CS206 Quiz 1 **Sep 28, 2021** **A**

Name: ______________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

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1. (10%) True/False

   (a) Let $\mathcal{P}(S)$ be the power set of a set S. If S is a finite set with n elements, i.e., $|S| = n$ and $n \in \mathbb{N}$, then $|\mathcal{P}(S)| = 2^n - 1$.

   Solution: FALSE. The correct answer is 2^n.

   (b) The contrapositive statement of "if you get math jokes, you don't have friends" is "if you have friends, you don't get math jokes."

   Solution: TRUE.

2. (20%) Using the method of **proof by contrapositive**, prove the following theorem:

$$\text{if } r \text{ is irrational, then } r^{\frac{1}{5}} \text{ is irrational.}$$

Solution: The contrapositive statement is "if $r^{\frac{1}{5}}$ is rational, then r is rational." To prove this, let $r^{\frac{1}{5}} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. The we have $r = (r^{\frac{1}{5}})^5 = (\frac{a}{b})^5 = \frac{a^5}{b^5}$. And we know $a^5 \in \mathbb{Z}$ and $b^5 \in \mathbb{Z}$ because $a, b \in \mathbb{Z}$, hence $r \in \mathbb{Q}$. ■

3. (20%) Using the method of **direct proof**, prove the following theorem:

$$\forall n \in \mathbb{Z}, \text{ if } n \text{ is an even number, then } n^2 + 3n \text{ is an even number.}$$

Solution: Let $n = 2x$ and $x \in \mathbb{Z}$, then we have the following proof:

$$\begin{aligned} n^2 + 3n &= (2x)^2 + 3(2x) \\ &= 4x^2 + 6x \\ &= 2(2x^2 + 3x) \end{aligned}$$

And we also know that $x \in \mathbb{Z} \Rightarrow 2x^2 \in \mathbb{Z} \land 3x \in \mathbb{Z} \Rightarrow 2x^2 + 3x \in \mathbb{Z}$, so $2(2x^2 + 3x)$ is an even number.■

4. (20%) Using the method of **contradiction**, prove the following theorem:

   there exists no combination of $x, y \in \mathbb{Z}$ such that $36y + 12x = 3$.

   Solution: Assume the statement P: "there exists at least one pair of $x, y \in \mathbb{Z}$ such that $36y + 12x = 3$" is true. Then we get $3y + x = \frac{3}{12} = \frac{1}{4}$. Since $3y + x$ is an integer, it can never equal $\frac{1}{4}$. Due to this contradiction, the statement P is false, so the original theorem is true. ■

5. (10%) For any set A, let $\mathcal{P}(A)$ be its power set. Let $\emptyset$ denote the empty set.

   (a) Write down all the elements of $\mathcal{P}(\{1,2,3\})$.

   Solution:

   $$\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

   (b) How many elements are there in $\mathcal{P}(\{1,2,3,4,5,6\})$?

   Solution: $|\mathcal{P}(1,2,3,4,5,6)| = 2^6 = 64$.

6. (20%) There are 120 students in a school, 50 don't play sports, 70 don't play music, and 30 don't do either. How many students do both?

   Solution:
   "50 students don't play sports" $\Rightarrow$ "$120 - 50 = 70$ student play sports"
   "70 students don't play music" $\Rightarrow$ "$120 - 70 = 50$ students play music"
   Let x be the number of students that do both. Then we have the equation that $120 = (70 + 50 - x) + 30$ and hence $x = 30$.
   (In conclusion, there are $70 - 30 = 40$ students exclusively play sports, $50 - 30 = 20$ students exclusively play music, 30 students do both, and 30 do neither.)

7. (Extra Credits - 20%) There should be 6 characters in a NJ plate that include uppercase or lowercase letters, digits, and three special characters (e.g., space or dash). Use the **product rule** to calculate how many NJ plates are available.
   Note 1: The English alphabet has 26 letters; Note 2: Yes, there could be a NJ plate with all spaces, for the sake of this exercise...

   Solution: Each place has $26 + 26 + 10 + 3 = 65$ options, so the answer is $65 \times 65 \times 65 \times 65 \times 65 \times 65 = 65^6$.

GOOD LUCK!

CS206 Quiz 1 **Sep 30, 2021** **B**

Name: ____________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

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1. (10%) True/False

   (a) Let $\mathcal{P}(S)$ be the power set of a set S. If S is a finite set with n elements, i.e., $|S| = n$ and $n \in \mathbb{N}$, then $|\mathcal{P}(S)| = 2^n + 1$.

   Solution: FALSE. The correct answer is 2^n.

   (b) The contrapositive statement of "if you get math jokes, you don't have friends" is "if you don't get math jokes, you have friends."

   Solution: FALSE. The correct contrapositive statement is "if you have friends, you don't get math jokes."

2. (20%) Using the method of **proof by contrapositive**, prove the following theorem:

$$\text{if } r \text{ is irrational, then } r^{\frac{1}{3}} \text{ is irrational.}$$

Solution: The contrapositive statement is "if $r^{\frac{1}{3}}$ is rational, then r is rational." To prove this, let $r^{\frac{1}{3}} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$. The we have $r = (r^{\frac{1}{3}})^3 = (\frac{a}{b})^3 = \frac{a^3}{b^3}$. And we know $a^3 \in \mathbb{Z}$ and $b^3 \in \mathbb{Z}$ because $a, b \in \mathbb{Z}$, hence $r \in \mathbb{Q}$. ■

3. (20%) Using the method of **direct proof**, prove the following theorem:

$$\forall n \in \mathbb{Z}, \text{ if } n \text{ is an even number, then } n^2 + 3n + 1 \text{ is an odd number.}$$

Solution: Let $n = 2x$ and $x \in \mathbb{Z}$, then we have the following proof:

$$\begin{aligned} n^2 + 3n + 1 &= (2x)^2 + 3(2x) + 1 \\ &= 4x^2 + 6x + 1 \\ &= 2(2x^2 + 3x) + 1 \end{aligned}$$

And we also know that $x \in \mathbb{Z} \Rightarrow 2x^2 \in \mathbb{Z} \land 3x \in \mathbb{Z} \Rightarrow 2x^2 + 3x \in \mathbb{Z}$, so $2(2x^2 + 3x) + 1$ is an odd number.■

4. (20%) Using the method of **contradiction**, prove the following theorem:

   there exists no combination of $x, y \in \mathbb{Z}$ such that $24y + 12x = 1$.

   Solution: Assume the statement P: "there exists at least one pair of $x, y \in \mathbb{Z}$ such that $24y + 12x = 1$" is true. Then we get $2y + x = \frac{1}{12}$. Since $2y + x$ is an integer, it can never equal $\frac{1}{12}$. Due to this contradiction, the statement P is false, so the original theorem is true. ■

5. (10%) For any set A, let $\mathcal{P}(A)$ be its power set. Let $\emptyset$ denote the empty set.

   (a) Write down all the elements of $\mathcal{P}(\{0, 2, 4\})$.

   Solution:

   $$\mathcal{P}(\{0,2,4\}) = \{\emptyset, \{0\}, \{2\}, \{4\}, \{0,2\}, \{0,4\}, \{2,4\}, \{0,2,4\}\}.$$

   (b) How many elements are there in $\mathcal{P}(\{1, 2, 3, 4, 5, 6, 7, 8\})$?

   Solution: $|\mathcal{P}(1,2,3,4,5,6,7,8)| = 2^8 = 256$.

6. (20%) There are 100 students in a school, 50 don't play sports, 70 don't play music, and 30 don't do either. How many students do both?

   Solution:
   "50 students don't play sports" $\Rightarrow$ "$100 - 50 = 50$ student play sports"
   "70 students don't play music" $\Rightarrow$ "$100 - 70 = 30$ students play music"
   Let x be the number of students that do both. Then we have the equation that $100 = (50 + 30 - x) + 30$ and hence $x = 10$.
   (In conclusion, there are $50 - 10 = 40$ students exclusively play sports, $30 - 10 = 20$ students exclusively play music, 10 students do both, and 30 do neither.)

7. (Extra Credits - 20%) There should be 6 characters in a NJ plate that include uppercase or lowercase letters, digits, and three special characters (e.g., space or dash). Use the **product rule** to calculate how many NJ plates are available.
Note 1: The English alphabet has 26 letters; Note 2: Yes, there could be a NJ plate with all spaces, for the sake of this exercise...

Solution: Each place has $26 + 26 + 10 + 3 = 65$ options, so the answer is $65 \times 65 \times 65 \times 65 \times 65 \times 65 = 65^6$.

GOOD LUCK!