September 20, 2021

1. **Proof by Contradiction:**
   Prove the following:
   There exists no combination of $x, y \in \mathcal{Z}$ such that $33x + 11y = 1$.

   Solution: Proof:
   Suppose there do exist integers $a, b \in Z$ that do satisfy $33x + 11y = 1$.
   Then,

   $$\begin{aligned} 33a + 11b &= 1 \\ 3a + b &= \frac{1}{11} \end{aligned}$$

   since expression $3a + b$ must be an integer, the equality can not be satisfied. Hence, we have a contradiction.
   Therefore, there do not exist any integers x, y that can satisfy $33x + 11y = 1$.

2. **Proof by Induction:**
   Show that $\forall n \geq 1$:

   $$1 + 2 + 3 + ... + n = \frac{1}{2}n(n+1)$$

   Solution: Proof:
   Base Case: $n = 1$

   $$1 = 1 \tag{8}$$

   Inductive Hypothesis: For an arbitrary $k \geq 1$

   $$1 + 2 + 3 + ... + k + (k+1) = \frac{1}{2}(k+1)((k+1)+1)$$

   Then:

   $$\begin{aligned} \frac{1}{2}k(k+1) + (k+1) &= (\frac{1}{2}k + 1)(k+1) \\ &= \frac{1}{2}(k+2)(k+1) \\ &= \frac{1}{2}((k+1)+1)(k+1) \end{aligned}$$

   QED

3. **Proof by Contrapositive:**

   Prove that $\forall x \in \mathcal{Z}, x^2 - 6x + 5$ is even, then x is odd.

   For contrapositive, prove $p \implies q$ by proving $-q \implies -p$.

   Solution: Proof:
   Suppose that x is even. Then want to show that $x^2 - 6x + 5$ is odd. By definition of even,

   $$\begin{aligned} x^2 - 6x + 5 &= (2a)^2 - 6(2a) + 5 && (9) \\ &= 4a^2 - 12a + 5 && (10) \\ &= 2(2a^2 - 6a + 2) + 1 && (11) \end{aligned}$$

   Then, by definition, $x^2 - 6x + 5$ is odd. QED

4. **Proof by Contrapositive:**
   Let $a, b, n \in \mathcal{Z}$. If $n \not| ab$, then $n \not| a$ and $n \not| b$.

   Solution: Proof:
   Negation of $n \not| a$ AND $n \not| b$ is $n \mid a$ OR $n \mid b$ by Demorgan's law. Initial hypothesis negated becomes $n \mid ab$.
   So want to prove that if $n \mid a$ OR $n \mid b$, then $n \mid ab$. Suppose that n divides a, then $a = nc$ for some $c \in \mathcal{Z}$:

   $$ab = ncb = n(cb) \tag{12}$$

   Suppose n divides b, then $b = nd$ for some $d \in \mathcal{Z}$:

   $$ab = and = n(ad) \tag{13}$$

   In both cases, $n \mid ab$, therefore the result is true. QED

5. **Proof by Case-Analysis:**
   Let the domain of x be the set of all integers. Prove that $\forall x \in \mathcal{Z}, x^2 \neq 5$:

Solution: Proof:
Split domain of x into to subsets:

- Case 1: $x \geq -2$ and $x \leq 2$
- Case 1: $x < -2$ and $x > 2$

If we show that Case 1 and Case 2 can not equal 5, then we have proved the above statement as true. Case 1:

$$x^2 \leq 4 < 5 \tag{14}$$

Case 2:

$$x^2 \geq 9 > 5 \tag{15}$$

QED

6. Product Rule Problems

- How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

  Solution: By the generalized version of the basic principle, the answer is $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$=175,760,000.

- Beethoven wrote 9 symphonies and Mozart wrote 27 piano concertos. If a university radio station announcer wishes to play first a Beethoven symphony and then a Mozart concerto, in how many ways can this be done?

  Solution: There are 9 options for the first music, and 27 for the second. Therefore there are $9 \cdot 27$ possibilities.

- The station manager decides that on each successive night (7 days per week), a Beethoven symphony will be played, followed by a Mozart piano concerto, followed by a Schubert string quartet (of which there are 15). For roughly how many years could this policy be continued before exactly the same program would have to be repeated?

  Solution: There are $9 \cdot 27 \cdot 15 = 3645$ possible sequences. Therefore after about 10 years the same program would have to be repeated.