CS 206

Recitation - Section 4

Oct. 26

Pirates Problem

$$\forall x_i \in Z, \quad \sum_{i=1}^{n} x_i = C$$

$$\text{s.t. } a_i \leq x_i \leq b_i, i = 1, ..., n$$

Problem 1

How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 13.$$

(An **integer solution** to an equation is a solution in which the unknown must have an integer value.)

1. where $x_i \geq 0$ for each x_i?
2. where $x_i > 0$ for each x_i?
3. where $x_i \geq 2$ for each x_i?

Solution 1

Solution. This problem is just like giving 13 cookies to 5 kids. We need to say how many of the 13 units go to each of the 5 variables. In other words, we have 13 stars and 4 bars (the bars are like the "+" signs in the equation).

1. If x_i can be 0 or greater, we are in the standard case with no restrictions. So 13 stars and 4 bars can be arranged in $\binom{17}{4}$ ways.
2. Now each variable must be at least 1. So give one unit to each variable to satisfy that restriction. Now there are 8 stars left, and still 4 bars, so the number of solutions is $\binom{12}{4}$.
3. Now each variable must be 2 or greater. So before any counting, give each variable 2 units. We now have 3 remaining stars and 4 bars, so there are $\binom{7}{4}$ solutions.

Problem 2

How many integer solutions to $x_1 + x_2 + x_3 + x_4 = 25$ are there for which $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$ and $x_4 \geq 4$?

Solution 2

The problem can be reduced to $y1 + y2 + y3 + y4 = 15$

$y1, y2, y3, y4 >= 0$

The answer is $C(15+3, 3)$

Problem 3

We roll 6 standard 6-sided dice.
Find the number of outcomes with at least two dice showing 6 if order matters

Solution 3

Solution: $6^6 - (5^6 + \binom{6}{1}5^5)$

Using difference method, we subtract from the total number of arrangements 6^6 the amount of arrangements that do not include 2 dice with a 6. In other words, 5^6 gives us the number of arrangements where the 6 dice do not have a value of 6. Finally, the second term, $\binom{6}{1}5^5$ accounts for all the arrangements without a value of six in 5 remaining dice(since we accounted for the first die with term, 5^6.

Problem 4

How many 7 digit phone numbers are there in which the digits are non-increasing? That is, every digit is less than or equal to the previous one.

Solution 4

Solution. We need to decide on 7 digits so we will use 7 stars. The bars will represent a switch from each possible single digit number down to the next smaller one. So the phone number 866-5221 is represented by the stars and bars chart

$$|*||**|*|||**|*|.$$

There are 10 choices for each digit (0-9) so we must switch between choices 9 times. We have 7 stars and 9 bars, so the total number of phone numbers is $\binom{16}{9}$.

Problem 5

Using the digits 2 through 8, find the number of different 5-digit numbers such that:

(a) Digits cannot be repeated and must be written in increasing order. For example, 23678 is okay, but 32678 is not.

(b) Digits *can* be repeated and must be written in *non-decreasing* order. For example, 24448 is okay, but 24484 is not.

Solution 5

(a)

x1+x2+x3+x4+x5+x6=8 ,

x1>=0(can not be larger than 8),

x2,x3,x4,x5>=1(increasing),

x6>=2 (can not be larger than 2)

C(7,5)

(b)

x1+x2+x3+x4+x5+x6=8 ,

x1,x2,x3,x4,x5>=0 (non-decreasing),

x6>=2,

C(11,5)

Please review and think about how we get these **constraints** and **equations**?

Inclusion-Exclusion Principle

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

Prove it?

Inductive Proof

A∪B =|A∪B| = |A|+|B| - |A∩B |

|A∪B∪C| = |A|+|B|+|C| - |A∩B| - |B∩C| - |C∩A| + |A∩B∩C|

Problem 6

From 1 to 1001, how many numbers can not be divided by 7 **OR** 11 **OR** 13

Solution 6

|can not be divided by 7 **OR** 11 **OR** 13| = 1001 - |can be divided by 7 **OR** 11 **OR** 13| (difference method)

A = {can be divided by 7}

B={can be divided by 11}

C = {can be divided by 13}

|A∩B| = {can be divided by 7 and 11}

|A∩C| = {can be divided by 7 and 13}

|B∩C| = {can be divided by 11 and 13}

|A∩B∩C| = {can be divided by 7 and 11 and 13}

|can be divided by 7 **OR** 11 **OR** 13| = |A|+|B|+|C|-|A∩B|-|A∩C|-|B∩C|+|A∩B∩C|

|A| = 1001/7=143

|B|=1001/11=91

|C|=1001/13=77

|A∩B|=1001/77=13

|A∩C|=1001/91=11

|B∩C|=1001/143=7

|A∩B∩C|=1001/1001=1

|can not be divided by 7 **OR** 11 **OR** 13| = 1001- [(143+91+77) - (13+11+7) +1] = 720