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Have you seen the extra problems on canvas?

206
Discrete Structures II

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Week 10: Combinatorial Proofs

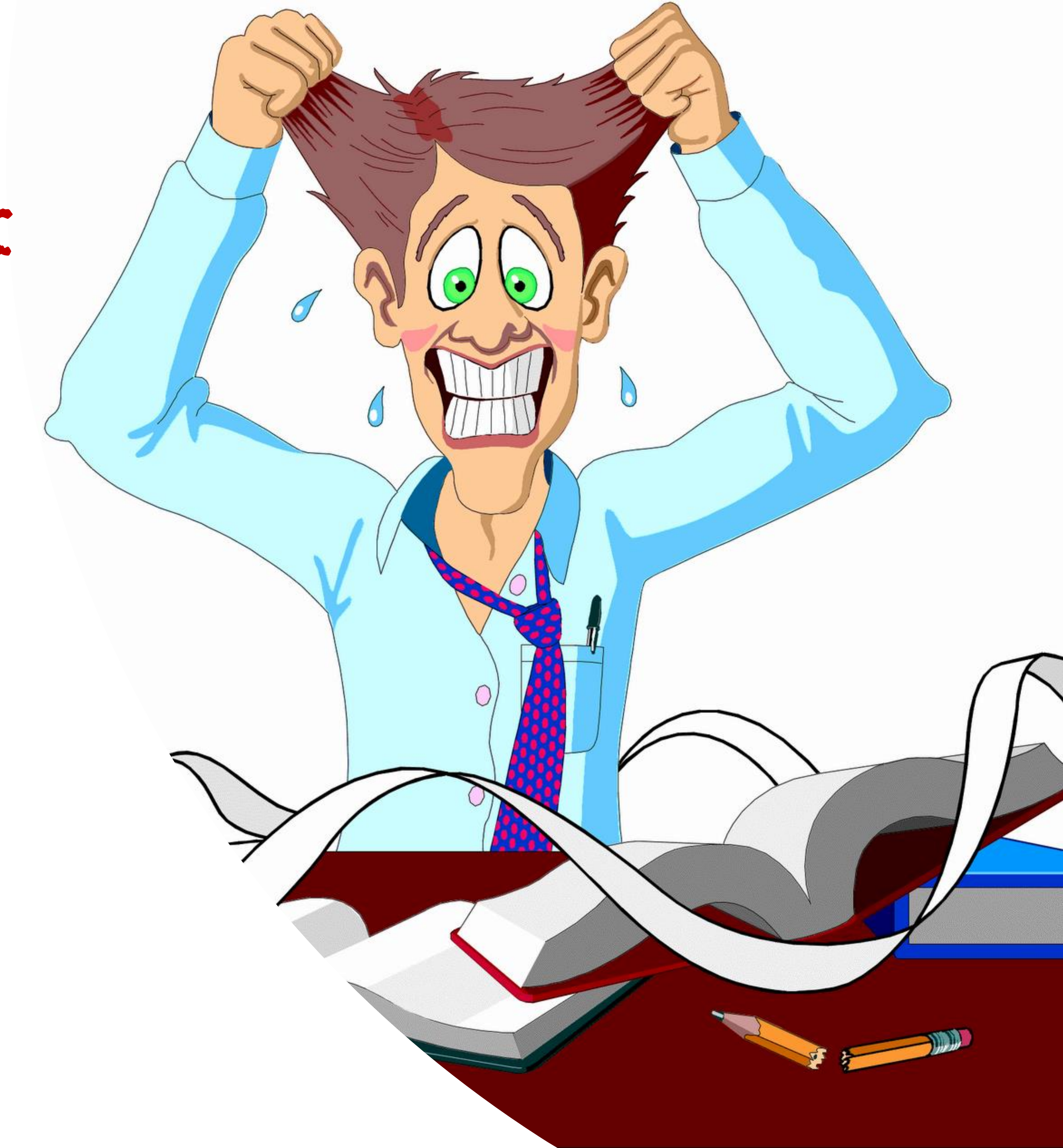
- Extra_Problems_4_Combinatorial Proofs.pdf
  - lecture_18.pdf
    - Recording 18 - Pass: uH2UpEj3
  - lecture_19.pdf
    - Recording 19 - Pass: FaMDgrV2

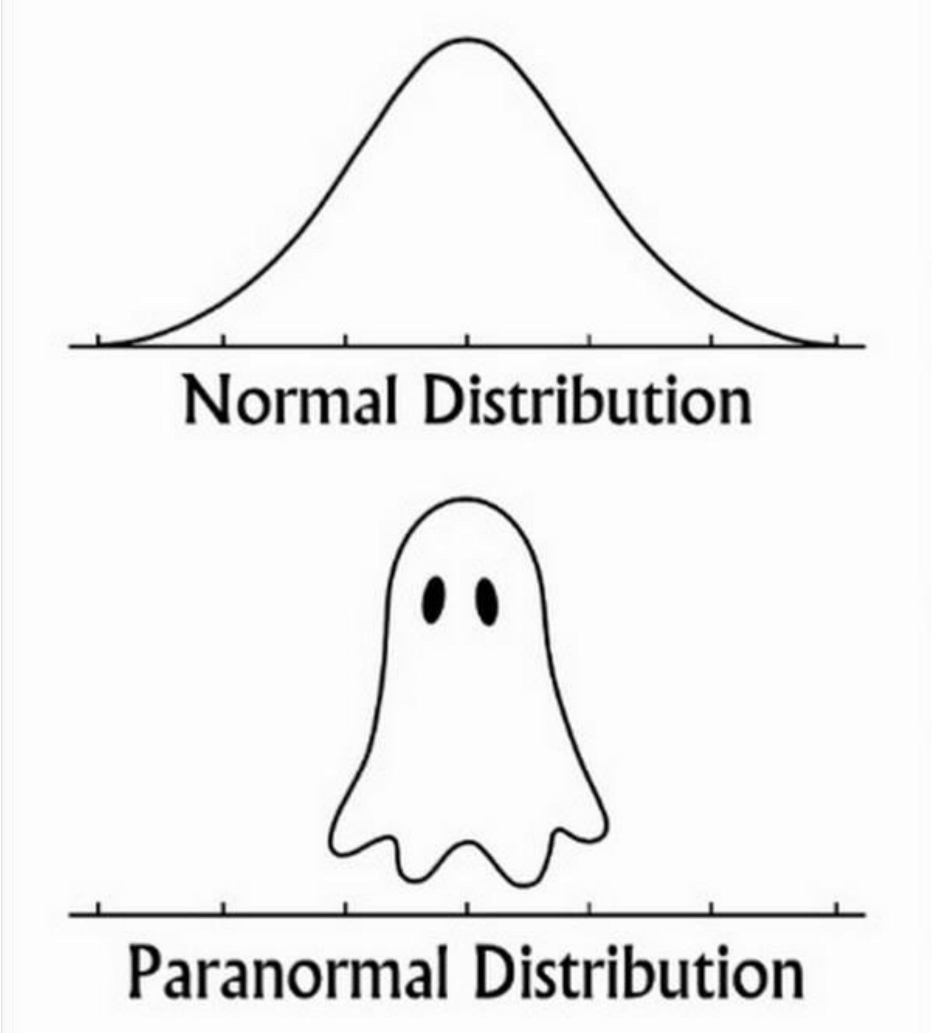
Week 11: Binomial Coefficients & Pascal Triangle

- Extra_Problems_5_Binomial_Coefficients.pdf
- Extra_Problems_6_Collective_Problems_Combinatorics.pdf
  - lecture_20.pdf
    - Recording 20 - Pass: Wpx9VVj6

Quiz 4 – Next Week

- Lectures 16 - 20
  - Inclusion-Exclusion / Pigeonhole Principle
  - Combinatorial Proofs and Binomial Coefficients
    - And everything else…
- **During recitation**





Today:

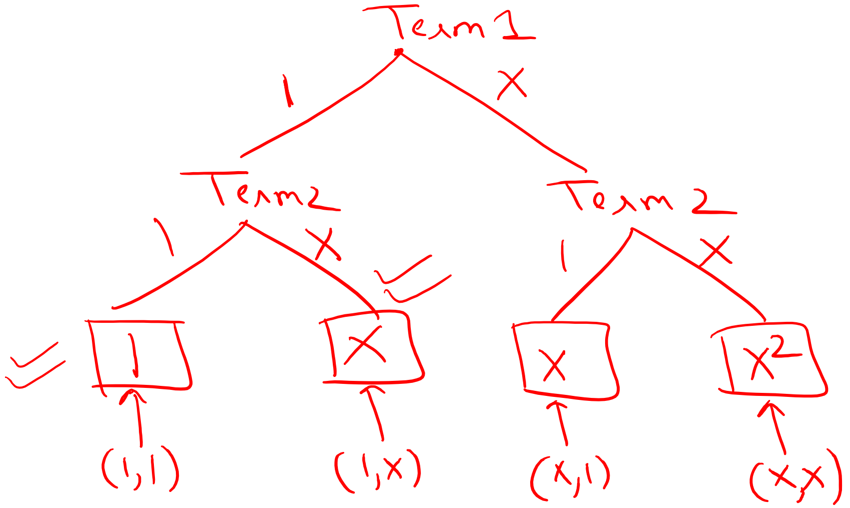
Probabilities !!!

Binomial Coefficients – Building insight

- $(1+x)^2 = 1 + 2x + x^2$

Given: $(1+x)^2 \rightarrow (1+x)\cdot(1+x) = 1 + x + x + x^2$
$= 1 + \underline{\underline{2}}x + x^2$

$(1+x)^2 \rightarrow \underset{\text{Term 1}}{(1+x)}\ \underset{\text{Term 2}}{(1+x)}$



Co-efficient of x = # ways to reach x from root node

$= 2$

ways to reach x

$=$ # ways to choose x once out of 2 terms $= \binom{2}{1}$

last class

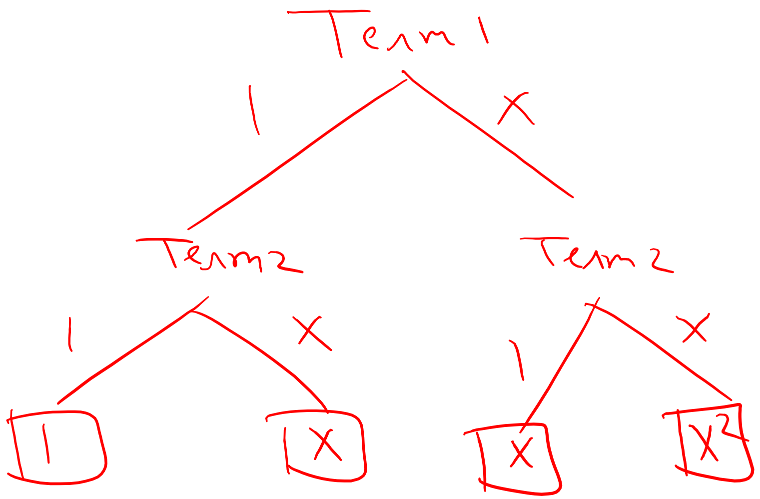
Binomial Coefficients

- $(1+x)^2 = 1 + 2x + x^2$

\# ways to reach $x = \binom{2}{1} = 2$

\# ways to reach $x^2 = \binom{2}{2} = 1$

\# ways to reach $1 = \binom{2}{2} = 1$



last class

Binomial Coefficients

- $(1+x)^3 = 1 + 3x + 3x^2 + x^3$

$$\underbrace{(1+x)}_{\text{Term1}}\ \underbrace{(1+x)}_{\text{Term2}}\ \underbrace{(1+x)}_{\text{Term3}}$$

Co-efficient of x

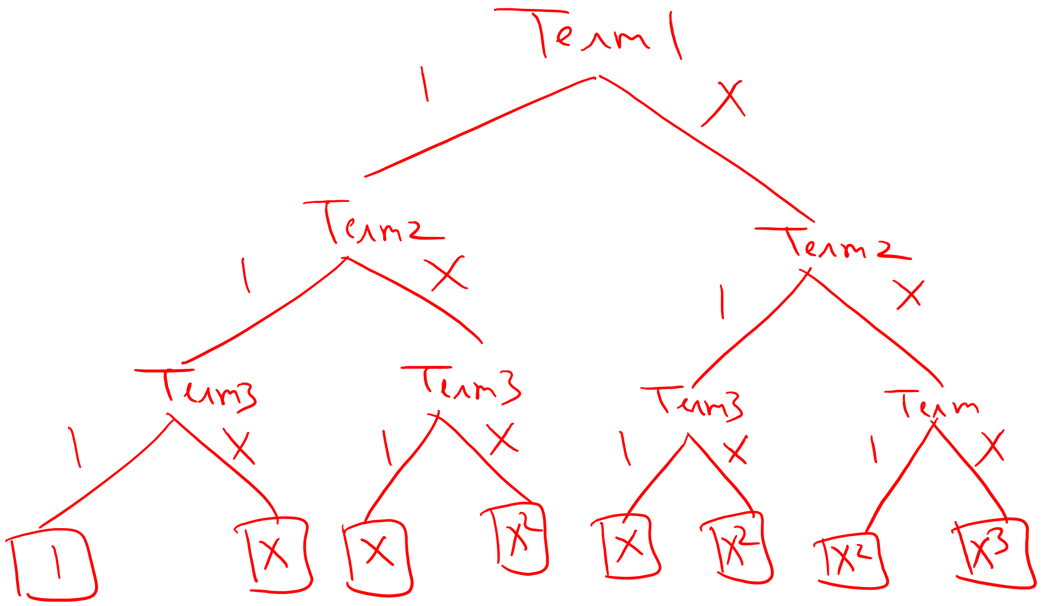
$=$ # ways to reach x

$= \binom{3}{1} = 3$

Co-efficient of x^2

$=$ # ways to reach x^2

$= \binom{3}{2} = 3$



last class

Binomial Coefficients

- $(1+x)^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$

C_n = # ways to reach $x^n = \binom{n}{n}$

C_{n-1} = # ways to reach $x^{n-1} = \binom{n}{n-1}$

$\vdots$

C_k = # ways to reach $x^k = \binom{n}{k}$ — Binomial Coefficients

$\vdots$

C_0 = # ways to reach $x^0 = \binom{n}{0} = 1$

last class

The Binomial Formula – Univariate Case

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

last class

The Binomial Formula – Multivariate Case

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

last class

The Multinomial Formula – 3 variables

$$(x+y+z)^n$$

$(x+y+z)^n = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-1} z - - - -$
$+ a_k x^a y^b z^c + - - -$

$x^a y^b z^c \rightarrow$ choice sequence must have a xs, b ys and c zs.

$\rightarrow$ any arrangement of a xs, b ys, c zs gives a valid way to get $x^a y^b z^c$

coefficient of $x^a y^b z^c$
$=$ # arrangements
$= \dfrac{n!}{a!\,b!\,c!}$

last class

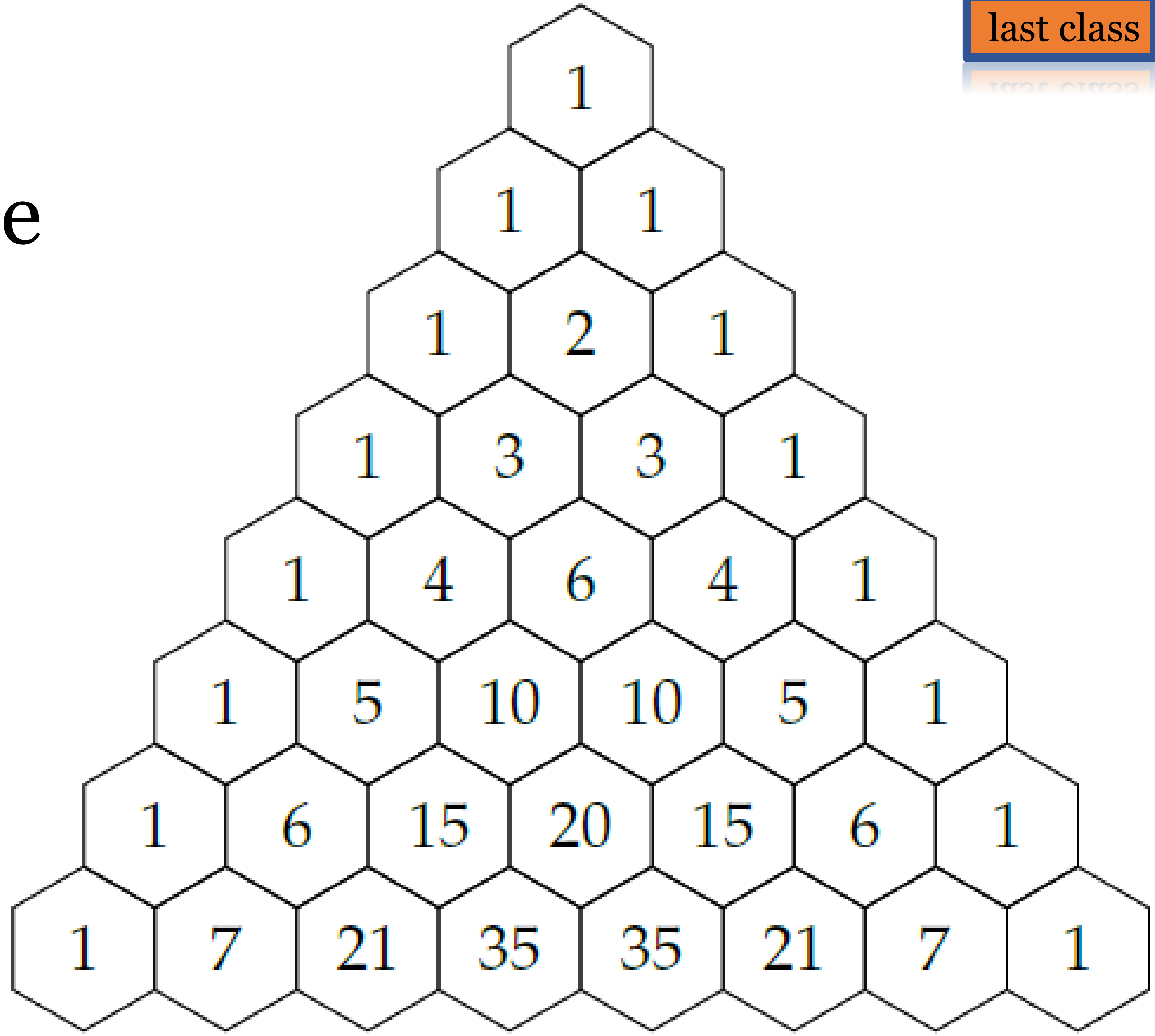
The Multinomial Formula – 3 variables

$$(x+y+z)^n = \sum_{k_1+k_2+k_3=n} \frac{n!}{k_1!\,k_2!\,k_3!} x^{k_1} y^{k_2} z^{k_3}$$

last class

Pascal’s Triangle

- 1. The entries on the border of the triangle are all 1.
- 2. Any entry not on the border is the sum of the two entries above it.
- 3. The triangle is symmetric. In any row, entries on the left side are mirrored on the right side.
- 4. The sum of all entries on a given row is a power of 2. (Check this!)



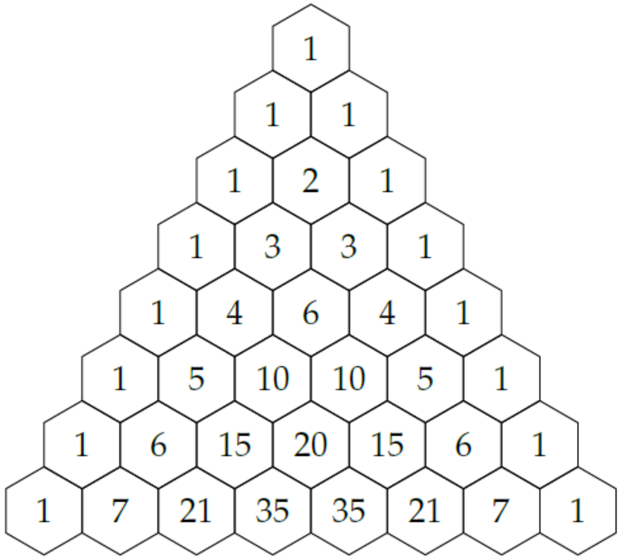
last class

Pascal’s Triangle

Line 4:

1, 4, 6, 4, 1

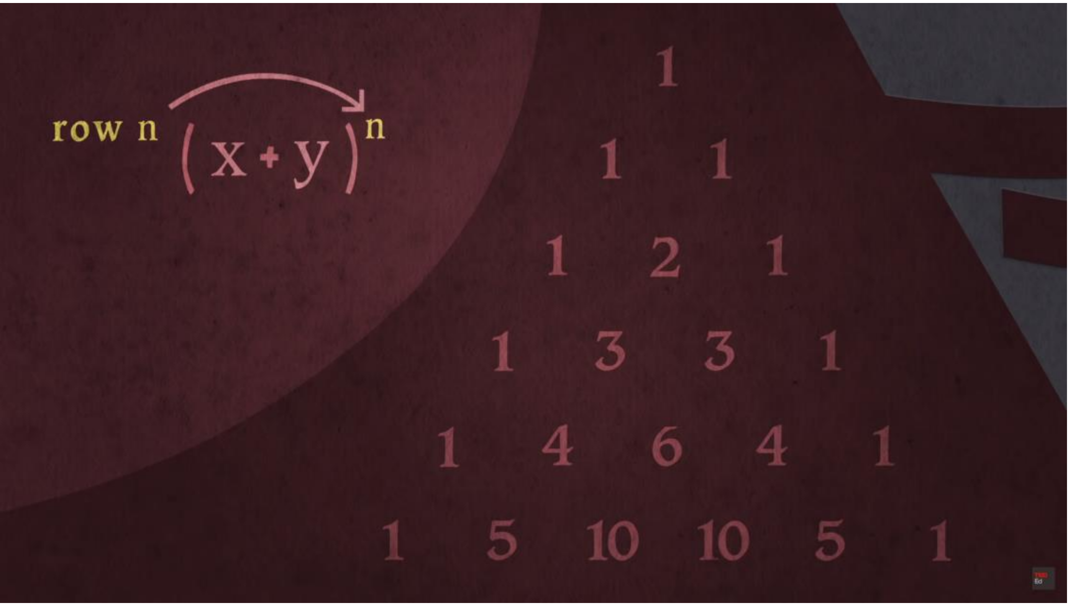
$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

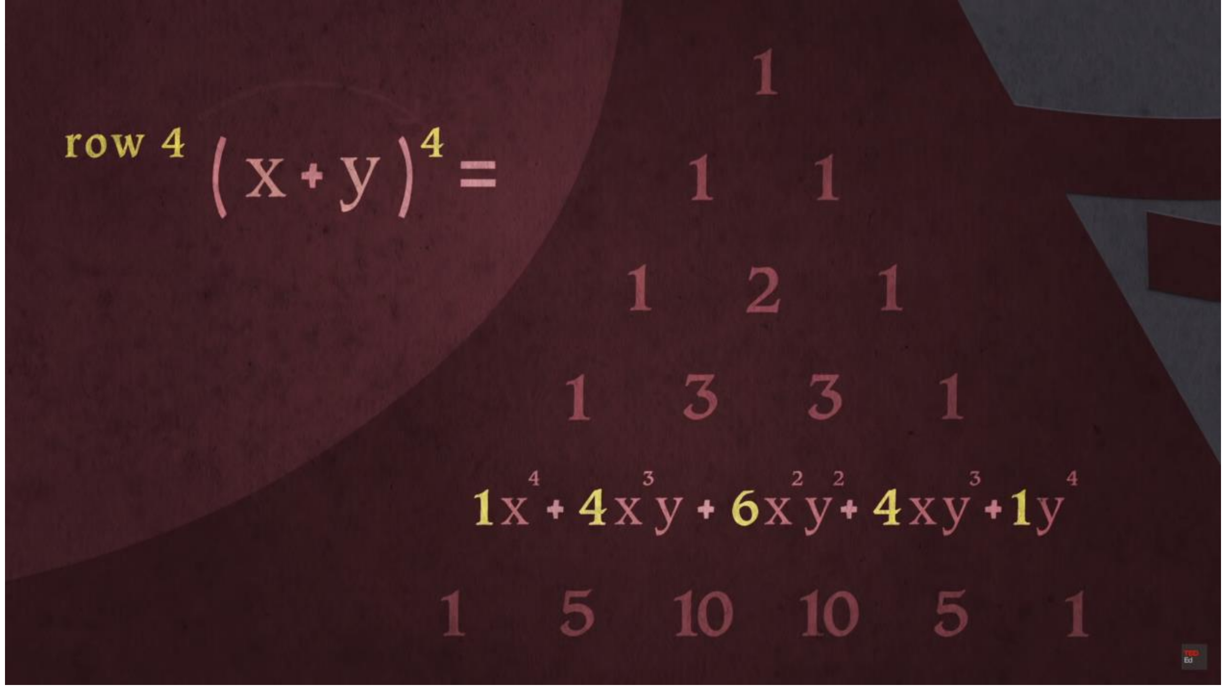


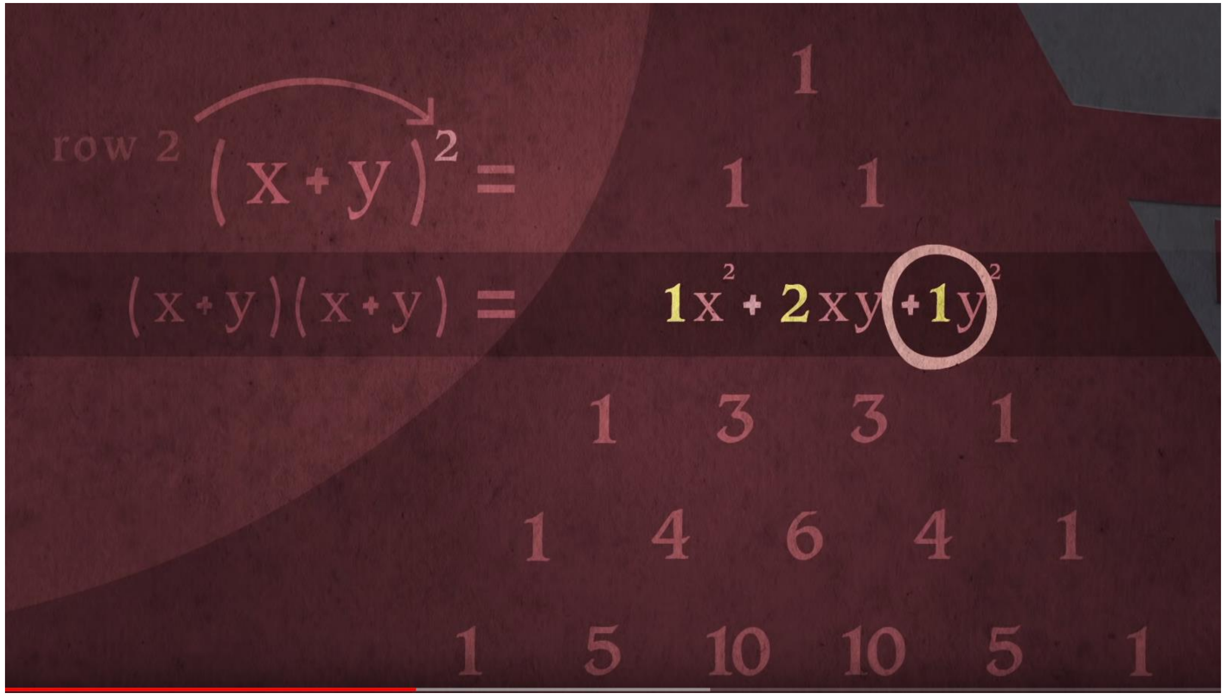
- Each entry in Pascal’s triangle is in fact a binomial coefficient.

- We can use Pascal’s triangle (and other counting methods we have learned) to prove binomial identities, i.e., equations that involve binomial coefficients

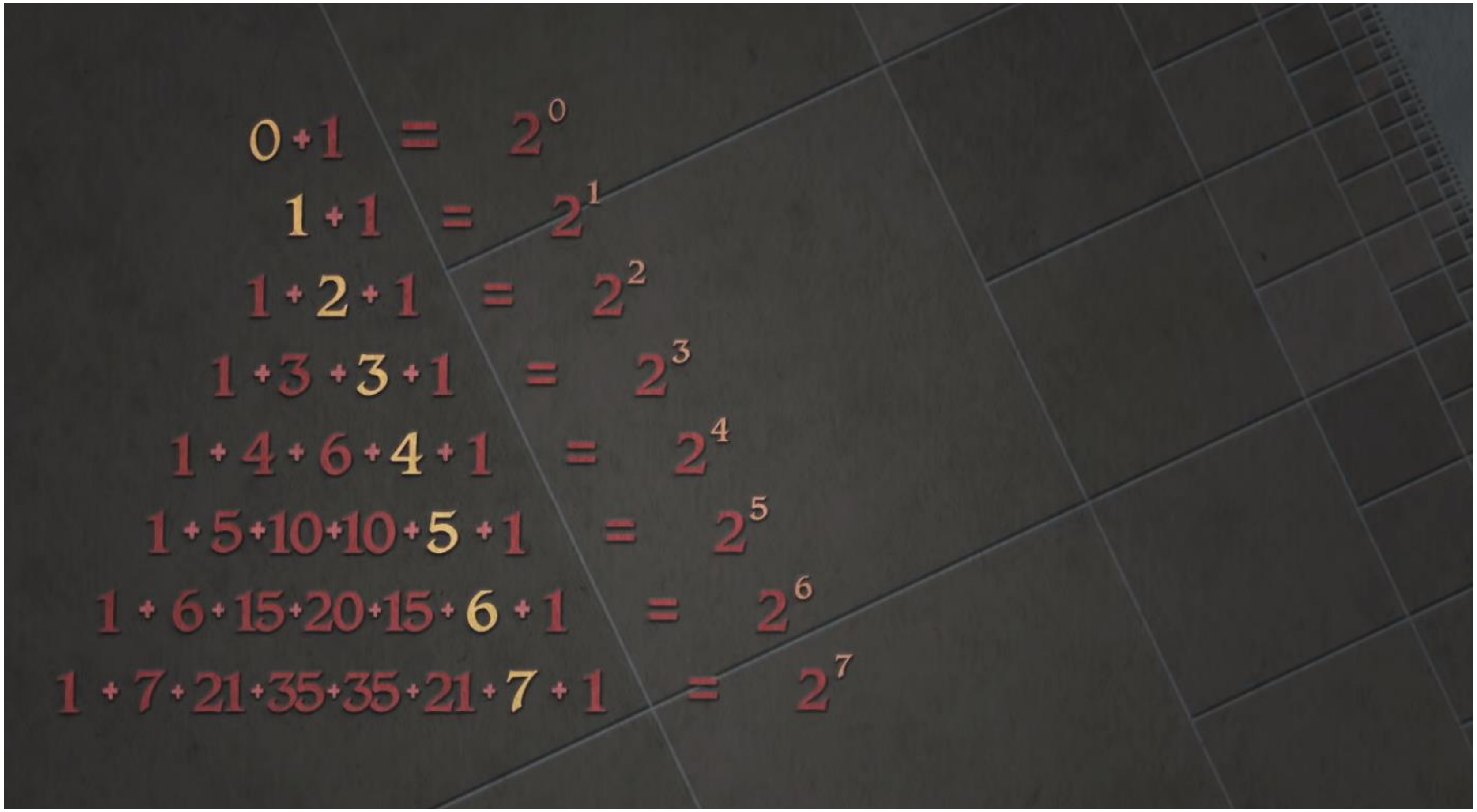
Pascal’s Triangle





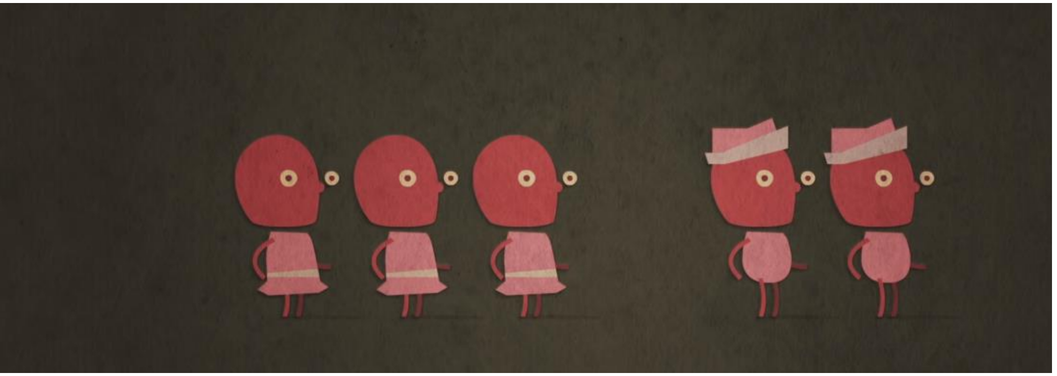


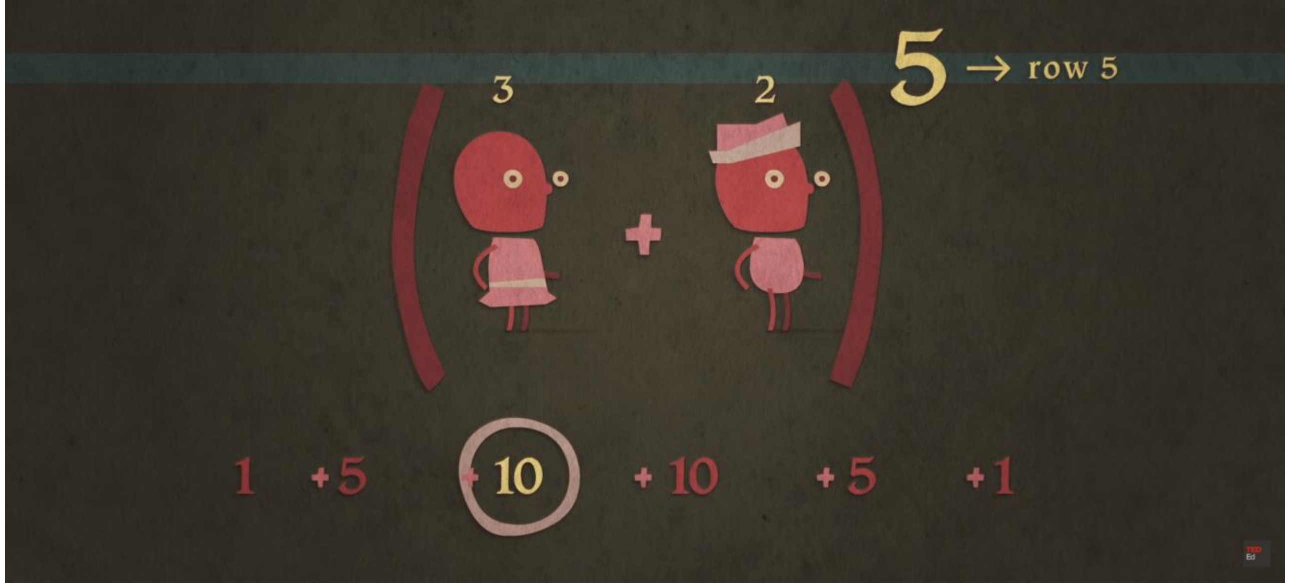
Pascal's Triangle

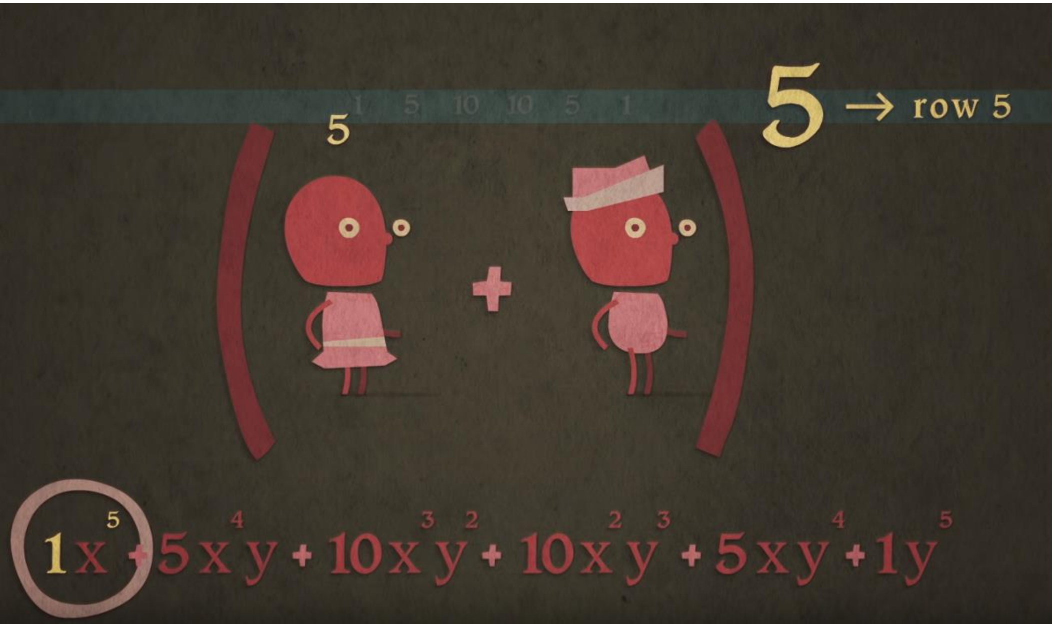


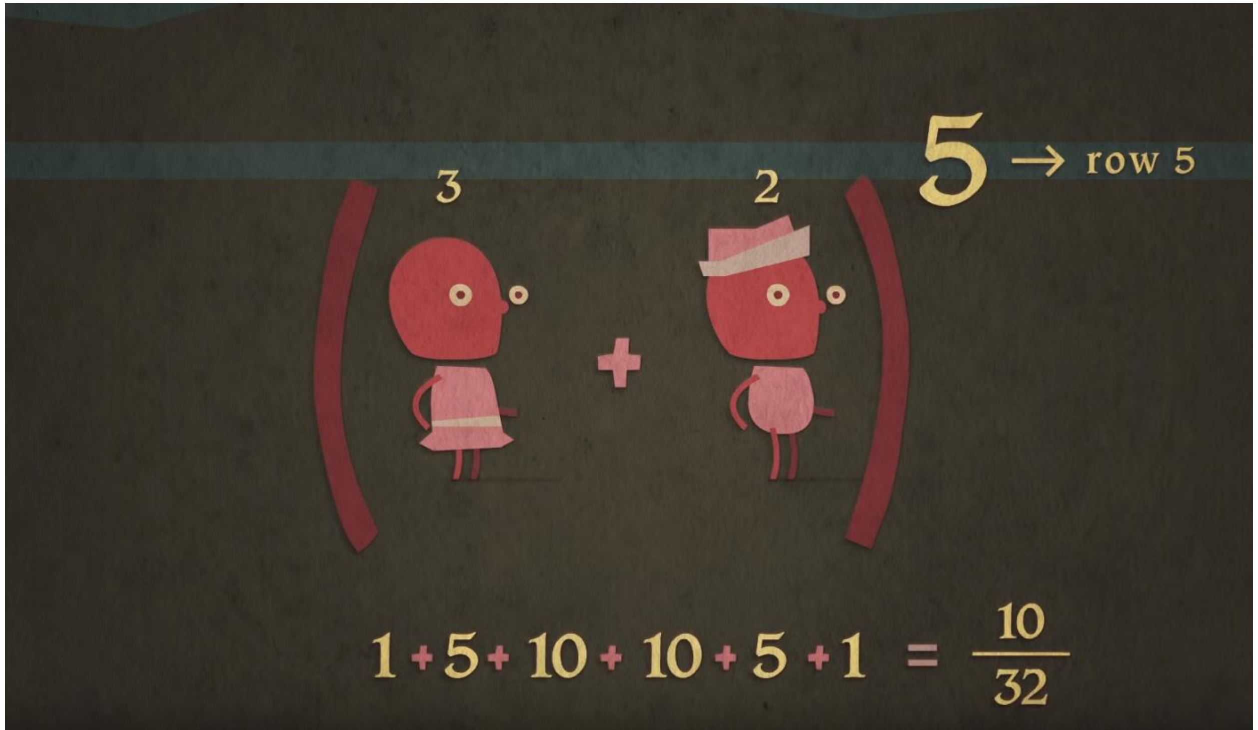
Look at Sierpinski Triangle…

Pascal’s Triangle









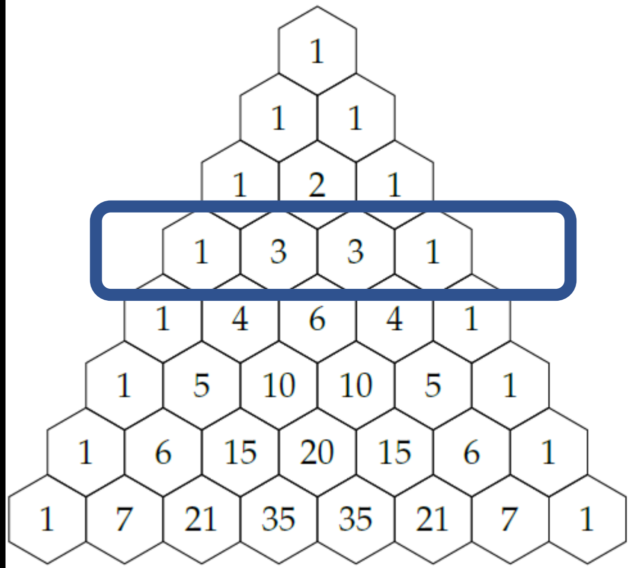
Binomial Theorem

$$(x-2)^3 =$$

$$1(x)^3(-2)^0 + 3(x)^2(-2)^1 +$$

$$3(x)^1(-2)^2 + 1(x)^0(-2)^3$$

$$\boxed{1 \quad 3 \quad 3 \quad 1}$$

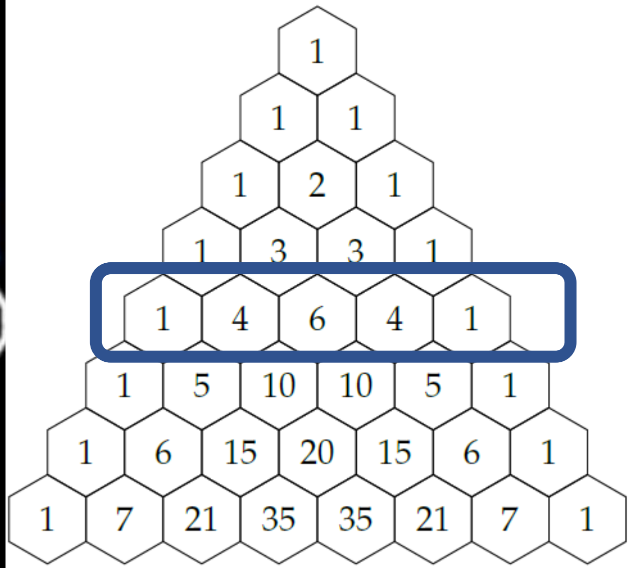


Binomial Theorem

$$(2x+3y)^4$$

$$1(2x)^4\cancel{(3y)^0} + 4(2x)^3(3y)^1 +$$

$$6(2x)^2(3y)^2 + 4(2x)^1(3y)^3 + 1(2x)^0(3y)^4$$



Take a Break

5 min

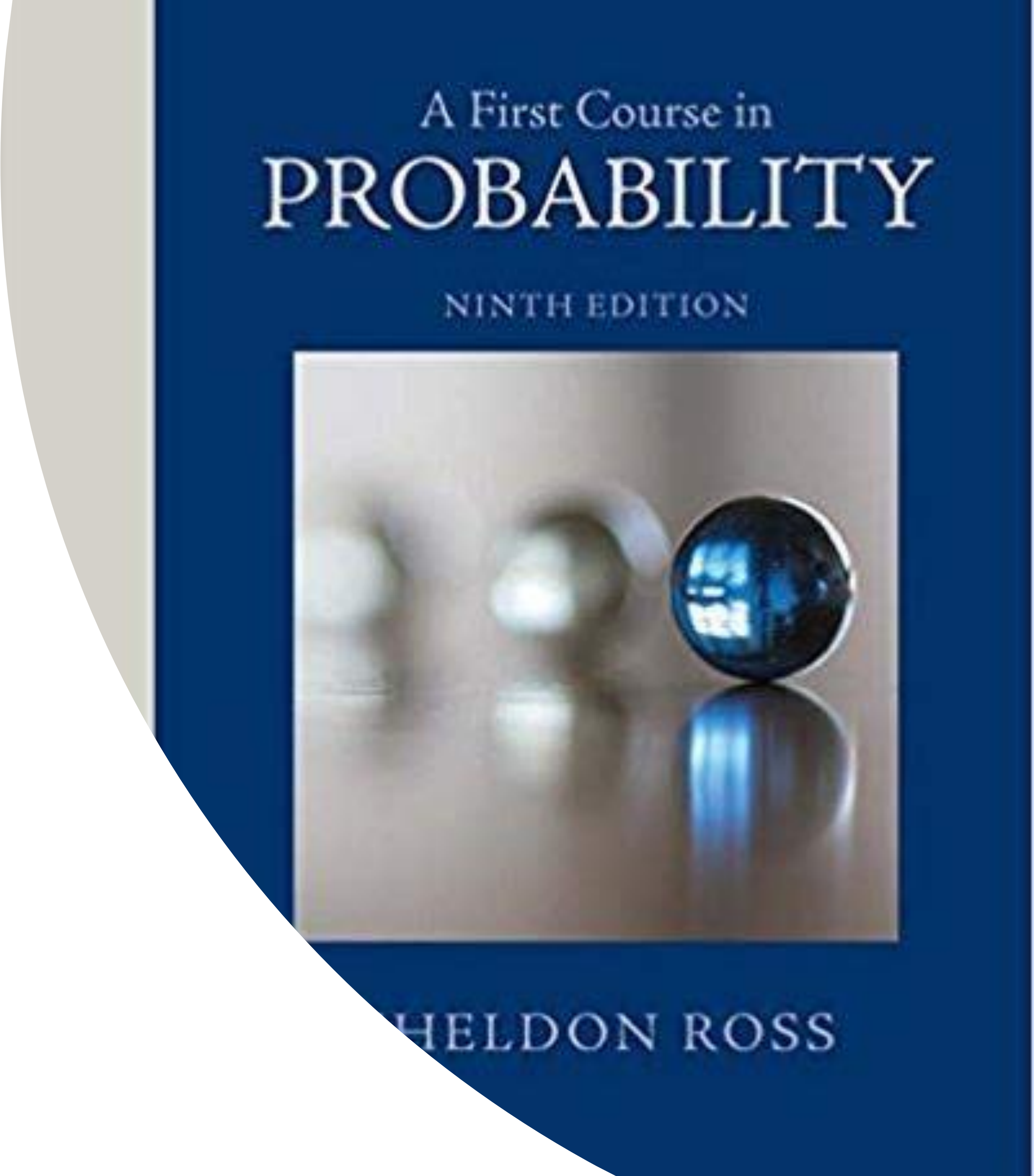
Today – Probabilities (!!!)

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Event
  - Outcome(s) that you are interested in understanding

Textbook #1

A First Course in Probability

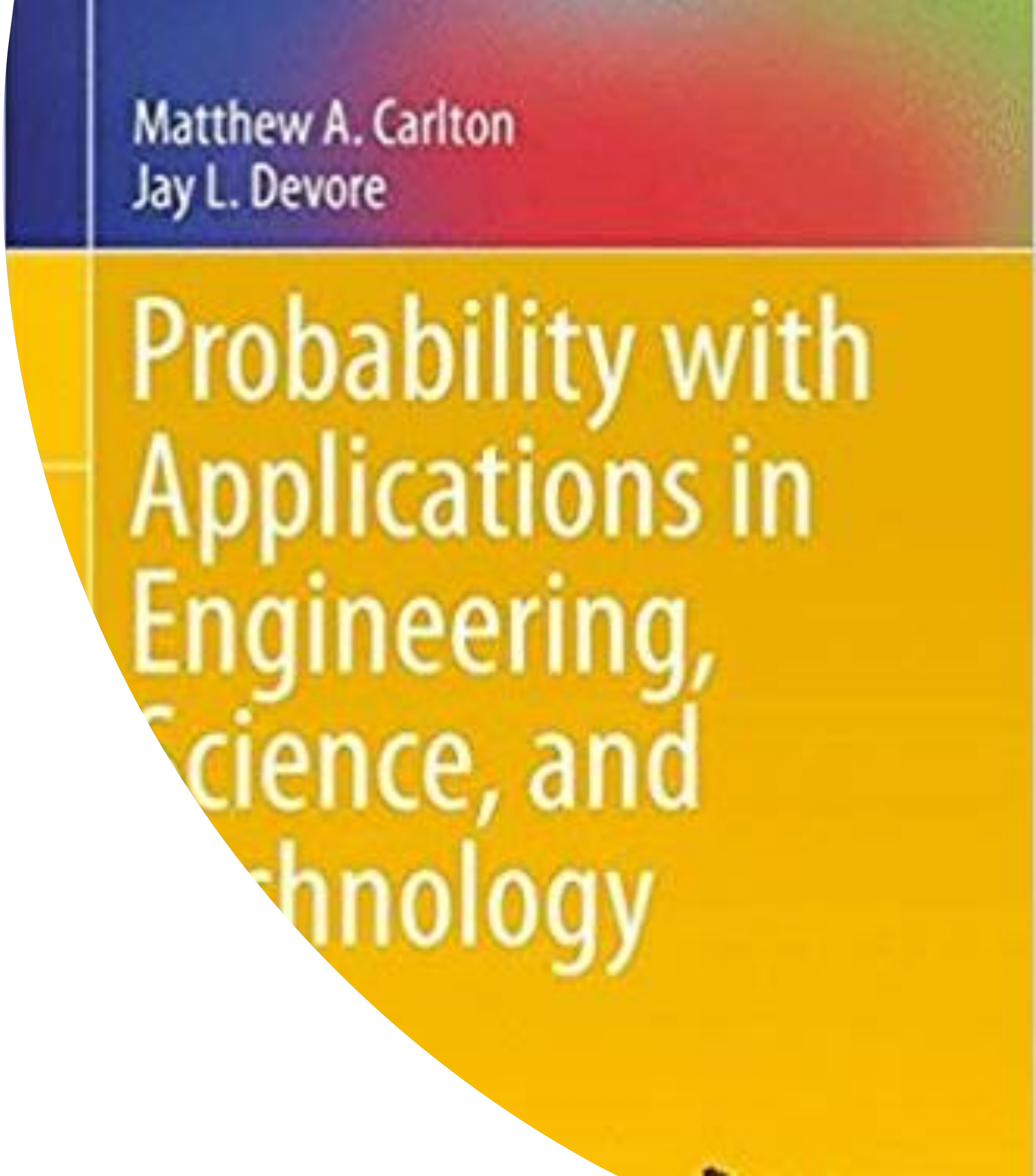
- S. Ross
- any edition



Textbook #2

Probability with Applications in Engineering, Science, and Technology

- M. A. Carlton and J. L. Devore
- Available for free through university library website.



Outline for this month

→ Basic building blocks

- Sample spaces and events
- Basics of probability

→ Intermediate

- Independence, conditional probability

→ Advanced

- Random variables, expectation, variance

Probabilities

- Experiment
  - Toss a fair coin 10 times

Probabilities

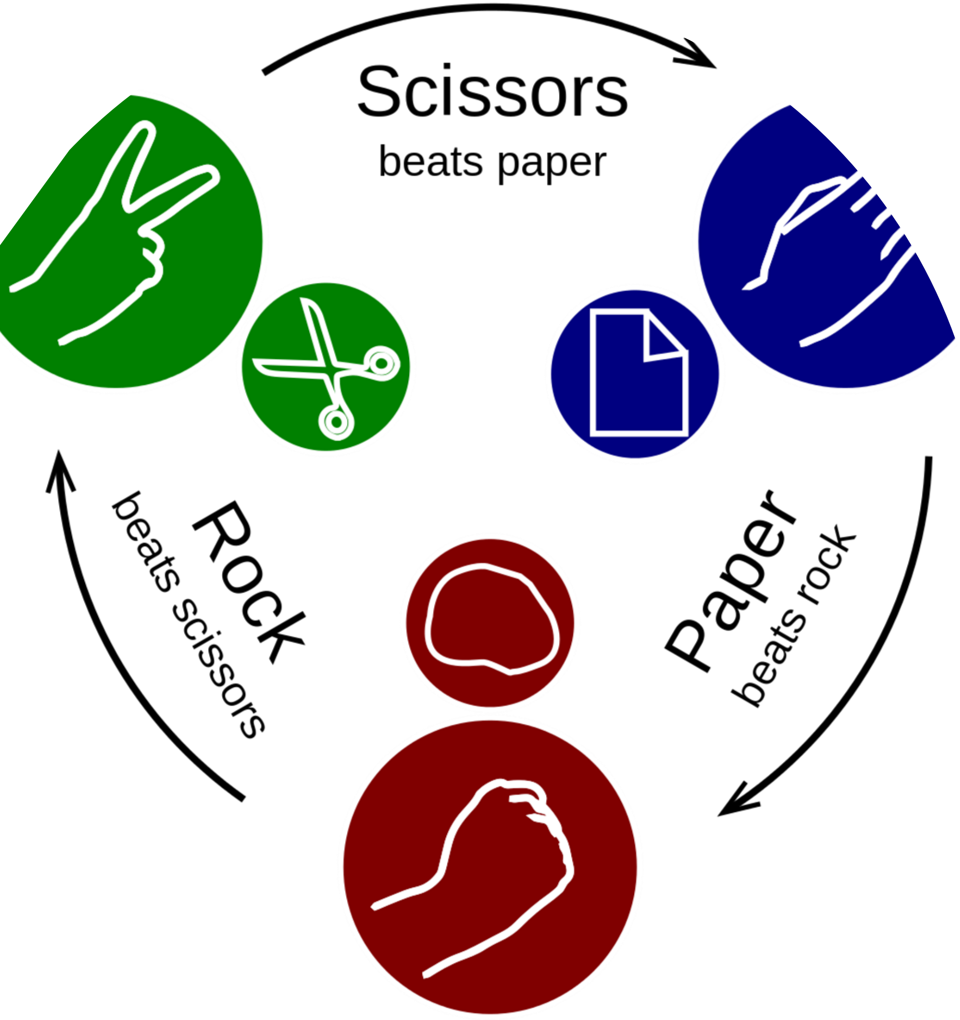
- Study of random/uncertain phenomena.
- Origins in gambling.
  - Pascal invented probability theory to come up with gambling strategies.
- Two dice are rolled 4 times. If (6,6) shows up I win. Else I lose. Should I play?

Probability's origin – The problem of points

…or the problem of the division of the stakes

- Two players are playing a best of 5 game
  - After 3 games, player 1 is leading 2 to 1
  - A fight breaks out and game cannot be finished
  - How should the prize money be divided?

led Blaise Pascal to the first explicit reasoning about what today is known as an *expected value*



Probability: Today's Applications

- Weather prediction
- Stock market prediction
- Inventory management
- Studying behavior of a virus
- Understanding rational behavior in economics
- Understanding the chance of failure of algorithms
- Cryptography
- Machine Learning

Probabilities – Real-life Example

- US population is ~350 million.
- Want to figure out if majority prefer Biden or Trump.

Probabilities – Real-life Example

independent of
350 million

Theorem:
Poll a random sample of 2000 people. Then, with probability > .99,

% preferring Biden over Trump = % in sample $\pm$ 2%

Probabilities

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment

Probability - Sample Space

- Consider an experiment whose outcome *is not predictable with certainty*.
- However, although the outcome of the experiment will not be known in advance, let us suppose that *the set of all possible outcomes is known*.
- This set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by S.

Probability – Sample Space Examples

- If the outcome of an experiment consists of the determination of the gender of a newborn child, then

$$S = \{g, b\}$$

- If the experiment consists of flipping two coins, then the sample space consists of the following four points,

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

- If the outcome of an experiment is the order of finish in a race among the 7 horses having post positions 1, 2, 3, 4, 5, 6, and 7, then

$$S = \{\text{all } 7! \text{ Permutations of } (1,2,3,4,5,6,7)\}$$

Probability – Sample Space Examples

- Toss a coin 10 times

$$\Omega = \left\{ \begin{array}{c} (H, H, H, \cdots, H) \\ (H, T, \cdots\cdots) \\ \\ (T, T, \cdots \cdots T) \end{array} \right\}, \quad |\Omega| = 2^{10}$$

Probability – Sample Space Examples

- Roll two dice

$$\Omega = \left\{ \begin{array}{l} (1,1), (1,2), \cdots (1,6) \\ (2,1), (2,2), \cdots (2,6) \\ \vdots \\ (6,1), (6,2) \cdots (6,6) \end{array} \right\}, \quad |\Omega| = 6 \cdot 6$$

Probability – Sample Space Examples

Experiment : people vote

$$\Omega = \left\{ \begin{array}{l} (H, H, \ldots H, T, \ldots T) \\ (T, T, \ldots \ldots T) \\ (H, H, H, T, \ldots T) \\ \vdots \end{array} \right\}, \quad |\Omega| = 2^{350\times10^6}$$

Probability – Sample Space Examples

- Toss a coin until you see a H

$$\Omega = \{ H, TH, TTH, TTTH, \ldots \}$$

$$|\Omega| = \infty$$

Probability

- Experiment
  - Toss a fair coin 10 times
- Sample Space (Ω)
  - All possible outcomes of the experiment
- Event
  - Outcome(s) that you are interested in understanding (or counting…)

Probability – Events Examples

Probability is the likelihood that an **event** will occur.

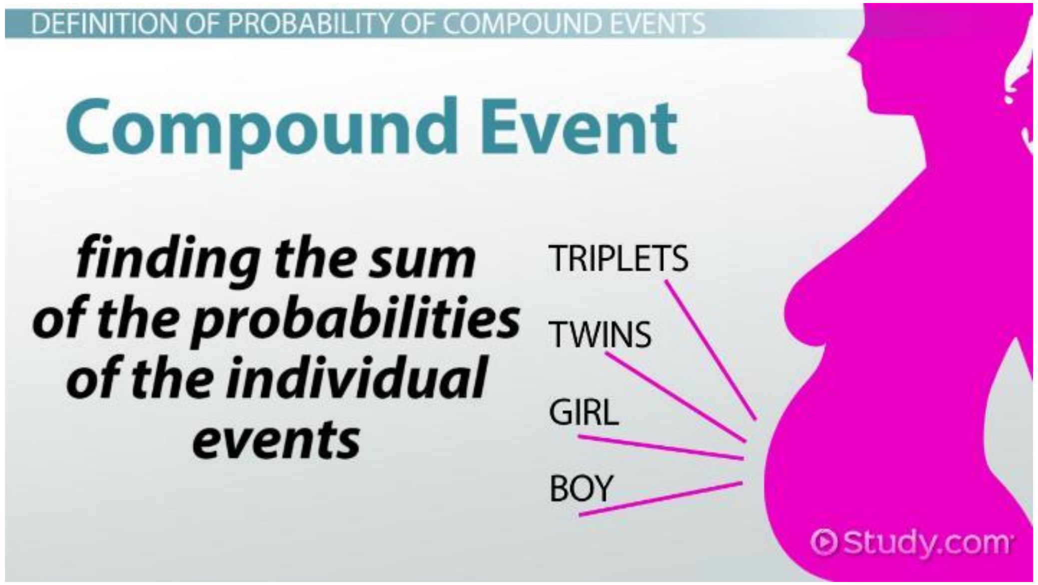
- Toss a coin 10 times

Any subset of Ω is an Event.

Events – Simple Event

- A **simple event** is an **event** where all possible outcomes are equally likely to occur.
- So the **probability** of **simple events** will have all possible outcomes equally likely to happen or occur.
- E.g., when you toss a coin, there are two possible outcomes – heads or tails, and the **probability** of heads or tails is equal.

Events - Compound Event



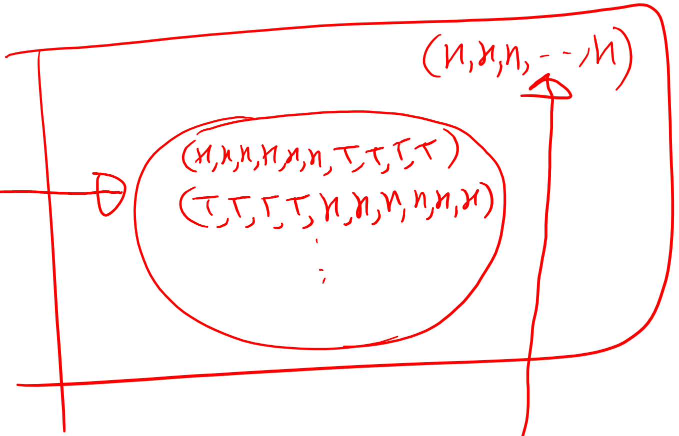
- A **compound event** is one in which there is more than one possible outcome.
- Determining the **probability** of a **compound event** involves finding the sum of the **probabilities** of the individual **events** and, if necessary, removing any overlapping **probabilities**.

Probability – Events Examples

- Toss a coin 10 times

Compound Event

Event 1: H appears 6 times



Event 2: only see Heads

Simple Event

Probability – Events Examples

- Roll two dice

$$\Omega = \left\{ \begin{array}{l} (1,1), (1,2), \ldots (1,6) \\ \vdots \\ (6,1), (6,2), \ldots (6,6) \end{array} \right\}$$

Simple Event : first die = 6, Second die = 5 $\rightarrow$ (6,5)

Compound Event : first die equals second die $\rightarrow$ (1,1), (2,2), (3,3), -- (6,6)

Probability – Events Examples

- Toss a coin until you see a H.

Simple Event: Get an H on first try.

Compound Event: Don't get an H on first try.