combra.cs.rutgers.edu

Character is simply habit long continued - Plutarch

206
Discrete Structures II

Konstantinos P. Michmizos

Computational Brain Lab

Computer Science | Rutgers University | NJ, USA

Counting

- In the next few lectures
  - Fundamental tools and techniques for counting
  - Sum Rule
  - Product Rule
  - Difference Method
  - Bijection Method
  - Permutations/Combinations
  - Inclusion Exclusion
  - Binomial/Multinomial coefficients

Fundamental Blocks

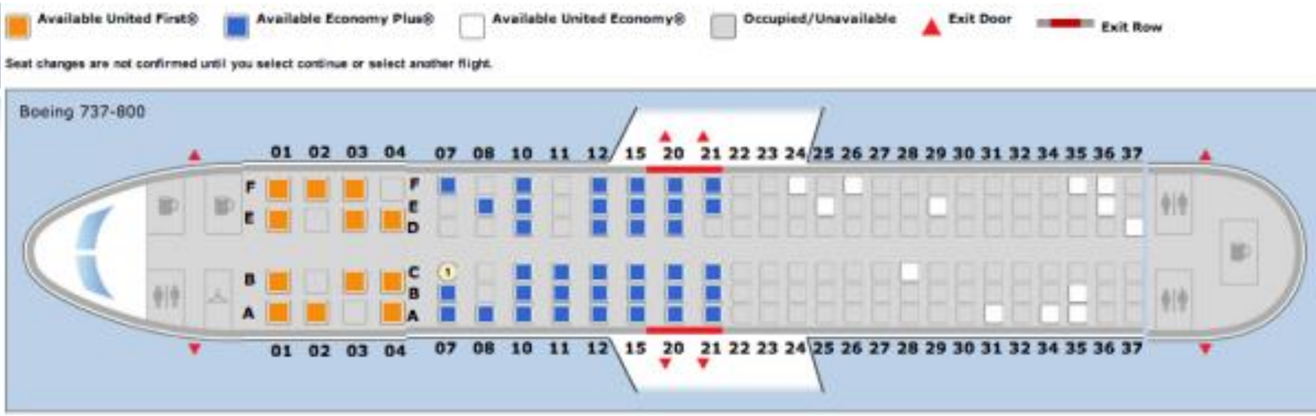
Intermediate

Advanced

Finishing this part today!

Generalized Product Rule

So far...



- How many ways to assign 100 passengers to 100 seats?

Let $P_1, \ldots, P_{100}$ be the passengers.

100 choices for seat of P_1

99 choices for seat of P_2

$\vdots$

1 choices for seat of P_{100}

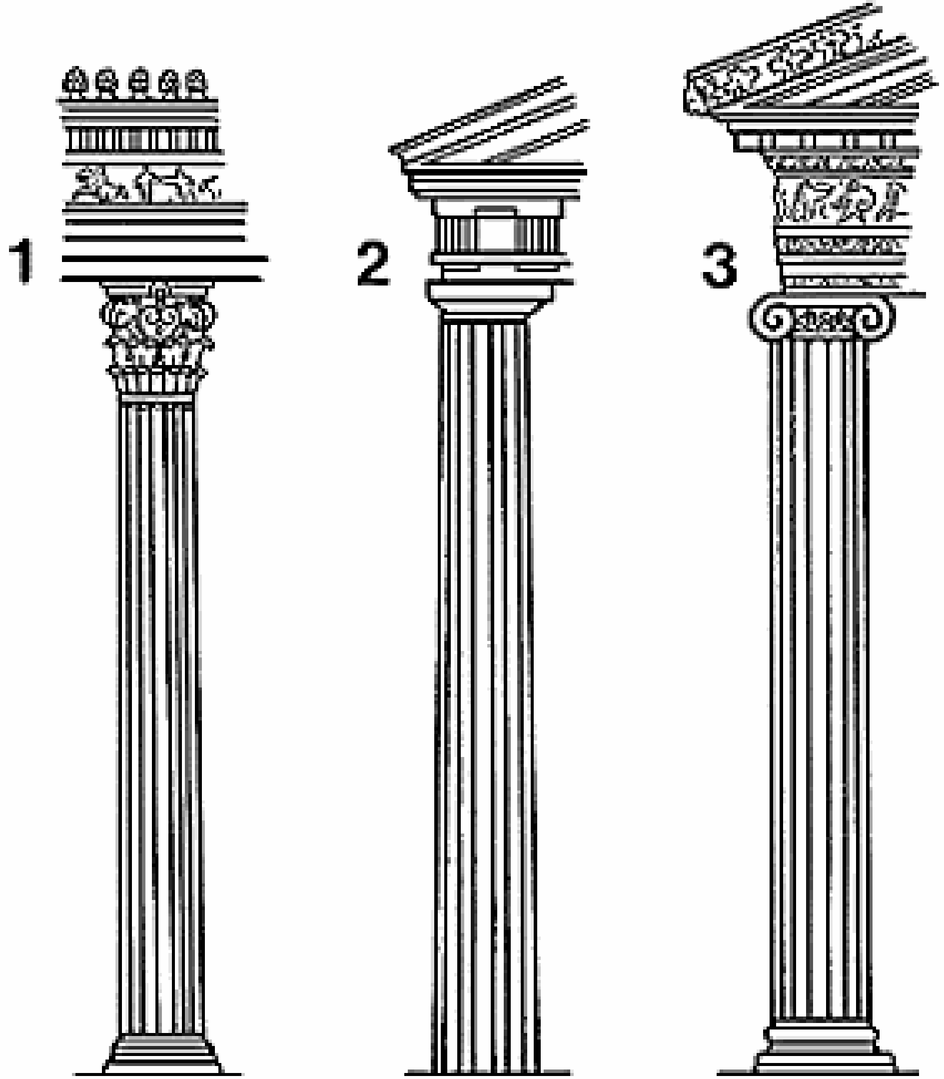
$\Rightarrow$ answer $= 100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot 1$

So far…

Generalized Product Rule – Order is important

- Suppose every object of a set S, can be constructed by a sequence of n choices with P_1 possibilities for the first choice, P_2 possibilities for the second choice, and so on
- **IF**
  - Each sequence of choices constructs an object in S.
  - No two different sequences **create the same** object
- THEN
  - $|S| = P_1 \times P_2 \times \cdots P_n$

Product Rule



order is important

So far…

Counting Pitfalls – and how to avoid them

- You are signing up for an account on FlixBiz.com. The password has the following requirements
  - The password must be 6 to 8 characters long.
  - Each password is an uppercase letter or digit.
  - Each password must contain at least one digit.

$A_6 \rightarrow$ all valid pass words of length 6

Can I use the Generalized Product Rule?

- Pick position of first digit $\rightarrow$ 6 ways
- Pick value $\rightarrow$ 10 ways $\rightarrow$ wrong
- Pick remaining values $\rightarrow 36^5$

$|A_6| = 6 \times 10 \times 36^5 \neq 36^6 - 26^6$

We are overcounting…Why?

So far...

Counting Pitfalls

Process 1

→ pick position →

→ pick value →

→ pick remaining

S = all valid passwords

AB4CDE ✓

A64BGZ

⋮

→ every choice sequence in Process 1 maps to a unique element of set

Can I use the Generalized Product Rule? → Given element of S, must be able uniquely decode how we got to it

→ In process 1 multiple ways to reach A64BGZ

So far...

Hint! When/How to use Product Rule

- If you are counting the size of a set S
  - For every object in S you should be able to reconstruct the unique sequence of choices that led to it.

- Ask yourself
  - Am I creating objects of the right type?
  - Can I reverse engineer my choice sequence from any given object?

Product Rule – Counting Pitfalls cont'd

- How many binary strings of length 8 with exactly two 0's?

Pick location of 1^{st} 0 → 8 ways

Pick location of 2^{nd} 0 → 7 ways

answer = $8 \cdot 7 = 56$

Pick location of 1^{st} 0

Pick location of 2^{nd} 0

Choice sequence

Two ways to reach this

00111111

01011111

⋮

Actual set

Product Rule

- How many binary strings of length 8 with exactly two 0's?

$A_1 =$ all strings with 2 0's such that 1st zero from left appears at position 1.

$A_2 =$ 1st zero is at position 2

$A_3 =$ 1st zero is at position 3

⋮

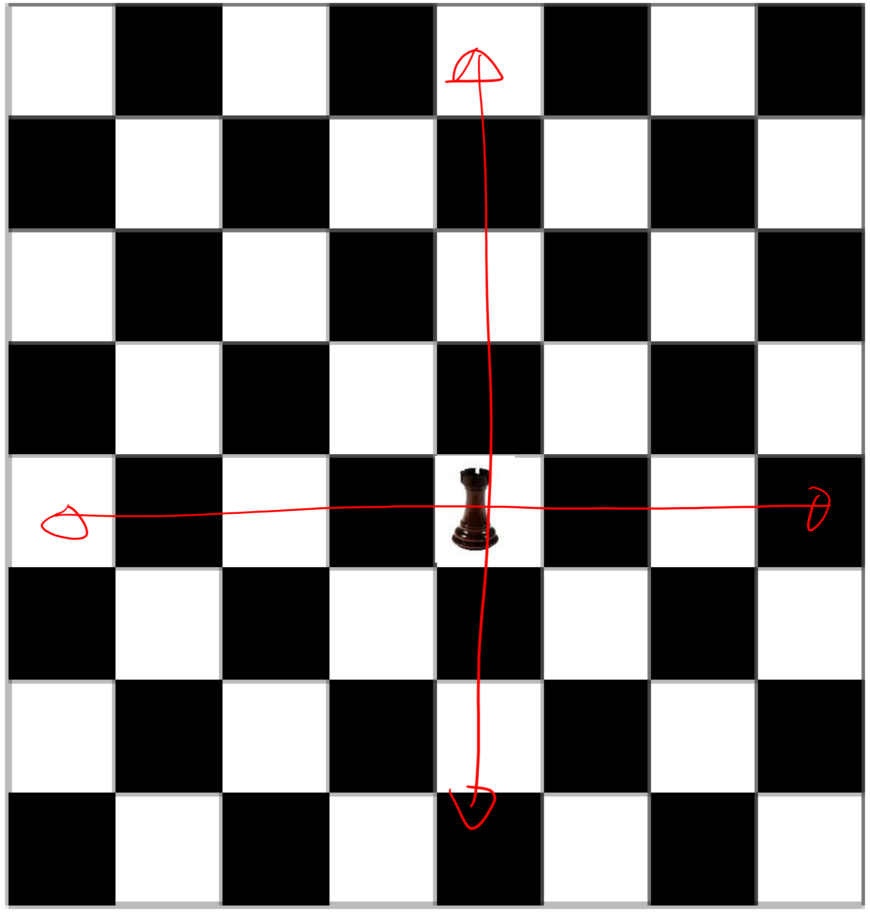
$A_8 =$ 1st zero is at position 8

$|A_1| = 7$
$|A_2| = 6$
$|A_3| = 5$
$|A_4| = 4$
$|A_5| = 3$
$|A_6| = 2$
$|A_7| = 1$
$|A_8| = 0$

28

So far...

Generalized Product Rule



- Given two rooks labeled 1 and 2
- How many ways to place them so that they don't threaten each other?

— 64 choices for first rook

— (64-15) choices for second rook

answer (64·49)

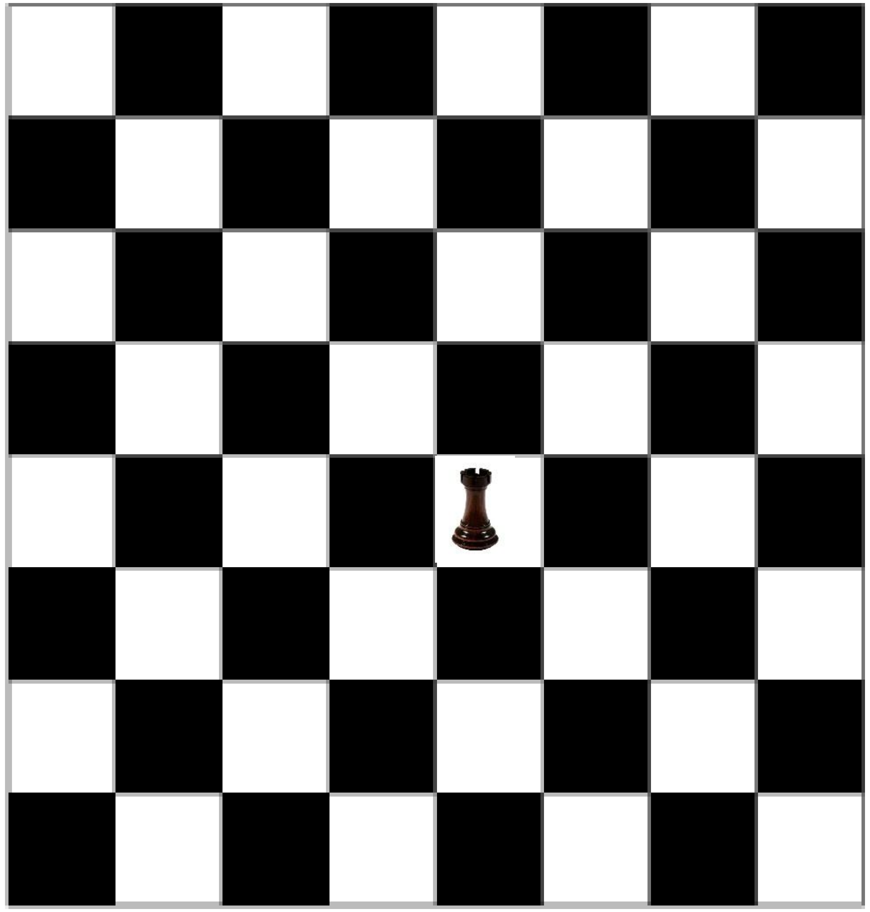
New Stuff

Difference Method

Generalized Product Rule

$B = S \setminus A$

= all ways where they threaten



$|S| = 64 \cdot 63$

$|B| = 64 \cdot 14$

How many ways to place two rooks ***so that they don't threaten each other?***

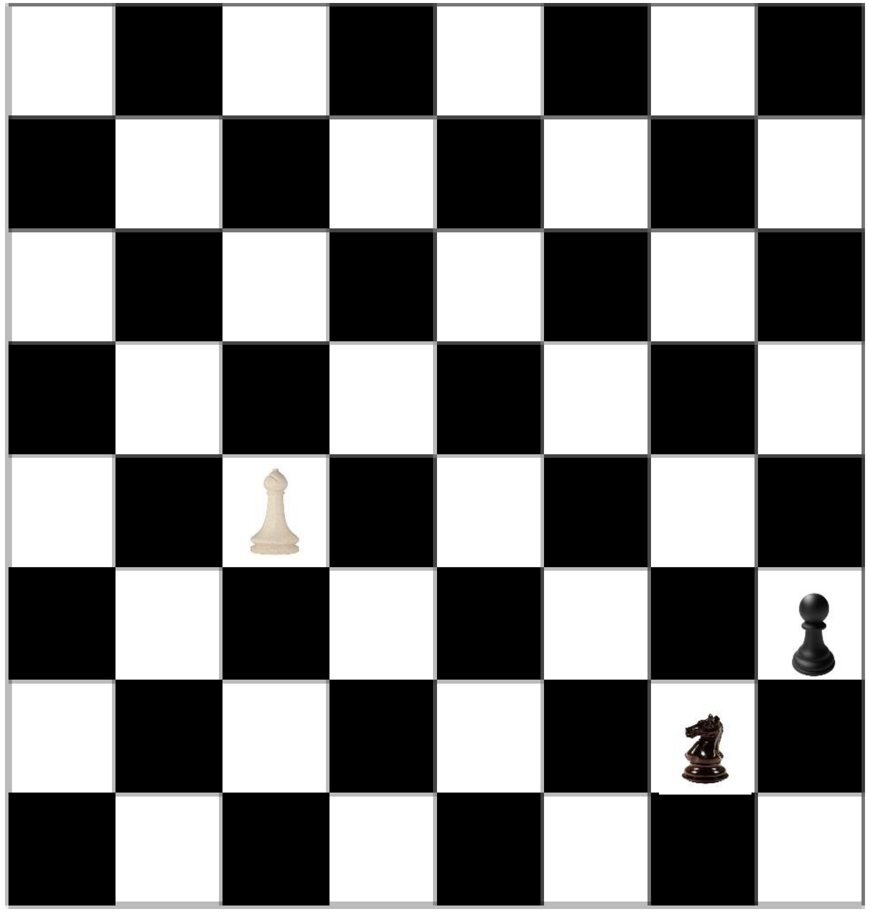
S = all ways

A = all ways such that they don't threaten

$\Rightarrow |A| = |S| - |B| = 64 \cdot 49$

Generalized Product Rule

Potential Pitfall????



YES! If we had two (interchangeable) knights!!!

How many ways to place a knight, bishop, and pawn so that no two share a row or column?

Pick row for bishop → 8
Pick colum for bishop → 8
Pick row for knight → 7 ways
Pick col for knight → 7 ways
Pick row for pawn → 6 ways
Pick col for pawn → 6 ways

ans = $8^2 7^2 6^2$

Intuition: "Knight", "bishop" and "pawn" are just words that *mathematically represent* unique entities (order matters)

Exercise – Bring everything together…

- An IP address is a string of 32 bits. It begins with a network number (netid) followed by a host number (hostid).
  - There are three forms of addresses.
  - Class A addresses consists of 0, followed by a 7-bit netid and a 24-bit hostid.
  - Class B addresses consists of 10, followed by a 14-bit netid and a 16-bit hostid.
  - Class C addresses consists of 110, followed by a 21-bit netid and a 8-bit hostid.
- Restrictions
  - 1111111 is not available as the netid of a class A network.
  - Hostids cannot be all 0s or all 1s.

How many IP addresses are there?

Exercise – Bring everything together…

- Class A addresses consists of 0, followed by a 7-bit netid and a 24-bit hostid.
- Restrictions
  - 1111111 is not available as the netid of a class A network.
  - Hostids cannot be all 0s or all 1s.

0 ___ ___

7 bits netid — 24 bits hostid

$|A| = (\text{\# choices of netid}) \cdot (\text{\# choices for hostid})$

$= (2^7 - 1)(2^{24} - 2)$

Exercise – Bring everything together…

- Class B addresses consists of 10, followed by a 14-bit netid and a 16-bit hostid.
- Restrictions
  - 1111111 is not available as the netid of a class A network.
  - Hostids cannot be all 0s or all 1s.

$$\underline{1}\ \underline{0}\ \underbrace{\qquad\qquad\qquad}_{14\text{ bit netid}}\ \underbrace{\qquad\qquad\qquad}_{16\text{ bit hostid}}$$

$$|B| = (2^{14})(2^{16}-2)$$

Exercise – Bring everything together…

- Class C addresses consists of 110, followed by a 21-bit netid and a 8-bit hostid.
- Restrictions
  - 1111111 is not available as the netid of a class A network.
  - Hostids cannot be all 0s or all 1s.

$$\underline{1}\ \underline{1}\ \underline{0}\ \underbrace{\qquad\qquad\qquad\qquad}_{\text{21 bit netid}}\ \underbrace{\qquad\qquad\qquad}_{\text{8 bit hostid}}$$

$$|C| = 2^{21} \cdot (2^8 - 2)$$

answer $= |A| + |B| + |C|$

Take a Break

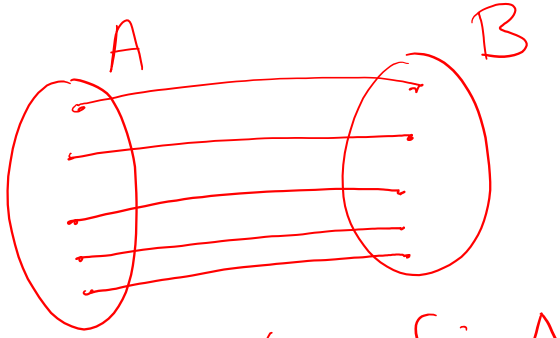
5 min

Bijection Method

Various Problems

Sequence Counting

Suppose want $|A|$



Suppose exists $f: A \to B$ and f is a bijection

Then conclude that, $|A| = |B|$

Bijection Method

- To find the size of a set A,
  - Find a set B with known size
  - Exhibit a bijection f from A to B.
  - $|A| = |B|$

- Possible outcomes where white die has a larger value than the black die?

Bijection Method

- Possible outcomes where white die has a larger value than the black die?

3 options

$A \leftarrow$ all outcomes where white die $>$ black die

$B \leftarrow$ all outcomes where black die $>$ white die

$C \leftarrow$ all outcomes where black die $=$ white die

$S \leftarrow$ all outcomes

Then, by sum rule

$|S| = |A| + |B| + |C|$

We know that $|S| = 36$
and $|C| = 6$

$\Rightarrow |A| + |B| = 30$

Bijection Method

- Possible outcomes where white die has a larger value than the black die?

$A =$ all outcomes where white die $>$ black die

$B =$ all outcomes where white die $<$ black die

Claim: $|A| = |B| \implies |A| + |B| = 30$

$\implies 2|A| = 30$

$\implies |A| = 15$

A B

$(2,1) \longrightarrow (1,2)$

$f: (a,b) \to (b,a)$

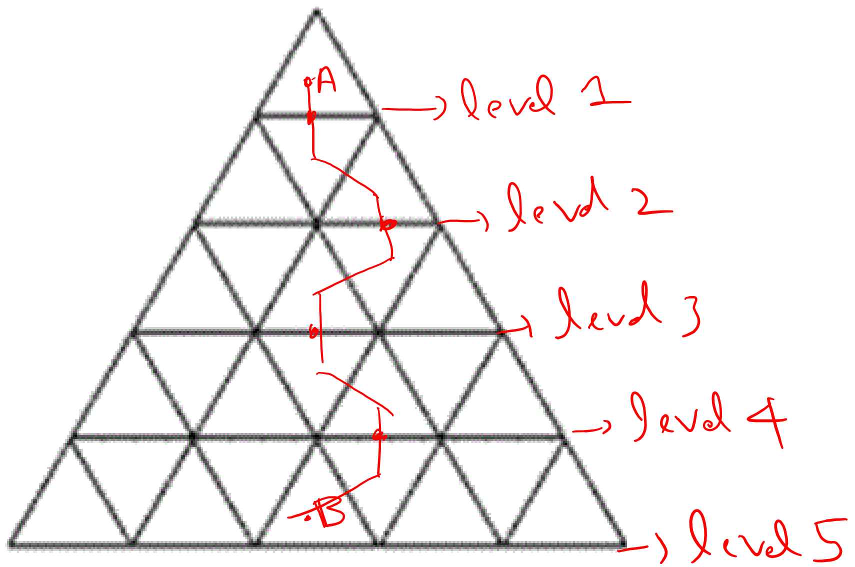
Bijection Method

- To find the size of a set A,
  - Find a set B with known size
  - Exhibit a bijection f from A to B.
  - $|A| = |B|$

Correspondence Principle:
If two finite sets can be placed into a bijection, then they **have the same size**.

Exercise

Bijection Method

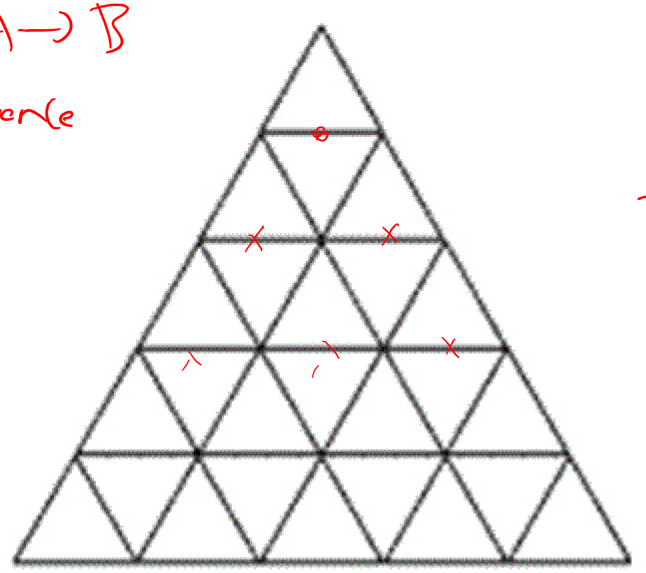


- Consider an equilateral triangle of side length 5, divided into unit length triangle. How many paths from point A to B?
  - Adjacent triangles in a valid path have to share a common edge.
  - A path can never go upwards or revisit a triangle.

How many paths from A to B?

Exercise

Each path from $A \to B$ has an exit sequence $(\underline{1}, \underline{2}, \underline{3}, \underline{3})$



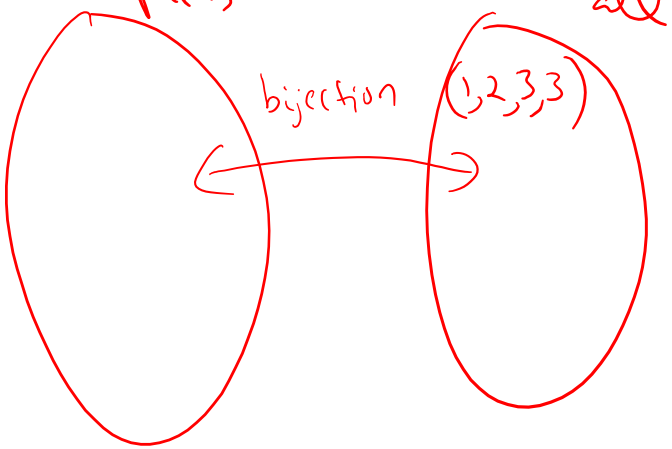
all valid exit sequences

$= 1 \cdot 2 \cdot 3 \cdot 4$

$= 24$

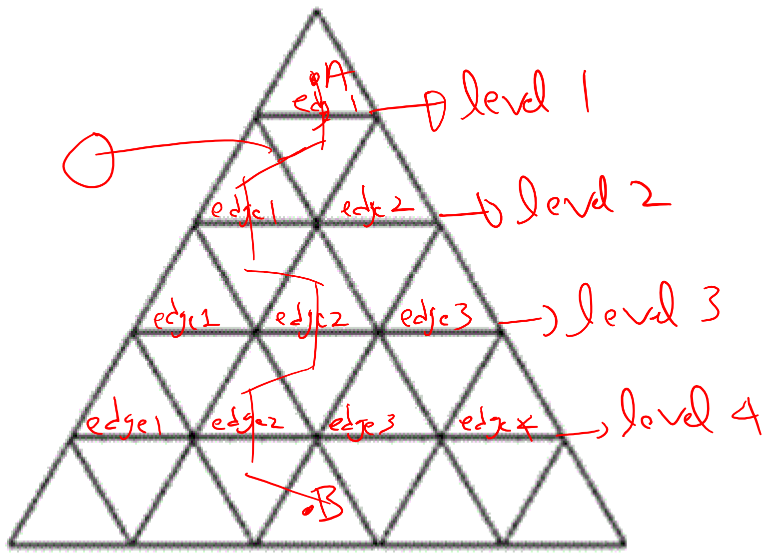
all valid paths

all exit sequences



Exercise

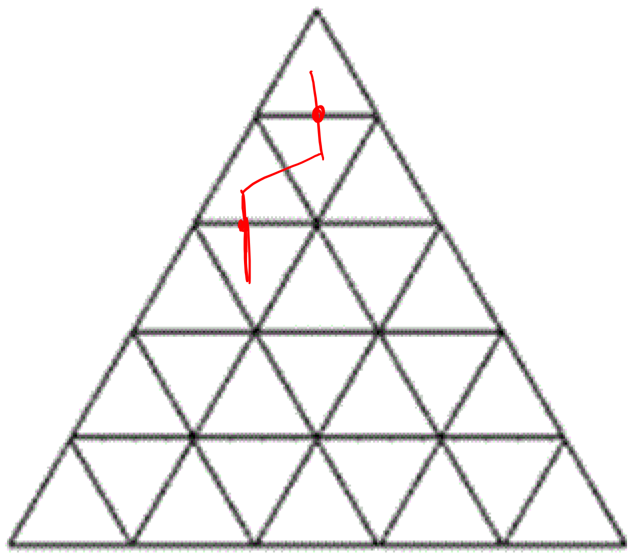
exit sequence = (1, 1, 2, 2)



Each path has an exit sequence and vice versa

- Consider an equilateral triangle of side length 5, divided into unit length triangle. How many paths from point A to B?
  - Adjacent triangles in a valid path have to share a common edge.
  - A path can never go upwards or revisit a triangle.

Exercise



\# valid paths = # exit sequences

$= 1 \cdot 2 \cdot 3 \cdot 4 = 24$ paths

If n levels answer $= 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) = (n-1)!$