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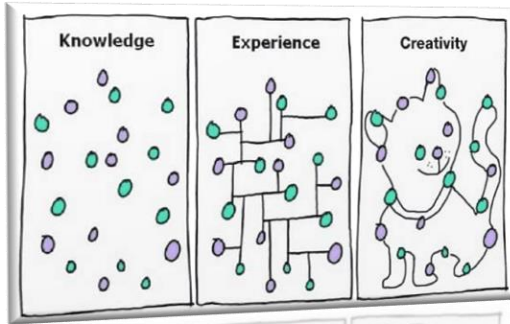


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Discrete Structures II



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CTAAR survey

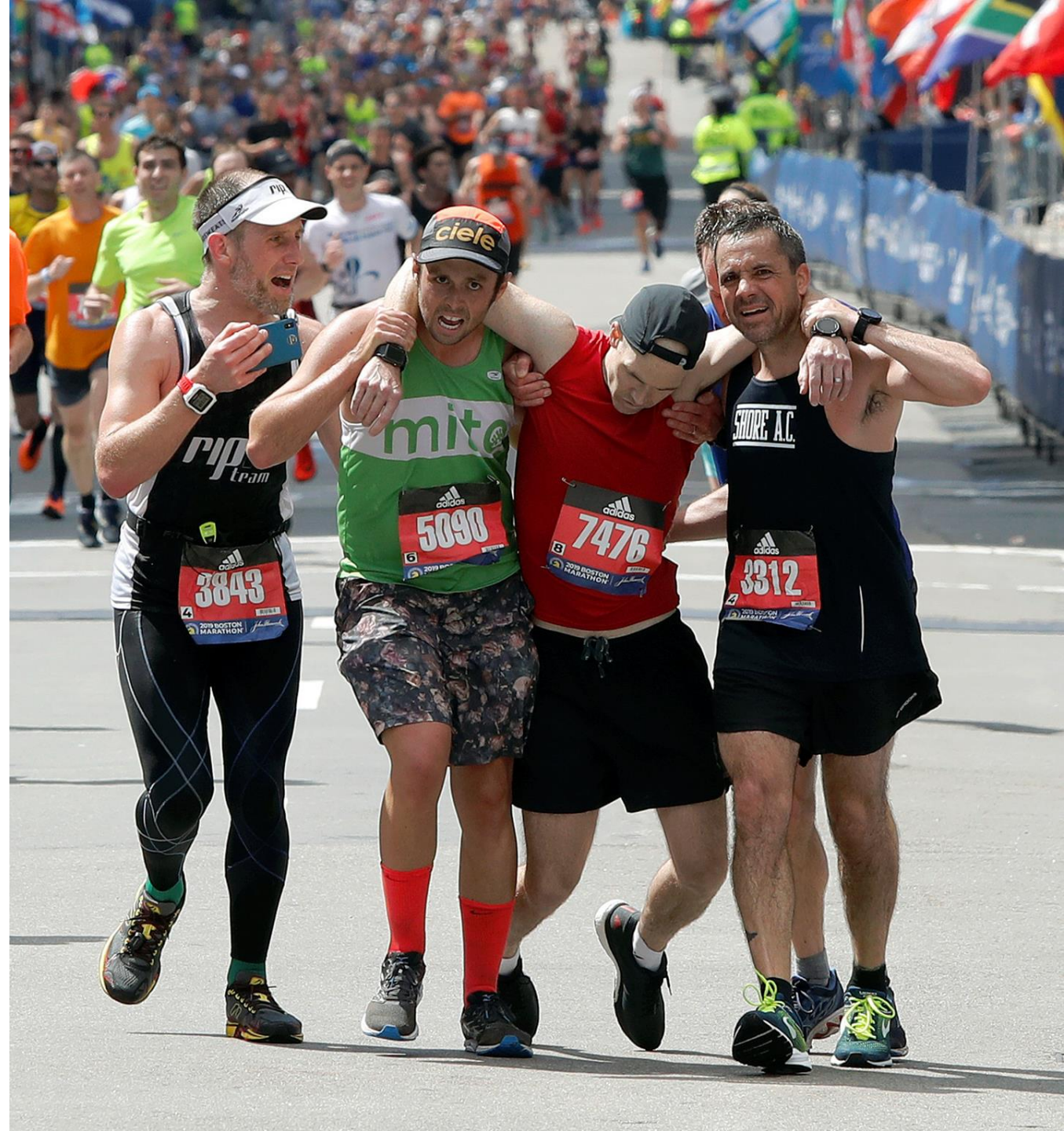
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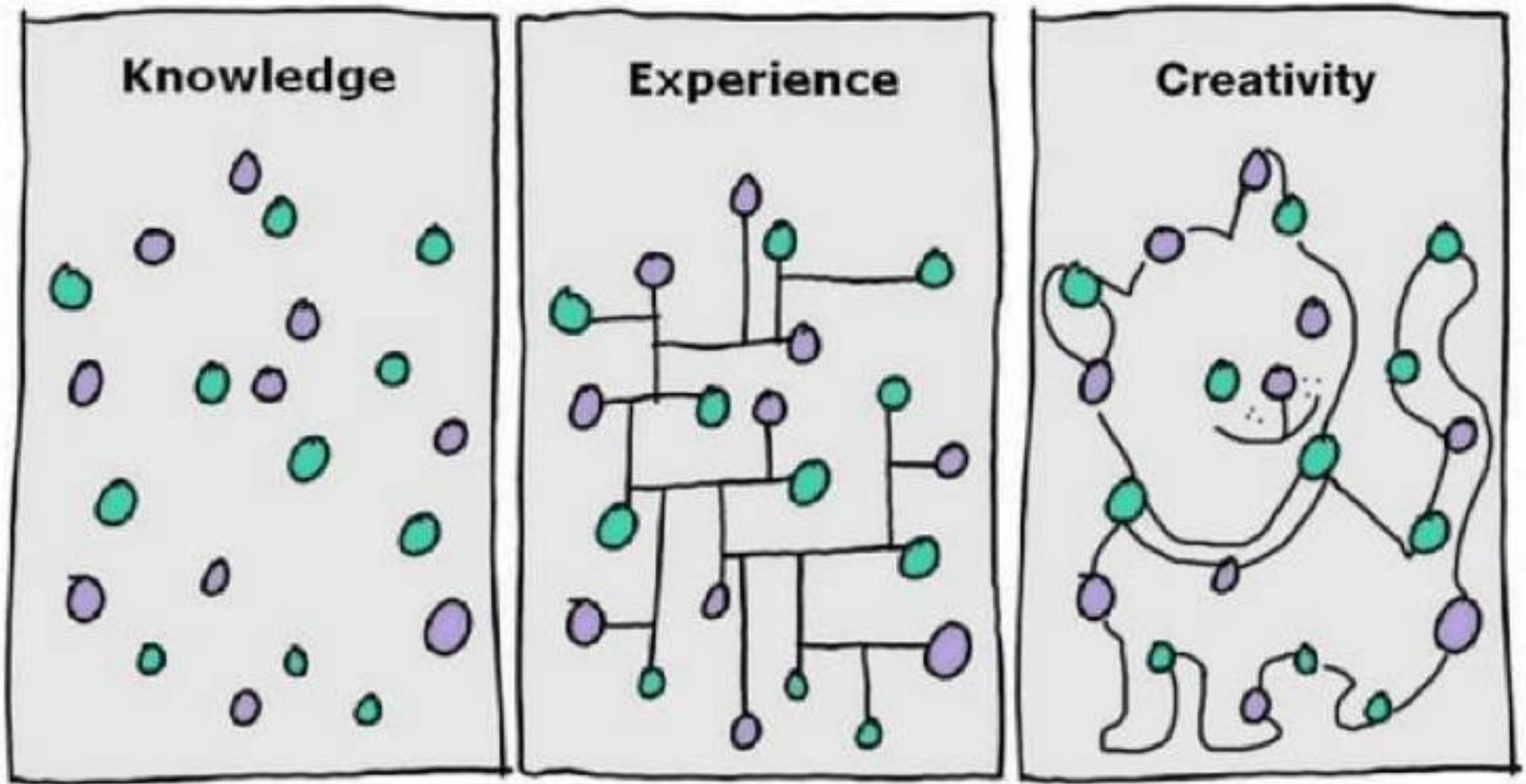


$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Announcements

- Assignment 2 is running
 - New Deadline Dec 8 (tomorrow)
- Quiz 6 → This week
 - Probabilities (Up to today's lecture)





Lectures

Quizzes

Homework

Assignments

Real-Life!

Outline

- Sample space
- Axioms of probability
- Conditional probability
- Independence
- Bayes rule
- Random Variables
- Expectation

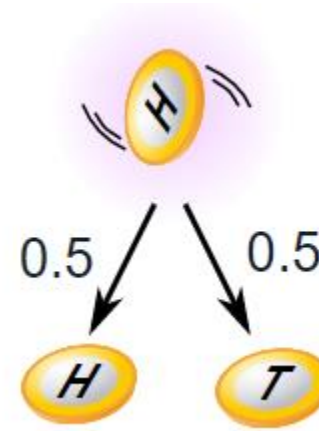
Probability - Events

Events can be:

- **Independent**

each event is not affected by other events

- e.g., a coin does not know that it came up “Heads” in the past and the chance is simply 50% in every toss of the coin



- **Dependent (or Conditional)**

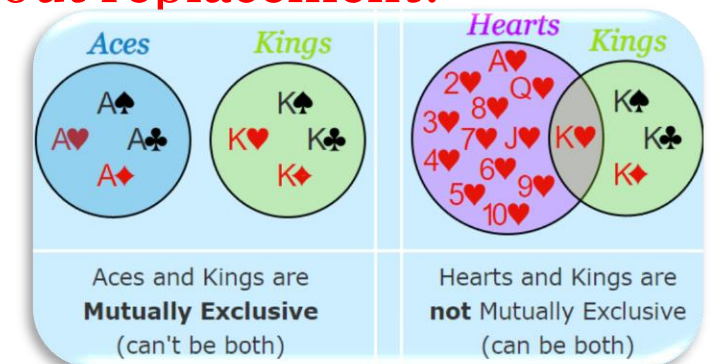
where an event is affected by other events

- e.g., after taking one card from the deck there are **less cards** available, so the probabilities change! How the probability of getting a King changes, after the 1st card was a King (less likely), and after the 1st card was not a King (more likely)?
- What would happen if we remove cards with and without replacement? (*independent vs. dependent*)

- **Mutually Exclusive**

events can't happen at the same time

- e.g., “Left or Right”, “Heads or Tails”, “Kings or Aces”



Probability of Independent Events

- We can calculate the chances of two or more independent events by multiplying the chances.

Example: Probability of 3 Heads in a Row

For each toss of a coin a "Head" has a probability of 0.5:

$$\begin{array}{c} \text{H} \\ 0.5 \end{array}$$

$$\begin{array}{cc} \text{H} & \text{H} \\ 0.5 \times 0.5 = 0.25 & \text{(or } \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \text{)} \end{array}$$

$$\begin{array}{ccc} \text{H} & \text{H} & \text{H} \\ 0.5 \times 0.5 \times 0.5 = 0.125 & \text{(or } \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \text{)} \end{array}$$

And so the chance of getting 3 Heads in a row is **0.125**

Probability of Independent Events

Question 1: What is the probability of 7 heads in a row?

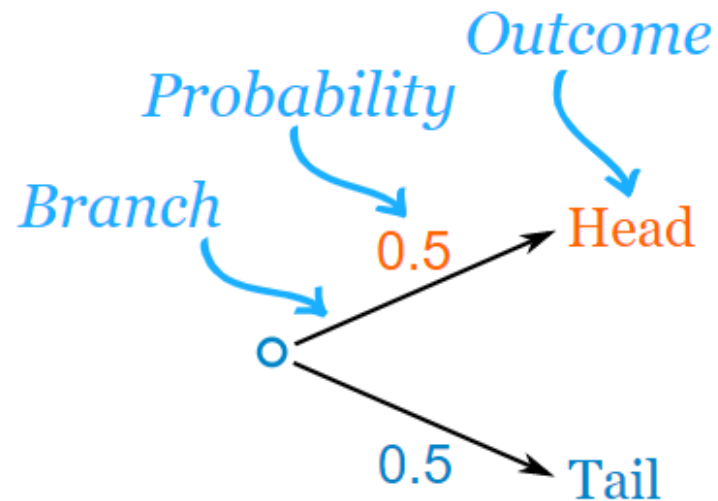
➡ Answer: $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 0.0078125$ (less than 1%).

Question 2: Given that **we have just got 6 heads** in a row, what is the probability that **the next toss** is also a head?

Probability Tree Diagrams



Here is a tree diagram for the toss of a coin:

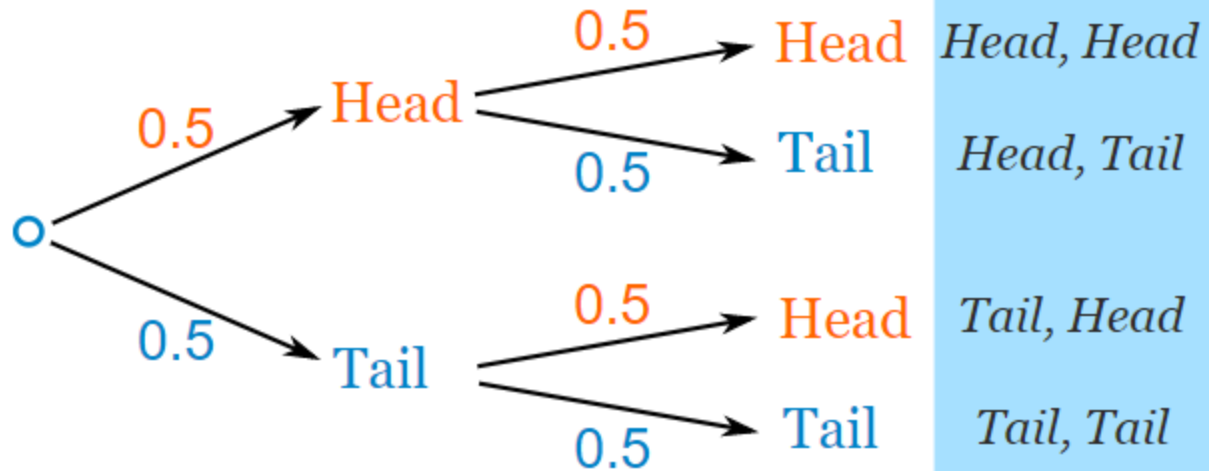


There are two "branches" (Heads and Tails)

- The probability of each branch is written on the branch
- The outcome is written at the end of the branch

Probability Tree Diagrams

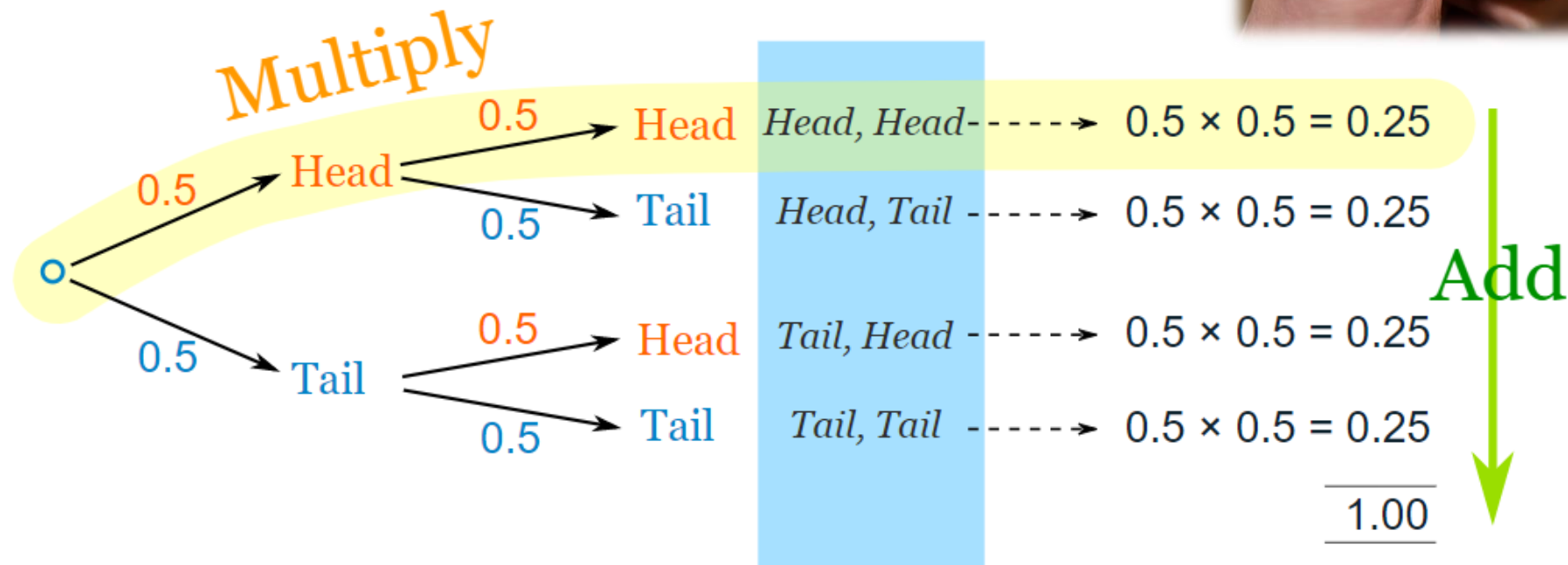
We can extend the tree diagram to two tosses of a coin:



Probability Tree Diagrams

Independent Events

- We **multiply** probabilities **along the branches**
- We **add** probabilities down **columns**





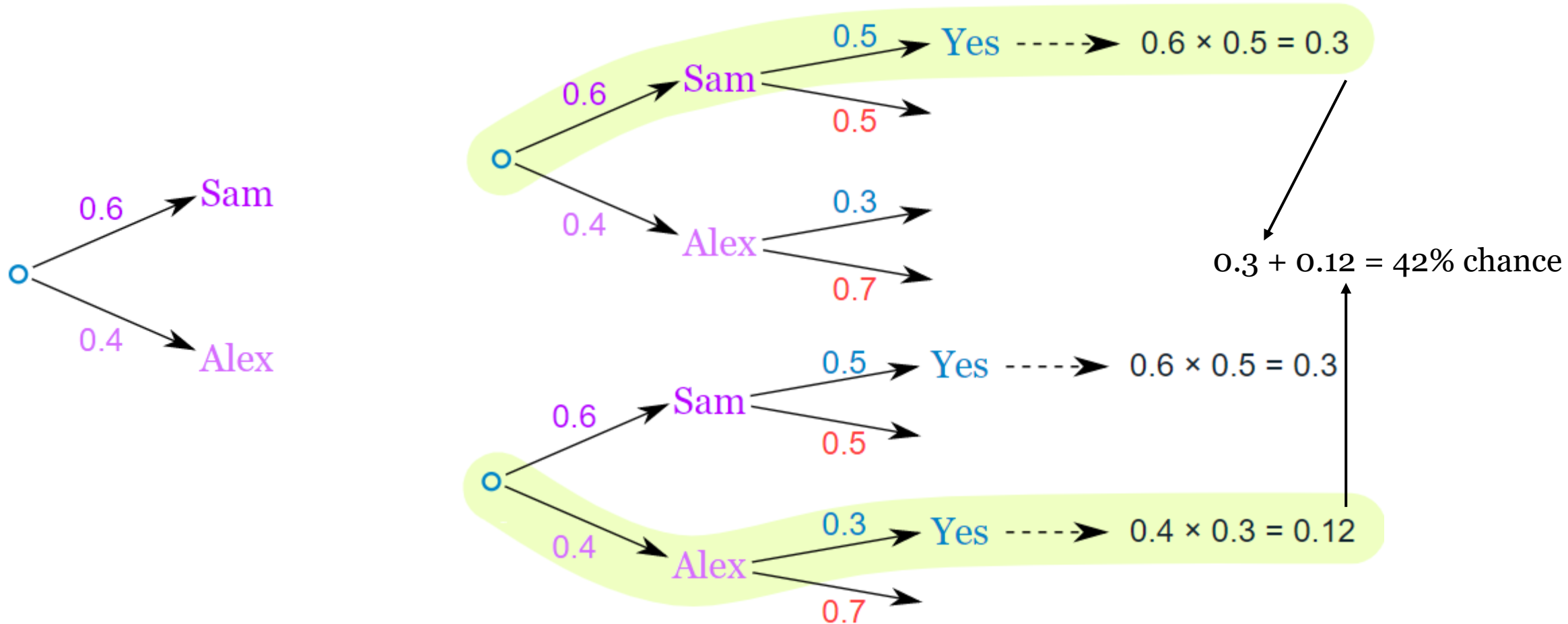
Probability Tree Diagrams

The chance of you being the Goalkeeper depends on who is the Coach:

- with Coach Sam the probability of being Goalkeeper is **0.5**
- with Coach Alex the probability of being Goalkeeper is **0.3**
- Sam is Coach more often ... about 6 out of every 10 games (a probability of **0.6**).
- So, what is the probability you will be a Goalkeeper today?



Probability Tree Diagrams



Probability Tree Diagrams

Example: Marbles in a Bag

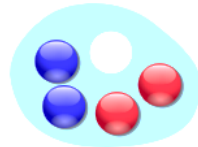
2 blue and 3 red marbles are in a bag.

What are the chances of getting a blue marble?

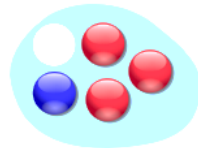
The chance is **2 in 5**

But after taking one out the chances change!

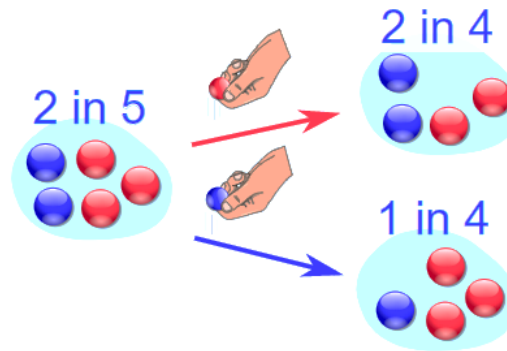
So the next time:



if we got a **red** marble before, then the chance of a blue marble next is **2 in 4**



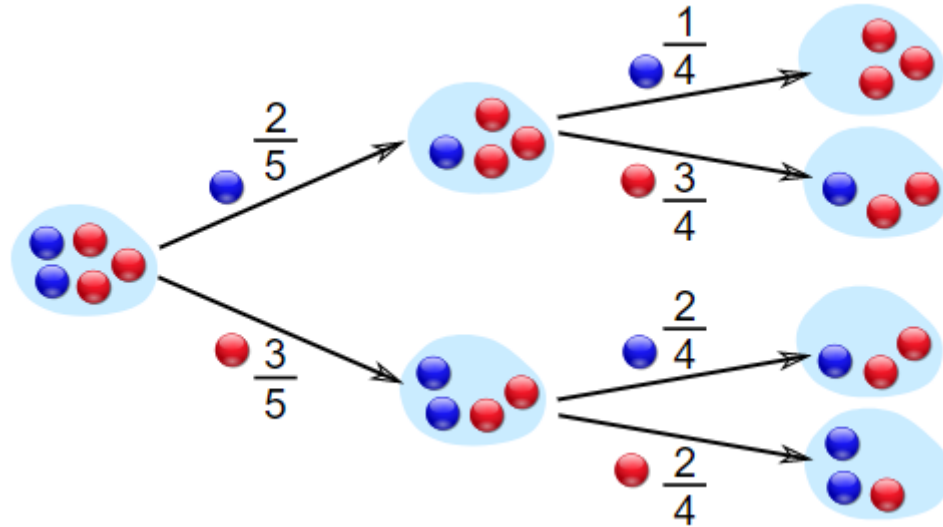
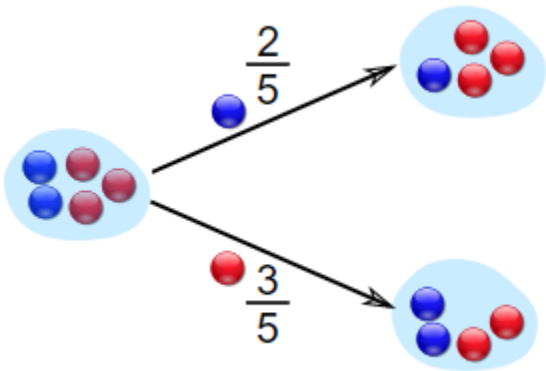
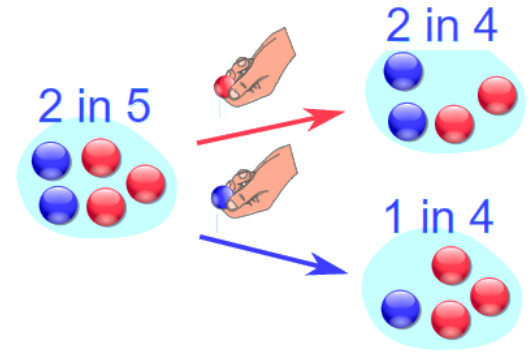
if we got a **blue** marble before, then the chance of a blue marble next is **1 in 4**



- The second event **depends on** what happened in the first event and is called **dependent**.
- If we **replace** the marbles in the bag each time, then the chances do **not** change and the events are independent:
- **With Replacement**: the events are **Independent** (the chances don't change)
- **Without Replacement**: the events are **Dependent** (the chances change)

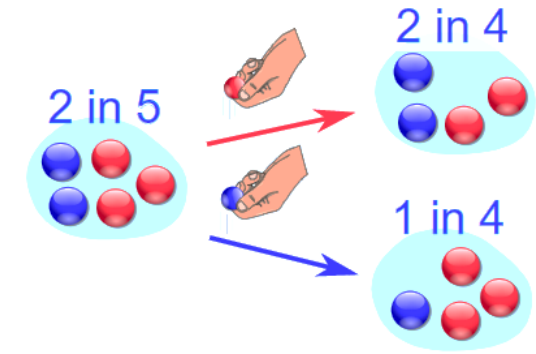
Dependent Events

Probability Tree Diagrams

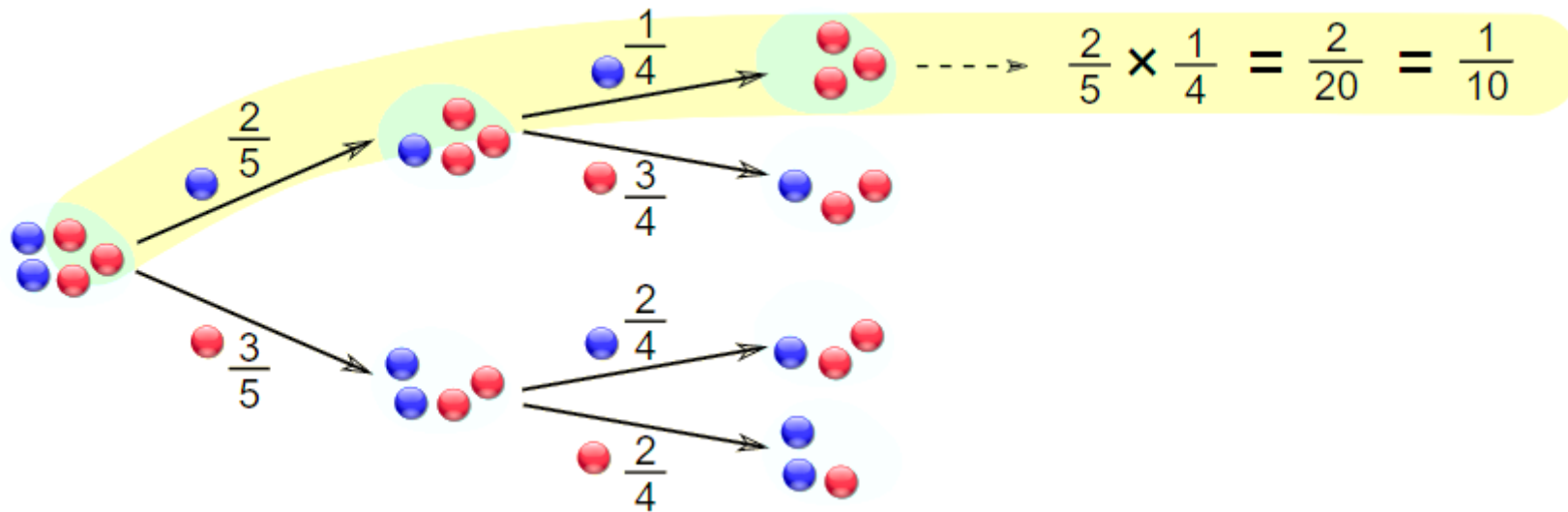


Dependent Events

Probability Tree Diagrams



What are the chances of drawing 2 blue marbles?



Conditional Probabilities–Formula to remember

- $P(A|B) = \frac{P(A \cap B)}{P(B)}$

If equally likely outcomes $\frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|}{|B|}$

$$P(A|B, C) = \frac{P(A \cap B \cap C)}{P(B \cap C)}$$

Independence

- A and B are independent events if $P(A|B) = P(A)$

$$P(A|B) = P(A), \quad A \text{ and } B \text{ are independent}$$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$$

Independence – more than 2 events...

- A_1, A_2, A_3 are independent events if $P(A_1), P(A_2), P(A_3)$ does not change by knowing any subset of the other.

$$P(A_1|A_2) = P(A_1)$$

$$P(A_1|A_3) = P(A_1)$$

$$P(A_1|A_2, A_3) = P(A_1)$$

$$\Rightarrow P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$$

$$P(A_1 \cap A_3) = P(A_1)P(A_3)$$

$$P(A_1 \cap A_2) = P(A_1)P(A_2)$$

⋮

Conditional Probabilities – Toy Example

- Two fair coins are flipped. $A = \{\text{first coin is } H\}$, $B = \{\text{second coin is } H\}$.
- Are A and B independent?

$$\Omega = \{(H,H), (T,T), (H,T), (T,H)\}$$

$$P(A|B) = P(A), \quad P(A) = \frac{|A|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}$$

$$P(A|B): \text{ new sample space } B = \{(H,H), (T,H)\}$$

$$P(A|B) = \frac{1}{2} = P(A)$$

Conditional Probabilities

- Two fair coins are flipped. $A = \{\text{first coin is } H\}$. $B =$ **two coins have different outcomes.**
- Are A and B independent?

$$\begin{aligned} P(A) &= \frac{1}{2} \\ \Omega &= \{(H,H), (H,T), (T,H), (T,T)\} \\ B &= \{(H,T), (T,H)\} \\ P(A|B) &= \frac{1}{2} = P(A) \end{aligned}$$

Conditional Probability

- Consider a family with two children. Given that one of the children is a boy, what is the probability that both children are boys?

→ children are equally likely

$$\Omega = \{(B,B), (B,G), (G,B), (G,G)\}$$

$B \rightarrow$ one of them is a boy $\rightarrow \{(B,B), (B,G), (G,B)\}$

$A \rightarrow$ both of them are boys

$$P(A|B) = \frac{1}{3}$$
$$P(A) = \frac{1}{4}$$

Conditional Probability

- Consider a family with two children. Given that the first child is a boy, what is the probability that both children are boys?

$$\Omega = \{(B, B), (B, G), (G, G), (G, B)\}$$
$$B \rightarrow \{(B, B), (B, G)\}$$

$A \rightarrow$ both are boys

$$P(A|B) = \frac{1}{2}$$

$$P(A) = \frac{1}{4}$$

Conditional Probability

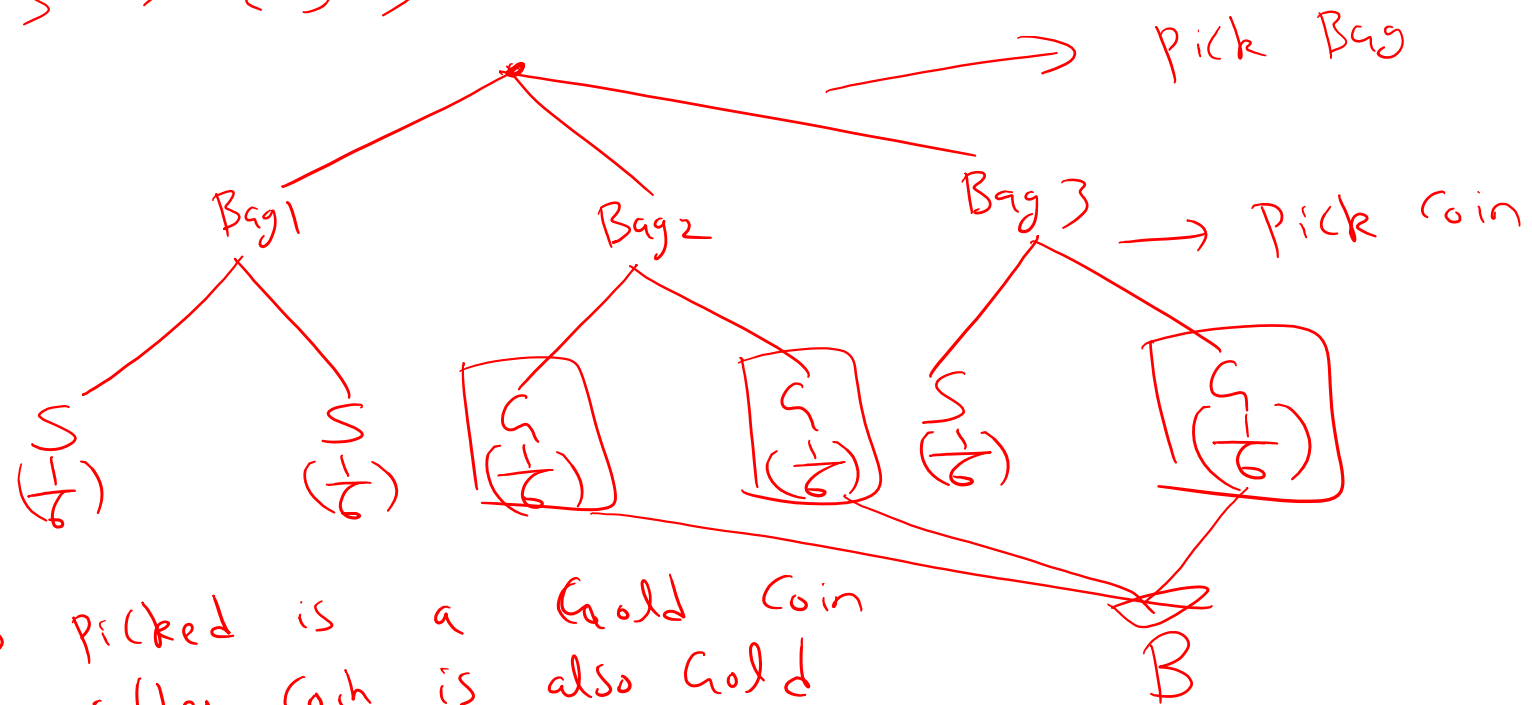
- One bag has two silver coins, another has two gold coins, and the third has one of each.
- One bag is selected at random. One coin from it is selected at random.
- It turns out to be gold What is the probability that the other coin is gold?

Conditional Probability

Bag 1 $\rightarrow \{S, S\}$

Bag 2 $\rightarrow \{G, G\}$

Bag 3 $\rightarrow \{S, G\}$



B \rightarrow picked is a Gold coin

A \rightarrow other coin is also Gold

$$P(A|B) = \frac{2}{3}$$

CTAAR survey

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5 min
Take a Break



Bayesian Inference



Dilemma at the movies

This person dropped their ticket in the hallway.





meaning of idk



All Images News Videos Shopping More Settings Tools

About 24,000,000 results (0.73 seconds)

Idk is an abbreviation of the phrase I don't know. **Idk** is most commonly used in informal communication, such as text messaging. There are no formal rules about the capitalization of words like **idk**.

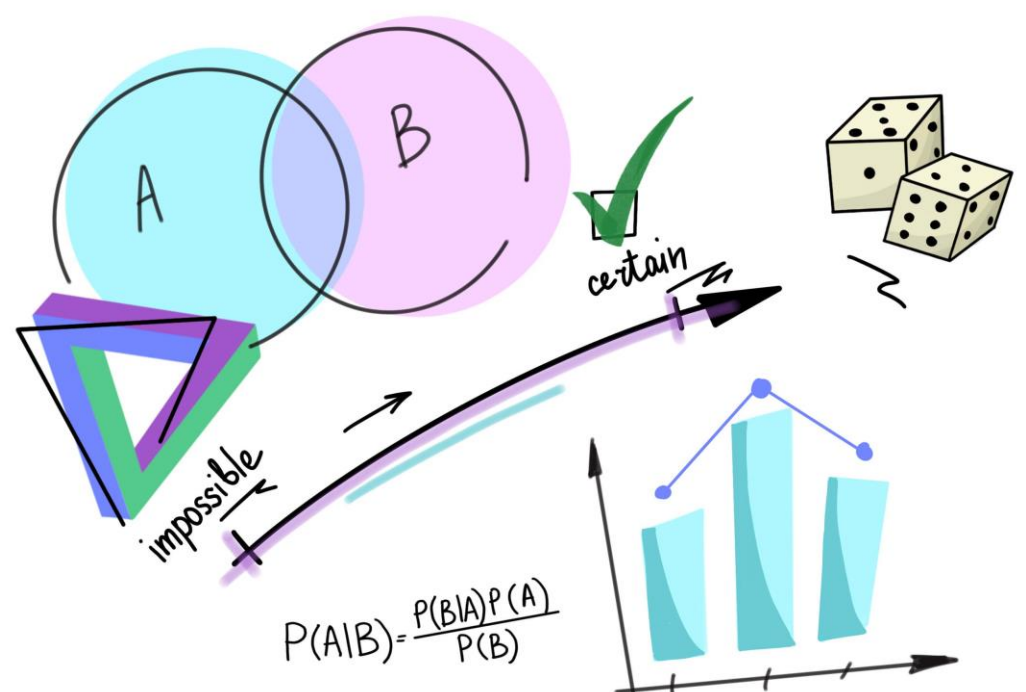


What Does Idk Mean? | Grammarly
<https://www.grammarly.com/blog/idk-meaning>

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People also ask

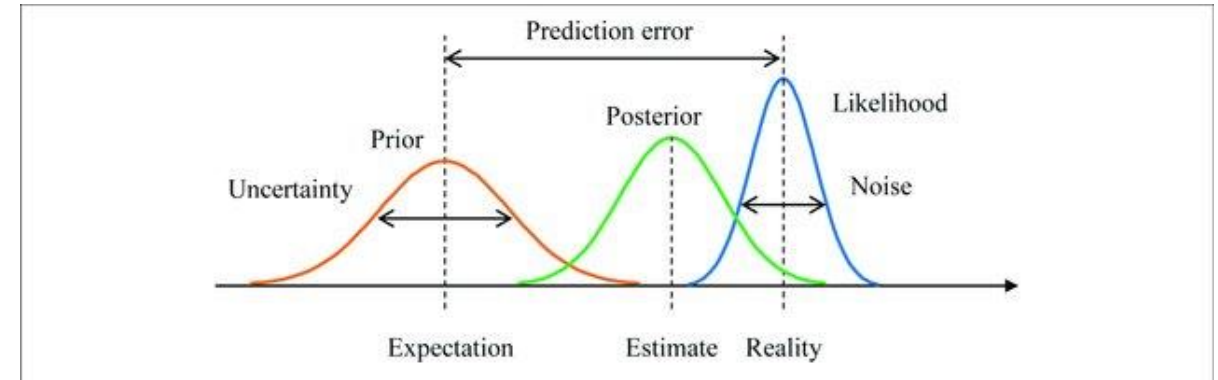
How does Google do this?



Bayesian Inference*

$$\begin{array}{c} \text{Posterior} \\ \downarrow \\ P(A|B) \end{array} = \frac{\begin{array}{c} \text{Likelihood} \\ \downarrow \\ P(B|A) \end{array} * \begin{array}{c} \text{Prior} \\ \downarrow \\ P(A) \end{array}}{\begin{array}{c} P(B) \\ \uparrow \\ \text{Evidence} \end{array}}$$

* *Inference = Educated guessing*



- Bayesian inference with a **prior distribution**, a **posterior distribution**, and a **likelihood function**.
- The prediction error is the difference between the **prior expectation** and the **peak of the likelihood function (i.e., reality)**.
- **Uncertainty** is the variance of the prior. **Noise** is the variance of the likelihood function.

Conditional Probabilities

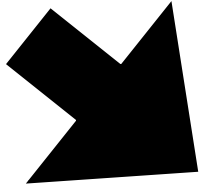
- $P(B|A)$ means “Probability of event B **given** event A”

In other words, event A has already happened, now what is the chance of event B?

"Probability Of" "Given"

$$P(\text{A and B}) = P(\text{A}) \times P(\text{B} | \text{A})$$

Event A *Event B*



"Probability of **event A and event B** equals
the probability of **event A** times the probability of **event B given event A**"

$$P(\text{B} | \text{A}) = \frac{P(\text{A and B})}{P(\text{A})}$$

"The probability of **event B given event A** equals
the probability of **event A and event B** divided by the probability of **event A**"

Bayes Rule

An internet search for "movie automatic shoe laces" brings up "Back to the future"

- Has the search engine watched the movie?

No, but it knows from lots of other searches what people are **probably** looking for.

- And it calculates that probability using Bayes' Theorem.

The screenshot shows a Google search for "movie automatic shoe laces". The search bar at the top contains the query. Below the search bar, there are tabs for "All", "Shopping", "Videos", "Images", "News", and "More". The "Shopping" tab is selected. The results show "About 13,100,000 results (0.79 seconds)". Below this, there is a section titled "See movie automatic shoe laces" with a "Sponsored" label. This section displays four product listings:

- HICKIES Tie-Free Laces H2 Black**: \$15.99, hickies.com, Special offer.
- Nike Mens Air Max 720 SATRN Casual...**: \$129.99, Eastbay, Special offer.
- Nike Fly.By Low II Men's Basketball...**: \$48.75, Kohl's, 4.5 stars (221).
- Christian Louboutin Spike Sock Donna...**: \$1,295.00, Neiman Marcus, 4.5 stars (27).

Below the shopping results, there is a snippet from a Fox article titled "Fox. In the **film**, Marty and Dr. Emmett 'Doc' Brown travel to the future where, in 2015, **shoes** have power **laces**. A small number of fans got their hands on some working Nike Mag **shoes** with power **laces** in 2016. Jul 2, 2018". Below this snippet is a link to a Hollywood Reporter article: "'Back to the Future: Part II' Film-Worn Sneaker Sells for Nearly ... https://www.hollywoodreporter.com > heat-vision > back-future-part-ii-film-...".

At the bottom, there is a "Videos" section with three video thumbnails:

- Back to the Future 2 - Nike Air 2015 Kicks**: 0:13, sleepy6uy, YouTube - Apr 11, 2007.
- 'Back to the Future' self-lacing shoes now a re...**: 2:02, CNN, YouTube - Oct 21, 2015.
- Back to the Future 2 - Power Laces [Movie Clip] English (1989)**: 0:27, Pro Movie Kino, YouTube - Oct 26, 2017.

Bayes Rule – One more example

Example: If dangerous fires are rare (1%) but smoke is fairly common (10%) due to barbecues, and 90% of dangerous fires make smoke then:

$$\begin{aligned} P(\text{Fire}|\text{Smoke}) &= \frac{P(\text{Fire}) P(\text{Smoke}|\text{Fire})}{P(\text{Smoke})} \\ &= \frac{1\% \times 90\%}{10\%} \\ &= 9\% \end{aligned}$$

So the "Probability of dangerous Fire when there is Smoke" is 9%

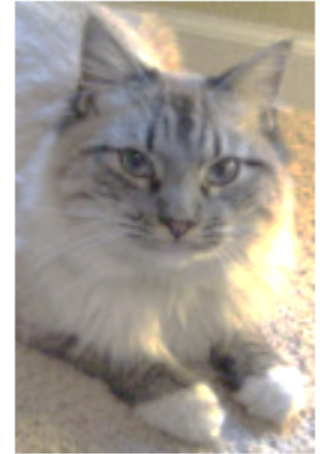
Bayes Rule – Yet another example

Example: Allergy or Not?

Hunter says she is itchy. There is a test for Allergy to Cats, but this test is not always right:

- For people that **really do** have the allergy, the test says "Yes" **80%** of the time
- For people that **do not** have the allergy, the test says "Yes" **10%** of the time ("false positive")

If 1% of the population have the allergy, and **Hunter's test says "Yes"**, what are the chances that Hunter really has the allergy?



Bayes Rule – Yet another example



$$P(\text{Allergy}|\text{Yes}) = \frac{P(\text{Allergy}) P(\text{Yes}|\text{Allergy})}{P(\text{Yes})}$$

$P(\text{Allergy})$ is Prob of Allergy = 1%

$P(\text{Yes}|\text{Allergy})$ is Prob of test saying "Yes" for people with allergy = 80%

$P(\text{Yes})$ is Prob of test saying "Yes" (to anyone) = ??%

- We **don't know** what the **general** chance of the test saying "Yes" is but we can calculate it by adding up those **with**, and those **without** the allergy:
 - 1% have the allergy, and the test says "Yes" to 80% of them
 - 99% do **not** have the allergy and the test says "Yes" to 10% of them
- $P(\text{Yes}) = 1\% \times 80\% + 99\% \times 10\% = 10.7\%$ of the population.

$$P(\text{Allergy}|\text{Yes}) = \text{about } 7\%$$