November 1 and 8, 2021

INCLUSION EXCLUSION PRINCIPLE PROBLEMS

1. How many positive integers not exceeding 1000 are divisible by 7 or 11?

> **Solution:** Using the inclusion-exclusion principle for 2 sets, $|A \cup B| = |A| + |B| - |A \cap B|$, we start by defining A as the set of positive integers not exceeding 1000 divisible by 7, B the set of positive integers not exceeding 1000 divisible by 11, and $A \cap B$ the set of positive integers not exceeding 1000 divisible by both 7 AND 11. Knowing the size of these three sets, we can calculate $A \cup B$ which is the set of positive integers not exceeding 1000 divisible by either 7 OR 11. The sizes of the sets are as follows: $|A| = \lfloor \frac{1000}{7} \rfloor = 142$,$|B| = \lfloor \frac{1000}{11} \rfloor = 90$,$|A \cap B| = \lfloor \frac{1000}{7 \times 11} \rfloor = \lfloor \frac{1000}{77} \rfloor = 12$. Therefore, by inclusion exclusion principle, we get $|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12$

2. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

> **Solution:** We use the inclusion-exclusion principle for 3 sets, $|R \cup F \cup S| = |R| + |F| + |S| - |R \cap F| - |R \cap S| - |S \cap F| + |R \cap F \cap S|$ where sets R,f,S represent sets of students taking languages Russian, French, and Spanish, respectively. Since we are asked to find number of students who have taken all three, we need to calculate the size of set $|R \cap F \cap S|$, or the intersection of the three sets. Also, sizes for all other sets are given in the problem, thus we get:$|R \cap F \cap S| = |R \cup F \cup S| - |R| - |F| - |S| + |R \cap F| + |R \cap S| + |S \cap F| = 2092 - 114 - 879 - 1232 + 14 + 23 + 103$

3. Out of a class of 20 students, how many ways are there to form a study group? Assume that a study group must have at least 2 students

> **Solution:** This is equal to the number of subsets $S' \subseteq S = \{x_1, x_2, \ldots, x_{20}\}$. There are $|\mathcal{P}(S)| = 2^{20}$ total subsets, where
>
> (a) 20 of which have size 1 — $\{x_1\}, \{x_2\}, \ldots, \{x_{20}\}$;
>
> (b) and 1 which has size 0 — an empty study group $\emptyset$.
>
> So the answer is $2^{20} - 20 - 1$.

4. Suppose you need to come up with a password that uses only the letters A, B, and C and which must use each letter at least once. How many such passwords of length 8 are there?

Solution: We will first find the number of passwords that leave out at least one of A, B, or C.
Let X be the set of passwords that doesn't contain A;
Let Y be the set of passwords that doesn't contain B;
Let Z be the set of passwords that doesn't contain C.
We want to find the size of $X \cup Y \cup Z$
Passwords that don't contain A just contain B and C. So there are 2^8 such passwords—i.e. $|X| = 2^8$. By the same reasoning $|Y| = |Z| = 2^8$.
Passwords that don't contain A or B just contain C. There is one such password (namely 'CCCCCCCC') so $|X \cap Y| = 1$. By the same reasoning $|X \cap Z| = |Y \cap Z| = 1$.
Passwords that don't contain A, B, or C can't exist because passwords in this problem only use the letters A,B,and C.So $|X \cap Y \cap Z| = 0$. So by inclusion-exclusion $X \cup Y \cup Z = 3 * 2^8 - 3 * 1 + 0 = 3 * 2^8 - 3$.
To find the answer to the original question, we need to subtract the number we just found from the total number of passwords, which is 3^8. This gives $3^8 - (3 * 2^8 - 3)$

5. Given equation $x_1 + x_2 + x_3 + x_4 = 15$ such that $x_i \geq 0$. How many solutions exist. Also, suppose the case where $x_1 < 4$.

Solution: $\frac{18!}{15!3!}$. General solution: $\frac{(n+k-1)!}{n!(k-1)!} = \binom{(n+k-1}{k-1)}$. If $x_1 < 4$, we need to subtract all combinations where $x_1 \leq 4$, which is simply the number of arrangements of $15-4 = 11$ dots(originally 18) and 3 bars(originally 3) which is $\frac{14}{11!3!}$. Then, the number of solutions is $\frac{18!}{15!3!} - \frac{14}{11!3!}$.

6. There are 50 patients at the hospital, 25 are diagnosed with pneumonia, 30 with bronchitis, and 10 with both.

   (a) How many patients are diagnosed with pneumonia OR bronchitis?

   (b) How many patients are not diagnosed with either pneumonia or bronchitis?

Solution: (a) Let S be patients at the hospital, A be people who are diagnosed with pneumonia, B be people who are diagnosed with bronchitis, $|A \cap B| = 10$. $|A \cup B| = |A| + |B| - |A \cap B| = 25 + 30 - 10 = 4$ (b) $|S| - |A \cup B| = 5$

7. Suppose we define prime-looking numbers as any composite number(defined as an integer c such that $a * b = c$ where $a, b < c$ and $a, b \in Z$) not divisible by 2,3, or 5. For example, the three smallest prime-looking numbers are 49,77,91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

   SOLUTION: `https://www.cut-the-knot.org/arithmetic/combinatorics/InclExclEx.shtml#solution`

Solution: Inclusion exclusion problem: Define sets: Set of numbers less than 1000 divisible by 2 is $|N_2| = \lfloor \frac{999}{2} \rfloor = 499$,divisible by 3 is $|N_3| = \lfloor \frac{999}{3} \rfloor = 333$,divisible by 5 is $|N_5| = \lfloor \frac{999}{5} \rfloor = 199$,divisible by 2 or 3 is $|N_2 \cap N_3| = \lfloor \frac{999}{2*3} \rfloor = 166$,divisible by 2 or 5 is $|N_2 \cap N_5| = \lfloor \frac{999}{2*5} \rfloor = 99$,divisible by 3 or 5 is $|N_3 \cap N_5| = \lfloor \frac{999}{3*5} \rfloor = 66$,divisible by 2 or 3 or 5 is $|N_2 \cap N_3 \cap N_5| = \lfloor \frac{999}{2*3*5} \rfloor = 33$. Then, by inclusion exclusion principle, we have 499+333+199-166-99-66+33=733 that are not prime-looking including 3 actual primes(2,3,5). To get number of prime-looking numbers under 1000, we subtract 999-733=266 to get number of prime-looking numbers and actual primes. Now subtract actual primes to get 266-165=101 prime-looking numbers. The reason we use 165 rather than 168 is to account for 2,3,5 primes in 733. Finally, we subtract integer 1 from our 101 prime-like numbers, since it is not a composite number.

8. Suppose that 4 people are standing in line. How many ways are there to rearrange the line so that nobody is standing in their original place?

Solution: We will first count the number of ways to arrange the line of people so that at least one person stays in the same place. For each i between 1 and 4, let A_i be the set of ways to order the people in line so that person i stays in the same place. We want to find the size of $A_1 \cup A_2 \cup A_3 \cup A_4$. The number of ways to order the line so that person i remains in the same place is just the number of ways to arrange the other three people—i.e. $3 \times 2 \times 1 = 3!$. In other words, for each i, $|A_i| = 3!$. Similar reasoning shows that the double intersections have size 2!, and that the triple and quadruple intersections have size 1.
Since there are 6 double intersections, 4 triple intersections, and 1 quadruple intersection, the inclusion-exclusion tells us that the number of ways to arrange the people so that someone stays in the same place is $4 \times 3! - 6 \times 2! + 4 \times 1 - 1 \times 1$. Subtracting this from the total number of ways to arrange four people, which is 4!, gives us $4! - (4 \times 3! - 6 \times 2! + 4 \times 1 - 1 \times 1)$

PIGEONHOLE PRINCPLE PROBLEMS

1. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

Solution: Based on pigeonhole principle, There are 26 letters as boxes and 30 student as objects, so $\lceil \frac{30}{26} \rceil = 2$

2. Show that there are at least 250 four digit numbers whose digits all sum to the same value.

Solution: The number of four digit numbers is $9 \times 10 \times 10 \times 10 = 9000$: there are 9 options for the first digit (since it cannot be 0) and 10 options for the remaining three digits. The number of possible sums of the digits of four digit numbers is 36: the lowest possible sum is 1, the highest possible sum is 36 (since that's the sum when every digit is 9), and all the possible sums are integers. Viewing the four digit numbers as objects and the possible sums as boxes, the pigeonhole principle implies that at least one box will end up with at least $\lceil \frac{9000}{36} \rceil = 250$ objects—in other words there is at least one value which is the digit sum of at least 250 four digit numbers.

3. Given a line segment of length L that contains n=1 points, let d be the length of the shortest segment between consecutive points. What is the maximum value of d over all configurations of points.

Solution: Starting with trivial example, let the n+1 points be equally spaced on the line, with n line segments of length $\frac{L}{n}$. Clearly, shifting any points will make the shortest d less than current $d = \frac{L}{n}$. Now if we assume each of the n line segments is a box, with the n+1 points as objects put into the box, we have, by pigeon hole principle that one of the boxes will have 2 points, hence with maximum $d = \frac{L}{n}$

4. A coffee shop sells five sizes of coffee. You buy 11 coffees.

   (a) How many ways are there to order 11 coffees?

   Solution: We can think of the five sizes of coffee as 5 distinguishable boxes and the 11 coffees we must order as indistinguishable balls. To throw a ball into a box means to order one more of that size of coffee (so every ball must go into some box). The balls are indistinguishable because all that matters is how many of each size we order and not which order we order them in. There are no restrictions on how many of each size we order, so this problem can be solved using the stars and bars method. So the solution is $\binom{11+5-1}{11}$

   (b) Prove that one can always find 3 coffees of the same size.

   Solution: There are 5 boxes to which you are assigning 11 coffees. Hence, by pigeonhole principle, $\lceil \frac{11}{5} \rceil = 3$

COMBINATORIAL PROBLEMS

5. Prove that:

$$\binom{2n}{n} = 2\binom{2n-1}{n-1} \tag{20}$$

Solution: The RHS can be reformulated as

$$2\binom{2n-1}{n-1} = \binom{2n-1}{n-1} + \binom{2n-1}{n-1} \tag{21}$$

$$= \binom{2n-1}{n-1} + \binom{2n-1}{n} \tag{22}$$

Then we can see that the first term of the reformulated RHS counts number of combinations of size $n-1$ formable from $2n-1$ elements assuming some specific element is included in the all combinations by default. And the second term of the reformulated RHS counts all combinations with that specific element missing from all combinations. Then we have included both cases, resulting in the same counting as happens on the LHS where all combinations of size n are counted from 2n elements.