combra.cs.rutgers.edu

Quality is not an act
it is a habit

206
Discrete Structures II

Konstantinos P. Michmizos

Computational Brain Lab

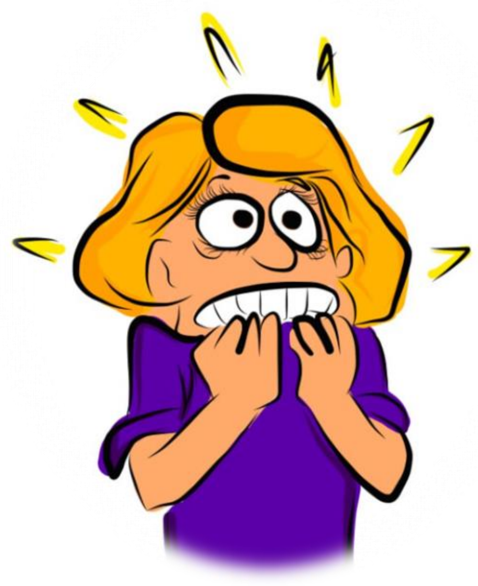
Computer Science | Rutgers University | NJ, USA

Lecture 15 | Inclusion/Exclusion & Pigeonhole Principle | Thursday 10/21 2021

Announcements

- **Quiz 3**
  - Next Tuesday/Thursday
- Assignment 1
  - Next Thursday

Quiz 3 – Next Tuesday/Thursday



- More time (35 minutes)
  - + more questions ☺
- What will cover
  - Permutations with/out repetition
  - Combinations
  - Pirates Problem
  - Pirates Problem
  - Pirates Problem
    - Have you seen the extra Pirates problems?

General Hint – Revisited

For each problem

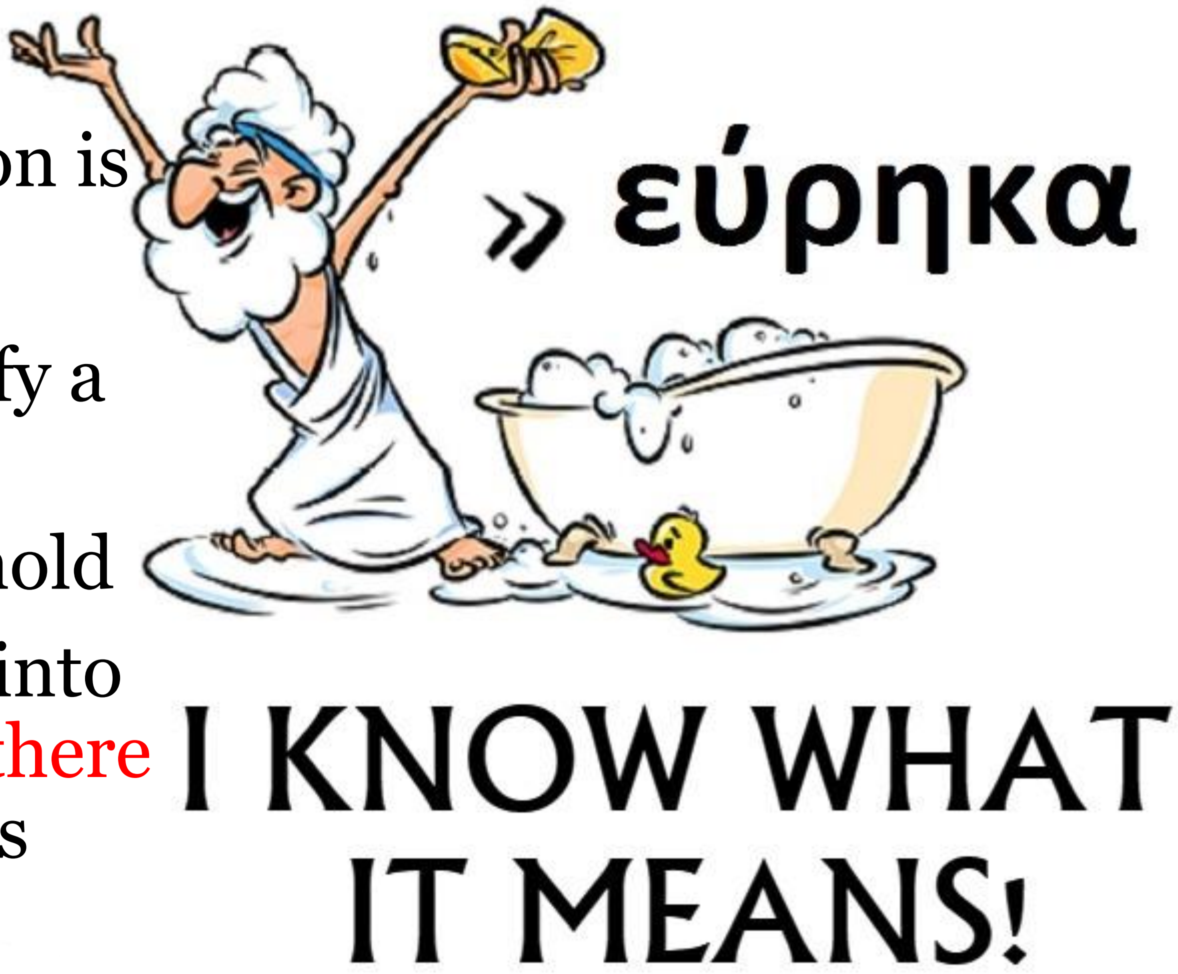
(1) Fully understand what the question is

(2) Fully understand what you know

(3) Based on the previous two, identify a method

(4) Make sure that the assumptions hold

(5) Turn the wording of the problem into the input to your method. Typically, there is a "key" thought that will unlock this part of the solution for you.



So Far

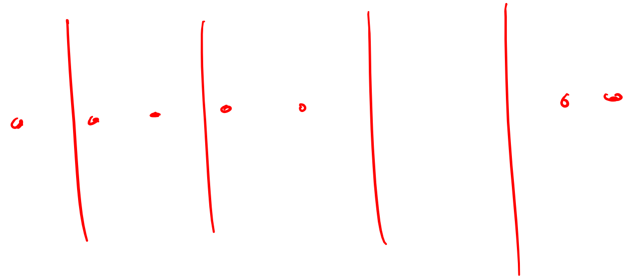
- ~~Proofs/Induction~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- ~~Permutation/Combinations~~
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

Combinations with Repetitions

- 5 distinct pirates want to divide up 20 identical, indivisible bars of gold. How many ways to divide the loot?

Count all sequences of (a,b,c,d,e) such that a+b+c+d+e=20

a = # pirate 1 gets
b = # 2 gets
c = # 3 gets
d = 4
e = 5 gets



answer = all ways to arrange 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)} = \binom{24}{4}$$

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_5 = 20$
  - $x_1, x_2, \ldots, x_5 \geq 0$

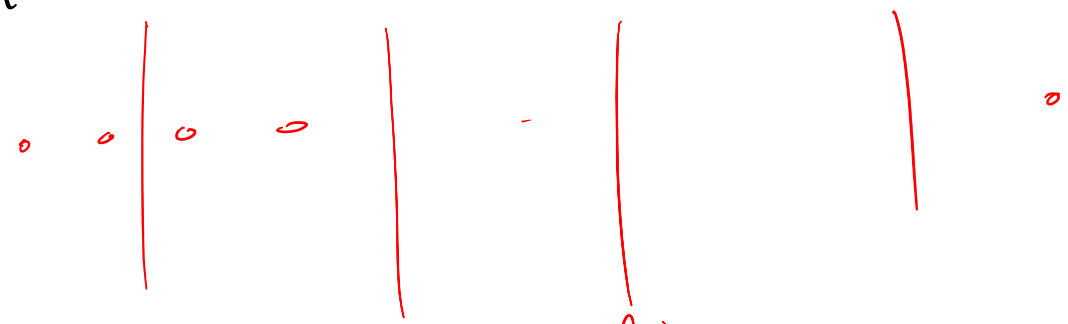
$(x_1, x_2, x_3, x_4, x_5)$ such that $\sum x_i = 20$

$\Rightarrow$ all arrangements of 20 dots and 4 lines

$$= \frac{(24)!}{(20!)(4!)}$$

Combinations with Repetitions

- How many integer solutions to the following equation?
  - $x_1 + x_2 + \cdots + x_k = n$
  - $x_1, x_2, \ldots, x_k \geq 0$



k-1 lines

→ all arrangements of n dots and k-1 lines

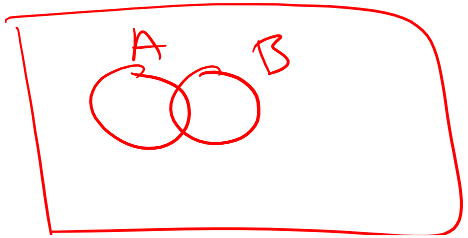
→ $\dfrac{(n+k-1)!}{n!\,(k-1)!} = \dbinom{n+k-1}{k-1}$

New Stuff...

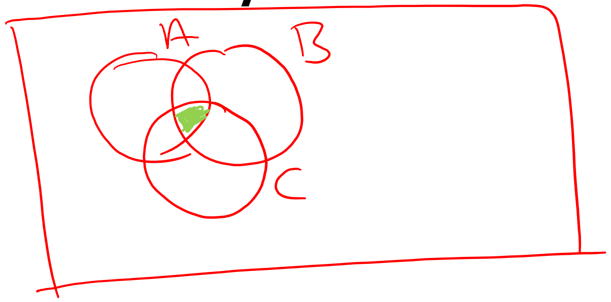
Inclusion/Exclusion

Sum Rule:
If A and B are disjoint sets, then $|A \cup B| = |A| + |B|$

- What if A and B are not disjoint? $|A \cup B| = ?$



Inclusion/Exclusion for 3 sets



$|A \cup B \cup C|$

$= |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

$$|A \cup B \cup C| = \underbrace{|A| + |B| + |C|}_{\text{Include}} \underbrace{- |A \cap B| - |B \cap C| - |A \cap C|}_{\text{Exclude}} + \underbrace{|A \cap B \cap C|}_{\text{Include}}$$

Direct Proof

Inclusion/Exclusion for 3 sets

$|A \cup B \cup C|$, Let $X = B \cup C$

$= |A \cup X| = |A| + |X| - |A \cap X|$ → follows from formula for 2 sets

$|X| = |B \cup C| = |B| + |C| - |B \cap C|$

$|A \cap X| = |A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)|$ → Apply formula for 2 sets

$= |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|$

$= |A \cap B| + |A \cap C| - |A \cap B \cap C|$

Hence,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Inclusion/Exclusion for 4 sets

$|A \cup B \cup C \cup D|$, $X = B \cup C \cup D$

$|A \cup X| = |A| + |X| - |A \cap X|$

$|X| = |B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |C \cap D| - |B \cap D| + |B \cap C \cap D|$

$|A \cap X| = |A \cap (B \cup C \cup D)| = |(A \cap B) \cup (A \cap C) \cup (A \cap D)|$

use formula for 3 sets

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |B \cap C \cap D| + |A \cap C \cap D| + |A \cap B \cap D| - |A \cap B \cap C \cap D|$$

General Inclusion/Exclusion

$$|A_1 \cup A_2 \cdots \cup A_n| = |A_1| + |A_2| + \cdots |A_n| \rightarrow n \text{ terms}$$

$$- |A_1 \cap A_2| - |A_2 \cap A_3| - \cdots \rightarrow \binom{n}{2} \text{ terms}$$

$$+ |A_1 \cap A_2 \cap A_3| + \cdots \rightarrow \binom{n}{3} \text{ terms}$$

$$- |A_1 \cap A_2 \cap A_3 \cap A_4| - \cdots \rightarrow \binom{n}{4} \text{ terms}$$

$$\vdots$$

$$(-1)^{n+1} |A_1 \cap A_2 \cap A_3 \cdots \cap A_n| \rightarrow \binom{n}{n} = 1 \text{ term}$$

General Inclusion/Exclusion

- In the set S={1,2,.....100} how many multiples of 6 or 7?

General Inclusion/Exclusion

- Solutions to $x + y + z = 15$ with $x \leq 3$ and $y \leq 4$? $(x,y,z) \geq 0$

General Inclusion/Exclusion

- Solutions to $x + y + z = 15$ with $x \leq 3$ and $y \leq 4$? $x, y, z \geq 0$

$A_1 = \#$ solutions with $x \leq 3$

$A_2 = \#$ solutions with $y \leq 4$

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$|A_1| =$

General Inclusion/Exclusion

$x, y, z \geq 0$

- Solutions to $x + y + z = 15$ with $x \leq 3$ and $y \leq 4$?

$|A_2|$ = # solutions to $x+y+z=15$ and $y \leq 4$

= (all solutions) − (solutions with $y \geq 5$)

$= \binom{17}{2} - \binom{12}{2}$

General Inclusion/Exclusion

$x, y, z \geq 0$

- Solutions to $x + y + z = 15$ with $x \leq 3$ and $y \leq 4$?

Hence, $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$$= \binom{17}{2} - \binom{13}{2} + \binom{17}{2} - \binom{12}{2}$$

$$- \binom{17}{2} + \binom{8}{2}$$

$$= 20$$

General Inclusion/Exclusion

- A group of 3 rabbits is playing outside their individual burrows when they are surprised by an eagle.
- Each rabbit escapes down a random hole. One rabbit per hole.
- How many ways to reorganize while avoiding their own hole.

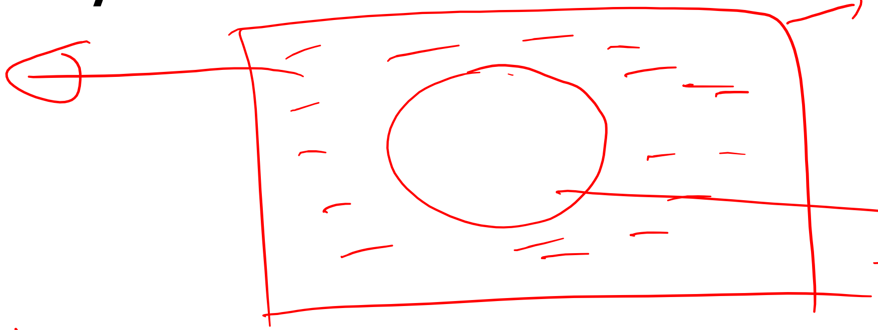
$A_1 =$ all ways to reorganize such that rabbit 1 goes to its own hole

$A_2 =$ all ways to " rabbit 2 goes to its own hole

$A_3 =$ all ways to " rabbit 3 goes to its own hole

General Inclusion/Exclusion

What we want



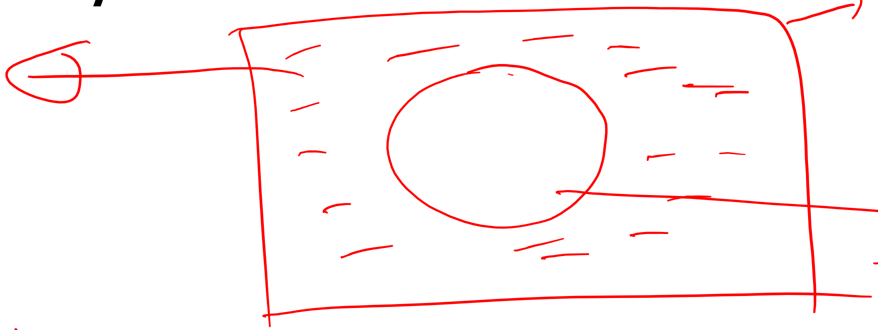
all ways to reorganize

$A_1 \cup A_2 \cup A_3$ = all outcomes where some rabbit goes to its own hole

$$\text{answer} = \text{all ways} - |A_1 \cup A_2 \cup A_3|$$

$$= 3! - |A_1 \cup A_2 \cup A_3|$$

General Inclusion/Exclusion



$$\text{answer} = \text{all ways} - |A_1 \cup A_2 \cup A_3|$$

$$= 3! - |A_1 \cup A_2 \cup A_3|$$

$|A_1| = 2!$, $|A_2| = 2!$, $|A_3| = 2!$

$|A_1 \cap A_2| = 1$, $|A_2 \cap A_3| = 1$, $|A_3 \cap A_1| = 1$

$|A_1 \cap A_2 \cap A_3| = 1$

Hence answer $= 3! - (2+2+2-1-1-1+1) = 2$

General Inclusion/Exclusion

- A group of n rabbits is playing outside their individual burrows when they are surprised by an eagle.
- Each rabbit escapes down a random hole. One rabbit per hole.
- How many ways to reorganize while avoiding their own hole.

$A_1 =$ all outcomes when rabbit 1 goes to own space

$\vdots$

$A_i =$ " rabbit i "

$\vdots$

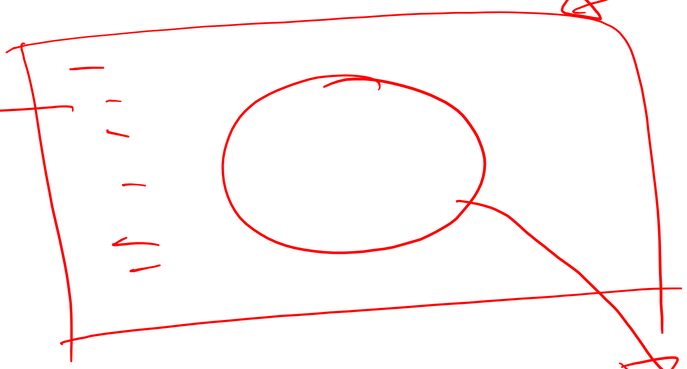
$A_n =$ " rabbit n "

General Inclusion/Exclusion

answer $= n! - |A_1 \cup \cdots A_n|$

$$|A_1 \cup A_2 \cdots \cup A_n|$$
$$= |A_1| + |A_2| + \cdots |A_n|$$
$$- |A_1 \cap A_2| - |A_2 \cap A_3| - \cdots$$
$$+ |A_1 \cap A_2 \cap A_3| + \cdots$$
$$\vdots$$
$$(-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|$$

all ways



$A_1 \cup A_2 \cdots \cup A_n$ = all ways where someone goes to its own hole