Sample Probability Problems

1 10 points each

- Three fair dice are rolled. What is the probability that the sum of their values is 17?

  Solution: $|\Omega| = 216$. There are three outcomes that sum up to 17, namely, $(6,6,5),(6,5,6),(5,6,6)$. Hence, probability $= \frac{3}{216}$.

- A deck of 52 cards is randomly permuted with each outcome being equally likely. What is the probability that the first four cards are all Aces?

  Solution: $|\Omega| = 52!$. If the first four cards have to be Aces then the number of possible permutations equals $4!48!$. This is because the four Aces can be permuted in the first four positions in $4!$ ways and then the remaining 48 cards can be permuted in the remaining positions in $48!$ ways. Hence, probability $= \frac{4!48!}{52!}$.

- A fair die is rolled until a 6 turns up. What is the expected number of rolls?

  Solution: Let p be the probability that when we roll a fair die a 6 turns up. Then $p = \frac{1}{6}$. Hence, in each roll there is a p chance of getting a 6. Computing the expected number of rolls is the same as the expected number of rolls till we see a H when flipping a biased coin of probability p. This number is $\frac{1}{p}$. Hence, expected number of rolls equals 6.

- The probability that a grader will make a marking error on any particular question of a multiple-choice exam is p. If there are ten questions on the exam and questions are marked independently, what is the probability that no errors are made?

  Solution: Since the errors are independent and in each step there is a $(1-p)$ chance that no error is made, the overall probability equals $(1-p)^{10}$.

  What is the probability that at least one error is made?

  Solution: This equals $1 -$ P(no error is made) $= 1-(1-p)^{10}$.

  What is the probability that exactly one error is made?

  Solution: Mark a question as 'C' if it is graded correctly and as 'I' if it is graded incorrectly. Then the possible outcomes where exactly one error is made is all sequences of the form $(C, C, \ldots, I, C, \ldots, C)$ with exactly one 'I'. There are 10 such sequences. The probability of each sequence is $p(1-p)^9$. Hence the total probability equals $10p(1-p)^9$.

- The probability of winning a lottery draw each week is $\frac{1}{N}$. Suppose you buy n tickets in a given week. What is the probability that you win a lottery that week?

  Solution: Probability of winning $=$ Probability that one of the n tickets is the right one $= \frac{n}{N}$.

  Suppose you buy one ticket for each of the n weeks. What is the probability that you win at least once?

  Solution: Probability of winning at least once $= 1-$ Probability of never winning $= 1-(1-\frac{1}{N})^n$.

- If 5 digits are selected at random from 0, 1, . . . , 9, what is the expected number of distinct digits?

  Solution: Define Bernoulli random variables $X_0, X_1, \ldots, X_9$ where $X_i = 1$ if digit i is selected at least once. Then expected number of distinct digits $= \sum_{i=0}^{9} E[X_i]$. Furthermore, $E[X_i] = P(i \text{ selected at least once}) = 1 - P(i \text{ never selected}) = 1-(\frac{9}{10})^5$.

  Hence, expected number of distinct digits $= 10(1-(\frac{9}{10})^5)$.

- A pair of dice is rolled many times
  - What is the probability that in 10 rolls a sum of 7 shows up 4 times?
    Solution: In a given roll, the probability of getting a sum of 7 equals $\frac{1}{6}$.
    Mark each roll as '1' if sum is 7. Otherwise mark it as '0'. Then the number of outcomes where 7 shows up 4 times in 10 rolls equals the number of length 10 binary strings with 4 1s. There are $\binom{10}{4}$ such outcomes. The probability of each outcome is $(\frac{1}{6})^4(\frac{5}{6})^6$ since each roll is independent. Hence, total probability equals $\binom{10}{4}(\frac{1}{6})^4(\frac{5}{6})^6$.
  - What is the expected number of rolls till a sum of 7 shows up 4 times?
    Solution: In a given roll the probability of seeing a sum of 7 equals $\frac{1}{6}$.
    Define random variables X_1, X_2, X_3, X_4 where X_1 equals the number of rolls required to see a sum of 7 for the first time. X_2 equals the number additional rolls required after seeing a sum of 7 to see the sum of 7 for the second time. Similarly, X_i equals the number additional rolls required after seeing a sum of 7 $i-1$ times to see the sum of 7 for the ith time. Then, the expected number of rolls to see a sum of 7 four times equals $E[X_1]+E[X_2]+E[X_3]+E[X_4]$.
    Furthermore, $E[X_i]$is the same as the number of times one needs to flip a coin of bias $\frac{1}{6}$ until a H is seen, where in this problem a H means seeing a sum of 7. This expectation equals 6. Hence, $E[X_1]+E[X_2]+E[X_3]+E[X_4]=24$.

- We have three urns, I, II, and III. Urn I has 3 black and 2 green chips. Urn II has 4 white and 6 red chips. Urn III has 3 white and 7 red chips. We draw a chip at random from urn I. If this is a black chip, we draw a chip from urn II at random. Otherwise, we draw a chip from urn III at random. Given that the second selected chip was red, what is the probability that the first selected chip was black?

  Solution: Let $A=$ first chip is black and $B=$ second chip is red. Then by Bayes theorem $P(A|B)=\frac{P(A)P(B|A)}{P(A)P(B|A)+P(A')P(B|A')}=\frac{.36}{.36+.28}=\frac{36}{64}$.

- Events A and B are disjoint if $A\cap B$ is empty, and independent if $P(A\cap B)=P(A)P(B)$. If A and B are independent events and $P(A)=1/3$, $P(B)=1/4$. Find
  (i) $P(A\cup B)$.
  Solution: $1/3+1/4-1/12=\frac{1}{2}$.
  (ii) $P(A\cap B)$.
  Solution: $\frac{1}{12}$.
  (iii) Same as (i), but assume that A and B are disjoint.
  Solution: $\frac{7}{12}$.

- Suppose you deal 13 cards from a standard deck of 52 cards. What is the probability that the distribution of suits is 5, 5, 2, 1? (That is, you have 5 cards of one suit, 5 cards of another, 2 of another, and 1 of the last.)

  Solution: Total outcomes, i.e., $|\Omega|=\binom{52}{13}$. Let's count favorable outcomes. We first pick which suits will have 5 cards and which ones will have 2 and 1. This can be done by picking the suit that will have 2 cards in 4 ways and then picking the suit that will have 1 card in 3 ways. Hence, total choices equals 12. Once the distribution is picked, for each such choice, the actual cards can be picked from different suits in $\binom{13}{5}^2\binom{13}{2}\binom{13}{1}$ ways. Hence, total favorable outcomes equals $12\binom{13}{5}^2\binom{13}{2}13$.

  This implies that the total probability equals $\frac{12\binom{13}{5}^2\binom{13}{2}13}{\binom{52}{13}}$.

- You have 10 pairs of socks (so 20 socks total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks, at random. What is the expected number of complete pairs you have left?

  Solution: Define Bernoulli random variables $X_1,\ldots,X_{10}$ where X_i equals 1 if pair i survives. Then, the expected number of surviving pairs equals $\sum_{i=1}^{10} E[X_i] = \sum_{i=1}^{10} P(i \text{ survives})$. Furthermore, $P(i \text{ survives}) = \frac{\binom{18}{4}}{\binom{20}{4}} = \frac{12}{19}$. Hence, the expected number of surviving pairs equals $\frac{120}{19}$.

- Suppose that every page in the chapter contains exactly 3000 characters, and there is an average of one typo per page. What is the probability that there are exactly 8 typos in the 10-page chapter?

  Solution: Probability of typo, $p = \frac{1}{3000}$. We have 30000 characters and want exactly 8 of them to have a typo. This is the same as flipping a coin of bias p n times and getting i Heads. Here $p = \frac{1}{3000}$, $n = 30000$ and $i = 8$. From the binomial distribution formula, this probability equals $\binom{n}{i} p^i (1-p)^{n-i} = \binom{30000}{8} (\frac{1}{3000})^8 (\frac{2999}{3000})^{29992}$.