CS206 Recitation Problem Sets Section 06

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1 Inclusion-exclusion principle

1. How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution:
Using the inclusion-exclusion principle for 2 sets, $|A \cup B| = |A| + |B| - |A \cap B|$.

We start by defining A as the set of positive integers not exceeding 1000 divisible by 7, B the set of positive integers not exceeding 1000 divisible by 11, and $A \cap B$ the set of positive integers not exceeding 1000 divisible by both 7 AND 11.

Knowing the size of these three sets, we can calculate $A \cup B$ which is the set of positive integers not exceeding 1000 divisible by either 7 OR 11. The sizes of the sets are as follows:

$$|A| = \lfloor \frac{1000}{7} \rfloor = 142$$
$$|B| = \lfloor \frac{1000}{11} \rfloor = 90$$
$$|A \cap B| = \lfloor \frac{1000}{7 \times 11} \rfloor = \lfloor \frac{1000}{77} \rfloor = 12$$

Therefore, by inclusion exclusion principle, we get

$$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12$$

2. A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian.

   Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian.

   If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution: We use the inclusion-exclusion principle for 3 sets

$$|R \cup F \cup S| = |R| + |F| + |S| - |R \cap F| - |R \cap S| - |S \cap F| + |R \cap F \cap S|$$

where sets R,f,S represent sets of students taking languages Russian, French, and Spanish, respectively.

Since we are asked to find number of students who have taken all three, we need to calculate the size of set $|R \cap F \cap S|$, or the intersection of the three sets.

Also, sizes for all other sets are given in the problem, thus we get:

$$\begin{aligned} |R \cap F \cap S| &= |R \cup F \cup S| - |R| - |F| - |S| + |R \cap F| + |R \cap S| + |S \cap F| \\ &= 2092 - 114 - 879 - 1232 + 14 + 23 + 103 \end{aligned}$$

2 Pigeonhole principle

1. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

   Solution: Based on pigeonhole principle, There are 26 letters as boxes and 30 student as objects, so $\lceil \frac{30}{26} \rceil = 2$

2. Show that there are at least 250 four digit numbers whose digits all sum to the same value.

   Solution:
   The number of four digit numbers is $9 \times 10 \times 10 \times 10 = 9000$: there are 9 options for the first digit (since it cannot be 0) and 10 options for the remaining three digits.

   The number of possible sums of the digits of four digit numbers is 36: the lowest possible sum is 1, the highest possible sum is 36 (since that's the sum when every digit is 9), and all the possible sums are integers.

   Viewing the four digit numbers as objects and the possible sums as boxes, the pigeonhole principle implies that at least one box will end up with at least $\lceil \frac{9000}{36} \rceil = 250$ objects—in other words there is at least one value which is the digit sum of at least 250 four digit numbers.