Did you bring your notepad?

206
Discrete Structures II

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Remember

Reading for Quiz 1 (and beyond…)

Lecture 2	Recap and Basics of Counting	Chapters 1, 2 and 5 of Rosen
Lecture 3	Basics of Counting	Chapters 1, 2 and 5 of Rosen Chapter 15 of Lehman
Lecture 4**+5**	Basics of Counting	Chapters 6 of Rosen Chapter 15 of Lehman

What we will cover today

Combinatorics

- Recap
  - Sets - Venn
- Today
  - Functions
  - Break
  - Proofs
    - Direct
    - Contrapositive
    - Case Analysis
    - Contradiction
    - Induction
  - [Next Week] Counting
    - Partition Method
    - Difference Method

Course Outline

- Part I
    - Recap of basics – sets, function, proofs, induction
    - Basic counting techniques
    - Pigeonhole principle
    - Generating functions
- Part II
    - Sample spaces and events
    - Basics of probability
    - Independence, conditional probability
    - Random variables, expectation, variance
    - Moment generating functions
- Part III
    - Graph Theory
    - Machine learning and statistical inference

So far…

Combinatorics

Your Password:

- Must be different from your User ID
- Must contain 8 to 20 characters, including one letter and number
- May include one of the following characters: %, &, _, ?, #, =, -
- Your new password cannot have any spaces and will not be case sensitive.

*REQUIRED FIELD

How many different passwords we can create?

Sets

- The order of elements is not significant, so $\{x, y\}$ and $\{y, x\}$ are the same set written two different ways.

- And what about $y = x$?
  - $\{x, x\} = \{x\}$

- The expression $e \in S$ asserts that e **is an element of** set S
  - E.g.,$32 \in S$ or $blue \notin S$

So far…

Sets - Set Operations

Example
$X ::= \{1,2,3\}$
$Y ::= \{2,3,4\}$

- Union: $X \cup Y$
  - All elements present in X or Y or both.

$X \cup Y = \{1,2,3,4\}$

- Intersection: $X \cap Y$
  - All elements present in *both* X and Y.

$X \cap Y = \{2,3\}$

- Difference: $X \setminus Y$
  - All elements present in X but not in Y.
  - Not symmetric!

$X \setminus Y = \{1\}$
$Y \setminus X = \{4\}$

- Product: $X \times Y$
  - Collection of all tuples (a, b) where $a \in X$ and $b \in Y$.

$X \times Y = \{(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)\}$

- Size: $|X|$
  - Number of elements in X.

$|X| = 3$
$|Y| = 3$

So far...

Power Set

$X = \{1,2,3\}$

$Power(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

- Let X be a set.
- $Power(X)$ = set of all subsets of X

- E.g., $Power(\{1,2\}) = \{1\}, \{2\}, and\ \{1,2\}$

- Is this correct?
  - NO!
  - $Power(\{1,2\}) = \{1\}, \{2\}, \{1,2\}, \boldsymbol{and}\ \{\}$

- Generally, if A has n elements, then there are 2^n sets in $Power(A)$

So far...

Set Builder Notation

- Often sets cannot be fully described by listing the elements explicitly or by taking unions, intersections, etc., of easily-described sets
- **Set builder notation** often comes to the rescue
- The idea is to define a set using a predicate; in particular, the set consists of all values that make the predicate true

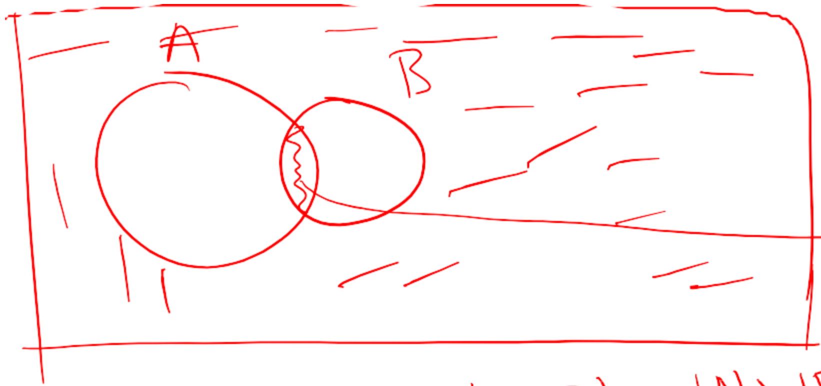
Examples:

- $X = \{n \in \mathbb{N}: n\ is\ prime\}$
- $Y = \{x \in \mathbb{R}: x^3 - 3x + 1 > 0\}$
- $Z = \{z \in YouTube\_videos: z\ is\ less\ than\ 3\ minutes\ long\}$

Venn Diagram

- There are 131 students in CS 206.
- 100 like chocolate ice cream. 50 like vanilla ice cream.
- 20 like both chocolate and vanilla ice cream.
- Draw a Venn diagram to represent this.
- How many students do not like either flavor of ice cream.

A = chocolate
B = vanilla



$|A| = 100$
$|B| = 50$
$|A \cap B| = 20$

Want $131 - |A \cup B|$, $|A \cup B| = |A| + |B| - |A \cap B| = 130$

$\Rightarrow 131 - |A \cup B| = 1$

Lehman, 73-75

Functions

- What is a *function?*
  - A function *assigns* an element of one set to an element of another set
  - The **mapping** is done from one set, called *domain*, to another set, called *codomain*
  - Notation $f: A \mapsto B$
- Examples
  - $f: \mathbb{R} \mapsto \mathbb{R}$
  - $x \mapsto 4x^2$

  The familiar notation $f(a) = b$ indicates that f assigns the element $b \in B$ to a. Here b would be called the value of f at argument a
  - Example using a formula for b: $f(x) = 4x^2$

Functions - Example

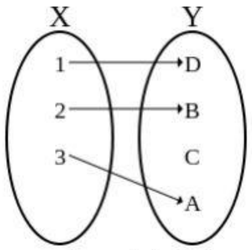
- **Algorithms are functions**
  - Example:
  - Let $X = set\ of\ all\ web\ pages$
  - PageRank: $X \mapsto \mathbb{R}$

Types of Functions

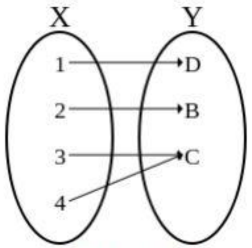
- Injection (one-to-one)
  - $f: X \mapsto Y$ is injective if each $x \in X$ is mapped to a *different* $y \in Y$.

  This function *preserves distinctness* as it never maps distinct elements of its domain to the same elements of its codomain.

- Subjection (onto)
  - $f: X \mapsto Y$ is subjective if each y $\in Y$, there exists $x \in X$ such that $f(x) = y$.

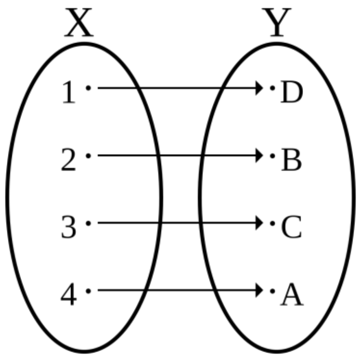


one-to-one (injective), but not onto



onto (surjective), but not one-to-one

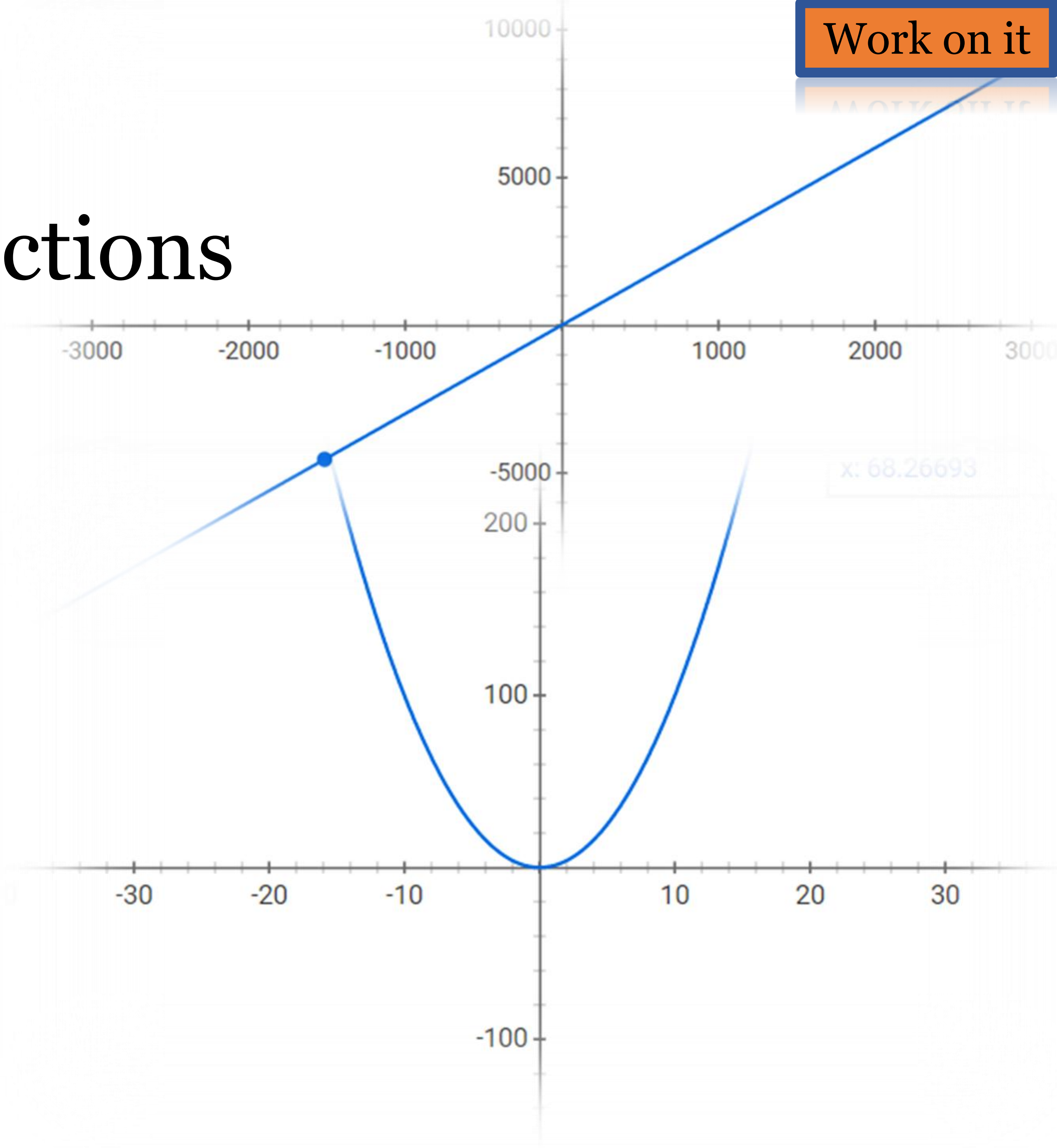
- Bijection
  - $f: X \mapsto Y$ is a bijection if it is both one-to-one and onto.



Work on it

2 Examples: Types of Functions

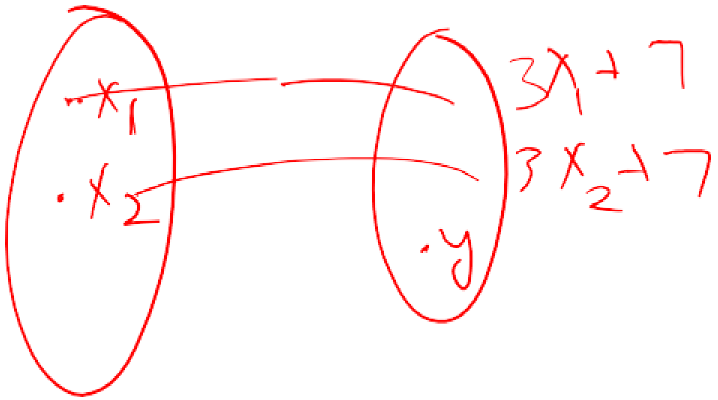
- $f: \mathbb{R} \mapsto \mathbb{R}, f(x) = 3x + 7$
- $f: \mathbb{R} \mapsto \mathbb{R}, f(x) = x^2$



Work on it

2 Examples: Types of Functions

- $f: \mathbb{R} \mapsto \mathbb{R}$, $f(x) = 3x + 7$



– $f(x) = 3x+7$ is one-to-one
Since if x_1 and x_2 are different then
$3x_1+7$ and $3x_2+7$ are also different

– $f(x) = 3x+7$ is on-to since for any real
value y, $f\left(\frac{y-7}{3}\right) = y$

Hence $f(x) = 3x+7$ is a bijection

Work on it

2 Examples: Types of Functions

- $f: \mathbb{R} \mapsto \mathbb{R}$, $f(x) = x^2$

- $f(x) = x^2$ is not one-to-one since
  $f(2) = f(-2) = 4$

- $f(x) = x^2$ is not onto since for
  $y = -3$, no real value x exists such
  that $f(x) = -3$

Take a Break

5 min

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Proofs

- A mathematical proof…
  - …of a **proposition** is a chain of logical deductions from axioms and previously proved statements.

A ***prime*** is an integer greater than one that is not divisible by any other integer greater than 1, e.g., 2, 3, 5, 7, 11, . . .

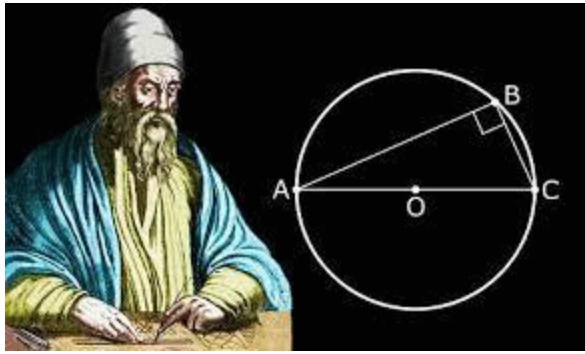
- Proposition
  - A statement that is either *true* or *false*
  - *e.g., Every even integer greater than 2 is the sum of two primes (Goldbach's Conjecture – remains unsolved since 1742…)*
- Predicates
  - A proposition whose truth depends on the value of variables
  - e.g., $P(n) ::= "n\ is\ a\ perfect\ square"$ – $P(4)$ is true but $P(5)$ is false

Proofs

The first two Proofs we will learn:

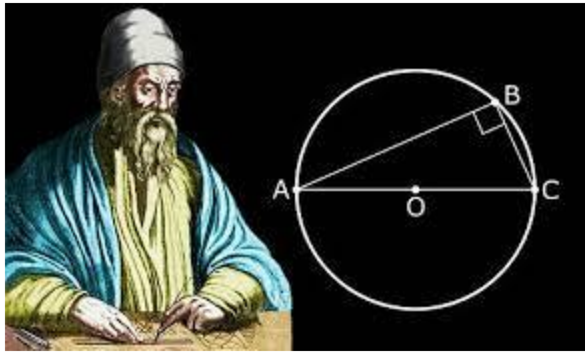
- Direct Proof
- Proof by Contrapositive

Axioms



- A standard procedure for establishing truth in mathematics was invented by Euclid, a Greek mathematician working in Alexandria, circa 300 BC.
- He began with 5 assumptions about geometry, which seemed undeniable based on direct experience. (For example, "There is a straight line segment between every pair of points.)
- Propositions like these that are **simply accepted as true** are called axioms
- Starting from these axioms, Euclid established the truth of many additional propositions by providing "proofs."

Axioms



- Euclid’s axiom-and-proof approach, now called the *axiomatic method*, remains the foundation for mathematics today.

- In fact, just a handful of axioms, called the axioms Zermelo-Frankel with Choice (ZFC), together with a few logical deduction rules, ***appear to be sufficient to derive essentially all of mathematics***

Logical Deductions (or Inference Rules)

Used to prove new propositions using previously proved ones

- $\dfrac{P, P \Rightarrow Q}{Q}$
  - If P is true and P implies Q, then Q is true.
- $\dfrac{P \Rightarrow Q, Q \Rightarrow R}{P \Rightarrow R}$
  - If P implies Q and Q implies R, then P implies R.
- $\dfrac{\neg P \Rightarrow \neg Q}{Q \Rightarrow P}$
  - If $\neg P$ implies $\neg Q$, then Q implies P

If we can prove this …

$$\frac{\boxed{\textit{antecedents}}}{\boxed{\textit{consequent}}}$$

…then this is true

Proving an Implication via Direct Proof

- To prove: $P \Rightarrow Q$
  - Assume that P is true.
  - Show that Q logically follows