**CS206 Quiz 3** **Oct 26, 2021** **A**

Name: ______________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

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1. (15%) How many 8 letter words contain only vowels?
   *Hint: There are 5 vowels*

   **Solution:** $5^8$. There are 5 vowels. Hence, for each letter there are 5 choices and these are repeatable, $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$.

   How many 8 letter words contain only consonants?

   **Solution:** $21^8$. Same as before. There are 21 consonants. Hence, for each letter there are 21 choices.

   How many 8 letter words contain both vowels and consonants?

   **Solution: DIFFERENCE METHOD**: $26^8 - 5^8 - 21^8$. Let $S$ be the set of all 8-letter words, $S_1$ be the set of all 8-letter words containing only vowels, and $S_2$ be the set of all 8-letters words containing only consonants. Then by the difference method, the number of 8-letter words that contain both vowels and consonants is $|S| - |S_1| - |S_2| = 26^8 - 5^8 - 21^8$.

2. (10%) How many ways to divide 10 identical Hershey bars among 3 kids?

   **Solution:** See slides 33–45 of lecture 7. This is the same as number of non-negative solutions to $x_1 + x_2 + x_3 = 10$. And the same as counting the ways for arranging 3-1 = 2 dividers within the 10 dots, as in lecture. Hence, $\binom{12}{2}$.

   How many ways to divide a subset of 10 identical Hershey bars among 3 kids when you can decide to keep some of the bars for yourself.

   **Solution:** See slides 33–45 of lecture 7. This is the same as number of non-negative solutions to $x_1 + x_2 + x_3 + x_4 = 10$. Hence, $\binom{10+3}{3}$.

3. (30%) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 20$ where $x_1, x_2, x_3, x_4$ are non-negative integers?

   **Solution:** See slides 33–45 of lecture 7. We use 3 dividers to divide 20 dots, and we have $\binom{23}{3}$ ways to place them within the 20 dots.

In each of the following cases, how many solutions are there to the above equation, when, in addition:

(a) $x_1 \geq 1$.

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| **Solution:** $\binom{22}{3}$. We give 1 (golden bar) out of 20 to $x_1$ and then we re-do everything we did before. |

(b) $x_i \geq 2$ for $i = 1, 2, \ldots, 4$.

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| **Solution:** $\binom{15}{3}$. We give 2 to each of the 4 "pirates". We are left with 12. We still have 3 dividers. |

(c) $0 \leq x_1 \leq 10$.

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| **Solution:** Difference Method: Total $\binom{23}{3}-$ Opposite Set $\binom{12}{3}$. For opposite set consider $x_1 \geq 11$, so we give to "$pirate_1$" 11 golden bars and then we re-do what we know to do. After giving 11 bars to $x_1$, we now have 9 golden bars to give away and still 3 dividers. |

(d) $0 \leq x_1 \leq 3$, $1 \leq x_2 \leq 4$, $x_3 \geq 15$.

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| **Solution:** Since $x_3 \geq 15$ and $x_2 \geq 1$, we give to $x_3$ 15 bars and to $x_2$ 1 bar. We are left with just 4 bars. So we are now looking at the number of solutions to $x_1 + x_2 + x_3 + x_4 = 20 - 15 - 1 = 4$ with $0 \leq x_1 \leq 3$ and $0 \leq x_2 \leq 3$. Difference Method: The number of solutions is the total number of solutions minus the special cases that do not hold based on our two constraints. That is, $\binom{4+3}{3} - 1 - 1$. The two "1"'s are for the two cases that need to be excluded, when $x_1 = 4$ and $x_2 = 4$, respectively. |

(e) $x_1 \geq x_2$.

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| **Solution:** Let $x_2 = i$. Since $x_1 \geq x_2$, $i$ ranges from 0 to 10 (WHY?). For each $i$, we are looking at number of solutions to $x_1 + x_3 + x_4 = 20 - 2i$. This is $\binom{20-2i+2}{2}$. Hence, the total number of solutions equals $\sum_{i=0}^{10} \binom{22-2i}{2}$. |

4. (10%) If we toss 20 numbered balls into 10 bins, how many outcomes are there?

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| **Solution:** Each of the 20 balls is different. For each ball, we have 10 choices. So $10^{20}$. |

5. (10%) In order to play a game of basketball, 6 children at a playground divide themselves into two teams of 3 each. How many different divisions are possible?

**Solution:** We can pick 3 children in $\binom{6}{3}$ ways. But the order of the two teams is irrelevant. That is, there is no A and B team, but just a division consisting of 2 groups of 3 each. Hence, to get the desired answer, we should divide by 2! (Generalized Bijection Method - see, e.g., lecture 7, slide 19). Total solution is $\binom{6}{3}/2!$.

6. (15%) We roll 10 standard 6-sided dice. Find the number of outcomes with at least two dice showing 6.

**Solution:** $6^{10} - (5^{10} + \binom{10}{1}5^9)$. Difference Method: Total outcomes minus "No 6" case minus "One 6" case and "No 6" case. For the "One 6" case, we choose 1 out of the 10 dice to be 6 in $\binom{10}{1}$ ways and the rest 9 dice will have any number between 1 and 5, that is $5^9$ ways.

7. (10%) Given the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, answer the following:

   (a) How many subsets of $X$ contain element $x_1$?

   **Solution:** $2^5$. Each element, other than $x_1$ can either be included or not. So we want to count sets with $6 - 1 = 5$ of those elements.

   (b) How many subsets of $X$ contain elements $x_2$ **and** $x_3$, **and do not** contain $x_5$?

   **Solution:** $2^3$. As before, since $x_2$ and $x_3$ are included while $x_5$ is not, we only have to worry about $x_1$, $x_4$ and $x_6$; that is three elements. Each of these three elements can, as before, be either present or not.

8. (Extra Credits - 25%) In lecture, we showed that the number of ways to divide 16 pieces of gold amongst 5 pirates is $\binom{20}{4}$ by counting all strings with 16 G's and 4 dividers, /'s. The following alternate method is suggested to count such strings: Write down 16 G's. Each of the 4 /'s can go to 17 places, for a total of $17^4$ possibilities. The /'s are indistinguishable, so their ordering does not matter. Thus, we divide by 4! since we could have placed the 4 /'s in any order. So the answer must be $17^4/4!$. However, this is certainly not correct as it is not equivalent to our answer $\binom{20}{4}$, and in fact it is not even an integer! Explain concisely what is wrong with this approach. In particular, does it overcount or undercount? Why? *Hint: Applying your methods requires adhering to their assumptions.*

**Solution:** The method is undercounting. If the 4 '/s go into different positions, then they are being overcounted by 4!. However, if say two of them go into the same position and two go to different positions, then such a configuration is counted only $\binom{4}{2}2$ times and not 4! times. Hence, one cannot uniformly divide by 4! as not every configuration is being counted the same number of times.

**CS206 Quiz 3** **Oct 28, 2021** **B**

Name: ____________________________

NetID: ____________(Please **PRINT**)

Section No.: ____________

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1. (15%) How many 7 letter words contain only vowels?
   *Hint: There are 5 vowels*

   **Solution:** $5^7$. There are 5 vowels. Hence, for each letter there are 5 choices. .

   How many 7 letter words contain only consonants?

   **Solution:** $21^7$.There are 21 consonants. Hence, for each letter there are 21 choices.

   How many 7 letter words contain both vowels and consonants?

   **Solution:** Difference Method: $26^7 - 5^7 - 21^7$. Let S = all 7 letter words, S1 = all 7 letter words containing only vowels and S2 = all 7 letters words containing only consonants. Then by the difference method, the number of 7 letter words that contain both vowels and consonants = $|S| - |S1| - |S2| = 26^7 - 5^7 - 21^7$.

2. (10%) How many ways to divide 8 identical Hershey bars among 3 kids?

   **Solution:** This is the same as number of non-negative solutions to x1 + x2 + x3 = 8. And the same as counting the ways for arranging 3 - 1 = 2 dividers within the 8 dots, as in the lectures. Hence, $\binom{10}{2}$ .

   How many ways to divide a subset of 8 identical Hershey bars among 3 kids, when you can decide to keep some of the bars for yourself?

   **Solution:** This is as simple as adding one more variable, say $x_4$ to the equation. We now have 4 "kids" and therefore 3 dividers, which is the same as number of non-negative solutions to $x_1 + x_2 + x_3 + x_4 = 8$. Hence, $\binom{11}{3}$

3. (30%) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$ where $x_1, x_2, x_3, x_4$ are non-negative integers?

**Solution:** Same as before, with 17 dots and $4-1=3$ dividers: $\binom{20}{3}$.

In each of the following cases, how many solutions are there to the above equation, when, in addition:

(a) $x_1 \geq 1$.

**Solution:** Here, you give one "bar"—point—dot to $x_1$ and you are left with 16 bars to give to all 4 of the "pirates". Hence $\binom{19}{3}$.

(b) $x_i \geq 2$ for $i = 1, 2, \ldots, 4$.

**Solution:** We give 2 to each of the 4 "pirates". We are left with 9. We still have 3 dividers. $\binom{9+4-1}{3}$.

(c) $0 \leq x_1 \leq 10$.

**Solution:** For opposite set consider $x_1 \geq 11$, so we give to "$pirate_1$" 11 golden bars and then we re-do what we know to do. After giving 11 bars to $x_1$, we now have 6 golden bars to give away and still 3 dividers. So $\binom{17+4-1}{3} - \binom{6+4-1}{3}$.

(d) $0 \leq x_1 \leq 3$, $1 \leq x_2 \leq 4$, $x_3 \geq 15$.

**Solution:** Since $x_3 \geq 15$ and $x_2 \geq 1$, we are looking at number of solutions to $x_1 + x_2 + x_3 + x_4 = 1$ with $0 \leq x_1 \leq 3$ and $0 \leq x_2 \leq 3$. The number of solutions is $\binom{1+4-1}{1}$.

(e) $x_1 \geq x_2$.

**Solution:** Let $x_2 = i$. Since $x_1 \geq x_2$, $i$ ranges from 0 to 8. For each $i$, we are looking at number of solutions to $x_1 + x_3 + x_4 = 17 - 2i$. This is $\binom{17-2i+2}{2}$. Hence, the total number of solutions equals $\sum_{i=0}^{8} \binom{19-2i}{2}$.

4. (10%) If we toss 15 numbered balls into 10 bins, how many outcomes are there?

**Solution:** Each of the 15 balls is different. For each ball, we have 10 choices. So $10^{15}$

5. (10%) In order to play a game of basketball, 6 children at a playground divide themselves into two teams of 3 each. How many different divisions are possible?

**Solution:** Because now the order of the two teams is irrelevant. That is, there is no A and B team, but just a division consisting of 2 groups of 3 each. Hence, the desired answer is $\frac{\binom{6}{3}\binom{3}{3}}{2!}$

6. (15%) We roll 8 standard 6-sided dice. Find the number of outcomes with at least two dice showing 6.

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| **Solution:** $6^8 - (5^8 + \binom{8}{1}5^7)$ Difference Method: Total outcomes minus "No 6" case minus "One 6" case and "No 6" case. For the "One 6" case, we choose 1 out of the 8 dice to be 6 in $\binom{8}{1}$ ways and the rest 7 dice will have any number between 1 and 5, that is $5^7$ ways. |

7. (10%) Given the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, answer the following:

   (a) How many subsets of $X$ contain element $x_1$?

   | |
   |---|
   | **Solution:** $2^7$ Each element, other than $x_1$ can either be included or not. So we want to count sets with $8 - 1 = 7$ of those elements. |

   (b) How many subsets of $X$ contain elements $x_2$ **and** $x_3$, **and do not** contain $x_6$?

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   | **Solution:** $2^5$. As before, since $x_2$ and $x_3$ are included while $x_6$ is not, we only have to worry about $x_1$,$x_4$, $x_5$, $x_7$ and $x_8$; that is five elements. Each of these three elements can, as before, be either present or not. |

8. (Extra Credits - 25%) In lecture, we showed that the number of ways to divide 20 pieces of gold amongst 5 pirates is $\binom{24}{4}$ by counting all strings with 20 G's and 4 dividers, /'s. The following alternate method is suggested to count such strings: Write down 20 G's. Each of the 4 /'s can go to 21 places, for a total of $21^4$ possibilities. The /'s are indistinguishable, so their ordering does not matter. Thus, we divide by 4! since we could have placed the 4 /'s in any order. So the answer must be $21^4/4!$. However, this is certainly not correct as it is not equivalent to our answer $\binom{24}{4}$, and in fact it is not even an integer! Explain concisely what is wrong with this approach. In particular, does it overcount or undercount? Why? *Hint: Applying your methods requires adhering to their assumptions.*

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| **Solution:** The method is undercounting. If the 4 '/s go into different positions, then they are being overcounted by 4!. However, if say two of them go into the same position and two go to different positions, then such a configuration is counted only $\binom{4}{2}2$ times and not 4! times. Hence, one cannot uniformly divide by 4! as not every configuration is being counted the same number of times. |