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Strive not to be a success, but rather to be of value — Albert Einstein

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Discrete Structures II

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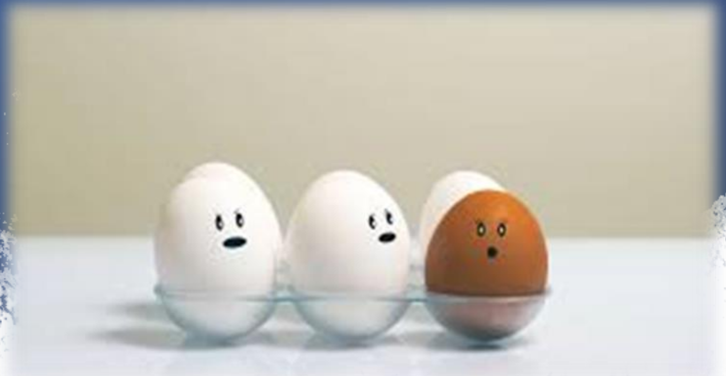
So Far

- ~~Sets / Functions~~
- ~~Proofs~~
- ~~Sum Rule~~
- ~~Partition Method~~
- ~~Difference Method~~
- ~~Bijection Method~~
- ~~Product Rule~~
- ~~Generalized product rule~~
- Permutation/Combinations
- Inclusion-Exclusion / Pigeonhole Principle
- Combinatorial Proofs and Binomial Coefficients

Difference between Permutations and Combinations



So far...

The **difference between combinations and permutations** is in

ordering

- With **permutations** we care about the order of the elements, whereas with **combinations** we don't care.

Examples:

- Permutation: Find a locker "combo" is 12345; Cellphone PIN is 5432
- Combination: Pick 5 students from a 180-student audience

So far...

Find 4-digit Permutations

of the numbers 2,3,4,5

____ ____ ____ ____

Find 4-digit Permutations

of the numbers 2,3,4,5

So far...

For the third position, we have two numbers left

$$\underline{4} \cdot \underline{3} \cdot \underline{2} \; \underline{\quad}$$

There is one number left for the last position

$$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$$

So far...

Permutations with Repetition

___ ___ ___ ___

- What if I now want to find the total number of permutations involving the numbers 2, 3, 4, and 5
- but want to include orderings such as 5555 or 2234 *where not all of the numbers are used, and some are used more than once*?

So far...

Permutations with Repetition

$$\underline{4} \cdot \underline{4} \cdot \underline{4} \cdot \underline{4} = 4^4 = 256$$

- What if I now want to find the total number of permutations involving the numbers 2, 3, 4, and 5
- but want to include orderings such as 5555 or 2234 *where not all of the numbers are used, and some are used more than once*?

So far…

Choosing a subset (a.k.a. Combinations)

_____ _____ _____ _____ _____

- *How many different 5-card hands can be made from a standard deck of cards?*
- In this problem the order is irrelevant since it doesn't matter what order we pick the cards.
- We'll begin with five lines to represent our 5-card hand.

So far...

Choosing a subset

$\underline{52} \cdot \underline{51} \cdot \underline{50} \cdot \underline{49} \cdot \underline{48}$

311,875,200 *permutations*

- *How many* ***<u>different</u>*** *5-card hands can be made from a standard deck of cards?*
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- *How many different 5-card hands can be made from a standard deck of cards?*
- In this problem the order is irrelevant since it doesn't matter what order we select the cards.
- We'll begin with five lines to represent our 5-card hand.
- **That's permutations, not combinations**
- To fix this we need to **divide by the number of hands that are different permutations but the same combination**

So far…

Choosing a subset

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

- That's permutations, not combinations.
- To fix this we need to divide by the number of hands that are different permutations but the same combination.

- This is the same as saying *how many different ways can I arrange 5 cards?*

So far...

Choosing a subset

$$\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$

- So the number of five-card hands combinations is:

Rewriting with Factorials

$$\frac{52!}{47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47} \cdot \cancel{46} \cdot \ldots \cdot \cancel{2} \cdot \cancel{1}}{\cancel{47} \cdot \cancel{46} \cdot \ldots \cdot \cancel{2} \cdot \cancel{1}}$$

- With a little ingenuity we can rewrite the above calculation using factorials.
- We know 52! = 52•51•50•…•3•2•1, but we only need the products of the integers from 52 to 48. How can we isolate just those integers?
- We'd like to divide out all the integers except those from 48 to 52. To do this divide by 47! since it's the product of the integers from 47 to 1.

So far...

Rewriting with Factorials

$$\frac{52!}{5!47!}$$

- Make sure to divide by **5!** to get rid of the extra permutations:

There we go!

So far...

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left

So far...

Combinations Formula

$$\binom{n}{k} = C^n_k = {}_nC_k$$

- Different Annotations

So far...

Permutations Formula

$$P_k^n = \frac{n!}{(n-k)!}$$

- The formula for permutations is similar to the combinations formula, except we needn't divide out the permutations, so we can remove k! from the denominator:

New stuff

Permutations

- A permutation of n objects is an ordering of the objects.

- The number of permutations of n distinct elements

$$n \cdot (n-1) \cdot (n-2) \cdots (1) = n!$$

$$P_k^n = \frac{n!}{(n-k)!}$$

Permutations

- A permutation of n objects is an ordering of the objects.

- How many different permutations of a deck of 52 cards?

answer = $52 \cdot 51 \cdot 50 \cdots 1 = 52!$

$$P^n_k = \frac{n!}{(n-k)!}$$

Permutations

- How many ways to assign 100 passengers to 100 seats?

answer = 100!

$$P^{n}_{k} = \frac{n!}{(n-k)!}$$

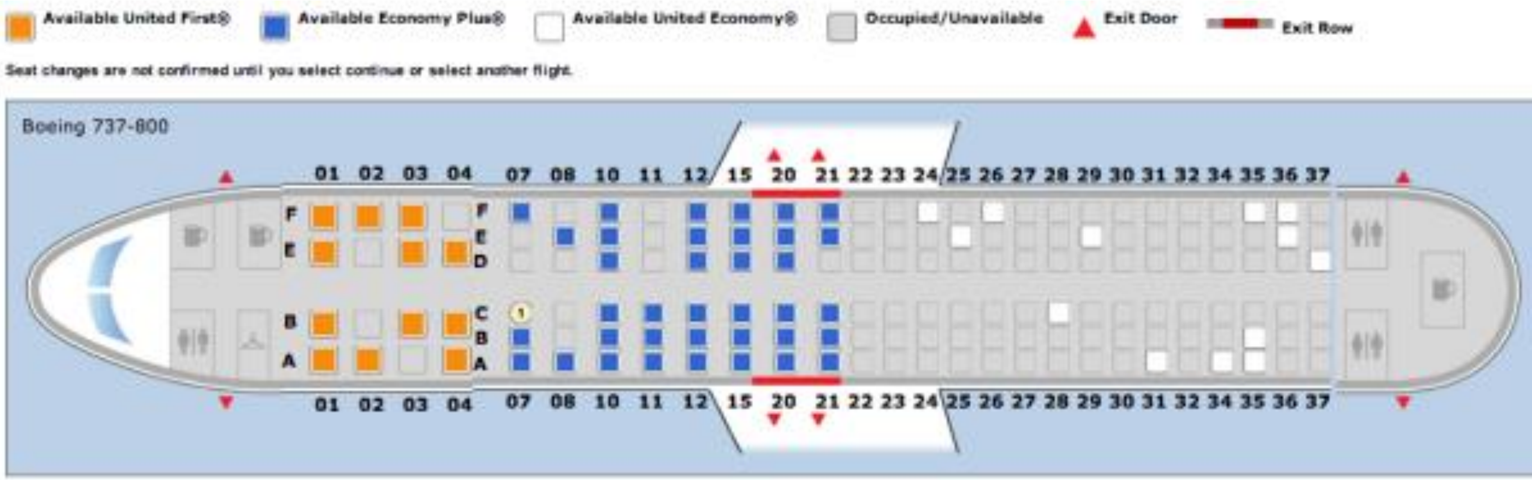
Permuting r out of n objects

- How many ways to assign 100 passengers to 20 first class seats?

100 99 98 ___ ___ 81

S1 S2 S20

answer = (100 · 99 · 98 · ... · 81) = $\frac{100!}{80!}$



$$P^n_k = \frac{n!}{(n-k)!}$$

Permutations Formula – One more time..

- Permuting r out of n distinct objects. $\underline{\underline{{}^{n}P_{r}}}$

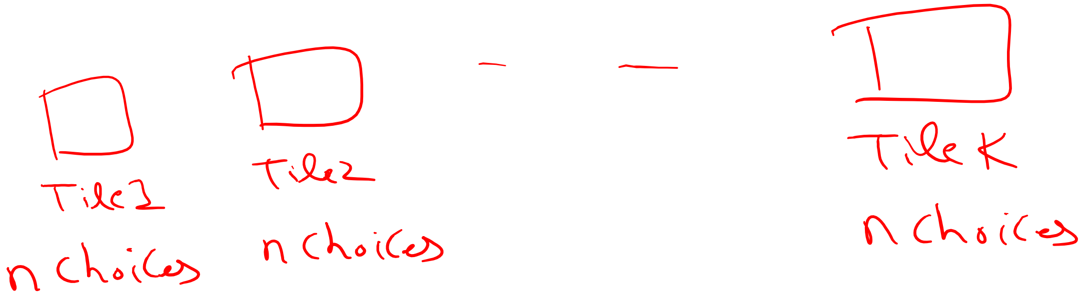
$$\underset{P1}{\underline{n}}\quad \underset{P2}{\underline{n-1}}\quad \underline{n-2}\quad \underline{}\quad \underset{Pr}{\underline{n-r+1}}$$

$$\text{answer} = n\cdot(n-1)\cdot(n-2)\cdot\ \ldots\ (n-r+1) = \frac{n!}{(n-r)!}$$

$$P_{k}^{n} = \frac{n!}{(n-k)!}$$

Repetitions

- Have n colors. Want to paint k tiles. How many ways?
  - Can reuse colors any number of times.



answer $= n^K$

So far we have seen 2 types of Permutations

- Permuting r out of n distinct objects.
  - Without repetition $\rightarrow$ ${}^nP_r = \frac{n!}{(n-r)!}$
  - With repetition $\rightarrow$ n^r

Example

- How many sequences of 7 letters are there (hint: 26 letters)?

$\underset{\text{choices}}{\underline{26}}$ _ _ _ _ _ $\underline{26}$

answer = 26^7

- Questions to ask:
  - Does order matter?
    - If yes, we can use the product rule (→ Permutation Formula)
  - Is repetition allowed?
    - This determines the number of options per "position"

One more Example

- If 10 horses race, how many orderings of the top 3 finishers are there?

$$\frac{10!}{7!} = {}^{10}P_3$$

Product Rule

Summary

- If one event can occur in m ways, a second event in n ways and a third event in r, then the three events can occur in $m \times n \times r$ ways.

- Example

Erin has 5 tops, 6 skirts and 4 caps from which to choose an outfit. In how many ways can she select one top, one skirt and one cap?

Solution: Ways = $5 \times 6 \times 4$

Product Rule – with Repetition

If one event with n outcomes occurs r times with repetition allowed, then the number of ordered arrangements is n^r

- Example

What is the number of arrangements if a die is rolled

(a) 2 times? 6×6

(b) 3 times? $6 \times 6 \times 6$

(c) r times? $6 \times 6 \times 6 \times 6 \times \ldots\ldots = 6^r$

Product Rule – Adv'ed Repetition Problems

- How many different car number plates are possible with 3 letters (hint: 26 letters) followed by 3 digits?

Solution: $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 263 \times 103$

- How many of these number plates begin with ABC

Solution: $1 \times 1 \times 1 \times 10 \times 10 \times 10 = 10^3$

- In how many ways can 6 people be arranged in a row?

Solution : $6 \text{ x } 5 \text{ x } 4 \text{ x } 3 \text{ x } 2 \text{ x } 1 = 6!$

- How many arrangements are possible if only 3 of them are chosen?

Solution: $6 \text{ x } 5 \text{ x } 4 = 120$

Permutations

- Distinctly ordered sets are called permutations (arrangements). The number of permutations of n distinct objects taken k at a time is given by:

$$P^n_k = \frac{n!}{(n-k)!}$$

n = number of distinct objects

k = number of positions

Permutations Formula – Remember!

$$P^n_k = \frac{n!}{(n-k)!}$$

The formula for permutations is similar to the combinations formula, except we needn't divide out the permutations, so we remove k! from the denominator

Combinations Formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

If we have *n* objects and we want to choose *k* of them, we can find the total number of combinations by using the formula on the left

Take a Break

5 min

Permutations - Examples

A maths debating team consists of 4 speakers.

- In how many ways can all 4 speakers be arranged in a row for a photo?

Solution : **$4 \times 3 \times 2 \times 1 = 4!$ or $^4P_4$**

- How many ways can the captain and vice-captain be chosen?

Solution : **$4 \times 3 = 12$ or $^4P_2$**

Permutations

<u>A flutter on the horses</u>
There are 7 horses in a race.

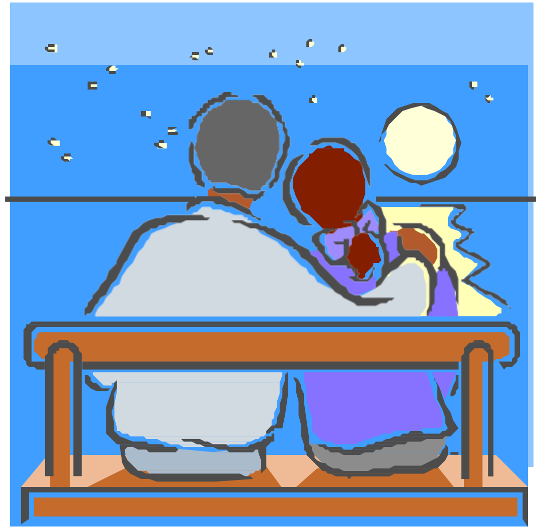
- In how many different orders can the horses finish?

Solution : $7x6x5x4x3x2x1 = 7! \text{ or } {}^7P_7$

- How many trifectas (1st , 2nd and 3rd) are possible?

Solution : $7x6x5 = 210 \text{ or } {}^7P_3$

Permutations with Restrictions



In how many ways can 5 boys and 4 girls be arranged on a bench if

- there are no restrictions?

  Solution : **$9!$ or $^{9}P_{9}$**

- boys and girls alternate?

Solution : A boy will be on each end

BGBGBGBGB = $5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1$

$= 5! \times 4!$ **or** $^{5}P_{5} \times {}^{4}P_{4}$