CS 206

Recitation - Section 4

Nov. 9

Inclusion-Exclusion Principle

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

Inductive Proof

A∪B =|A∪B| = |A|+|B| - |A∩B |

|A∪B∪C| = |A|+|B|+|C| - |A∩B| - |B∩C| - |C∩A| + |A∩B∩C|

Problem 1

Suppose you need to come up with a password that uses only the letters A, B, and C and which must use each letter at least once. How many such passwords of length 8 are there?

Solution 1

Solution: We will first find the number of passwords that leave out at least one of A, B, or C.
Let X be the set of passwords that doesn't contain A;
Let Y be the set of passwords that doesn't contain B;
Let Z be the set of passwords that doesn't contain C.
We want to find the size of $X \cup Y \cup Z$
Passwords that don't contain A just contain B and C. So there are 2^8 such passwords—i.e. $|X| = 2^8$. By the same reasoning $|Y| = |Z| = 2^8$.
Passwords that don't contain A or B just contain C. There is one such password (namely 'CCCCCCCC') so $|X \cap Y| = 1$. By the same reasoning $|X \cap Z| = |Y \cap Z| = 1$.
Passwords that don't contain A, B, or C can't exist because passwords in this problem only use the letters A,B,and C.So $|X \cap Y \cap Z| = 0$. So by inclusion-exclusion $X \cup Y \cup Z = 3 * 2^8 - 3 * 1 + 0 = 3 * 2^8 - 3$.
To find the answer to the original question, we need to subtract the number we just found from the total number of passwords, which is 3^8. This gives $3^8 - (3 * 2^8 - 3)$

Problem 2

Suppose we define prime-looking numbers as any composite number(defined as an integer c such that $a * b = c$ where $a, b < c$ and $a, b \in Z$) not divisible by 2,3, or 5. For example, the three smallest prime-looking numbers are 49,77,91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

Solution 2

Solution: Inclusion exclusion problem: Define sets: Set of numbers less than 1000 divisible by 2 is $|N_2| = \lfloor\frac{999}{2}\rfloor = 499$,divisible by 3 is $|N_3| = \lfloor\frac{999}{3}\rfloor = 333$,divisible by 5 is $|N_5| = \lfloor\frac{999}{5}\rfloor = 199$,divisible by 2 or 3 is $|N_2 \cap N_3| = \lfloor\frac{999}{2*3}\rfloor = 166$,divisible by 2 or 5 is $|N_2 \cap N_5| = \lfloor\frac{999}{2*5}\rfloor = 99$,divisible by 3 or 5 is $|N_3 \cap N_5| = \lfloor\frac{999}{3*5}\rfloor = 66$,divisible by 2 or 3 or 5 is $|N_2 \cap N_3 \cap N_5| = \lfloor\frac{999}{2*3*5}\rfloor = 33$. Then, by inclusion exclusion principle, we have 499+333+199-166-99-66+33=733 that are not prime-looking including 3 actual primes(2,3,5). To get number of prime-looking numbers under 1000, we subtract 999-733=266 to get number of prime-looking numbers and actual primes. Now subtract actual primes to get 266-165=101 prime-looking numbers. The reason we use 165 rather than 168 is to account for 2,3,5 primes in 733. Finally, we subtract integer 1 from our 101 prime-like numbers, since it is not a composite number.

Problem 3

Suppose that 4 people are standing in line. How many ways are there to rearrange the line so that nobody is standing in their original place?

Solution 3

Solution: We will first count the number of ways to arrange the line of people so that at least one person stays in the same place. For each i between 1 and 4, let A_i be the set of ways to order the people in line so that person i stays in the same place. We want to find the size of $A_1 \cup A_2 \cup A_3 \cup A_4$. The number of ways to order the line so that person i remains in the same place is just the number of ways to arrange the other three people—i.e. $3 \times 2 \times 1 = 3!$. In other words, for each i, $|A_i| = 3!$. Similar reasoning shows that the double intersections have size 2!, and that the triple and quadruple intersections have size 1.
Since there are 6 double intersections, 4 triple intersections, and 1 quadruple intersection, the inclusion-exclusion tells us that the number of ways to arrange the people so that someone stays in the same place is $4 \times 3! - 6 \times 2! + 4 \times 1 - 1 \times 1$. Subtracting this from the total number of ways to arrange four people, which is 4!, gives us $4! - (4 \times 3! - 6 \times 2! + 4 \times 1 - 1 \times 1)$

Pigeonhole Principle

If n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

Item? Container?

Problem 4

Show that there are at least 250 four digit numbers whose digits all sum to the same value.

Solution 4

Solution: The number of four digit numbers is $9 \times 10 \times 10 \times 10 = 9000$: there are 9 options for the first digit (since it cannot be 0) and 10 options for the remaining three digits. The number of possible sums of the digits of four digit numbers is 36: the lowest possible sum is 1, the highest possible sum is 36 (since that's the sum when every digit is 9), and all the possible sums are integers. Viewing the four digit numbers as objects and the possible sums as boxes, the pigeonhole principle implies that at least one box will end up with at least $\lceil \frac{9000}{36} \rceil = 250$ objects—in other words there is at least one value which is the digit sum of at least 250 four digit numbers.

Problem 5

100 balls in are placed in a circle. 41 red, 59 blue. Prove that we can find two red balls, such that there are 19 balls between these two red balls.

Solution 5

Solution: Label these balls from 1 to 100 in sequence (clockwise). Divide these balls into 20 groups. We have 41 red balls, according to the pigeonhole principle, at least 1 group having at least 3 (or more) red balls. In that group, 2 of these 3 red balls are *'next to'* each other (pigeonhole principle again!). Between these two red balls, there are 19 balls.

(1, 21, 41, 61, 81) ;

(2, 22, 42, 62, 82) ;

……

(20, 40, 60, 80, 100)

Problem 6

Given m integers $a_1, a_2, ..., a_m$, there exists a range $[i : j]$, such that $\sum_{k=i}^{j} a_k$, say the range sum, can be divided by m.

Solution 6

Solution: Define $a_0 = 0$. The range sum $\sum_{k=i+1}^{j} a_k$ can be expressed as the difference of two prefix sums,

$$\sum_{k=i+1}^{j} a_k = \sum_{k=0}^{j} a_k - \sum_{k=0}^{i} a_k.$$

There are $m+1$ prefix sums of $a_0, a_1, a_2, ..., a_m$. According to the pigeonhole principle, at least two of these $m+1$ prefix sums are congruent modulo m. Assume these two prefix sums are $\sum_{k=0}^{x} a_k$ and $\sum_{k=0}^{y} a_k$, then $\sum_{k=x+1}^{y} a_k$ can be divided by m.