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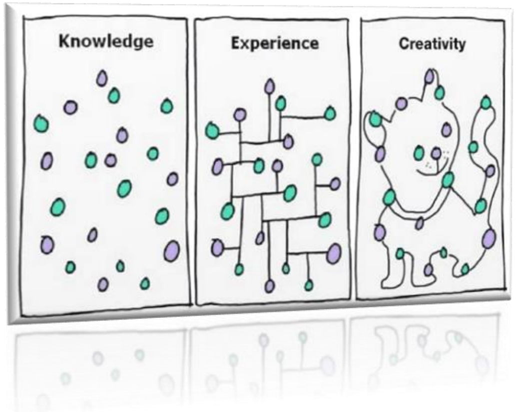

206
Discrete Structures II

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Lecture 25 | Conditional Probabilities - Bayes| Tuesday 12/7/2021

CTAAR survey

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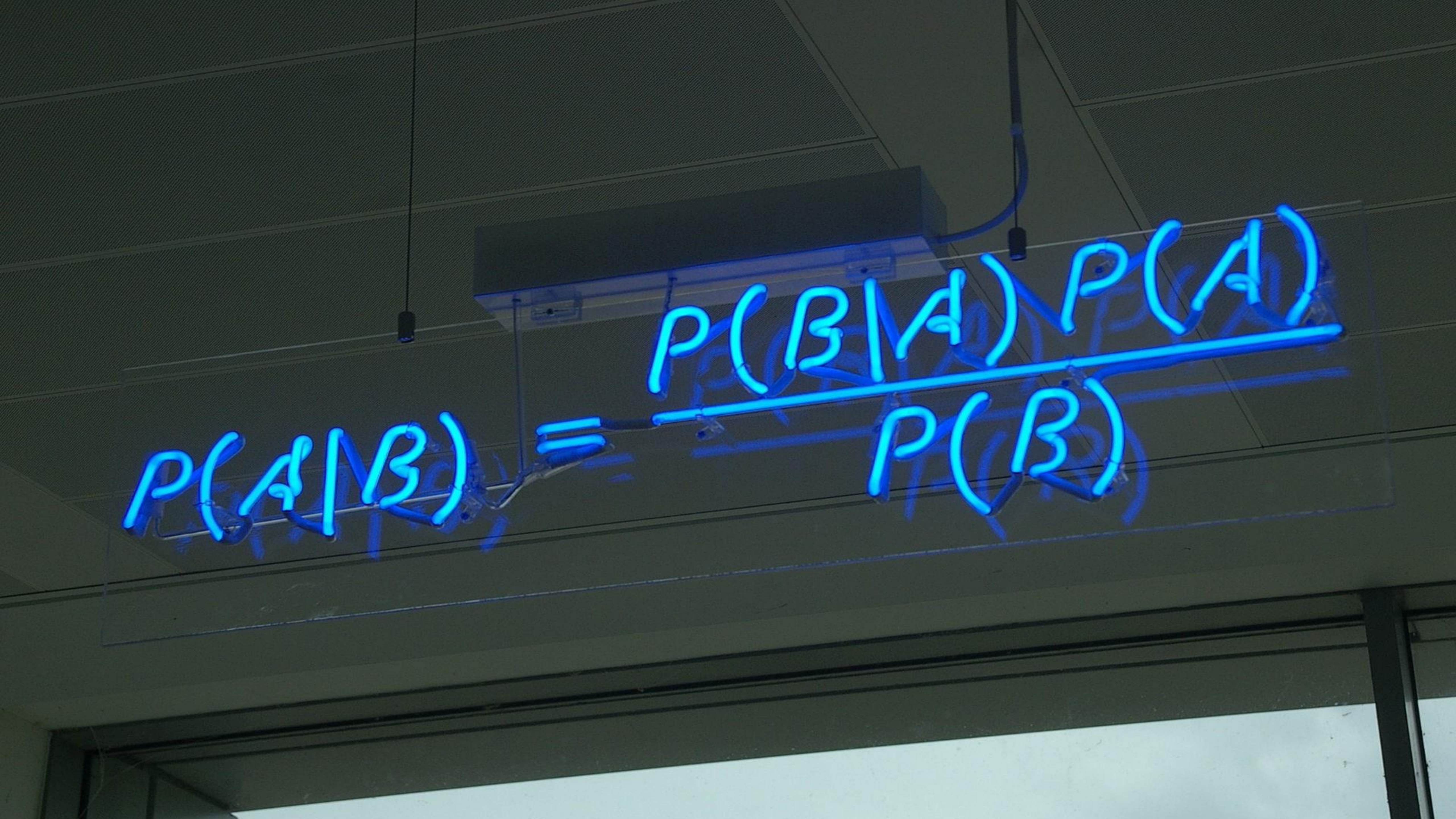
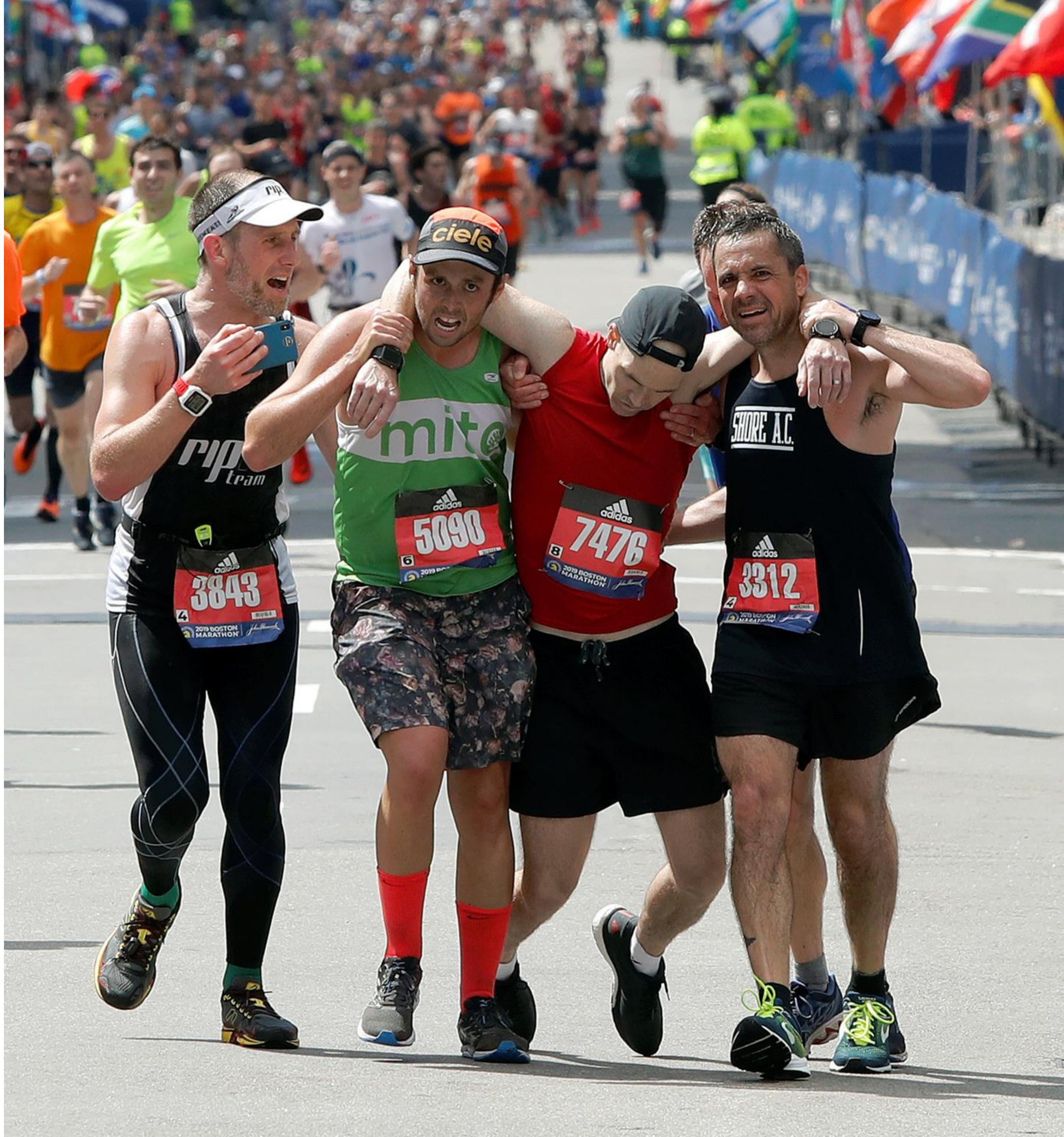
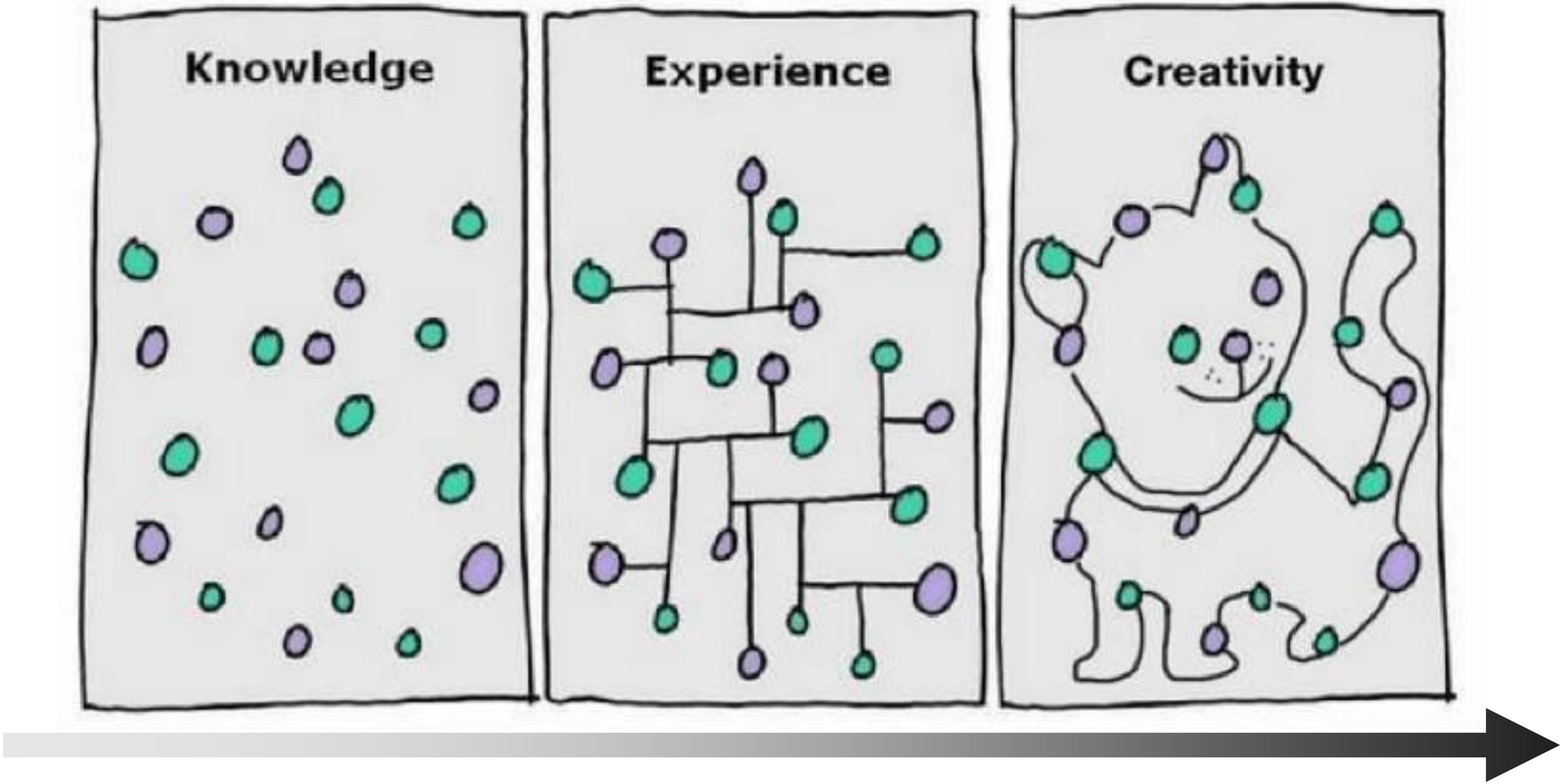

$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

Announcements

- Assignment 2 is running
  - New Deadline Dec 8 (tomorrow)
- Quiz 6 → This week
  - Probabilities

  (Up to today'ish lecture)



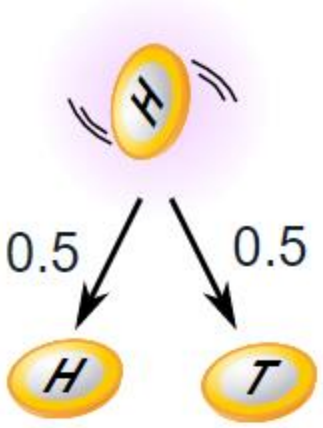


Lectures Quizzes Homework Assignments Real-Life!

Outline

- Sample space
- Axioms of probability
- Conditional probability
- Independence
- Bayes rule
- Random Variables
- Expectation

Probability - Events



Events can be:

- **Independent**

  each event **is not affected** by other events

  - *e.g., a coin does not know that it came up "Heads" in the past and the chance is simply 50% in every toss of the coin*

- **Dependent** (or **Conditional**)

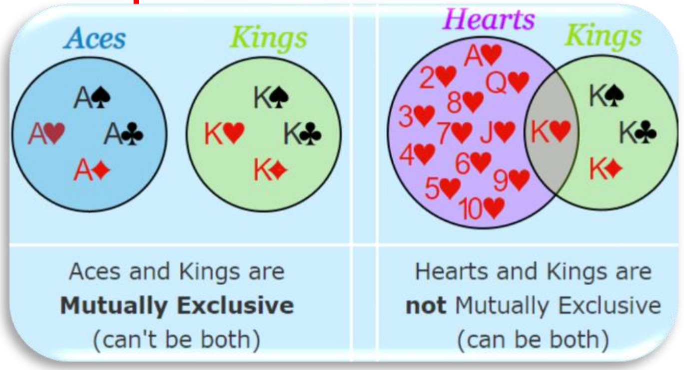
  where an event **is affected** by other events

  - *e.g., after taking one card from the deck there are* ***less cards*** *available, so the probabilities change! How the probability of getting a King changes, after the 1st card was a King (less likely), and after the 1st card was not a King (more likely)?*
  - What would happen if we remove cards with and without replacement? (*independent vs. dependent*)

- **Mutually Exclusive**

  events can't happen at the same time

  - e.g., "Left or Right", "Heads or Tails", "Kings or Aces"



Last Class

Probability of Independent Events

- We can calculate the chances of two or more independent events by ***multiplying the chances***.

Example: Probability of 3 Heads in a Row

For each toss of a coin a "Head" has a probability of 0.5:

0.5

$0.5 \times 0.5 = 0.25$ $\left(or\ \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\right)$

$0.5 \times 0.5 \times 0.5 = 0.125$ $\left(or\ \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$

And so the chance of getting 3 Heads in a row is **0.125**

Last Class

Probability of Independent Events

Question 1: What is the probability of 7 heads in a row?

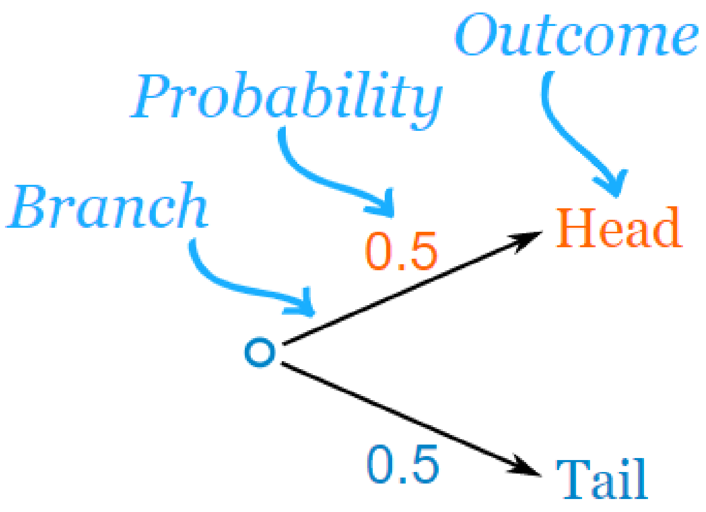
Answer: $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 0.0078125$ (less than 1%).

Question 2: Given that **we have just got 6 heads** in a row, what is the probability that **the next toss** is also a head?

Last Class

Probability Tree Diagrams

Here is a tree diagram for the toss of a coin:



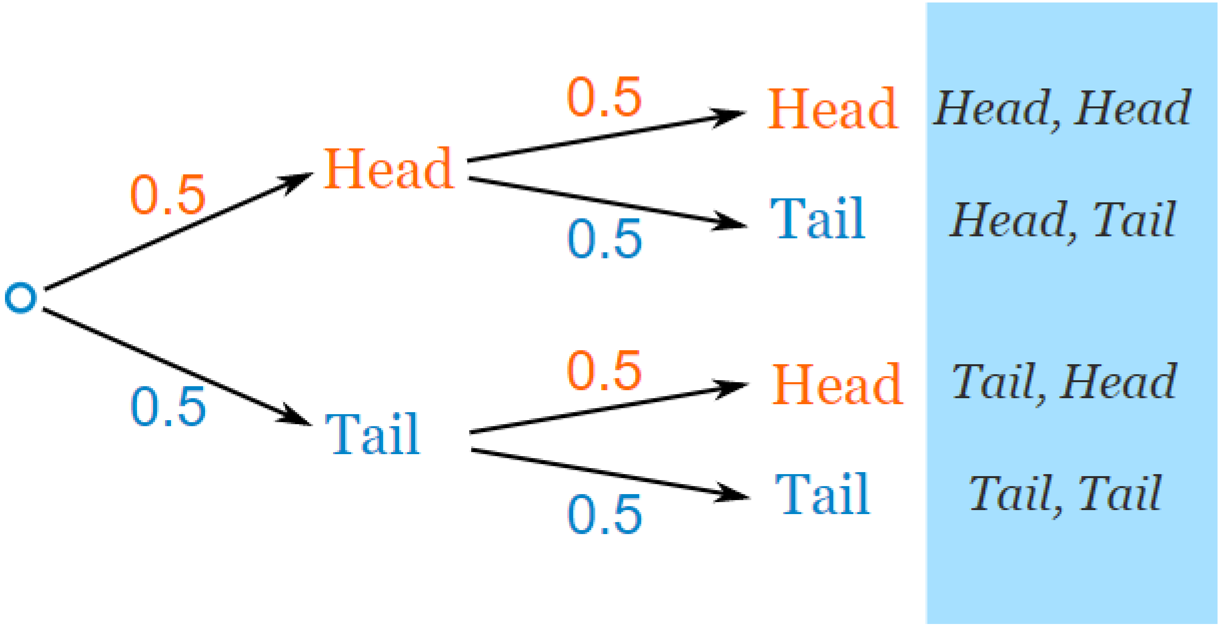
There are two "branches" (Heads and Tails)

- The probability of each branch is written on the branch
- The outcome is written at the end of the branch

Last Class

Probability Tree Diagrams

We can extend the tree diagram to two tosses of a coin:

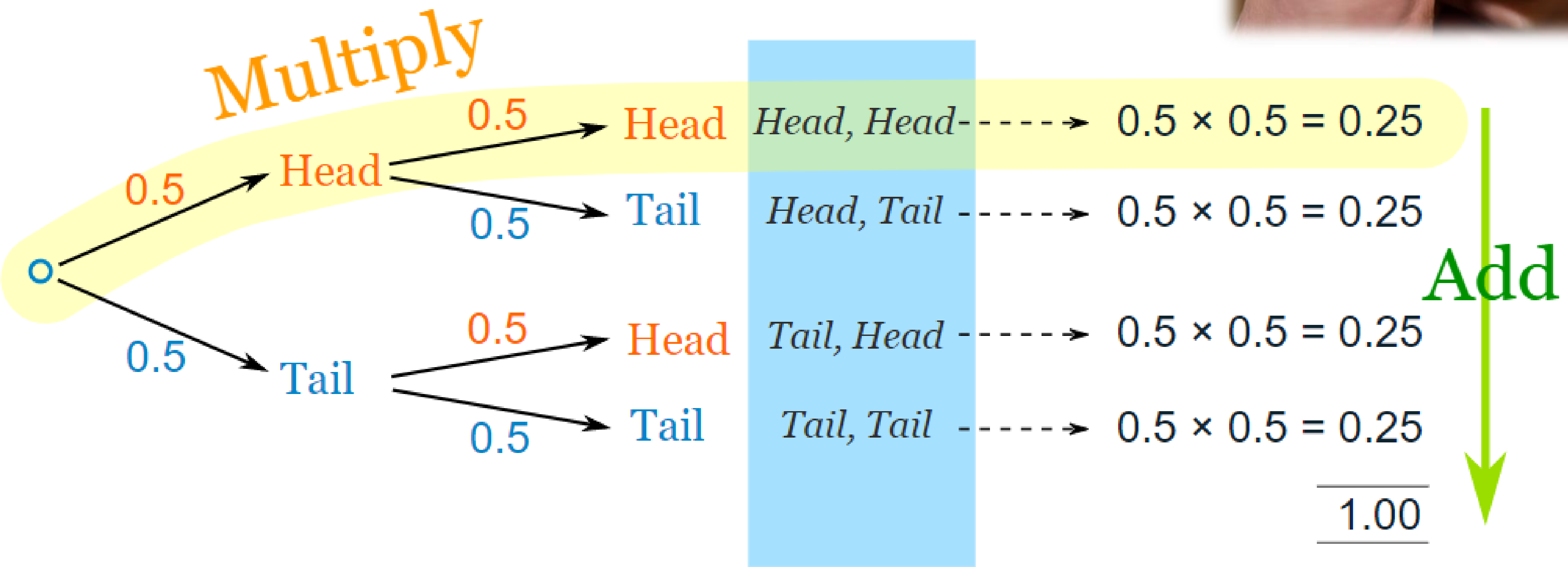


Last Class

Probability Tree Diagrams

Independent Events

- We **multiply** probabilities **along the branches**
- We **add** probabilities down **columns**



Probability Tree Diagrams

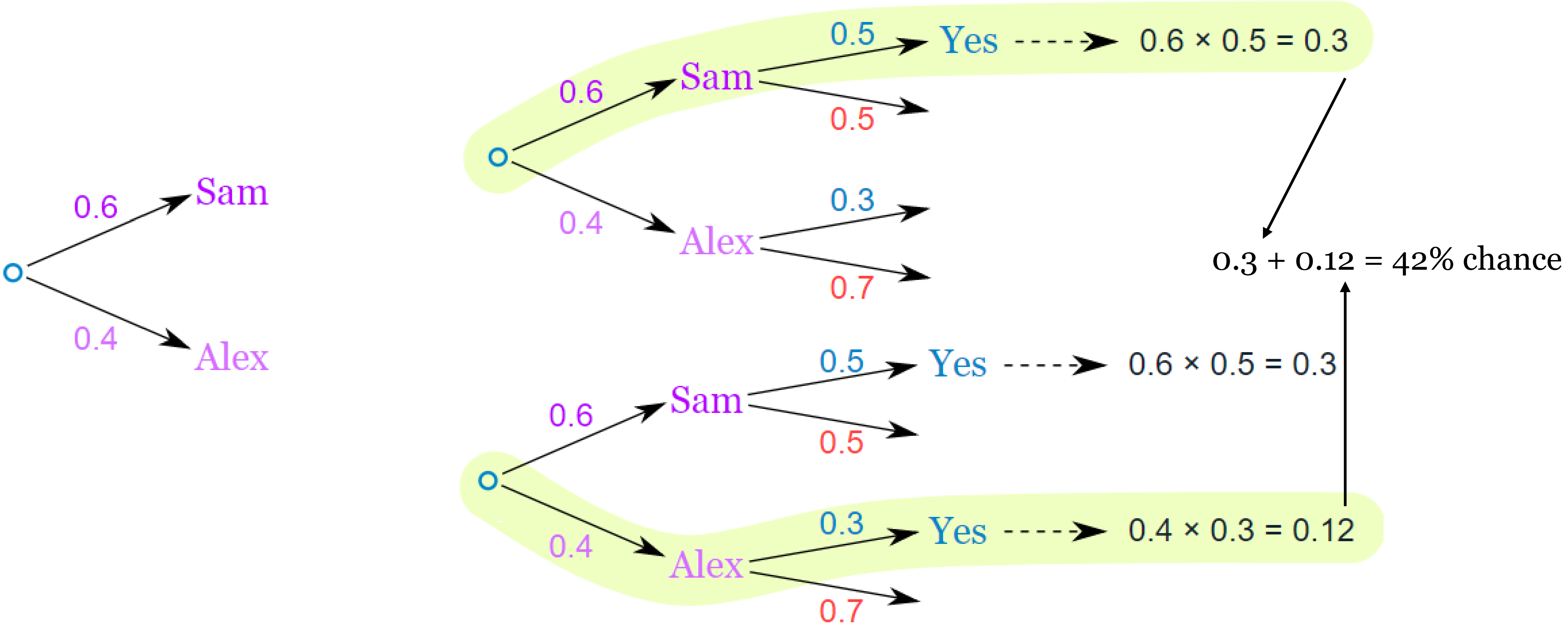
The chance of you being the Goalkeeper depends on who is the Coach:

- with Coach Sam the probability of being Goalkeeper is **0.5**
- with Coach Alex the probability of being Goalkeeper is **0.3**
- Sam is Coach more often ... about 6 out of every 10 games (a probability of **0.6**).
- So, <u>what is the probability you will be a Goalkeeper today</u>?

Dependent Events

Last Class

Probability Tree Diagrams



Dependent Events

Last Class

Probability Tree Diagrams

Example: Marbles in a Bag

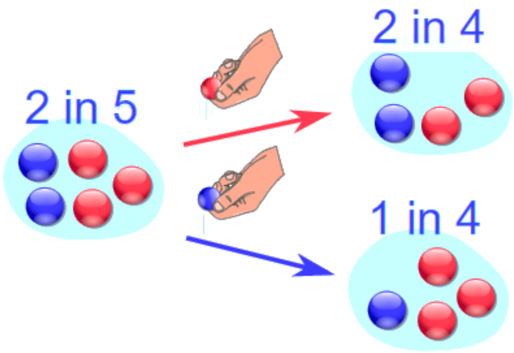
2 blue and 3 red marbles are in a bag.

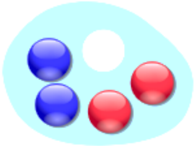
What are the chances of getting a blue marble?

The chance is **2 in 5**

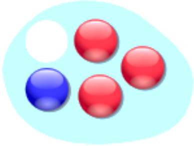
But after taking one out the chances change!

So the next time:





if we got a **red** marble before, then the chance of a blue marble next is **2 in 4**

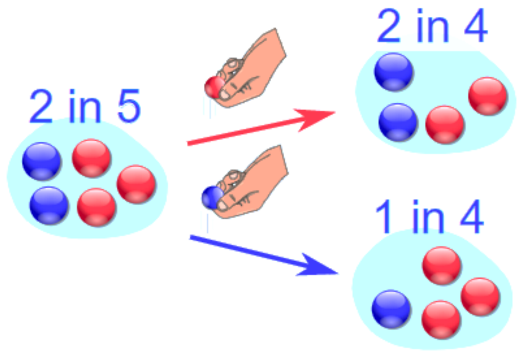


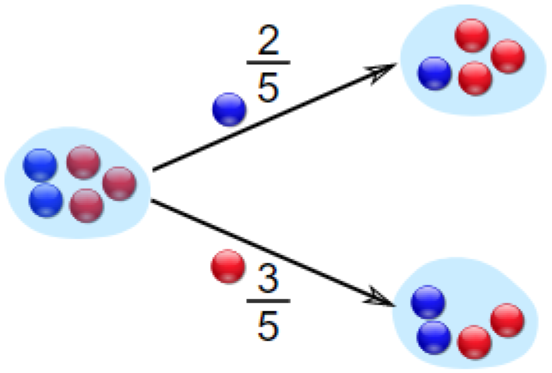
if we got a **blue** marble before, then the chance of a blue marble next is **1 in 4**

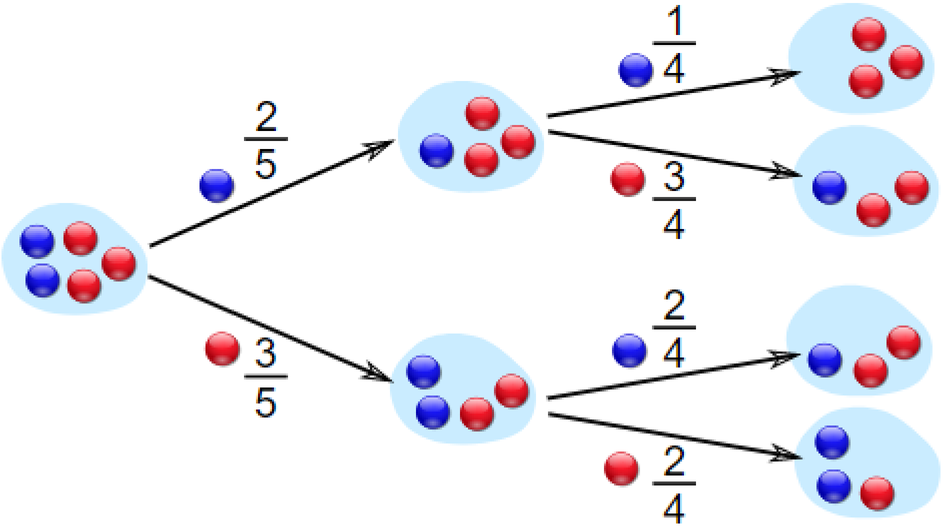
- *The second event* ***depends on*** *what happened in the first event and is called* ***dependent****.*
- *If we* ***replace*** *the marbles in the bag each time, then the chances do* ***not*** *change and the events are independent:*
- ***With*** *Replacement: the events are* ***Independent*** *(the chances don't change)*
- ***Without*** *Replacement: the events are* ***Dependent*** *(the chances change)*

Dependent Events

Probability Tree Diagrams

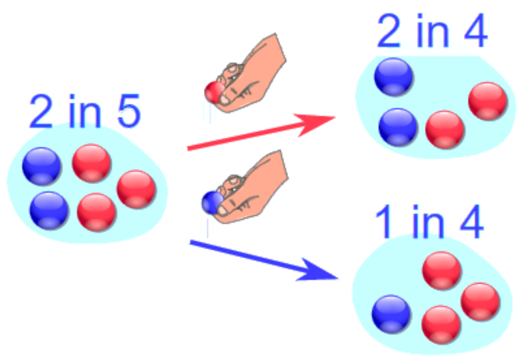




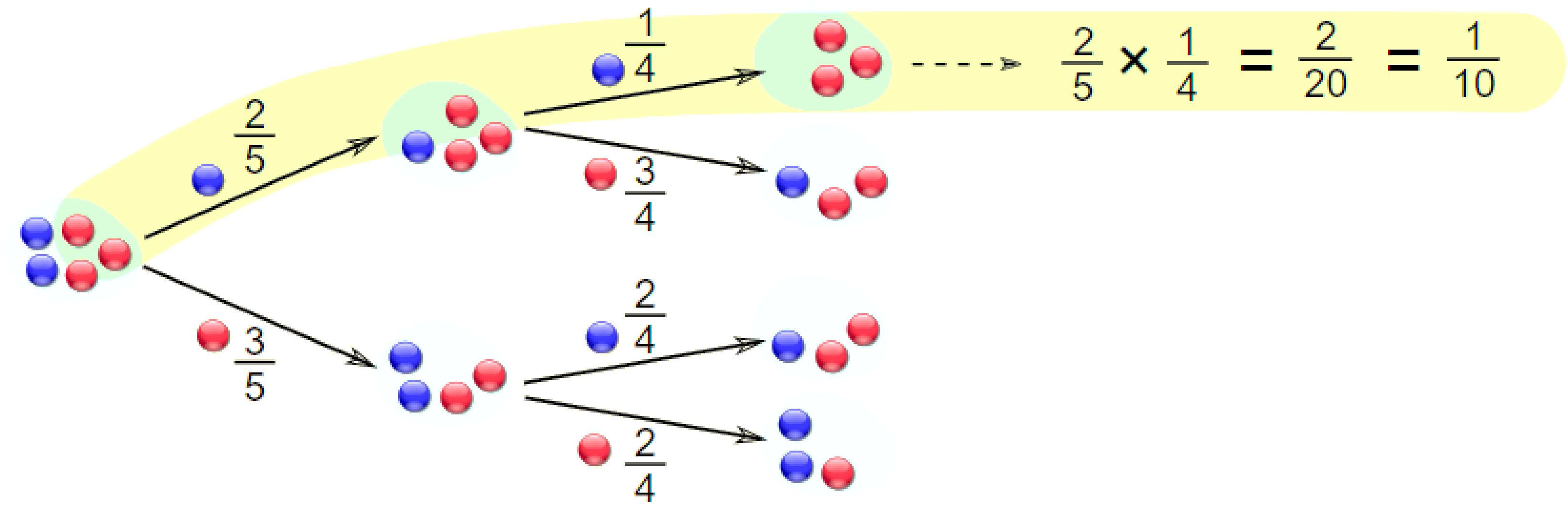


Probability Tree Diagrams

Dependent Events



What are the chances of drawing 2 blue marbles?



Last Class

Conditional Probabilities–Formula to remember

- $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

If equally likely outcomes $\dfrac{P(A \cap B)}{P(B)} = \dfrac{|A \cap B|}{|B|}$

$$P(A|B,C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$

Independence

- A and B are independent events if $P(A|B) = P(A)$

$P(A|B) = P(A)$, A and B are independent

$\Rightarrow \dfrac{P(A\cap B)}{P(B)} = P(A)$

$\Rightarrow P(A\cap B) = P(A)\cdot P(B)$

Independence – more than 2 events…

- A_1, A_2, A_3 are independent events if $P(A_1), P(A_2), P(A_3)$ does not change by knowing any subset of the other.

$P(A_1 | A_2) = P(A_1)$

$P(A_1 | A_3) = P(A_1)$

$P(A_1 | A_2, A_3) = P(A_1)$

$\Rightarrow \quad P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

$P(A_1 \cap A_3) = P(A_1)P(A_3)$

$P(A_1 \cap A_2) = P(A_1)P(A_2)$

$\vdots$

Conditional Probabilities – Toy Example

- Two fair coins are flipped. $A = \{first\ coin\ is\ H\}, B = \{second\ coin\ is\ H\}$.
- Are A and B independent?

$$\Omega = \{(H,H),(T,T),(H,T),(T,H)\}$$

$$P(A|B) = P(A), \quad P(A) = \frac{|A|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}$$

$P(A|B)$: new sample space $B = \{(H,H),(T,H)\}$

$$P(A|B) = \frac{1}{2} = P(A)$$

Conditional Probabilities

- Two fair coins are flipped. $A = \{first\ coin\ is\ H\}$. $B =$ two coins have different outcomes.
- Are A and B independent?

$P(A) = \frac{1}{2}$

$\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$

$B = \{(H,T),(T,H)\}$

$P(A|B) = \frac{1}{2} = P(A)$

Conditional Probability

- Consider a family with two children. Given that one of the children is a boy, what is the probability that both children are boys?

$\rightarrow$ children are equally likely

$\Omega = \{(B,B), (B,G), (G,B), (G,G)\}$

$B \rightarrow$ one of them is a boy $\rightarrow \{(B,B), (B,G), (G,B)\}$

$A \rightarrow$ both of them are boys

$P(A|B) = \frac{1}{3}$

$P(A) = \frac{1}{4}$

Conditional Probability

- Consider a family with two children. Given that the first child is a boy, what is the probability that both children are boys?

$\Omega = \{(B,B), (B,G), (G,G), (G,B)\}$

$A \to$ both are boys

$B \to \{(B,B), (B,G)\}$

$P(A|B) = \frac{1}{2}$

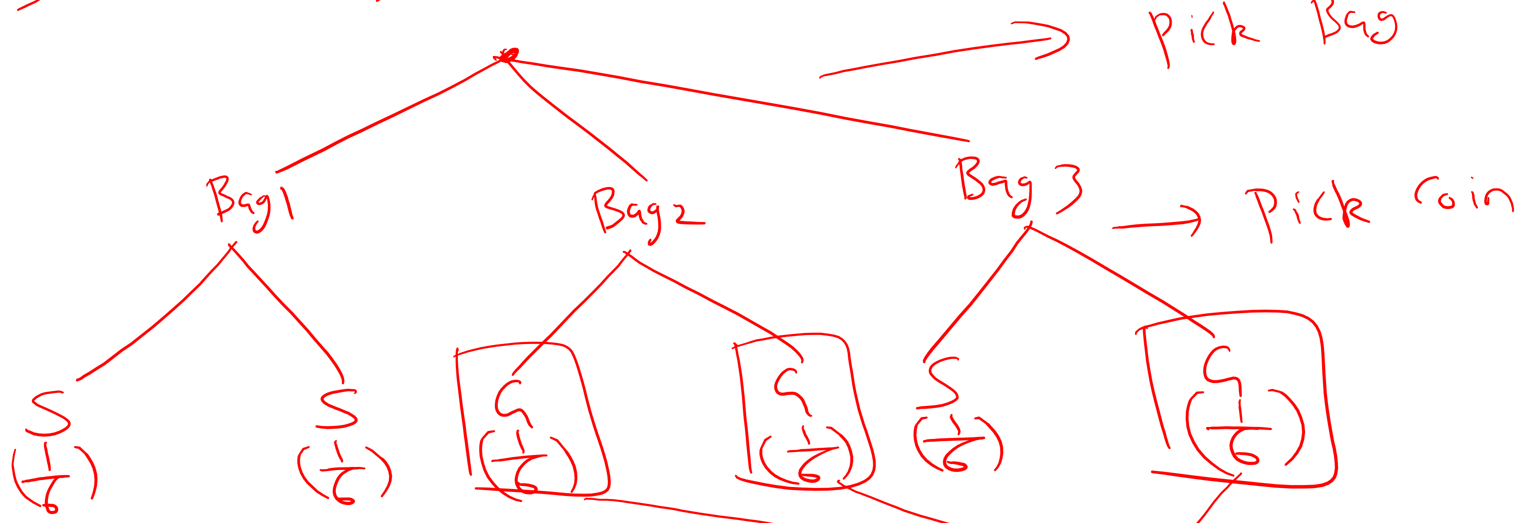
$P(A) = \frac{1}{4}$

Conditional Probability

- One bag has two silver coins, another has two gold coins, and the third has one of each.
- One bag is selected at random. One coin from it is selected at random.
- It turns out to be gold What is the probability that the other coin is gold?

Conditional Probability

Bag 1 → {S, S}
Bag 2 → {G, G}
Bag 3 → {S, G}



B → picked is a Gold Coin
A → other Coin is also Gold

$$P(A|B) = \frac{2}{3}$$

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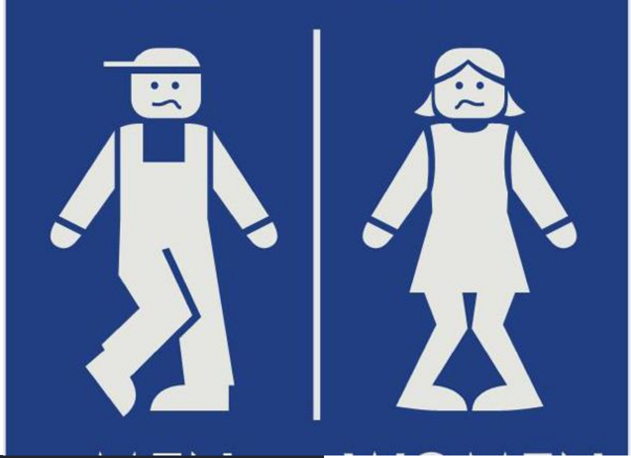
Take a Break

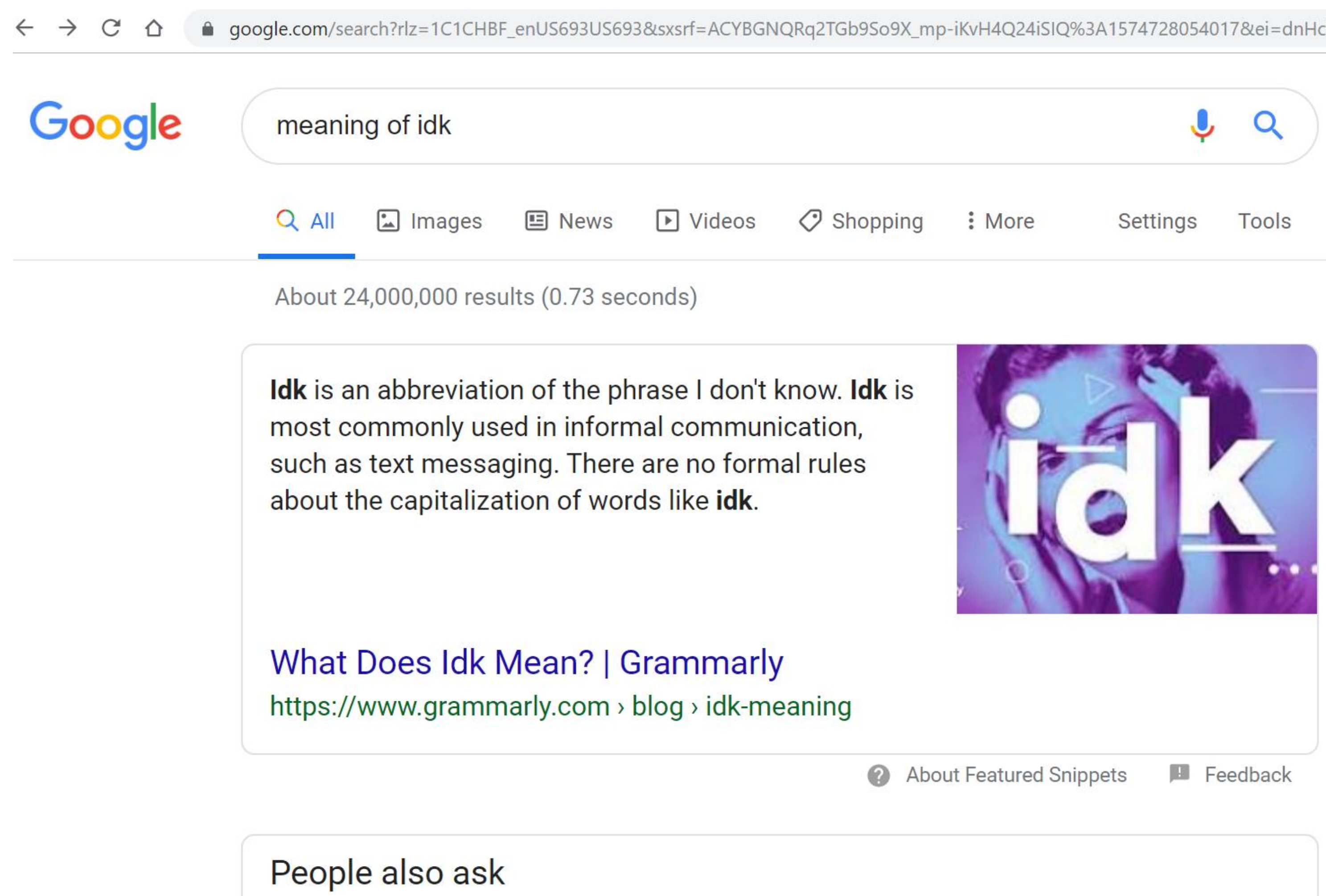
5 min

Bayesian Inference

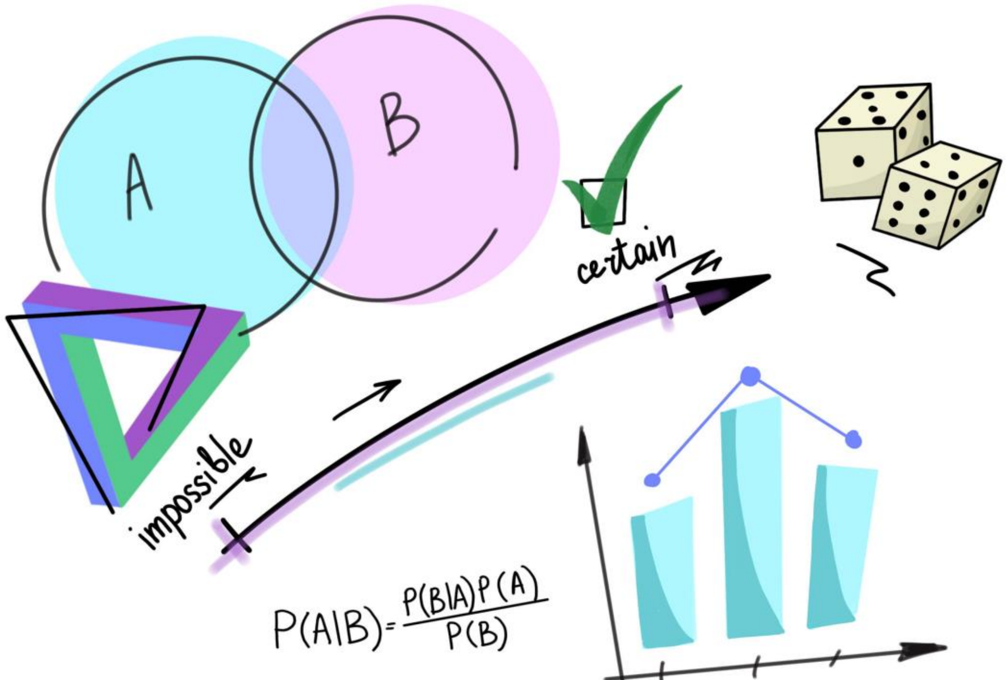


Dilemma at the movies

This person dropped their ticket in the hallway.



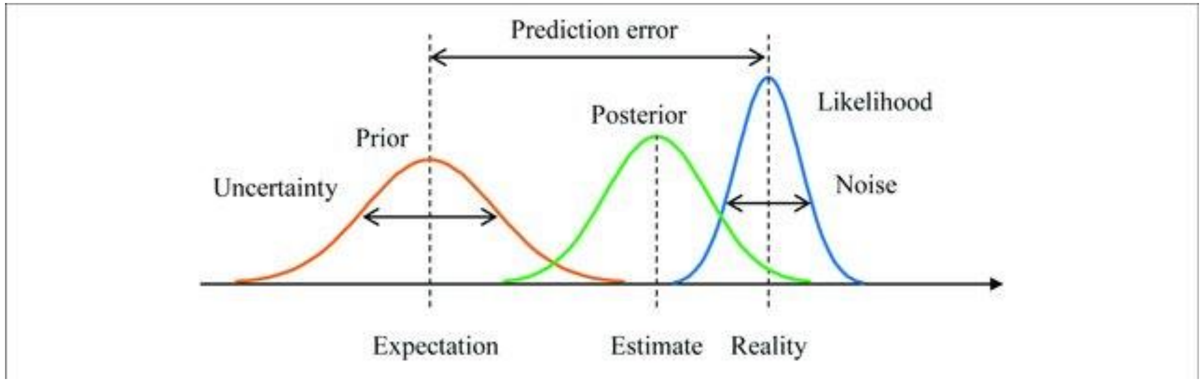
How does Google do this?



Bayesian Inference*

$$\overset{\text{Posterior}}{P(A|B)} = \frac{\overset{\text{Likelihood}}{P(B|A)} * \overset{\text{Prior}}{P(A)}}{\underset{\text{Evidence}}{P(B)}}$$

* *Inference = Educated guessing*



- Bayesian inference with a prior distribution, a posterior distribution, and a likelihood function.
- The prediction error is the difference between the prior expectation and the peak of the likelihood function (i.e., reality).
- Uncertainty is the variance of the prior. Noise is the variance of the likelihood function.

Conditional Probabilities

- P(B|A) means “Probability of event B **given** event A"

In other words, event A has already happened, now what is the chance of event B?

"Probability Of" *"Given"*

$$P(A \text{ and } B) = P(A) \times P(B \mid A)$$

Event A *Event B*

"Probability of ***event A and event B*** *equals*
the probability of ***event A*** *times the probability of* ***event B given event A***"

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

"The probability of ***event B given event A*** *equals*
the probability of ***event A and event B*** *divided by the probability of* ***event A***

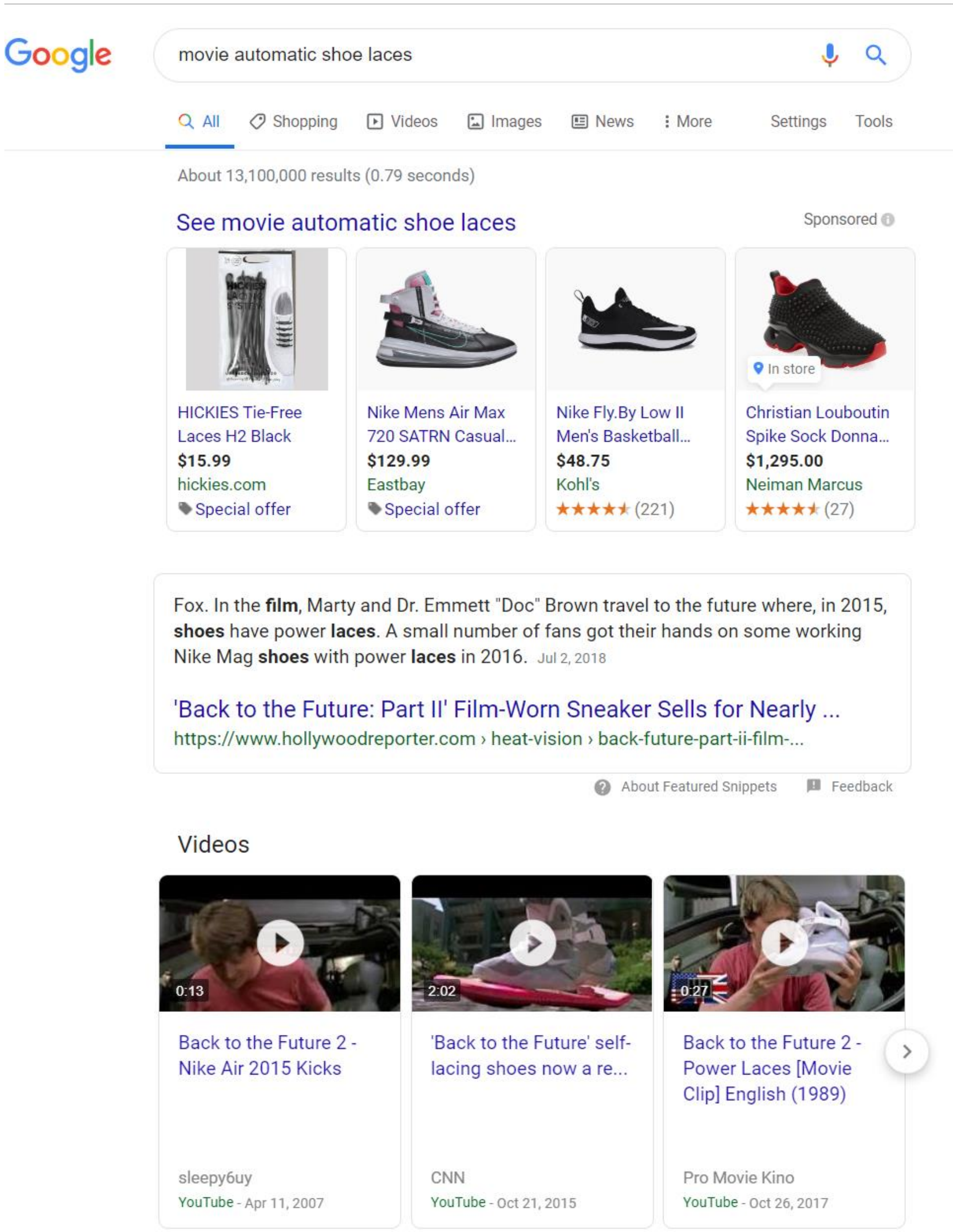
Bayes Rule

An internet search for "movie automatic shoe laces" brings up "Back to the future"

- Has the search engine watched the movie?

No, but it knows from lots of other searches what people are **probably** looking for.

- And it calculates that probability using Bayes' Theorem.



Bayes Rule – One more example

Example: If dangerous fires are rare (1%) but smoke is fairly common (10%) due to barbecues, and 90% of dangerous fires make smoke then:

$$P(\text{Fire}|\text{Smoke}) = \frac{P(\text{Fire})\, P(\text{Smoke}|\text{Fire})}{P(\text{Smoke})}$$

$$= \frac{1\% \times 90\%}{10\%}$$

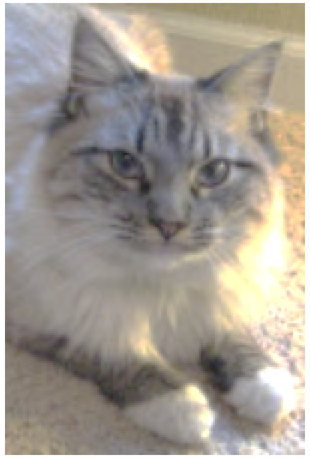
$$= 9\%$$

So the "Probability of dangerous Fire when there is Smoke" is 9%

Bayes Rule – Yet another example

Example: Allergy or Not?

Hunter says she is itchy. There is a test for Allergy to Cats, but this test is not always right:



- For people that **really do** have the allergy, the test says "Yes" **80%** of the time
- For people that **do not** have the allergy, the test says "Yes" **10%** of the time ("false positive")

If 1% of the population have the allergy, and **Hunter's test says "Yes"**, what are the chances that Hunter really has the allergy?

Bayes Rule – Yet another example

$$P(\text{Allergy}|\text{Yes}) = \frac{P(\text{Allergy})\ P(\text{Yes}|\text{Allergy})}{P(\text{Yes})}$$

P(Allergy) is Prob of Allergy = 1%

P(Yes|Allergy) is Prob of test saying "Yes" for people with allergy = 80%

P(Yes) is Prob of test saying "Yes" (to anyone) = ??%

- We **don't know** what the **general** chance of the test saying "Yes" is but we can calculate it by adding up those **with**, and those **without** the allergy:
- 1% have the allergy, and the test says "Yes" to 80% of them
- 99% do **not** have the allergy and the test says "Yes" to 10% of them

P(Yes) = 1% × 80% + 99% × 10% = 10.7% of the population.

P(Allergy|Yes) = about **7%**