Sample Combinatorics Problems

1 10 points each

- How many different numbers can you make from the digits 11122337?
  Solution: $\frac{8!}{3!2!2!}$.
- Determine the coefficient of x^7 in $(-5x+2)^9$.
  Solution: $\binom{9}{7}(-5)^7 2^2$.
- Determine the coefficient of $x^{23}y^{17}z^{60}$ in $(x+y+z)^{100}$?
  Solution: $\frac{100!}{23!17!60!}$.
- How many non-negative integer solutions to equation $x_1+x_2+x_3+x_4=15$?
  Solution: $\binom{18}{3}$.
- Consider 10 digit ternary numbers (each digit is 0 or 1 or 2).
  (5 pts) How many 10 digit ternary numbers are there?
  Solution: 3^{10}.
  (10 pts) How many with at least one zero?
  Solution: $3^{10}-2^{10}$.
- In a survey on the chewing gum preferences of baseball players, it was found that
  - 22 like fruit.
  - 25 like spearmint.
  - 39 like grape.
  - 9 like spearmint and fruit.
  - 17 like fruit and grape.
  - 20 like spearmint and grape.
  - 6 like all flavors.
  - 4 like none.

  How many players were surveyed?
  Solution:
  Let x be the total number of playes surveyed. Let $A = \#$ playes who like fruit, $B = \#$ playes who like spearmint, $C = \#$ playes who like grape. The we have that $|A| = 22$, $|B| = 25$, $|C| = 39$, $|A\cap B| = 9$, $|A\cap C| = 17$, $|B\cap C| = 20$, $|A\cap B\cap C| = 6$ and $x - |A\cup B\cup C| = 4$. Using the inclusion/excluison formula

  $$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |B\cap C| - |A\cap C| + |A\cap B\cap C|$$

  we get that $x-4 = 22+25+39-9-17-20+6$. Hence $x = 50$.
- In the movie Cheaper by the Dozen, there are 12 children in the family.
  (a) Prove that at least two of the children were born on the same day of the week;
  Solution: There are 7 days in a week, hence by the pigeonhole at least $\lceil\frac{12}{7}\rceil = 2$ children must be born on the same day.
  (b) Prove that at least two family members (including mother and father) are born in the same month;

Solution: There are 12 months and 14 family members, hence by the pigeonhole at least $\lceil \frac{14}{12} \rceil = 2$ children must be born in the same month.

(c) Assuming there are 4 childrens bedrooms in the house, show that there are at least 3 children sleeping in at least one of them.

Solution: There are 12 children and 4 bedrooms and hence by the pigeonhole at least $\lceil \frac{12}{4} \rceil = 3$ children must be sleeping in the same room.

- Prove that 531!472! is a divisor of 1003!.

  Solution: $\frac{1003!}{531!472!} = \binom{1003}{531}$ = number of ways to choose 531 objects from 1003 distinct objects. Hence, $\frac{1003!}{531!472!}$ is an integer.

- The Fibonacci numbers, $F_0, F_1, F_2, \ldots$, are defined recursively by the equations $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n > 1$. Prove by induction that

  $$F_1^2 + F_2^2 + \ldots F_n^2 = F_n F_{n+1}$$

  for all positive integers $n \geq 1$.

  Solution: Base case: Since $F_1 = F_2 = 1$, we have $F_1^2 = F_1 F_2$. Hence the base case is true.

  Inductive step: Assume the statement is true for n, i.e., $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$. Consider $F_1^2 + F_2^2 + \ldots + F_{n+1}^2$. Using the induction hypothesis for n this can be simplified to $F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2}$. Hence if the statement is true for n then it is also true for $n + 1$.

- There are five points inside an equilateral triangle of side length 2. Show that at least two of the points are within 1 unit distance from each other.

  Solution: Draw another equilateral triangle inside the given triangle by joining the mid points of each side. This divides the triangle into 4 smaller triangles, each of side length 1. By pigeonhole principle, at least 2 of the 5 points must fall in the same smaller triangle and hence will be at a distance at most 1 from each other.

- Prove that $\binom{n+1}{r+1} = \sum_{k=r+1}^{n} \binom{k-1}{r}$.

  Solution: LHS = number of binary strings of length $n + 1$ with exactly $r + 1$ ones.

  Here is another way to count the number of binary strings of length $n+1$ with exactly $r+1$ ones. Let k be the position of the rightmost 1. Notice that k must range from $r + 1$ to n. For a give choice of k, r ones need to be chosen from $k - 1$ positions. This can be done in $\binom{k-1}{r}$ ways. Hence, the total number of binary strings = $\sum_{k=r+1}^{n} \binom{k-1}{r}$.

- Prove that $2^n - 1 = \sum_{k=1}^{n} 2^{k-1}$.

  Solution: LHS = number of binary strings of length n with at least one 0.

  Here is a another way to count. Let k be the position of the rightmost 0. Notice that k ranges from 1 to n. For each k, the $k - 1$ positions to the left of the rightmost 0 can be filled in 2^k ways. Hence, total number of strings = $\sum_{k=1}^{n} 2^{k-1}$.