CS 206: Sample Final Exam

1 10 points each

- How many ways are there to permute the letters of MISSOURI?

  Solution: $\frac{8!}{2!2!}$.

- How many integers in 1 to 3000 are not divisible by any of 2, 3, or 7?

  Solution: Let A be the set of integers divisible by 2, B be the set divisible by 3 and C be the set divisible by 7. Then we want $|A \cup B \cup C|$. By inclusion-exclusion this equals $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$. Furthermore we have that $|A| = \lfloor \frac{3000}{2} \rfloor = 1500$, $|B| = \lfloor \frac{3000}{3} \rfloor = 1000$, $|C| = \lfloor \frac{3000}{7} \rfloor = 428$, $|A \cap B| = \lfloor \frac{3000}{6} \rfloor = 500$, $|B \cap C| = \lfloor \frac{3000}{21} \rfloor = 142$, $|C \cap A| = \lfloor \frac{3000}{14} \rfloor = 214$ and $|A \cap B \cap C| = \lfloor \frac{3000}{42} \rfloor = 71$. Hence, $|A \cup B \cup C| = 1500 + 1000 + 428 - 500 - 142 - 214 + 71 = 2143$

- 100 students are sick. 1000 students are tired. 10000 students are antsy for the break. Determine the number of students who are sick or tired or antsy for the break under each of the following conditions:

  i. Every sick student is tired, and every tired student is antsy for the break.

  Solution: Let A be the set of sick students, B be the tired students and C be the antsy ones. Here, we are given that $A \subset B$ and $B \subset C$. We want $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| = |A| + |B| + |C| - |A| - |B| - |A| + |A| = 10000$.

  ii. The sets of students are pairwise disjoint

  Solution: Here we are given that $A \cap B = \emptyset$ and $B \cap C = \emptyset$ and $C \cap A = \emptyset$. Hence, $|A \cup B \cup C| = |A| + |B| + |C| = 11100$.

  iii. There are two students who are both sick and tired, two who are both tired and antsy of the break, two who are both sick and antsy for the break, and one who is sick and tired and antsy for the break.

  Solution: Here we are given that $|A \cap B| = 2$, $|B \cap C| = 2$, $|C \cap A| = 2$ and $|A \cap B \cap C| = 1$. Substituting in the formula we get that $|A \cup B \cup C| = 100 + 1000 + 10000 - 2 - 2 - 2 + 1 = 11095$.

- The final exam of a discrete mathematics course consist of 50 true/false questions, each worth one point, and 25 multiple choice questions, each worth two points.

  1. In how many ways can a student get a grade of 96 on the test?

  Solution: To get a score of 96 there are three options. (a) Answer 23 multiple choice correctly and all 50 true/false correctly. This can be done in $\binom{25}{23}$ ways. (b) Answer 24 multiple choice correctly and 48 true/false correctly. This can be done in $\binom{25}{24} \cdot \binom{50}{48}$ ways. (c) Answer 25 multiple choice correctly and 46 true/false correctly. This can be done in $\binom{50}{46}$ ways. Hence, total ways equals $\binom{25}{23} + \binom{25}{24} \cdot \binom{50}{48} + \binom{50}{46}$.

  2. Suppose the questions are numbered 1 . . . 75. In how many ways can a student get 2 even-numbered questions wrong?

  Solution: There are 37 even numbered questions. Hence total ways equals $\binom{37}{2}$.

  3. Suppose a student has answered three questions wrong. Suppose a lazy professor chooses 10 questions to be graded at random. What is the probability that at most one of the chosen questions is wrong?

Solution: P(at most one chosen question is wrong) = P(no chosen question is wrong) + P(exactly one chosen question is wrong) $= \frac{\binom{72}{10}}{\binom{75}{10}} + 3\frac{\binom{72}{9}}{\binom{75}{10}}$.

- Prove the identity $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

  [Hint: Use the expansion of $(x+y)^n$.]

  Solution: We know that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. Substituting $x=-1$ and $y=1$ we get the identity.

- How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats?

  Solution: $10!$.

- How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if all couples are to get adjacent seats?

  Solution: $5! \cdot 2^5$.

- Six people get into an elevator at the ground floor of a hotel which has 10 upper floors. Assuming each person gets off at a randomly chosen floor from 1 to 10, what is the probability that no two people get off at the same floor?

  Solution: The total number of ways in which the six people can make choices is 10^6. The total choices where no two pick the same floor are $\binom{10}{6}\cdot 6!$. P(no two people get off at the same floor) $= \frac{\binom{10}{6}\cdot 6!}{10^6}$.

- An urn contains 30 distinguishable balls, of which 10 are red and the other 20 blue. Suppose you take out 8 balls from this urn with each outcome being equally likely. What is the probability that among the 8 balls in this sample exactly 3 are red and 5 are blue?

  Solution: $\frac{\binom{10}{3}\cdot\binom{20}{5}}{\binom{30}{8}}$.

- Assume a committee of 10 has to be selected from a group of 100 people, of which 40 are men and 60 are women.

  (a) How many ways are there to choose such a committee?

  Solution: $\binom{100}{10}$.

  (b) How many ways are there to choose the committee so that exactly half of the members are men?

  Solution: $\binom{40}{5}\cdot\binom{60}{5}$.

  (c) What is the probability that a randomly selected committee of 10 consists of exactly 5 men and 5 women?

  Solution: $\frac{\binom{40}{5}\cdot\binom{60}{5}}{\binom{100}{10}}$.

- A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease, and a probability 0.10 of giving a (false) positive when applied to a non-sufferer. It is estimated that 0.5% of the population are sufferers. Suppose that the test is now administered to a person about whom we have no relevant information relating to the disease (apart from the fact that he/she comes from this population). Calculate the following probabilities:

  (a) that the test result will be positive

  Solution: $P(+) = P(\text{sufferer})P(+|\text{sufferer}) + P(\text{non-sufferer})P(+|\text{non-sufferer}) = 0.005\cdot 0.95 + 0.995\cdot 0.1$.

  (b) that, given a positive result, the person is a sufferer

  Solution: $P(\text{sufferer}|+) = \frac{P(\text{sufferer})P(+|\text{sufferer})}{P(+)} = \frac{0.005\cdot 0.95}{0.005\cdot 0.95+0.995\cdot 0.1}$.

  (c) that, given a negative result, the person is a non-sufferer

  Solution: $P(\text{non-sufferer}|-) = \frac{P(\text{non-sufferer})P(-|\text{non-sufferer})}{P(-)} = \frac{0.995\cdot 0.}{1-0.005\cdot 0.95-0.995\cdot 0.1}$.

  (d) that the person will be misclassified.

  Solution: $P(\text{misclassified}) = P(\text{sufferer})P(\text{misclassified}|\text{sufferer}) + P(\text{non-sufferer})P(\text{misclassified}|\text{non-sufferer}) = 0.005\cdot 0.05 + 0.995\cdot 0.1$.

- Calculate the coefficient of x^2 in the expansion of $(x^3 + \frac{1}{2x})^{10}$.

  Solution: Using the expansion of $(x+y)^n$ we get that $(x^3 + \frac{1}{2x})^{10} = \sum_{k=0}^{10} \binom{10}{k} x^{3k} (\frac{1}{2x})^{10-k}$. In this expansion, x^2 will appear when $k = 3$ and the coefficient will be $\frac{\binom{10}{3}}{2^7}$.

- Consider the following events when drawing two cards from a standard deck of 52 cards.

  A. The first card is a Spade

  B. The second card is a Heart

  C. The first card is the Ace of Clubs

  Compute the following probabilities

  - $P(B)$

    Solution: $\frac{39 \cdot 13 + \binom{13}{2}}{\binom{52}{2}}$.
  - $P(A \cap B)$

    Solution: $\frac{13 \cdot 13}{\binom{52}{2}}$.
  - $P(C|A)$

    Solution: 0.
  - $P(A \cup C)$

    Solution: $\frac{1}{4} + \frac{51}{\binom{52}{2}}$.
  - $P(A|B)$

    Solution: $\frac{13 \cdot 13}{39 \cdot 13 + \binom{13}{2}}$.
  - $P(B|A)$

    Solution: $\frac{4 \cdot 13 \cdot 13}{\binom{52}{2}}$.

- Show that among any 4 numbers one can find 2 numbers so that their difference is divisible by 3.

  Solution: There are three possible remainders when divided by 3. Hence, by the pigeonhole principle, among 4 numbers there must exist two numbers a and b that have the same remainder. We can write $a = 3k_1 + r$ and $b = 3k_2 + r$ for some integers k_1, k_2 and r. It is easy to see that $a - b$ is divisible by 3.

- Prove that if two events A and B are independent then A' and B' are also independent.

  Solution: $P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{1-P(A \cup B)}{P(B')} = \frac{1-P(A)-P(B)+P(A)P(B)}{P(B')} = \frac{(1-P(A))(1-P(B))}{P(B')} = \frac{P(A')P(B'))}{P(B')} = P(A')$. Hence, A' and B' are independent.

- n people numbered 1 to n enter a room. Each pair (i, j) decides to shake hands independently with probability 0.5. A triple of people (a, b, c) is called a trio if a, b shook hands, b, c shook hands and a, c shook hands. Let T be the total number of trios. Compute $E[T]$.

  Solution: For each triplet (a,b,c), define a Bernoulli random variable $X_{a,b,c}$ that is 1 if (a,b,c) form a trio. Let X be the total number of trios. Then $X = \sum_{a,b,c} X_{a,b,c}$ and $E[X] = \sum_{a,b,c} E[X_{a,b,c}] = \sum_{a,b,c} = P(X_{a,b,c} = 1)$. There are $\binom{n}{3}$ triplets and for each of them $P(X_{a,b,c} = 1) = \frac{1}{8}$. Hence, $E[X] = \frac{\binom{n}{3}}{8}$.

- At a grocery store there are X number people that shop each day. Here X is a random variable with the following probability mass function: $P(X = k) = \frac{\mu^k e^{-\mu}}{k!}$ for any integer $k \geq 0$. Here $\mu > 0$ is a fixed constant. Compute $E[X]$ and $Var[X]$.

  You can use the fact that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ for any x.

  Solution: $E[X] = \sum_{k=0}^{\infty} kP(X = k) = \sum_{k=0}^{\infty} k \frac{\mu^k e^{-\mu}}{k!} = e^{-\mu}[\mu + 2 \cdot \frac{\mu^2}{2!} + 3 \cdot \frac{\mu^3}{3!} + \ldots] = e^{-\mu}[\mu + \frac{\mu^2}{1!} + \frac{\mu^3}{2!} + \ldots]$.

  Next we use the fact that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. Multiplying both sides by x we get that $xe^x = x + \frac{x^2}{1!} + \frac{x^3}{2!} + \ldots$. Substituting $x = \mu$ we get that $\mu + \frac{\mu^2}{1!} + \frac{\mu^3}{2!} + \ldots = \mu e^{\mu}$. Hence, $E[X] = e^{-\mu} \cdot \mu e^{\mu} = \mu$.

Next, $E[X^2] = \sum_{k=0}^{\infty} k^2 P(X = k) = \sum_{k=0}^{\infty} k^2 \frac{\mu^k e^{-\mu}}{k!} = e^{-\mu}[\mu + 2^2 \cdot \frac{\mu^2}{2!} + 3^2 \cdot \frac{\mu^3}{3!} + \ldots] = e^{-\mu}[\mu + 2 \cdot \frac{\mu^2}{1!} + 3 \cdot \frac{\mu^3}{3!} + \ldots]$.

Now again, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$. Multiplying by x on both sides we get that $xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots$. Differentiating both sides with respect to x we get that $xe^x + e^x = 1 + 2\frac{x}{1!} + 3\frac{x^2}{2!} + 4\frac{x^3}{3!} + \ldots$. Multiplying both side by x we get that $x(xe^x + e^x) = x + 2\frac{x^2}{1!} + 3\frac{x^3}{3!} + \ldots$. Substituting $x = \mu$ we get that $\mu(\mu e^{\mu} + e^{\mu}) = \mu + 2 \cdot \frac{\mu^2}{1!} + 3 \cdot \frac{\mu^3}{3!} + \ldots$. Hence, $E[X^2] = e^{-\mu} \cdot \mu(\mu e^{\mu} + e^{\mu}) = \mu^2 + \mu$. This implies that $Var[X] = E[X^2] - \mu^2 = \mu$.

- Alice is playing blackjack at a casino. She wins each round with probability 1/3, and loses the round with probability 2/3. For every round that she wins, she makes 9 dollars and for each round that she loses, she loses 3 dollars. Suppose Alice plays 100 rounds of blackjack.

  (a) What is the total amount of money that Alice will have at the end of 100 rounds in expectation?

  Solution: Let X_i be the amount earned in round i and Z be the total money earned in 100 rounds. Then $E[X_i] = \frac{1}{3} \cdot 9 + \frac{2}{3} \cdot (-3) = 1$. Hence total expect money in 100 rounds $E[Z] = \sum_{i=1}^{100} E[X_i] = 100$.

  (b) Given the best upper bound you can on the probability that Alice will end up owing money at the end of 100 rounds.

  We will use Chebychev's inequality. Let's compute $Var[Z] = \sum_{i=1}^{100} Var[X_i]$. Furthermore, $Var[X_i] = E[X_i^2] - 1 = \frac{1}{3} \cdot 9^2 + \frac{2}{3} \cdot 3^2 - 1 = 32$. Hence, $\sigma^2 = Var[Z] = 3200$. Now $P(\text{ Alice owes money after 100 rounds}) = P(Z < 0) = P(Z - 100 < -100) \leq P(|Z - 100| > 100) = P(|Z - 100| > \frac{100}{\sigma} \cdot \sigma) \leq \frac{\sigma^2}{100^2} = \frac{3200}{100^2} = \frac{32}{100} = \frac{8}{25}$.

- At all times, a pipe-smoking mathematician carries 2 matchboxes – 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match, he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly k matches in the other box? All the matches are identical.

  Solution: Let's compute the probability that the mathematician discovers that the left pocket is empty and the right pocket has k matches. The total probability will be twice this value since it could also happen that the right pocket is empty and the left one has k matches.

  For the left pocket to be empty and the right one to have k matches, the mathematician must choose the left pocket N times and the right pocket N-k times within the first 2N-k trials. Then in the $2N - k + 1$st trial he must choose the left pocket again to realize that it is empty. This happens with probability $\binom{2N-k}{N}(\frac{1}{2})^{2N-k+1}$.

  Hence total probability equals $2\binom{2N-k}{N}(\frac{1}{2})^{2N-k+1}$.

- A communication system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function. For what values of p is a 5-component system more likely to operate effectively than a 3-component system?

  Solution: $P(\text{5 component system works}) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5$. $P(\text{3 component system works}) = \binom{3}{2}p^2(1-p) + p^3$. We want $\binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p) + p^5 > \binom{3}{2}p^2(1-p) + p^3$. Canceling out p^2 we get that $10p(1-p)^2 + 5p^2(1-p) + p^3 > 3(1-p) + p$. Rearranging, we get that $10p(1-p)^2 + (5p^2 - 3)(1-p) - p(1-p)(1+p) > 0$. Taking out the common $(1-p)$ term, this simplifies to $10p(1-p) + 5p^2 - 3 - p(1+p) > 0$. Further simplification gives $-6p^2 + 9p - 3 > 0$ or in other words $2p^2 - 3p + 1 < 0$. This is the same as $(p - \frac{3}{4})^2 - \frac{1}{16} < 0$. Hence, we need $p > \frac{1}{2}$.