

- remember all past conditions + weather outcomes

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- remember all past conditions + weather outcomes
- predict weather according to the most similar conditions in memory

The nearest neighbor rule

SETTING

Let $(\mathcal{X}, \rho)$ be a metric space.

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$$\mathrm{NN}(x) = \arg\min_{\tau} \rho(x, x_\tau)$$

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$$\hat{y}(x) = y_{\mathrm{NN}(x)}$$

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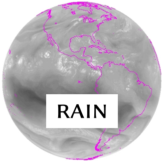

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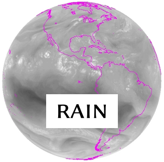

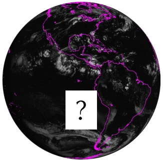

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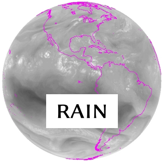

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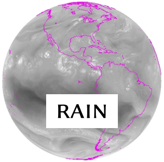

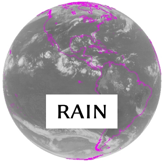

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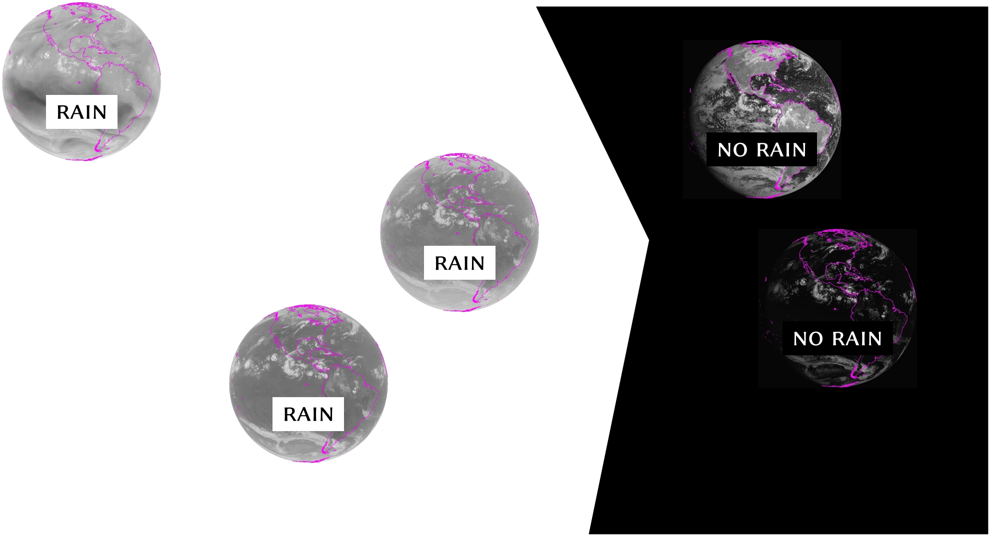

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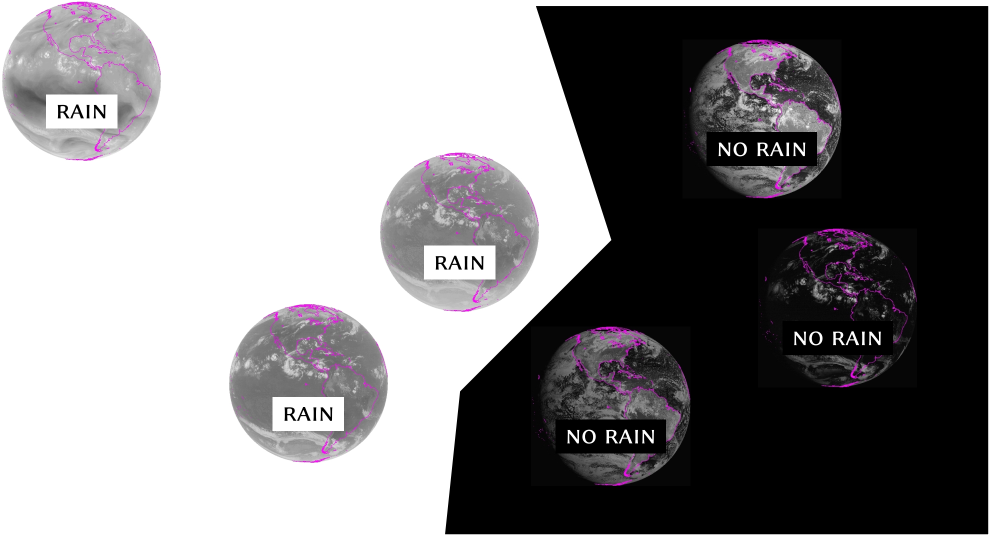

Behavior of online nearest neighbor

QUESTION

When is the *nearest neighbor rule* a reasonable online prediction strategy?

Online learning setting

ONLINE LEARNING LOOP

For $t = 1, 2, \ldots$

- receive instance x_t

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- observe true label y_t
- incur loss $\ell(x_t, y_t, \hat{y}_t)$

Online learning setting

REALIZABILITY ASSUMPTION

The true labels are generated by some underlying function $f : \mathcal{X} \to \mathcal{Y}$,

$$y_t = f(x_t).$$

Online learning setting

GOAL

Make fewer and fewer mistakes over time.

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Make fewer and fewer mistakes over time. Formally:

$$\underbrace{\mathrm{er}_T := \frac{1}{T}\sum_{t=1}^{T} \ell\big(x_t, y_t, \hat{y}_t\big) \to 0 \,.}_{\text{achieve vanishing error rate}}$$

Connection to regret

In the usual goal in the online learning setting is to achieve sublinear regret:

$$\text{regret}_T := \sum_{t=1}^{T} \ell\big(x_t, y_t, \hat{y}_t\big) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell\big(x_t, y_t, h(x_t)\big).$$

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- In the realizable setting, if $\mathcal{H}$ is non-parametric (e.g. all nearest neighbor classifiers), no mistakes are made by any optimal $h \in \mathcal{H}$ on $(x_1, y_1), \ldots, (x_T, y_T)$.

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- Thus, sublinear regret is equivalent to vanishing error rate.

Difficulty of realizable online learning

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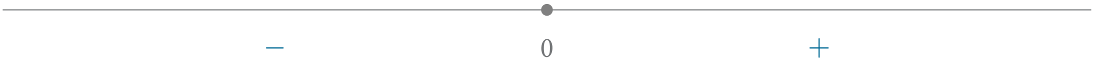
- The sequence of instances x_t do not come i.i.d. from some distribution.
- In the worst-case, each x_t is selected so that learner makes a mistake each time.

Negative example: learning the sign function

GOAL

Learn the sign function $\quad f(x) := \begin{cases} + & x \geq 0 \\ - & x < 0 \end{cases}$



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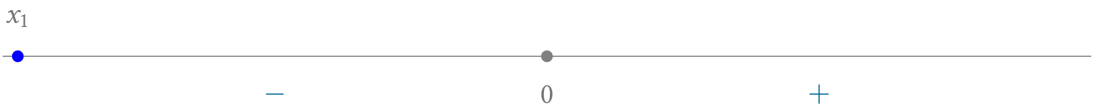

EXAMPLE. A worst-case sequence where the nearest neighbor rule errs every time.

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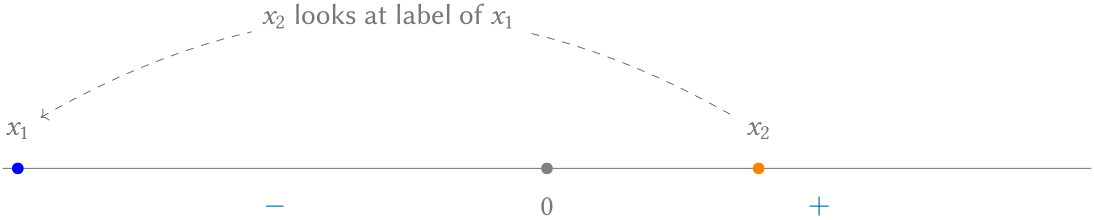

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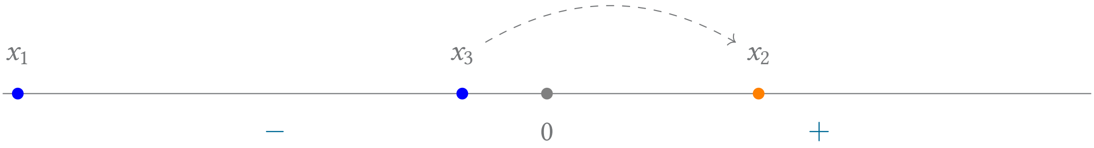

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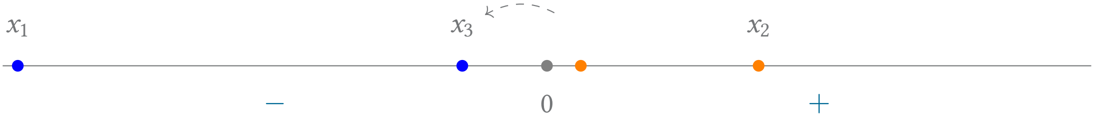

EXAMPLE. A worst-case sequence where the nearest neighbor rule errs every time.

- The sequence alternate signs and the nearest neighbor of x_{t+1} is x_t out of $x_1, \ldots, x_t$.
- Mistake rate fails to go to zero despite the mistake set shrinking exponentially fast.

Generalized negative result

SETTING

Let $(\mathcal{X}, \rho)$ be a totally bounded metric space and $f : \mathcal{X} \to \{-, +\}$.

Proposition (Non-convergence in the worst-case)

There is a sequence of instances $(x_t)_t$ on which the nearest neighbor error rate is bounded away from zero if and only if there is no positive separation between classes:

$$\inf_{f(x) \neq f(x')} \rho(x, x') = 0.$$

- **Proof idea:** can always find arbitrarily close pairs (x, x') with opposite signs
  - can select sequence so that x_{2t} is closest to x_{2t-1}, which has the opposite sign

Implications of negative result

The worst-case adversary is too powerful—learning may not be possible in this setting.

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This work

RESEARCH QUESTION

Under what *general conditions* is realizable online learning possible?

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Under what *general conditions* is realizable online learning possible?

- How much do we need to relax the worst-case adversary?

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  - Introduce a (probability) measure over problem instances.
  - Show that almost all problems are easy (the hard instances have measure zero).
    - Or, problems are easy with high probability/on average.

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The smoothed online setting is also studied by Rakhlin et al. (2011); Haghtalab et al. (2020).

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GAUSSIAN-SMOOTHED ADVERSARY:

- adversary selects $\overline{x}$
- test instance x is a perturbed version $\overline{x} + \xi$ where $\xi \sim \mathcal{N}(0, \sigma^2 I)$, so:

$$\mu = \mathcal{N}(\bar{x}, \sigma^2 I).$$

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σ-SMOOTHED ADVERSARY:

- let ν be an underlying distribution over $\mathcal{X}$
- the adversary can select any distribution μ satisfying:

$$\mu(A) \leq \frac{1}{\sigma} \cdot \nu(A),$$

for all $A \subset \mathcal{X}$ measurable.

Dominated adversary

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Definition (Dominated adversary)

The measure ν uniformly dominates a family $\mathcal{M}$ of probability distributions on $\mathcal{X}$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that:

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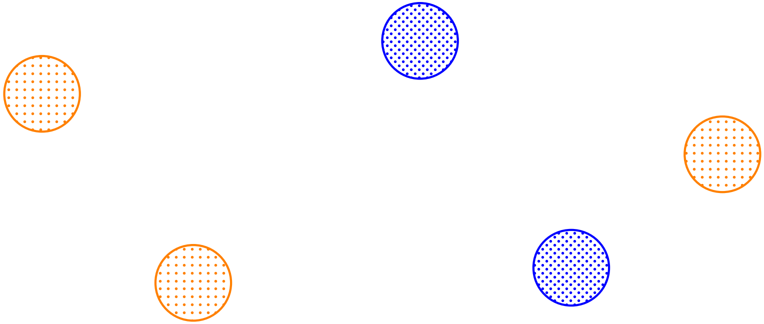
for all $A \subset \mathcal{X}$ measurable and distribution $\mu \in \mathcal{M}$. We say that adversary is ν-dominated if at all times t it selects μ_t from a family of distributions uniformly dominated by ν.

Example: learning labels for well-separated clusters

SETTING

Suppose that the instance space $\mathcal{X}$ consists of countably many well-separated clusters



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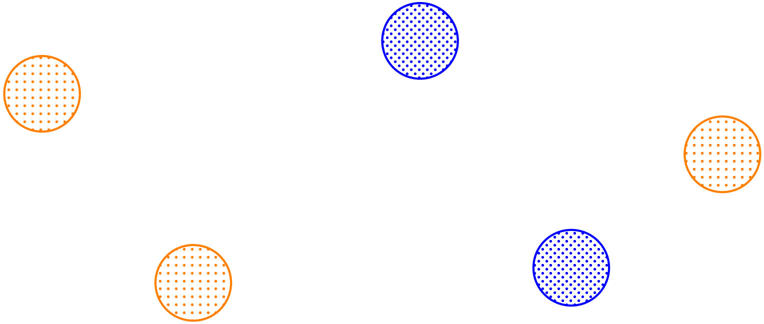

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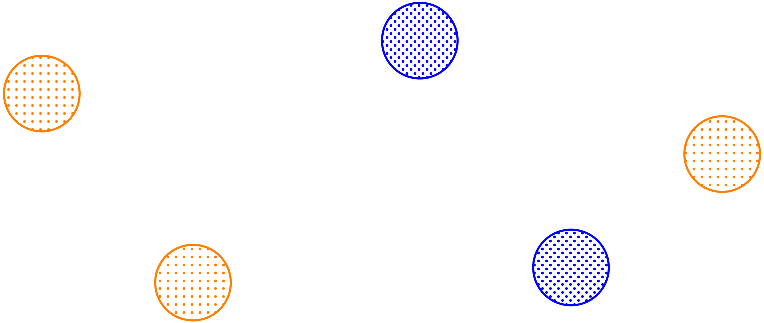

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THE NEAREST NEIGHBOR LEARNER

makes at most one mistake made per cluster

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CONVERGENCE RESULT FOR WELL-SEPARATED CLUSTERS

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CONVERGENCE RESULT FOR WELL-SEPARATED CLUSTERS

Let ν be a finite measure on $\mathcal{X}$. The nearest neighbor learner achieves vanishing error rate against any ν-dominated adversary.

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- The ν-dominated adversary selects points from $\mathcal{X}_{\text{small}}$ at rate $\mu(\mathcal{X}_{\text{small}}) < \varepsilon$.
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  - These mistakes contribute nothing to the asymptotic mistake rate.
- The ν-dominated adversary selects points from $\mathcal{X}_{\text{small}}$ at rate $\mu(\mathcal{X}_{\text{small}}) < \varepsilon$.
  - By the law of large number, at most an ε-fraction of $(x_t)_t$ comes from $\mathcal{X}_{\text{small}}$.
- Thus, the asymptotic mistake rate is upper bounded by ε almost surely.

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    - Such a decomposition exists for any $\delta > 0$ by the finiteness of ν.
- Nearest neighbor makes finitely many mistakes on $\mathcal{X}_{\text{easy}}$.
  - These mistakes contribute nothing to the asymptotic mistake rate.
- The ν-dominated adversary selects points from $\mathcal{X}_{\text{small}}$ at rate $\mu(\mathcal{X}_{\text{small}}) < \varepsilon$.
  - By the law of large number, at most an ε-fraction of $(x_t)_t$ comes from $\mathcal{X}_{\text{small}}$.
- Thus, the asymptotic mistake rate is upper bounded by ε almost surely.
  - Simultaneously apply upper bound for a countable collection of $\varepsilon_k \downarrow 0$.

Example: learning labels for well-separated clusters

Proof sketch.

- Split $\mathcal{X}$ into two pieces $\mathcal{X}_{\text{easy}} \cup \mathcal{X}_{\text{small}}$, where:
  - $\mathcal{X}_{\text{easy}}$ is a finite union of clusters
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The asymptotic mistake rate is zero. $\square$

Generalizing the argument

The argument works even if the clusters are not well-separated.

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KEY PROPERTY USED

The nearest neighbor learner makes at most one mistake per cluster.

Generalizing the argument

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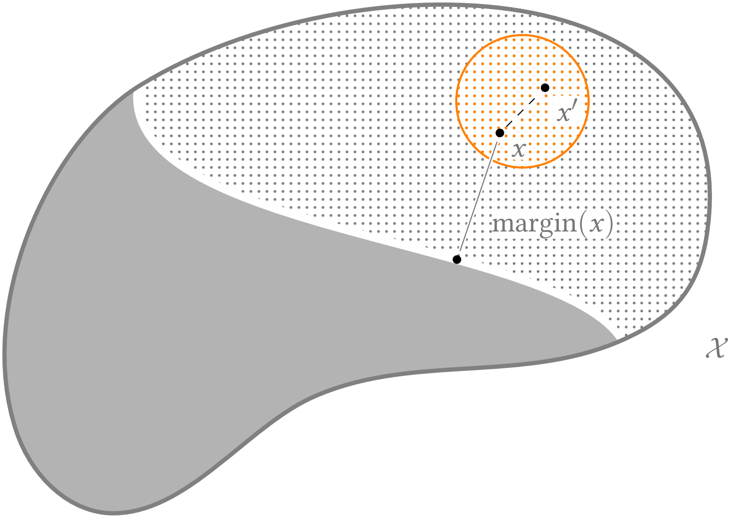
KEY PROPERTY USED

The nearest neighbor learner makes at most one mistake per mutually-labeling set.

- We introduce the device of mutually-labeling sets $U \subset \mathcal{X}$ satisfying the property:

$$\text{interpoint distances in } U < \text{distance to points with different labels.}$$

Mutually-labeling set



Generalizing argument

Definition (Mutually-labeling set)

A subset $U \subset \mathcal{X}$ is ***mutually labeling*** *if for all $x, x' \in U$:*

$$\underbrace{\rho(x, x')}_{\textit{interpoint distances}} \quad < \quad \underbrace{\mathrm{margin}(x)}_{\textit{distance to decision boundary}}$$

where $\mathrm{margin}(x)$ *is the smallest distance between x and points with different labels:*

$$\mathrm{margin}(x) = \inf\{\rho(x, \overline{x}) : f(x) \neq f(\overline{x})\}.$$

Convergence result

SETTING

Let $(\mathcal{X}, \rho, \nu)$ be a space equipped with a separable metric ρ and a finite Borel measure ν.

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  - e.g. The decision boundary is not a space-filling curve.

Theorem (Convergence of nearest neighbor)

The nearest neighbor rule achieves vanishing mistake rate against a ν-dominated adversary:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{1}\{\hat{y}_t \neq y_t\} = 0 \quad \text{a.s.}$$

Convergence result

Proof sketch.

- Sufficiently small open balls around non-boundary points are mutually-labeling.

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By prior argument, the mistake rate converges to zero almost surely. $\square$

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   - Quantify geometry of the the instance space and concept to be learned
     - doubling dimension of space and Minkowski content of the boundary
   - Quantify the strength of the adversary
     - Smoothness rate in definition of a dominated adversary $\varepsilon(\delta)$

Further work

Open questions

QUESTIONS

- Does the ν-dominated adversary balance between generality and tractability well?

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- Does the ν-dominated adversary balance between generality and tractability well?
- Is smoothed online learning possible when there is benign label noise?

Online learning with noise

ONLINE LEARNING LOOP

For $t = 1, 2, \ldots$

- receive instance x_t
- predict label $\hat{y}_t$
- observe label $y_t \sim P_{Y|X=x_t}$ drawn from a fixed conditional distribution
- incur loss $\ell(x_t, y_t, \hat{y}_t)$

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QUESTION

How should this data be used to construct a classifier?

Online learning with noise: binary search on noise

BINARY SEARCH SAMPLING ALGORITHM

Online learning with noise: binary search on noise

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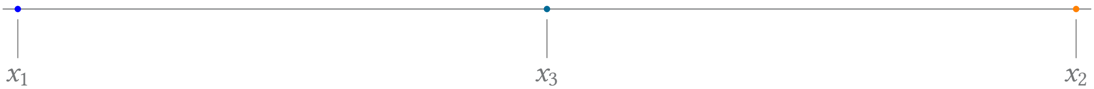

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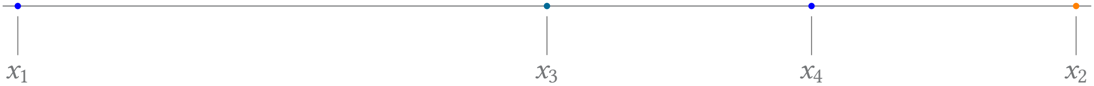

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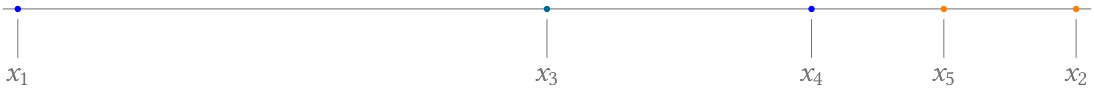

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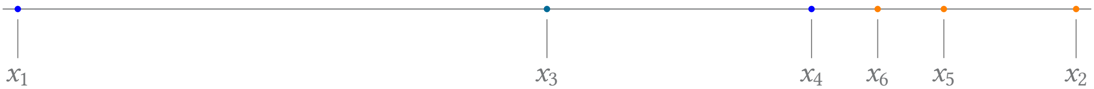

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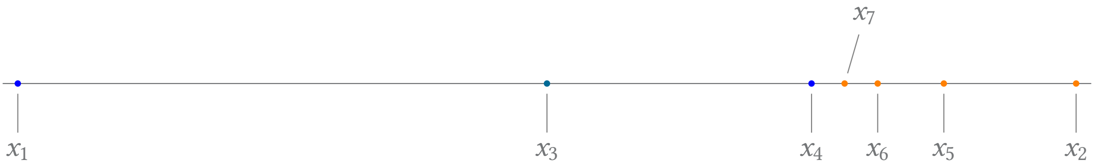

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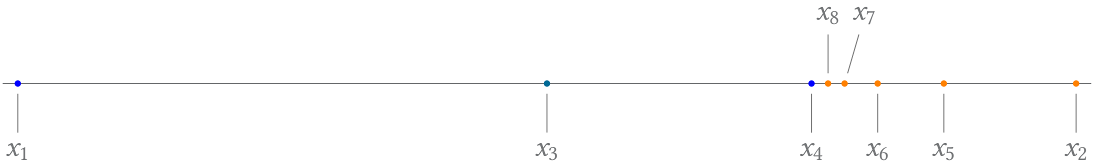

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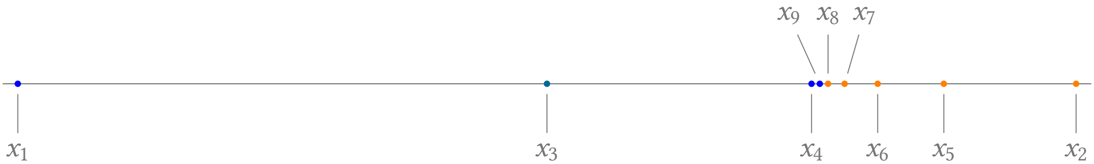

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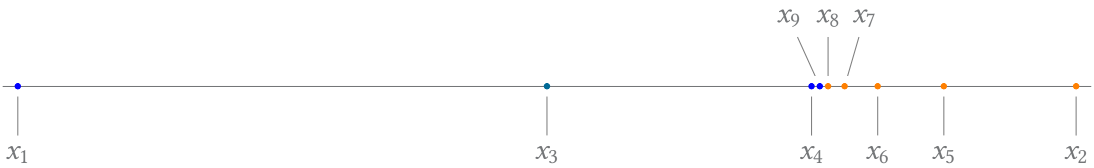

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For $t = 1, 2, \ldots$

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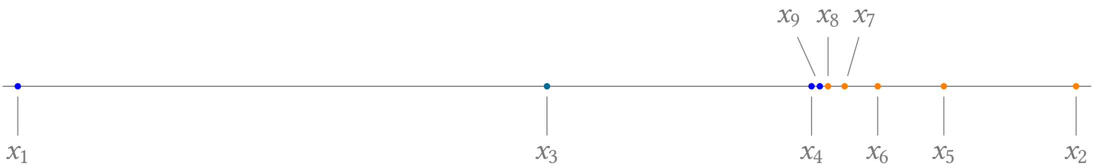

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For $t = 1, 2, \ldots$

- $x_- \leftarrow$ max negative data point in data set

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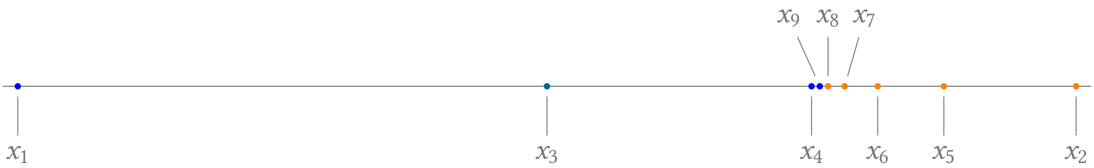

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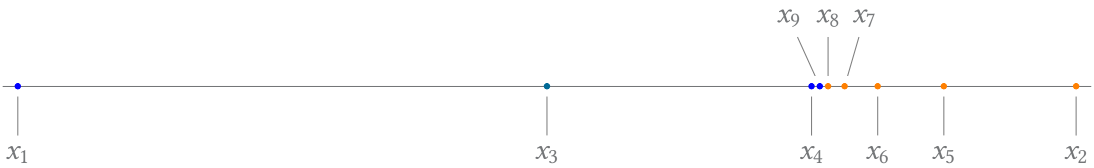

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For $t = 1, 2, \ldots$

- $x_- \leftarrow$ max negative data point in data set
- $x_+ \leftarrow$ min positive data point in data set
- $x_{t+1} \leftarrow \mathrm{mean}(x_-, x_+)$

Online learning with noise: binary search on noise

TWO INDISTINGUISHABLE WORLDS

An unbounded adversary can select points in a way so that the learner can't distinguish:

Online learning with noise: binary search on noise

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An unbounded adversary can select points in a way so that the learner can't distinguish:

1. the labels are generated by a threshold function $f(x) = \mathbb{1}\{x \geq \theta\}$
2. the labels are drawn from $\mathrm{Ber}(\frac{1}{2} + \Delta)$

Online learning with noise: binary search on noise

Suppose labels are drawn from pure noise $\mathrm{Ber}(\frac{1}{2})$.

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where $\#$ data points in $I \approx \frac{1}{2}t$.

- In fact, for a vast majority of intervals with $< \frac{1}{2}t$ points, the average label is far from $\frac{1}{2}$.

Challenge of online learning with noise

MAKING PATTERNS OUT OF NOISE

In the sequential setting, the uniform law of large number can fail

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Challenge of online learning with noise

MAKING PATTERNS OUT OF NOISE

In the sequential setting, the uniform law of large number can fail

- there can be many balls/intervals whose average label is far from correct
- finite VC dimension does not imply sequential uniform Glivenko-Cantelli property

The k_n-nearest neighbor rule

k_N-NEAREST NEIGHBOR ALGORITHM

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- given query x, find the k_n most similar data point in memory
- predict using majority vote over k_n nearest neighbors

Online learning with noise

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Let $\mathcal{X} = [0,1]$ be the unit interval and let $P_{Y|X=x} = \mathrm{Ber}(\frac{1}{2} + \Delta)$.

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Proposition

Suppose that $k_n = o(n)$. An unbounded adversary can adaptively generate a sequence $(x_n)_n$ such that the mistake rate of the k_n-nearest neighbor rule never converges to zero:

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$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}\{\hat{y}_{k_n\text{-}NN}(x_n) \neq 1\} = \Omega(1).$$

Smoothed online learning with noise

QUESTION

How does the k_n-nearest neighbor rule perform against a dominated adversary?

Preliminary result

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Let $\mathcal{X} = [0, 1]$ be the unit interval and let $P_{Y|X=x} = \mathrm{Ber}(\frac{1}{2} + \Delta)$.

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Let $\mathcal{X} = [0,1]$ be the unit interval and let $P_{Y|X=x} = \mathrm{Ber}(\frac{1}{2} + \Delta)$.

- Let the dominated adversary μ be smoothed at rate $\varepsilon : [0,1] \to [0,1]$ so that:

$$\nu(A) < \delta \qquad \Longrightarrow \qquad \mu(A) < \varepsilon(\delta).$$

  Here, ν is the Lebesgue measure on $\mathcal{X}$.

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Let $k_n = \omega\big((\log \varepsilon^{-1}(\frac{1}{n}))^2\big)$. The k_n-nearest neighbor rule eventually makes no mistakes:

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Proof idea

1. Define the finite family of simple intervals of depth $\ell_n \in \mathbb{N}$ of the form:

$$\left[j_0 2^{-\ell_n}, j_1 2^{-\ell_n}\right], \qquad j_0, j_1 \in \mathbb{N}$$

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2. Approximate any k_n-nearest neighbor interval $I_{k_n\text{-NN}}$ by simple intervals:

$$I_{\text{inner}} \subset I_{k_n\text{-NN}} \subset I_{\text{outer}}$$

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   - They can be chosen so that $I_{\text{outer}} \setminus I_{\text{inner}}$ is a union of two dyadic intervals of length $2^{-\ell_n}$

3. I_{outer} contains at least k_n points: most are correctly labeled points (uniform LLN)

Proof idea

1. Define the finite family of simple intervals of depth $\ell_n \in \mathbb{N}$ of the form:

$$\left[j_0 2^{-\ell_n}, j_1 2^{-\ell_n}\right], \qquad j_0, j_1 \in \mathbb{N}$$

   - this class satisfies uniform law of large numbers because it is finite

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Our choice of k_n leads to $\Pr(\mathbb{1}\{\text{mistake}_n\}) = o(n^{-1})$. Apply Borel-Cantelli. □

Big picture: smoothed analysis of online learning

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- Is the ν-dominated online learning setting realistic and tractable?

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OPPORTUNITY: we might not live in the worst-case adversarial setting

- Is the ν-dominated online learning setting realistic and tractable?
- If so, can we design and analyze algorithms specifically for this setting?
  - e.g. a minimax optimal algorithm might not be optimal in this setting

Thank you

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Joint work with Sanjoy Dasgupta and Robi Bhattacharjee.

Paper is available at *https://arxiv.org/abs/2307.01170*.

Additional slides

Related work: realizable online learning

LEARNABILITY OF A CONCEPT CLASS

Let $\mathcal{F}$ be a concept class. When is it learnable under worst-case online setting?

- Littlestone (1988): if $\mathcal{F}$ has finite Littlestone dimension d, it is possible to make at most d mistakes (uniform bound over all $f \in \mathcal{F}$)
- Bousquet et al. (2021): if $\mathcal{F}$ does not have an infinite Littlestone tree, it is possible to make finitely many mistakes (no uniform-bound over $f \in \mathcal{F}$)

NON-PARAMETRIC ONLINE LEARNING

Non-parametric classes have infinite Littlestone trees. Any deterministic learner makes a mistake every round in the worst-case.

- We show that online learning is possible under mild smoothing of adversary.
  - Finite Littlestone dimension not needed!

Related work: uniform convergence

I.I.D. UNIFORM CONVERGENCE

- Balsubramani et al. (2019): uniform convergence for empirical conditional measures
  - Let $\mathcal{A}, \mathcal{B} \subset 2^{\mathcal{X}}$ have VC dimensions at most d. At time n, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$:

$$\left|\hat{\mu}_n(A|B) - \mu(A|B)\right| < O\left(\sqrt{\frac{d\log(n)}{\#\text{ data points in } B}}\right) \quad \text{w.h.p.}$$

SEQUENTIAL UNIFORM CONVERGENCE

- Rakhlin et al. (2015): finite VC dimension is not sufficient for sequential uniform convergence; finite Littlestone dimension necessary and sufficient.
  - Let $(X_n)_n$ be an $(\mathcal{F}_n)_n$-stochastic process and μ_n the conditional law of X_n given $\mathcal{F}_{n-1}$.

$$\forall \varepsilon > 0, \quad \lim_{N\to\infty} \sup_{\mu} \Pr\left(\sup_{n>N} \sup_{A\in\mathcal{A}} \left|\hat{\mu}_n(A) - \frac{1}{n}\sum_{k=1}^{n} \mu_k(A)\right| > \varepsilon\right) = 0$$

Open questions: sequential uniform convergence

1. Sequential uniform convergence for (adaptive) sequences $(\mathcal{A}_n)_n$ of classes $A_n \subset 2^{\mathcal{X}}$?

$$\forall \varepsilon > 0, \quad \lim_{N \to \infty} \sup_{\mu} \Pr\left(\sup_{n > N} \boxed{\sup_{A \in \mathcal{A}_n}} \left| \hat{\mu}_n(A) - \frac{1}{n} \sum_{k=1}^{n} \mu_k(A) \right| > \varepsilon \right) = 0$$

2. Sequential uniform convergence for smoothed processes?
   - Suppose $\mathcal{A}$ is well-approximated by some class $\mathcal{B}$ with finite Littlestone dimension:

$$\sup_{A \in \mathcal{A}} \inf_{B \in \mathcal{B}} \nu\big(A \Delta B\big) < \delta.$$

   Can smoothness extend uniform convergence for $\mathcal{B}$ to $\mathcal{A}$?

Related work: smoothed online learning

EXISTING RESULTS

- Haghtalab et al. (2022) and Block et al. (2022) show that in the smoothed online setting where the adversary also controls labels, finite VC dimension is sufficient
  - Assumes $\frac{1}{\sigma}$-Lipschitz smoothing: $\mu(A) < \frac{1}{\sigma} \cdot \nu(A)$ for all $A \subset \mathcal{X}$ measurable.
  - Requires knowledge of underlying base measure ν.

OUR RESULT

- Generalize smoothed adversary to dominated adversary.
- Does not require finite VC dimension.
- Does not need knowledge of base measure ν.
- But, labels are not chosen adaptively (chosen adversarially at beginning of time).

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