combra.cs.rutgers.edu

The last lecture on Combinatorics

206
Discrete Structures II

Konstantinos P. Michmizos

Computational Brain Lab

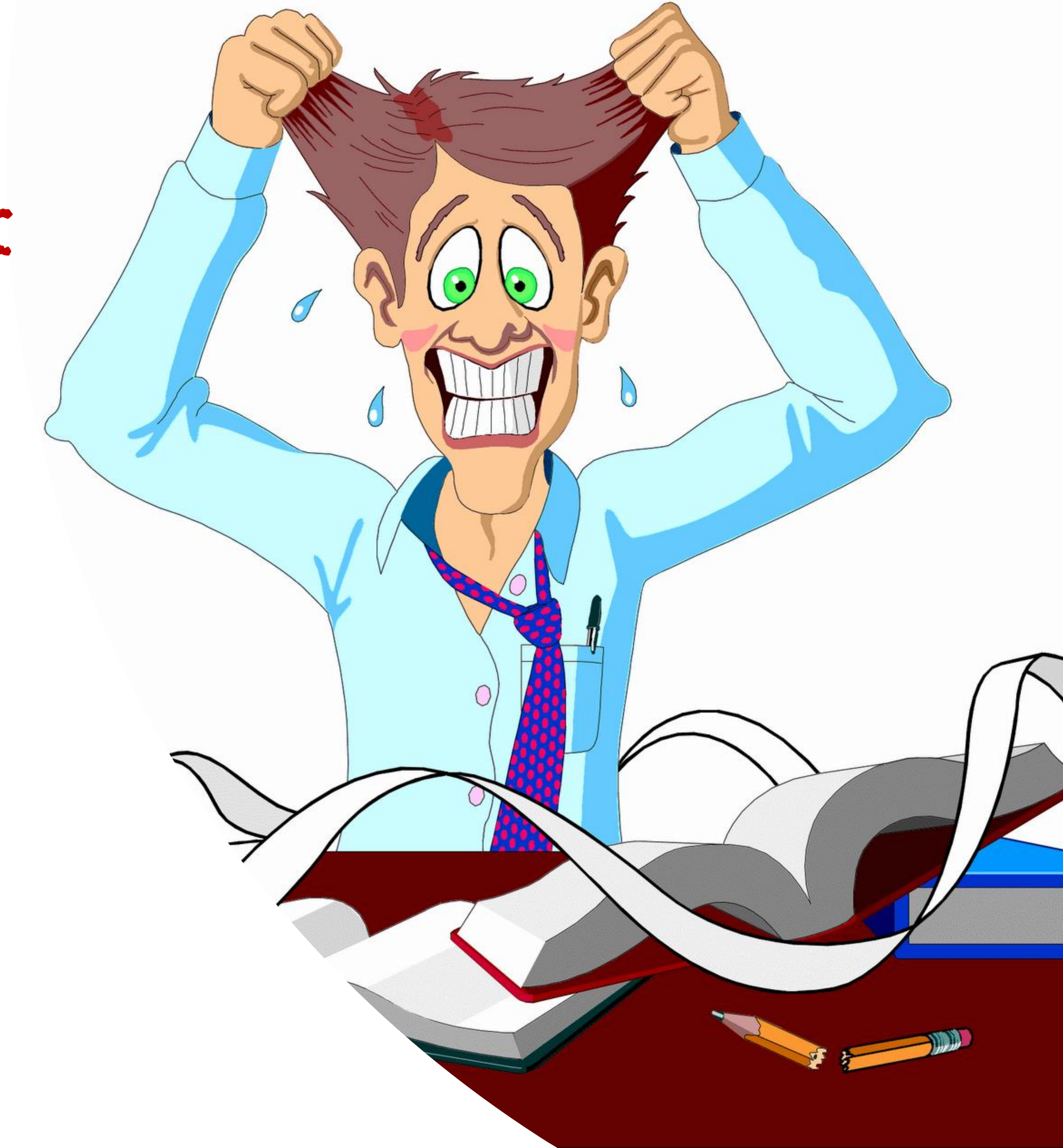
Computer Science | Rutgers University | NJ, USA

The Binomial Formula – **Preview!**

Find coefficient of $x^{10}y^{5}$ in $(19x+4y)^{15}$

Quiz 4 – Next Week

- Lectures 16 - 20
  - Inclusion-Exclusion / Pigeonhole Principle
  - Combinatorial Proofs and Binomial Coefficients
    - And everything else…
- During recitation



So far...

Combinatorial Proofs – Hints *Revisited*!

- Define a set S.
- Show that $|S| = n$ by counting one way.
- Show that $|S| = m$ by counting ***another way***.
- Conclude that $n = m$.

So far...

Combinatorial Proofs

In general, to give a combinatorial proof for a binomial identity, say A = B you do the following:

1. **Find a counting problem** *you will be able to answer in two ways*.
2. Explain why one answer to the counting problem is A.
3. Explain why the other answer to the counting problem is B.

Since both A and B are the answers to the same question, we must have A=B.

The tricky thing is coming up with the question. This is not always obvious, but it gets easier the more counting problems you solve.

Not so fast...

Combinatorial Proofs

Why

$$\binom{n}{0} = 1$$

- The number of ways to select 0 objects from a collection of n objects. And there is only 1 way to do this, to not select any of the objects.

$$\binom{n}{n} = 1$$

- The number of ways to select n objects from a collection of n objects. There is only 1 way to do this, to select all objects.

Hint! Σ → consider sum rule

So far...

Combinatorial Proofs – Example 2

- Prove that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

Let $S = \{1, 2, \cdots n\}$

Counting problem: How many subsets of S?

RHS = 2 choices for each element
→ Hence, # subsets $= 2^n$

LHS = use partition rule
→ Count all subsets of size 0 → $\binom{n}{0}$
→ " " size 1 → $\binom{n}{1}$ $= \sum_{k=0}^{n}\binom{n}{k}$

→ " size n → $\binom{n}{n}$

Combinatorial Proofs

- Prove that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

- Define the problem: If a pizza joint offers *n toppings*, how many pizzas can we build using any number of toppings from no toppings to all toppings, using each topping at most once?
- On one hand, the answer is 2^n. For each topping you can say “yes” or “no,” so you have two choices for each topping.
- On the other hand, divide the possible pizzas into disjoint groups: the pizzas with no toppings, the pizzas with one topping, the pizzas with two toppings, etc.
  - If we want no toppings, there is only one pizza like that
  (the empty pizza, if you will)

Combinatorial Proofs

RHS

Pizzas with 0 toppings: $\binom{n}{0}$

Pizzas with 1 topping: $\binom{n}{1}$

Pizzas with 2 toppings: $\binom{n}{2}$

$\vdots$

Pizzas with n toppings: $\binom{n}{n}$.

The total number of possible pizzas will be the sum of these, which is exactly the left-hand side of the identity we are trying to prove.

Combinatorial Proofs - Correction

- Prove that $\sum_{k=0}^{\cancel{n}\ n/2}\binom{n}{2k} = 2^{n-1}$

Problem: # even sized subsets of n elements

RHS = 2^{n-1} $[\quad]$

LHS = Use partition method

- subsets of size 0 → $\binom{n}{0}$
- subsets of size 2 → $\binom{n}{2}$ → $\sum_{K=0}^{n/2}\binom{n}{2K} = 2^{n-1}$
- subsets of size 4 → $\binom{n}{4}$

So far...

Combinatorial Proofs – Example 3

- Prove that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Counting Problem: # ways to choose k out of n people

— LHS = $\binom{n}{k}$

RHS: Use partition method

Case1: # ways to choose k out of n such that element 1 is chosen

Case2: # ways to chose k out of n such that element 1 is not chosen

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Combinatorial Proofs

So far...

- Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof. By the definition of $\binom{n}{k}$, we have

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(n-1-(k-1))!(k-1)!} = \frac{(n-1)!}{(n-k)!(k-1)!}$$

and

$$\binom{n-1}{k} = \frac{(n-1)!}{(n-1-k)!k!}.$$

Thus, starting with the right-hand side of the equation:

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(n-k)!(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!} \\
&= \frac{(n-1)!k}{(n-k)!\,k!} + \frac{(n-1)!(n-k)}{(n-k)!\,k!} \\
&= \frac{(n-1)!(k+n-k)}{(n-k)!\,k!} \\
&= \frac{n!}{(n-k)!\,k!} \\
&= \binom{n}{k}.
\end{aligned}$$

The second line (where the common denominator is found) works because $k(k-1)! = k!$ and $(n-k)(n-k-1)! = (n-k)!$. QED

Certainly a valid proof, but also entirely useless

Why? Even if you understand the proof perfectly,

it does not tell you why the identity is true.

New...

Binomial Coefficients

- $\binom{n}{k}$, known as the ***Binomial Coefficient***.
  - Number of ways to pick k out of n distinct objects.
  - Intimately connected to algebraic polynomials.

Examples:

$$\left.\begin{array}{l} 1+x \\ 1+x+3x^2 \\ 5x^3-2x^2+7x-8 \end{array}\right\} \rightarrow \text{univariate polynomials}$$

General: $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_nx^n \rightarrow$ degree n polynomial

Basic Problem: Given a polynomial, infer the co-efficients

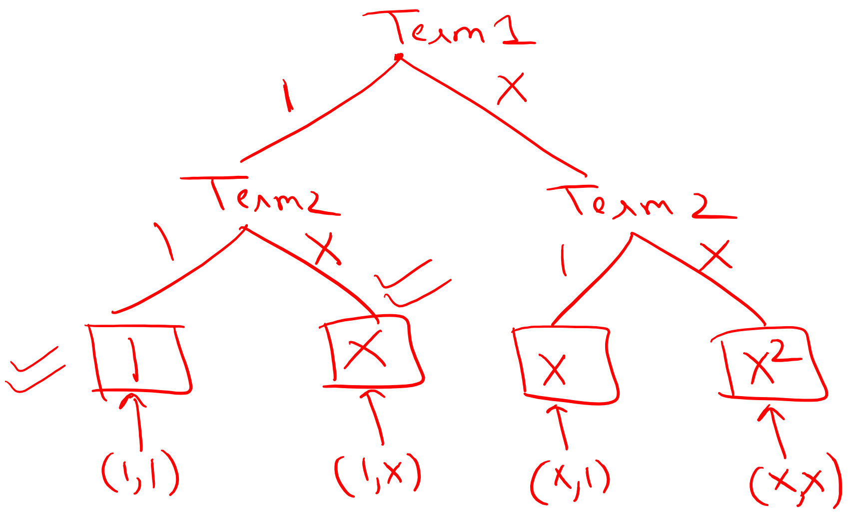
Binomial Coefficients – Building insight

- $(1+x)^2 = 1 + 2x + x^2$

Given: $(1+x)^2 \to (1+x)\cdot(1+x) = 1 + x + x + x^2$

$= 1 + \underline{2x} + x^2$

$(1+x)^2 \to \underbrace{(1+x)}_{\text{Term 1}}\ \underbrace{(1+x)}_{\text{Term 2}}$



Co-efficient of x = # ways to reach x from root node

$= 2$

ways to reach x

= # ways to choose x once out of 2 terms $= \binom{2}{1}$

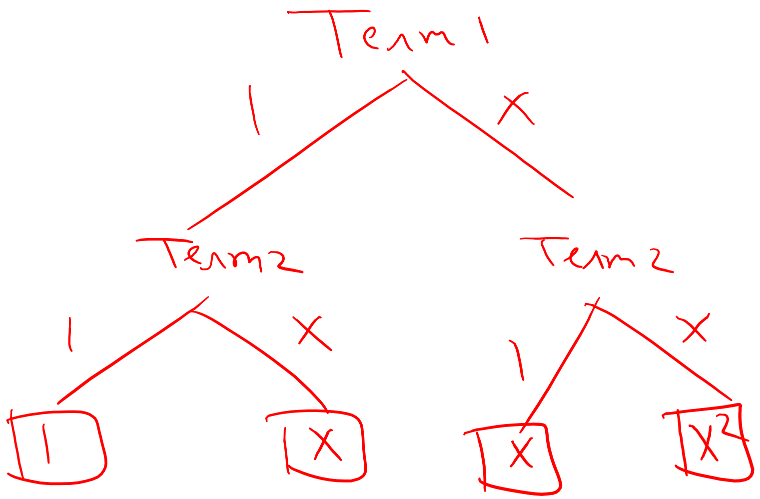
Binomial Coefficients

- $(1 + x)^2 = 1 + 2x + x^2$

\# ways to reach $x = \binom{2}{1} = 2$

\# ways to reach $x^2 = \binom{2}{2} = 1$

\# ways to reach $1 = \binom{2}{2} = 1$



Binomial Coefficients

- $(1+x)^3 = 1 + 3x + 3x^2 + x^3$

$$\underbrace{(1+x)}_{\text{Term1}}\ \underbrace{(1+x)}_{\text{Term2}}\ \underbrace{(1+x)}_{\text{Term3}}$$

Co-efficient of x

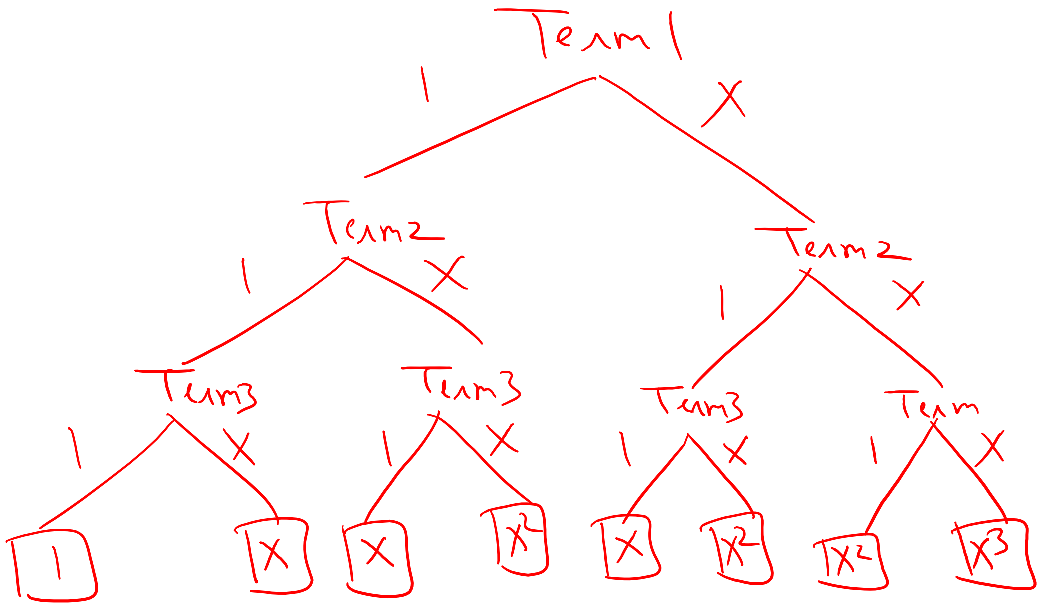
$= \#$ ways to reach x

$= \binom{3}{1} = 3$

Co-efficient of x^2

$= \#$ ways to reach x^2

$= \binom{3}{2} = 3$



Binomial Coefficients

- $(1+x)^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$

$c_n = \#$ ways to reach $x^n = \binom{n}{n}$

$c_{n-1} = \#$ ways to reach $x^{n-1} = \binom{n}{n-1}$

$\vdots$

$c_k = \#$ ways to reach $x^k = \binom{n}{k}$ → Binomial Coefficients

$\vdots$

$c_0 = \#$ ways to reach $x^0 = \binom{n}{0} = 1$

Binomial Coefficients

- $(1+x)^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$

- What is c_k?
    - Number of paths in the choice tree with exactly k x's.
    - $= \binom{n}{k}$

The Binomial Formula – Univariate Case

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

The Binomial Formula – Example 1

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

- $x = 1$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n}$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

$\Rightarrow$ Proof that the size of the powerset is 2^n

The Binomial Formula – Example 2

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

- $x = -1$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots$$

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \binom{n}{7} \cdots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

$\hookrightarrow$ # subsets of odd size = # subsets of even size

The Binomial Formula - Example 3

- Prove that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

$$\frac{\partial}{\partial x}\left[(1+x)^n\right] = \frac{\partial}{\partial x}\left[\binom{n}{0} + x\binom{n}{1} + x^2\binom{n}{2} + \cdots x^k\binom{n}{k} + \cdots x^n\binom{n}{n}\right]$$

$$= n(1+x)^{n-1} = \binom{n}{1} + 2x\binom{n}{2} + \cdots kx^{k-1}\binom{n}{k} + \cdots nx^{n-1}\binom{n}{n}$$

Set $x=1$

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots k\binom{n}{k} + \cdots n\binom{n}{n}$$

$$\Rightarrow \quad n \cdot 2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

Take a Break

5 min

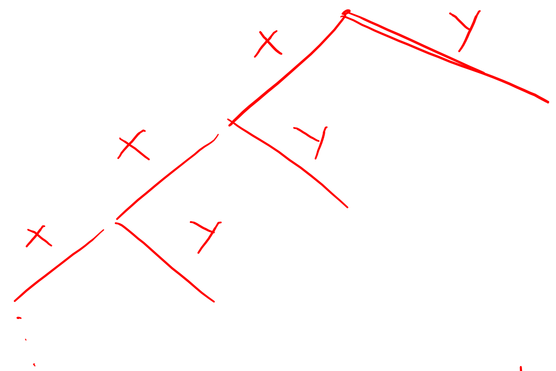
The Binomial Formula – Multivariate Case

$$(x+y)^n$$

multivariate polynomial

$$(x+y)^n = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots\ a_ny^n$$

$$\underbrace{(x+y)}_{T1}\underbrace{(x+y)}_{T2}\cdots\underbrace{(x+y)}_{Tn}$$



$a_0 = \#$ ways to reach x^n

$$= \binom{n}{n} = \binom{n}{0}$$

$a_1 = \#$ ways to reach $x^{n-1}y$

$$= \binom{n}{n-1} = \binom{n}{1}$$

$a_k = \#$ ways to reach $x^{n-k}y^k$

$$= \binom{n}{n-k} = \binom{n}{k}$$

The Binomial Formula - Multivariate

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} x^0 y^n$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

The Binomial Formula – Example 4

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Find coefficient of $x^{10}y^5$ in $(x+y)^{15}$

— 15 terms

— want to get $x^{10}y^5$

— # ways $= \binom{15}{10} = \binom{15}{5}$

The Binomial Formula – Example 5

Find coefficient of $x^{10}y^5$ in $(19x+4y)^{15}$

$(19x+4y)(19x+4y) - - - (19x+4y)$

— Want to get $x^{10}y^5$

— # ways $= \binom{15}{10} = \binom{15}{5}$

$\to$ Coefficient $= \binom{15}{10}(19)^{10}(4)^5$

Coefficient of $x^{10}y^5$ in $(19x-4y)^{15}$

Co-efficient $= \binom{15}{10}(19)^{10}(-4)^5$

The Multinomial Formula – 3 variables

$$(x+y+z)^n$$

$(x+y+z)^n = a_0x^n + a_1x^{n-1}y + a_2x^{n-1}z - - - -$

$+ a_k x^a y^b z^c + - - -$

$x^a y^b z^c \rightarrow$ choice sequence must have a xs, b ys and c zs.

$\rightarrow$ any arrangement of a xs, b ys, c zs gives a valid way to get $x^a y^b z^c$

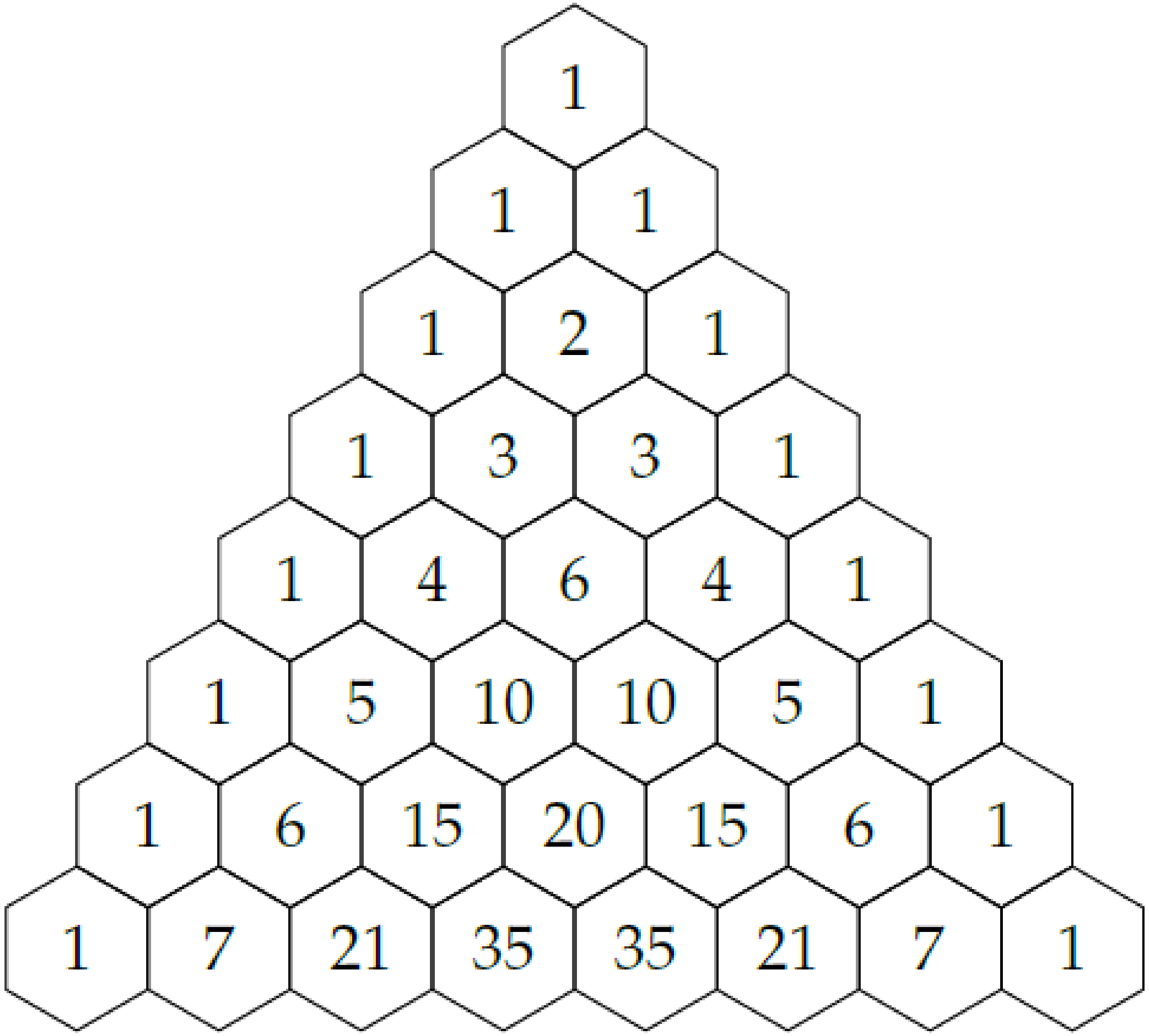
coefficient of $x^a y^b z^c$

= # arrangements

$= \dfrac{n!}{a!\,b!\,c!}$

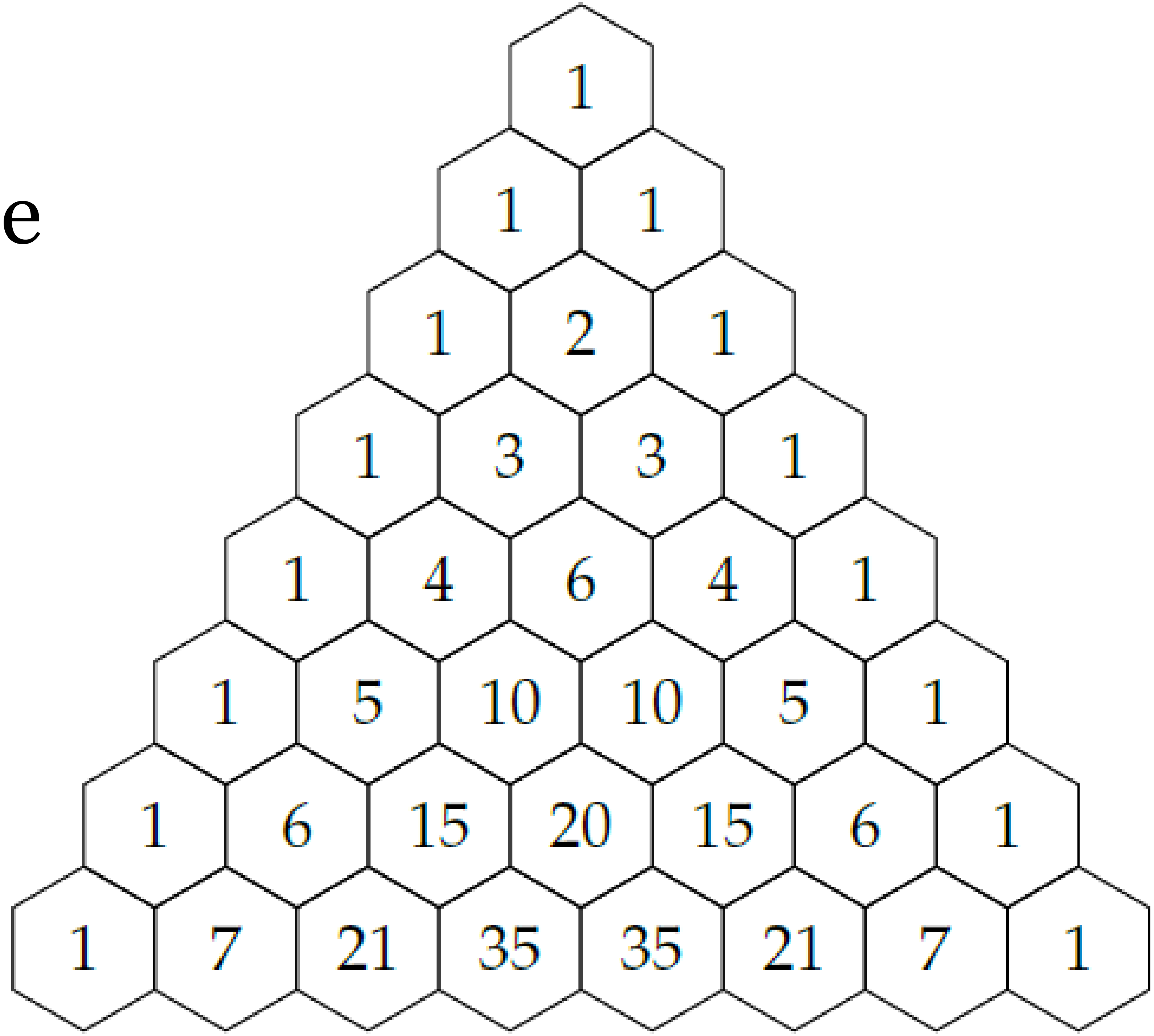
The Multinomial Formula – 3 variables

$$(x+y+z)^n = \sum_{k_1+k_2+k_3=n} \frac{n!}{k_1!\,k_2!\,k_3!} x^{k_1} y^{k_2} z^{k_3}$$


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Pascal's Triangle

- 1. The entries on the border of the triangle are all 1.
- 2. Any entry not on the border is the sum of the two entries above it.
- 3. The triangle is symmetric. In any row, entries on the left side are mirrored on the right side.
- 4. The sum of all entries on a given row is a power of 2. (Check this!)



Line 4:

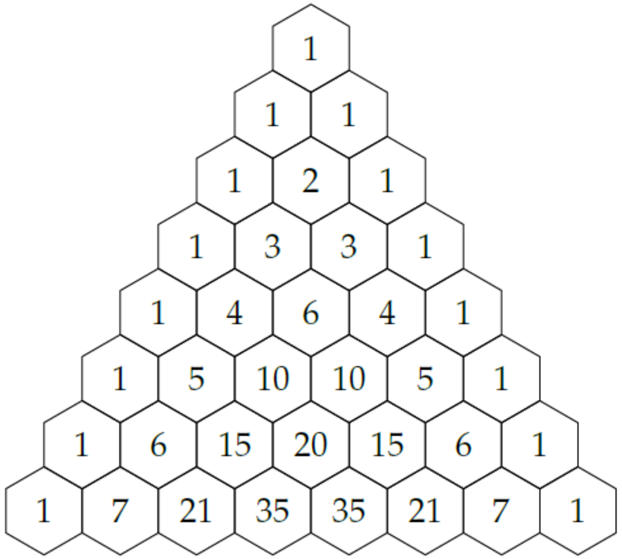
1, 4, 6, 4, 1

$$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$$

Pascal’s Triangle

- Each entry in Pascal’s triangle is in fact a binomial coefficient.

- We can use Pascal’s triangle (and other counting methods we have learned) to prove binomial identities, i.e., equations that involve binomial coefficients



Pascal's Triangle