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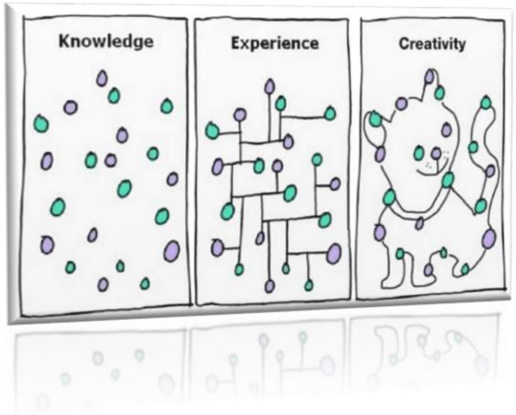

206 Discrete Structures II

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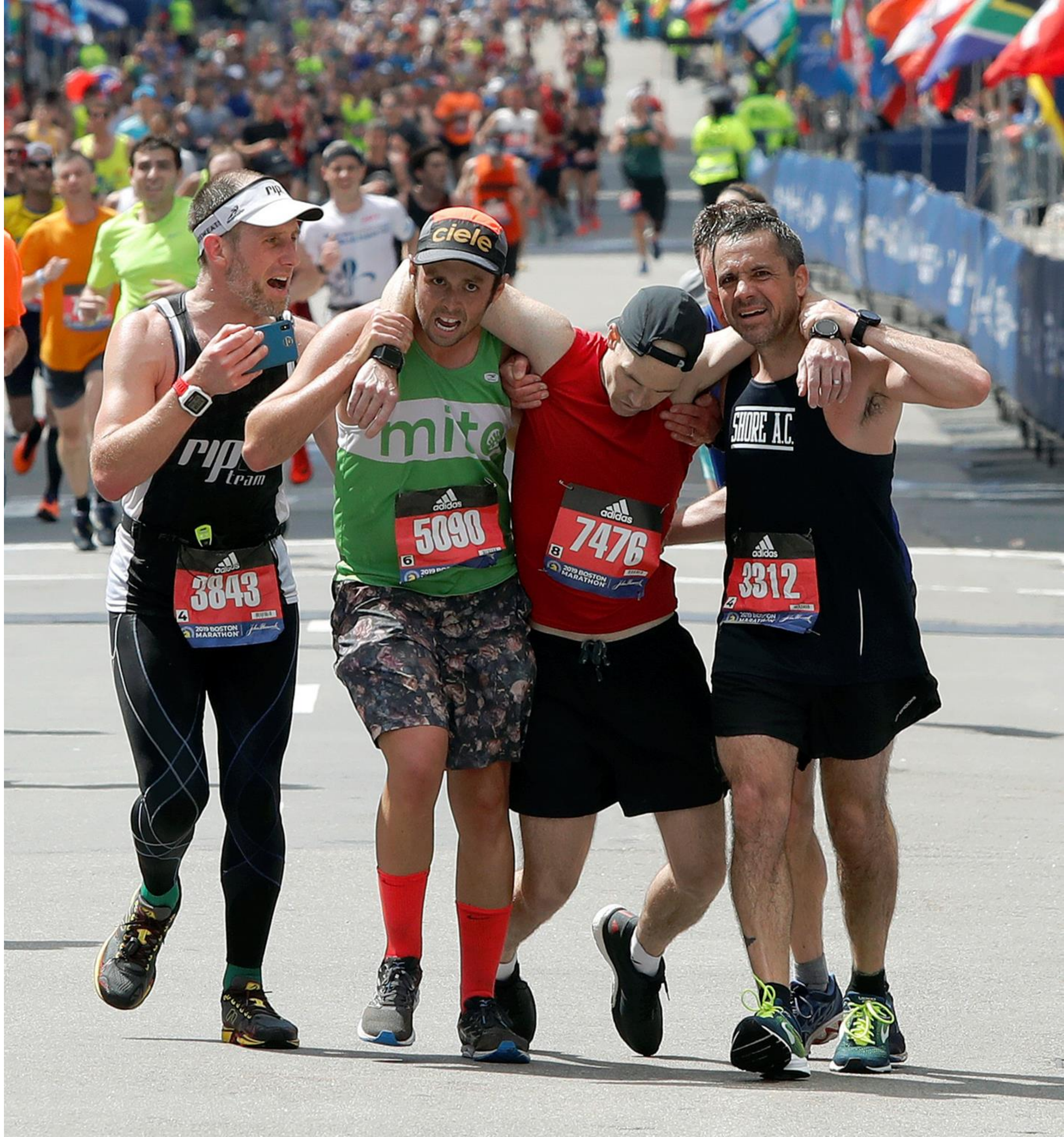
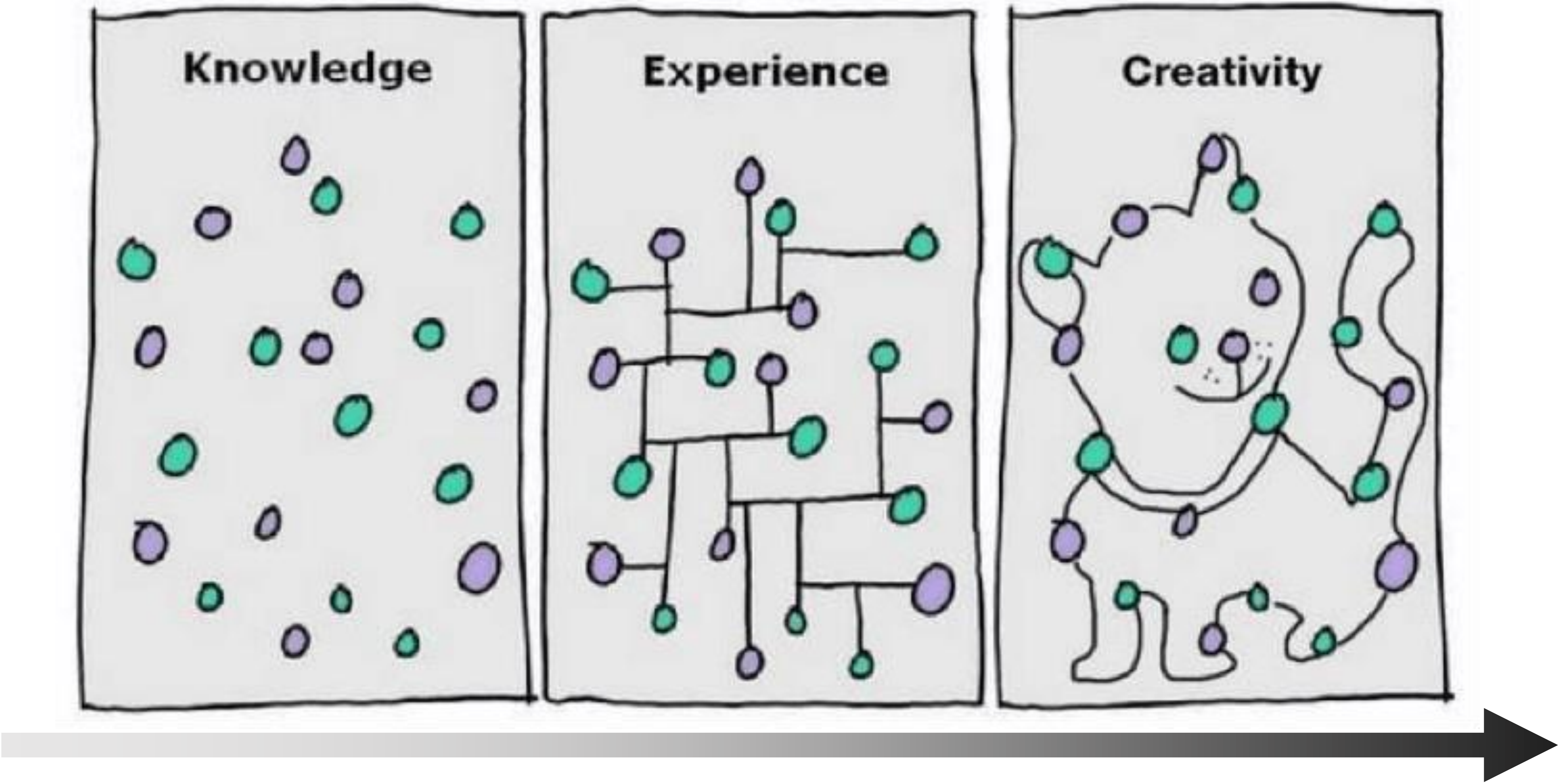
Announcements

- Assignment 2 is running
  - New Deadline Dec 8
- Quiz 5 → This week
  - Probabilities (Basic)





Lectures Quizzes Homework Assignments Real-Life!

Outline

- Sample space
- Axioms of probability
- Conditional probability
- Independence
- Bayes rule
- Random Variables
- Expectation

Last Class

Probability - Basics

$$\text{Probability of an event happening} = \frac{\text{Number of ways it can happen}}{\text{Total number of outcomes}}$$

Example: the chances of rolling a "4" with a die

Number of ways it can happen: 1 (there is only 1 face with a "4" on it)

Total number of outcomes: 6 (there are 6 faces altogether)

$$\text{So the probability} = \frac{1}{6}$$

Last Class

Probability - Basics

$$\text{Probability of an event happening} = \frac{\text{Number of ways it can happen}}{\text{Total number of outcomes}}$$

Example: there are 5 marbles in a bag: 4 are blue, and 1 is red. What is the probability that a blue marble gets picked?

Number of ways it can happen: 4 (there are 4 blues)

Total number of outcomes: 5 (there are 5 marbles in total)

$$\text{So the probability} = \frac{4}{5} = 0.8$$

Last Class

Probability - Basics

Experiment: a repeatable procedure with a set of possible results.

Example: Throwing dice

We can throw the dice again and again, so it is repeatable.

The set of possible results from any single throw is {1, 2, 3, 4, 5, 6}

Probability - Basics

Outcome: A possible result of an experiment.

Example: Getting a "6"

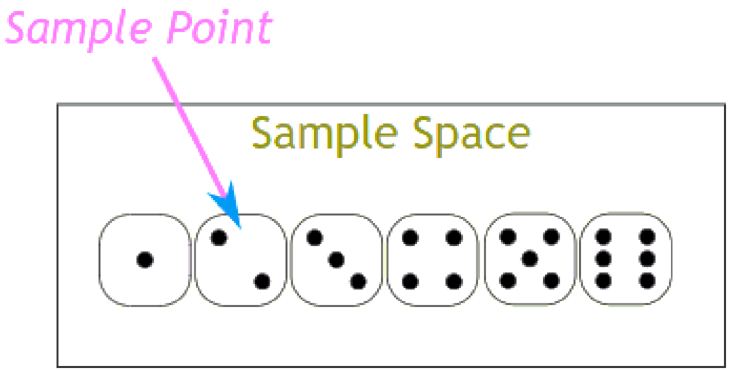
Last Class

Probability - Basics

Sample Point: just one of the possible outcomes

Example: Throwing dice

There are 6 different sample points in the sample space.



Last Class

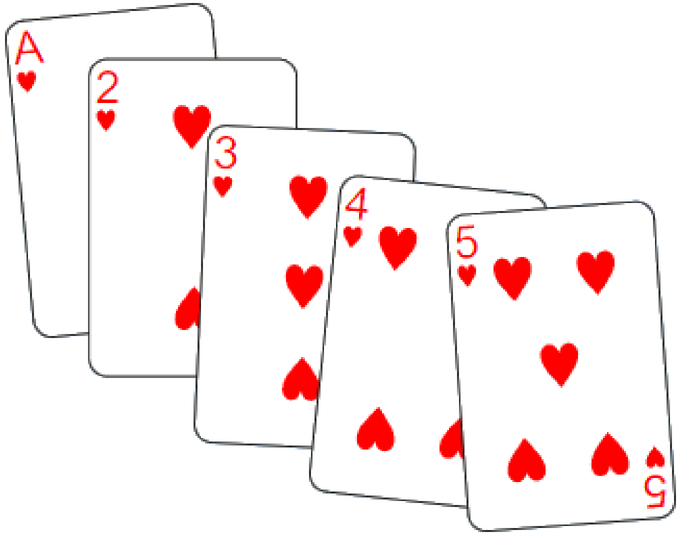
Probability - Basics

Sample Space: all the possible outcomes of an experiment.

Example: choosing a card from a deck

There are 52 cards in a deck (not including Jokers)

So the **Sample Space is all 52 possible cards**: {Ace of Hearts, 2 of Hearts, etc… }



Last Class

Probability - Basics

Event: one **or more** outcomes of an experiment

Example Events:

An event can be just one outcome:

- Getting a Tail when tossing a coin
- Rolling a "5"

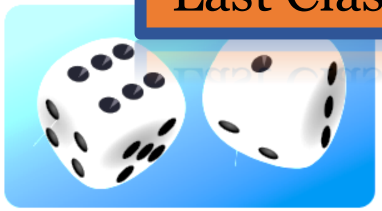
An event can include more than one outcome:

- Choosing a "King" from a deck of cards (any of the 4 Kings)
- Rolling an "even number" (2, 4 or 6)

Last Class

Probability - Basics

Example: Alex wants to see how many times a "double" comes up when throwing 2 dice.



The **Sample Space** is all possible **Outcomes** (36 Sample Points):

{1,1} {1,2} {1,3} {1,4} … {6,3} {6,4} {6,5} {6,6}

The **Event** Alex is looking for is a "double", where both dice have the same number. It is made up of these **6 Sample Points**:

{1,1} {2,2} {3,3} {4,4} {5,5} and {6,6}

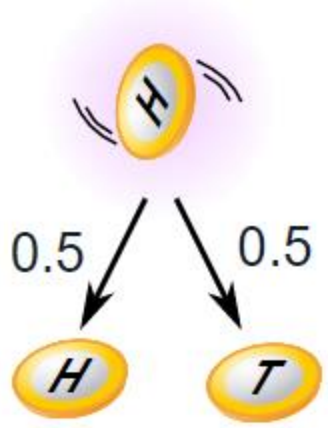
These are Alex's Results:

Experiment	Is it a Double?
{3,4}	No
{5,1}	No
{2,2}	**Yes**
{6,3}	No
...	...

After 100 **Experiments**, Alex has 19 "double" **Events** ... is that close to what you would expect?

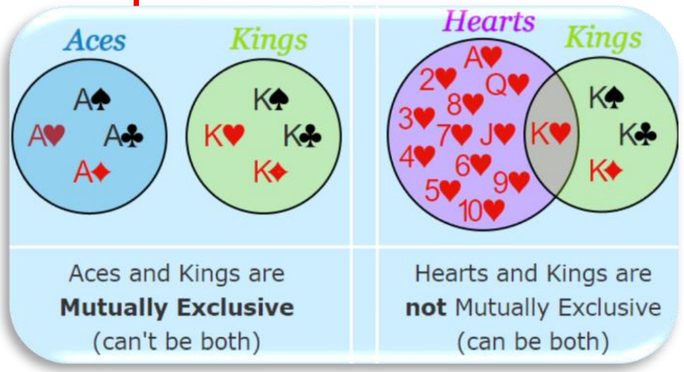
Last Class

Probability - Events



Events can be:

- **Independent**
  each event **is not affected** by other events
  - *e.g., a coin does not know that it came up "Heads" in the past and the chance is simply 50% in every toss of the coin*
- **Dependent** (or **Conditional**)
  where an event **is affected** by other events
  - *e.g., after taking one card from the deck there are **less cards** available, so the probabilities change! How the probability of getting a King changes, after the 1st card was a King (less likely), and after the 1st card was not a King (more likely)?*
  - What would happen if we remove cards with and without replacement? (*independent vs. dependent*)
- **Mutually Exclusive**
  events can't happen at the same time
  - e.g., "Left or Right", "Heads or Tails", "Kings or Aces"



Last Class

Probability of Independent Events

- We can calculate the chances of two or more independent events by ***multiplying the chances***.

Example: Probability of 3 Heads in a Row

For each toss of a coin a "Head" has a probability of 0.5:

0.5

$0.5 \times 0.5 = 0.25$ $\left(\text{or } \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\right)$

$0.5 \times 0.5 \times 0.5 = 0.125$ $\left(\text{or } \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}\right)$

And so the chance of getting 3 Heads in a row is **0.125**

Last Class

Probability of Independent Events

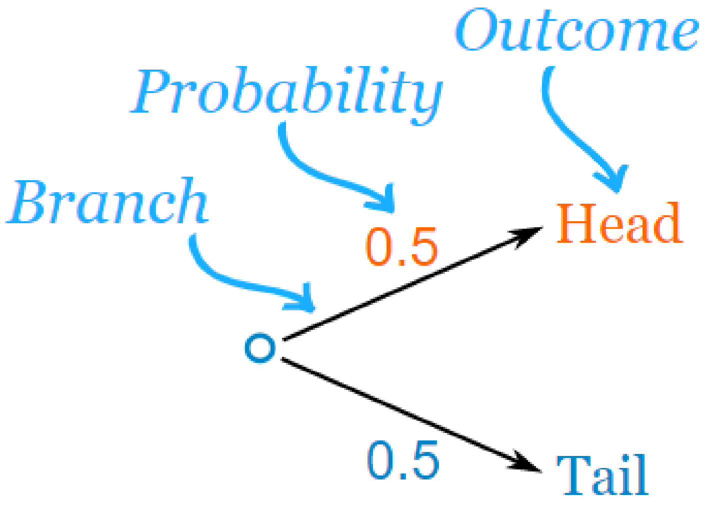
Question 1: What is the probability of 7 heads in a row?

Answer: $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 0.0078125$ (less than 1%).

Question 2: Given that **we have just got 6 heads** in a row, what is the probability that **the next toss** is also a head?

Probability Tree Diagrams

Here is a tree diagram for the toss of a coin:

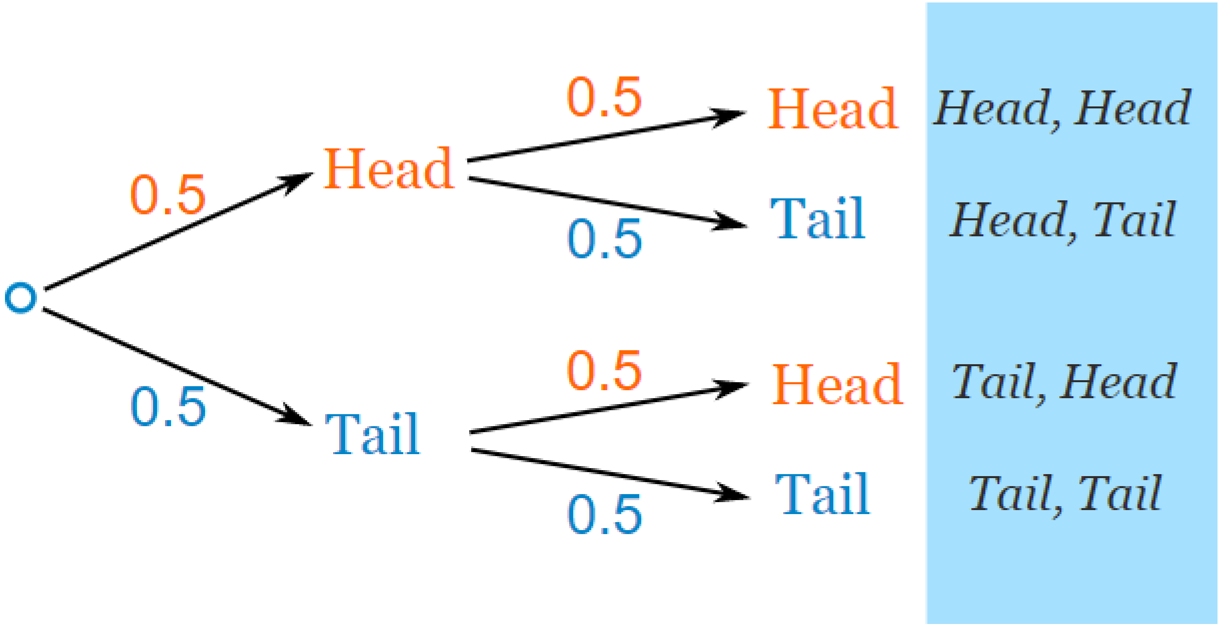


There are two "branches" (Heads and Tails)

- The probability of each branch is written on the branch
- The outcome is written at the end of the branch

Probability Tree Diagrams

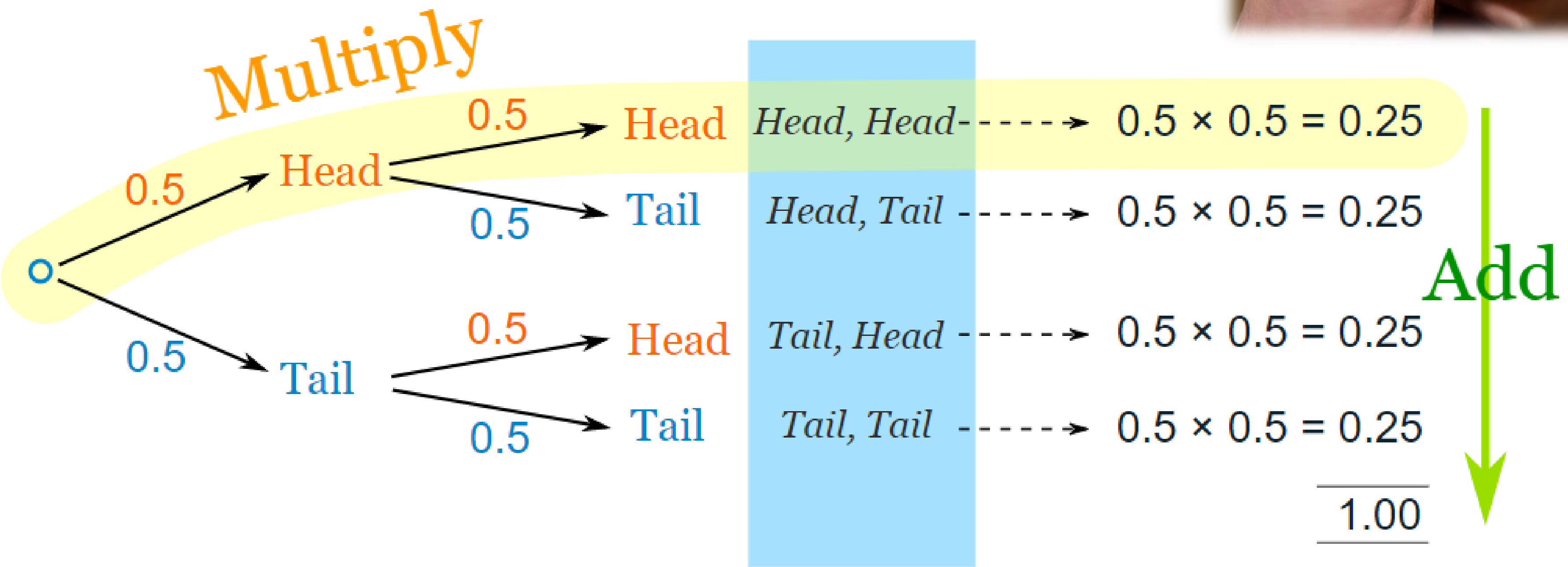
We can extend the tree diagram to two tosses of a coin:



Probability Tree Diagrams

Independent Events

- We **multiply** probabilities **along the branches**
- We **add** probabilities down **columns**



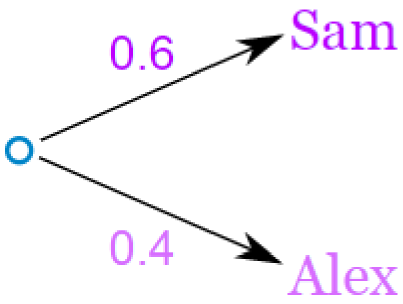
Probability Tree Diagrams

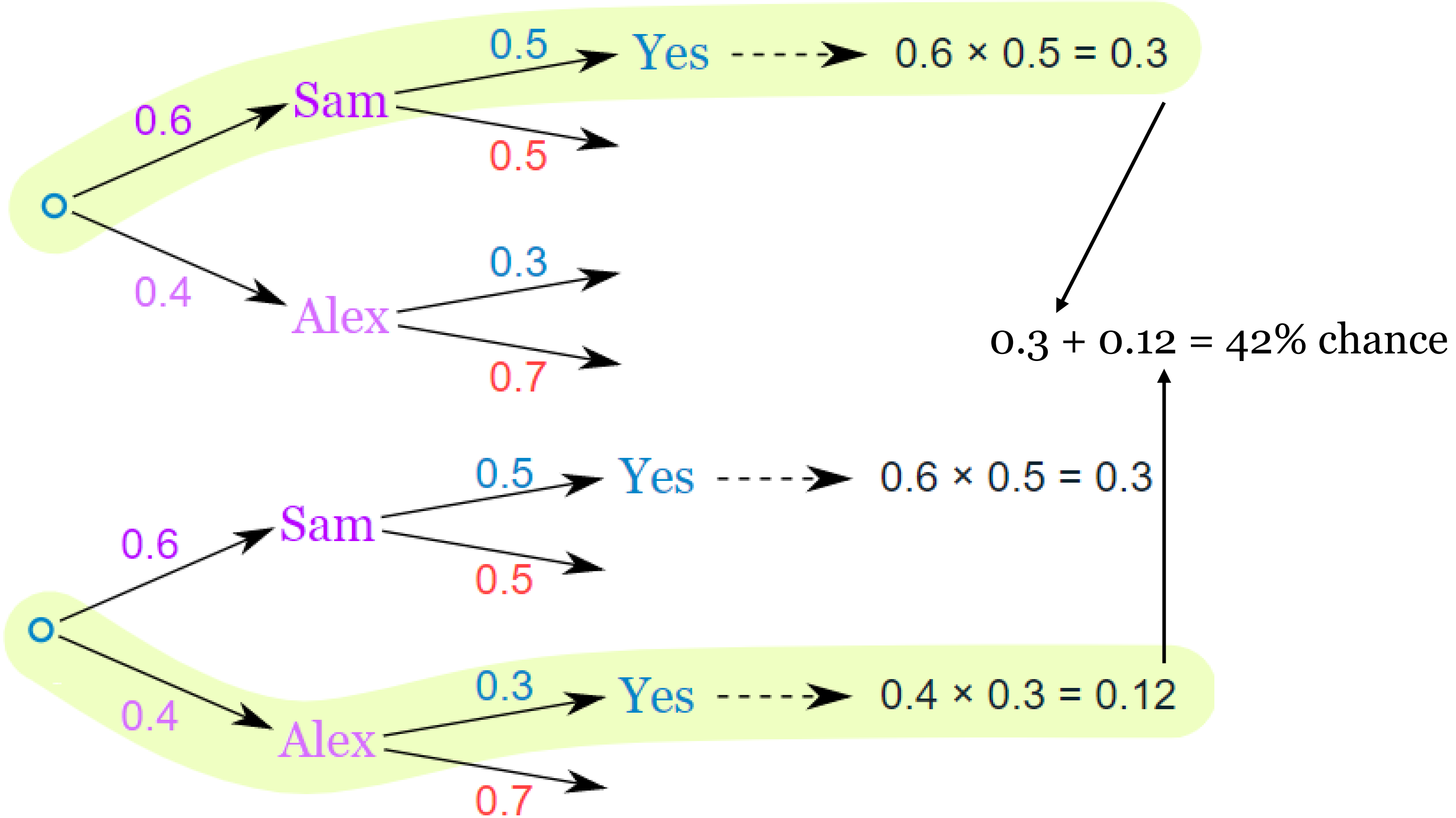
The chance of you being the Goalkeeper depends on who is the Coach:

- with Coach Sam the probability of being Goalkeeper is **0.5**
- with Coach Alex the probability of being Goalkeeper is **0.3**
- Sam is Coach more often ... about 6 out of every 10 games (a probability of **0.6**).
- So, what is the probability you will be a Goalkeeper today?

Dependent Events

Probability Tree Diagrams





Dependent Events

Probability Tree Diagrams

Example: Marbles in a Bag

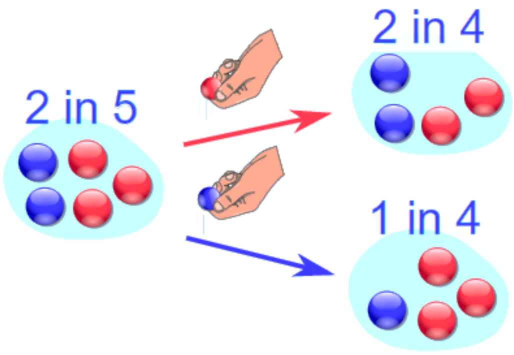
2 blue and 3 red marbles are in a bag.

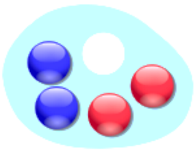
What are the chances of getting a blue marble?

The chance is **2 in 5**

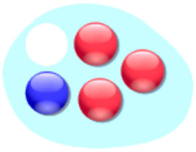
But after taking one out the chances change!

So the next time:





if we got a **red** marble before, then the chance of a blue marble next is **2 in 4**

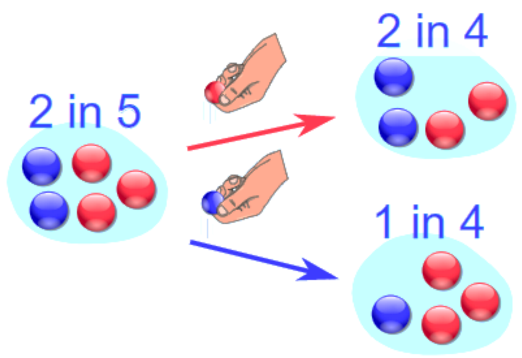


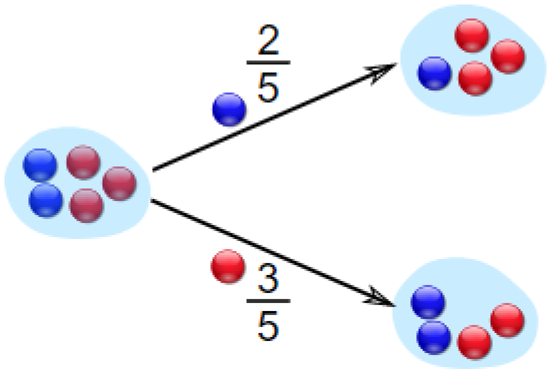
if we got a **blue** marble before, then the chance of a blue marble next is **1 in 4**

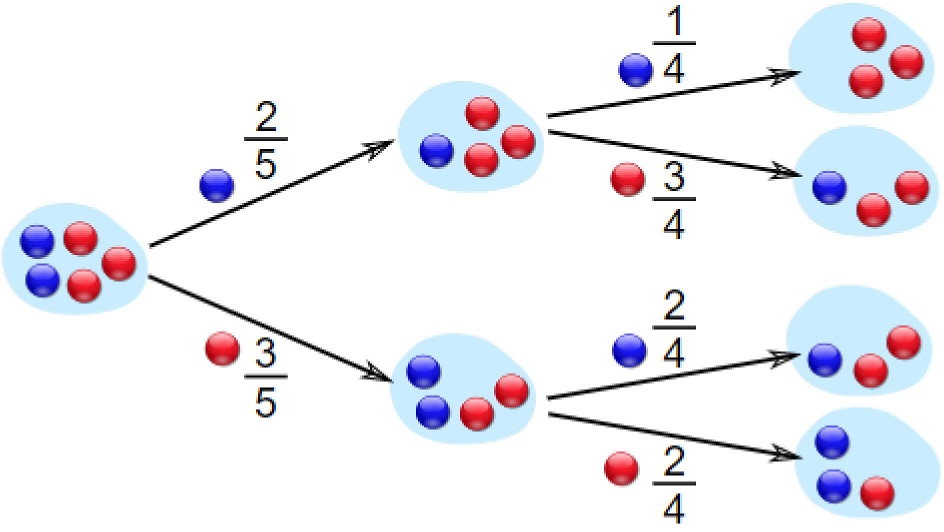
- *The second event **depends on** what happened in the first event and is called **dependent**.*
- *If we **replace** the marbles in the bag each time, then the chances do **not** change and the events are independent:*
- ***With** Replacement: the events are **Independent** (<u>the chances don't change</u>)*
- ***Without** Replacement: the events are **Dependent** (<u>the chances change</u>)*

Dependent Events

Probability Tree Diagrams

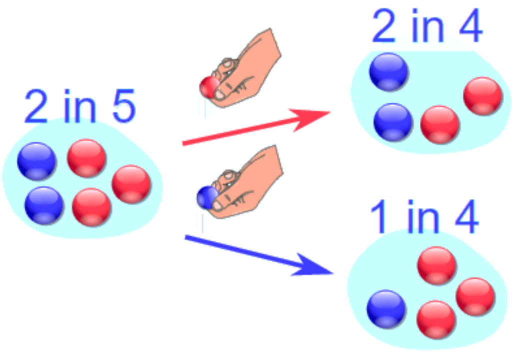




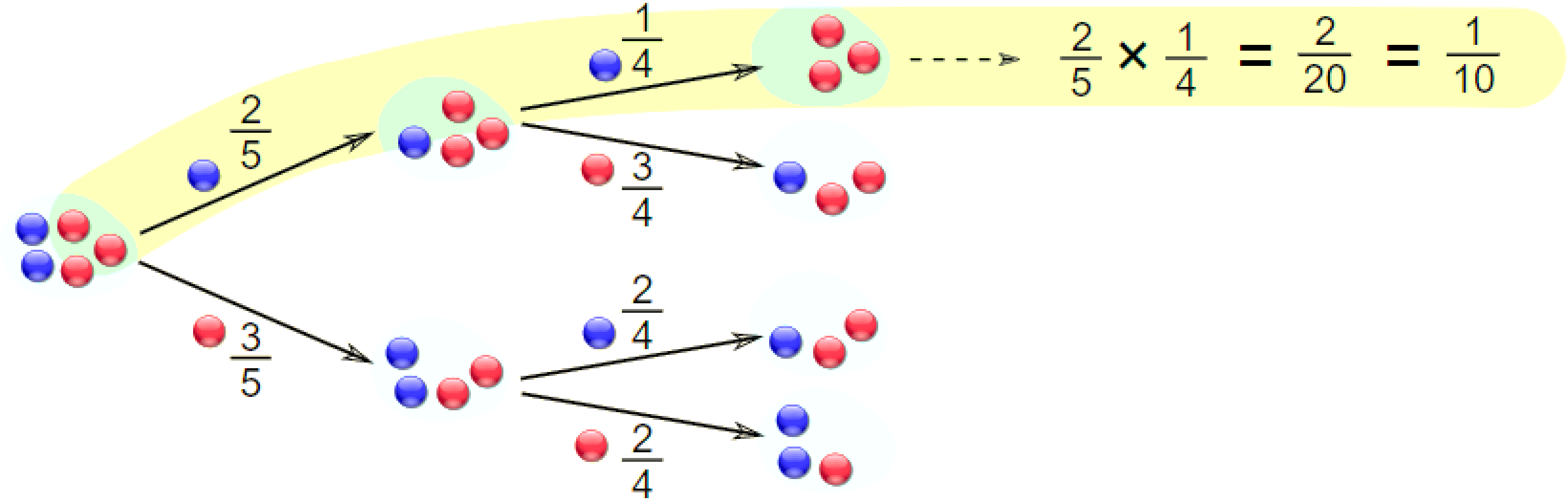


Dependent Events

Probability Tree Diagrams



What are the chances of drawing 2 blue marbles?



Tree Enumeration

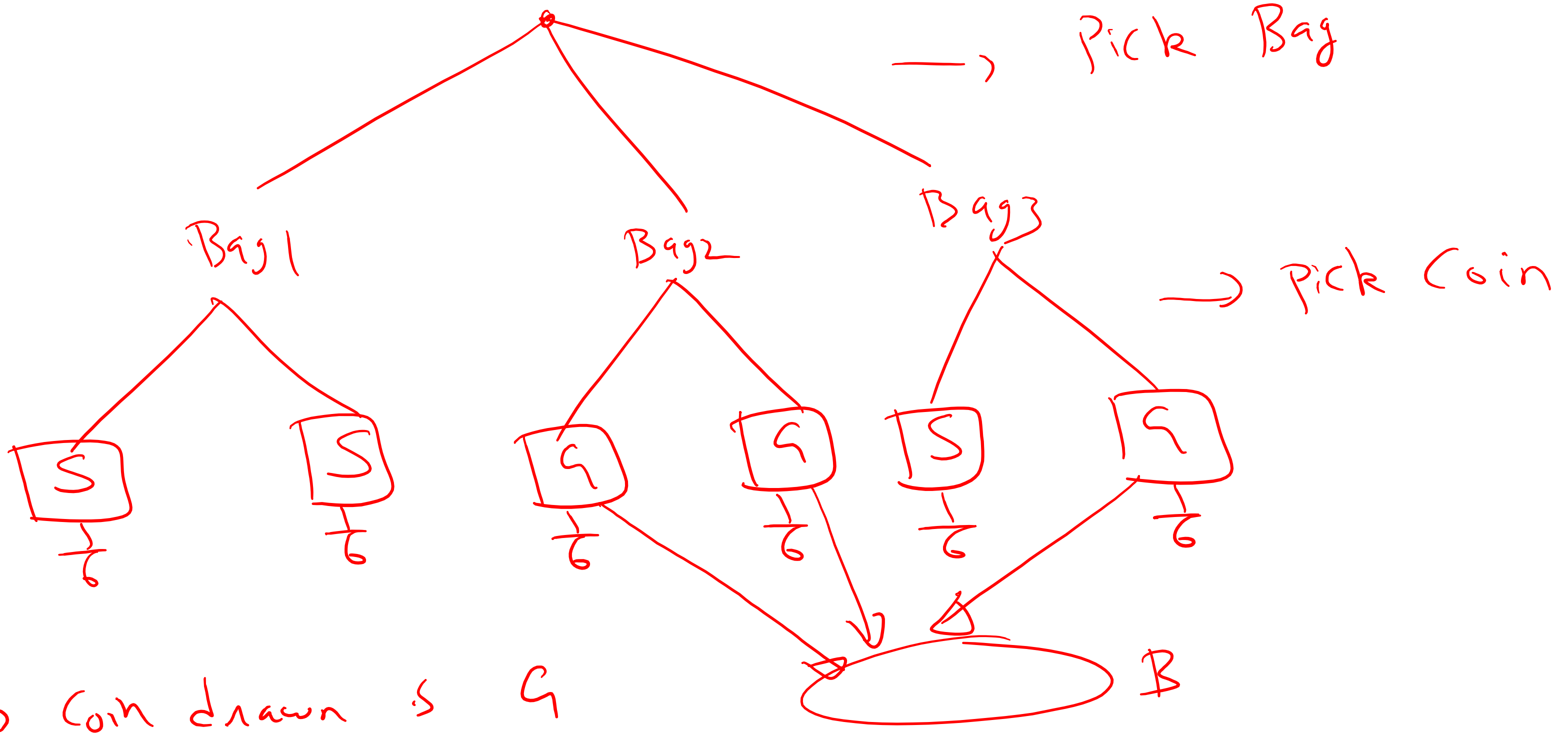
- One bag has two silver coins, another has two gold coins, and the third has one of each.
- One bag is selected at random. One coin from it is selected at random.
- It turns out to be gold What is the probability that the other coin is gold?

$A \rightarrow$ other coin in bag is G

$B \rightarrow$ Coin Drawn is G

$P(A|B)$

Tree Enumeration



B → Coin drawn is G

A → other coin is also G

$$P(A|B) = \frac{2}{3}$$

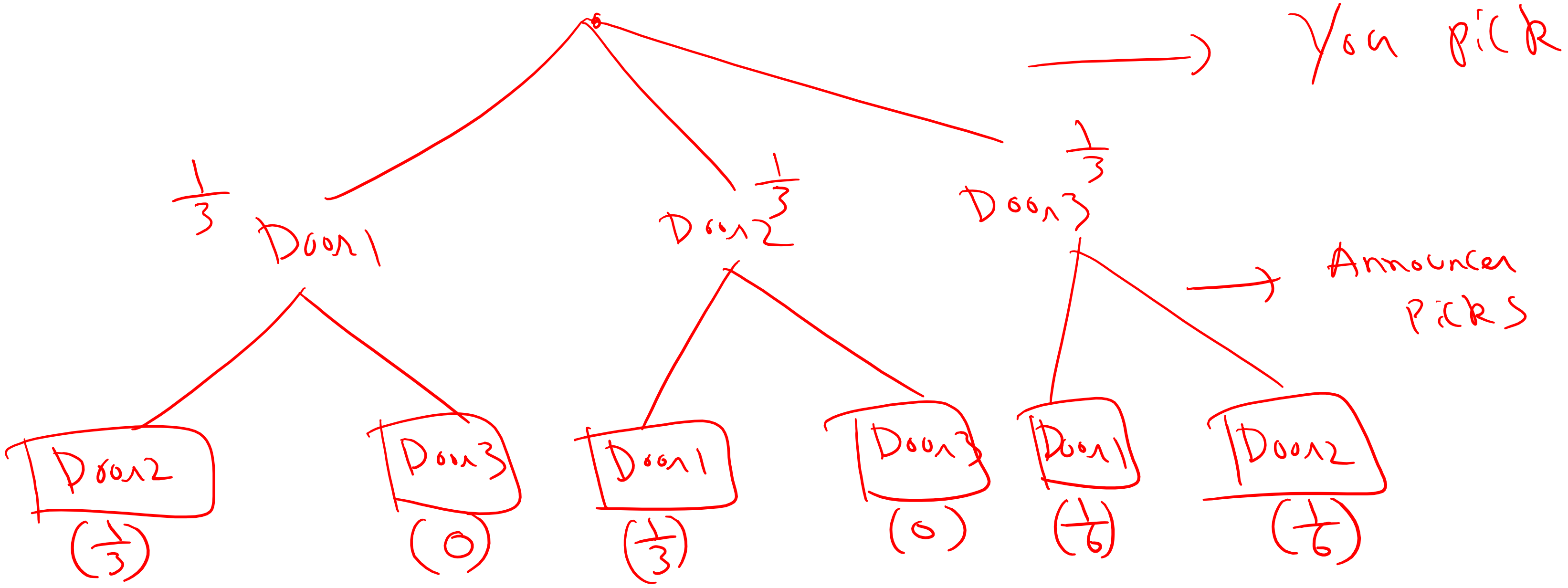
Monty Hall Problem

Door 1 → G
Door 2 → G
Door 3 → Car

P(win by switching)

Monty Hall Problem

Door 1 → G
Door 2 → G
Door 3 → Car



$$P(\text{win by switching}) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

CTAAR survey

https://sirs.ctaar.rutgers.edu/blue

Take a Break

5 min

Conditional Probabilities–Formula to remember

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$

If equally likely outcomes $\frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|}$

$$P(A|B,C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$

Independence

- A and B are independent events if $P(A|B) = P(A)$

$P(A|B) = P(A)$, A and B are independent

$\Rightarrow \dfrac{P(A \cap B)}{P(B)} = P(A)$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

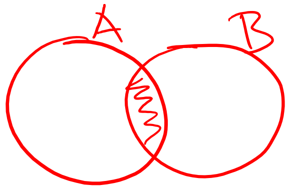
Conditional Probabilities

- Suppose we roll a white and a black die. What is the probability that the white die is 1 given that the sum is 7?
- A = white die is 1
- B = sum if 7

→ Conditional Probability

- We want $P(A|B)$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$



We know B has happened

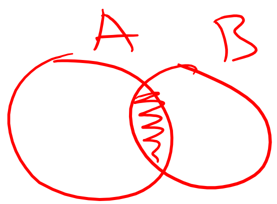
$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$ → Formula for conditional probability

$$= \frac{|A\cap B|}{|B|}$$ → If equally likely outcomes

Conditional Probabilities

- Suppose we roll a white and a black die. What is the probability that the white die is 1 given that the sum is 7?
- A = white die is 1
- B = sum if 7
- We want $P(A|B)$

$$= \frac{|A \cap B|}{|B|}$$



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{6}$$

Conditional Probabilities

- Two fair coins are flipped. $A = \{first\ coin\ is\ H\}, B = \{second\ coin\ is\ H\}$.
- Are A and B independent?

$\Omega = \{(H,H),(T,T),(H,T),(T,H)\}$

$P(A|B) = P(A)$, $P(A) = \frac{|A|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}$

$P(A|B)$: new sample space $B = \{(H,H),(T,H)\}$

$P(A|B) = \frac{1}{2} = P(A)$

Conditional Probabilities

- Two fair coins are flipped. $A = \{first\ coin\ is\ H\}$. $B =$ two coins have different outcomes.
- Are A and B independent?

$P(A) = \frac{1}{2}$

$\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$

$B = \{(H,T),(T,H)\}$

$P(A|B) = \frac{1}{2} = P(A)$

Independence

- A_1, A_2, A_3 are independent events if $P(A_1), P(A_2), P(A_3)$ does not change by knowing any subset of the other.

$$P(A_1 | A_2) = P(A_1)$$
$$P(A_1 | A_3) = P(A_1)$$
$$P(A_1 | A_2, A_3) = P(A_1)$$
$$\Rightarrow P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$
$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$
$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$
$$\vdots$$

Independence

- $A_1, A_2, A_3, \ldots A_n$ are independent events if $P(A_i)$ does not change by knowing any subset of the other.

Independence

- $A_1, A_2, A_3, \dots A_n$ are independent events if for all $k = 2,3, \dots n$, and for all indices $i_1, i_2, \dots i_k$
- $P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right) = P\left(A_{i_1}\right)P(A_{i_2}) \dots P(A_{i_k})$

Conditional Probability

- Consider a family with two children. Given that one of the children is a boy, what is the probability that both children are boys?

$\longrightarrow$ Children are equally likely

$\Omega = \{(B,B), (B,G), (G,B), (G,G)\}$

$B \rightarrow$ one of them is a boy $\rightarrow \{(B,B), (B,G), (G,B)\}$

$A \longrightarrow$ both of them are boys

$P(A|B) = \frac{1}{3}$

$P(A) = \frac{1}{4}$

Conditional Probability

- Consider a family with two children. Given that the first child is a boy, what is the probability that both children are boys?

$\Omega = \{(B,B), (B,G), (G,G), (G,B)\}$

$B \to \{(B,B), (B,G)\}$

$A \to$ both are boys

$P(A|B) = \frac{1}{2}$

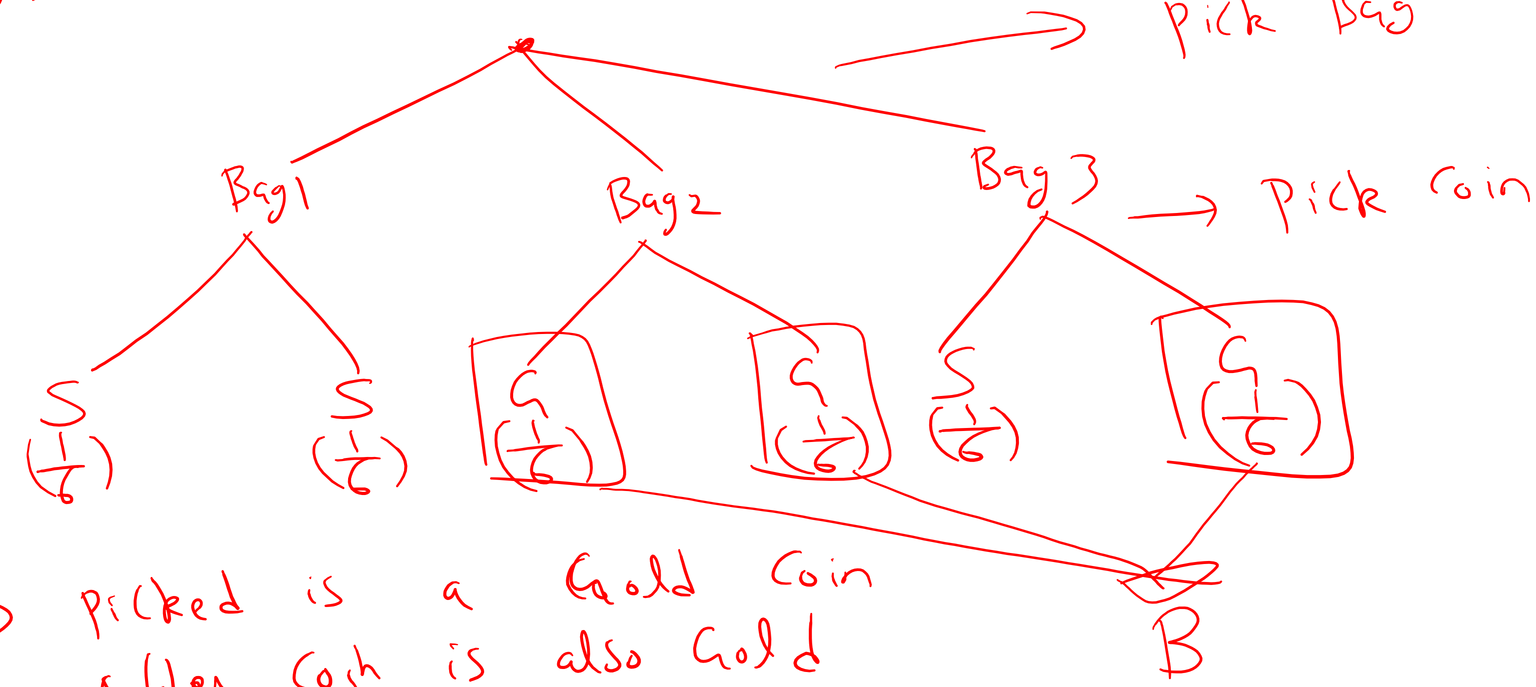
$P(A) = \frac{1}{4}$

Conditional Probability

- One bag has two silver coins, another has two gold coins, and the third has one of each.
- One bag is selected at random. One coin from it is selected at random.
- It turns out to be gold What is the probability that the other coin is gold?

Conditional Probability

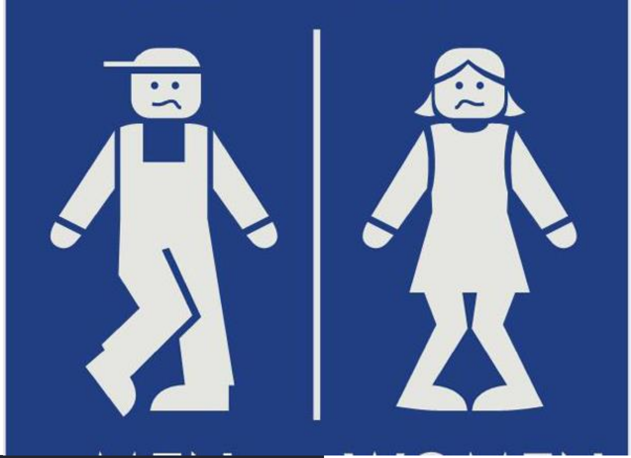
Bag 1 → {S, S}
Bag 2 → {G, G}
Bag 3 → {S, G}



B → picked is a Gold Coin
A → other coin is also Gold

$$P(A|B) = \frac{2}{3}$$

Bayesian Inference



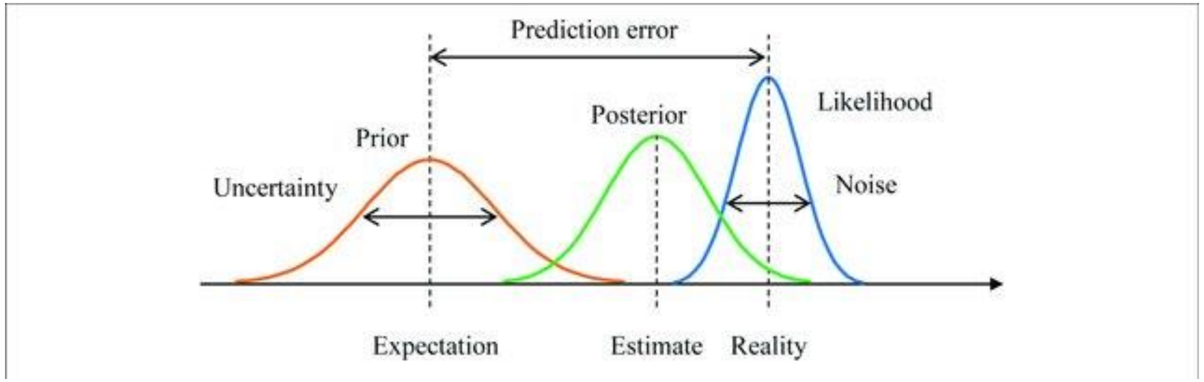
Dilemma at the movies

This person dropped their ticket in the hallway.

Bayesian Inference*

$$\overset{\text{Posterior}}{P(A|B)} = \frac{\overset{\text{Likelihood}}{P(B|A)} * \overset{\text{Prior}}{P(A)}}{\underset{\text{Evidence}}{P(B)}}$$

** Inference = Educated guessing*



- Bayesian inference with a prior distribution, a posterior distribution, and a likelihood function.
- The prediction error is the difference between the prior expectation and the peak of the likelihood function (i.e., reality).
- Uncertainty is the variance of the prior. Noise is the variance of the likelihood function.