Topic: Number of solutions to the linear system

Question: How many solutions does the following linear system have?

$$-x - 2y + 3z = -30$$

$$-2x - 3y - 5z = 22$$

$$x + 5y + 5z = -11$$

Answer choices:

A One solution

B No solutions

C Infinitely many solutions

Solution: A

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & -2 & 3 & | & -30 \\ -2 & -3 & -5 & | & 22 \\ 1 & 5 & 5 & | & -11 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row. Perform $-R_1 \to R_1$.

$$\begin{bmatrix} 1 & 2 & -3 & | & 30 \\ -2 & -3 & -5 & | & 22 \\ 1 & 5 & 5 & | & -11 \end{bmatrix}$$

Zero out the rest of the first column, first with $2R_1 + R_2 \to R_2$, and then $-R_1 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 2 & -3 & | & 30 \\ 0 & 1 & -11 & | & 82 \\ 1 & 5 & 5 & | & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & | & 30 \\ 0 & 1 & -11 & | & 82 \\ 0 & 3 & 8 & | & -41 \end{bmatrix}$$

Zero out the rest of the second column, first with $-2R_2 + R_1 \to R_1$, and then $-3R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & 19 & | & -134 \\ 0 & 1 & -11 & | & 82 \\ 0 & 3 & 8 & | & -41 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 19 & | & -134 \\ 0 & 1 & -11 & | & 82 \\ 0 & 0 & 41 & | & -287 \end{bmatrix}$$

Find the pivot entry in the third row with $(1/41)R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & 19 & | & -134 \\ 0 & 1 & -11 & | & 82 \\ 0 & 0 & 1 & | & -7 \end{bmatrix}$$

Zero out the rest of the third column, first with $-19R_3 + R_1 \to R_1$, then $11R_3 + R_2 \to R_2$.

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & -11 & | & 82 \\ 0 & 0 & 1 & | & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & -7 \end{bmatrix}$$

Therefore, there's one unique solution to the system, $(x, y, z) = (-1,5,-7)$.

Topic: Number of solutions to the linear system

Question: Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$3a + 9b - 3c = 24$$

$$a - 3b + 11c = -2$$

$$-2a + 5b - 20c = -5$$

Answer choices:

A One solution

B No solutions

C Infinitely many solutions

Solution: B

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 3 & 9 & -3 & | & 24 \\ 1 & -3 & 11 & | & -2 \\ -2 & 5 & -20 & | & -5 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row. Perform $(1/3)R_1 \to R_1$.

$$\begin{bmatrix} 1 & 3 & -1 & | & 8 \\ 1 & -3 & 11 & | & -2 \\ -2 & 5 & -20 & | & -5 \end{bmatrix}$$

Zero out the rest of the first column, first with $-R_1 + R_2 \to R_2$, then $2R_1 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 3 & -1 & | & 8 \\ 0 & -6 & 12 & | & -10 \\ -2 & 5 & -20 & | & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & | & 8 \\ 0 & -6 & 12 & | & -10 \\ 0 & 11 & -22 & | & 11 \end{bmatrix}$$

Find the pivot entry in the second row with $-(1/6)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 3 & -1 & | & 8 \\ 0 & 1 & -2 & | & \frac{5}{3} \\ 0 & 11 & -22 & | & 11 \end{bmatrix}$$

Zero out the rest of the second column, first with $-3R_2 + R_1 \to R_1$, then $-11R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & 5 & | & 3 \\ 0 & 1 & -2 & | & \frac{5}{3} \\ 0 & 0 & 0 & | & -\frac{22}{3} \end{bmatrix}$$

The third row shows us that $0 = -22/3$, which can't be true. Therefore, the system has no solutions.

Topic: Number of solutions to the linear system

Question: Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$w + 2x - 3y + 7z = 4$$

$$3w - x - 2y = 12$$

$$-2w + 5x + 2z = 5$$

$$2w + 3x - 5y + 11z = 8$$

Answer choices:

A One solution

B No solutions

C Infinitely many solutions

Solution: C

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 3 & -1 & -2 & 0 & | & 12 \\ -2 & 5 & 0 & 2 & | & 5 \\ 2 & 3 & -5 & 11 & | & 8 \end{bmatrix}$$

Zero out the rest of the first column, first with $-3R_1 + R_2 \to R_2$, then $2R_1 + R_3 \to R_3$, and finally with $-2R_1 + R_4 \to R_4$.

$$\begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 0 & -7 & 7 & -21 & | & 0 \\ -2 & 5 & 0 & 2 & | & 5 \\ 2 & 3 & -5 & 11 & | & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 0 & -7 & 7 & -21 & | & 0 \\ 0 & 9 & -6 & 16 & | & 13 \\ 2 & 3 & -5 & 11 & | & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 0 & -7 & 7 & -21 & | & 0 \\ 0 & 9 & -6 & 16 & | & 13 \\ 0 & -1 & 1 & -3 & | & 0 \end{bmatrix}$$

Find the pivot entry in the second row with $-(1/7)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 0 & 1 & -1 & 3 & | & 0 \\ 0 & 9 & -6 & 16 & | & 13 \\ 0 & -1 & -1 & -3 & | & 0 \end{bmatrix}$$

Zero out the rest of the second column, first with $-2R_2+R_1 \to R_1$, then $-9R_2+R_3 \to R_3$, and finally with $R_2+R_4 \to R_4$.

$$\begin{bmatrix} 1 & 0 & -1 & 1 & | & 4 \\ 0 & 1 & -1 & 3 & | & 0 \\ 0 & 9 & -6 & 16 & | & 13 \\ 0 & -1 & 1 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 & | & 4 \\ 0 & 1 & -1 & 3 & | & 0 \\ 0 & 0 & 3 & -11 & | & 13 \\ 0 & -1 & 1 & -3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & | & 4 \\ 0 & 1 & -1 & 3 & | & 0 \\ 0 & 0 & 3 & -11 & | & 13 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since the entire last row has only zeros, the linear system has infinitely many solutions.