CS 344: Design and Analysis of Computer Algorithms

(Spring 2022 — Sections 5,6,7,8)

Lecture 8: Hashing, Dynamic Programming

Hash Tables: Chaining

How to handle the collisions?

Chaining

- The hash table is an array with each cell being a linked-list
- Given the array A[1:n]:
  - For every i, we compute b(i) = h(A[i]) and add A[i] to the tail of the linked-list at T[b(i)]
- Given x to be searched:
  - We iterate over elements of the linked-list T[h(x)] to find x or output it does not exist

Chaining : Example

- $h(x) = (x \mod 4)$+1

n=8 **m=4**

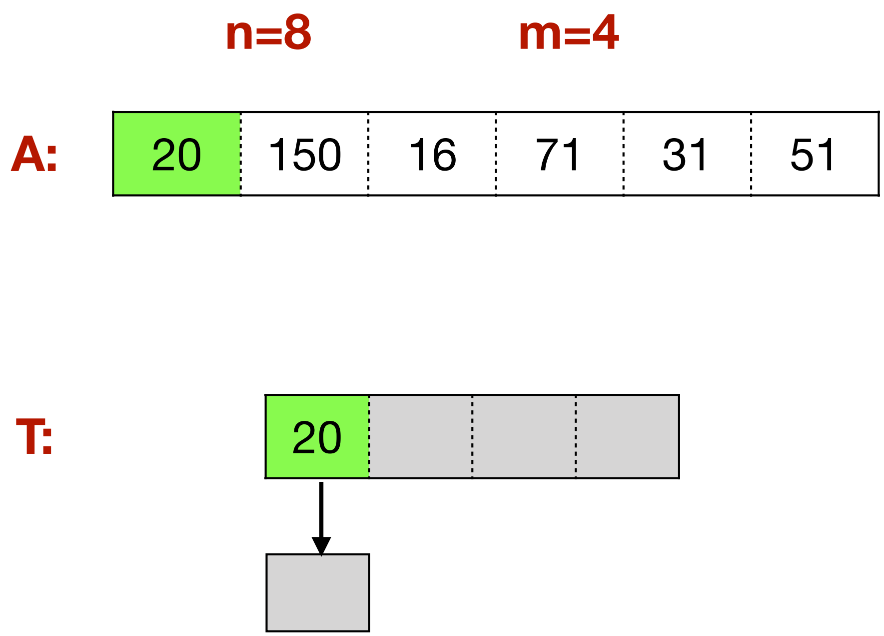
A:

20	150	16	71	31	51

T:

Chaining : Example

- $h(x) = (x \mod 4)+1$
- h(20) = 1



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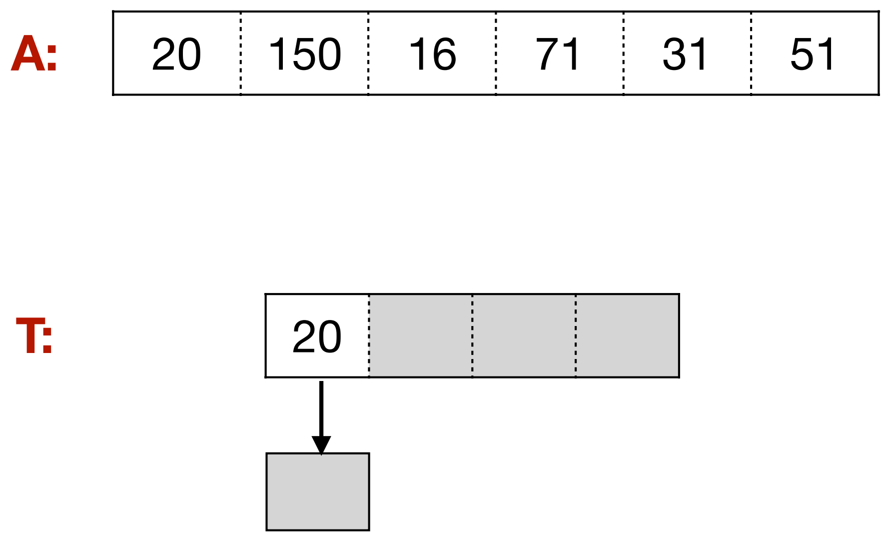
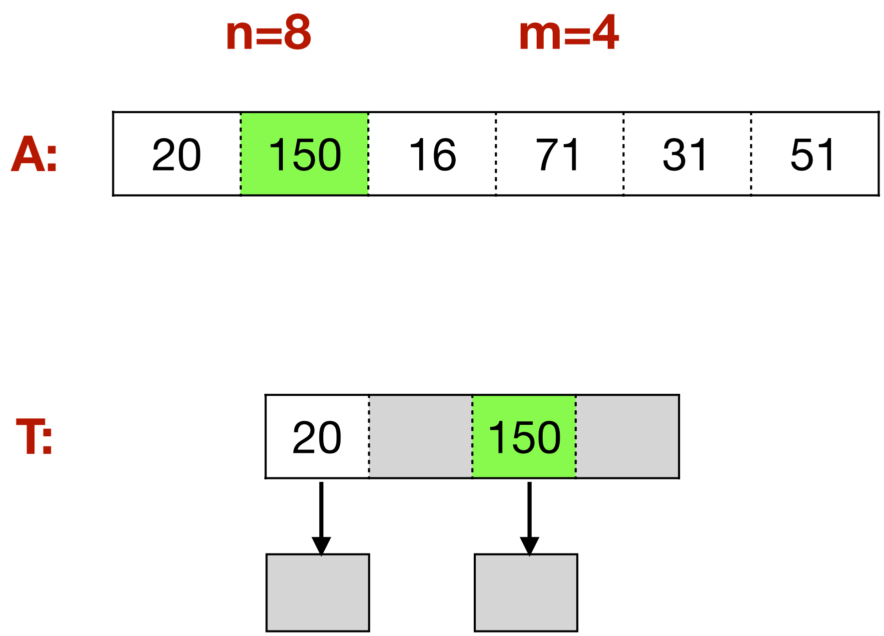


Chaining : Example

- $h(x) = (x \mod 4)$+1
- h(20) = 1
- h(150) = 3



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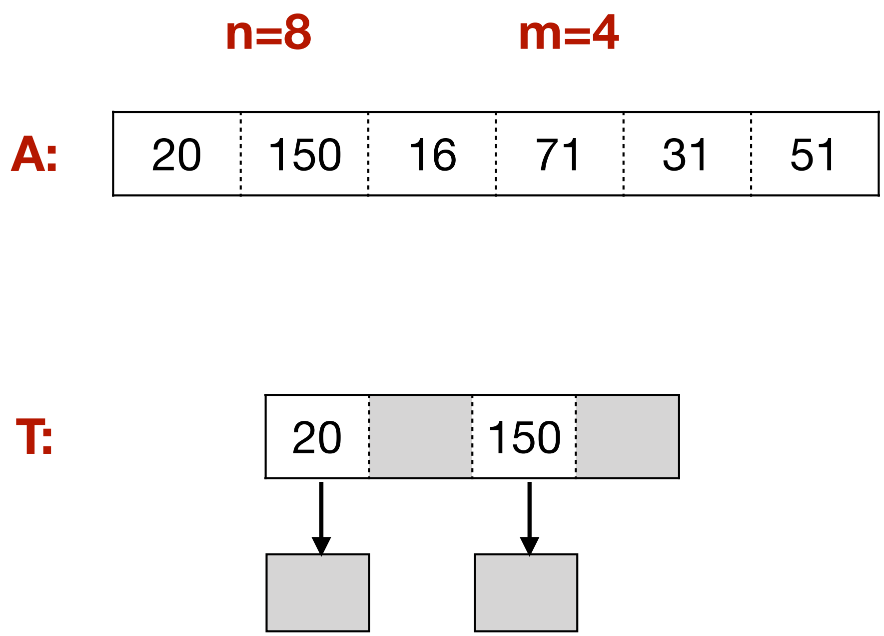

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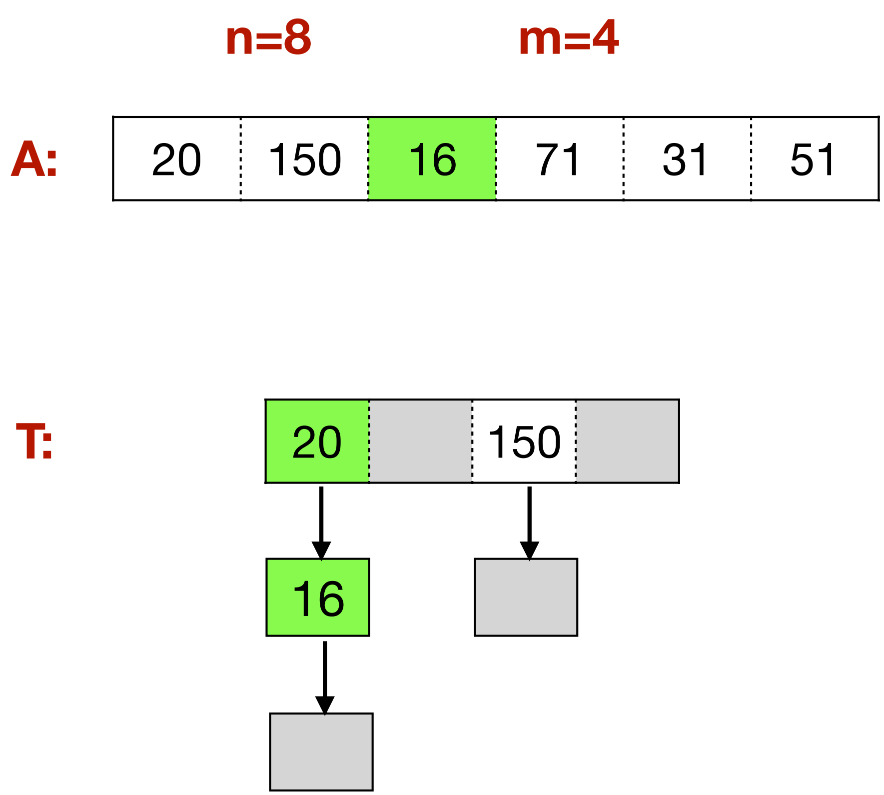

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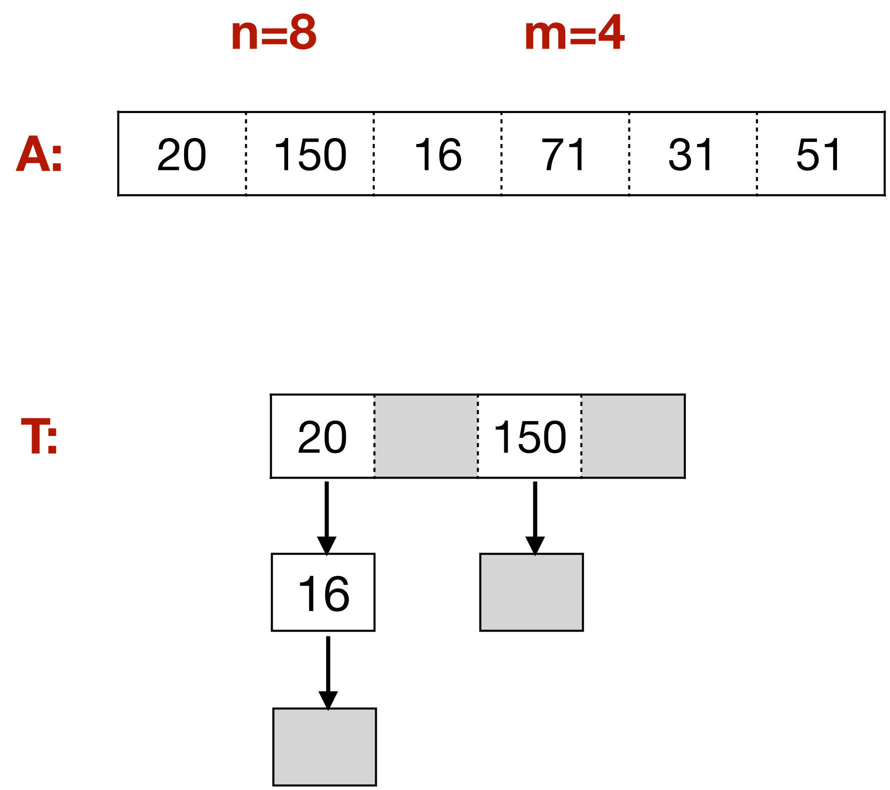
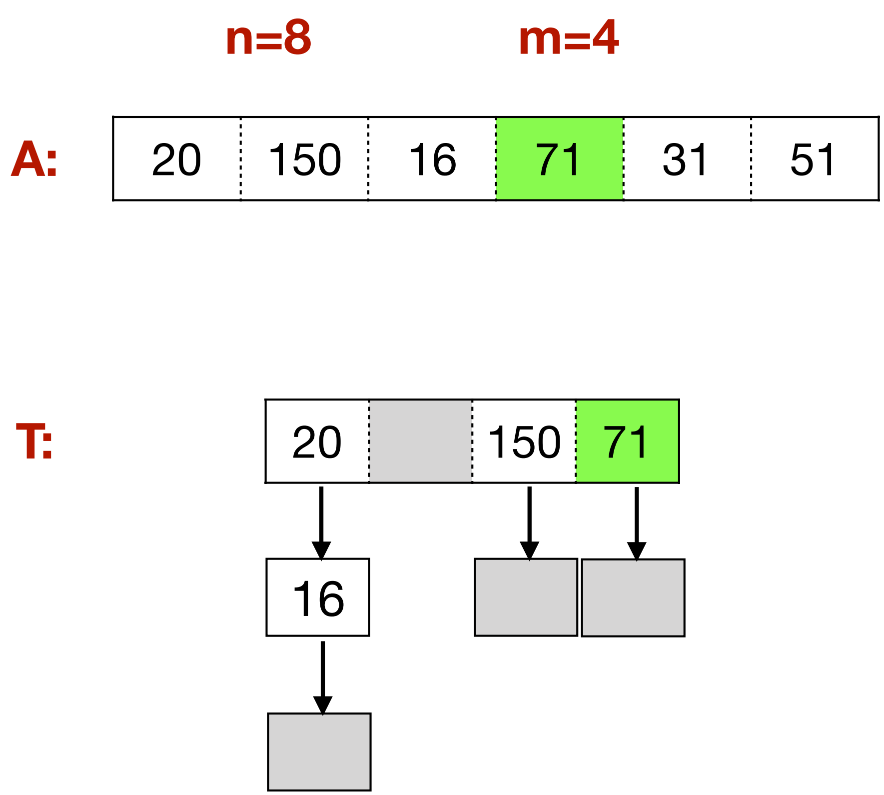


Chaining : Example

- $h(x) = (x \mod 4)+1$
- h(20) = 1
- h(150) = 3
- h(16) =1
- h(71) = 4



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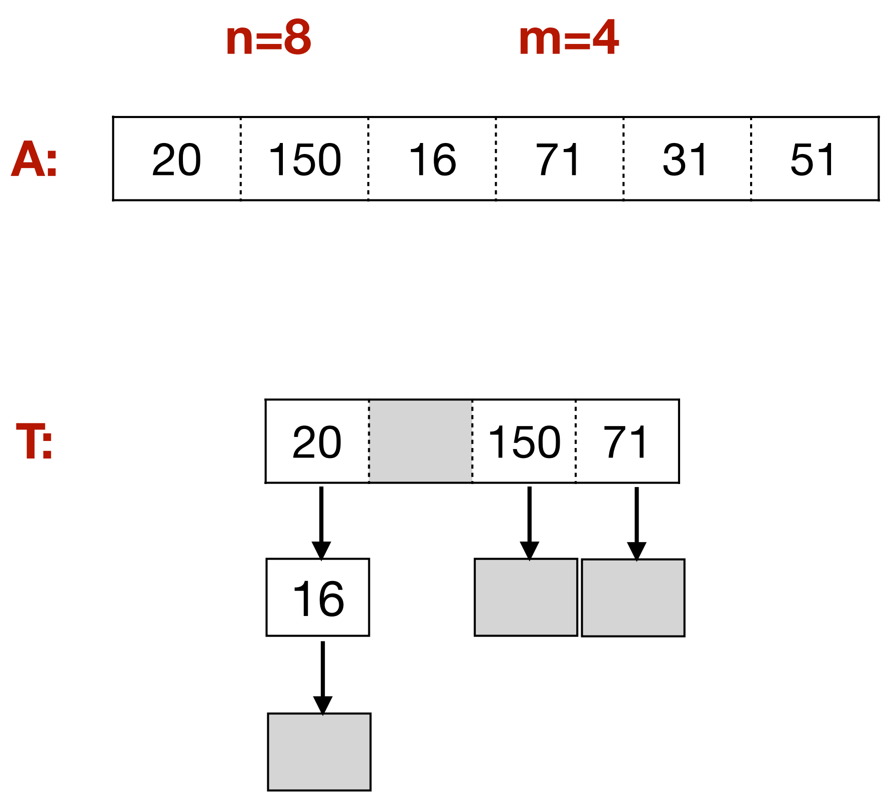

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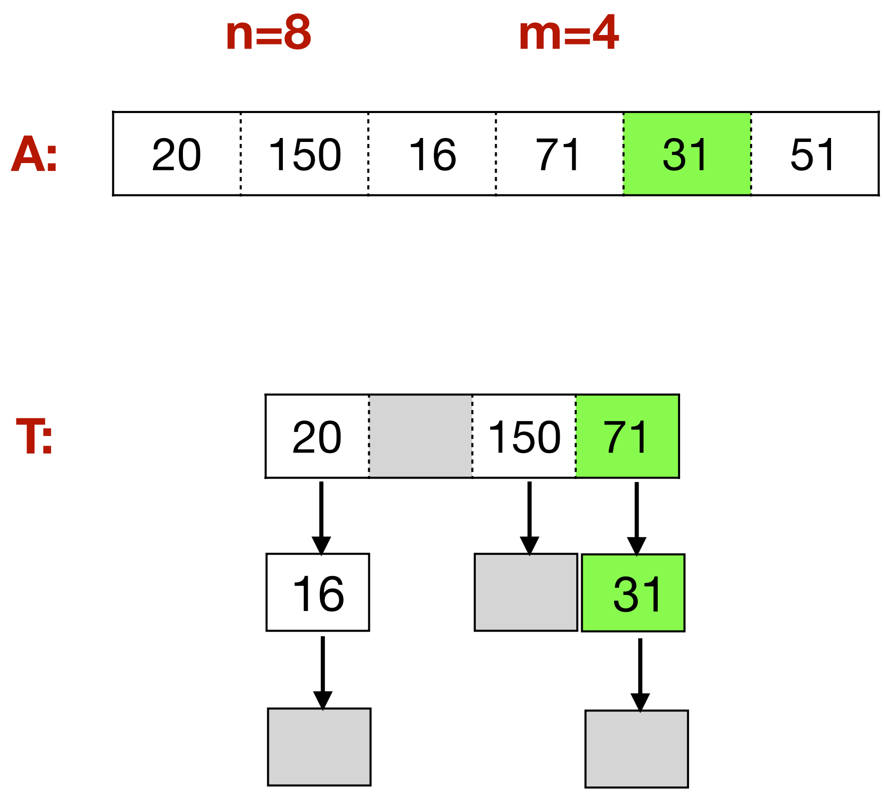

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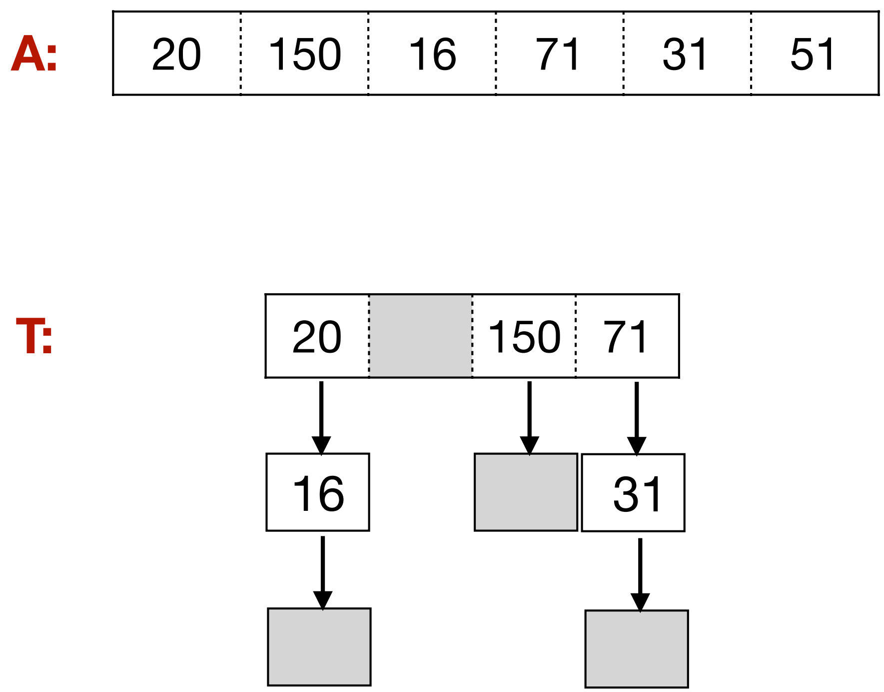
n=8 **m=4**



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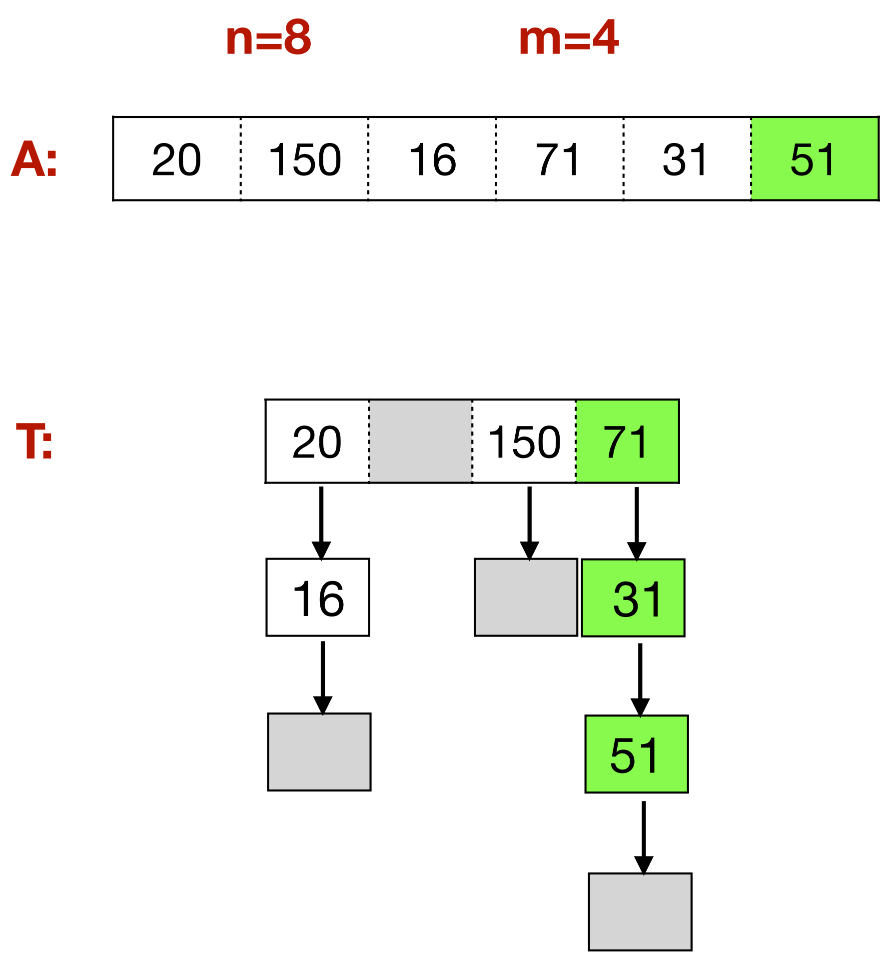

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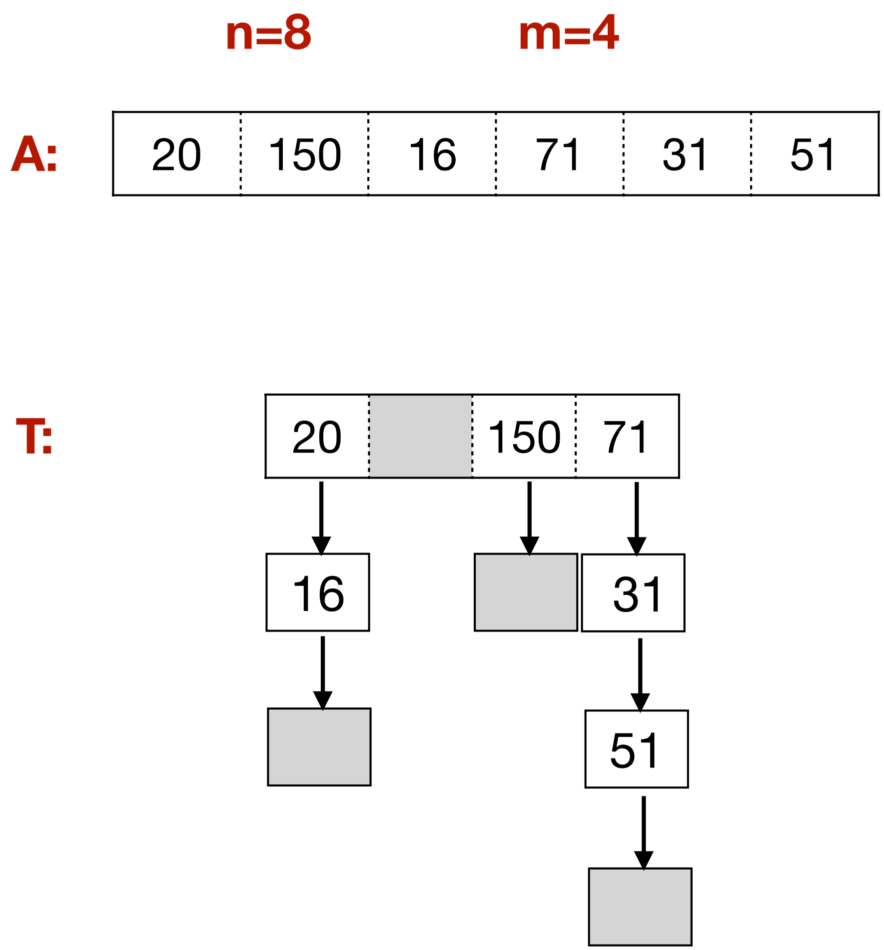
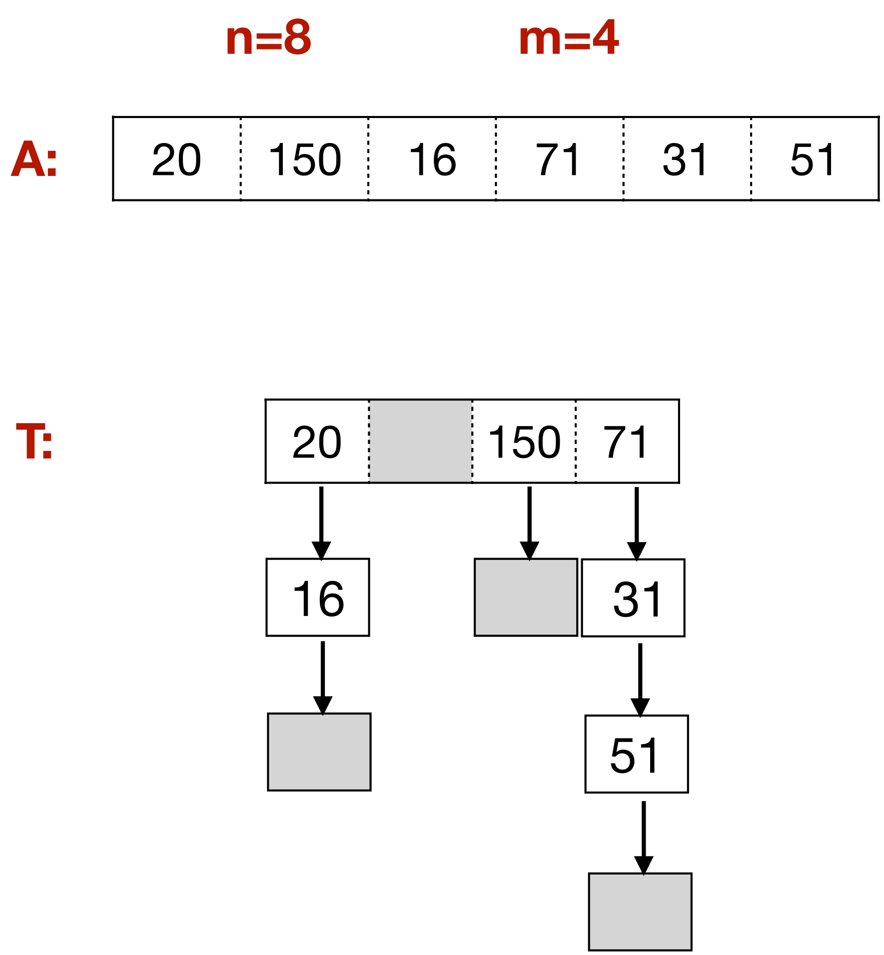


Chaining : Example

- $h(x) = (x \mod 4)+1$
- Search for x=31
  - h(31) = 4



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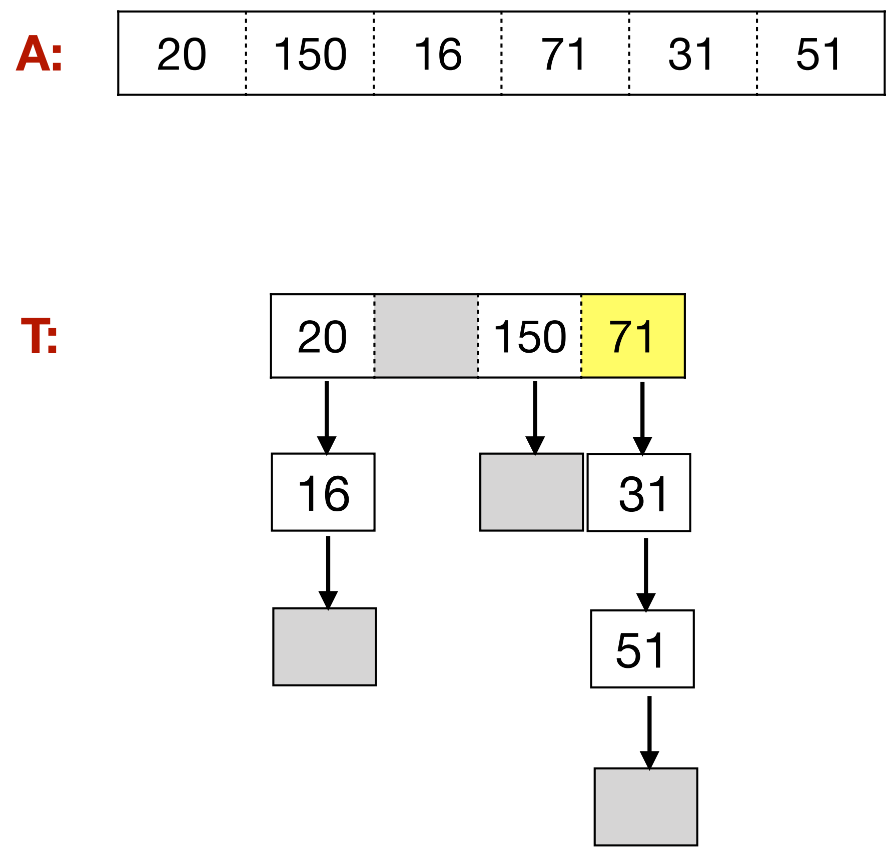

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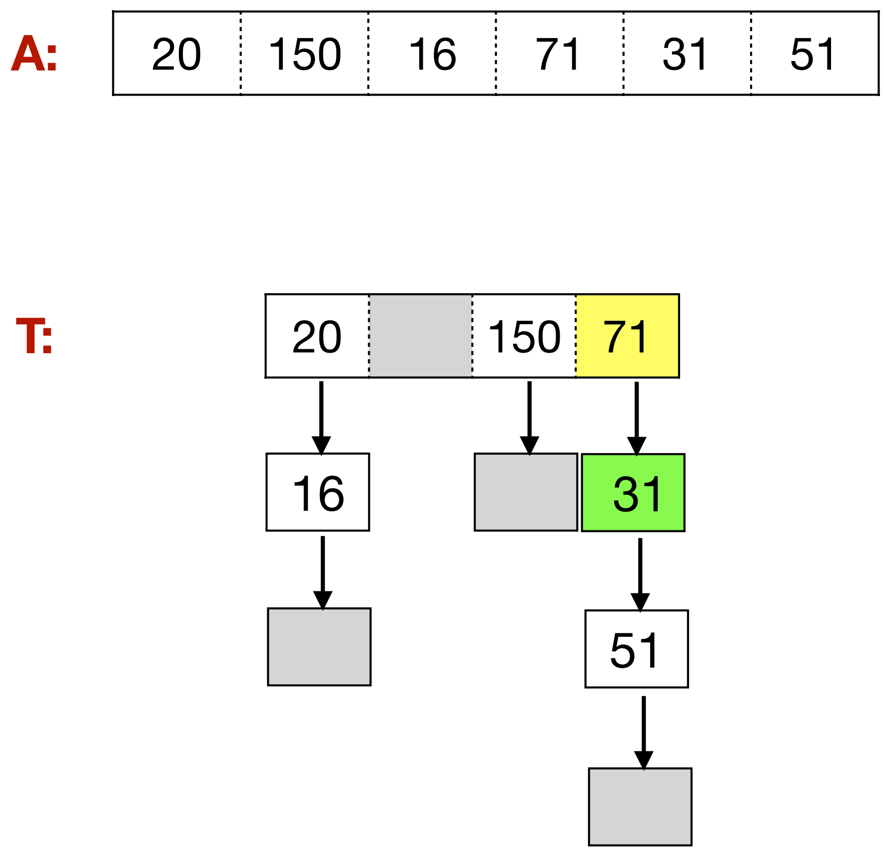
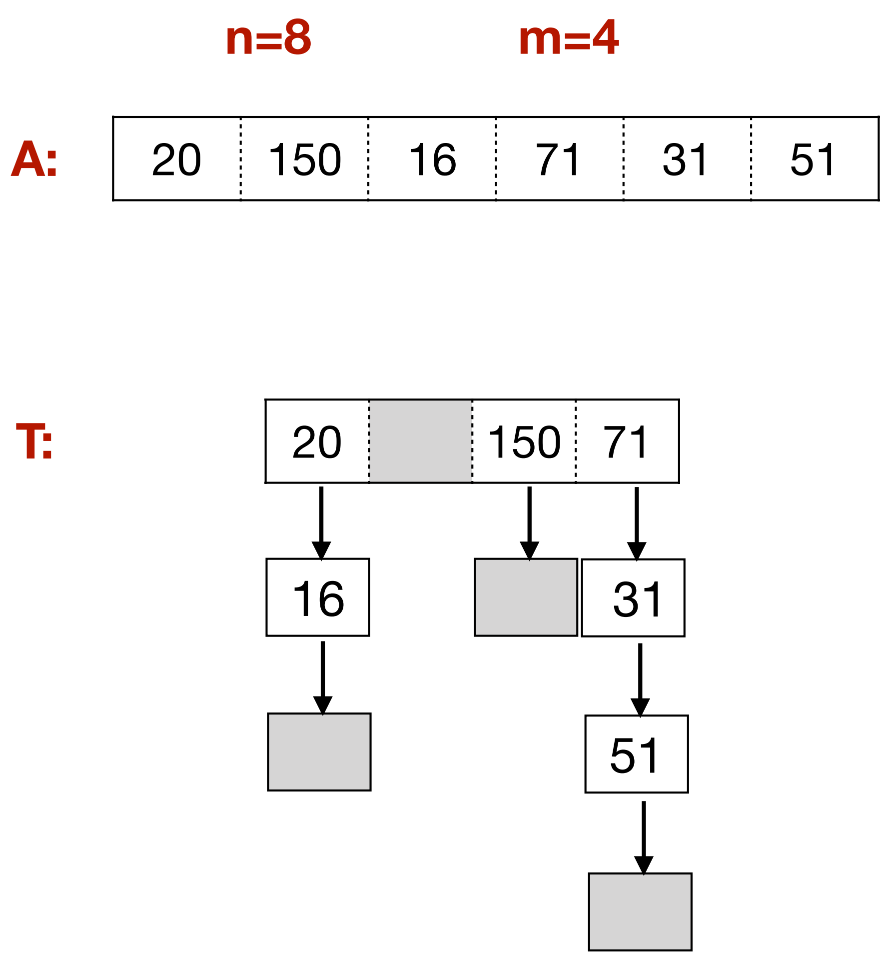


Chaining : Example

- $h(x) = (x \mod 4)+1$
- Search for x=31
  - h(31) = 4
- Search for x=64
  - h(64)=1



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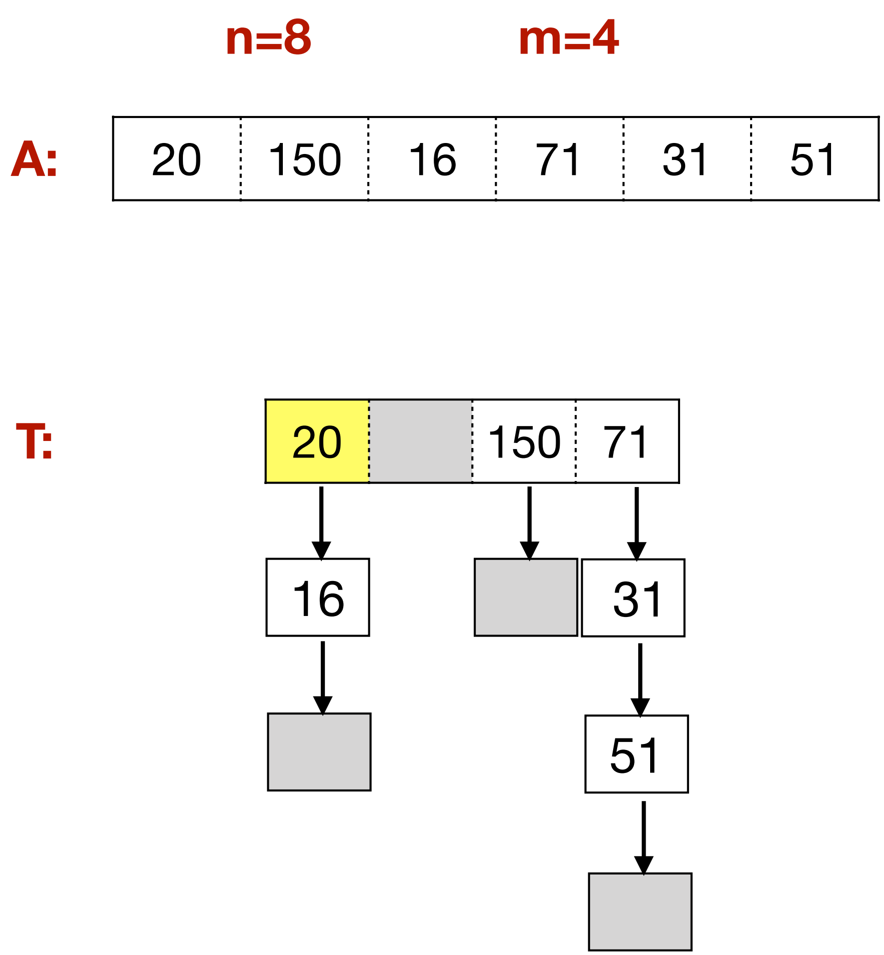

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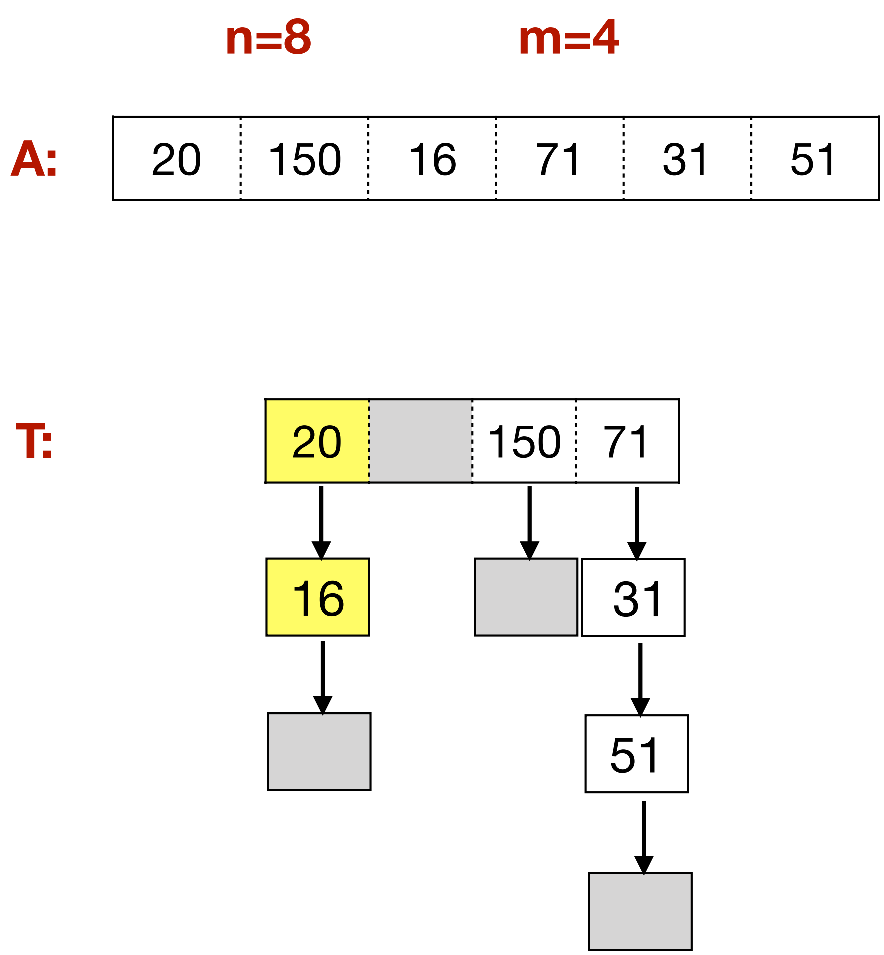

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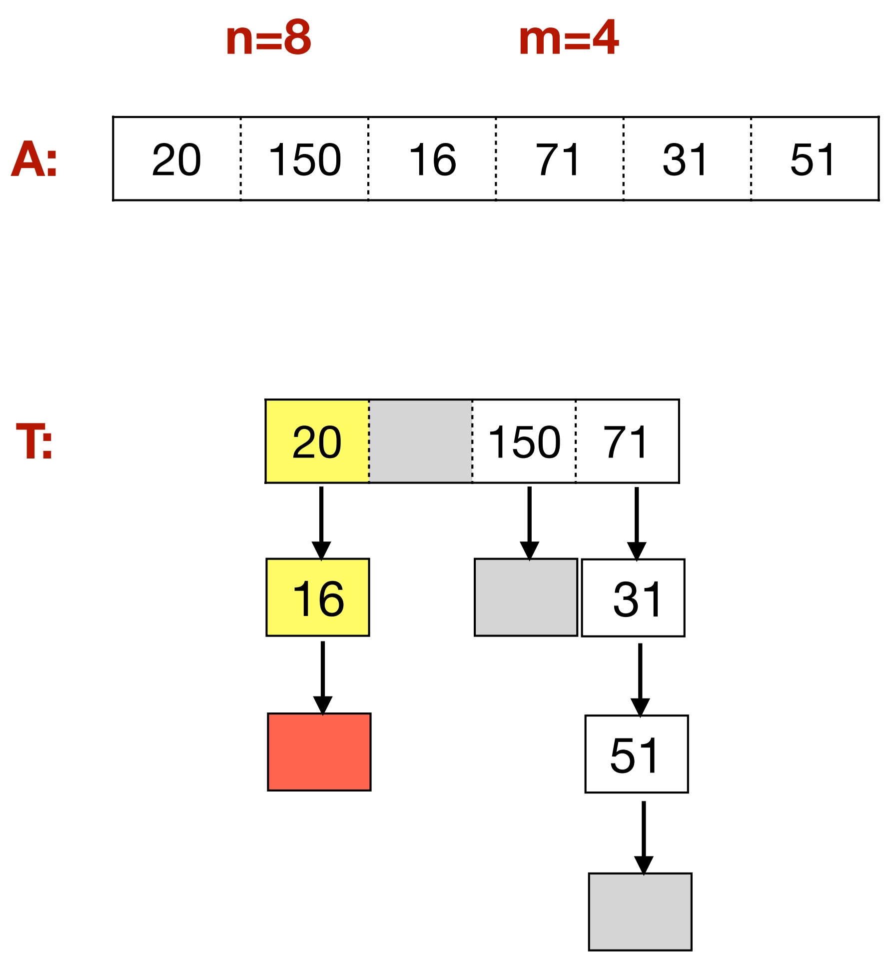

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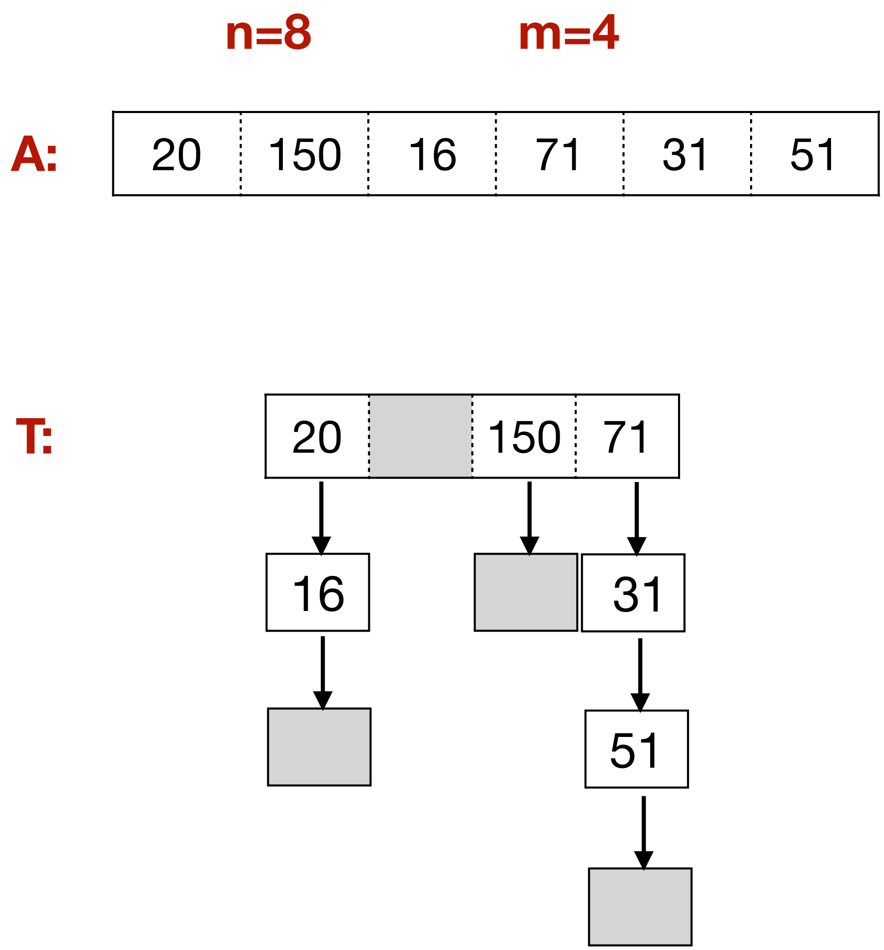


Chaining: Proof of Correctness

- Every element A[i] of is added to the linked-list of T[h(A[i])] and no other number appears in the linked-list
- For any number x, it can only be in the linked-list T[h(x)] and we search the entire list for it

Chaining: Runtime Analysis

- What is worst-case runtime of **search**(x)?

Chaining: Runtime Analysis

- What is worst-case expected runtime of **search**(x)?

Chaining: Runtime Analysis

- Worst-case expected runtime of **search**(x) using near-universal hash functions:
  - Define $\ell(x)$ as the number of elements in A[1:n] mapped to x by the hash function h
  - Runtime of **search**(x) is $O(1 + \ell(x))$
- Worst-case expected runtime of **search**(x) is $O(1 + \mathbf{E}_{h \in \mathscr{H}}[\ell(x)])$

Chaining: Runtime Analysis

- So worst-case expected runtime is:

$$O(1 + \mathbf{E}_{h \in \mathscr{H}}[\ell(x)]) = O(1 + \frac{n}{m})$$

Called **Load**
ratio of number of elements to hash to the size of hash table

Dynamic Programming:
It is just Smart Recursion

Dynamic Programming?

- A technique for speeding up recursive algorithms
- Proof of correctness: exactly the same as recursive algorithms
- Runtime? Will become much faster typically

Fibonacci Numbers

Recursive Algorithm

- **RecFibo**(n):
  - If n=1 or n=2, return 1
  - Otherwise, return **RecFibo**(n-1) + **RecFibo**(n-2)

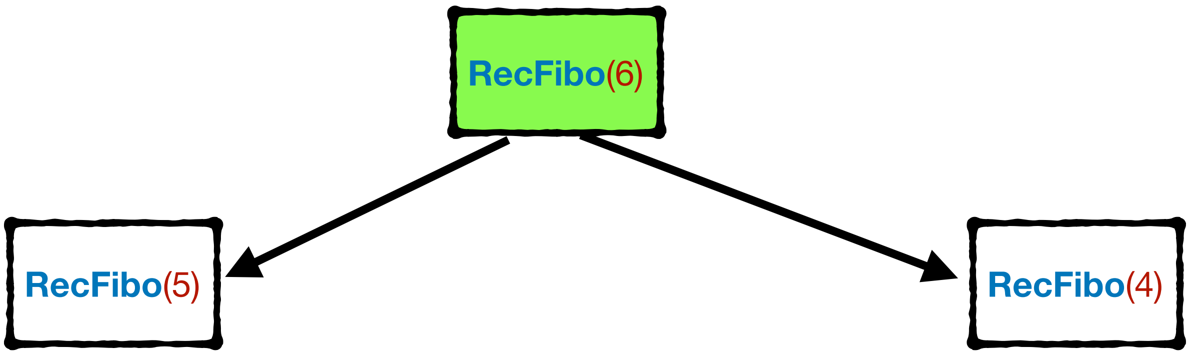
Recursive Algorithm

- **RecFibo**(n):
  - If n=1 or n=2, return 1
  - Otherwise, return **RecFibo**(n-1) + **RecFibo**(n-2)
- Proof of correctness: This is literally the same formula as F(n)!
- Runtime analysis?

RecFibo(6):

RecFibo(6)

RecFibo(6):



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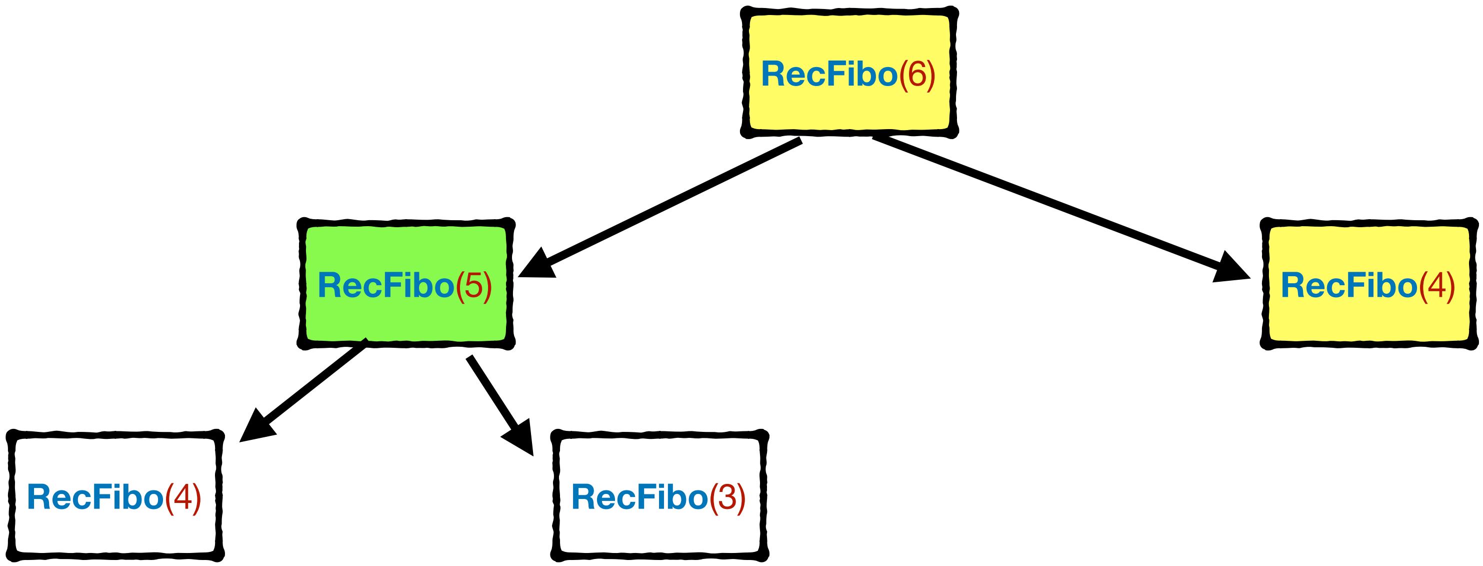

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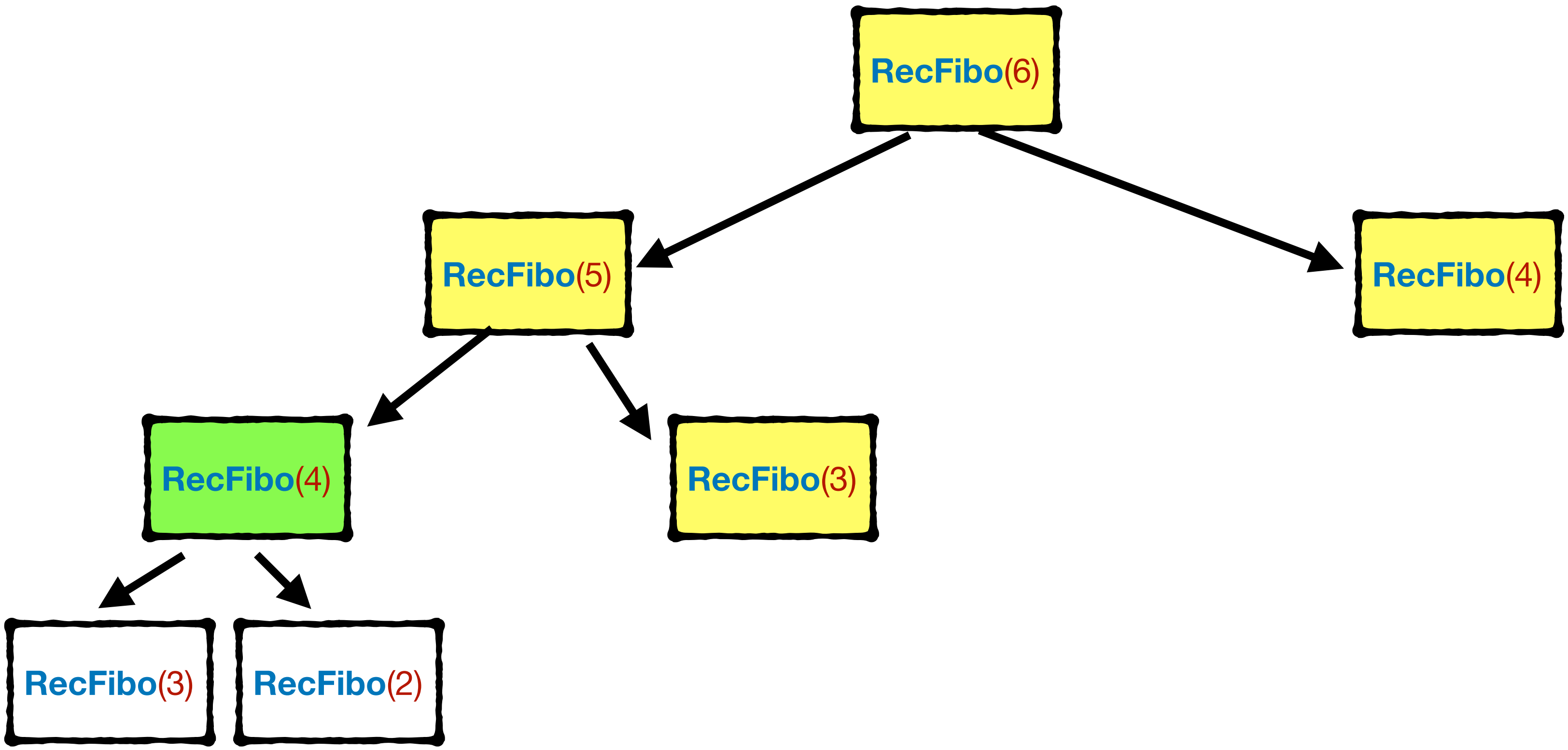
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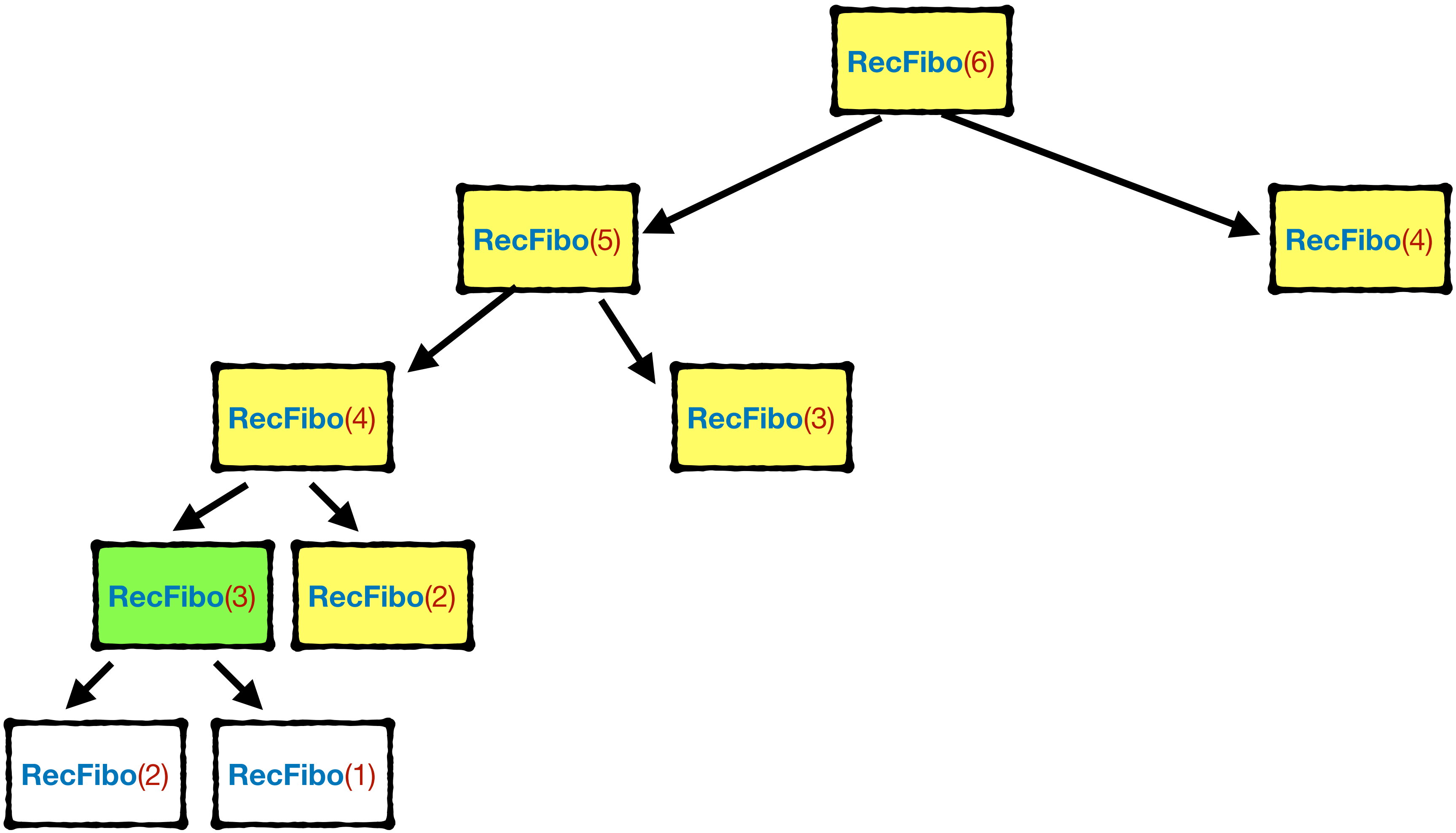
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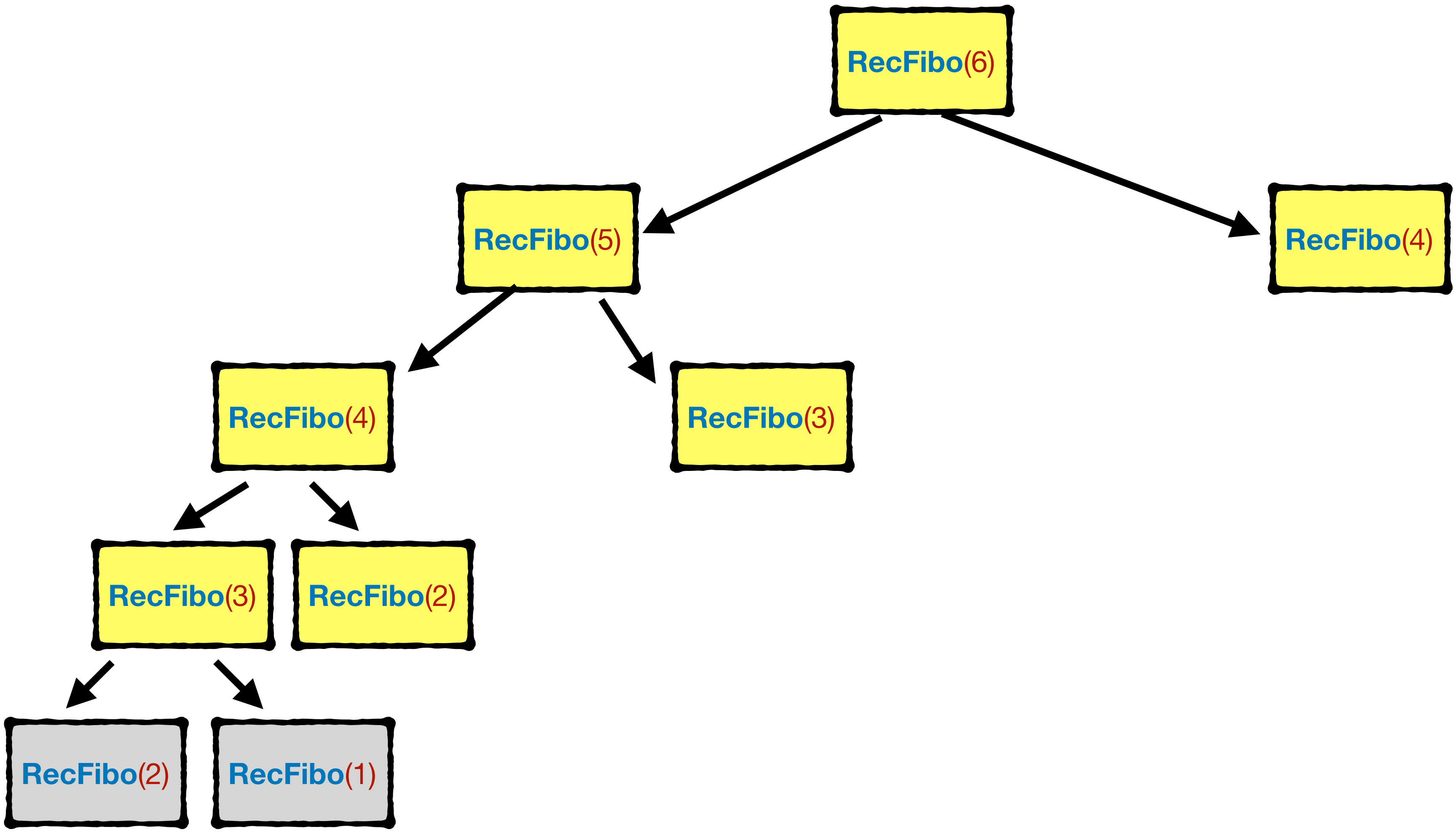
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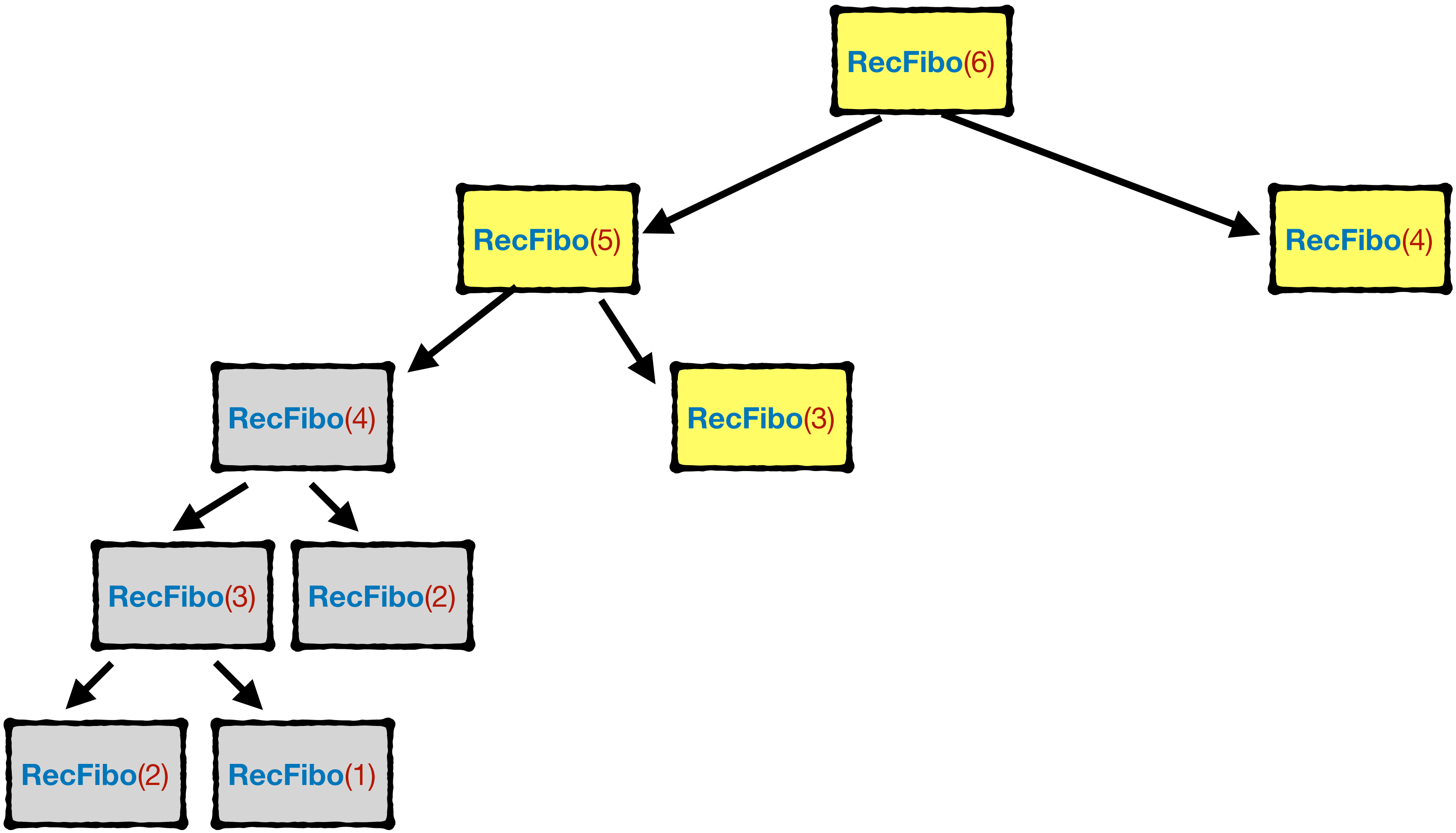
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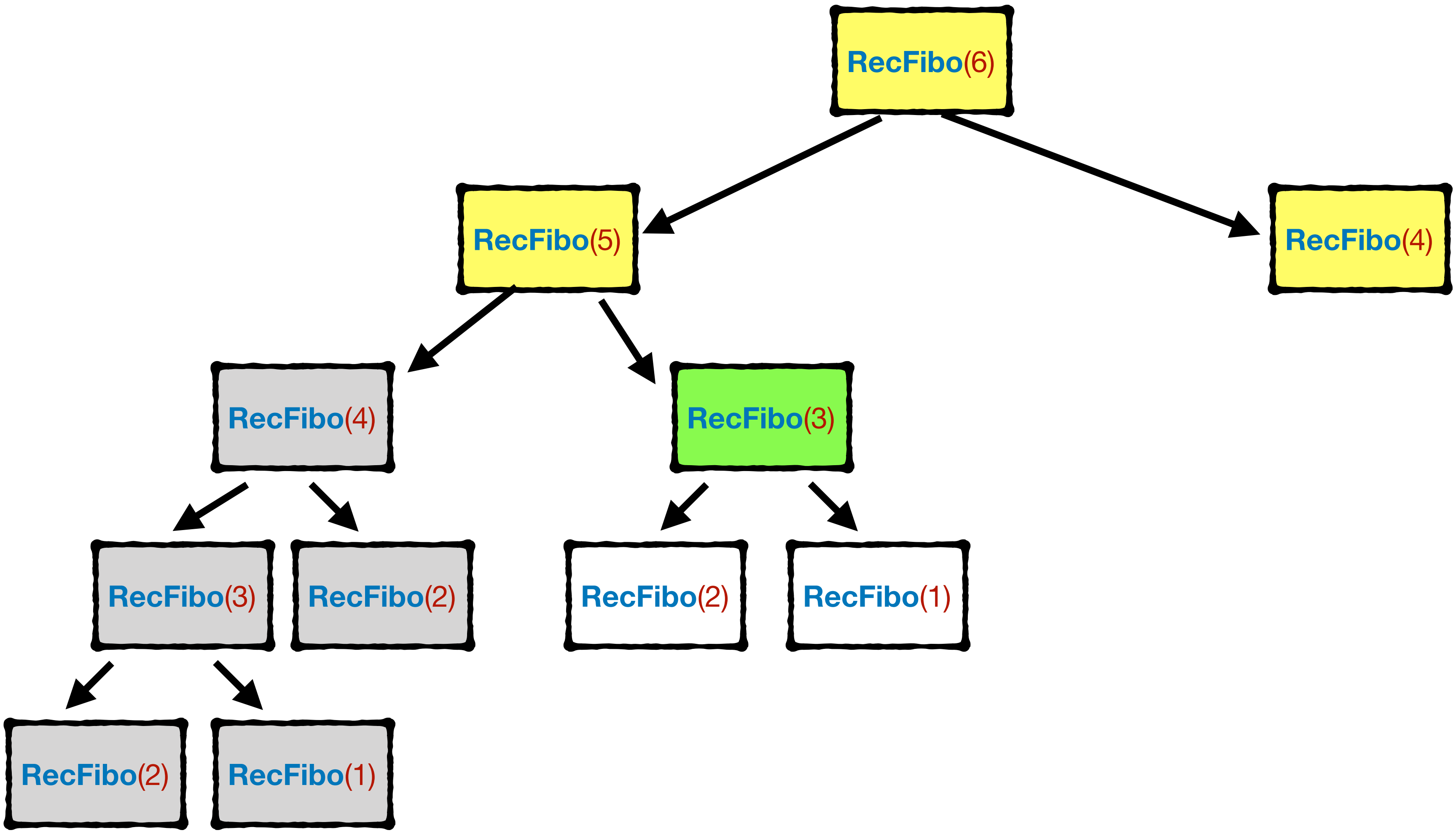

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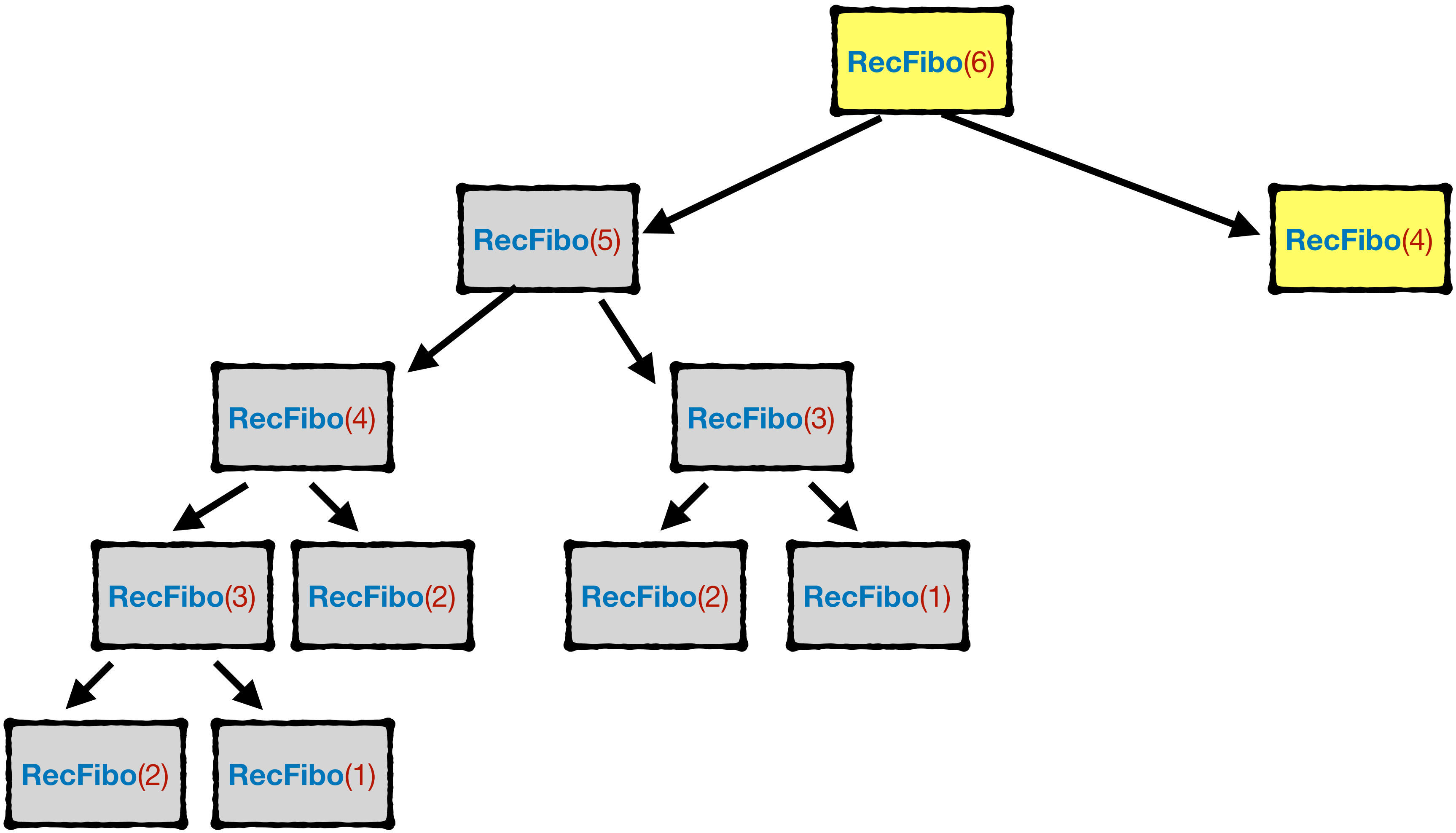
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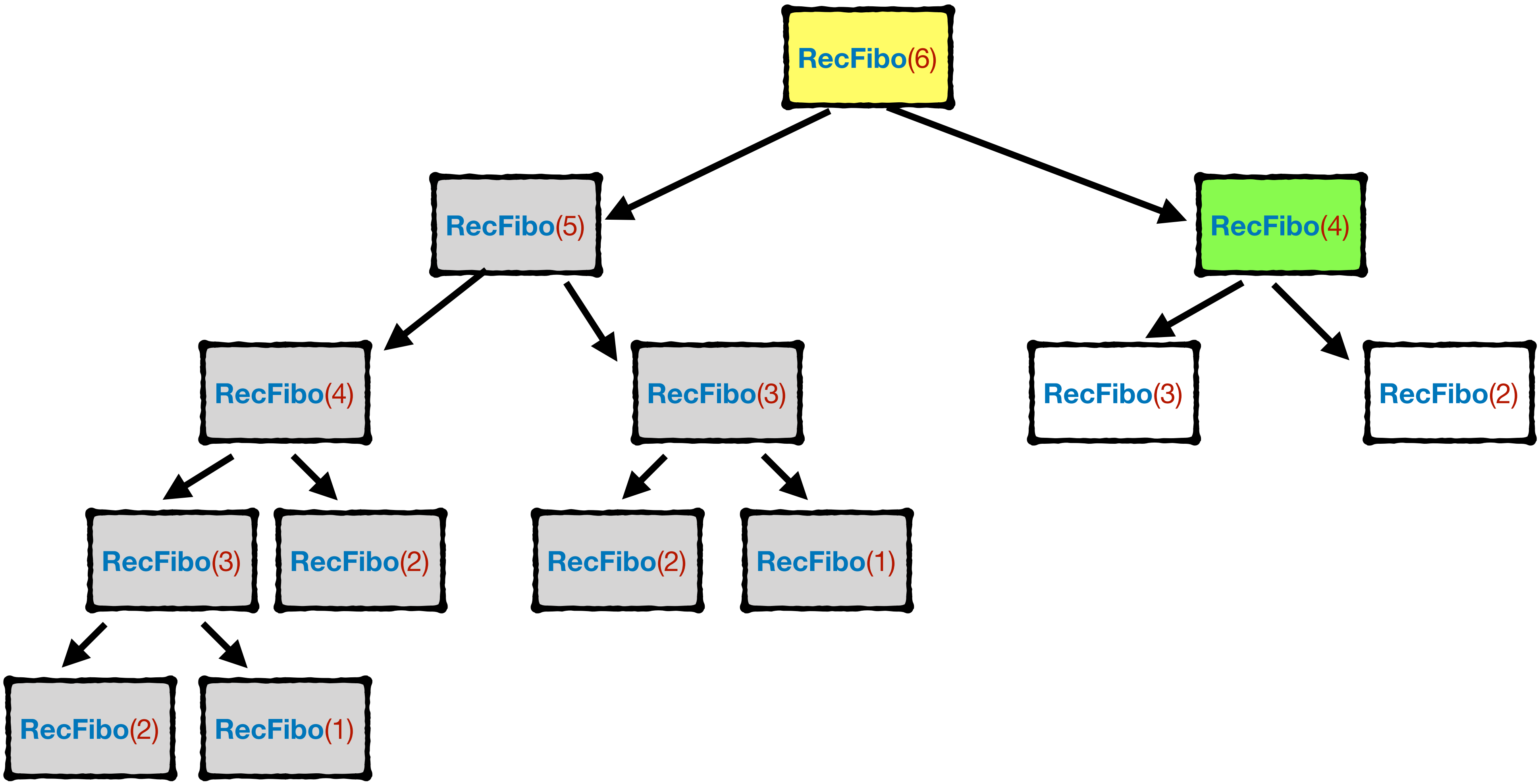
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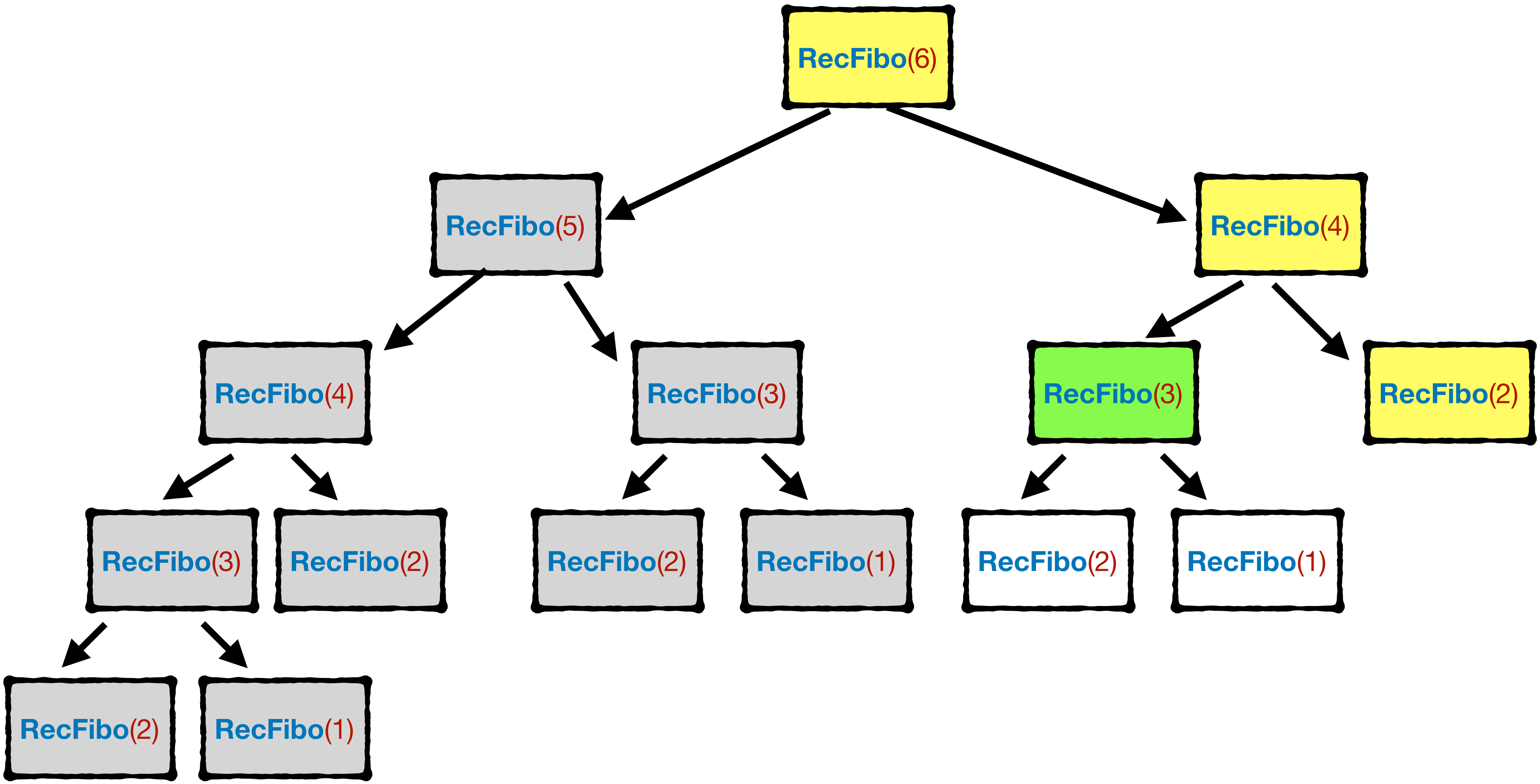
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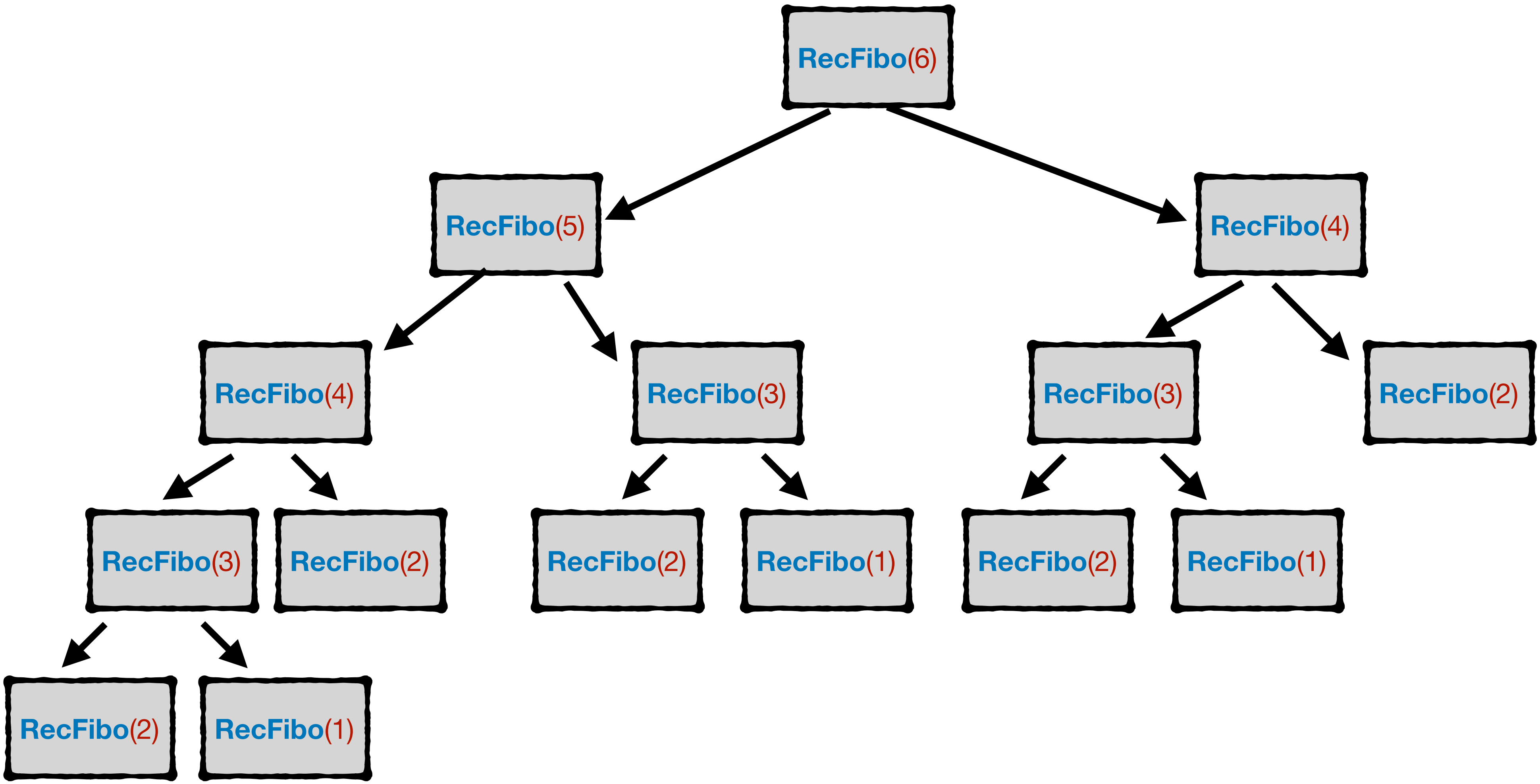
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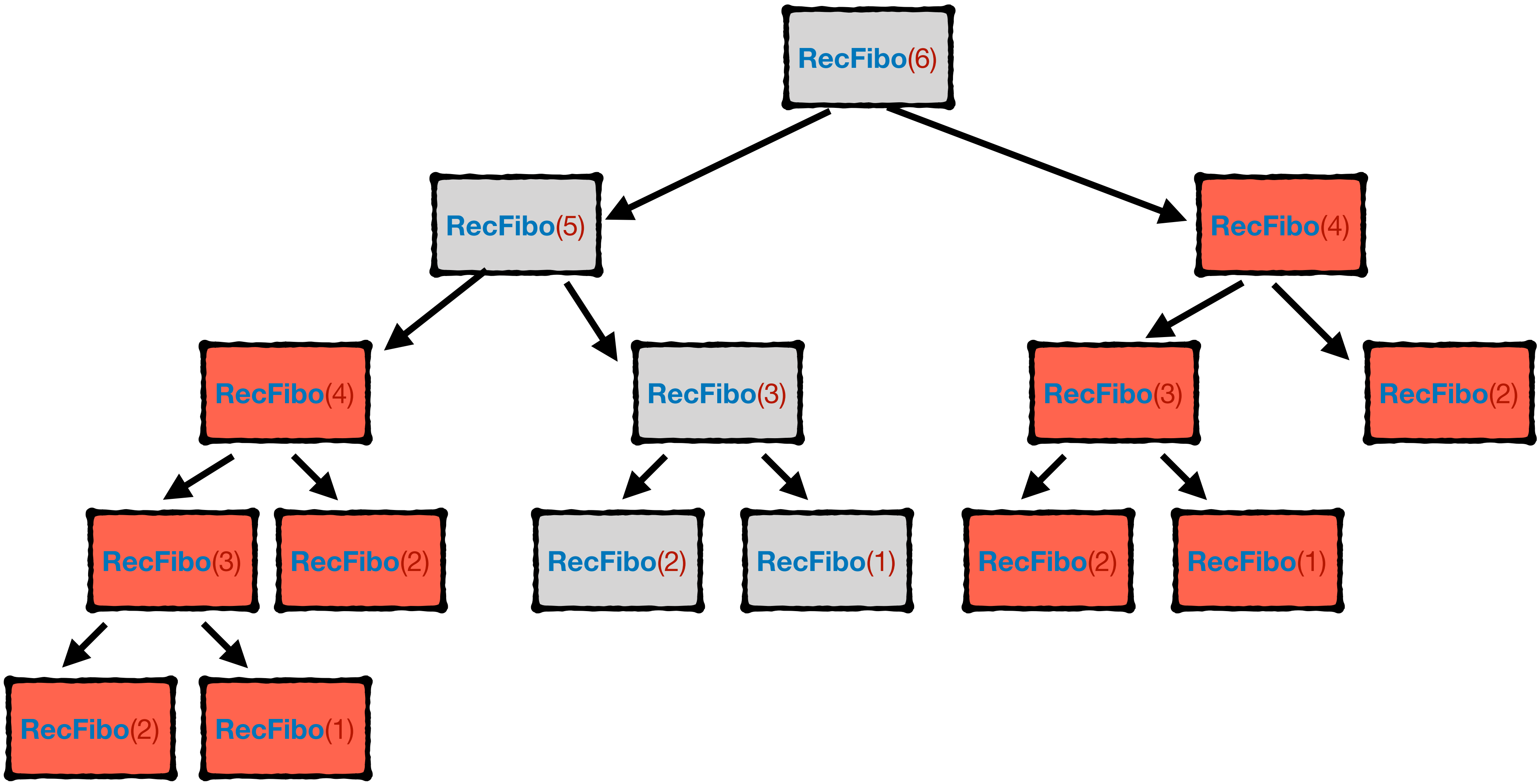
What is wrong with this figure?



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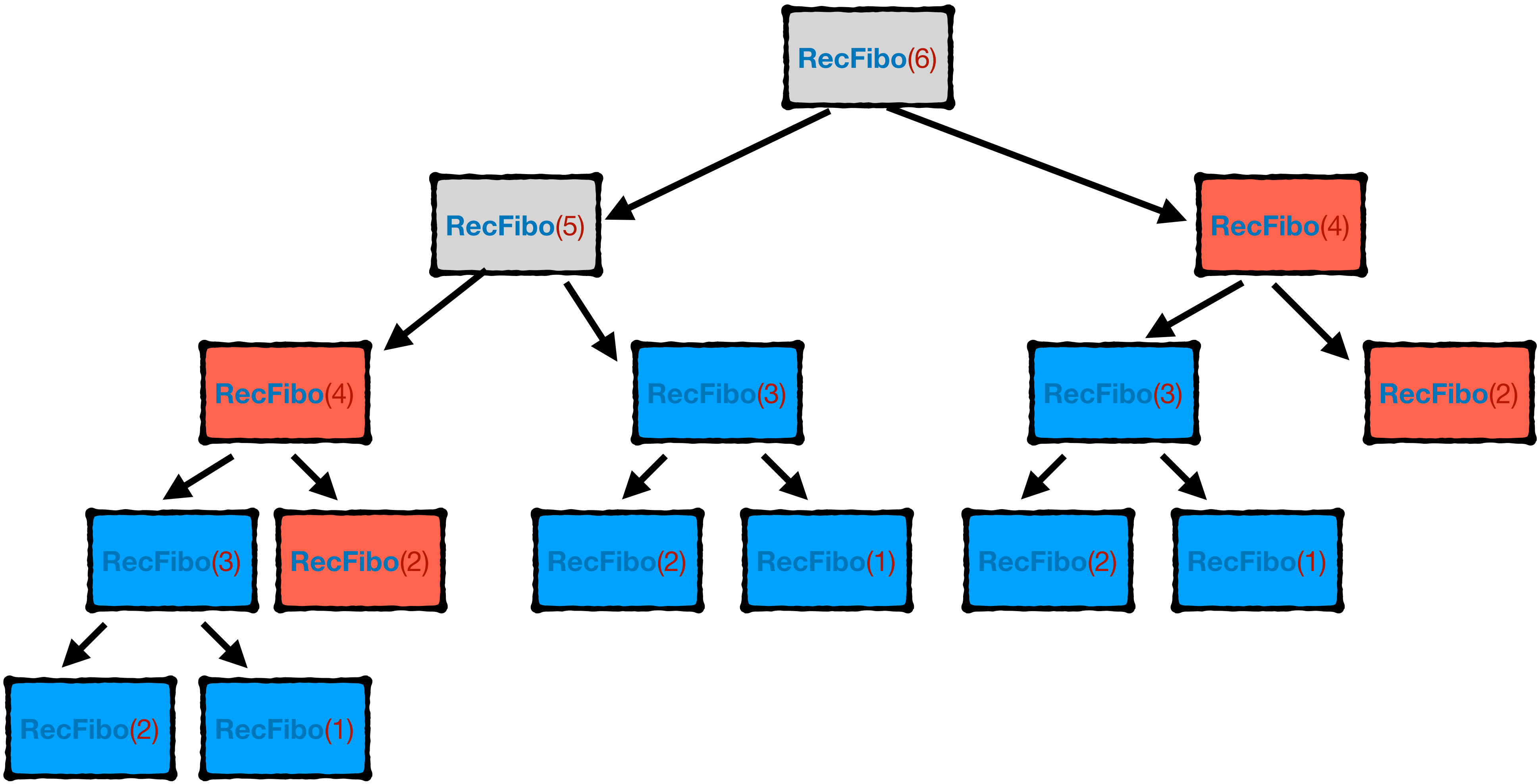
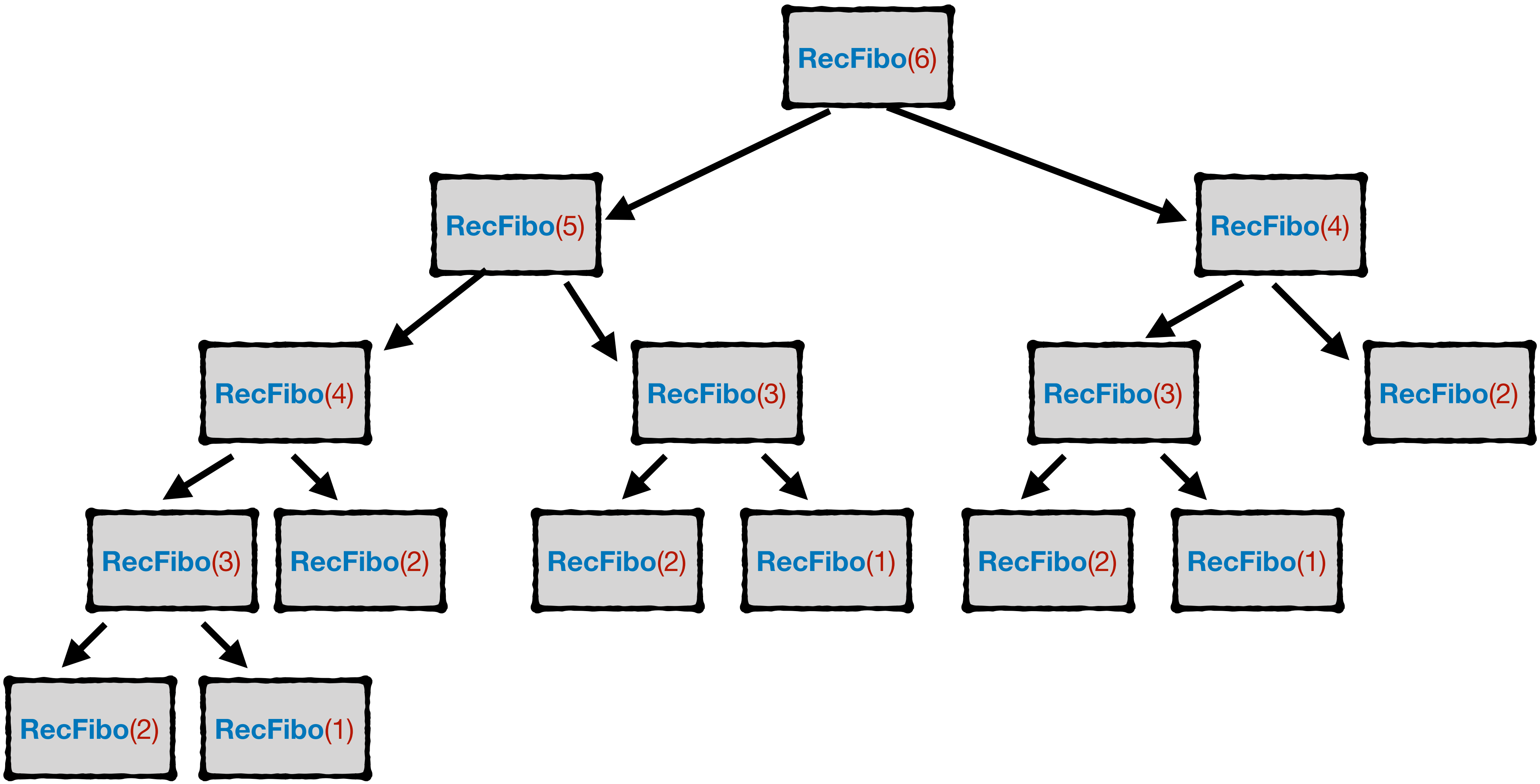


An "ideal" figure?



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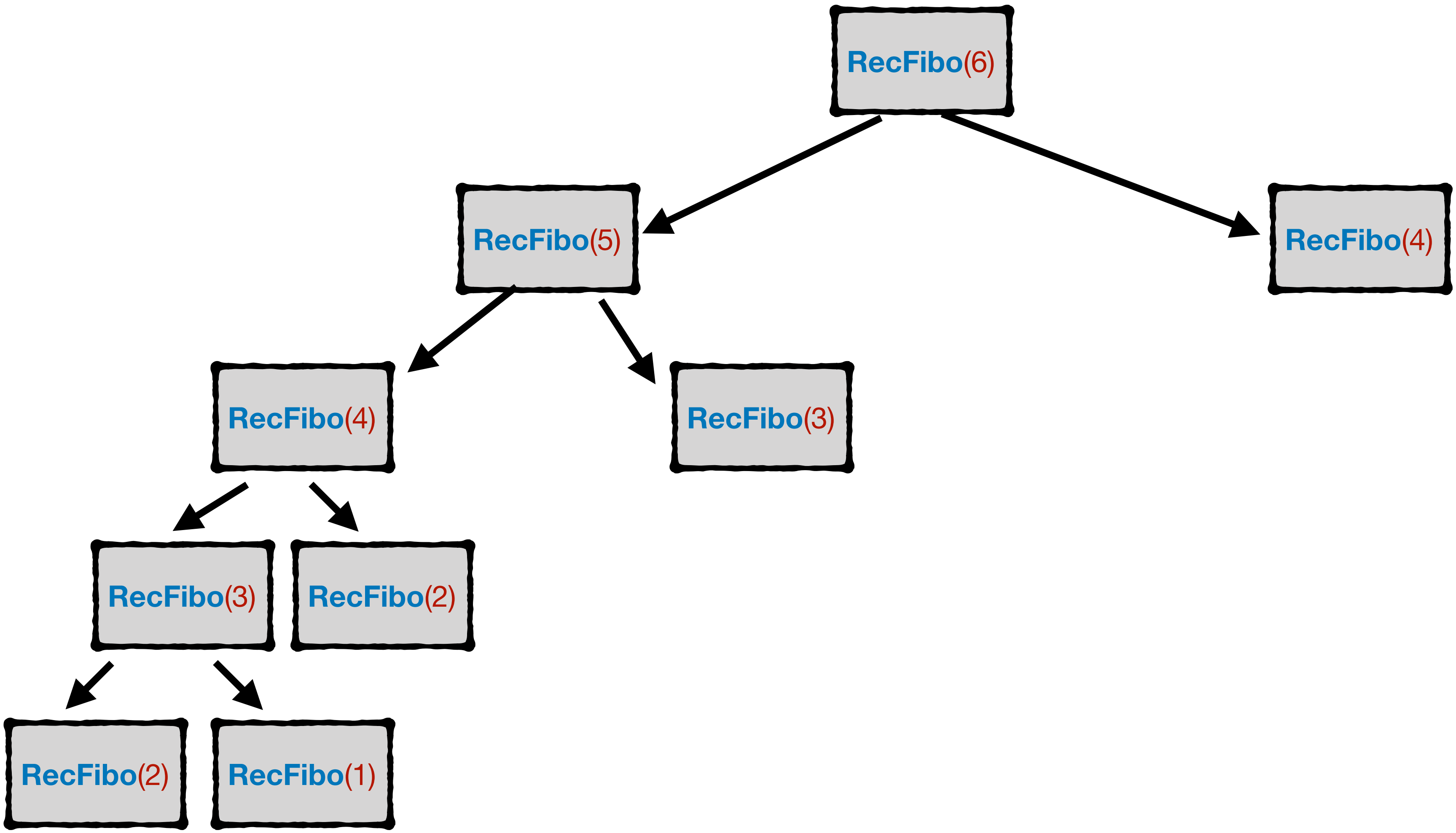

Memoization

- Pick an array S[1:n] initialized with ‘undefined’ in every entry
- Whenever we compute F(i), store it in S[i]
- Next time, instead of recomputing F(i), just return S[i]

Fibonacci Numbers with Memoization

- Initialize an array S[1:n] with 'undefined' in every entry
- **MemFibo**(n):
  - If S[n] $\neq$ 'undefined', return S[n]
  - If n=1 or n=2, let S[n] = 1
  - Otherwise, let S[n] = MemFibo(n-1) + MemFibo(n-2)
  - Return S[n]

Fibonacci Numbers with Memoization

- Proof of Correctness:
  - Nothing has changed in the logic of algorithm between **MemFibo** and **RecFibo**
  - So **MemFibo** is also correct

Fibonacci Numbers with Memoization

- Runtime Analysis:
  - We spend $O(1)$ time, ignoring recursive calls, for each subproblem **MemFibo**(m) for any $1 \leq m \leq n$
  - We will **not** compute a subproblem more than once
  - So the runtime is O(n)

MemFibo(6)

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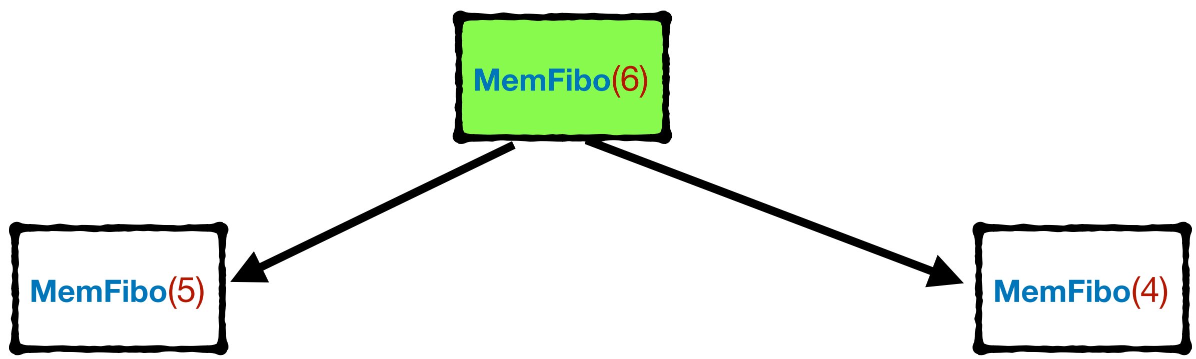

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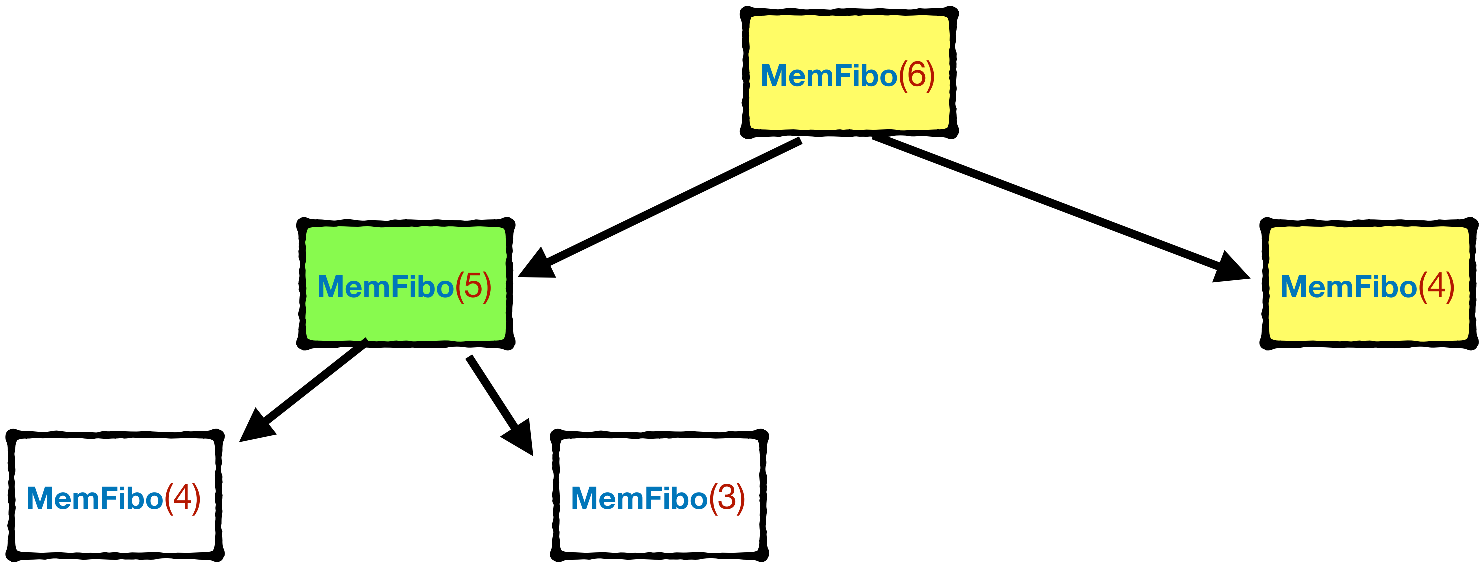

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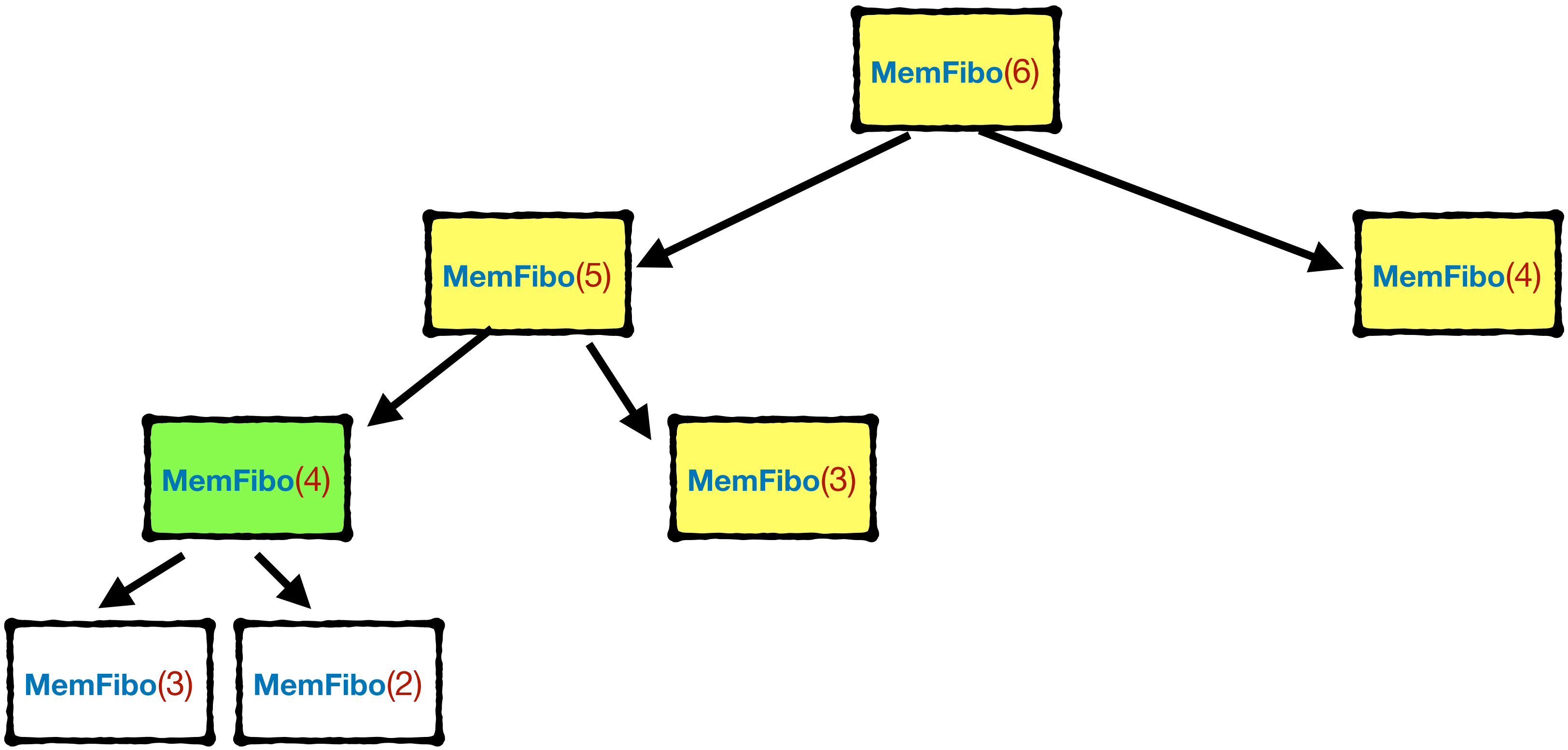

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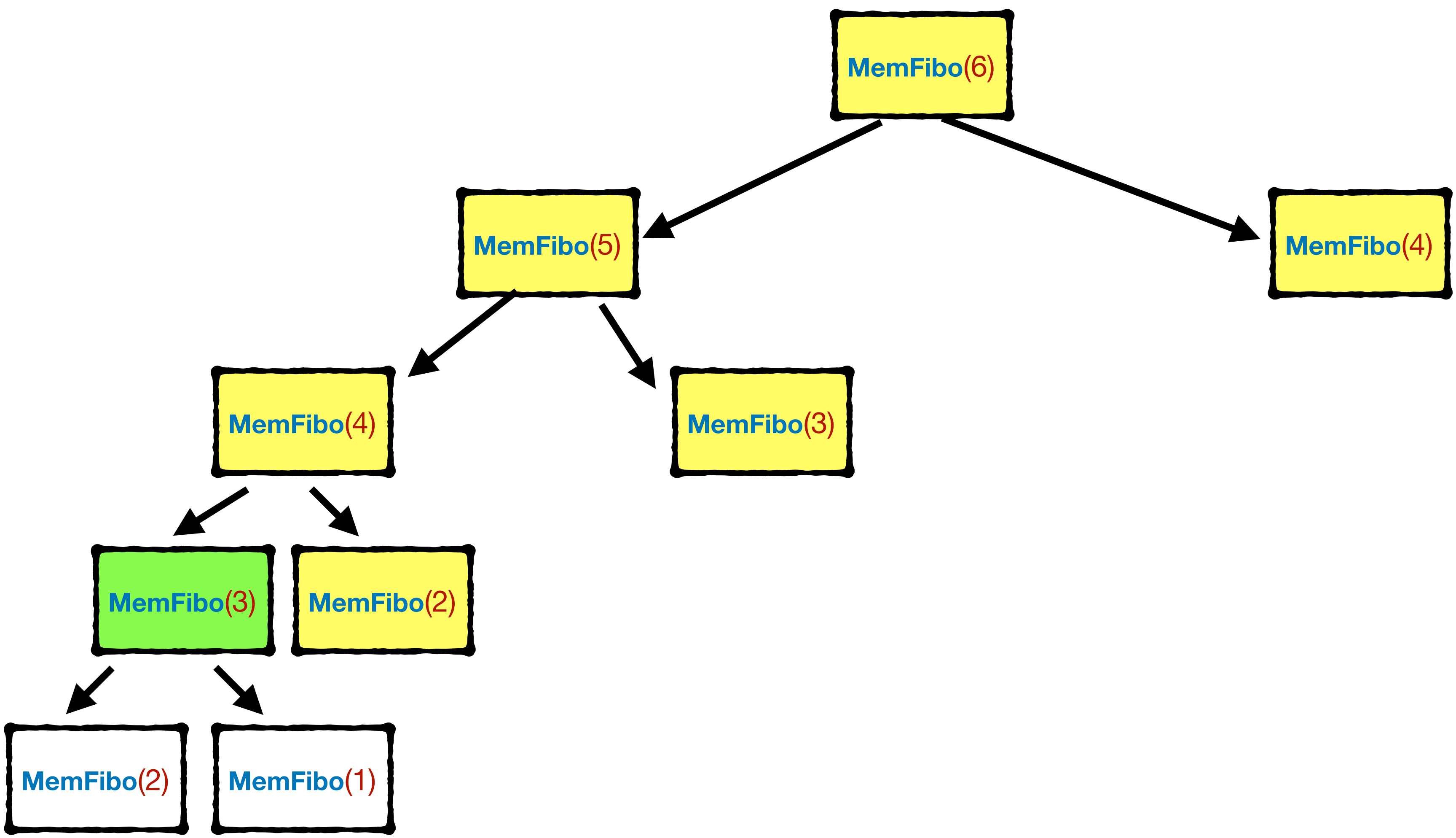
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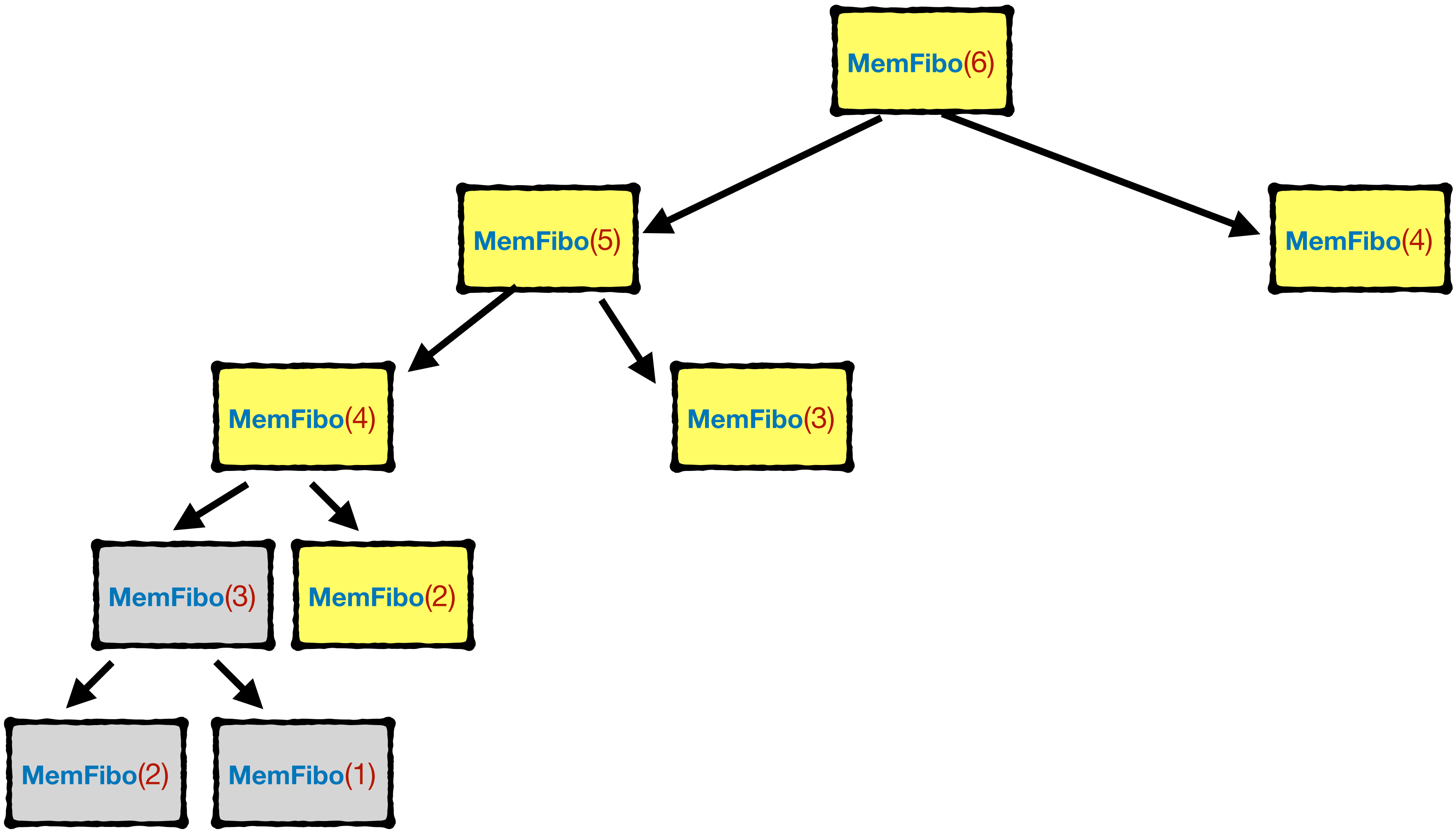
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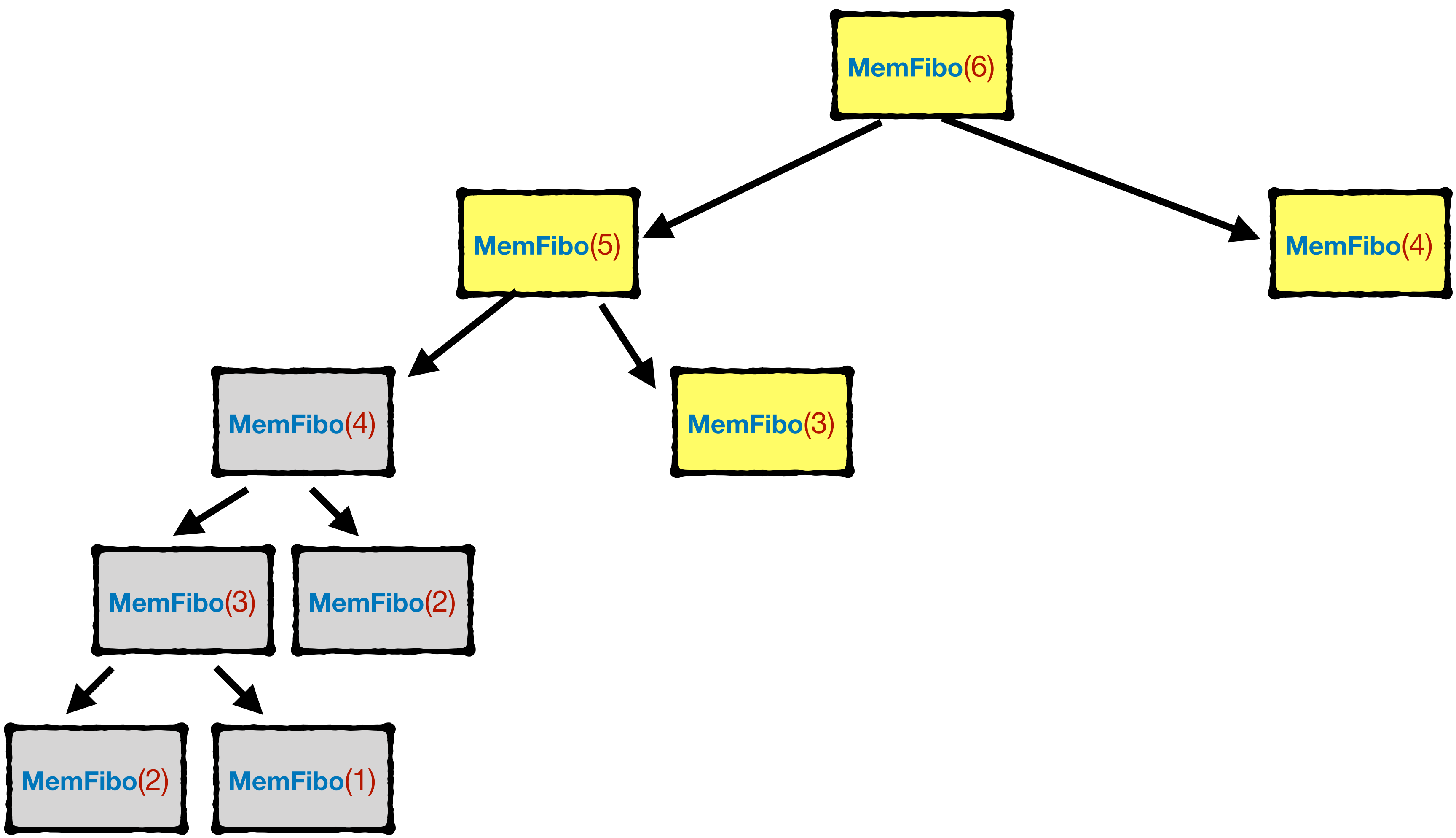
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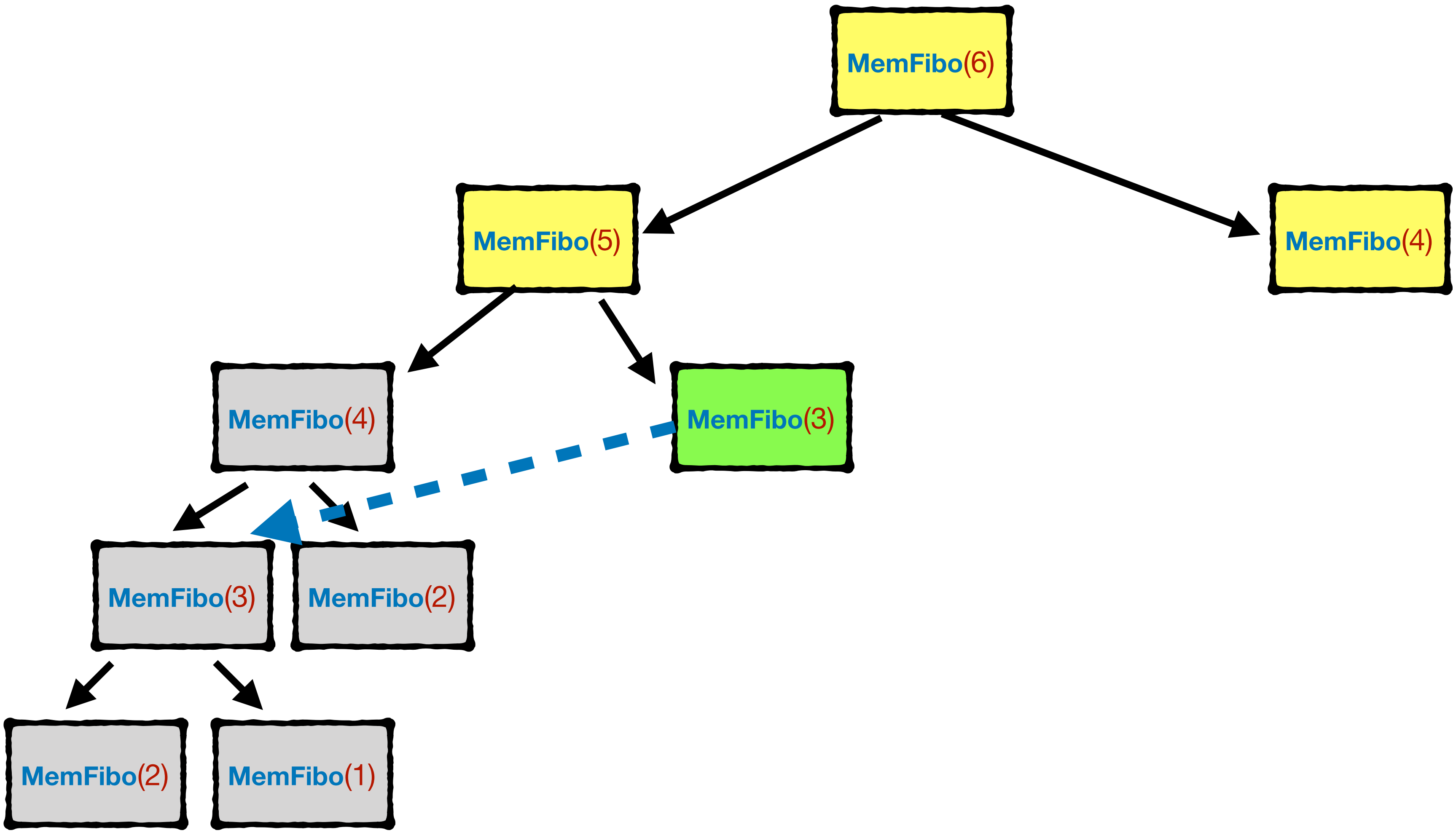

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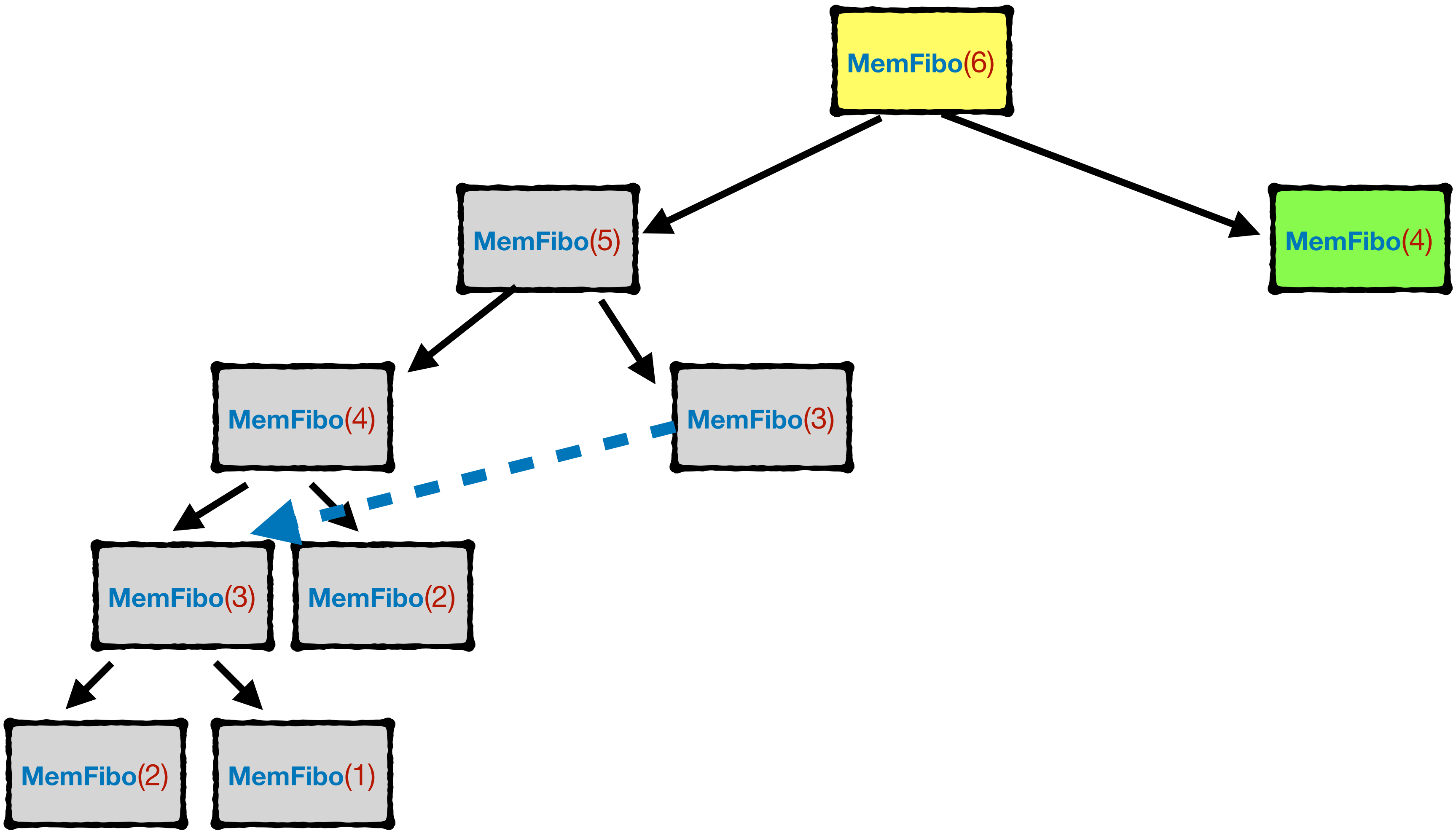

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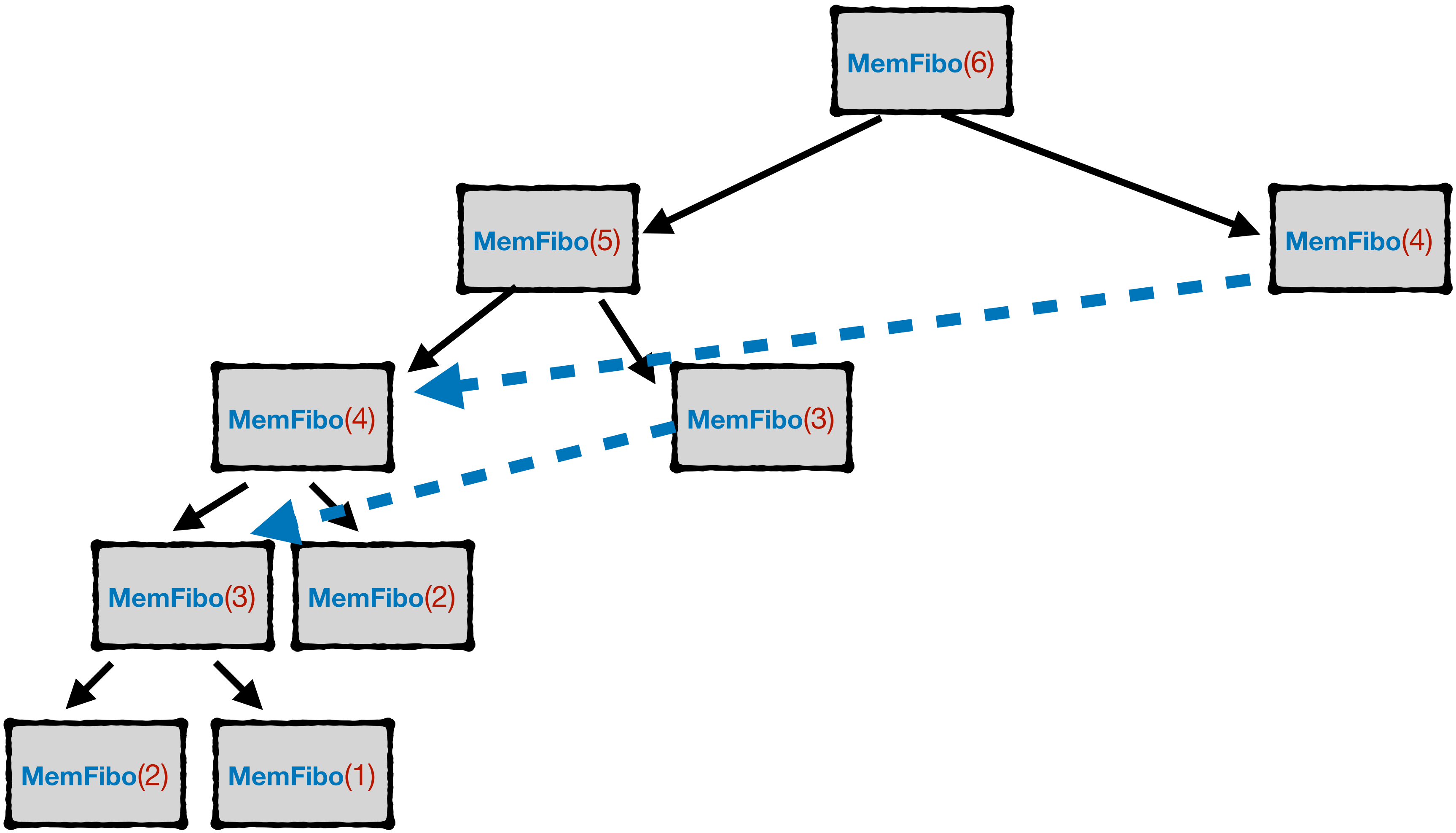
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MemFibo(1)

Beyond Memoization?

- Memoization is a form of dynamic programming
- We literally only need to store the answers to our recursive calls so we do not recompute them
- It is usually called **top-down** dynamic programming
- There is also a notion of **bottom-up** dynamic programming
  - A way of stating dynamic programming without recursive calls

Bottom-Up Dynamic Programming

- We can literally fill up the array of stored answers ourselves
- **DynFibo**(n):
  - Create an empty array S[1:n]
  - Let S[1] = S[2] = 1
  - For i = 3 to n: let S[i] = S[i-1] + S[i-2]
  - Return S[n]

Bottom-Up Dynamic Programming

- Runtime analysis?
  - A for-loop of length $O(n)$
  - $O(1)$ runtime per each iteration of for-loop
  - $O(n)$ time in total then

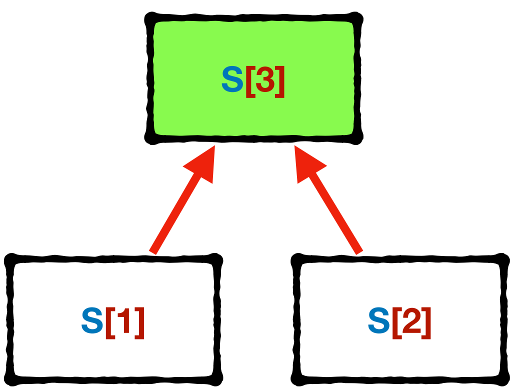
DynFibo(6)

S[1]

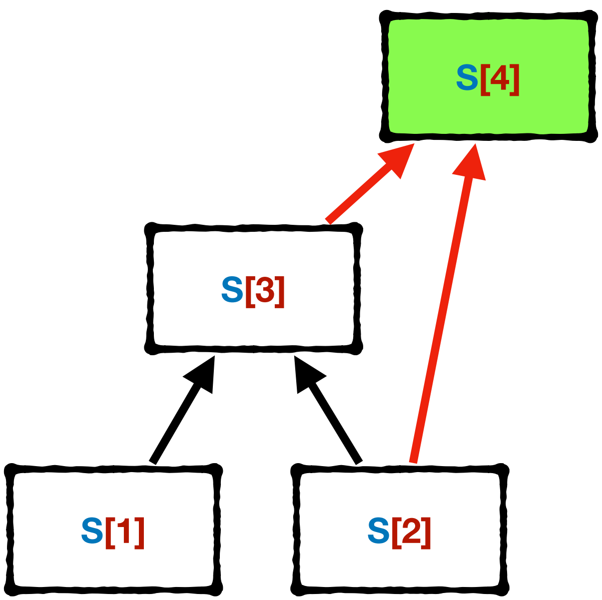
DynFibo(6)

S[1] S[2]

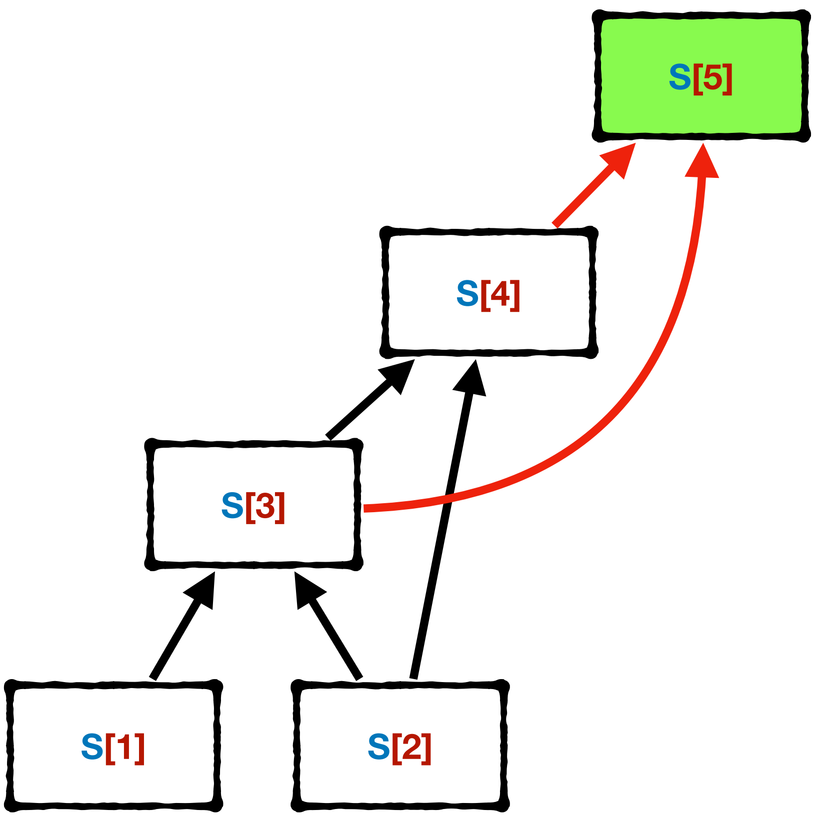
DynFibo(6)



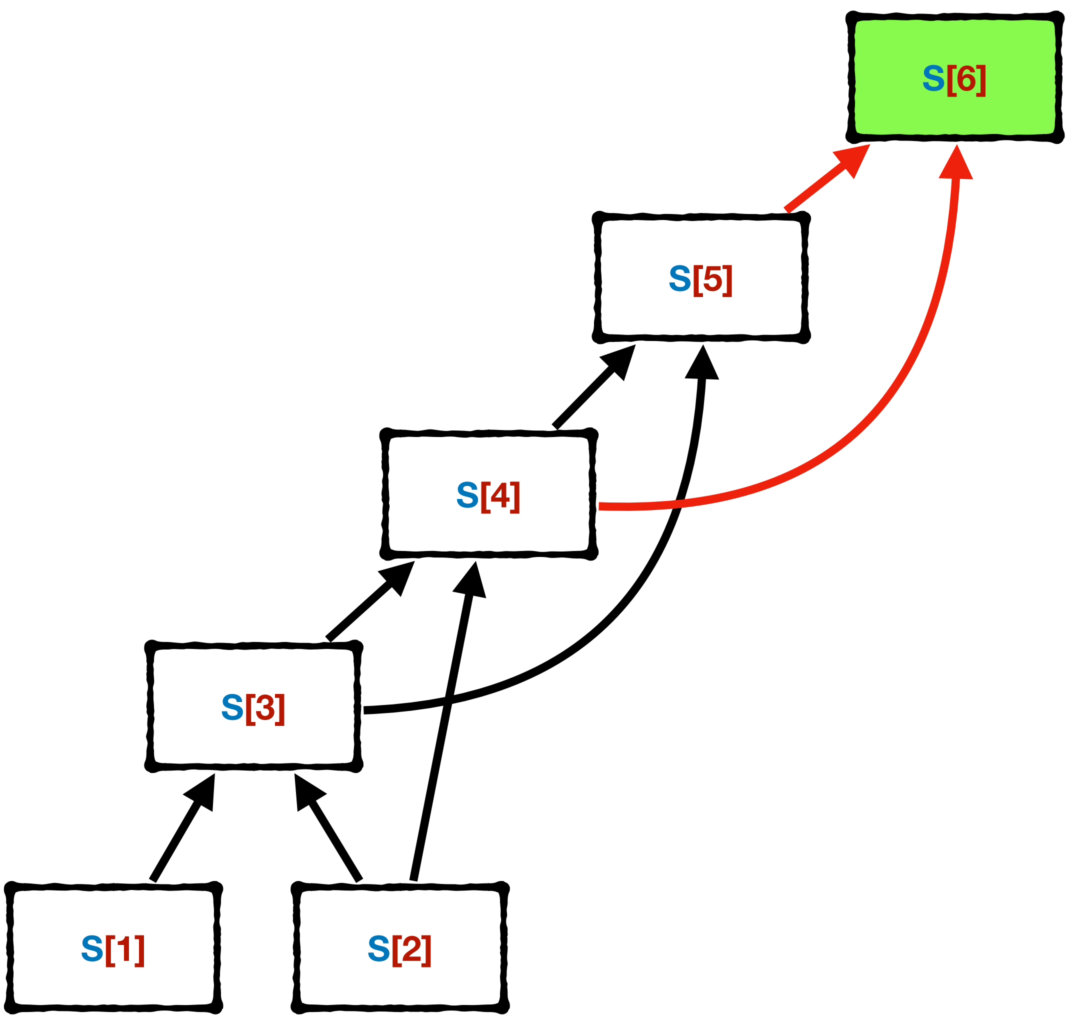
DynFibo(6)



DynFibo(6)



DynFibo(6)



Elements of Dynamic Programming

Dynamic Programming?

- Write a recursive formula for the problem we want to solve
- Write a recursive function that computes the recursive formula
- Use a table to store the answer of recursive function for each recursive call and do not recompute them

Dynamic Programming?

- Write a recursive formula for the problem we want to solve
- Write a recursive function that computes the recursive formula
- Use a table to store the answer of recursive function for each recursive call and do not recompute them
- Two ways:
  - Memoization: Top-Down Dynamic Programming
  - Iterative: Bottom-Up Dynamic Programming

Dynamic Programming?

- Write a recursive formula for the problem we want to solve
- Write a recursi… …sive formula
- Us…
  n…
- Tw…
  - Memoiza…
  - Iterative: Bottom-Up Dynamic Programming

Reminder: Goal of dynamic programming is to speed up computation nothing more

Writing Recursive Formula

- Step One: **Specification**
  - Answer to the question of **“What?”**
  - Describe the problem you want to solve using your formula in plain English
    - Example: “For every $1 \leq i \leq n$, the formula F(i) is supposed to be the i-th Fibonacci number”
  - Describe how the answer to the original problem can be obtained IF we have a solution to the recursive formula
    - Example: Return F(n)”

Writing Recursive Formula

- Step Two: **Solution**
  - Answer to the question of **"How?"**
  - Give a recursive formula for solving for the problem you described in specification by solving the instances of the same problem
    - Example: "$F(1) = F(2) = 1$; for any $i > 2$, $F(i) = F(i-1)+F(i-2)$"
  - Prove that this recursive formula indeed matches the specification provided in the previous step
    - Answer to the question of **"Why?"**

Writing Recursive Formula

- Prove that this recursive formula indeed matches the specification provided in the previous step:
  - Prove that base case of recursive formula is correct
  - Prove that larger values of formula are computed correctly from the smaller values

- Note: this is just induction in disguise

Next Steps?

- Step Three:
  - Use either memoization or bottom-up dynamic programming
  - The choice is entirely up to you
    - Just remember in bottom-up dynamic programming, you have to specify the order of evaluation also

Next Steps?

- Step Four:
  - Analyze the runtime of the algorithm
    - How many subproblems are there?
    - How much time solving each one takes?

- We are done! This is all there is to dynamic programming
- The only thing remained for us is to practice

Dynamic Programming

- Every time you do dynamic programming:
  - Clearly specify subproblems in plain English
  - Design a recursive formula for solving subproblems from smaller ones
  - Prove the correctness of your recursive formula
  - Turn the formula into a dynamic programming algorithm using either memoization or bottom-up approach
  - Analyze the runtime of your algorithm