CS206 Quiz 4 **Nov 16, 2021** **A**

Name: ______________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

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1. (20%) How many positive integers no greater than 100 are divisible by 2, 3, or 5, respectively? And how many are divisible by all of them?

   Solution: Define A as the set of positive integers not exceeding 100 divisible by 2, B the set of positive integers not exceeding 100 divisible by 3, C the set of positive integers not exceeding 100 divisible by 5, and $A \cap B \cap C$ the set of positive integers not exceeding 100 divisible by 2, 3 AND 5. The sizes of the sets are as follows: $|A| = \lfloor \frac{100}{2} \rfloor = 50$,$|B| = \lfloor \frac{100}{3} \rfloor = 33$,$|C| = \lfloor \frac{100}{5} \rfloor = 20$,$|A \cap B \cap C| = \lfloor \frac{100}{2 \times 3 \times 5} \rfloor = \lfloor \frac{100}{30} \rfloor = 3$.

2. (20%) How many 6-digit even numbers can be formed using $1, 2, 3, \ldots, 6$, if each number is used once, and 5 is never next to 1 or 3? For reference, **51**2346, 6**35**214, and 12346**5** are all invalid numbers, while 132546 is valid.

   Solution: Start with placing "5". Let the rightmost digit be the 1st slot, there are 5 slots (2, 3, 4, 5, and 6) for "5" due to the even number constraint. If "5" is placed in the 2nd or the 6th slot, then "1" and "3" have 3 slots to place (slot 3, 4 and 5), and the rest even numbers ("2", "4", and "6") takes the rest 3 slots, so $2 \times 3! \times 3!$. If "5" is placed in slot 3, 4, or 5, then "1" and "3" have only 2 slots, and slots for "2", "4", and "6" remains unchanged, so $3 \times 2! \times 3!$. In total, 108.

3. (20%) Suppose you need to come up with a password that uses only the letters A, B, C, and D and which must use each letter at least once. How many such passwords of length 9 are there?

Solution: We will first find the number of passwords that leave out at least one of A, B, or C.
Let X be the set of passwords that doesn't contain A;
Let Y be the set of passwords that doesn't contain B;
Let Z be the set of passwords that doesn't contain C.
Let W be the set of passwords that doesn't contain D.
We want to find the size of $X \cup Y \cup Z \cup W$
Passwords that don't contain A just contain B and C and D. So there are 3^9 such passwords—i.e. $|X| = 3^9$. By the same reasoning $|Y| = |Z| = |W| = 3^9$.
Passwords that don't contain A or B just contain C,D. $|X \cap Y| = 2^9$. By the same reasoning $|X \cap Z| = |Y \cap Z| = |X \cap W| = |Y \cap W| = |Z \cap W| = 2^9$.
Passwords that don't contain A or B or C just contain D. There is one such password (namely 'DDDDDDDDD') so $|X \cap Y \cap Z| = 1$. By the same reasoning $|X \cap Y \cap W| = |X \cap Z \cap W| = |Y \cap Z \cap W| = 1$.
Passwords that don't contain A, B, C or D can't exist because passwords in this problem only use the letters A,B,C,and D.So $|X \cap Y \cap Z \cap D| = 0$. So by inclusion-exclusion $X \cup Y \cup Z \cup W = 4 * 3^9 - 6 * 2^9 + 4 - 0 = 4 * 3^9 - 6 * 2^9 + 4$.
To find the answer to the original question, we need to subtract the number we just found from the total number of passwords, which is 4^9. This gives $4^9 - (4 * 3^9 - 6 * 2^9 + 4)$

4. (20%) Suppose that at a certain college there are only three majors: math, biology and CS. Also suppose that there are:

   (a) 85 students total in the college
   (b) 5 students who have not declared a major yet
   (c) 50 math majors, 30 biology majors, and 40 CS majors
   (d) 10 double majors in math and biology, 20 in math and CS, and 10 in biology and CS.

   How many declared triple majors are there?

Solution: We have the following sets: M,B,C for students with a declared majors Math, Biology, CS. The sizes of the following sets are given in the problem: $|M| = 50, |B| = 30, |C| = 40, |M \cap B| = 10, |M \cap C| = 20, |B \cap C| = 10$. The number of students with atleast a single declared major is $|M \cup B \cup C| = 85 - 5$. The question asks for the size of set $|M \cap B \cap C|$, the number of students with all 3 majors declared as their majors. Using the principle of inclusion-exclusion, we get $|M \cap B \cap C| = |M \cup B \cup C| - |M| - |B| - |C| + |M \cap B| + |M \cap C| + |B \cap C| = 80 - 50 - 30 - 40 + 10 + 20 + 10 = 0$. Thus, there are no declared triple majors.

5. (20%) Suppose S is a set of $n + 1$ integers. Using the pigeonhole principle, prove that there exist distinct $a, b \in S$ such that $a - b$ is a multiple of n.

Solution: Since there are n residuals of the elements of S, modulo n, but S consists of $n + 1$ elements, then by the pigeonhole principle, there must be distinct elements $a, b \in S$ such that $a \equiv b (modn)$. In other words, the $n + 1$ element must have the same residual as one of the 0 to n elements.

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6. (extra 15 credits for Quiz 3) Consider the following information regarding three sets A, B, and C. Suppose that $|A| = 14$, $|B| = 10$ and $|A \cup B \cup C| = 24$ and $|A \cap B| = 6$. Consider the following assertions:
1. C has at most 24 members.
2. C has at least 6 members.
3. $A \cup B$ has exactly 18 members.
Which ones are true?

Solution: 1. True since $|C| \leq |A \cup B \cup C| = 24$.
2. True since $|A \cup B| = 18$ and hence $|C| \geq 6$ for $|A \cup B \cup C| = 24$ and $|A \cap B| = 6$
3. True since by inclusion/exclusion we have $|A \cup B| = |A| + |B| - |A \cap B|$ = 14+10-6 = 18.

7. (extra 15 credits for Quiz 3) How many ways are there to distribute 5 different toys among 3 different children if every child must get at least one toy? *Hint: Use the Difference Method and (then) the Inclusion/Exclusion principle.*

Solution: $3^5 - 3 \times 31$.
Let A be outcomes where kid 1 gets nothing, B be outcomes where kid 2 gets nothing, and C be outcomes where kid 3 gets nothing. Then answer is total outcomes - $|A \cup B \cup C|$. Total outcomes equals 3^5. Furthermore, $|A| = |B| = |C| = 2^5$, $|A \cap B| = |B \cap C| = |B \cap C| = 1$, and $|A \cup B \cup C| = 0$.

8. (extra 15 credits for Quiz 3) From an unlimited supply of blocks that are blue, red, and yellow, in how many ways can 7 be selected if there must be more blue than red and more red than yellow. *Hint: How many yellow blocks can we really choose?*

   Solution: The number of yellow blocks can be either 0 or 1. When number of yellow blocks is 0 we are looking at the number of non-negative solutions to $x_1 + x_2 = 7$ with $x_1 > x_2$ and $x_2 \geq 1$. This equals 3.
   When number of yellow blocks is 1 we are looking at the number of non-negative solutions to $x_1 + x_2 = 6$ with $x_1 > x_2$ and $x_2 \geq 2$. This equals 1. Therefore, we have 3+1 = 4 total solutions.

9. (extra 15 credits for Quiz 3) Next week, I'm going to get really fit! On day 1, I will exercise for 5 minutes. On each subsequent day, I will exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

   Solution: 4^6. I have 4 choices for day-2, 4 choices for day-3 etc. Day 1 is fixed to 5 minutes (1 choice.)

CS206 Quiz 4 **Nov 18, 2021** **B**

Name: ____________________________

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Section No.: ____________

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1. (20%) How many positive integers no greater than 100 are divisible by 2, 5, or 7 , respectively? And how many are divisible by all of them?

   Solution: Define A as the set of positive integers not exceeding 100 divisible by 2, B the set of positive integers not exceeding 100 divisible by 3, C the set of positive integers not exceeding 100 divisible by 5, and $A \cap B \cap C$ the set of positive integers not exceeding 100 divisible by 2, 3 AND 5. The sizes of the sets are as follows: $|A| = \lfloor \frac{100}{2} \rfloor = 50$,$|B| = \lfloor \frac{100}{5} \rfloor = 20$,$|C| = \lfloor \frac{100}{7} \rfloor = 14$,$|A \cap B \cap C| = \lfloor \frac{100}{2\times5\times7} \rfloor = \lfloor \frac{100}{70} \rfloor = 1$.

2. (20%) How many 6-digit even numbers can be formed using $1, 2, 3, \ldots, 6$, if each number is used once, and 5 is never next to 1 or 3? For reference, **51**2346, 6**35**214, and 12346**5** are all invalid numbers, while 132546 is valid.

   Solution: Start with placing "5". Let the rightmost digit be the 1st slot, there are 5 slots (2, 3, 4, 5, and 6) for "5" due to the even number constraint. If "5" is placed in the 2nd or the 6th slot, then "1" and "3" have 3 slots to place (slot 3, 4 and 5), and the rest even numbers ("2", "4", and "6") takes the rest 3 slots, so $2 \times 3! \times 3!$. If "5" is placed in slot 3, 4, or 5, then "1" and "3" have only 2 slots, and slots for "2", "4", and "6" remains unchanged, so $3 \times 2! \times 3!$. In total, 108.

3. (20%) Suppose you need to come up with a password that uses only the letters A, B, C, and D and which must use each letter at least once. How many such passwords of length 7 are there?

Solution: We will first find the number of passwords that leave out at least one of A, B, or C.
Let X be the set of passwords that doesn't contain A;
Let Y be the set of passwords that doesn't contain B;
Let Z be the set of passwords that doesn't contain C.
Let W be the set of passwords that doesn't contain D.
We want to find the size of $X \cup Y \cup Z \cup W$
Passwords that don't contain A just contain B and C and D. So there are 3^7 such passwords—i.e. $|X| = 3^7$. By the same reasoning $|Y| = |Z| = |W| = 3^7$.
Passwords that don't contain A or B just contain C,D. $|X \cap Y| = 2^7$. By the same reasoning $|X \cap Z| = |Y \cap Z| = |X \cap W| = |Y \cap W| = |Z \cap W| = 2^7$.
Passwords that don't contain A or B or C just contain D. There is one such password (namely 'DDDDDDD') so $|X \cap Y \cap Z| = 1$. By the same reasoning $|X \cap Y \cap W| = |X \cap Z \cap W| = |Y \cap Z \cap W| = 1$.
Passwords that don't contain A, B, C or D can't exist because passwords in this problem only use the letters A,B,C,and D.So $|X \cap Y \cap Z \cap D| = 0$. So by inclusion-exclusion $X \cup Y \cup Z \cup W = 4 * 3^7 - 6 * 2^7 + 4 - 0 = 4 * 3^7 - 6 * 2^7 + 4$.
To find the answer to the original question, we need to subtract the number we just found from the total number of passwords, which is 4^7. This gives $4^7 - (4 * 3^7 - 6 * 2^7 + 4)$

4. (20%) Suppose that at a certain college there are only three majors: math, biology and CS. Also suppose that there are:

   (a) 95 students total in the college

   (b) 5 students who have not declared a major yet

   (c) 50 math majors, 30 biology majors, and 40 CS majors

   (d) 5 double majors in math and biology, 20 in math and CS, and 5 in biology and CS.

   How many declared triple majors are there?

Solution: We have the following sets: M,B,C for students with a declared majors Math, Biology, CS. The sizes of the following sets are given in the problem: $|M| = 50$,$|B| = 30$,$|C| = 40$,$|M \cap B| = 5$,$|M \cap C| = 20$,$|B \cap C| = 5$. The number of students with atleast a single declared major is $|M \cup B \cup C| = 95 - 5$. The question asks for the size of set $|M \cap B \cap C|$, the number of students with all 3 majors declared as their majors. Using the principle of inclusion-exclusion, we get $|M \cap B \cap C| = |M \cup B \cup C| - |M| - |B| - |C| + |M \cap B| + |M \cap C| + |B \cap C| = 90 - 50 - 30 - 40 + 5 + 20 + 5 = 0$. Thus, there are no declared triple majors.

5. (20%) Suppose S is a set of $n + 1$ integers. Using the pigeonhole principle, prove that there exist distinct $a, b \in S$ such that $a - b$ is a multiple of n.

Solution: Since there are n residuals of the elements of S, modulo n, but S consists of $n + 1$ elements, then by the pigeonhole principle, there must be distinct elements $a, b \in S$ such that $a \equiv b (mod n)$. In other words, the $n + 1$ element must have the same residual as one of the 0 to n elements.

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2. True since $|A \cup B| = 18$ and hence $|C| \geq 6$ for $|A \cup B \cup C| = 24$ and $|A \cap B| = 6$
3. True since by inclusion/exclusion we have $|A \cup B| = |A| + |B| - |A \cap B|$ = 14+10-6 = 18.

7. (extra 15 credits for Quiz 3) How many ways are there to distribute 5 different toys among 3 different children if every child must get at least one toy? *Hint: Use the Difference Method and (then) the Inclusion/Exclusion principle.*

Solution: $3^5 - 3 \times 31$.
Let A be outcomes where kid 1 gets nothing, B be outcomes where kid 2 gets nothing, and C be outcomes where kid 3 gets nothing. Then answer is total outcomes - $|A \cup B \cup C|$. Total outcomes equals 3^5. Furthermore, $|A| = |B| = |C| = 2^5$, $|A \cap B| = |B \cap C| = |B \cap C| = 1$, and $|A \cup B \cup C| = 0$.

8. (extra 15 credits for Quiz 3) From an unlimited supply of blocks that are blue, red, and yellow, in how many ways can 7 be selected if there must be more blue than red and more red than yellow. *Hint: How many yellow blocks can we really choose?*

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When number of yellow blocks is 1 we are looking at the number of non-negative solutions to $x_1 + x_2 = 6$ with with $x_1 > x_2$ and $x_2 \geq 2$. This equals 1. Therefore, we have 3+1 = 4 total solutions.

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Solution: 4^6. I have 4 choices for day-2, 4 choices for day-3 etc. Day 1 is fixed to 5 minutes (1 choice.)