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The mind is not a vessel to be filled but a fire to be kindled. – Plutarch

206
Discrete Structures II

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Preview:

Did you know you can solve this?

Prove that

$$\sum_{k=0}^{n} \binom{n}{2k} = 2^{n-1}$$

So Far

- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- ~~Permutation/Combinations~~
- ~~Inclusion-Exclusion / Pigeonhole Principle~~
- Combinatorial Proofs and Binomial Coefficients

Pigeonhole Principle

If m pigeons are in n holes and $m > n$, then at least $\left\lceil \frac{m}{n} \right\rceil$ pigeons are in the same hole.

$\left\lceil \frac{m}{n} \right\rceil$ = nearest integer higher than $\frac{m}{n}$

Ceiling of m over n rounds the ratio to the larger integer

$m = 20$
$n = 9$
$\left\lceil \frac{20}{9} \right\rceil = 3$

So far...

PHP – Example 3

- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

Let 6 People be $P_1, P_2, P_3, P_4, P_5, P_6$

Define 2 boxes

friends to P1

Strangers to P1

— Every remaining person goes to one of these boxes depending on whether she/he knows P1 or not.

— By pigeonhole principle one of the two boxes must have at least $\left\lceil \frac{5}{2} \right\rceil = 3$ People

PHP – Example 3

- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

Let 6 People be $P_1, P_2, P_3, P_4, P_5, P_6$

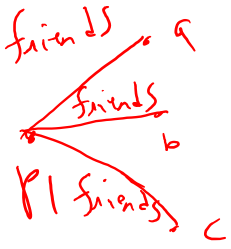
Define 2 boxes

friends of P1

Strangers to P1

— Case 1: Friends of P1 box has ≥ 3 people.

Let a, b, c be any 3 people in the box



— If any of a, b, c know each other then together with P1 they form a group of 3 mutual friends

— otherwise a, b, c is a group of 3 mutual strangers

PHP – Example 3

- In a group of 6 people there are either 3 mutual friends or 3 mutual strangers.

Let 6 People be $P_1, P_2, P_3, P_4, P_5, P_6$

Define 2 boxes

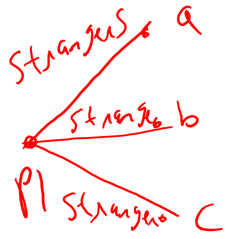
friends of P1

Strangers to P1

— Case 2: **Strangers to** P1 box has ≥ 3 people.

Let a, b, c be any 3 people in the box

— If any two of a, b, c are strangers then together with P1 they form a group of 3 mutual strangers.



— Otherwise a, b, c form a group of 3 mutual friends.

So far...

Combinatorial Proofs

In general, to give a combinatorial proof for a binomial identity, say A = B you do the following:

1. **Find a counting problem** *you will be able to answer in two ways*.
2. Explain why one answer to the counting problem is A.
3. Explain why the other answer to the counting problem is B.

Since both A and B are the answers to the same question, we must have A=B.

The tricky thing is coming up with the question. This is not always obvious, but it gets easier the more counting problems you solve.

So far...

Combinatorial Proofs – Example 1

- Prove that $\binom{n}{k} = \binom{n}{n-k}$

Alternate Proof

— Define a counting problem

— In this case counting problem = # ways to select k out of n people

— One way to count $= \binom{n}{k}$ = LHS

— Another way to count = choose n-k people to not select

$= \binom{n}{n-k}$

— Both ways solving the same problem, Hence $\binom{n}{k} = \binom{n}{n-k}$

Hint! Σ → consider sum rule

Combinatorial Proofs – Example 2

- Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Let $S = \{1, 2, \dots n\}$

Counting problem: How many subsets of S?

RHS = 2 choices for each element

→ Hence, # subsets $= 2^n$

LHS = use partition rule

→ Count all subsets of size $0 \to \binom{n}{0}$

→ ,, ,, size $1 \to \binom{n}{1}$ $= \sum_{k=0}^{n} \binom{n}{k}$

→ ,, size $n \to \binom{n}{n}$

Take a Break

5 min

Combinatorial Proofs – Example 3

- Prove that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Counting Problem: # ways to choose k out of n people

– LHS = $\binom{n}{k}$

RHS: Use partition method

Case1: # ways to chose k out of n such that element 1 is chosen

Case2: # ways to chose k out of n such that element 1 is not chosen

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Combinatorial Proofs – Example 4

- Prove that $\sum_{k=0}^{n}\binom{n}{2k} = 2^{n-1}$

Problem: # even sized subsets of n elements

RHS = 2^{n-1}

LHS = Use partition method

— subsets of size 0 → $\binom{n}{0}$

— subsets of size 2 → $\binom{n}{2}$

— subsets of size 4 → $\binom{n}{4}$

⋮

→ $\sum_{k=0}^{n}\binom{n}{2k} = 2^{n-1}$

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Combinatorial Proofs – Hints *Revisited*!

- Define a set S.
- Show that $|S| = n$ by counting one way.
- Show that $|S| = m$ by counting ***another way***.
- Conclude that $n = m$.

Binomial Coefficients

- $\binom{n}{k}$, known as the ***Binomial Coefficient***.
  - Number of ways to pick k out of n distinct objects.
  - Intimately connected to algebraic polynomials.

Examples:

$$\left.\begin{array}{l} 1+x \\ 1+x+3x^2 \\ 5x^3-2x^2+7x-8 \end{array}\right\} \to \text{univariate polynomials}$$

General: $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots a_nx^n \to$ degree n polynomial

Basic Problem: Given a polynomial, infer the co-efficients

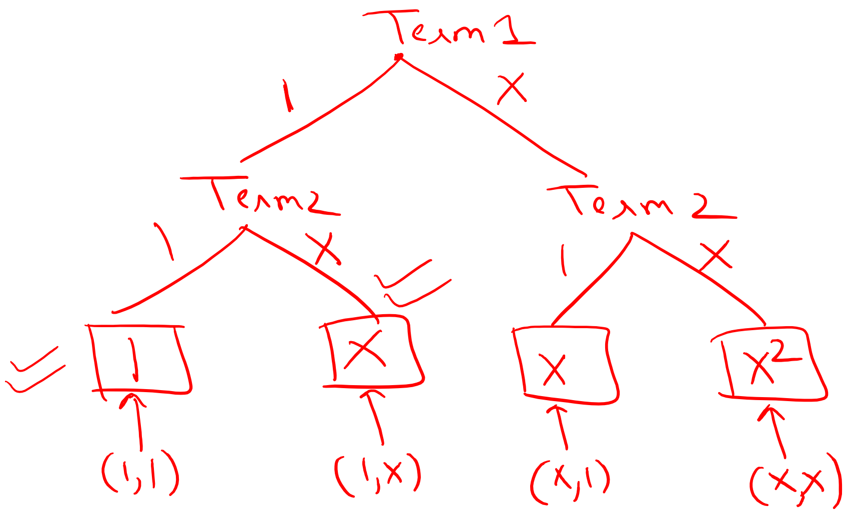
Binomial Coefficients – Building insight

- $(1+x)^2 = 1 + 2x + x^2$

Given: $(1+x)^2 \rightarrow (1+x)\cdot(1+x) = 1 + x + x + x^2$

$= 1 + \underline{\underline{2x}} + x^2$

$(1+x)^2 \rightarrow \underset{\text{Term1}}{(1+x)}\ \underset{\text{Term2}}{(1+x)}$



Co-efficient of x = # ways to reach x from root node

$= 2$

ways to reach x

= # ways to choose x once out of 2 terms $= \binom{2}{1}$