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206
Discrete Structures II

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Preview…

k distinct pirates want to divide up n identical, indivisible bars of gold. How many ways to divide the loot when each must get at least r bars?

Preview...

How many integer solutions to the following equation?

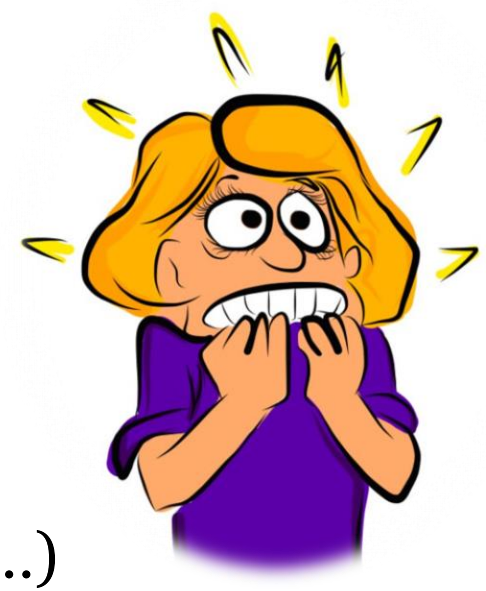
$$x_1 + x_2 + \cdots + x_k = n$$

$$x_1, x_2, \ldots, x_k \geq 0$$

So Far

- ~~Sets / Functions~~
- ~~Proofs~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- Permutation/Combinations
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

Quiz 2 – When and What?



- When
  - Sections 5 and 6: Thursday 10/14, during recitation (today…)
  - Section 4: Tuesday 10/19, during recitation
- What will cover
  - Sum rule (Week 4 Lectures)
  - Product rule (Week 4-5 Lectures)
  - Permutations with and without repetitions (Week 6 & Tuesday's Lectures)

Quiz 2 – Have you seen the Extra Problems?

- Week 7: Advanced Counting - Pirates Problem
  - Extra_Problems_1_Sum and Product Rules.pdf
  - Extra_Problems_2_Combinations_Permutations.pdf

General Hint

For each problem

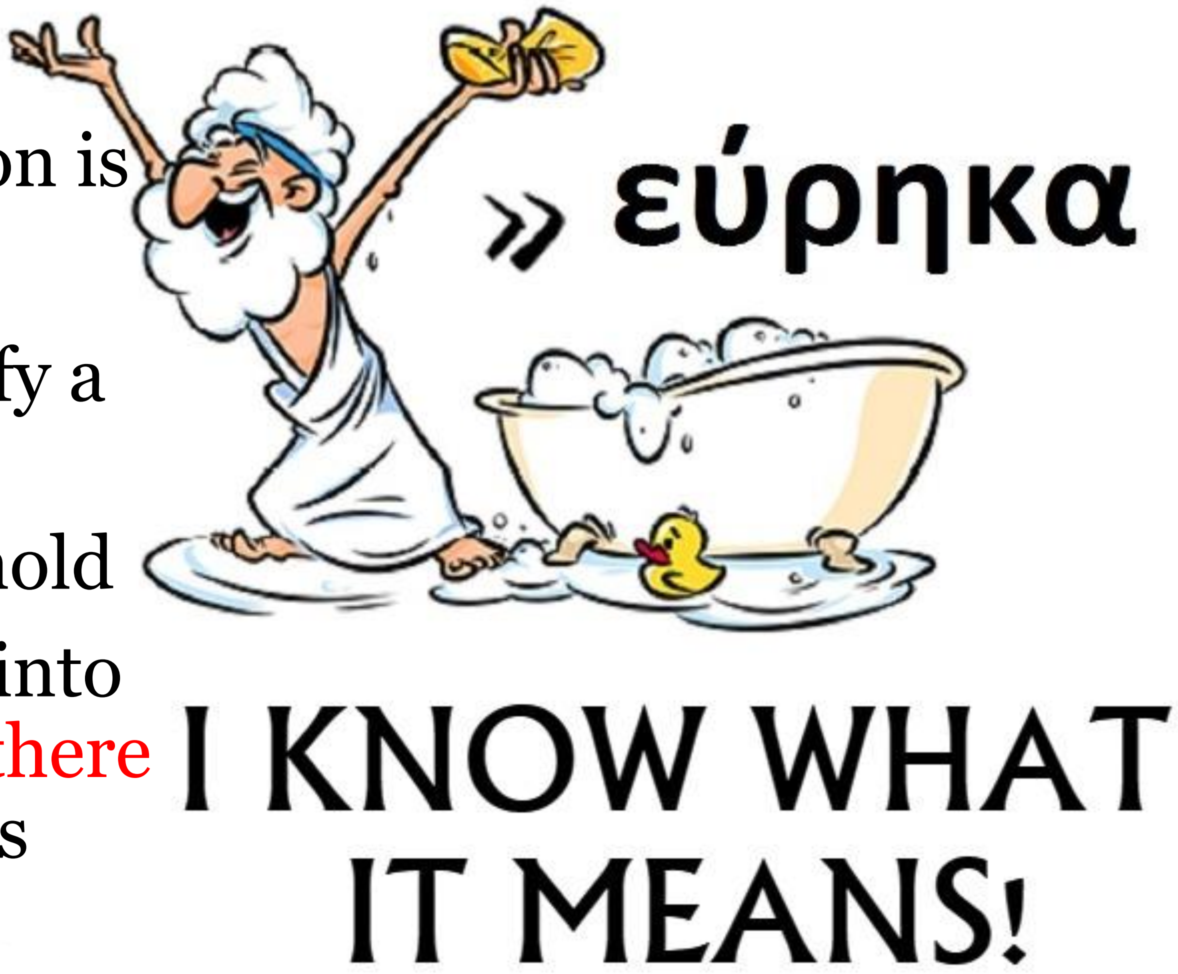
(1) Fully understand what the question is

(2) Fully understand what you know

(3) Based on the previous two, identify a method

(4) Make sure that the assumptions hold

(5) Turn the wording of the problem into the input to your method. Typically, there is a “key” thought that will unlock this part of the solution for you.



So far...

Permutations

- Distinctly ordered sets are called permutations (arrangements). The number of permutations of n distinct objects taken k at a time is given by:

$$P^n_k = \frac{n!}{(n-k)!}$$

n = number of distinct objects
k = number of positions

So far...

Permutations Formula – Remember!

$$P^n_k = \frac{n!}{(n-k)!}$$

The formula for permutations is similar to the combinations formula, except we needn't divide out the permutations, so we remove k! from the denominator

So far...

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left

So far...

Permutations without Repetitions

A maths debating team consists of 4 speakers.

- In how many ways can all 4 speakers be arranged in a row for a photo?

Solution : $4 \times 3 \times 2 \times 1 = 4! \text{ or } {}^4P_4$

- How many ways can the captain and vice-captain be chosen?

Solution : $4 \times 3 = 12 \text{ or } {}^4P_2$

Permutations without Repetitions

<u>A flutter on the horses</u>

There are 7 horses in a race.

- In how many different orders can the horses finish?

Solution : $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7!$ or ${}^{7}P_{7}$

- How many trifectas (1st , 2nd and 3rd) are possible?

Solution : $7 \times 6 \times 5 = 210$ or ${}^{7}P_{3}$

Permutations with Restrictions

In how many ways can 5 boys and 4 girls be arranged on a bench if

- there are no restrictions?

Solution : $9!$ or $^9P_9$

- boys and girls alternate?

_

Solution : A boy will be on each end

BGBGBGBGB = $5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1$

$= 5! \times 4!$ or $^5P_5 \times {}^4P_4$

Permutations with Restrictions

In how many ways can 5 boys and 4 girls be arranged on a bench if

- boys and girls are in separate groups?

Solution : **Boys & Girls or Girls & Boys**

$$= 5! \times 4! + 4! \times 5! = 5! \times 4! \times 2$$

$$\text{or } {}^5P_5 \times {}^4P_4 \times 2$$

d) Anne and Jim wish to stay together?

Solution : **(AJ) _ _ _ _ _ _ _**

$$= 2 \times 8! \text{ or } 2 \times {}^8P_8$$

Permutations with Repetitions

If we have **n** elements of which x are alike of one kind, y are alike of another kind, z are alike of another kind, ... then the **number of ordered selections or permutations** is given by:

$$\boxed{\frac{n!}{x!\,y!\,z!}}$$

So far...

Permutations with Repetitions

How many permutations of the word **PARRAMATTA** are possible?

Solution :

10 letters but note repetition (4 A's, 2 R's, 2 T's)

P

A A A A

R R

M

T T

No. of arrangements $= \dfrac{10!}{4!\,2!\,2!}$

$= 37\,800$

So far...

Permutations with Restrictions

How many arrangements of the letters of REMAND are possible if:

- there are no restrictions?

Solution : $^{6}P_{6} = 720$ or $6!$

- they begin with RE?

Solution : $\underline{R\ E}$ _ _ _ _ $= {}^{4}P_{4} = 24$ or $4!$

- they do not begin with RE?

Solution : Total – (b) = $6! - 4! = 696$

So far...

Permutations with Restrictions

How many arrangements of the letters of REMAND are possible if:

- they have RE together in order?

Solution : **(<u>RE</u>) _ _ _ _ = $^5P_5$ = 120 or 5!**

- they have REM together in any order?

Solution : **(REM) _ _ _ = $^3P_3 \times {}^4P_4$ = 144**

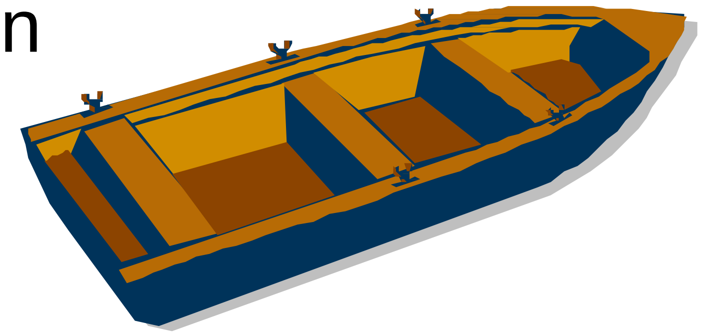
- R, E and M are not to be together?

Solution : **Total – (e) = 6! – 144 = 576**

So far...

Permutations with Restrictions

There are 6 boys who enter a boat with 8 seats, 4 on each side. In how many ways can



- they sit anywhere? **Solution :** $^{8}P_{6}$

- two boys A and B sit on the port side and another boy W sit on the starboard side?

Solution : $A \,\&\, B = {}^{4}P_{2}$

$W = {}^{4}P_{1}$

$\text{Others} = {}^{5}P_{3}$

$\text{Total} = {}^{4}P_{2} \times {}^{4}P_{1} \times {}^{5}P_{3}$

So far...

Permutations with Restrictions

From the digits 2, 3, 4, 5, 6

- how many numbers greater than 4,000 can be formed?

Solution : **5 digits (any) = $^5P_5$**

4 digits (must start with digit $\geq$ 4) = $^3P_1 \times {}^4P_3$

Total = $^5P_5 + {}^3P_1 \times {}^4P_3$

- how many 4 digit numbers would be even?

Even (ends with 2, 4 or 6) = _ _ _ $^3P_1$

$= {}^4P_3 \times {}^3P_1$

Take a Break

5 min

Combinations with Repetitions

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot?

Count all sequences of (a,b,c,d,e) such that $a+b+c+d+e=20$

$a = \#$ pirate 1 gets
$b = \#$ 2 gets
$c = \#$ 3 gets
$d =$ 4
$e =$ 5 gets

answer = all ways to arrange 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)} = \binom{24}{4}$$

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_5 = 20$
  - $x_1, x_2, \dots, x_5 \geq 0$

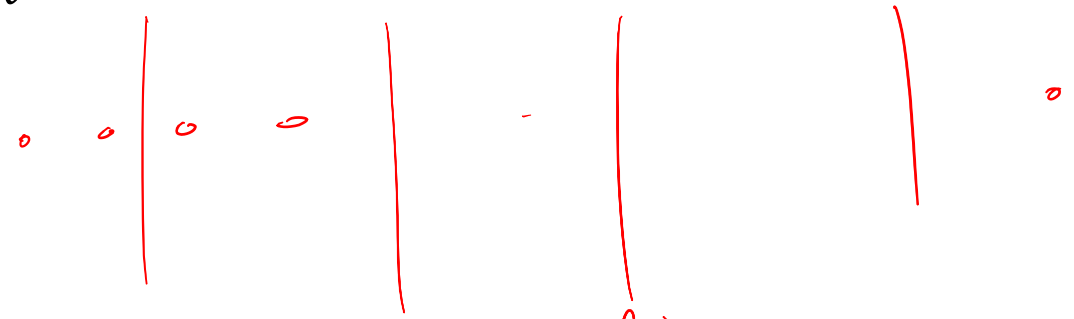
$(x_1, x_2, x_3, x_4, x_5)$ such that $\sum x_i = 20$

$\Rightarrow$ all arrangements of 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)}$$

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_k = n$
  - $x_1, x_2, \ldots, x_k \geq 0$



k-1 lines

→ all arrangements of n dots and k-1 lines

→ $\dfrac{(n+k-1)!}{n!\,(k-1)!} = \dbinom{n+k-1}{k-1}$