PROBLEM 1

G: 6 pennies, 4 dimes $\qquad P(\text{Green}) = 3/4$

$\qquad\qquad\qquad\qquad\quad P(\text{Black}) = 1/4$

B: 8 pennies, 2 dimes

Probability ~~of~~ that the coins were ~~picked~~ pulled from the green wallet:

dime, penny, penny

$$P(G \mid D,P,P) = \frac{P(G) \cdot P(D,P,P \mid G)}{P(D,P,P)}$$

$$= \frac{3/4 \cdot 4/10 \cdot 6/9 \cdot 5/8}{\underbrace{3/4 \cdot 4/10 \cdot 6/9 \cdot 5/8}_{\text{from green wallet}} \underset{\text{or}}{+} \underbrace{1/4 \cdot 2/10 \cdot 8/9 \cdot 7/8}_{\text{from black wallet}}} = \frac{\left(\frac{3.4.5.6}{4.\cancel{10}.9.8}\right)}{\left(\frac{3.4.5.6}{4.\cancel{10}.9.8} + \frac{1.2.8.7}{4.\cancel{10}.9.8}\right)} = \frac{360}{472} = 0.76$$

thus, $P(B \mid D,P,P) = 1 - 0.76 = 0.23$.

So is more likely that I picked the green wallet.

$$P(\text{error} \mid x) = \min\left[P(w_1 \mid x), P(w_2 \mid x)\right]$$

$$= \min\left[0.76, 0.23\right]$$

$$= 0.23 .$$

PROBLEM 2

$$\lambda(\alpha_i|\omega_j) = \begin{cases} 0 & i=j \\ \lambda_r & i=c+1 \\ \lambda_s & \text{otherwise} \end{cases}$$

Overall risk for $i=1,\ldots,c$.

$$R(\alpha_i|x) = \sum_{j=1}^{j=c} \lambda(\alpha_i|\omega_j)\, P(\omega_j|x)$$

we know $\lambda(\alpha_i|\omega_i)$, so we can write

$$R(\alpha_i|x) = \cancelto{0}{\lambda(\alpha_i|\omega_i)} P(\omega_i|x) + \sum_{\substack{j=1 \\ j\neq i}}^{j=c} \underbrace{\lambda(\alpha_i|\omega_j)}_{\lambda_s \text{ for } i\neq j} P(\omega_j|x)$$

$$R(\alpha_i|x) = \lambda_s \underbrace{\boxed{\sum_{\substack{j=1 \\ j\neq i}}^{j=c} P(\omega_j|x)}}_{1-P(\omega_i|x)}$$

$$\boxed{R(\alpha_i|x) = \lambda_s(1-P(\omega_i|x))}$$

$R(\alpha_i|x)$ is the minimum risk if it's less than the cost of rejection, thus:

$$R(\alpha_i|x) < \lambda_r$$

$$\lambda_s(1-P(\omega_i|x)) < \lambda_r$$

$$\lambda_s - \lambda_s(P(\omega_i|x)) < \lambda_r$$

$$-P(\omega_i|x) < \frac{\lambda_r - \lambda_s}{\lambda_s}$$

$$\boxed{P(\omega_i|x) \leq 1 - \frac{\lambda_r}{\lambda_s}}$$

If $\lambda_r = 0$, the cost of rejection would be minimum, and we would reject everything.

If $\lambda_r > \lambda_s$, we would always prefer to substitute and we would never reject.

PROBLEM 3

$p(x|\omega_1) \propto N(0, I)$
$p(x|\omega_2) \propto N\left(\begin{bmatrix}2\\1\end{bmatrix}, I\right)$
$P(\omega_1) = P(\omega_2) = 1/2.$

(a) $g_1(x) = g_2(x)$

Let's find the two discriminant functions

$$g_1(x) = \left(\frac{\mu_1}{\sigma^2}\right)^t x \xrightarrow{} 0 \quad - \cancel{\frac{1}{2\sigma^2}\mu_1^t\mu_1} \xrightarrow{} 0 \quad + \ln P(\omega_1)$$

$$\boxed{g_1(x) = \ln 1/2}$$

$$g_2(x) = \left(\frac{\mu_2}{\sigma^2}\right)^t x - \frac{1}{2\sigma^2}\mu_2^t\mu_2 + \ln P(\omega_2)$$

$\sigma^2 \to 1$

$$= [2\ 1]x - \frac{1}{2}[2\ 1]\begin{bmatrix}2\\1\end{bmatrix} + \ln 1/2$$

$$\boxed{g_2(x) = [2\ 1]x - 5/2 + \ln 1/2}$$

thus,

$$g_1(x) = g_2(x)$$
$$\cancel{\ln 1/2} = [2\ 1]x - 5/2 + \cancel{\ln 1/2}$$
$$[2\ 1]x = 5/2$$
$$[2\ 1]\begin{bmatrix}x_1\\x_2\end{bmatrix} = 5/2 \rightarrow \boxed{2x_1 + x_2 = 5/2}$$

(b)
$p(x|\omega_1) \propto N\left(0, \begin{bmatrix}2&0\\0&2\end{bmatrix}\right)$
$p(x|\omega_2) \propto N\left(\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}4&3\\2&4\end{bmatrix}\right)$

$$g_1(x) = \left(\frac{\mu_1}{\sigma^2}\right)^t x \xrightarrow{} 0 \quad - \cancel{\frac{1}{2\sigma^2}\mu_1^t\mu_1} \xrightarrow{} 0 \quad + \ln P(\omega_1)$$
$$= \ln P(\omega_1) = \boxed{\ln 1/2}$$

$\rightarrow$

$$g_2(x) = x^t W_2 x + w_2^t x + w_{20}$$

$$\Sigma^{-1} = \begin{bmatrix} 2/5 & -3/10 \\ -1/5 & 2/5 \end{bmatrix} = 1/10 \begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}$$

$$|\Sigma| = 10$$

$$W_2 = -\tfrac{1}{2}\Sigma_2^{-1}$$

$$w_2 = \Sigma_2^{-1}\mu_2$$

$$w_{20} = \frac{-1}{2}\mu_2^t\Sigma_2^{-1}\mu_2 - \tfrac{1}{2}\ln|\Sigma_2| + \ln P(\omega_2)$$

$$\boxed{W_2 = \frac{-1}{20}\begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}} \quad \boxed{w_2 = \frac{1}{10}\begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}}$$

$$w_{20} = \frac{-1}{2}[2\ 1]\ 1/10\begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \tfrac{1}{2}\ln 10 + \ln \tfrac{1}{2}$$

$$\boxed{w_{20} = -\tfrac{1}{2} - \tfrac{1}{2}\ln 10 + \ln \tfrac{1}{2}}$$

$$g_2(x) = x^t\left(\frac{-1}{20}\right)\begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}x + [1/2\ 0]x - 1/2 - 1/2\ln 10 + \ln 1/2$$

So, the decision boundary is found when

$$g_1(x) = g_2(x)$$

$$\cancel{\ln 1/2} = x^t(-1/20)\begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}x + [1/2\ 0]x - 1/2 - 1/2\ln 10 + \cancel{\ln 1/2}$$

$$x^t(-1/20)\begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}x + [1/2\ 0]x = 1/2 + 1/2\ln 10$$

$$[x_1\ x_2](-1/20)\begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1/2\ 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1/2 + 1/2\ln 10$$

$$(-1/20)[x_1\ x_2]\begin{bmatrix} 4x_1 - 3x_2 \\ -2x_1 + 4x_2 \end{bmatrix} + 1/2\,x_1 = 1/2 + 1/2\ln 10$$

$$\left(\frac{-1}{20}\right)(4x_1^2 - 3\cancel{x_2}x_1 - 2\cancel{x_1}x_2 + 4x_2^2) + 1/2\,x_1 = 1/2 + 1/2\ln 10$$

$$\boxed{-1/5\,x_1^2 + 1/4\,x_2x_1 - 1/5\,x_2^2 + 1/2\,x_1 = 1/2 + 1/2\ln 10}$$

PROBLEM 4

PART 1

```
function data = getRandn(N, mean, var)
    data = mean + sqrt(var).* randn(N,1);
end
```

PART 2

$N_1 = 1000$ $\quad$ $N_2 = 2000$ $\quad$ $N_3 = 3000$

$\mu_1 = 2$ $\quad$ $\mu_2 = 4$ $\quad$ $\mu_3 = ?$

$\sigma_1^2 = 4$ $\quad$ $\sigma_2^2 = 9$ $\quad$ $\sigma_3^2 = ?$

$$\text{mean}_1 = \frac{\text{sum of elts } N_1}{1000} \rightarrow \text{sum of elts } N_1 = 1000 . \text{mean}_1$$

$$\text{mean}_2 = \frac{\text{sum of eltos } N_2}{2000} \rightarrow \text{sum of eltos } N_2 = 2000 . \text{mean}_2$$

$$\text{mean}_3 = \frac{\text{sum of eltos } N_3}{3000} = \frac{\text{sum of eltos } N_1 + \text{sum of eltos } N_2}{3000}$$

$$= \frac{1\cancel{000}.\text{mean}_1 + 2\cancel{000}\ \text{mean}_2}{3\cancel{000}} = \frac{2 + 2.4}{3} = 3.33$$

$$\text{Var}_3 = \sum_{i=1}^{3000} (x_i - \text{mean}_3)^2$$

$$= \sum_{i=1}^{3000} x_i^2 - 6.66x_i + 3.33^2$$

$$= \sum_{i=1}^{3000} x^2 - 6.66\sum_{i=1}^{3000} x_i + 3000 . 3.33^2$$