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Any fool can know. The point is to understand — Albert Einstein

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Discrete Structures II

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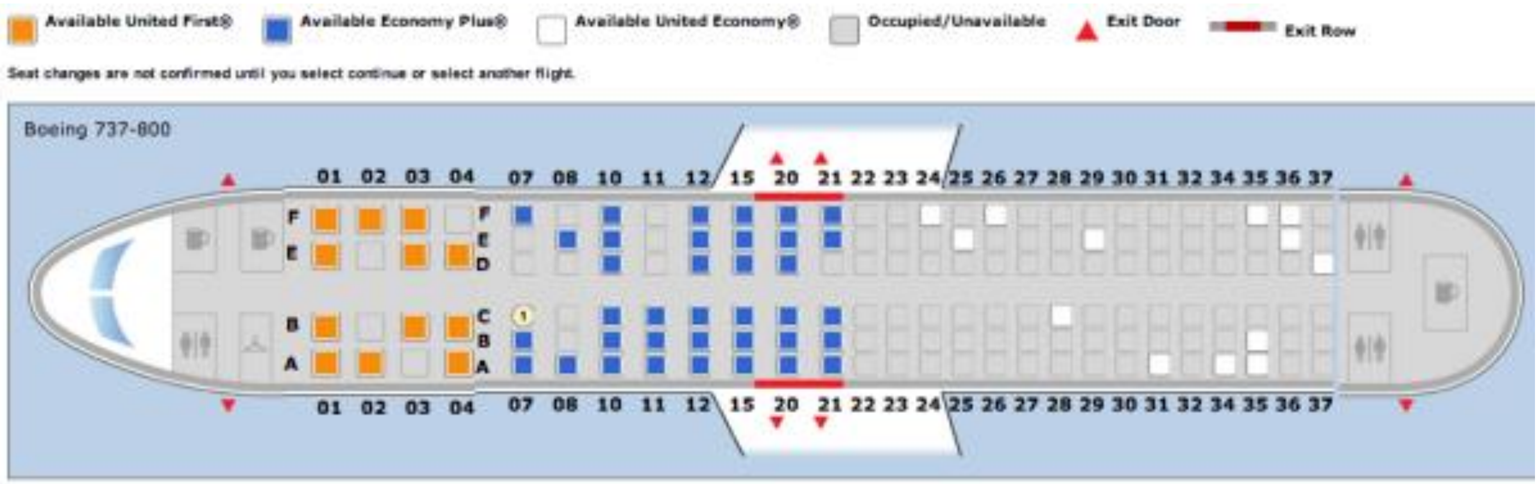
Lecture 10 | Permutations - Combinations | Tuesday October 5th 2021

So Far

- ~~Sets / Functions~~
- ~~Proofs~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- Permutation/Combinations
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

Permutations - Example

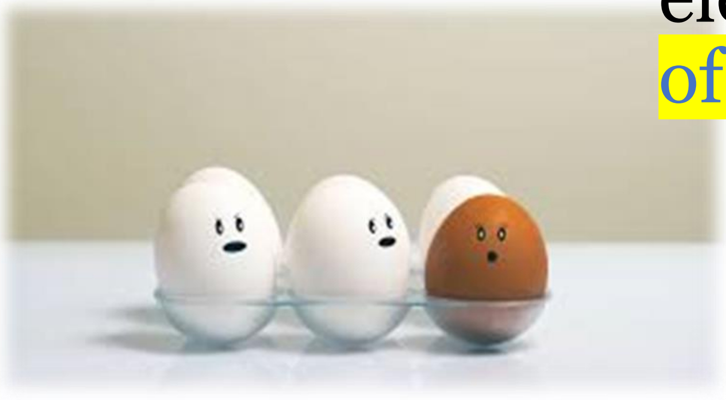
- How many ways to assign 100 passengers to 20 first class seats?



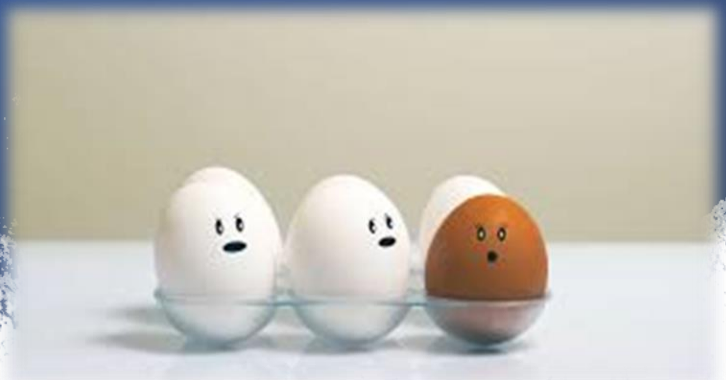
Permutations and Combinations

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters.

- Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected **does not** matter.



Difference between Permutations and Combinations



The **difference between combinations and permutations** is in

ordering

- With **permutations** we care about the order of the elements, whereas with **combinations** we don't care.

Examples:

- Permutation: Find a locker "combo" is 12345; Cellphone PIN is 5432
- Combination: Pick 5 students from a 180-student audience

Find 4-digit Permutations

of the numbers 2,3,4,5

____ ____ ____ ____

Find 4-digit Permutations

of the numbers 2,3,4,5

___ ___ ___ ___

The first digit can be any of the 4 numbers

4 ___ ___ ___

Find 4-digit Permutations

of the numbers 2,3,4,5

$\underline{\ 4\ }\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$

Now there are 3 options left for the second blank

$\underline{\ 4\ } \bullet \underline{\ 3\ }\ \underline{\quad}\ \underline{\quad}$

Find 4-digit Permutations

of the numbers 2,3,4,5

For the third position, we have two numbers left

$$\underline{4} \cdot \underline{3} \cdot \underline{2} \quad \underline{\quad}$$

There is one number left for the last position

$$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$$

Find 4-digit Permutations

of the numbers 2,3,4,5

$$4! = 4 \bullet 3 \bullet 2 \bullet 1 = 24$$

Permutations **with Repetition**

____ ____ ____ ____

- What if I now want to find the total number of permutations involving the numbers 2, 3, 4, and 5
- but want to include orderings such as 5555 or 2234 *where not all of the numbers are used, and some are used more than once*?

Permutations with Repetition

$$\underline{4} \cdot \underline{4} \cdot \underline{4} \cdot \underline{4} = 4^4 = 256$$

- What if I now want to find the total number of permutations involving the numbers 2, 3, 4, and 5
- but want to include orderings such as 5555 or 2234 *where not all of the numbers are used, and some are used more than once*?

Choosing a subset (a.k.a. Combinations)

_____ _____ _____ _____ _____

- *How many different 5-card hands can be made from a standard deck of cards?*
- In this problem the order is irrelevant since it doesn’t matter what order we pick the cards.
- We’ll begin with five lines to represent our 5-card hand.

Choosing a subset

$\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48}$

311,875,200 *permutations*

- *How many **different** 5-card hands can be made from a standard deck of cards?*
- In this problem the order is irrelevant since it doesn't matter what order we select the cards.
- We'll begin with five lines to represent our 5-card hand.

Choosing a subset

$$\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48}$$

- *How many different 5-card hands can be made from a standard deck of cards?*
- In this problem the order is irrelevant since it doesn't matter what order we select the cards.
- We'll begin with five lines to represent our 5-card hand.
- **That's permutations, not combinations**
- To fix this we need to **divide by the number of hands that are different permutations but the same combination**

Choosing a subset

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

- That's permutations, not combinations.
- To fix this we need to divide by the number of hands that are different permutations but the same combination.
- This is the same as saying *how many different ways can I arrange 5 cards?*

Choosing a subset

$$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$

- So the number of five-card hands combinations is:

Rewriting with Factorials

$$\frac{52!}{47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47} \cdot \cancel{46} \cdot \ldots \cdot \cancel{2} \cdot \cancel{1}}{\cancel{47} \cdot \cancel{46} \cdot \ldots \cdot \cancel{2} \cdot \cancel{1}}$$

- With a little ingenuity we can rewrite the above calculation using factorials.
- We know 52! = 52•51•50•…•3•2•1, but we only need the products of the integers from 52 to 48. How can we isolate just those integers?
- We'd like to divide out all the integers except those from 48 to 52. To do this divide by 47! since it's the product of the integers from 47 to 1.

Rewriting with Factorials

$$\frac{52!}{5!47!}$$

- Make sure to divide by **5!** to get rid of the extra permutations:

There we go!

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left

Combinations Formula

$$\binom{n}{k} = C_k^n = {}_nC_k$$

- Different Annotations

Permutations Formula

$$P_k^n = \frac{n!}{(n-k)!}$$

- The formula for permutations is similar to the combinations formula, except we needn't divide out the permutations, so we can remove k! from the denominator:

Take a Break

5 min

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=1$, choose 1 out of n elements, n ways

easy… order is not important here

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=2$, choose 2 out of n elements

Let A = all ways to choose 2 out of n elements

Let B = all ways to permute 2 out of n elements

Order is important

We know $|B| = {}^nP_2 = n(n-1)$

How to go from $|B|$ to $|A|$?

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=2$, choose 2 out of n elements

Associated choice sequence for counting $|B|$

↓

Pick 1st element

Pick 2nd element

Hence $n(n-1)$ ways

Order is important

Consider a pair $\{e_1, e_2\}$ in A. In B $\{e_1, e_2\}$ is counted twice. Either e_1 can be picked as 1st element and e_2 as second, or vice versa

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=2$, Choose 2 out of n elements

Associated choice sequence for counting $|B|$

Pick 1st element

Pick 2nd element

Hence $n(n-1)$ ways

Order is important

In other words, for each element of A there are 2 choices sequences in B that generate the same outcome.

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=2$, Choose 2 out of n elements

Associated choice sequence for Counting $|B|$

↓

Pick 1st element

Pick 2nd element

Hence $n(n-1)$ ways

Order is important

Hence, $|A| = \frac{|B|}{2} = \frac{n(n-1)}{2} = \binom{n}{2} \leftarrow$ n choose 2

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=3$, Choose 3 out of n elements

$A =$ all ways to choose 3 out of n elements

$B =$ all ways to permute 3 out of n elements

$|B| = n(n-1)(n-2)$

choice sequence:

Pick 1st element

Pick 2nd element

Pick 3rd element

3! ways to reach

$\{e_1, e_2, e_3\}$

A

Order is important

Every element in A can be reached via 3! choice sequences in B

Combinations – Permutations Better explained

- Choosing r out of n elements in no specific order. $\binom{n}{r}$

$r=3$, choose 3 out of n elements

$A =$ all ways to choose 3 out of n elements

$B =$ all ways to permute 3 out of n elements

$|B| = n(n-1)(n-2)$

choice sequence: pick 1st element, pick 2nd element, pick 3rd element

3! ways to reach $\{e_1, e_2, e_3\}$ in A

Order is important

Hence, $|A| = \frac{|B|}{3!} = \frac{n(n-1)(n-2)}{3!} = \binom{n}{3}$