CS206 Assignment
<u>Due Date</u>: Thursday, November 18, 2021, 3:00 pm

Score: /160 *pts*

Name: ________________________________

NetID: ______________

Section No.: ______________

Instructions: You are expected to solve all problems on your own. You may discuss a limited number (no more than 3) of problems with others, but not earlier than a week before the deadline. (This should give you some to think, not to procrastinate). Discussions are meant to help you understand the problem better. You should not ask someone for solutions. Any evidence of cheating needs to be reported to the university and could be subject to disciplinary action. Be prepared to come and explain your solution to us in person if we feel the need. You can discuss up to 3 problems in groups of 1, 2 or 3. In the end you must write up your own solution and write names and RUIDs of students you collaborated with. For each problem, justify your answer. Any scientifically correct answer, given reasonable assumptions, will be accepted as correct. Write formal proofs when asked for. Submit a single .pdf file containing your homework. The pdf could be a latex/docx generated file or a scan of a handwritten document. If we can't read your text, we have to assume that it is wrong. Late submissions will **not** be accepted, especially when a request for extension is received 48 hours prior to the deadline.

1. (10 *pts*) True/False

   (a) When using inclusion-exclusion to find the size of the union of 5 sets, you need to subtract the size of the triple intersections.

   Solution: False. The number of sets is actually irrelevant here. When using inclusion-exclusion to find the size of the union of sets $A_1, A_2, ..., A_n$, you first sum the sizes of all the sets individually then subtract the sizes of all the double intersections, then add the sizes of all the triple intersections and so on, alternating the sign each time.

   (b) The number of nine digit numbers that start with the digit 3 is greater than the number of nine digit numbers in which no digit is repeated.

   Solution: True. There are 10^8 nine digit numbers that start with 3 (one choice for the first digit and 10 choices for all other digits) and $9 \cdot 9 \cdot 8 \cdot 7 \cdots 3 \cdot 2$ nine digit numbers in which no digits are repeated (nine choices for the first digit since it can't start with 0, nine choices for the second digit since you can't repeat the first digit, and so on). It is straightforward to confirm that the first product is larger.

2. (10 *pts*) Using the Pigeonhole Principle, show that in every set of 100 integers, there exist two whose difference is a multiple of 37.

   Solution: There are 37 possible remainders from $0, 1, \ldots, 36$ when divided by 37. Hence, among any 100 integers, there must exist a and b that have the same remainder. Then $a - b$ will be a multiple of 37.

3. (10 *pts*) Your TAs are helping the students to form homework groups, so they have every student fill out a form listing all of the other students who they would be willing to work with. There are 251 students in the class, and every student lists exactly 168 other students who they would be willing to work with. For any two students in the class, if student A puts student B on their list, then student B will also have student A on their list. Using the Pigeonhole Principle, show that there must be some group of 4 students who are all willing to work with one another.

   Solution: Number the students from 1 to 251. Let A be the set of students willing to work with student 1 and B be the set of students who are not willing to work with 1. We have $|A| = 168$ and $|B| = 82$. Consider students i and j that are in set A and are willing to work with one another. Such a pair must exist as each student has to choose 168 other people and there are not enough options outside A. Now outside A there are only 83 students. Hence, both i and j need to pick at least 84 other students inside the remaining part of A that consists of 166 students. Hence, by the pigeonhole principle, there exists k in A that is picked by both i and j. Then 1, i, j, k are all willing to work with one another..

4. (10 *pts*) How many ways are there to arrange 20 books on a bookcase with 3 shelves? Assume, as in real life, that books are distinguishable and that the order of the books on each shelf matters.

   Solution: To put 20 books on 3 shelves, you can first put the books in order and then decide how many go on each shelf. There are P(20,20) = 20! ways to put the books in order and, by stars and bars, $\binom{20+3-1}{20}$ ways to choose how many books go on each shelf. 20 So the final answer is $20!\binom{22}{20}$.

5. (10 pts) How many ways are there to arrange the letters a, b, c, d, e, and f such that a is not directly followed by either b or c? For example, "$abdefc$" and "$acdefb$" are both invalid, but "$adbcef$" is valid.

   Solution: We will first count the number of ways to arrange the six letters so that a is followed directly by either b or c. To find the number of arrangements where a is directly followed by b, we can consider ab as a single letter. Thus there are 5! such arrangements (since we can just put the 5 letters ab, c, d, e, f in any order). And by the same reasoning, the number of arrangements where a is directly followed by c is the same: 5!. Since it is impossible for a to be directly followed by b and c at the same time, there are $2 \cdot 5!$ arrangements where a is directly followed by either b or c. Since there are 6! total ways to arrange the six letters when there are no restrictions, the answer to the original question is $6! - 2 * 5!$

6. (a) (10 pts) How many ways are there to arrange a deck of 52 cards so that for each suit, all cards of that suit are together? Recall that we have 13 ranks (Ace to King), and 4 suits (spades, hearts, diamonds and clubs).

   Solution: One method is to imagine first choosing which order to put the 4 suits in and then, for each suit, choosing how to arrange the 13 cards in the suit. There are 4! ways the 4 suits could be put arranged and for each suit there are 13! ways to arrange the cards in that suit (and we have to repeat this 4 times, once for each suit). So there are $4!(13!)^4$ ways to arrange the cards.

   (b) (10 pts) Using cards of all 13 ranks and 4 suits, how many different Full House sets can be formed? In contrary to real games, we take suits into consideration, i.e., $(\clubsuit 8, \heartsuit 8, \heartsuit Q, \spadesuit Q, \diamondsuit Q)$ and $(\heartsuit 8, \spadesuit 8, \clubsuit Q, \spadesuit Q, \diamondsuit Q)$ are different Full House sets. The order of the cards, however, doesn't matter.

Figure 1: Full House is a card set that contains three cards of one rank and two cards of another rank

   Solution: To count the number of full houses, we may first pick a denomination for the "3 of a kind": there are 13 choices. Pick 3 cards from this denomination (out of 4 suites): there are $\binom{4}{3} = 4$ choices. Next pick a denomination for the "2 of a kind": there are 12 choices left (different from the "3 of a kind"). Pick 2 cards from this denomination: there are $\binom{4}{2} = 6$ choices. So in total there are $13 * 4 * 12 * 6 = 3744$ possible full houses.

7. Suppose you have 8 boxes labelled 1 through 8 and 16 indistinguishable red balls. How many ways are there to put the balls into the boxes if:

   (a) (10 pts) No odd box can be empty.

   Solution: To find all the ways to put the balls into the boxes, we can first put a ball into every odd box–leaving $16 - 4 = 12$ balls–and then distribute the remaining balls into the 8 boxes in any way. At this point, it is just a standard stars and bars style problem with 12 balls and 8 boxes. The answer is $\binom{12+8-1}{12}$

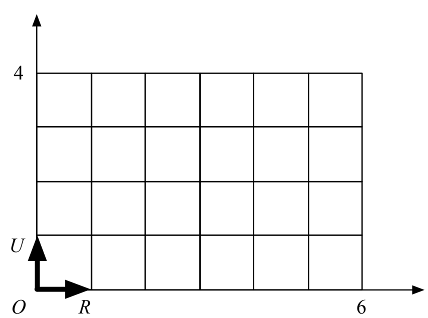
(b) (10 *pts*) Odd boxes must have an odd number of balls and even boxes must have an even number of balls.

Solution: There are two tricks here: first put one ball into every odd box, as in part (a). Second, after putting one ball into each odd box, group the balls into pairs and decide how many pairs to put into each box. After putting one ball into each odd box, there are 12 balls remaining, or 6 pairs. Finding the number of ways to put the pairs into the boxes is a standard stars and bars style problem with 6 stars (the pairs) and $8-1=7$ bars. So the final answer is $\binom{6+7}{6}$

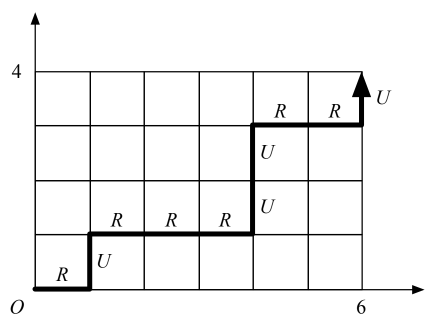
(c) (10 *pts*) You also have 16 indistinguishable green balls and want to distribute both the red and green balls into the boxes.

Solution: We can think of this as two separate tasks: first put the green balls into the boxes and then put the red balls into the boxes. Since the way we choose to complete the first task does not affect how many ways there are to complete the second task, we can just multiply together the number of ways to do each task on its own. And each task on its own is just a standard stars and bars style problem with 16 stars and $8-1=7$ bars. So the final answer is $\binom{16+7}{16}\binom{16+7}{16}$

8. (Extra Credits: You can get up to 60% more!) A grid in Cartesian coordinates with size 6×4 is shown in Figure 2a. We start from the original point $(0,0)$ and repeat moving 1 unit length to the next grid point by either moving up (denoted as U-action) or right (denoted as R-action), until we reach the destination point $(6,4)$. Each sequence of such R- and U-actions forms a path. Figure 2b shows an example path of this sequence: $R, U, R, R, R, U, U, R, R, U$.



(a) Case setup



(b) Example path sequence

Figure 2

(a) (10 *pts*) Try writing down a few more sequences by yourself, and find the pattern among them. How many paths can we obtain in total? Explain how you get the result.

Solution: The "pattern" is, no matter how you choose the path, it always contains 4 U-actions and 6 R-actions, and each permutation of the $4+6=10$ actions forms a valid path. Therefore, the answer is the full permutation of the 10 actions, divided by "the full permutation of 4 U-actions times the full permutation of the 6 R-actions," which is $\frac{10!}{4!\times 6!}$.
Alternatively, you can imagine there are $6+4=10$ action slots and you need to place 4 U-actions and 6 R-actions into the slots. So you choose 4 out of 10 for the U-actions, then choose 6 out of the rest 6 for R-actions (or vice versa), so the answer is $\binom{6+4}{4}\times\binom{6}{6}=\binom{10}{4}$=210.

(b) (10 pts) With a given grid point $(2,3)$ shown in Figure 3, how many paths are there if they must cover this point? Explain why.

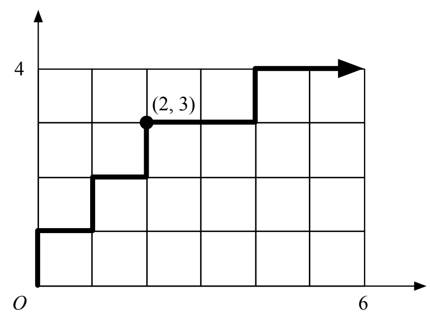


Figure 3: An example path that covers point $(2,3)$

Solution: Divide the problem into 2 and use the product rule. Let A be the set of paths that goes from $(0,0)$ to $(2,3)$, and B be the set of paths that goes from $(2,3)$ to $(6,4)$. The complete path that covers point $(2,3)$ is formed by choosing a path in A, then choosing one in B. Based on the result of (a), we can quickly get $|A|=\binom{2+3}{2}$ and $|B|=\binom{4+1}{4}$, so the answer is $|A|\times|B|=\binom{5}{2}\binom{5}{4}=50$.

(c) (10 pts) With two grid points $(2,3)$ and $(4,1)$ shown in Figure 4, how many paths are there if they are not allowed to cover either point? Explain why.

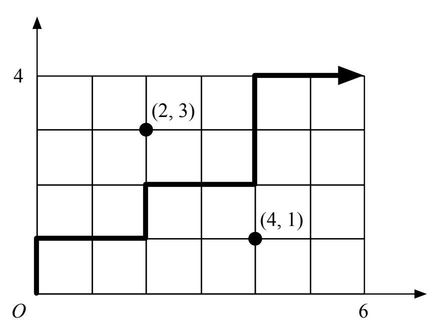


Figure 4: An example path that covers neither $(2,3)$ nor $(4,1)$

Solution: Based on (b), there are $\binom{2+3}{2}\binom{4+1}{4}$ paths covering point $(2,3)$ and $\binom{4+1}{4}\binom{2+3}{2}$ paths covering point $(4,1)$. Using the inclusion-exclusion method, the paths that don't cover either is
$|\text{all paths}|$
$-|\text{paths covering } (2,3)|$
$-|\text{paths covering } (4,1)|$
$+|\text{paths covering both } (2,3) \text{ and } (4,1)|$.
And because of the relative position of the 2 points, there are no paths covering both points (otherwise you have to go either down or left, which is not allowed). So the answer is $\binom{10}{6} - \binom{5}{2}\binom{5}{4} - \binom{5}{4}\binom{5}{2} + 0 = 110$.

(d) (10 *pts*) With another two grid points $(2,1)$ and $(5,3)$ shown in Figure 5, how many paths are there if they are not allowed to cover either point? Explain why. Note that the relative position of the points is different from that in question (c).

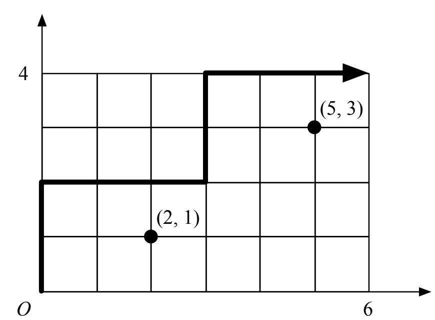


Figure 5: An example path that covers neither $(2,1)$ nor $(5,3)$

Solution: This is a similar question with (c), but due to the relative position of the 2 points, there are paths that goes from $(2,1)$ to $(5,3)$ so we need to consider this case. Similar to (b), paths covering both points can be calculated by dividing the grid into 3 parts and apply the product rule, and the answer is $\binom{3}{2}\binom{5}{3}\binom{2}{1}$. Using the inclusion-exclusion method, the final answer is
$|\text{all paths}|$
$-|\text{paths covering } (2,1)|$
$-|\text{paths covering } (5,3)|$
$+|\text{paths covering both } (2,1) \text{ and } (5,3)|$
$= \binom{10}{6} - \binom{3}{2}\binom{7}{4} - \binom{8}{5}\binom{2}{1} + \binom{3}{2}\binom{5}{3}\binom{2}{1} = 53.$

(e) (20 *pts*) Now we generalize question (d) to an $m \times n$ grid and i points (x_i, y_i) on the grid such that $0 < x_1 < x_2 < \cdots < x_i < m$ and $0 < y_1 < y_2 < \cdots < y_i < n$, as is shown in Figure 5. Devise the number of paths that don't cover any of these i points.

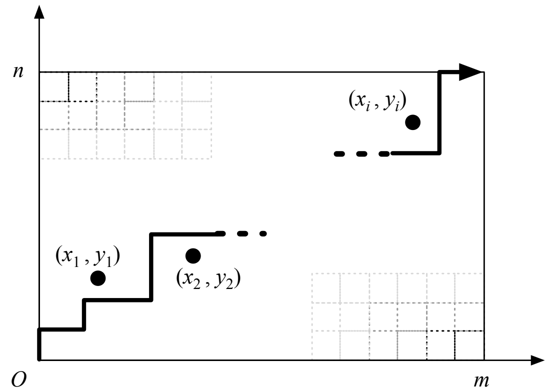


Figure 6: Some grid lines are omitted to improve readability; all the paths and points are aligned to the grid

Solution: In total, there are $\binom{m+n}{m}$ paths. For each point (x_i, y_i), there are $\binom{x_i+y_i}{x_i}\binom{m-x_i+n-y_i}{m-x_i}$ paths that cover it. Using the inclusion-exclusion rule, the answer is

$$\binom{m+n}{m} - \sum_{k\in[1,i]} \binom{x_k+y_k}{x_k}\binom{m-x_k+n-y_k}{m-x_k}$$

$$+ \sum_{\substack{k_1,k_2\in[1,i]\\ k_1\neq k_2}} \binom{x_{k_1}+y_{k_1}}{x_{k_1}}\binom{x_{k_2}-x_{k_1}+y_{k_2}-y_{k_1}}{x_{k_2}-x_{k_1}}\binom{m-x_{k_2}+n-y_{k_2}}{m-x_{k_2}}$$

$$- \sum_{\substack{k_1,k_2,k_3\in[1,i]\\ k_1\neq k_2\neq k_3}} \binom{x_{k_1}+y_{k_1}}{x_{k_1}}\binom{x_{k_2}-x_{k_1}+y_{k_2}-y_{k_1}}{x_{k_2}-x_{k_1}}\binom{x_{k_3}-x_{k_2}+y_{k_3}-y_{k_2}}{x_{k_3}-x_{k_2}}\binom{m-x_{k_3}+n-y_{k_3}}{m-x_{k_3}}$$

$$+\cdots$$

$$+ (-1)^i \binom{x_1+y_1}{x_1}\binom{x_2-x_1+y_2-y_1}{x_2-x_1}\cdots\binom{m-x_i+n-y_i}{m-x_i}.$$