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Discrete Structures II

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New – Extra Problems Set 7

▾ Week 12: Probabilities (Intro)

[Extra_Problems_7_probabilities.pdf]

[lecture_22.pdf]

[Recording 22 - Pass: vJ2xkQdm]

Quiz 5 on Tue 11/30 & Thu 12/2

- Extra Points (as always)
- No need to answer all questions (but you should at least try)
- Start from what you know better
- Typically, the higher the points, the more difficult the problem is
  - Problem Difficulty = synthesize multiple methods
- Don't Panic! ! !

last class

Probabilities - Outline for this month

- Sample spaces and events
- Basics of probability
- Independence, conditional probability
- Random variables, expectation, variance

→ Basic building blocks

→ Intermediate

→ Advanced

last class

Probability – so far...

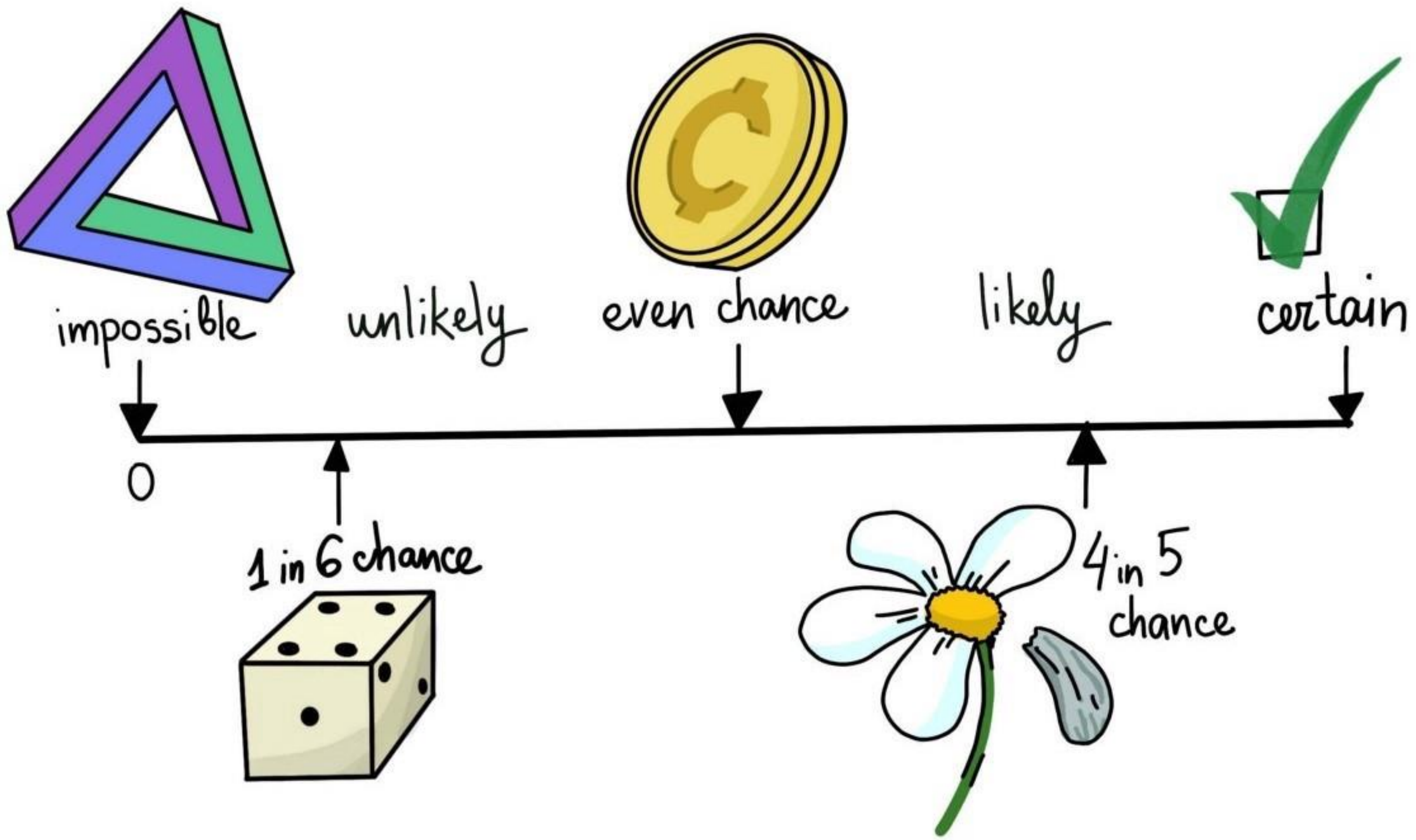
- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Simple Event
  - Any element of the sample space
- Compound Event
  - Subsets of the sample space
- Probability Distribution - Axioms

Probability

- Fix experiment and sample space Ω.

A probability distribution P assigns a number $P(A)$ to each event A.

- P needs to satisfy certain basic axioms.



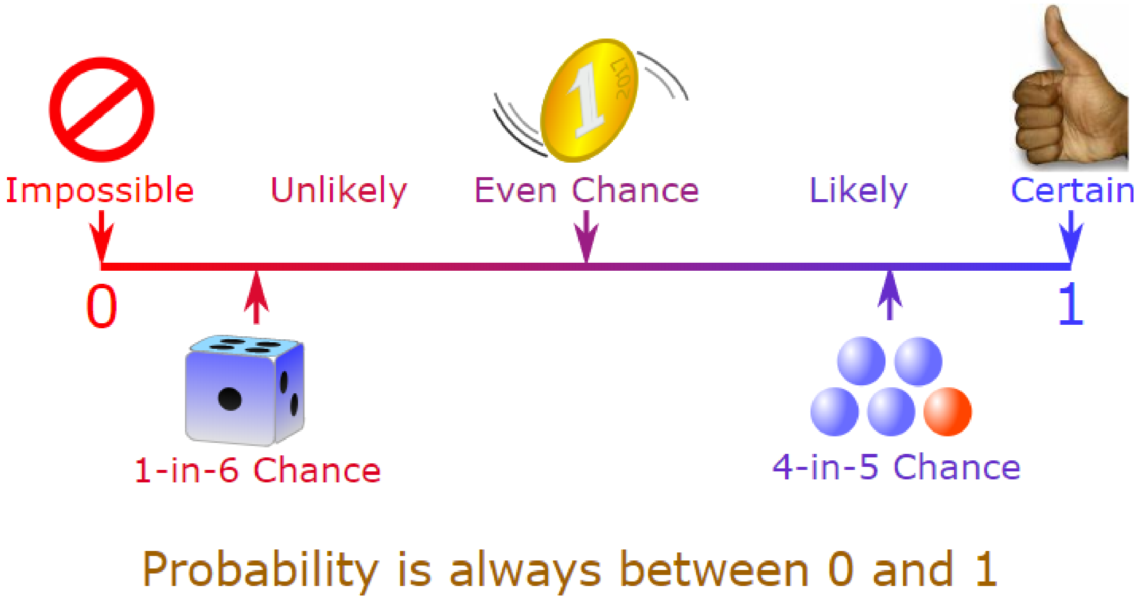
last class

Axioms of Probability

- $P(A) \geq 0$

- $P(\Omega) = 1$

- For a collection of disjoint events $A_1, A_2, \ldots$
  - $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$



last class

Equally Likely Outcomes

Consider experiment and a finite sample space Ω

- For every simple event $e \in \Omega$, assign $P(e) = \frac{1}{|\Omega|}$
- For every compound event A, assign $P(A) = \frac{|A|}{|\Omega|}$

- Then, P is a valid probability distribution.

(Proof on next slide)

last class

Equally Likely Outcomes - Proofs

- Proof:
- $P(A) \geq 0$ since $|A| \geq 0$
- $P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$
- Let $A_1, A_2, \ldots$ be disjoint events. Then
- $P(A_1 \cup A_2 \ldots) = \frac{|A_1 \cup A_2 \cup \cdots|}{|\Omega|}$
$$= \frac{|A_1|}{\Omega} + \frac{|A_2|}{\Omega} + \cdots = P(A_1) + P(A_2) \ldots$$
- We have proved that all 3 axioms are true.

last class

Probability

- Roll two dice. For any compound event A, of size |A| …

$|\Omega| = 36$

For equally likely outcomes

$P(A) = \frac{|A|}{36}$

last class

More Implications – Prove it!

- $P(A') = 1 - P(A)$

— A and A' are disjoint

$\Rightarrow P(A) + P(A') = P(A \cup A') = P(\Omega) = 1$

last class

More Implications – Prove it!

- $P(\mathrm{A}) \leq 1$

— $P(A) + P(A') = 1$

$\Rightarrow$ $P(A) \leq 1$

$P(A') \leq 1$

last class

More Implications

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

– Inclusion/Exclusion for Probabilities

– Extends to more than 2 sets

last class

Union Bound

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow P(A \cup B) \leq P(A) + P(B)$$

$\longrightarrow$ Union bound

$\longrightarrow$ Boole's inequality

last class

Uniform Distribution

- A fair coin is tossed 100 times. What is the probability that we get exactly 50 heads.

$\Omega \rightarrow \left\{ \begin{matrix} (H, H, - \text{---} H) \\ (T, T, \;-\; T) \\ \vdots \end{matrix} \right\}$, $|\Omega| = 2^{100}$

$A \rightarrow$ all outcomes with exactly 50 Heads, $|A| = \binom{100}{50}$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\binom{100}{50}}{2^{100}}$$

last class

Uniform Distribution

- If we roll a white die and a black die (both fair), what is the probability that the sum is 7 or 11?

$A \rightarrow$ sum is 7

$B \rightarrow$ sum is 11

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} = \frac{6}{36} + \frac{2}{36} = \frac{8}{36}$$

$$|\Omega| = 36$$

$$|A| = |\{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}| = 6$$

$$|B| = |\{(6,5),(5,6)\}| = 2$$

last class

Uniform Distribution

- If we roll a white die and a black die (both fair), what is the probability that the sum is 7 or die 1 is more than 3?

$A \to$ sum is 7

$B \to$ die 1 more than 3

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

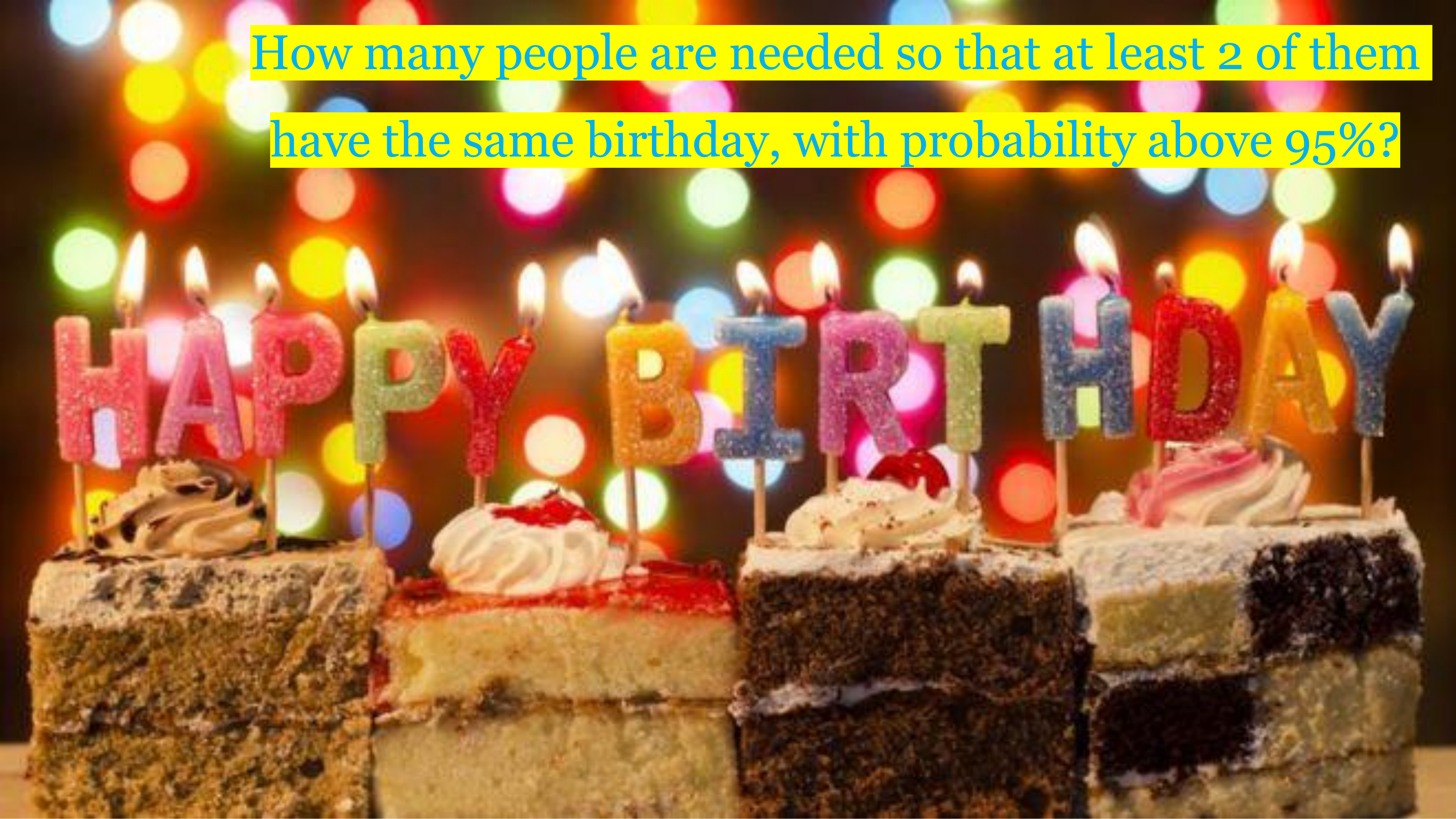
$$= \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} - \frac{|A \cap B|}{|\Omega|} = \frac{6}{36} + \frac{18}{36} - \frac{3}{36}$$

$|A| = 6$

$|B| = |\{(4,1) \ldots (4,6), (5,1) \ldots (5,6), (6,1) \ldots (6,6)\}| = 18$

$|A \cap B| = |\{(4,3), (5,2), (6,1)\}| = 3$

How many people are needed so that at least 2 of them have the same birthday, with probability above 95%?



New stuff

Uniform Distribution

- 23 people are in a room. Suppose all birthdays are equally likely. What is the probability that two will have the same birthday?

Assume $\rightarrow$ 365 possible birth days

$\Omega \rightarrow$ all possible assignment of birthdays to 23 people

$|\Omega| = 365^{23}$

$A \rightarrow$ all outcomes where at least two have same birthday

Uniform Distribution

- 23 people are in a room. Suppose all birthdays are equally likely. What is the probability that two will have the same birthday?

$$P(A) = 1 - \frac{|B|}{|\Omega|}$$

$B \rightarrow$ all outcomes where no two have same birthday

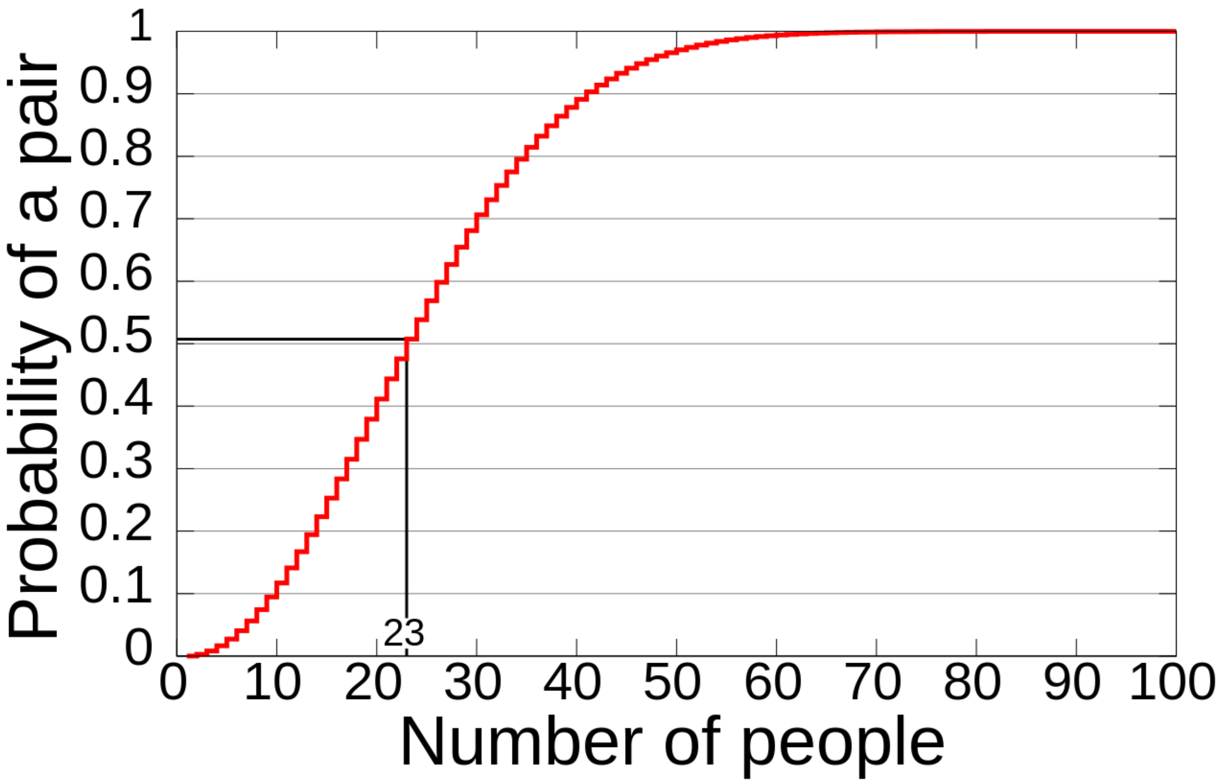
$|B| = {}_{365}P_{23} = 365 \cdot 364 \cdot 363 \cdots$

$$P(A) = 1 - \frac{{}_{365}P_{23}}{(365)^{23}} \approx .5027$$

Uniform Distribution

- 23 people are in a room. Suppose all birthdays are equally likely. What is the probability that two will have the same birthday?

Birthday Paradox!!



Take a Break

5 min

Uniform Distribution

- 55% of adults consume coffee regularly, 45% consume soda and 70% regularly consume at least one of the two.
- What is probability that a randomly selected adult consumes both coffee and soda?

$A \rightarrow$ an adult consumes coffee regularly

$B \rightarrow$ an adult consumes soda regularly

$P(A) = .55$, $P(B) = .45$, $P(A \cup B) = .7$

Want: $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= .55 + .45 - .7$

Uniform Distribution

- 55% of adults consume coffee regularly, 45% consume soda and 70% regularly consume at least one of the two.
- What is the probability that a randomly selected individual doesn't consume either of the two.

$A \cup B$ = people who consume at least one of two

$(A \cup B)'$ = people who don't consume either

$P((A \cup B)') = 1 - P(A \cup B) = 1 - .7$

Uniform Distribution

identical identical identical

- A box contains six 40W bulbs, five 60W bulbs and four 75W bulbs. If bulbs are selected one by one in a random order, what is the probability that at least two bulbs must be selected in order to get one that is rated 75W?

$A \rightarrow$ at least 2 tries for seeing 75W

$A' \rightarrow$ see 75 W bulb on first try

$$P(A) = 1 - P(A') = 1 - \frac{|A'|}{|\Omega|}$$

$$|\Omega| = \frac{15!}{6!\,5!\,4!}, \quad |A'| = \frac{14!}{6!\,5!\,3!}$$