CS206 Recitation Problem Sets Section 06

James Fu

November 11

1 Inclusion-exclusion principle

1. Suppose that 4 people are standing in line. How many ways are there to rearrange the line so that nobody is standing in their original place?

Solution:
We will first count the number of ways to arrange the line of people so that at least one person stays in the same place.

For each i between 1 and 4, let A_i be the set of ways to order the people in line so that person i stays in the same place. We want to find the size of $A_1 \cup A_2 \cup A_3 \cup A_4$.

The number of ways to order the line so that person i remains in the same place is just the number of ways to arrange the other three people—i.e. $3 \times 2 \times 1 = 3!$. In other words, for each i, $|A_i| = 3!$.

Similar reasoning shows that the double intersections have size 2!, and that the triple and quadruple intersections have size 1.

Since there are 6 double intersections, 4 triple intersections, and 1 quadruple intersection, the inclusion-exclusion tells us that the number of ways to arrange the people so that someone stays in the same place is $4 \times 3! - 6 \times 2! + 4 \times 1 - 1 \times 1$.

Subtracting this from the total number of ways to arrange four people, which is 4!, gives us

$$4! - (4 \times 3! - 6 \times 2! + 4 \times 1 - 1 \times 1)$$

2 Pigeonhole principle

1. A coffee shop sells five sizes of coffee. You buy 11 coffees.

   (a) How many ways are there to order 11 coffees?

   Solution:
   We can think of the five sizes of coffee as 5 distinguishable boxes and the 11 coffees we must order as indistinguishable balls.

   To throw a ball into a box means to order one more of that size of coffee (so every ball must go into some box). The balls are indistinguishable because all that matters is how many of each size we order and not which order we order them in. There are no restrictions on how many of each size we order, so this problem can be solved using the stars and bars method. So the solution is $\binom{11+5-1}{11}$

   (b) Prove that one can always find 3 coffees of the same size.

   Solution: There are 5 boxes to which you are assigning 11 coffees. Hence, by pigeonhole principle, $\lceil \frac{11}{5} \rceil = 3$

3 Combinatorial problems

1. Prove that:

$$\binom{2n}{n} = 2\binom{2n-1}{n-1} \tag{1}$$

Solution:
The LFS can be interpreted as

$$\text{Choose } n \text{ item from set } \{x_1, x_2, ..., x_{2n}\} \tag{2}$$

The RHS can be reformulated as

$$2\binom{2n-1}{n-1} = \binom{2n-1}{n-1} + \binom{2n-1}{n-1}$$

$$= \binom{2n-1}{n-1} + \frac{(2n-1)!}{(n-1)!n!}$$

$$= \binom{2n-1}{n-1} + \binom{2n-1}{n}$$

Then we can see that the first term of the reformulated RHS counts number of combinations of size $n-1$ formable from $2n-1$ elements assuming some specific element is included in the all combinations by default.

$$\begin{cases} \text{Choose } x_1 \\ \text{Choose } n-1 \text{ element from set } \{x_2, x_3, ..., x_{2n}\} \end{cases}$$

And the second term of the reformulated RHS counts all combinations with that specific element missing from all combinations.

$$\begin{cases} \text{Don't choose } x_1 \\ \text{Choose } n \text{ element from set } \{x_2, x_3, ..., x_{2n}\} \end{cases}$$

Then we have included both cases, resulting in the same counting as happens on the LHS where all combinations of size n are counted from 2n elements.

2. Prove that $\forall n \geq k \geq m \geq 0$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \tag{3}$$

Solution: Combinatorial proof, we show that both sides count the same thing in different ways. Both sides consist of 2 terms.

The first LHS term counts the number of k-combinations one can form from n elements, since its given that $n \geq k$. Then, for every k-combination of the first LHS term, the second LHS term counts how many m-combinations one can form from that specific k-combination. By multiplying the sizes of these two subsets, we get the total number of k-combinations formable form n elements and recursively the further m-combinations formable from each k-combination.

The first RHS term starts off counting the number of m-combinations formable from n elements. Since the first RHS term effectively counts all the possible m-combinations, what remains is to count the combinations for the remaining elements $k - m$. The second RHS term counts the number of ways one can combine the remaining $k - m$ elements from the remaining $n - m$ elements for each m-combination given by the first RHS term. Multiplying these two terms together, again we get total number of k-combinations we can form of paired with m-combinations in dependent manner.

Therefore, both sides count the same thing, but in different order/way.
Clearly, this can be shown algebraically, how it is important to understand the intuition behind the different counting methods. Here is the algebraic proof:

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \tag{4}$$

$$\frac{n!}{(n-k)!k!}\frac{k!}{(k-m)!m!} = \frac{n!}{(n-m)!m!}\frac{(n-m)!}{(n-m-k+m)!(k-m)!} \tag{5}$$

$$\frac{n!}{(n-k)!(k-m)!m!} = \frac{n!}{(n-k)!(k-m)!m!} \tag{6}$$

$$\tag{7}$$

QED