CS 206

Recitation - Section 4

Nov. 23

Combinatorial Problems

Problem 1

Prove that:

$$\binom{2n}{n} = 2\binom{2n-1}{n-1}$$

Solution 1

Solution: The RHS can be reformulated as

$$\begin{aligned} 2\binom{2n-1}{n-1} &= \binom{2n-1}{n-1} + \binom{2n-1}{n-1} \\ &= \binom{2n-1}{n-1} + \binom{2n-1}{n} \end{aligned}$$

Then we can see that the first term of the reformulated RHS counts number of combinations of size $n-1$ formable from $2n-1$ elements assuming some specific element is included in the all combinations by default. And the second term of the reformulated RHS counts all combinations with that specific element missing from all combinations. Then we have included both cases, resulting in the same counting as happens on the LHS where all combinations of size n are counted from 2n elements.

Problem 2

Prove that $\forall n \geq k \geq m \geq 0$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

Solution 2

The first LHS term counts the number of k-combinations one can form from n elements, since its given that $n \geq k$. Then, for every k-combination of the first LHS term, the second LHS term counts how many m-combinations one can form from that specific k-combination. By multiplying the sizes of these two subsets, we get the total number of k-combinations formable form n elements and recursively the further m-combinations formable from each k-combination.

The first RHS term starts off counting the number of m-combinations formable from n elements. Since the first RHS term effectively counts all the possible m-combinations, what remains is to count the combinations for the remaining elements $k - m$. The second RHS term counts the number of ways one can combine the remaining $k - m$ elements from the remaining $n - m$ elements for each m-combination given by the first RHS term. Multiplying these two terms together, again we get total number of k-combinations we can form of paired with m-combinations in dependent manner.

Therefore, both sides count the same thing, but in different order/way.

Solution 2

Clearly, this can be shown algebraically, how it is important to understand the intuition behind the different counting methods. Here is the algebraic proof:

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

$$\frac{n!}{(n-k)!k!}\frac{k!}{(k-m)!m!} = \frac{n!}{(n-m)!m!}\frac{(n-m)!}{(n-m-k+m)!(k-m)!}$$

$$\frac{n!}{(n-k)!(k-m)!m!} = \frac{n!}{(n-k)!(k-m)!m!}$$

Binomial Coefficients

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$(x+y+z)^n = \sum_{k_1+k_2+k_3=n} \frac{n!}{k_1!\,k_2!\,k_3!} x^{k_1} y^{k_2} z^{k_3}$$

Problem 3

What is the coefficient of x^8 in the expansion of $(x+1)^{14} + x^3(x+2)^{15}$?

Solution 3

Solution:

$$(x+1)^{14} = a_0 + a_1 x + ... + a_{14} x^{14},$$

$$(x+2)^{15} = b_0 + b_1 x + ... + b_{15} x^{15},$$

$$(x+1)^{14} + x^3 (x+2)^{15} = c_0 + c_1 x + ... + c_{18} x^{18},$$

then $c_8 = a_8 + b_5 = \binom{14}{8} + \binom{15}{5} 2^{10}$.

Problem 4

If $(1+x)^{10} = a_0 + a_1(1-x) + a_2(1-x)^2 + ... + a_{10}(1-x)^{10}$, calculate a_8.

Solution 4

Solution: $(1+x)^{10} = [2-(1-x)]^{10},\ a_8 = C(10,8)(-1)^8(-2)^2 = 180$

Problem 5

Prove $3^{2n+2} - 8n - 9$ can be divided by 64.

Solution 5

$$
\begin{aligned}
&3^{2n+2} - 8n - 9 \\
=&3^2 \cdot 3^{2n} - 8n - 9 \\
=&9(3^2)^n - 8n - 9 \\
=&9(8+1)^n - 8n - 9 \\
=&9[\binom{n}{0}8^n + \binom{n}{1}8^{n-1} + \binom{n}{2}8^{n-2} + \ldots \\
&+ \binom{n}{n-2}8^2 + \binom{n}{n-1}8 + \binom{n}{n}8^0] - 8n - 9 \\
=&9(64k + 8n + 1) - 8n - 9 \\
=&9 \times 64k + 72n + 9 - 8n - 9 \\
=&64m + 8n + 9 - 8n - 9 = 64m.
\end{aligned}
$$

Probability

- Experiment
    - Toss a fair coin 10 times
- Sample Space (Ω)
    - All possible outcomes of the experiment
- Simple Event
    - Any element of the sample space
- Compound Event
    - Subsets of the sample space

Problem 6

You toss a coin 10 times. In that sequence of 10 tosses,

(a) what is the probability of seeing exactly 5 heads?

(b) what is the probability of seeing exactly 5 heads in a row?

Solution 6

(a) what is the probability of seeing exactly 5 heads?

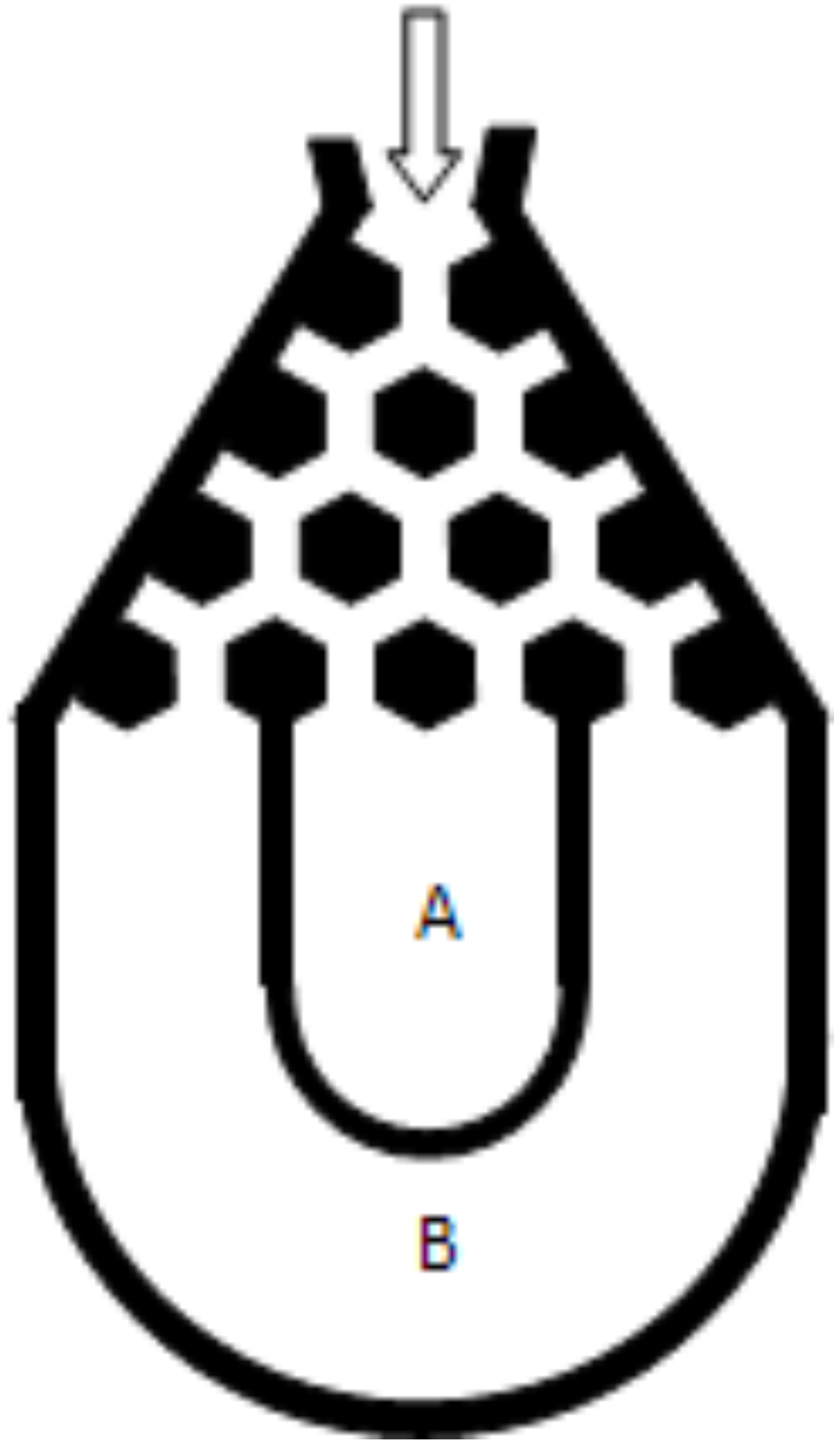
Solution: $\frac{\binom{10}{5}}{2^{10}}$

(b) what is the probability of seeing exactly 5 heads in a row?

Solution: $\frac{6}{2^{10}}$

Problem 6

Put a ball into this device.
The probability of the ball chooses the left equals to that of the ball chooses the right.
Calculate the probability of the ball falls into A.
Calculate the probability of the ball falls into B.



Solution 6

The ball has to make 3 decisions.
Sample space: 0-left 1-right
{000, 001, 010, 011, 100, 101, 110, 111}
The ball falls into B:
{000, 111}
P(B) = 2/8 = 1/4
P(A) = 1- P(B) = 3/4