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No quiz today!

# 206
# Discrete Structures II


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**Quiz 1**

# September

2021

| Sun | Mon | Tue | Wed | Thu | Fri | Sat |
|---|---|---|---|---|---|---|
| | | | 1 | 2 | 3 | 4 |
| 5 | 6<br>Labor Day | 7 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 26 | 27 | 28 | 29 | 30 | | |

# Quiz 1

- What will Quiz 1 cover?
  - Sets (Lecture 2)
  - Venn (Lecture 2)
  - Functions (Lecture 3)
  - Proofs (Lectures 3-5)
  - + What we will cover today (Sum and Product rules)

# Reading for Quiz 1

| | |
|---|---|
| Recap and Basics of Counting | Chapters 1, 2 and 5 of Rosen |
| Basics of Counting | Chapters 1, 2 and 5 of Rosen<br>Chapter 15 of Lehman |
| Basics of Counting | Chapters 6 of Rosen<br>Chapter 15 of Lehman |

# What we will cover today

Combinatorics

- Recap
  - Proofs (Direct, Contrapositive, Case Analysis, Contradiction, Induction)
- Today
  - Counting
    - Product Rule
    - Bijection Rule
- Next
  - Permutations/Combinations
  - Pigeonhole Principle

So far...

# Direct Proof

- To prove: $P \Rightarrow Q$
  - Assume that $P$ is true.
  - Show that $Q$ logically follows

The sum of two even numbers is even.

Proof $x = 2m, y = 2n$

$x+y = 2m+2n$

$= 2(m+n)$

The product of two odd numbers is odd.

Proof $x = 2m+1, y = 2n+1$

$xy = (2m+1)(2n+1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn+m+n) + 1$

proving "if P, then Q" is equivalent to proving "if not Q, then not P"

# Example of Proof by Contrapositive

- Theorem: *If r is irrational, then $\sqrt{r}$ is irrational.*

Proof:

We shall prove the contrapositive –

*"if √r is rational, then r is rational."*

Since √r is rational, √r = a/b for some integers a,b.

So $r = a^2/b^2$. Since a,b are integers, $a^2,b^2$ are integers.

Therefore, r is rational. Q.E.D.

(Q.E.D.) "which was to be demonstrated", or "quite easily done". ☺

*quod erat demonstrandum*

*Intuition: Square roots and absolute values are our worst enemies in proofs*

So far...

# Example of Proof by Case Analysis

- Theorem: *For all* $x \in \mathbb{R}$, $-5 \leq |x+2| - |x-3| \leq 5$

$|x+2| = x+2$, if $x+2 \geq 0$ or $x \geq -2$

$|x+2| = -(x+2)$, if $x+2 < 0$ or $x < -2$

$|x-3| = x-3$ if $x \geq 3$

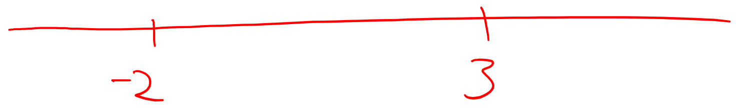
$|x-3| = -(x-3)$ if $x < 3$

$x < -2$ $\quad -2 \leq x < 3$ $\quad x \geq 3 \rightarrow$ Case I

Values of $x$ $\quad$ Case III $\quad -2 \quad$ Case II $\quad 3$

# Example of Proof by Case Analysis

- Theorem: *For all* $x \in \mathbb{R}$, $-5 \leq |x+2| - |x-3| \leq 5$




Case I: $x \geq 3$, Want $-5 \leq (x+2) - (x-3) \leq 5$

Want $-5 \leq 5 \leq 5$ $\square$

Case II: $-2 \leq x < 3$, Want $-5 \leq (x+2) - -(x-3) \leq 5$

Want $-5 \leq x+2+x-3 \leq 5$

Want $-5 \leq 2x-1 \leq 5$ $\square$

Case III $x < -2$, Want $-5 \leq -(x+2) - -(x-3) \leq 5$

Want $-5 \leq -(x+2) + x-3 \leq 5$

Want $-5 \leq -5 \leq 5$ $\square$

So far…

# Example of Proof by Contradiction

- Theorem: *There are infinitely many primes*

Assume: **There are finitely many primes** – And let $p_1, p_2, \ldots, p_N$ be all the primes.

Now we construct a new number, $p = p_1 p_2 \ldots p_N + 1$

Clearly, *p* is larger than any of the primes, so it does not equal one of them.
Therefore it cannot be prime and must be **composite**, i.e., divisible by at least one of the primes.

But our assumption was that *p* is not prime and therefore divisible by any prime number.

On the other hand, we know that any number must be divisible by *some* prime (*fundamental theorem of arithmetic or the unique factorization theorem or the unique-prime-factorization theorem*)

This leads to a **contradiction**, and therefore the assumption must be false.

So there must be infinitely many primes.

# Example of Induction

- Theorem: *For all* $n \in \mathbb{N}$,
  - $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

$P(n)$

<u>Base Case</u>: $P(0)$ is true, for $n = 0$
$LHS = 0$
$RHS = 0$

<u>Inductive Step</u>: $P(n) \Rightarrow P(n+1)$

Assume $P(n)$ is true. $1+2+\cdots+n = \frac{n(n+1)}{2}$

Want: $1+2+\cdots n + n+1 = \frac{(n+1)(n+2)}{2}$

$1+2+\cdots n + n+1 = \frac{n(n+1)}{2} + n+1 = \frac{(n+1)(n+2)}{2} = RHS$ $\square$

*Intuition: During induction, my goal is to construct what I have assumed as true*

# Course Outline

- Part I
    - Recap of basics – sets, function, proofs, induction
    - Basic counting techniques
    - Pigeonhole principle
    - Generating functions
- Part II
    - Sample spaces and events
    - Basics of probability
    - Independence, conditional probability
    - Random variables, expectation, variance
    - Moment generating functions
- Part III
    - Graph Theory
    - Machine learning and statistical inference

New Stuff

# Counting

New Stuff

# Counting

- Basic Question: What is the size of a given set?

- Easy when the set is explicitly defined.
  - $X = \{1,2,3,4\}$, what is $|X|$?

- Tricky when set is implicit or a defined via set operations.
  - How many ways to get flush in the game of poker?
  - How many ways to assign time slots to courses at Rutgers?
  - How many operations before my algorithm terminates?

New Stuff

# Counting

- In the next few lectures
  - Fundamental tools and techniques for counting
  - Sum Rule
  - Product Rule
  - Difference Method
  - Bijection Method
  - Permutations/Combinations
  - Inclusion Exclusion
  - Binomial/Multinomial coefficients

Fundamental Blocks

→ Intermediate

→ Advanced

# Sum Rule

- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- How many students are there in total in both sections?

Sum Rule:
If $A$ and $B$ are **disjoint** sets, then $|A \cup B| = |A| + |B|$

$A =$ all students in section 5
$B =$ all students in section 6

$|A \cup B| = |A| + |B| = 60 + 71 = 131$

# Example: Sum Rule

- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- There are 80 students in section 1 of 206.
- There are 80 students in section 2 of 206.
- How many students are there in total?

$60 + 71 + 80 + 80 = 291$

# Example: Sum Rule

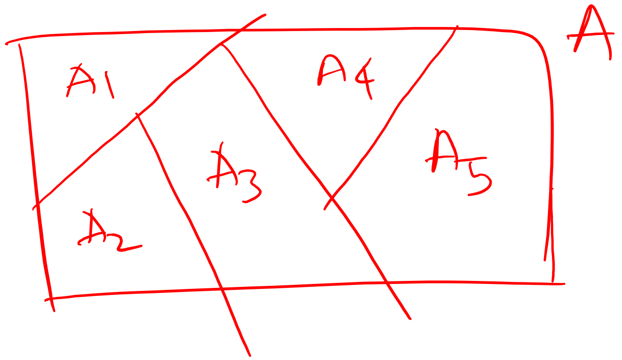
- There are 60 students in section 5 of 206.
- There are 71 students in section 6 of 206.
- There are 80 students in section 1 of 206.
- There are 80 students in section 2 of 206.
- How many students are there in total?

Sum Rule:
If $A_1, A_2, \ldots A_n$ are **disjoint** sets, then
$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$

# Partition Method – How to

- To find the size of a set A,
  - Partition it into a union of disjoint sets $A_1, A_2, \ldots, A_n$
  - Use sum rule

- Example: How many students are there in total in 206?



# Partition Method – Example

- To find the size of a set A,
  - Partition it into a union of disjoint sets $A_1, A_2, \ldots, A_n$
  - Use sum rule

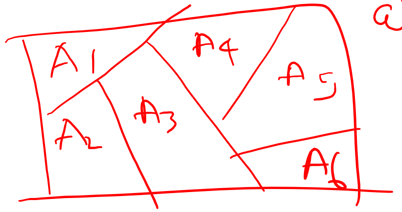
- If I roll a white and black die, how many possible outcomes do I see?

$$S = \left\{ \begin{matrix} (1,1), & (1,2), & - & - & (1,6) \\ (2,1), & (2,2), & & - & (2,6) \\ \vdots & & & & \\ (6,1), & (6,2), & & - & (6,6) \end{matrix} \right\}$$

$$|S| = 36$$

# Partition Method

- If I roll a white and black die, how many possible outcomes do I see?




all outcomes

$S$

$A_1$ = all outcomes with black die = 1

$A_2$ = all outcomes with black die = 2

⋮

$A_6$ = all outcomes with black die = 6

$|S| = |A_1| + |A_2| + \dots + |A_6|$

$= 6 \cdot 6 = 36$

# Take a Break

5 min

# Partition Method

- Possible outcomes where white and black die have different values?

$S =$ all possible outcomes

$A_1 =$ all outcomes with black die $= 1$

$A_2 =$ black die $= 2$

$\vdots$

$A_6 =$ black die $= 6$

$|A_1| = 5, \ |A_2| = 5,$

$|S| = 5 + 5 + 5 + \cdots 5 = 30$

# Partition Method

- Possible outcomes where white die has a larger value then the black die?

$S =$ all possible outcomes

$A_1 =$ all outcomes with black die $= 1$

$\vdots$

$A_6 =$ black die $= 6$

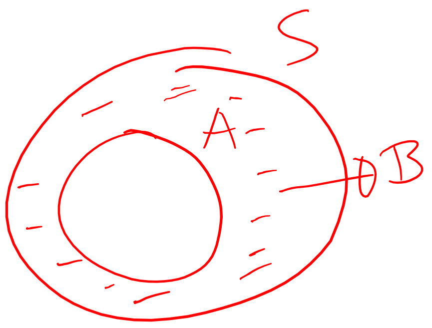
$|A_1| = 5, \ |A_2| = 4, \ |A_3| = 3$

$|A_4| = 2, \ |A_5| = 1, \ |A_6| = 6$

$$|S| = 5 + 4 + 3 + 2 + 1 = \frac{5(5+1)}{2} = 15$$

# Difference Method

- To find the size of a set A,
  - Find a larger set $S$ such that $S = A \cup B$ and
  - $A$ and $B$ are disjoint.
  - $|A| = |S| - |B|$




Want : $|A|$

Find $S$ that contains $A$

$B = S \setminus A$

$|S| = |A| + |B|$

$\Rightarrow |A| = |S| - |B|$

# Difference Method

- To find the size of a set A,
  - Find a larger set $S$ such that $S = A \cup B$ and
  - $A$ and $B$ are disjoint.
  - $|A| = |S| - |B|$

- Possible outcomes where white and black die have different values?

  - Find S with all possible outcomes $|S|$=36
  - Subtract B with the same values |B|=6
  - $|A| = |S| - |B| = 36 - 6 = 30$

# Partition Method

- Possible outcomes where white and black die have different values?

S = all possible outcomes

$A_1$ = all outcomes with black die = 1

$A_2$ = black die = 2

$\vdots$

$A_6$ = black die = 6

$|A_1| = 5, \ |A_2| = 5,$

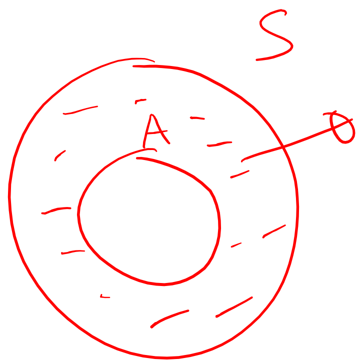
$|S| = 5 + 5 + 5 + \cdots 5 = 30$

# …or we can use the Difference Method

- Possible outcomes where white and black die have different values?

A = all outcomes where white die $\neq$ black die

S = all outcomes , $|S| = 36$




$B = S \setminus A$
= all outcomes where white die = black die

$|B| = 6$

$\Rightarrow \; |A| = 36 - 6 = 30$