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206
Discrete Structures II

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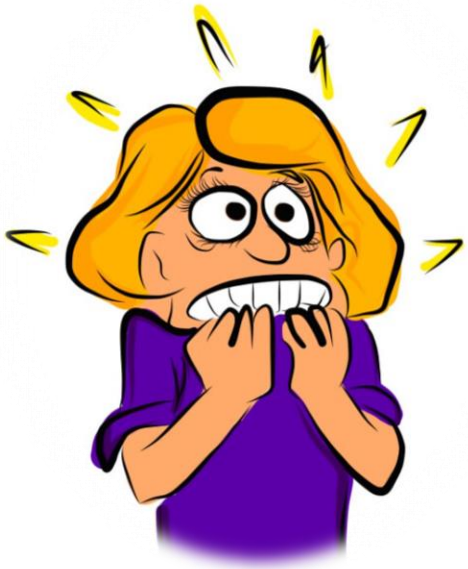
So Far

- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- ~~Permutation/Combinations~~
- ~~Inclusion-Exclusion~~ / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

Quiz 3 – Next ~~Tuesday~~/Thursday



- More time (35 minutes)
  - + more questions ☺
- What will cover
  - Permutations with/out repetition
  - Combinations
  - Pirates Problem
  - Pirates Problem
  - Pirates Problem
    - Have you seen the extra Pirates problems?

So far...

Pigeonhole Principle

If m pigeons are in n holes and $m > n$, then at least $\left\lceil \frac{m}{n} \right\rceil$ pigeons are in the same hole.

Ceiling of m over n rounds the ratio to the larger integer

$\left\lceil \frac{m}{n} \right\rceil$ = nearest integer higher than $\frac{m}{n}$

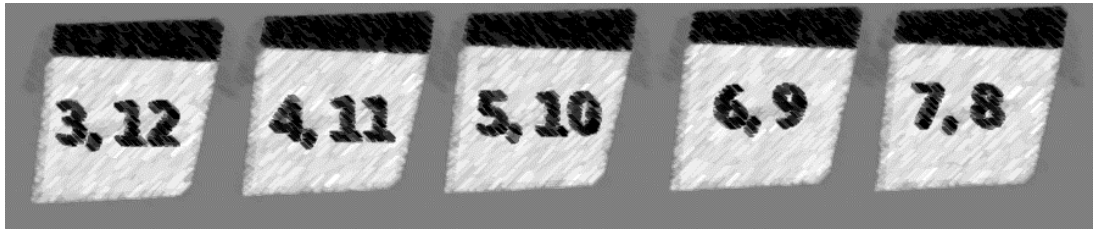
$m = 20$

$n = 9$

$\left\lceil \frac{20}{9} \right\rceil = 3$

PHP – Example 1

- Prove that if 6 integers are selected from {3,4,5,6,7,8,9,10,11,12}, there must be 2 integers whose sum is 15.
- Solution: Label 5 boxes



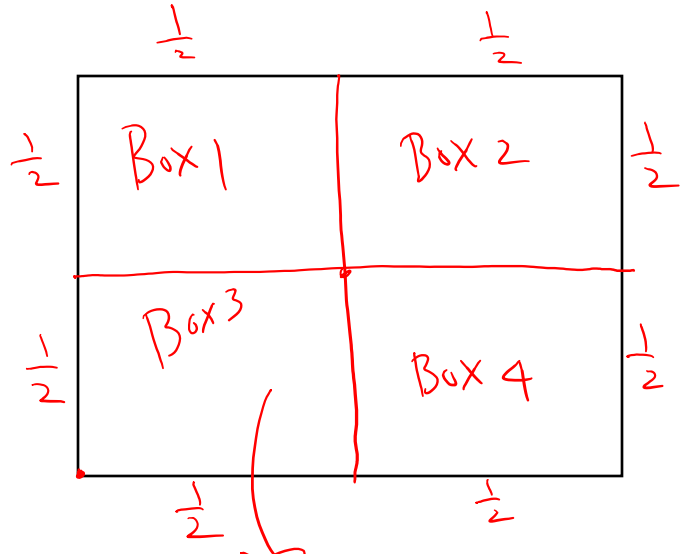
- We select 6 integers and place them in one of the boxes above, based on its label

- By PHP: One box must have at least 2 integers

So far...

PHP – Example 2

- Consider any 5 points in the interior of a square of unit length. Show that one can find two points that are at a distance of at most $\frac{\sqrt{2}}{2}$.



– If we pick 5 points Then by PHP some box must have ≥2 points

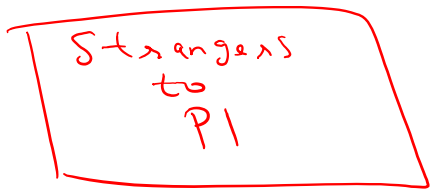
– Maximum distance between any two points in a box is when they are diagonally opposite. The distance is $\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{\sqrt{2}}{2}$

PHP – Example 3

- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

Let 6 people be $P_1, P_2, P_3, P_4, P_5, P_6$

Define 2 boxes



— Every remaining person goes to one of these boxes depending on whether she/he knows P1 or not.

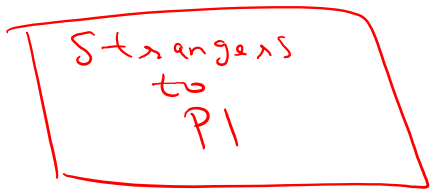
— By pigeonhole principle one of the two boxes must have at least $\lceil \frac{5}{2} \rceil = 3$ people

PHP – Example 3

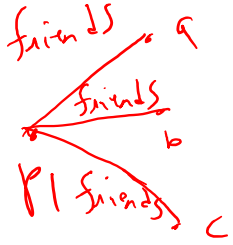
- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

Let 6 people be $P_1, P_2, P_3, P_4, P_5, P_6$

Define 2 boxes



— Case 1: Friends of P1 box has ≥ 3 people.
Let a, b, c be any 3 people in the box



— If any of a, b, c know each other then together with P1 they form a group of 3 mutual friends

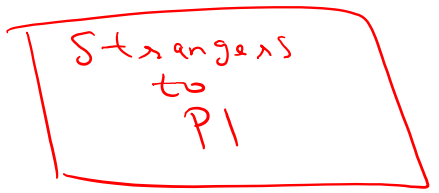
— otherwise a, b, c is a group of 3 mutual strangers

PHP – Example 3

- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

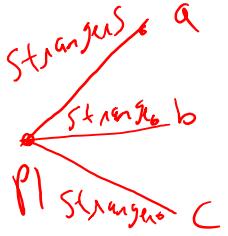
Let 6 people be $P_1, P_2, P_3, P_4, P_5, P_6$

Define 2 boxes



— Case 2: **Strangers to** P1 box has ≥3 people.

Let a, b, c be any 3 people in the box



— If any two of a, b, c are strangers then together with P1 they form a group of 3 mutual strangers.

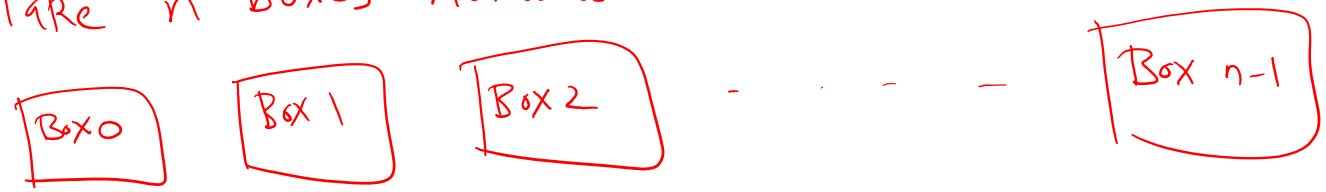
— Otherwise a, b, c form a group of 3 mutual friends.

PHP – Example 4

(PHP)

- There are n people in a room. Show that there must exist two people with the same number of acquaintances.

Make n boxes numbered 0 to n-1



— Box i has people who have i acquaintances

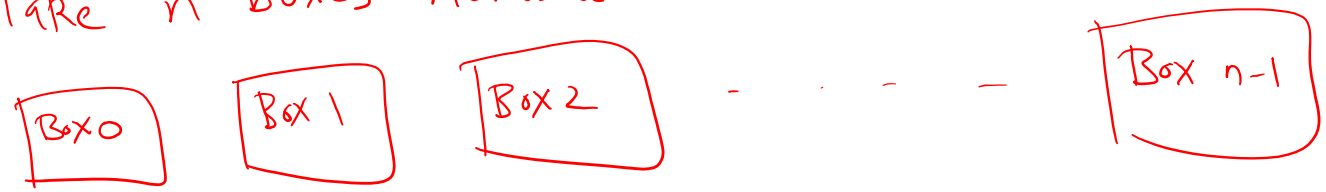
— We have n boxes and n people so can't directly apply PHP and infer something useful.

PHP – Example 4

(PHP)

- There are n people in a room. Show that there must exist two people with the same number of acquaintances.

Make n boxes numbered 0 to n-1



Case 1: Box 0 is empty

Then n people go to remaining n-1 boxes and by PHP some box must have ≥ 2 people

Case 2: Box n-1 is empty

Then again n people go to n-1 boxes and by PHP some box must have ≥ 2 people

Take a 5 min Break

PHP – Example 5

- There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

Apples are Pigeons
and
Baskets are Boxes

One way to model the process

Doesn't Work

1. We don't know how many apples in total
2. Even if we know, apples can't independently go to any box. There is a constraint that each box has ≤ 24 apples

PHP – Example 5

- There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

Baskets are pigeons and Apples are Boxes

Another Way

Works

- Create 24 boxes
- Each pigeon (basket) goes to the box corresponding to number of apples that the basket has
- By PHP, must exist a box with at least $\lceil \frac{50}{24} \rceil = 3$ pigeons

PHP – Example 6

- Suppose S is a set of $n + 1$ distinct integers. Show that there must exist $a, b \in S$ such that $a - b$ is divisible by n.

— n+1 integers as pigeons

— Lets create n boxes

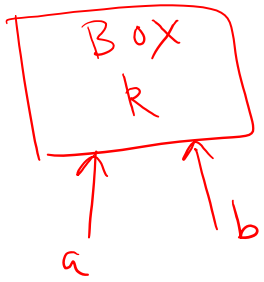
Any $a \in S$ goes to box i if a divided by n leaves i as remainder

By PHP there exists Box k that has ≥ 2 integers

Let k be one such box that contains a, b

PHP – Example 6

- Suppose S is a set of $n + 1$ distinct integers. Show that there must exist $a, b \in S$ such that $a - b$ is divisible by n.



Then, we must have that $a = r_1 n + k$ and $b = r_2 n + k$

for integers r_1 and r_2.

But then $a - b = (r_1 - r_2)n$ is divisible by n.