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Quiz 4 today
Good luck!

206
Discrete Structures II

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last class

Probabilities - Outline for this month

- Sample spaces and events
- Basics of probability
- Independence, conditional probability
- Random variables, expectation, variance

→ Basic building blocks

→ Intermediate

→ Advanced

Probability – Last time and today…

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Simple Event
  - Any element of the sample space
- Compound Event
  - Subsets of the sample space
- Probability Distribution - Axioms

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Probabilities

- Experiment
  - Toss a fair coin 10 times
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  - All possible outcomes of the experiment
- Event
  - Outcome(s) that you are interested in understanding

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Probability - Sample Space

- Consider an experiment whose outcome *is not predictable with certainty*.
- However, although the outcome of the experiment will not be known in advance, let us suppose that *the set of all possible outcomes is known*.
- This set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by S.

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Probability – Sample Space Examples

- If the outcome of an experiment consists of the determination of the gender of a newborn child, then

$$S = \{g, b\}$$

- If the experiment consists of flipping two coins, then the sample space consists of the following four points,

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

- If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, and 7, then

$$S = \{\text{all } 7! \text{ Permutations of } (1,2,3,4,5,6,7)\}$$

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Probability – Sample Space Examples

- Toss a coin 10 times

$$\Omega = \begin{Bmatrix} (H, H, H, \ldots, H) \\ (H, T, \ldots) \\ \vdots \\ (T, T, \ldots, T) \end{Bmatrix}, \quad |\Omega| = 2^{10}$$

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Probability – Sample Space Examples

- Roll two dice

$$\Omega = \begin{Bmatrix} (1,1), (1,2), \cdots (1,6) \\ (2,1), (2,2), \cdots (2,6) \\ \vdots \\ (6,1), (6,2) \cdots (6,6) \end{Bmatrix}, \quad |\Omega| = 6 \cdot 6$$

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Probability – Sample Space Examples

Experiment : people vote

$$\Omega = \left\{ \begin{array}{l} (H, H, - H, T, - - T) \\ (T, T, - - - T) \\ (H, H, H, T, - T) \\ \vdots \end{array} \right\}, \quad |\Omega| = 2^{350 \times 10^6}$$

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Probability – Sample Space Examples

- Toss a coin until you see a H

$$\Omega = \{ H, TH, TTH, TTTH, \ldots \}$$

$$|\Omega| = \infty$$

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Probability

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Event
  - Outcome(s) that you are interested in understanding (or counting…)

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Probability – Events Examples

Probability is the likelihood that an **event** will occur.

- Toss a coin 10 times

Any subset of Ω is an Event.

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Events – Simple Event

- A **simple event** is an **event** where all possible outcomes are equally likely to occur.
- So the **probability** of **simple events** will have all possible outcomes equally likely to happen or occur.
- E.g., when you toss a coin, there are two possible outcomes – heads or tails, and the **probability** of heads or tails is equal.

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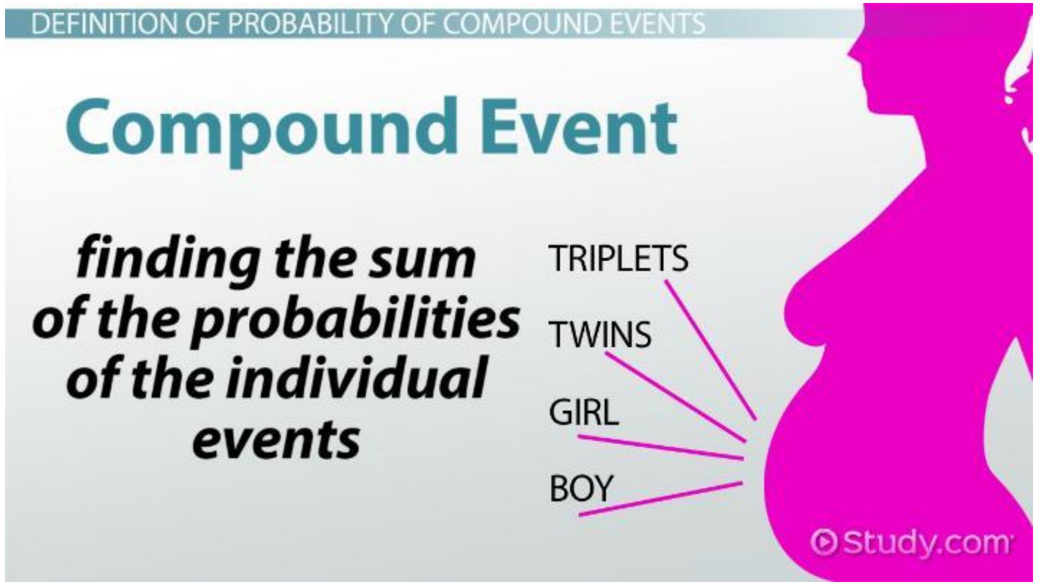
Events - Compound Event



- A **compound event** is one in which there is more than one possible outcome.
- Determining the **probability** of a **compound event** involves finding the sum of the **probabilities** of the individual **events** and, if necessary, removing any overlapping **probabilities**.

Probability – Events Examples

- Toss a coin 10 times

Compound Event

Event 1: H appears 6 times

(H,H,H,...,H)

(H,H,H,H,H,H,T,T,T,T)
(T,T,T,T,H,H,H,H,H,H)
⋮

Event 2: only see Heads

Simple Event

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Probability – Events Examples

- Roll two dice

$$\Omega = \left\{ \begin{array}{l} (1,1), (1,2), \ldots (1,6) \\ \vdots \\ (6,1), (6,2), \ldots (6,6) \end{array} \right\}$$

Simple Event : first die = 6, Second die = 5 → (6,5)

Compound Event : first die equals second die → (1,1), (2,2), (3,3), ... (6,6)

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Probability – Events Examples

- Toss a coin until you see a H.

Simple Event: Get an H on first try.

Compound Event: Don't get an H on first try.

New stuff...

Events - Operations

- A' → Complement of A
- $A \cap B$ → intersection
- $A \cup B$ → union

Disjoint Events

- A and B are disjoint events if $A \cap B = \phi$

Roll 2 dice

— A : die 1 = 1 , die 2 = 1

— B : die 1 = 2 , die 2 = 2

$A \cap B = \phi$

— A : Sum of dice = 2 → (1,1)

— B : Sum of dice = 3 → (1,2), (2,1)

$A \cap B = \phi$

Probability

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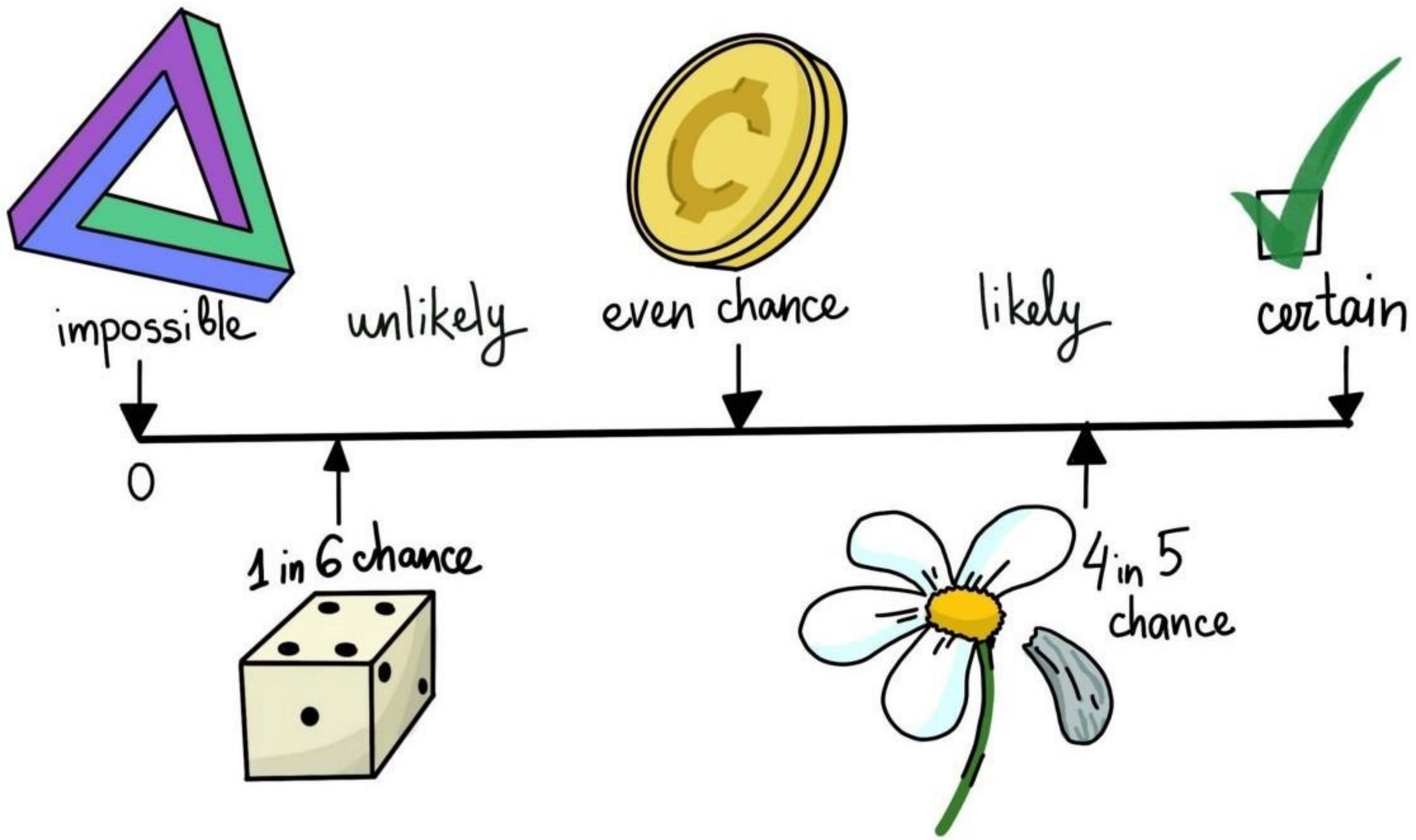
- Probability Distribution - Axioms

Probability

- Fix experiment and sample space Ω.

A probability distribution P assigns a number $P(A)$ to each event A.

- P needs to satisfy certain basic axioms.

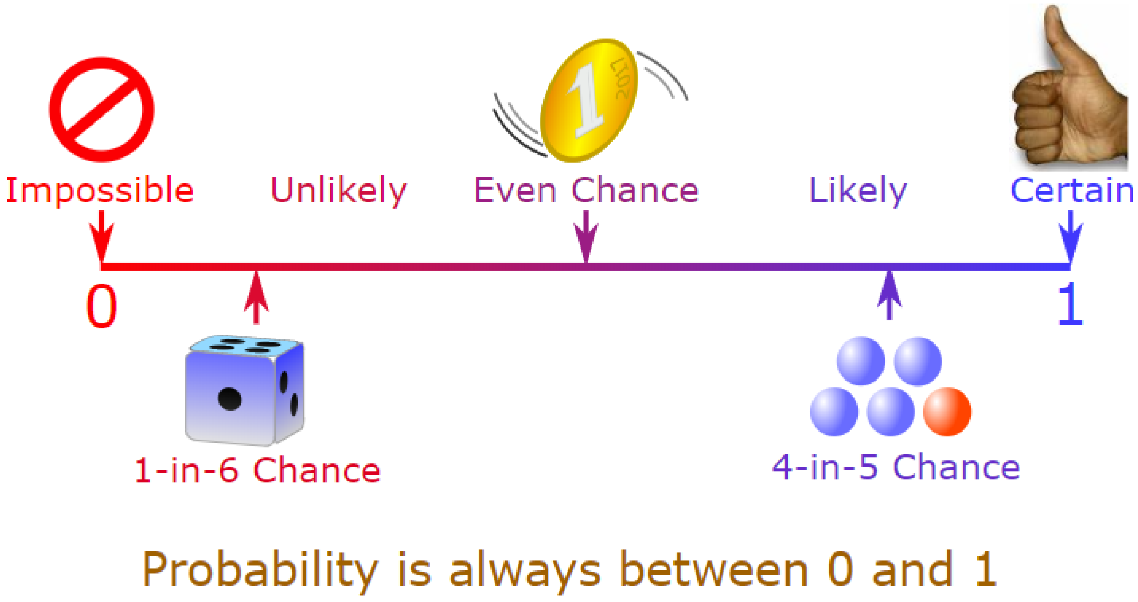


Axioms of Probability

- $P(A) \geq 0$

- $P(\Omega) = 1$

- For a collection of disjoint events $A_1, A_2, \ldots$
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$



Equally Likely Outcomes

Consider experiment and a finite sample space Ω

- For every simple event $e \in \Omega$, assign $P(e) = \frac{1}{|\Omega|}$
- For every compound event A, assign $P(A) = \frac{|A|}{|\Omega|}$

- Then, P is a valid probability distribution.

(Proof on next slide)

Equally Likely Outcomes

- Proof:
- $P(A) \geq 0$ since $|A| \geq 0$
- $P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$
- Let $A_1, A_2, \ldots$ be disjoint events. Then
- $P(A_1 \cup A_2 \ldots) = \frac{|A_1 \cup A_2 \cup \cdots|}{|\Omega|}$
$$= \frac{|A_1|}{\Omega} + \frac{|A_2|}{\Omega} + \cdots = P(A_1) + P(A_2) \ldots$$
- We have proved that all 3 axioms are true.

Probability – Calculate it

- Toss a coin.

$\Omega = \{H, T\}$

For equally likely outcome

$P(H) = \frac{1}{2}$

$P(T) = \frac{1}{2}$

$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{2}$

Probability

- Roll two dice. For any compound event A, of size |A| …

$|\Omega| = 36$

For equally likely outcomes

$$P(A) = \frac{|A|}{36}$$

More Implications – Prove it!

- $P(A') = 1 - P(A)$

— A and A' are disjoint

$\Rightarrow P(A) + P(A') = P(A \cup A') = P(\Omega) = 1$

More Implications – Prove it!

- $P(\mathrm{A}) \leq 1$

— $P(A) + P(A') = 1$

$\Rightarrow P(A) \leq 1$

$P(A') \leq 1$

More Implications

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

— Inclusion / Exclusion for Probabilities

— Extends to more than 2 sets

Union Bound

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) \leq P(A) + P(B) \longrightarrow$ Union bound

$\longrightarrow$ Boole's inequality

Take a Break

5 min

Interpretation of Probability

- For a collection of disjoint events $A_1, A_2, \ldots$
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

sum rule for Probability

Interpretation of $P(A)$

If $P(A) = .6$

- If we repeat experiment N times (N is very large)
- Then the out come will lie in A $.6N$ of the times

Uniform Distribution

- A fair coin is tossed 100 times. What is the probability that we get exactly 50 heads.

$$\Omega \to \left\{ \begin{array}{c} (H, H, \ldots \ldots H) \\ (T, T, \ldots \ldots T) \\ \vdots \end{array} \right\}, \quad |\Omega| = 2^{100}$$

$A \to$ all outcomes with exactly 50 Heads, $\quad |A| = \binom{100}{50}$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{100}{50}}{2^{100}}$$

Uniform Distribution

- If we roll a white die and a black die (both fair), what is the probability that the sum is 7 or 11?

$A \to$ sum is 7

$B \to$ sum is 11

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} = \frac{6}{36} + \frac{2}{36} = \frac{8}{36}$$

$|\Omega| = 36$

$|A| = |\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}| = 6$

$|B| = |\{(6,5),(5,6)\}| = 2$

Uniform Distribution

- If we roll a white die and a black die (both fair), what is the probability that the sum is 7 or die 1 is more than 3?

A → sum is 7

B → die 1 more than 3

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} - \frac{|A \cap B|}{|\Omega|} = \frac{6}{36} + \frac{18}{36} - \frac{3}{36}$

$|A| = 6$

$|B| = |\{(4,1) \ldots (4,6), (5,1) \ldots (5,6), (6,1) \ldots (6,6)\}| = 18$

$|A \cap B| = |\{(4,3), (5,2), (6,1)\}| = 3$

How many people are needed so that at least 2 of them have the same birthday, with probability above 95%?