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Much learning does not teach understanding
- Heraclitus

206
Discrete Structures II

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Preview:

Did you know you can solve this?

Prove that

$$\sum_{k=0}^{n} \binom{n}{2k} = 2^{n-1}$$

Quiz 3 – Next ~~Tuesday~~/Thursday

- More time (35 minutes)
  - + more questions ☺
- What will cover
  - Permutations with/out repetition
  - Combinations
  - Pirates Problem
  - Pirates Problem
  - Pirates Problem
    - Have you seen the extra Pirates problems?

So Far

- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- ~~Permutation/Combinations~~
- ~~Inclusion-Exclusion / Pigeonhole Principle~~
- Combinatorial Proofs and Binomial Coefficients

Pigeonhole Principle

If m pigeons are in n holes and $m > n$, then at least $\left\lceil \frac{m}{n} \right\rceil$ pigeons are in the same hole.

$\left\lceil \frac{m}{n} \right\rceil$ = nearest integer higher than $\frac{m}{n}$

Ceiling of m over n rounds the ratio to the larger integer

$m = 20$
$n = 9$

$\left\lceil \frac{20}{9} \right\rceil = 3$

New (kind of)...

PHP – Example 7

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub-groups of people (with no common person) the sum of whose ages is the same.

Option 1: View people as boxes and age as pigeons

Then we have 10 boxes and 60 pigeons

– Multiple pigeons going to a box means a person having multiple ages

– Doesn't make sense and violates the constraint of the problem ☹

PHP – Example 7

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub-groups of people (with no common person) the sum of whose ages is the same.

<u>Option 2:</u> View ages as boxes and people as pigeons

Then we have 60 boxes and 10 pigeons

— # pigeons < # boxes, hence PHP is not very useful

PHP – Example 7

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub-groups of people (with no common person) the sum of whose ages is the same.

— We really care about subgroups of people and their total age.

— Let's call pigeons as subgroups of people and put them in a box corresponding to total age

PHP – Example 7

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub-groups of people (with no common person) the sum of whose ages is the same.

Why -1?

How many subgroups of 10 people $= 2^{10} - 1 = 1023$

Box 1   Box 2 – Box i — — Box 600

— Subgroup S goes to box i if sum of ages in S = i

— By PHP there must exist a box that has at least $\left\lceil \frac{1023}{600} \right\rceil = 2$ pigeons

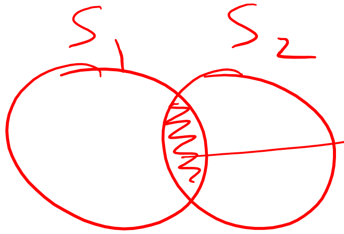
Are we done?

PHP – Example 7

- In a room there are 10 people, none of whom are older than 60, but each of whom is at least 1 year old. Prove that one can always find two sub-groups of people (with no common person) the sum of whose ages is the same.

Box i ← S_1, S_2

Sum of ages in S_1 = Sum of ages in S_2



Let $A = S_1 \cap S_2$

— Then sum of ages in $S_1 \setminus A$ = sum of ages in $S_2 \setminus A$

Take a Break

5 min

Combinatorial Proofs

- Proving algebraic identities via counting

IDENTITY

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a \pm b)^3 = a^3 \pm 3a^2b + +3ab^2 \pm b^3$

$(a \pm b)^4 = a^4 \pm 4a^3b + +6a^2b^2 \pm 4ab^3 + +b^4$

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$

$(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$

$$(a + b + c)^3 = a^3 + b^3 + c^3 + 6abc$$
$$+3(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2)$$

$$(a_1 + a_2 + \cdots a_n)^2 =$$
$$= {a_1}^2 + {a_2}^2 + \cdots {a_n}^2 + 2(a_1a_2 + a_1a_3 + \cdots a_{n-1}a_n)$$

$a^2 - b^2 = (a - b)(a + b)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2$$
$$= (a^2 + \sqrt{2}ab + b^2)(a^2 - \sqrt{2}ab + b^2)$$

$$a^4 - b^4 = (a^2 - b^2)(a^2 + b^2)$$
$$= (a + b)(a - b)(a^2 + b^2)$$

$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

Combinatorial Proofs

In general, to give a combinatorial proof for a binomial identity, say A = B you do the following:

1. **Find a counting problem** *you will be able to answer in two ways*.
2. Explain why one answer to the counting problem is A.
3. Explain why the other answer to the counting problem is B.

Since both A and B are the answers to the same question, we must have A=B.

The tricky thing is coming up with the question. This is not always obvious, but it gets easier the more counting problems you solve.

Combinatorial Proofs – Hints!

- Define a set S.
- Show that $|S| = n$ by counting one way.
- Show that $|S| = m$ by counting ***another way***.
- Conclude that $n = m$.

Combinatorial Proofs – Example 1

- Prove that $\binom{n}{k} = \binom{n}{n-k}$

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

$$LHS = \frac{n!}{(n-k)!\,k!} \longrightarrow \text{use formula}$$

$$RHS = \frac{n!}{k!\,(n-k)!} \longrightarrow \text{use formula}$$

$$= LHS$$

Combinatorial Proofs – Example 1

- Prove that $\binom{n}{k} = \binom{n}{n-k}$

Alternate Proof

— Define a Counting problem

— In this case Counting Problem = # ways to select k out of n people

— One way to count $= \binom{n}{k}$ = LHS

— Another way to count = choose n-k people to not select

$= \binom{n}{n-k}$

— Both ways solving the same problem, Hence $\binom{n}{k} = \binom{n}{n-k}$