Problem 1

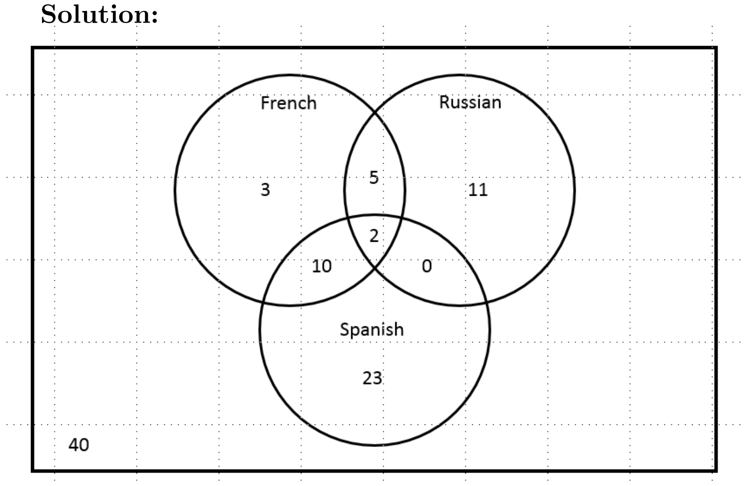
[a] Students in CS 206 class are known multiple languages:

- 20 students know French
- 18 students know Russian
- 35 students know Spanish
- 5 students know both Russian and French
- 2 students know French, Russian Spanish
- 10 students know Spanish and French
- 40 students know only English

Draw a Venn diagram for the above. How many students are there in the class?

[a]Still remember the equation $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$?

Solution:



There are 40+3+11+23+10+5+2+0=94 students in class

Problem 2

For any set A, let $\mathcal{P}(A)$ be its power set. Let $\emptyset$ denote the empty set.

1. Write down all the elements of $\mathcal{P}(\{\emptyset, \{\emptyset\}\})$.
2. How many elements are there in $\mathcal{P}(\{1,2,3,4,5,6,7,8\})$?

Solution:

$$\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\{\emptyset\}\}, \emptyset\}$$

$$|\mathcal{P}(\{1,2,3,4,5,6,7,8\})| = 2^8$$

Problem 3

[a] Given the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, answer the following:

(a) How many subsets of X contain element x_1 ?

(b) How many subsets of X contain element x_2 and x_3, and DO NOT contain x_4 ?

[a] Still remember how we get $|\mathcal{P}(A)| = 2^{|A|}$? If not, see the appendix.

Solution:

(a) Let Y ={y: y is a subset of X containing element x_1},
$Y = \{x_1\} \times \mathcal{P}(X - \{x_1\})$,

$$\begin{aligned}|Y| &= |\mathcal{P}(X - \{x_1\})| \\ &= |\mathcal{P}(\{x_2, x_3, x_4, x_5, x_6, x_7, x_8\})| \\ &= 2^7.\end{aligned}$$

(b) Let Y ={y: y is a subset of X containing element x_2 and x_3, and not containing x_4},
$Y = \{x_2, x_3\} \times \mathcal{P}(X - \{x_2, x_3, x_4\})$,

$$\begin{aligned}|Y| &= |\mathcal{P}(X - \{x_1\})| \\ &= |\mathcal{P}(\{x_1, x_5, x_6, x_7, x_8\})| \\ &= 2^5.\end{aligned}$$

Problem 4

[a] Let $f : A \to B$ and $g : B \to C$ be functions and $h : A \to C$ be their composition, namely, $h(a) ::= g(f(a))$, $\forall a \in A$.

(a) Prove that if f and g are surjective, then so is h.

(b) Prove that if f and g are bijective, then so is h.

[a] Dig deeper in this problem. If h is bijective/surjective/injective, what can we know about f and g?

Solution:

(a) If g is surjective, for each $c \in C$, there exists a $b \in B$ s.t. $g(b) = c$.
If f is surjective, for each $b \in B$, there exists an $a \in A$ s.t. $f(a) = b$.
Therefore, for each $c \in C$, we can find an $a \in A$ s.t. $h(a) = g(f(a)) = c$.
Hence, h is surjective. QED

(b) According to the definition, if a function is bijective, it is both injective and.
To prove h is bijective, we can prove h is both injective and surjective.
If f is injective, each $a \in A$ is mapped to a different $b \in B$.
If g is injective, each $b \in B$ is mapped to a different $c \in C$.
Therefore, for each $a \in A$, we can find a different $c \in C$, namely, $\forall a_1, a_2 \in A, a_1 \neq a_2, h(a_1) \neq h(a_2)$, h is injective.
In (a), we have proved that if f and g are surjective, h is surjective.
Hence, h is bijective. QED

Problem 5

Direct Proof

Prove: $\forall n \in \mathbb{Z}$, if n is odd, then $n^2 + 2n + 1$ is even.

Solution:
Assume n is odd.
By definition of an odd number,

$$n = 2j + 1,$$

where $j \in Z$.
Then,

$$\begin{aligned} n^2 + 2n + 1 &= (2j+1)(2j+1) + 2(2j+1) + 1 \\ &= 4j^2 + 4j + 1 + 4j + 2 + 1 \\ &= 4j^2 + 8j + 4 \\ &= 2(2j^2 + 4j + 2). \end{aligned}$$

since $(2j^2 + 4j + 2) \in Z$ it must be that $2(2j^2 + 4j + 2)$ is an even number by definition, hence the above statement must be true. QED

Appendix

A The proof of $|\mathcal{P}(A)| = 2^{|A|}$

Base case: When $|A| = 0$, $A = \emptyset$, $|\mathcal{P}(A)| = |\{\emptyset\}| = 1 = 2^0 = 2^{|A|}$.
Induction step: Let $k \in \mathbb{N}$ be given and suppose $|\mathcal{P}(A)| = 2^{|A|}$ is true for $|A| = k$. When $|A| = k + 1$, select an element x from A arbitrarily.

$$\begin{aligned}
|\mathcal{P}(A)| &= |\{x\} \times \mathcal{P}(A - \{x\})| + |\emptyset \times \mathcal{P}(A - \{x\})| \\
&= |\mathcal{P}(A - \{x\})| + |\mathcal{P}(A - \{x\})| \\
&= 2^k + 2^k \\
&= 2^{k+1}.
\end{aligned}$$

[1] Thus, the equation $|\mathcal{P}(A)| = 2^{|A|}$ holds for $|A| = k + 1$, and the proof of the induction step is complete.
Conclusion: By the principle of induction, $|\mathcal{P}(A)| = 2^{|A|}$ is true for all $|A| \in \mathbb{N}$.

[1] Please notice that $\mathcal{P}(A) \neq (\{x\} \times \mathcal{P}(A - \{x\})) \cup (\emptyset \times \mathcal{P}(A - \{x\}))$, because the product operation returns a set of pairs rather than a set of union sets.