CS 344: Design and Analysis of Computer Algorithms

(Spring 2022 — Sections 5,6,7,8)

Lecture 18:
Minimum Spanning Trees:
Kruskal’s and Prim’s Algorithms

The Minimum Spanning Tree Problem

The Minimum Spanning Tree Problem

- **Input:**
  - An undirected connected graph $G = (V, E)$
  - Positive weights on edges of G: edge e has weight $w_e > 0$
- **Output:**
  - A spanning tree T in G with minimum weight
    - Weight of T = $\sum_{e \in T} w_e$

A Generic "Algorithm" for MST

A Generic Meta-Algorithm

- Let $F = \emptyset$ be an empty forest initially
- For i = 1 to n-1 steps:
  - **Find a safe edge e for the current forest F**
  - Update F = F + e
- Output the final F as an MST

- This is NOT really an algorithm

Proof of Correctness

A Generic Meta-Algorithm

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- For i = 1 to n-1 steps:
  - **Find a safe edge e for the current forest F**
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- We should now find a way of finding safe edges

An Approach for Finding Safe Edges

- **Theorem:**
  - Suppose F is MST-good but not a tree yet
  - Let (S,V-S) be any cut with no cut edge in F
  - Then edge e in G-F with minimum weight among cut edges of (S,V-S) is safe for F

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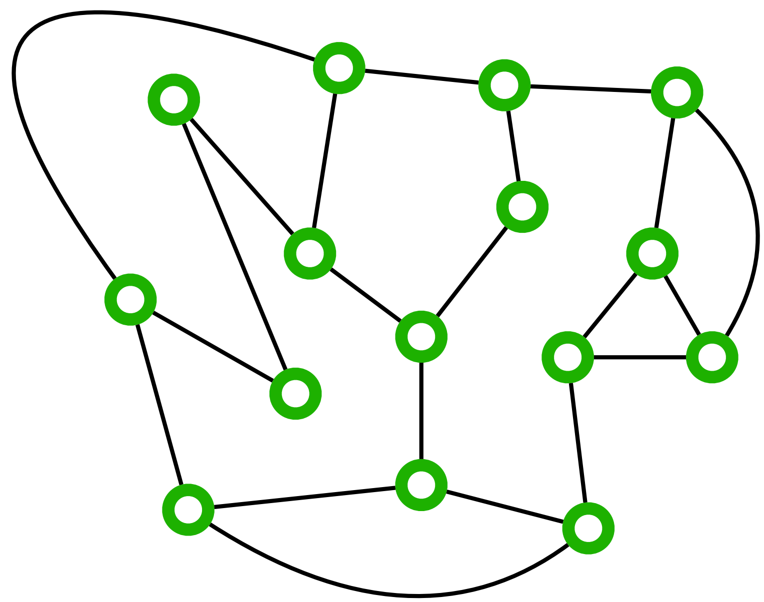

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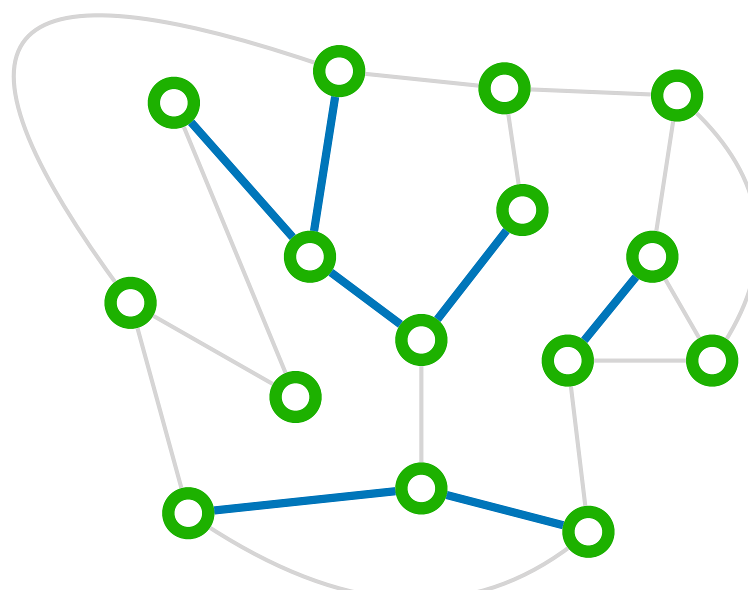

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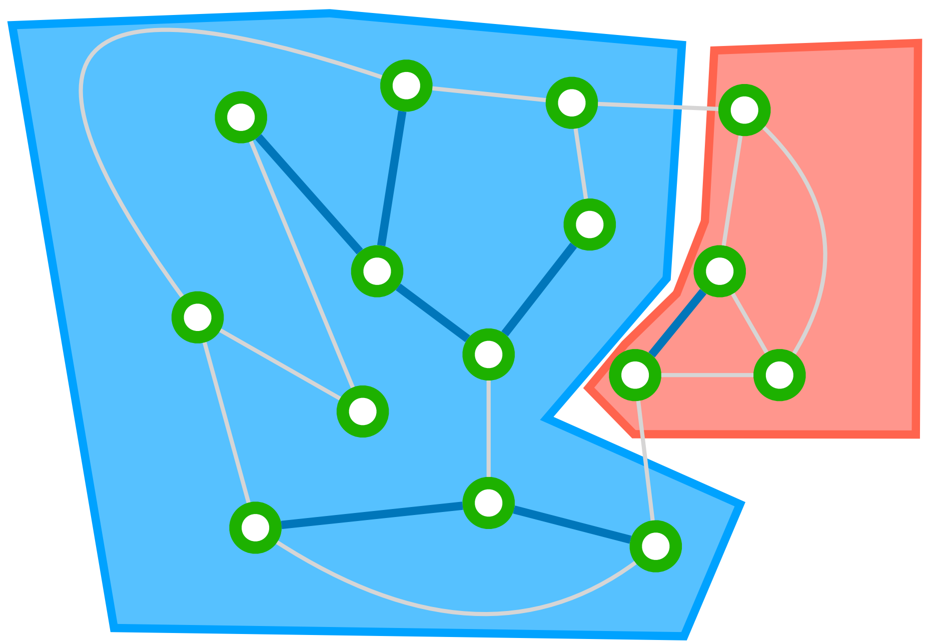

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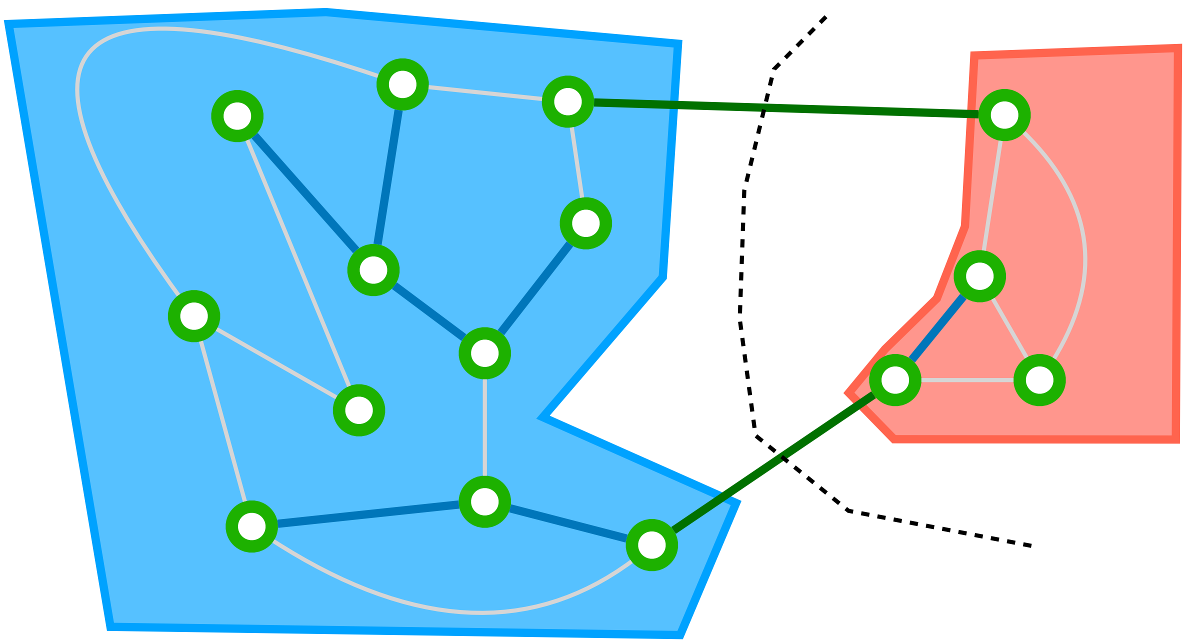

Kruskal’s Algorithm

Kruskal’s Algorithm

- A canonical and efficient algorithm for MST
- Implements the strategy of on the generic meta-algorithm

A Generic Meta-Algorithm

- Let $F = \emptyset$ be an empty forest initially
- For i = 1 to n-1 steps:
  - **Find a safe edge e for the current forest F**
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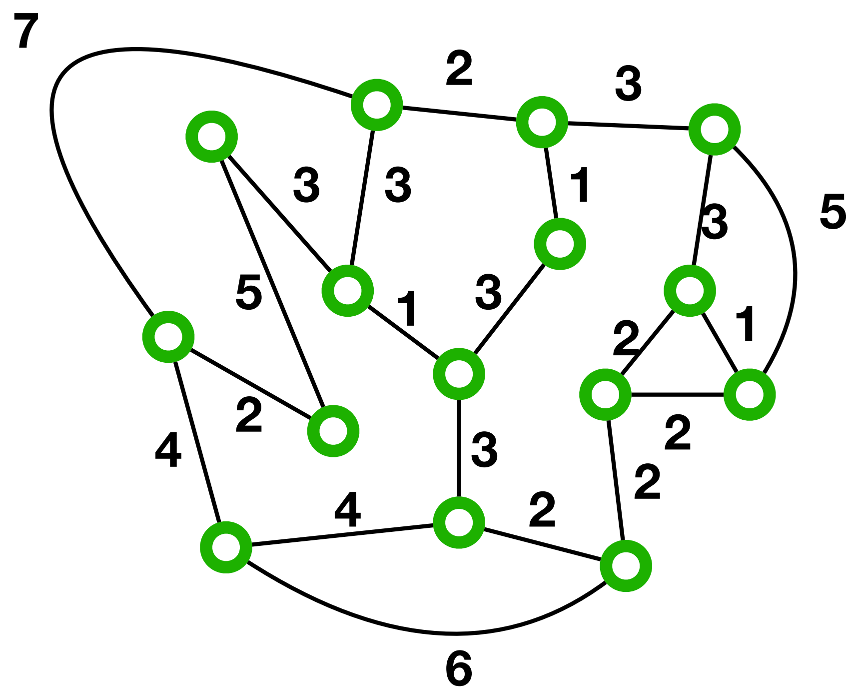
Kruskal's Algorithm

- Sort the edges in increasing (non-decreasing) order of weights
- Let $F = \emptyset$ be an empty forest initially
- For i = 1 to m:
  - if adding e_i to F does not create a cycle, let F = F + e_i
- Output F as an MST of the input

Example



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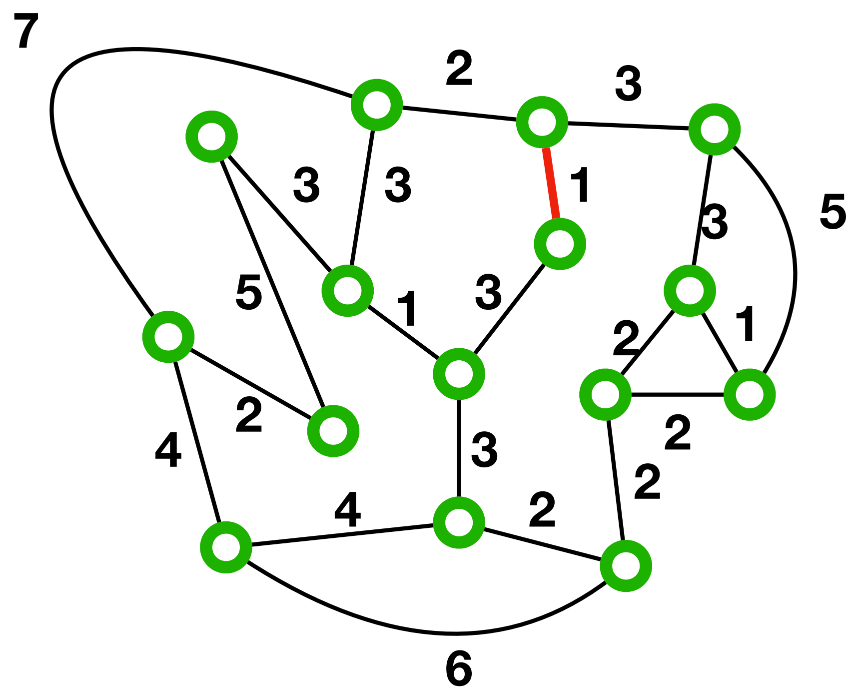

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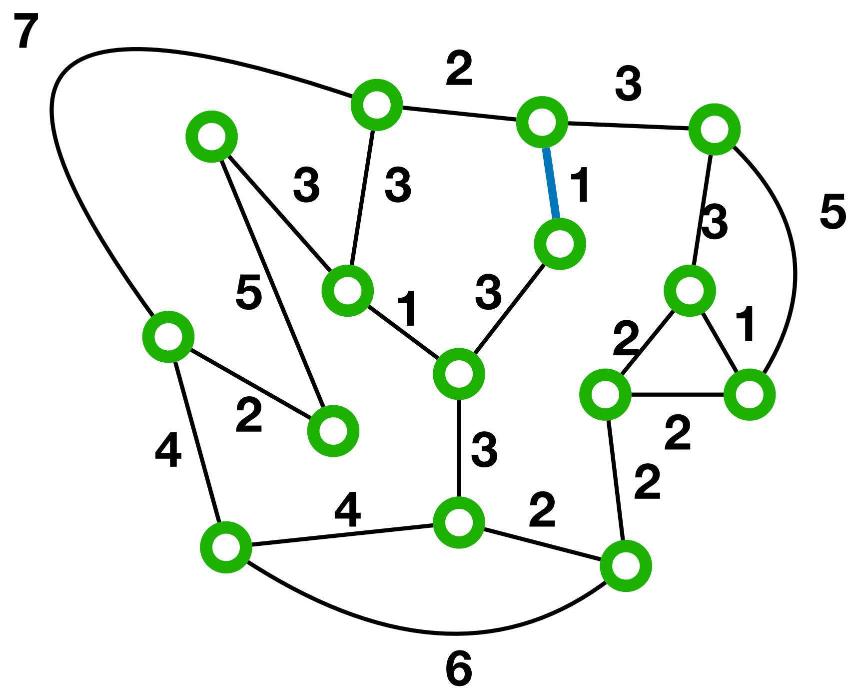

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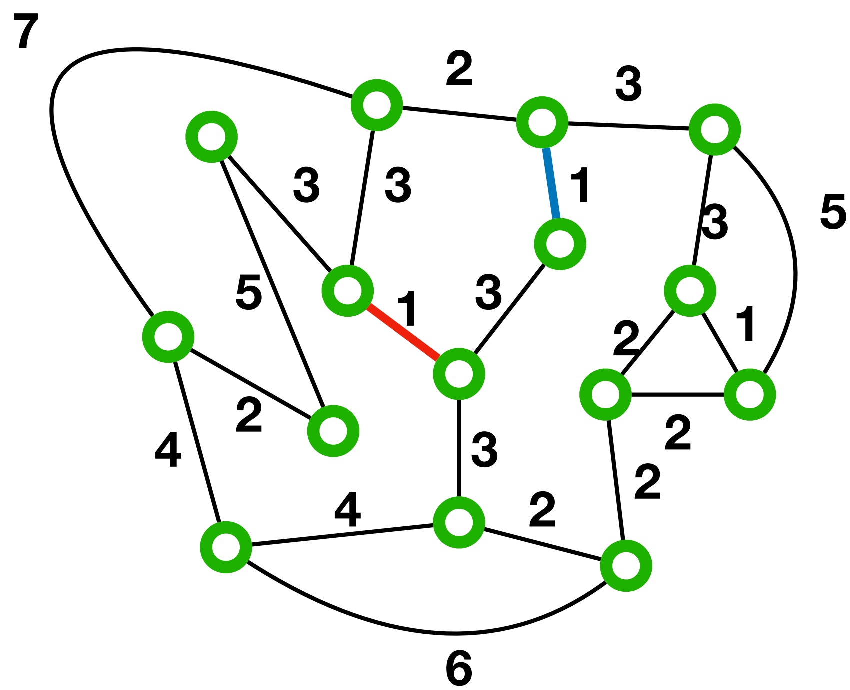

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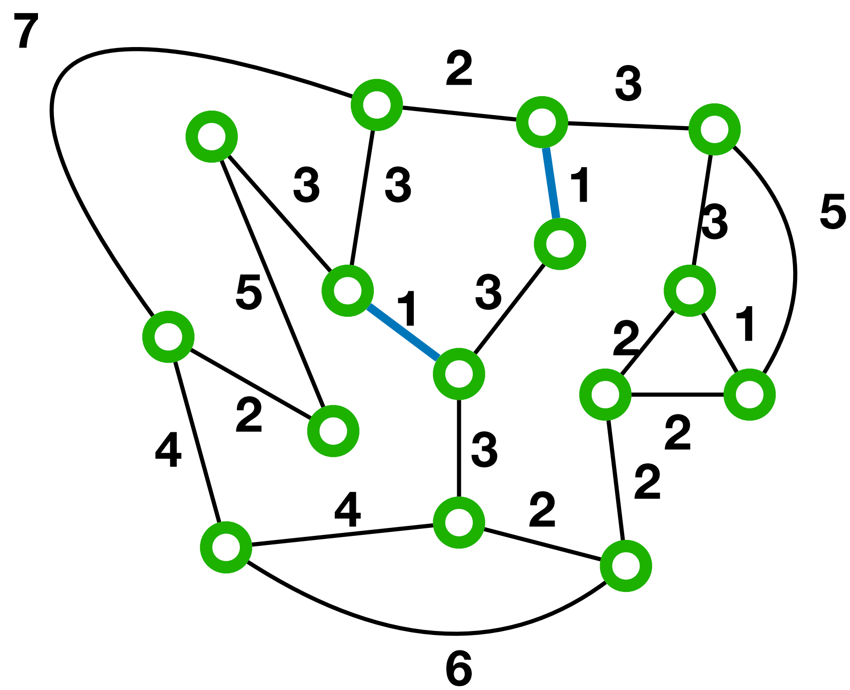

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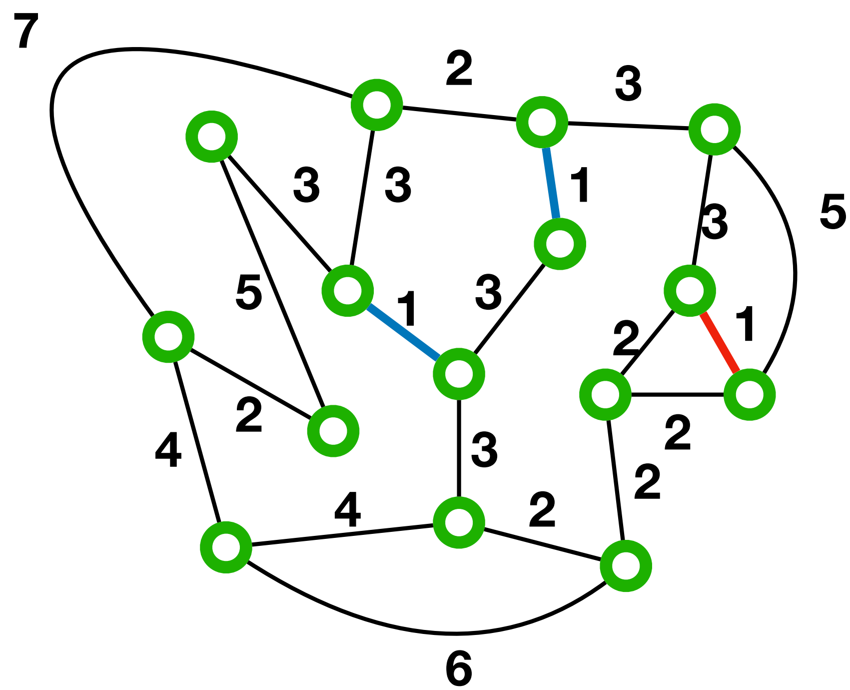

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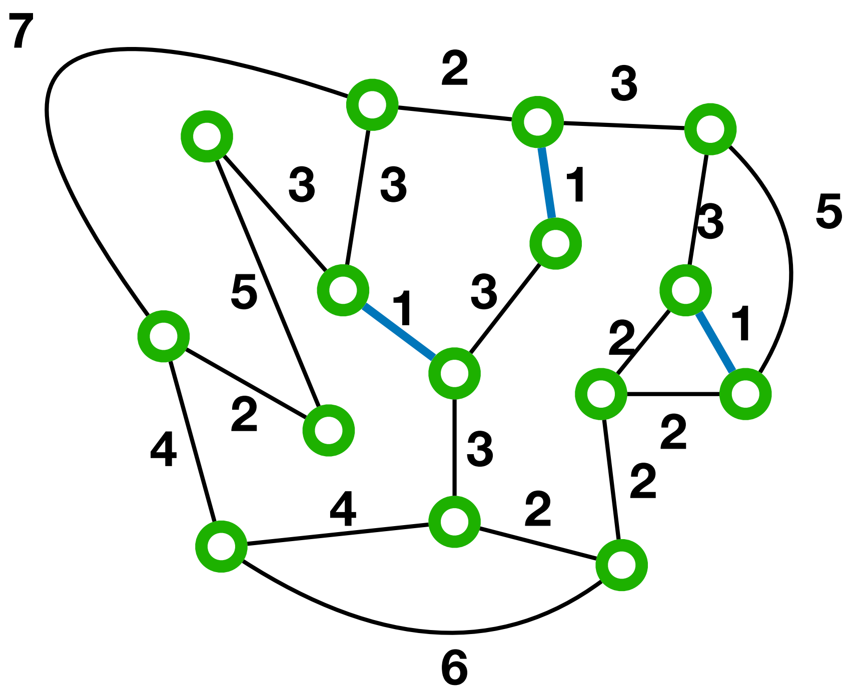

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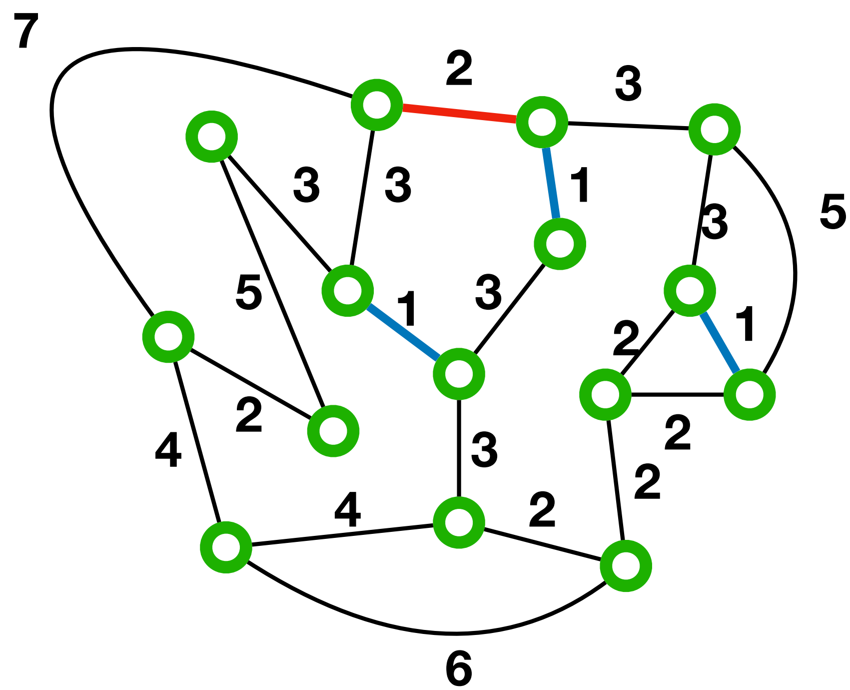

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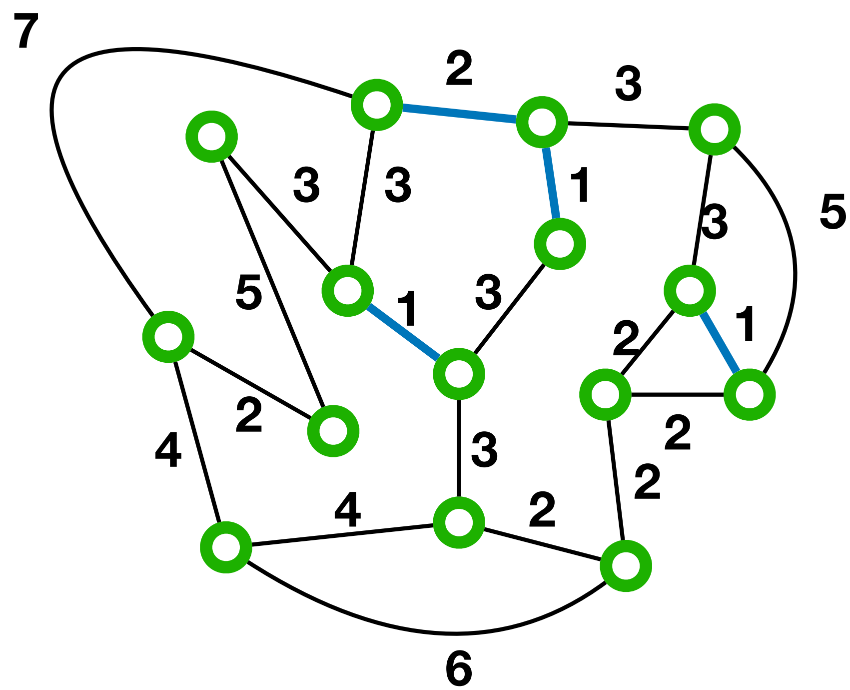

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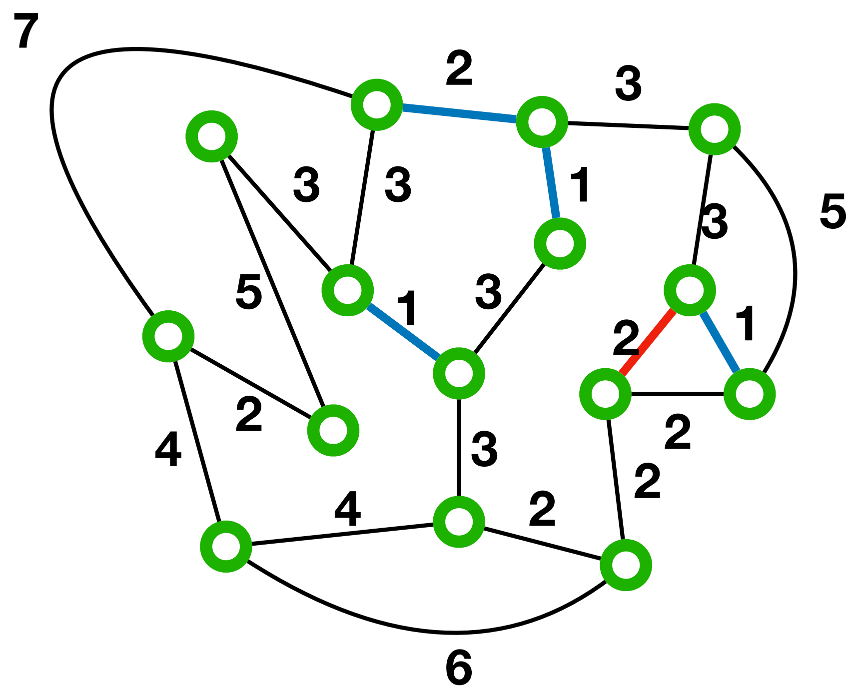

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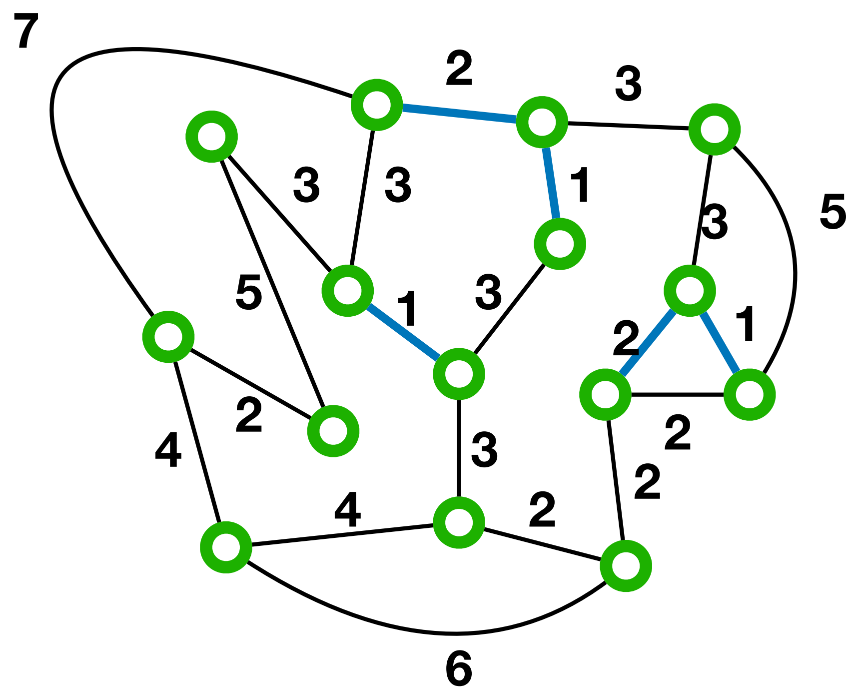

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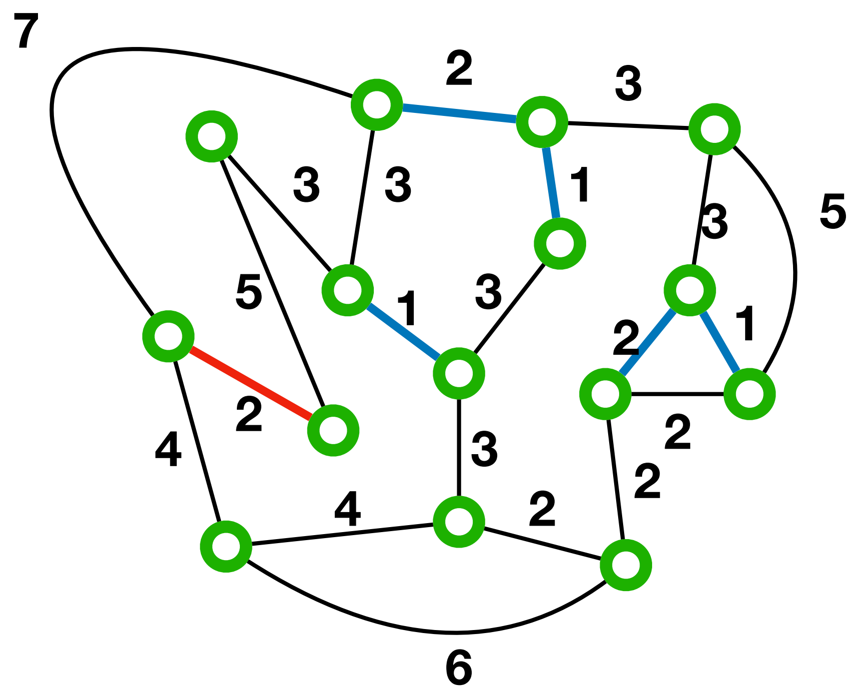

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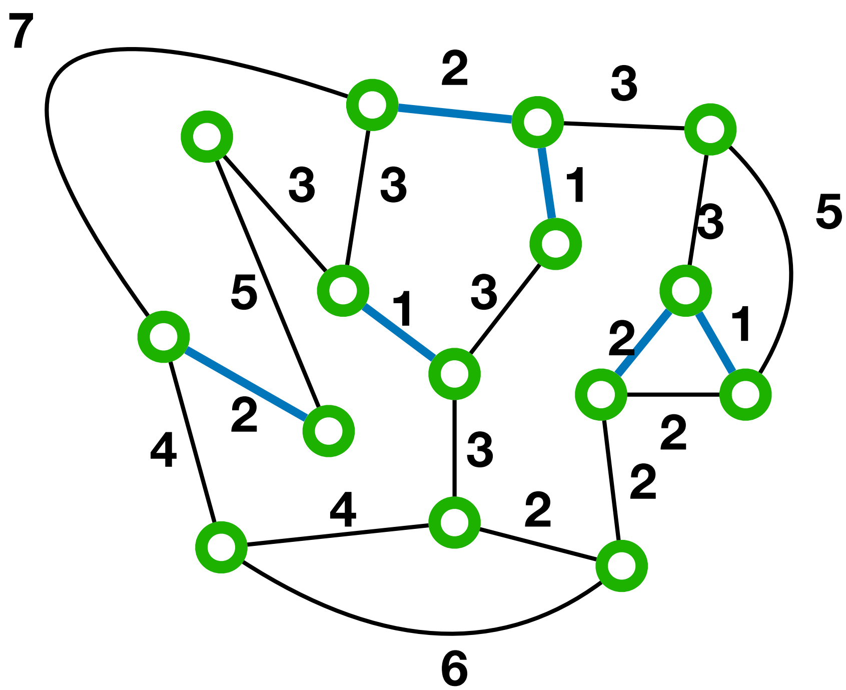

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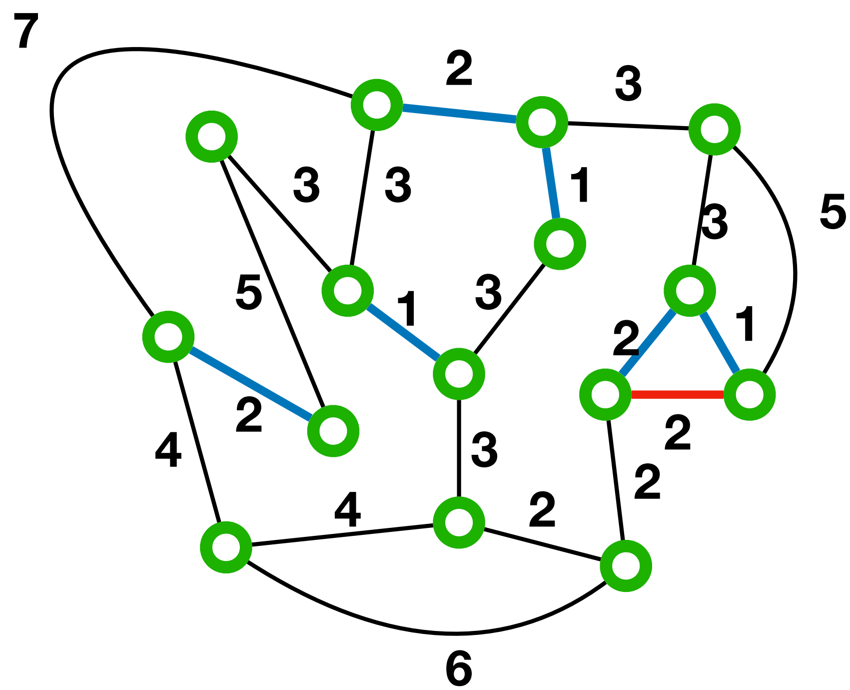

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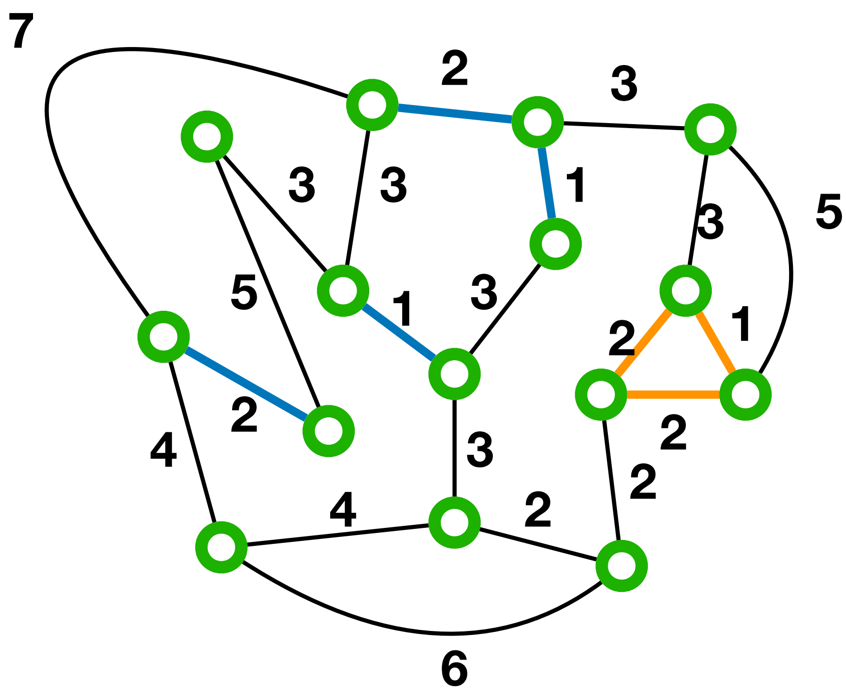

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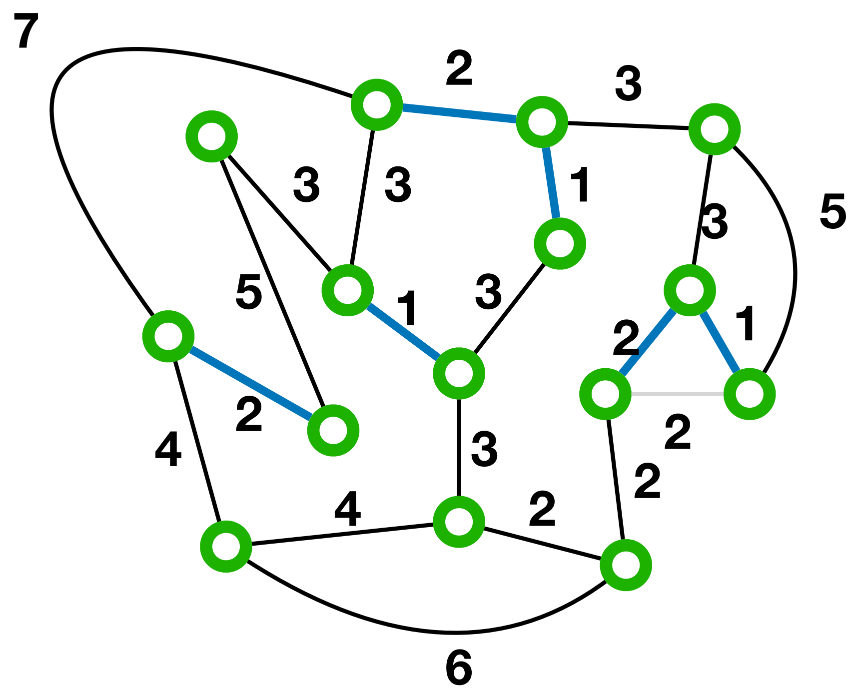

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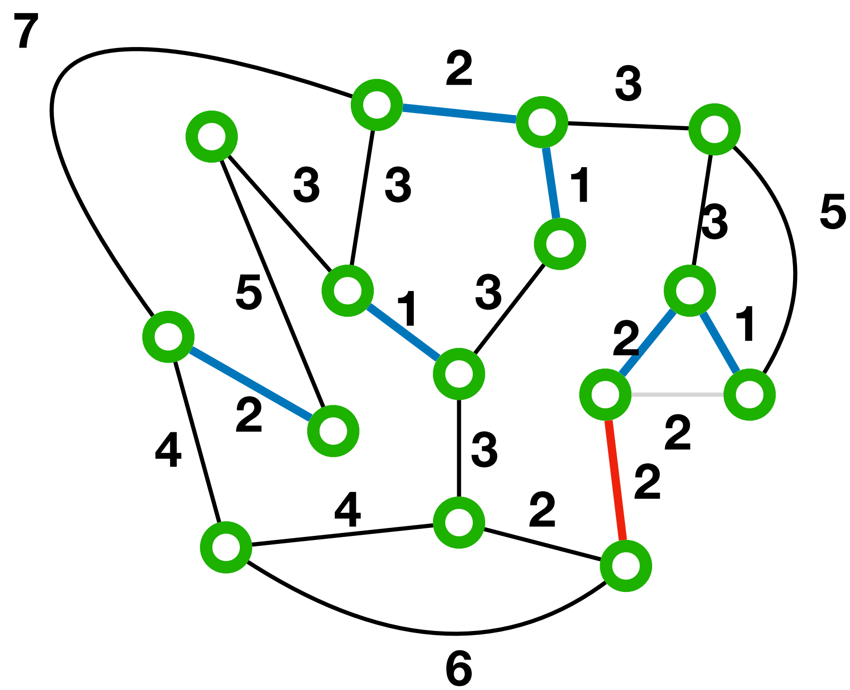

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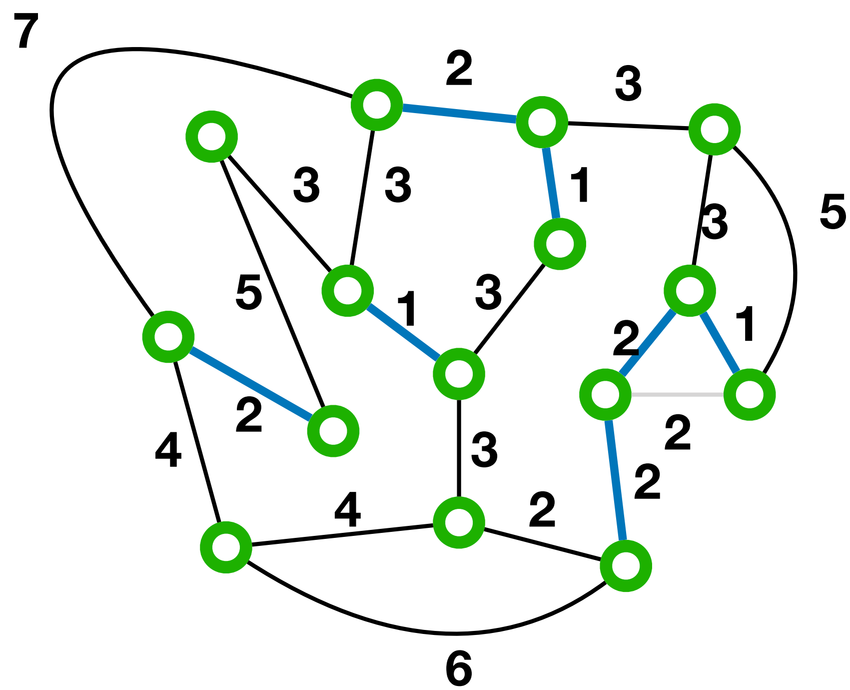

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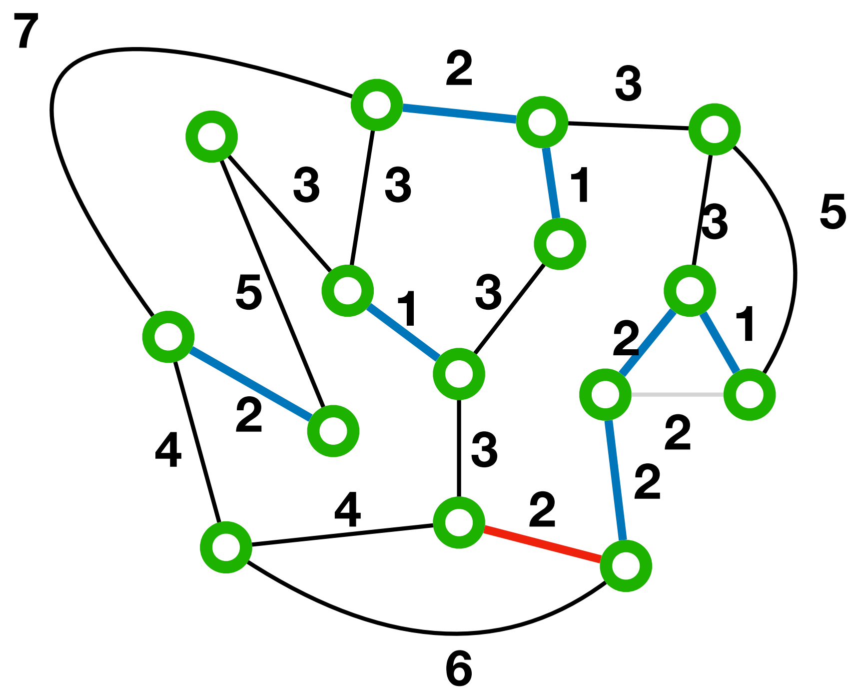

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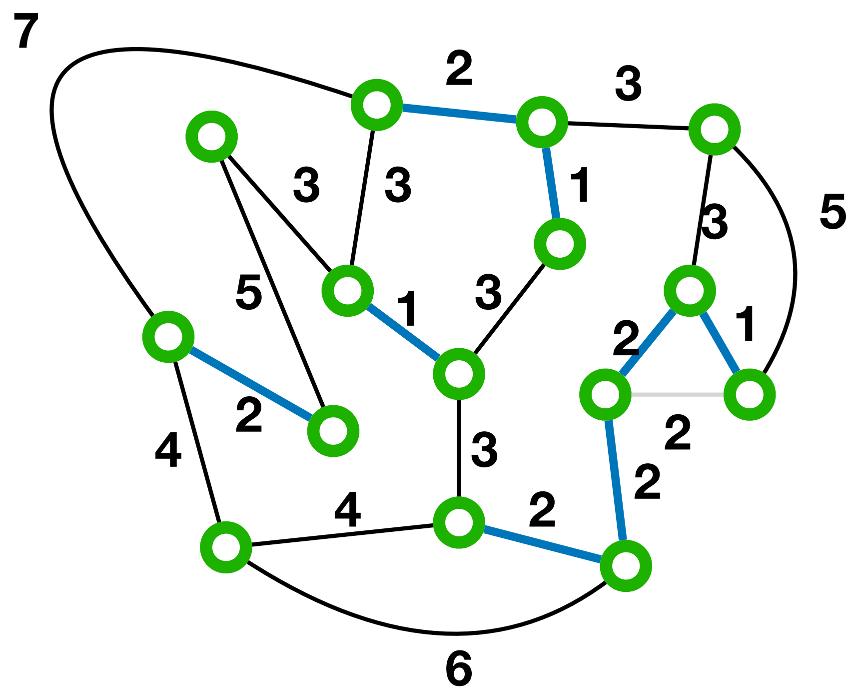

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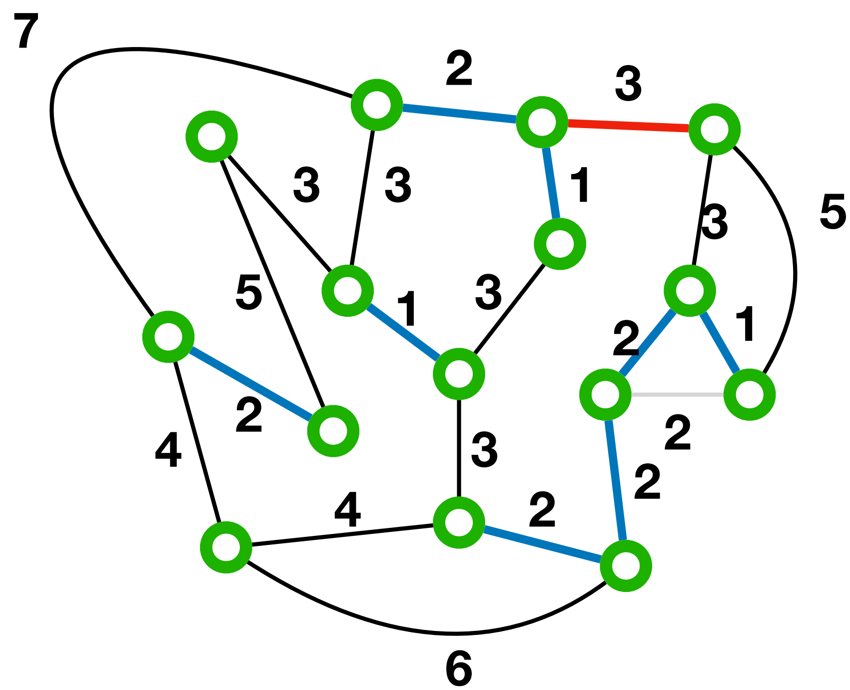

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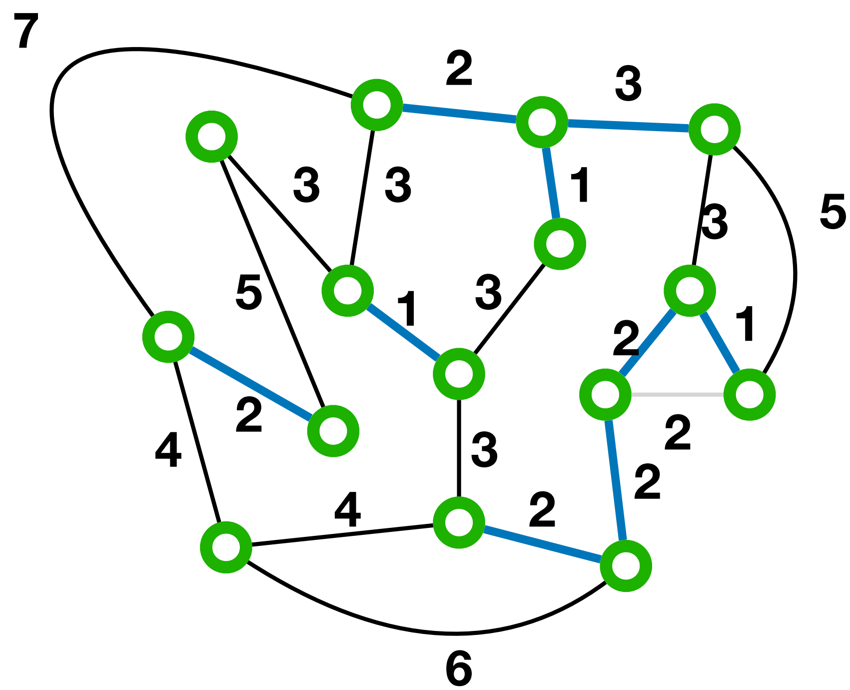

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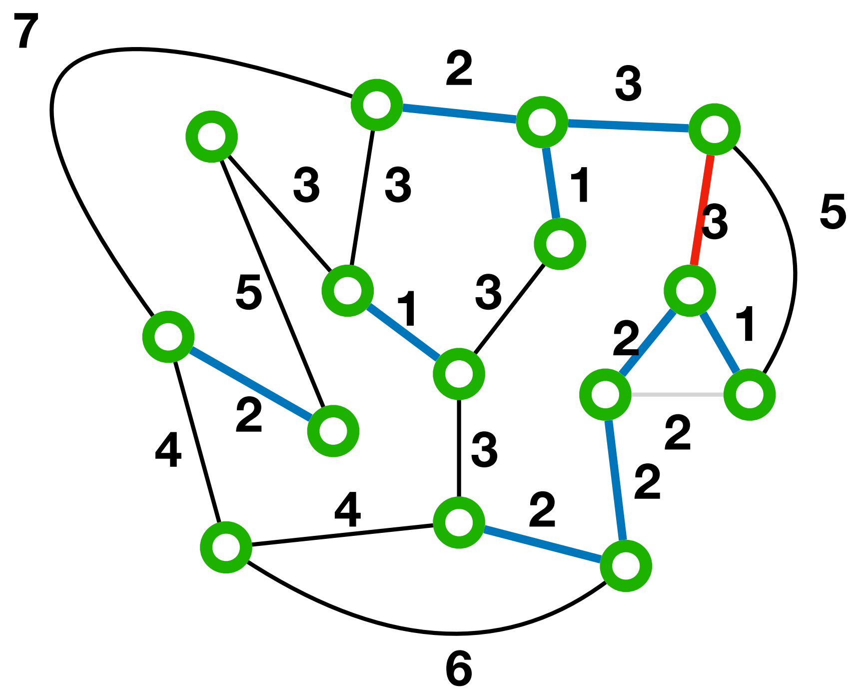

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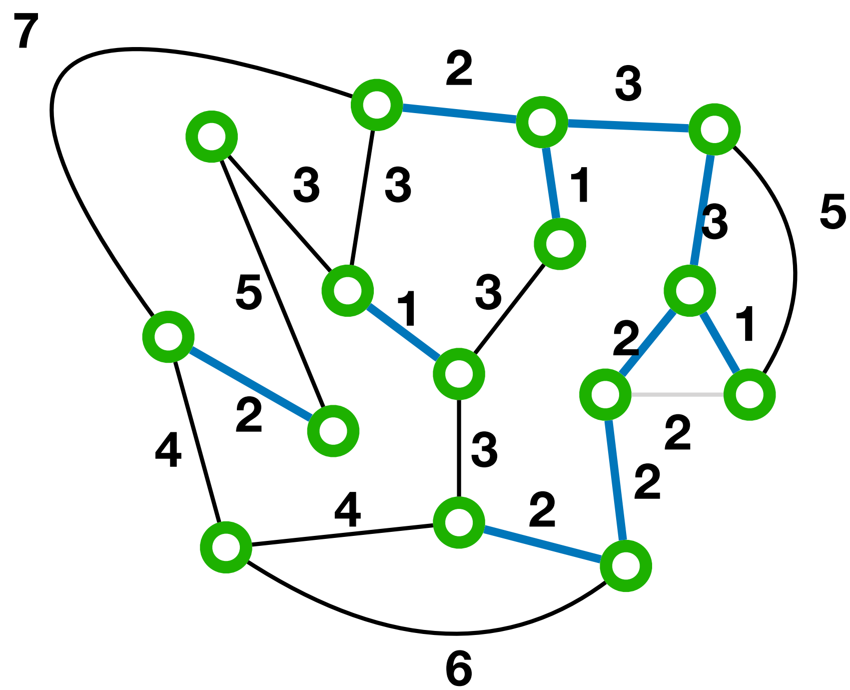

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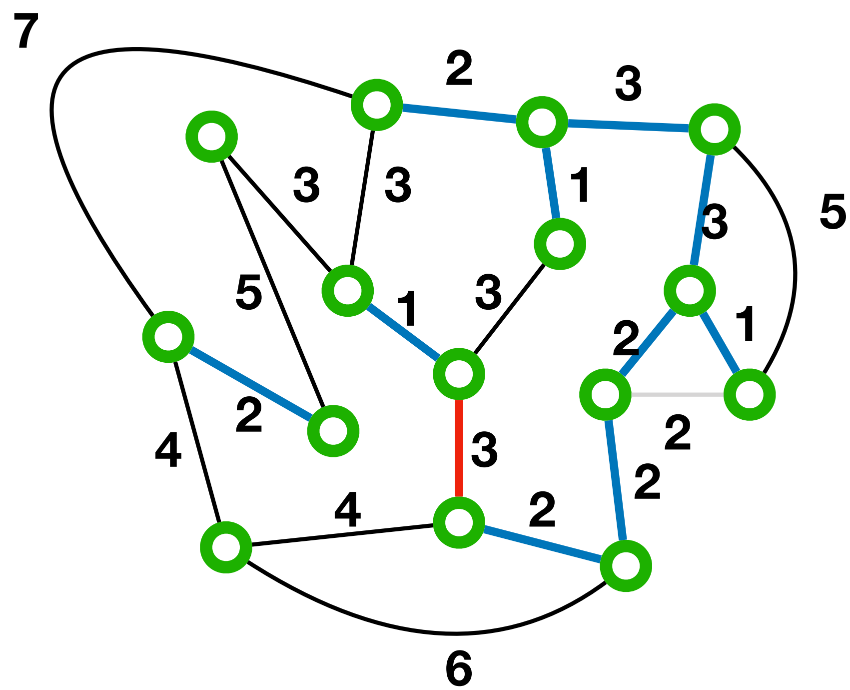

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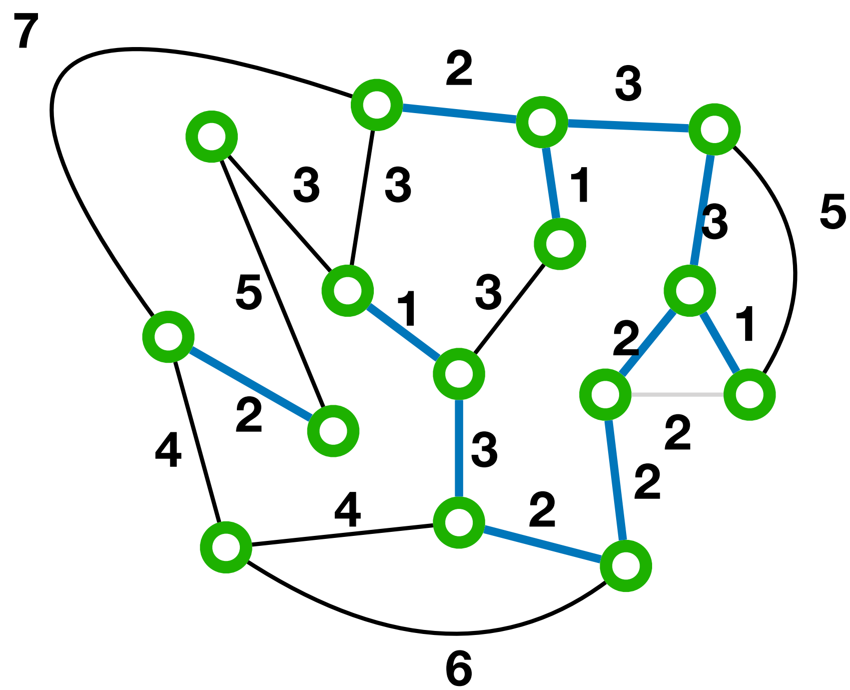

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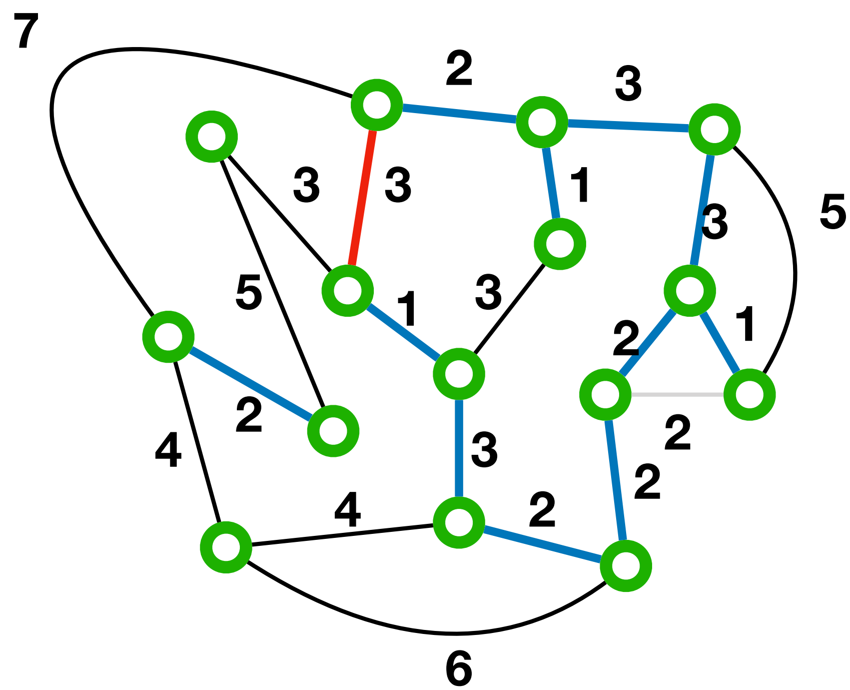

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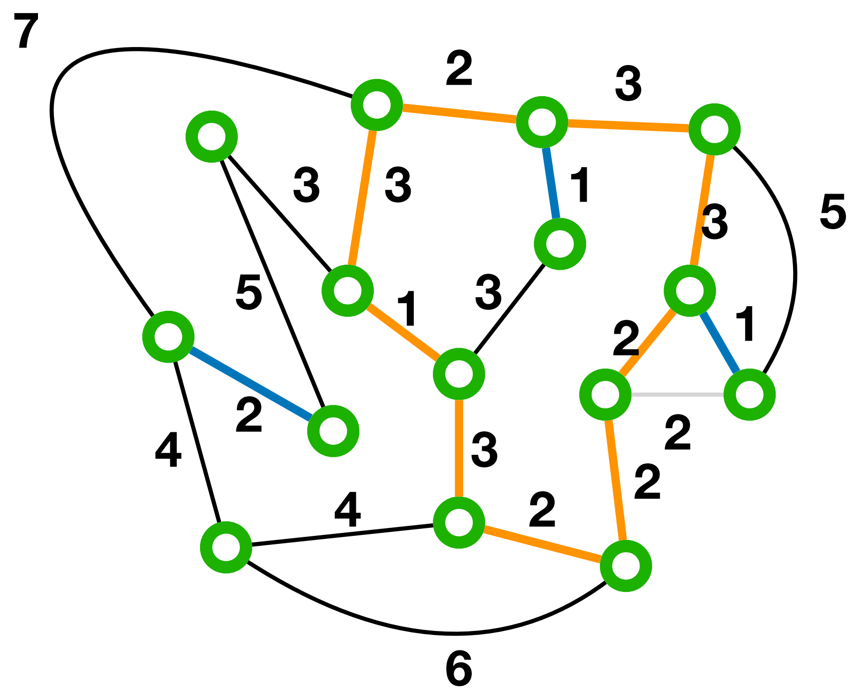

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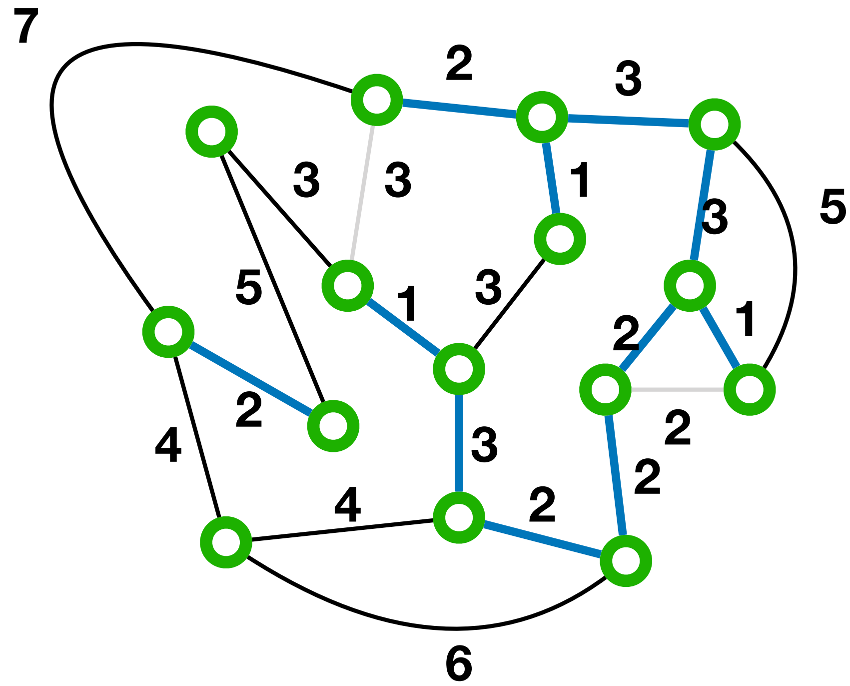

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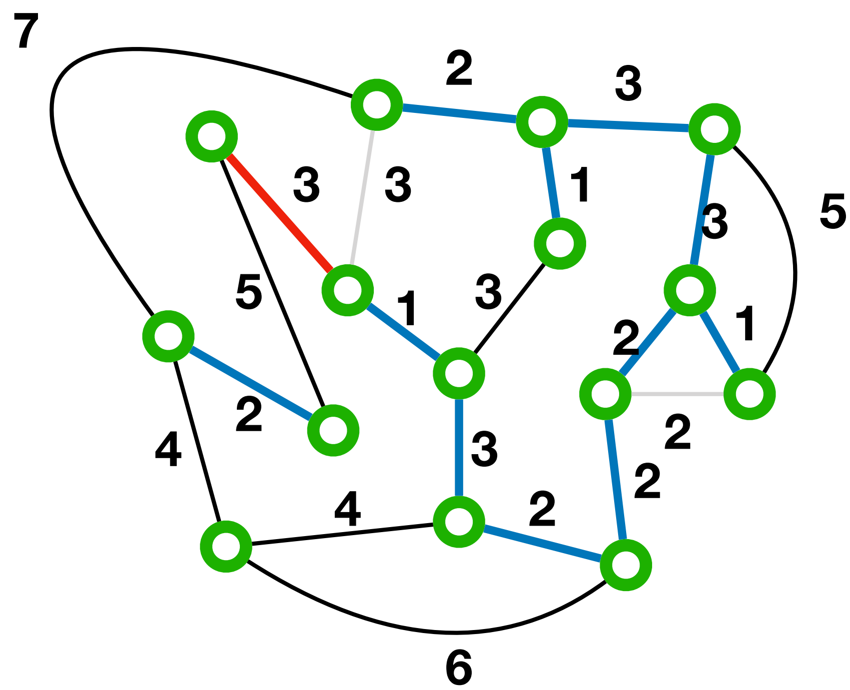

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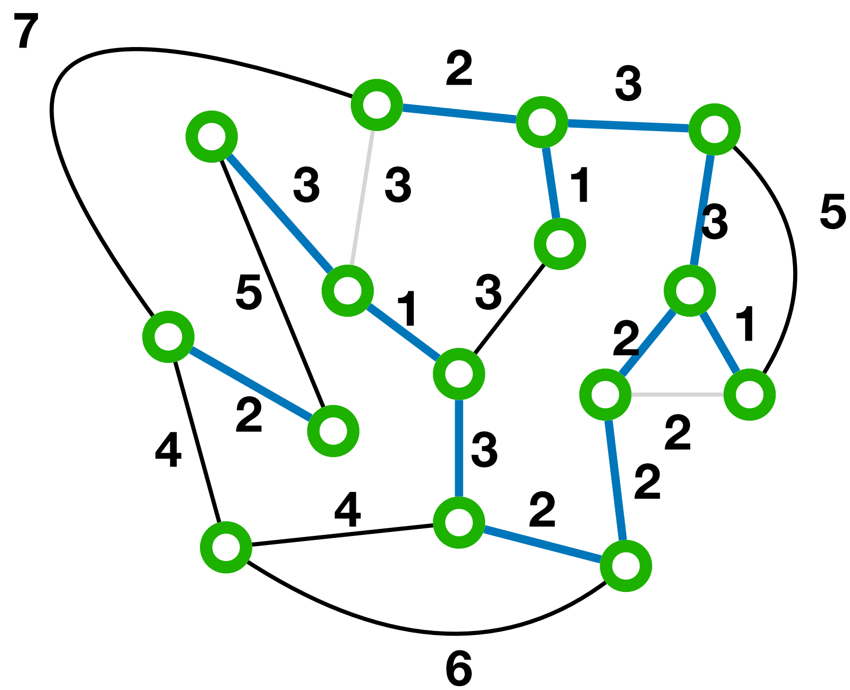

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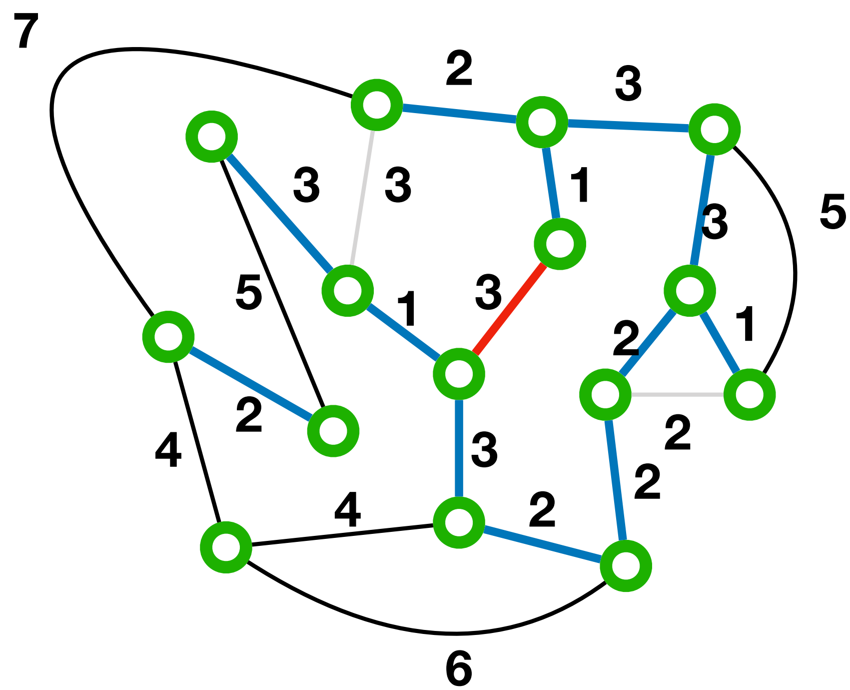

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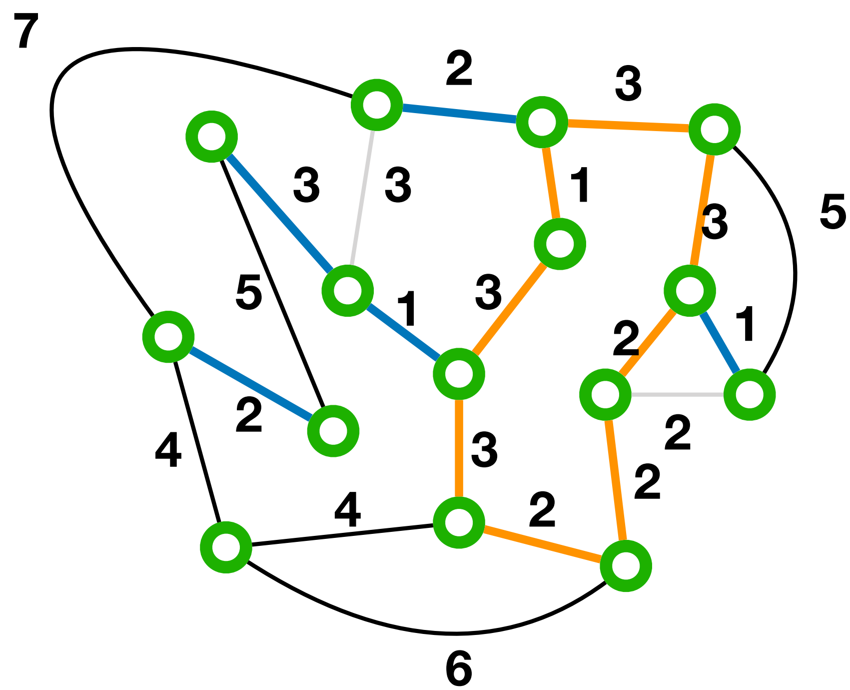

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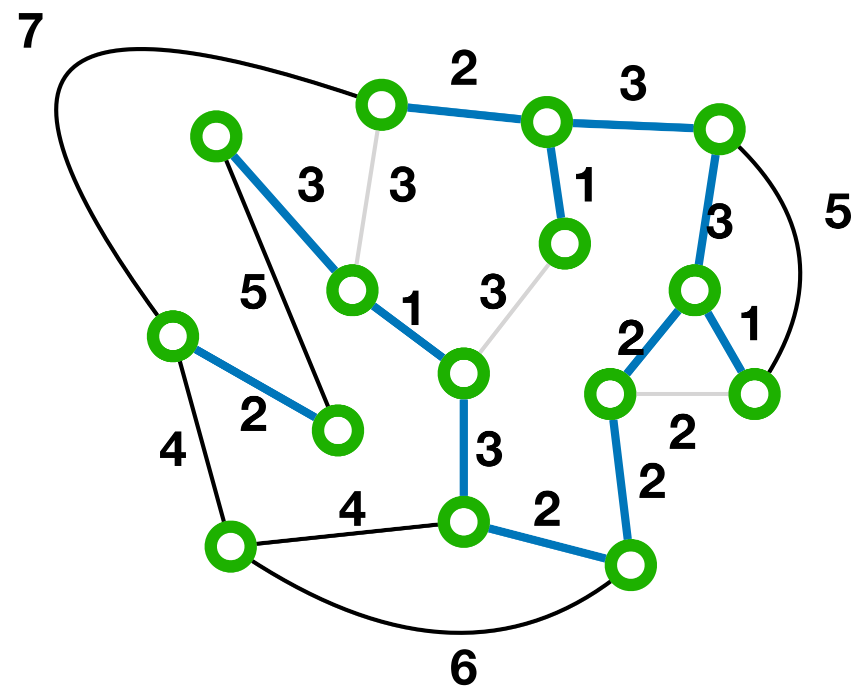

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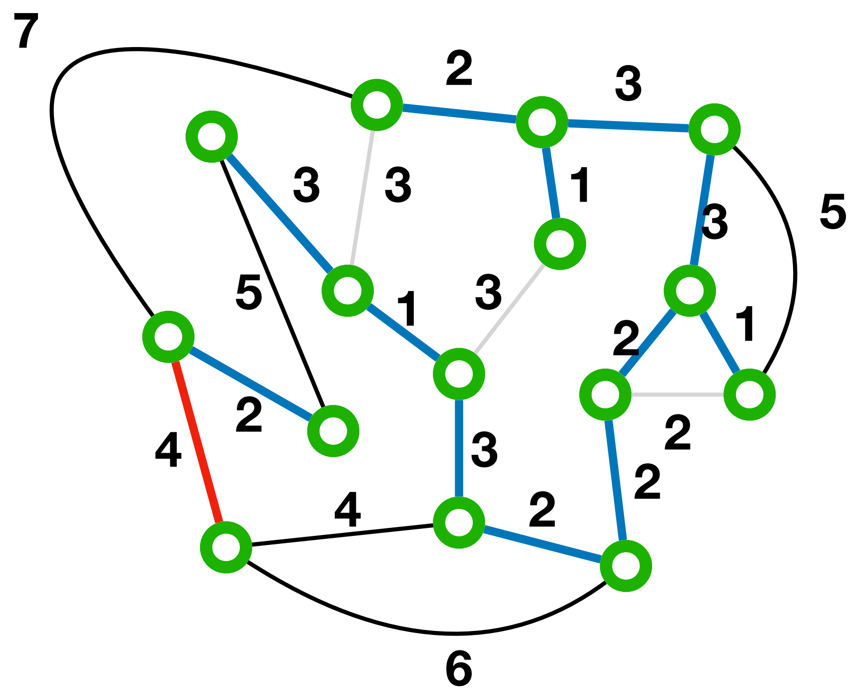

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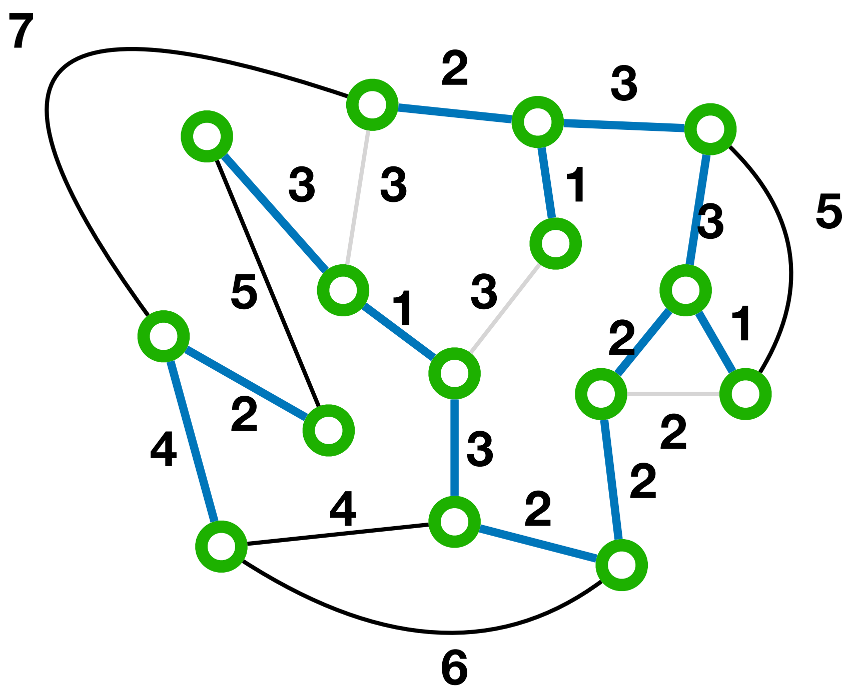

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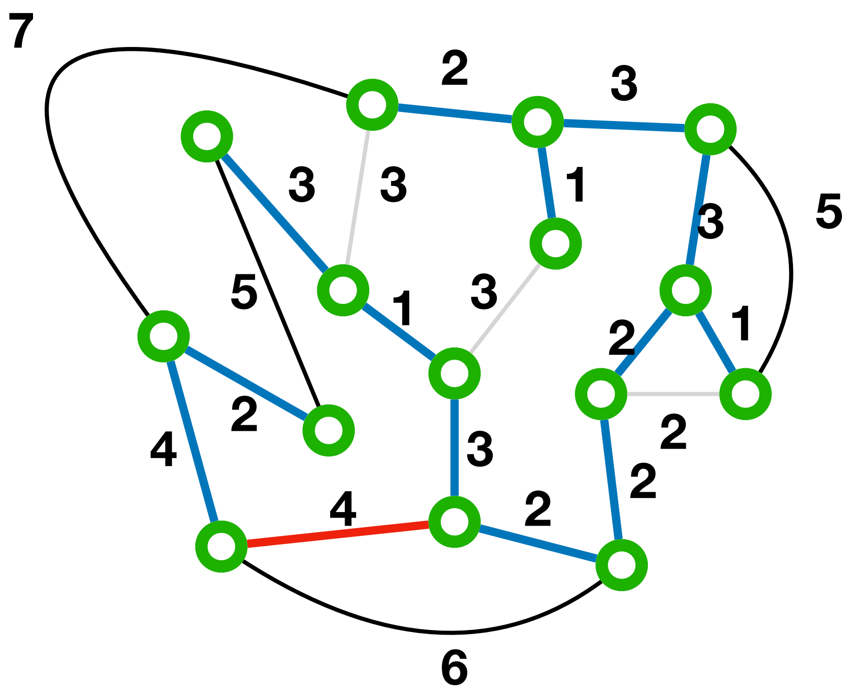

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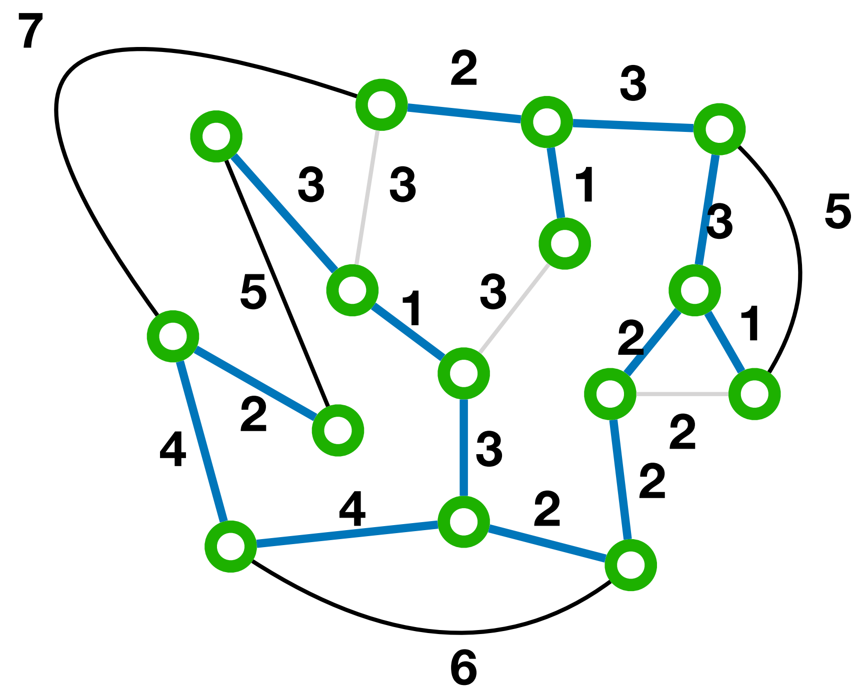

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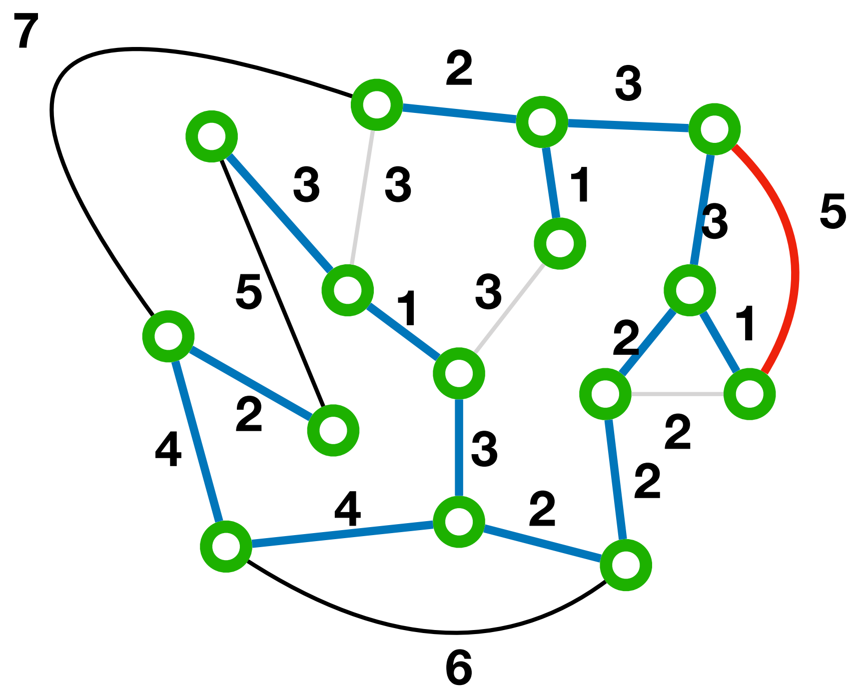

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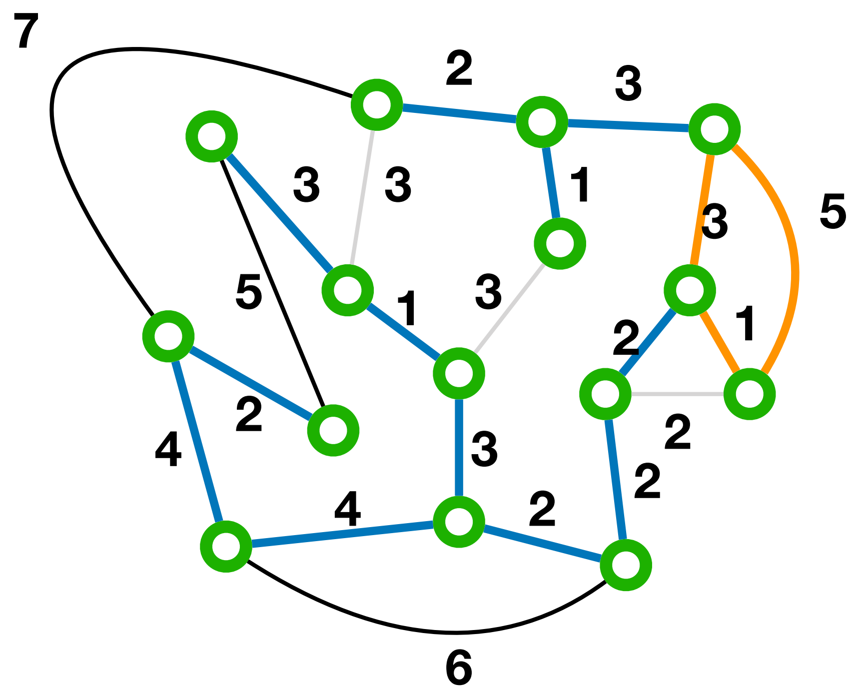

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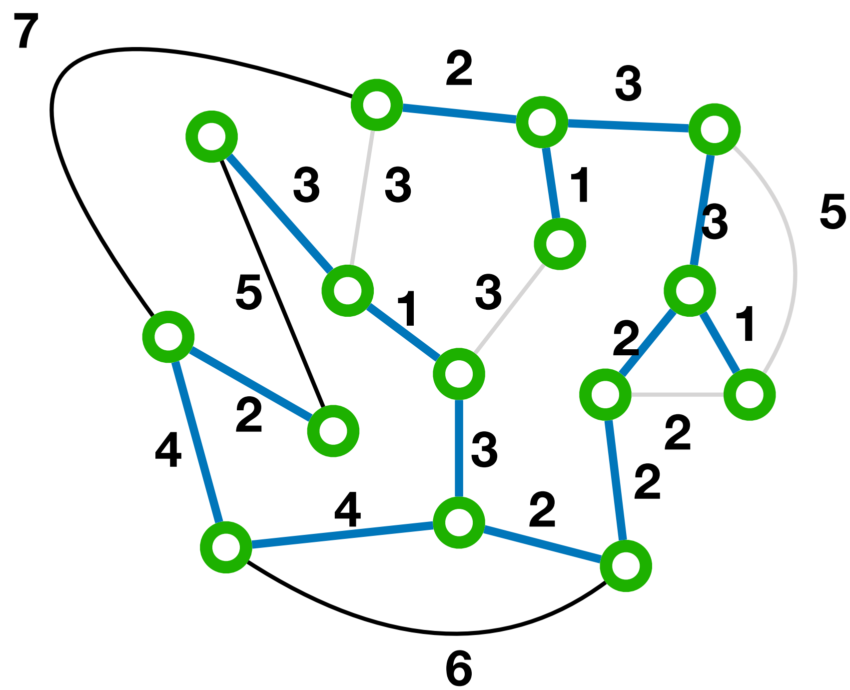

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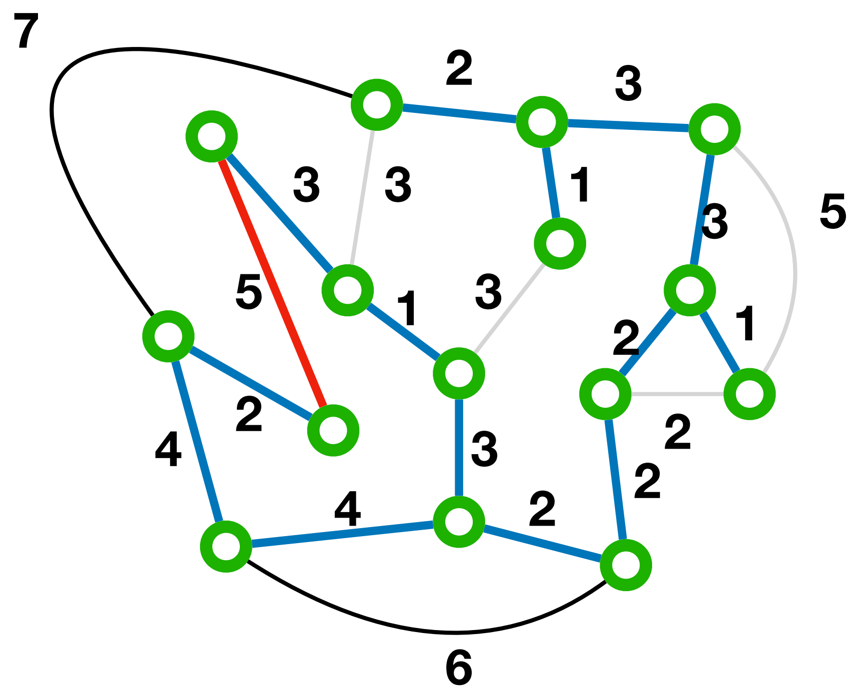

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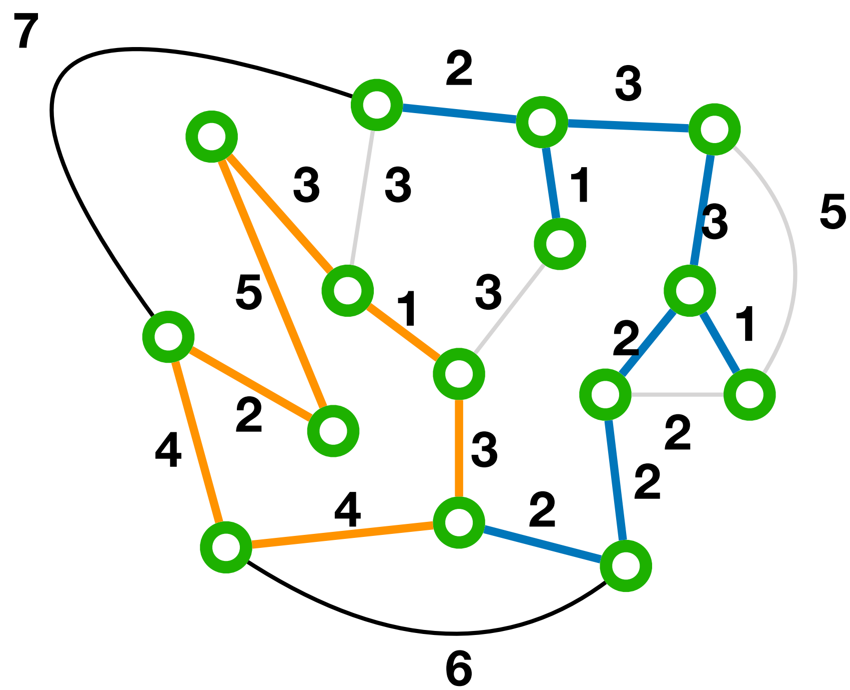

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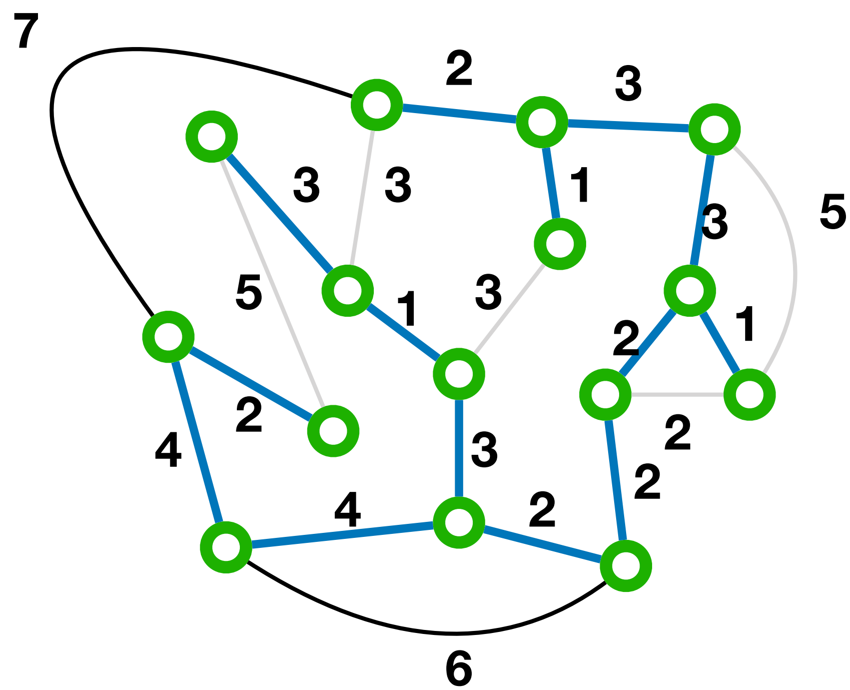

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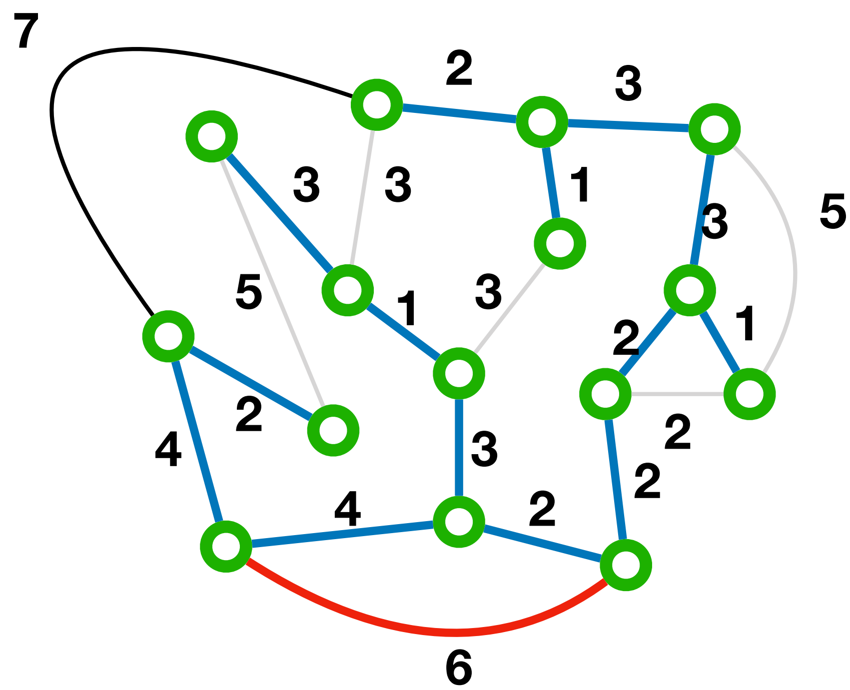

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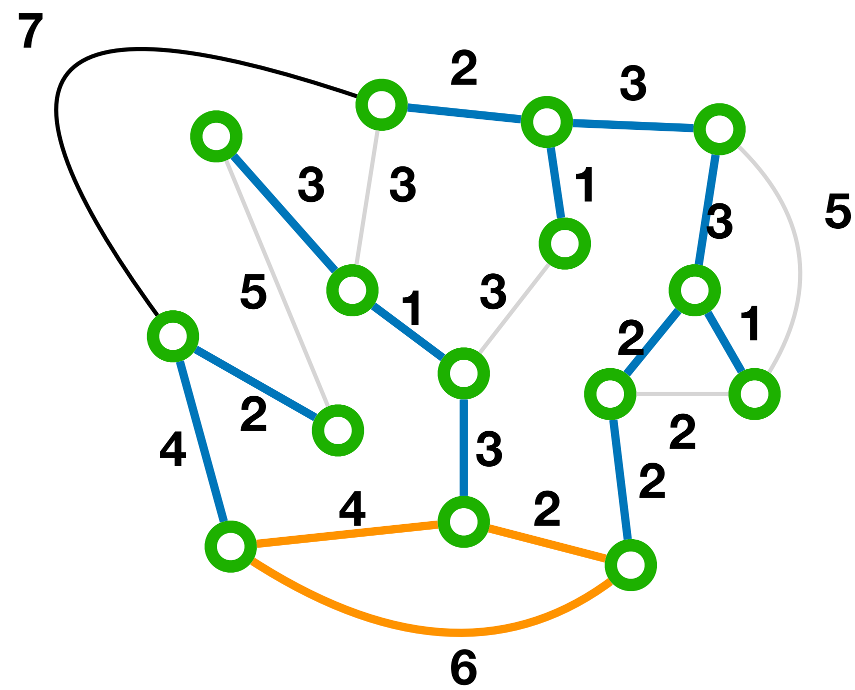

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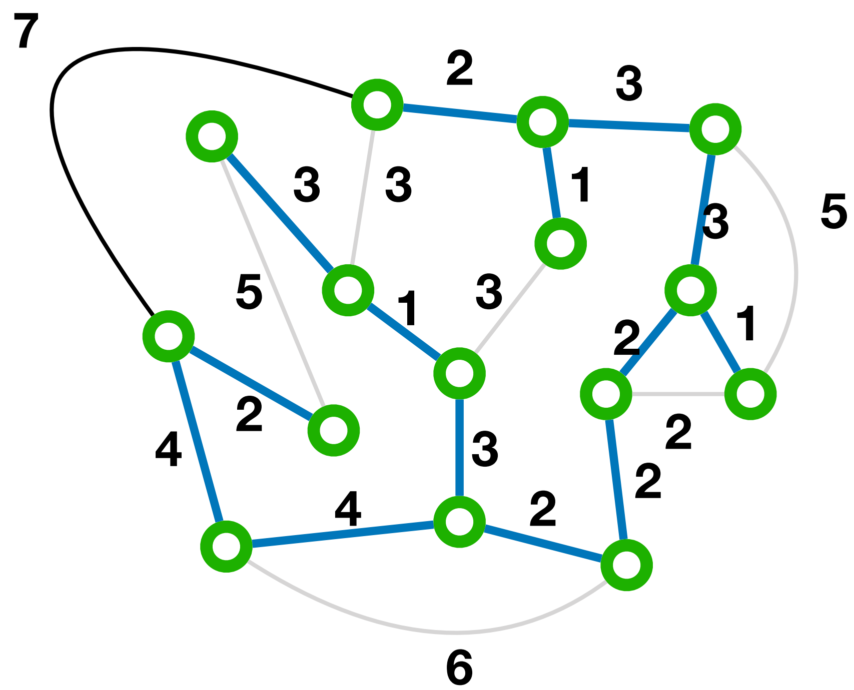

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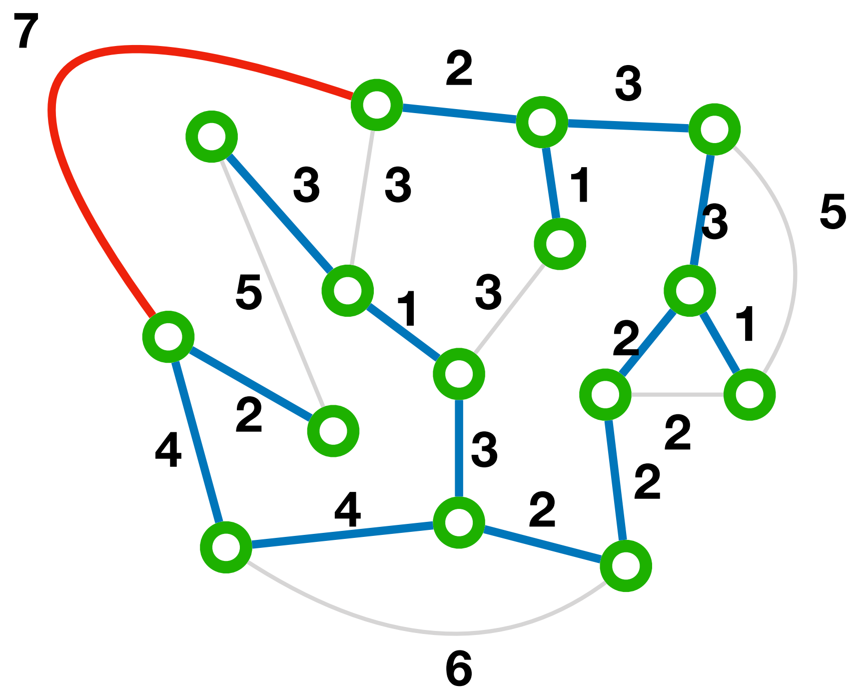

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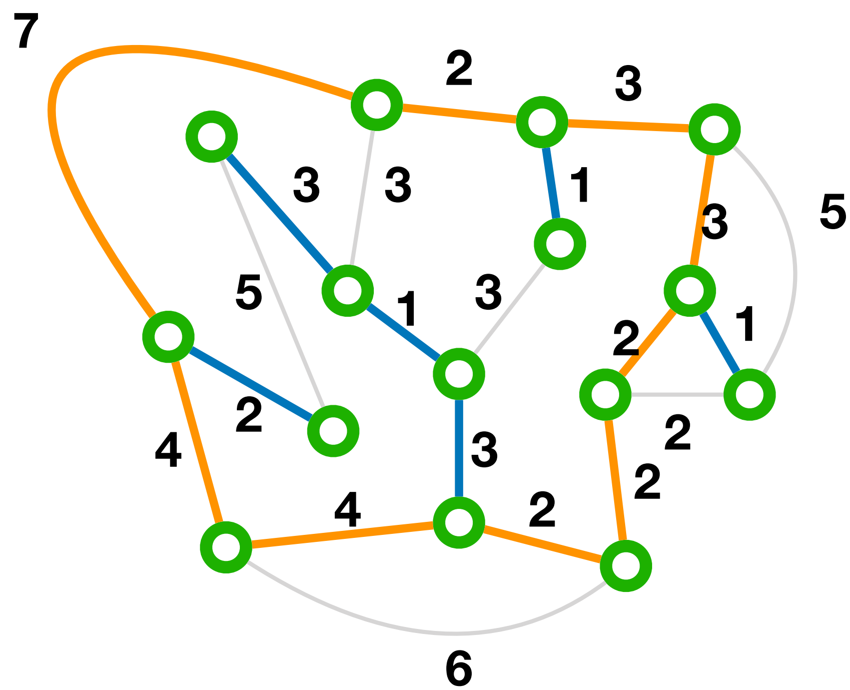

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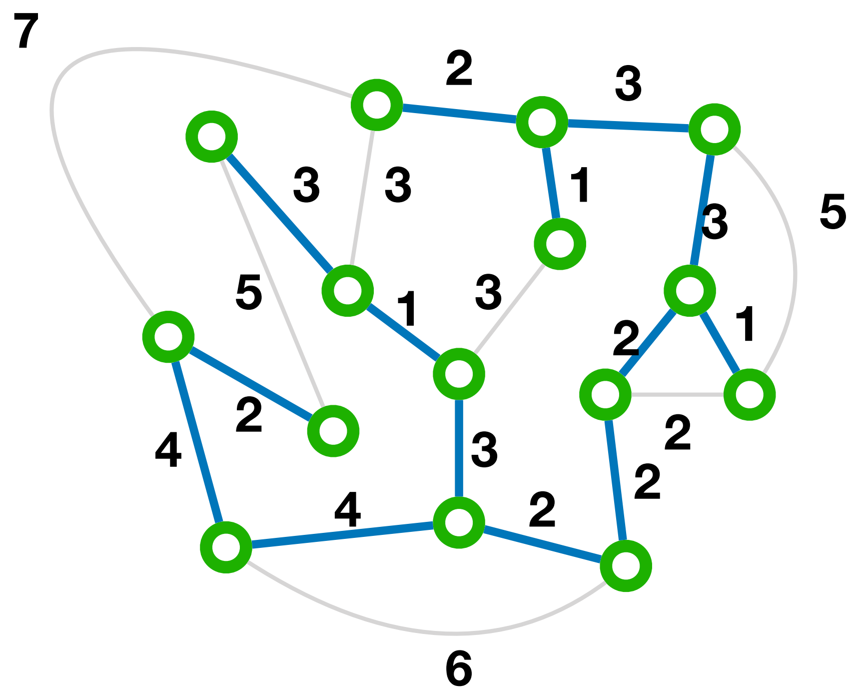

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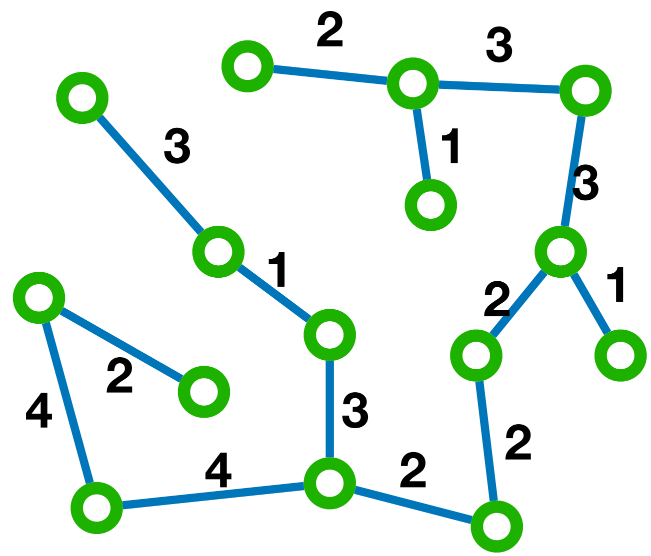

Proof of Correctness

- Sort the edges in increasing (non-decreasing) order of weights
- Let $F = \varnothing$ be an empty forest initially
- For i = 1 to m:
  - if adding e_i to F does not create a cycle, let F = F + e_i
- Output F as an MST of the input

- **Theorem:**
  - Suppose F is MST-good but not a tree yet
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Runtime Analysis

- Sort the edges in increasing (non-decreasing) order of weights
- Let $F = \emptyset$ be an empty forest initially
- For i = 1 to m:
  - if adding e_i to F does not create a cycle, let F = F + e_i
- Output F as an MST of the input

- First step: O(n+m logm) time
- Checking whether e creates a cycle can be done with a DFS or BFS (check if its endpoints are connected already)
- So each iteration can be done in O(n+m) time
- This way, the algorithm takes $O(m^2 \log m)$ time in total
- This is too slow however

Runtime Analysis

- Sort the edges in increasing (non-decreasing) order of weights

- Let $F = \emptyset$ be an empty forest initially

- For i = 1 to m:

  - **if adding e_i to F does not create a cycle, let F = F + e_i**

- Output F as an MST of the input

- This step can be implemented in only $O(\log m)$ time using proper a data structure

- The data structure:

  - **Disjoint-Union-Find**

- As it will be too much of a distraction, we will not cover this data structure in the course

- They are available in your notes however

Runtime Analysis

- Sort the edges in increasing (non-decreasing) order of weights
- Let $F = \varnothing$ be an empty forest initially
- For i = 1 to m:
  - **if adding e_i to F does not create a cycle, let F = F + e_i**
- Output F as an MST of the input

- Summary:
- Kruskal can be implemented in only O(m log m) time
- This is our first efficient (actual) algorithm for MST

Prim’s Algorithm

Prim's Algorithm

- Another canonical and efficient algorithm for MST
- Another way of implementing the strategy of the meta-algorithm
- Very closely related to Dijkstra's algorithm for the shortest path problem

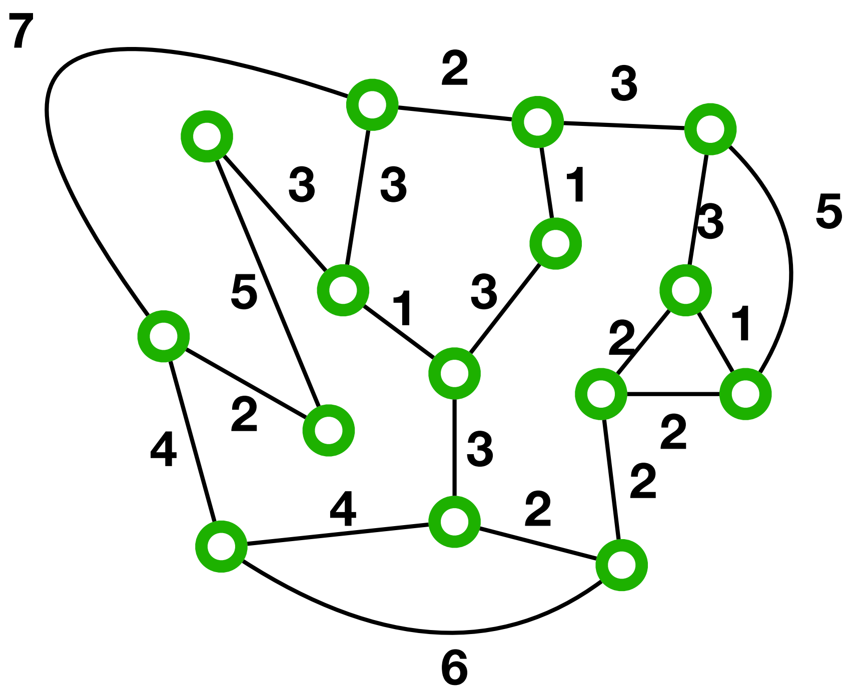
A Generic Meta-Algorithm

- Let $F = \emptyset$ be an empty forest initially
- For i = 1 to n-1 steps:
  - **Find a safe edge e for the current forest F**
  - Update F = F + e
- Output the final F as an MST

Prim's Algorithm

- Let mark[1:n] = false for all vertices and s be any arbitrary vertex
- Let F = ∅, mark[s] = true and H be the set of edges incident on s
- While H is not empty:
  - Remove the minimum weight edge e=(u,v) from H
  - If mark[u]=mark[v] = true, ignore this edge and go to the next iteration of the while-loop
  - Otherwise, let us assume by symmetry mark[u] = true only
  - Add the edge (u,v) to F and all edges incident on v to H; set mark[v] = true.
- Return F as an MST of the input graph

Example



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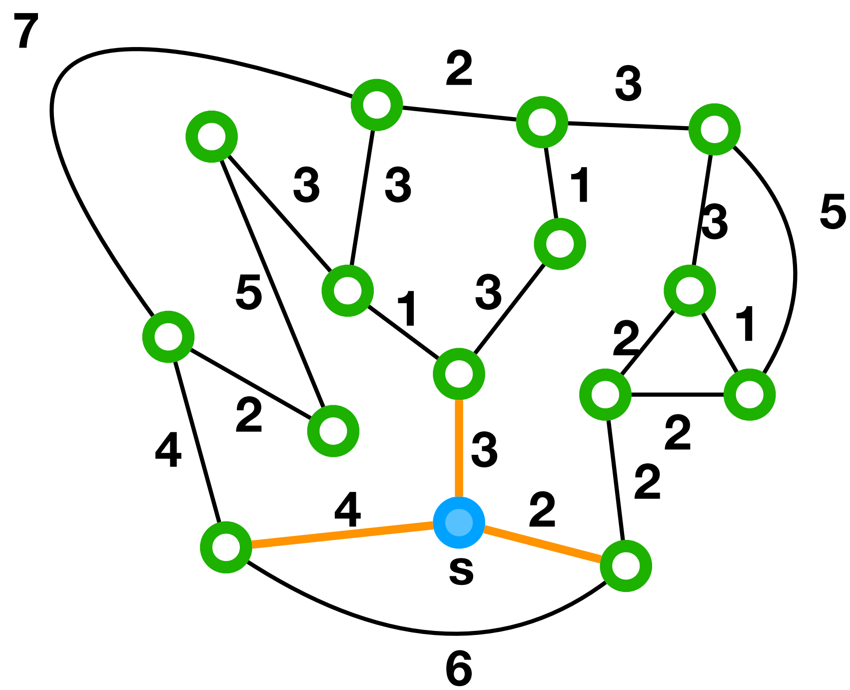

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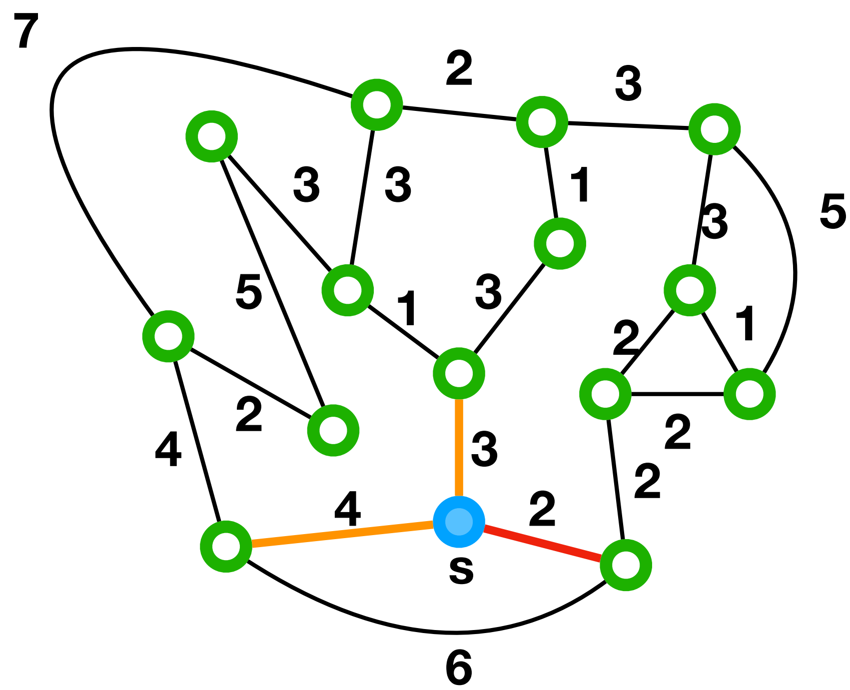
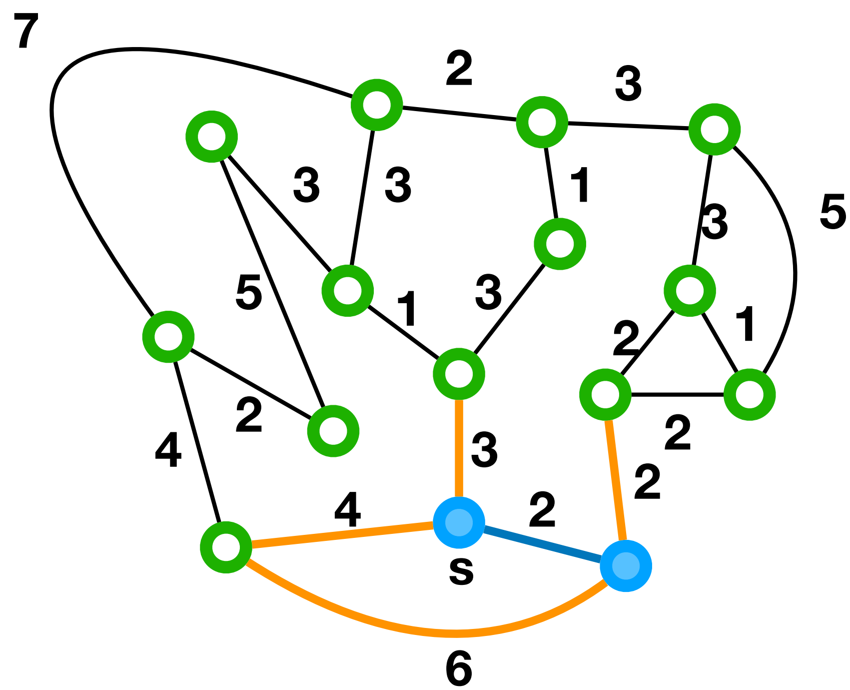


Example



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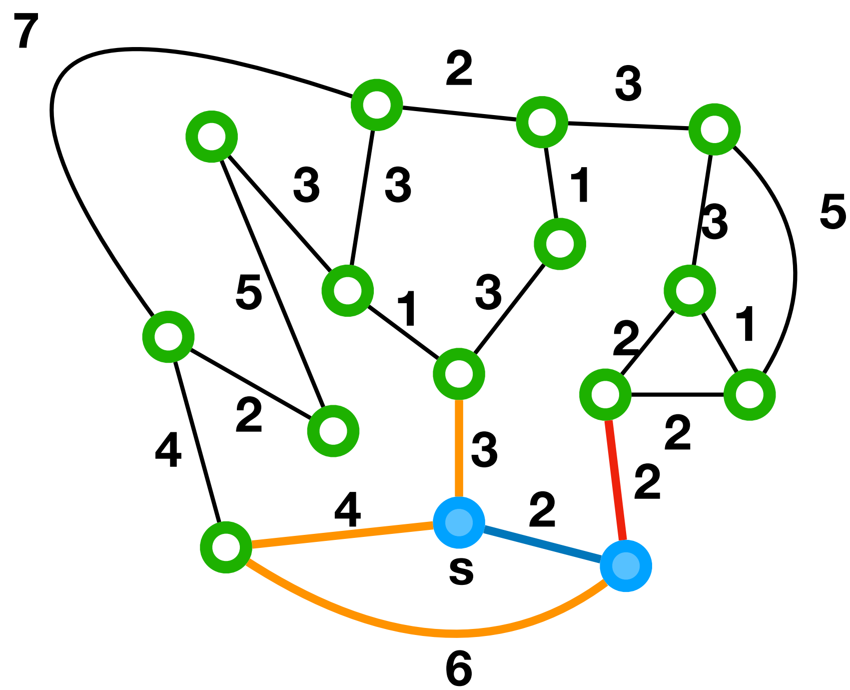

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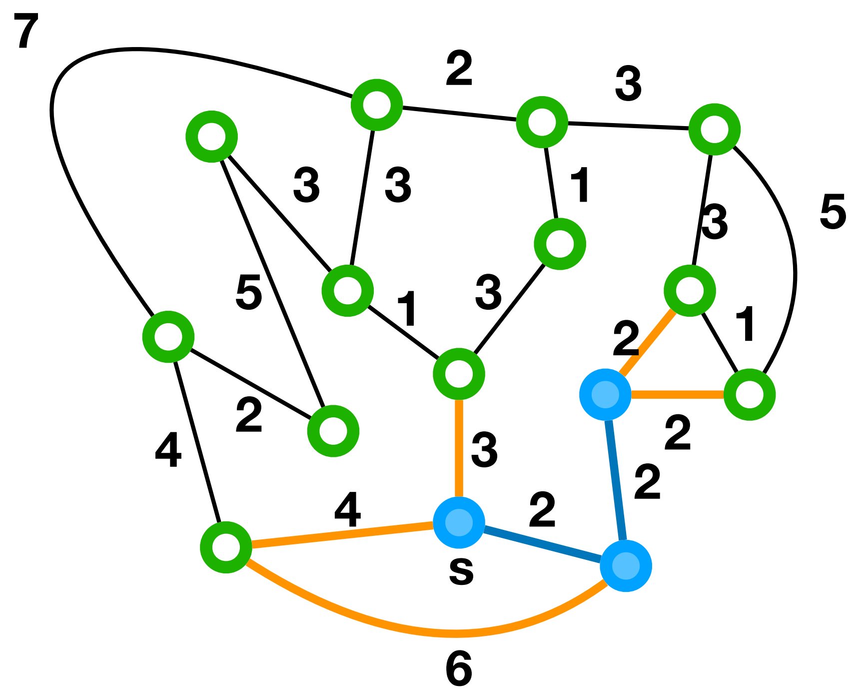

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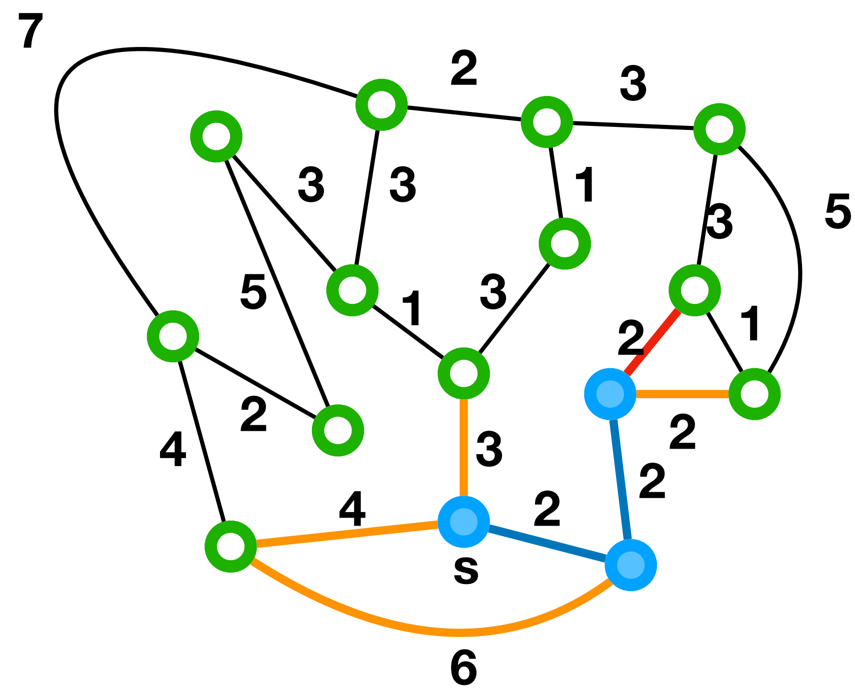

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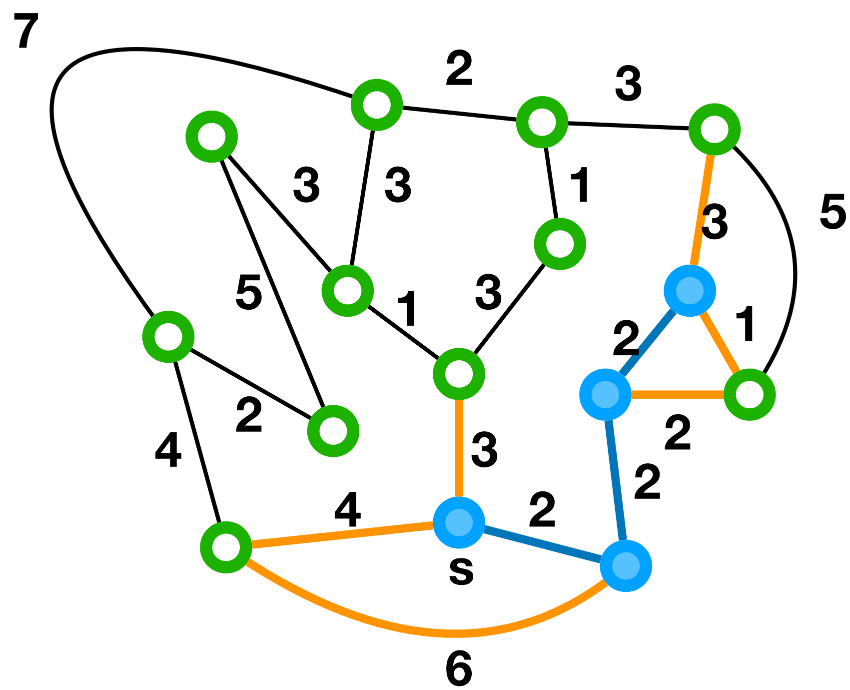

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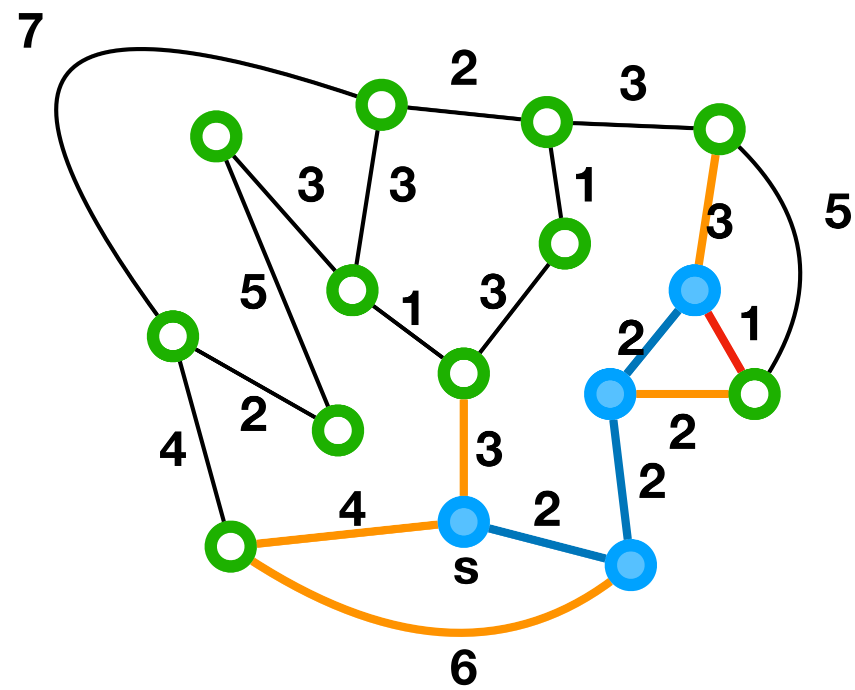

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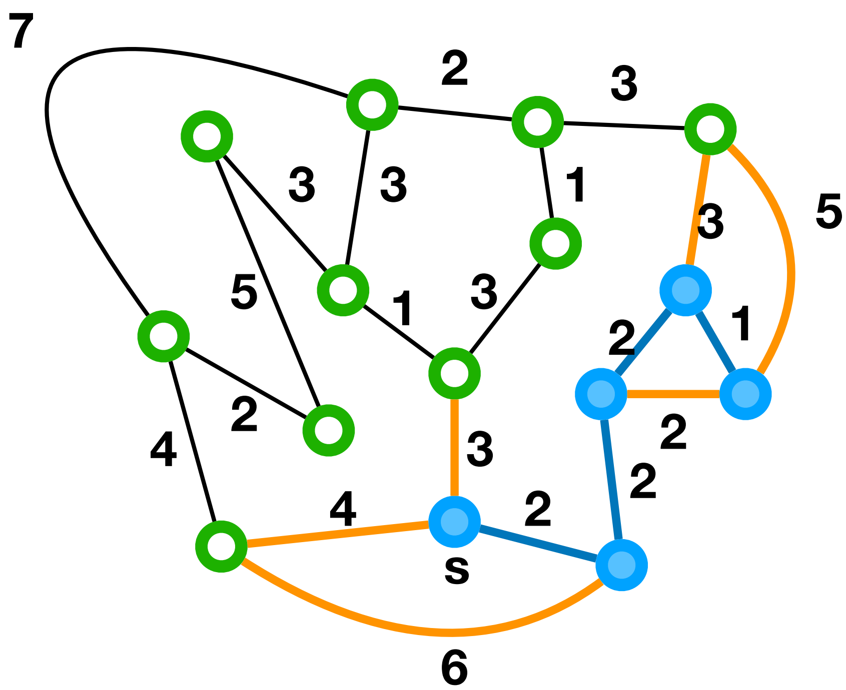

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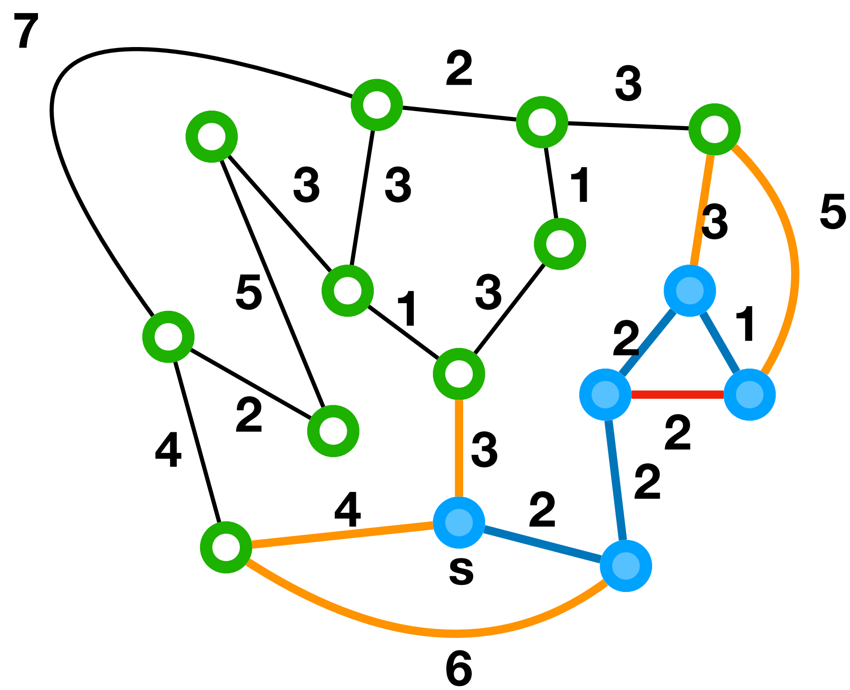

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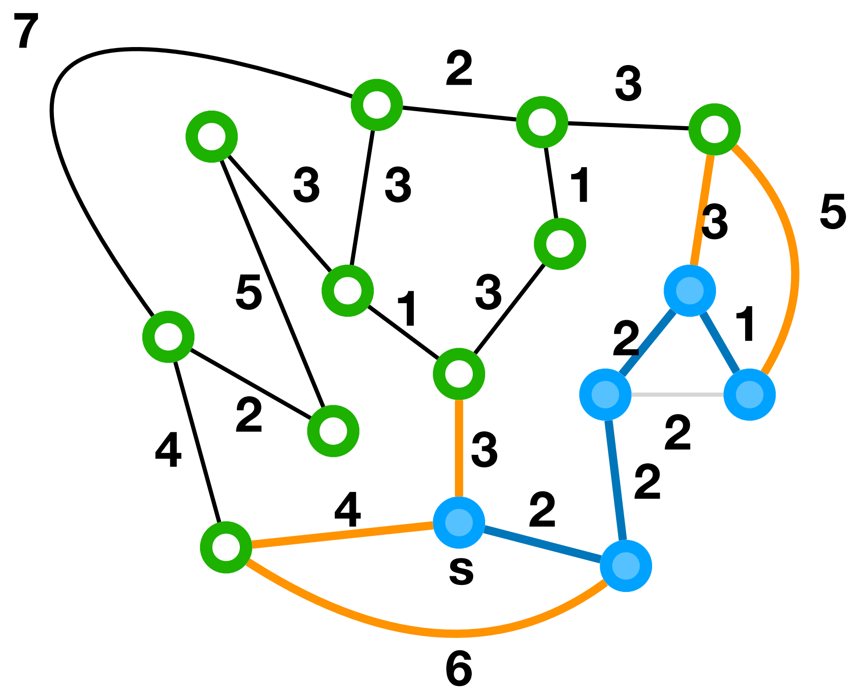

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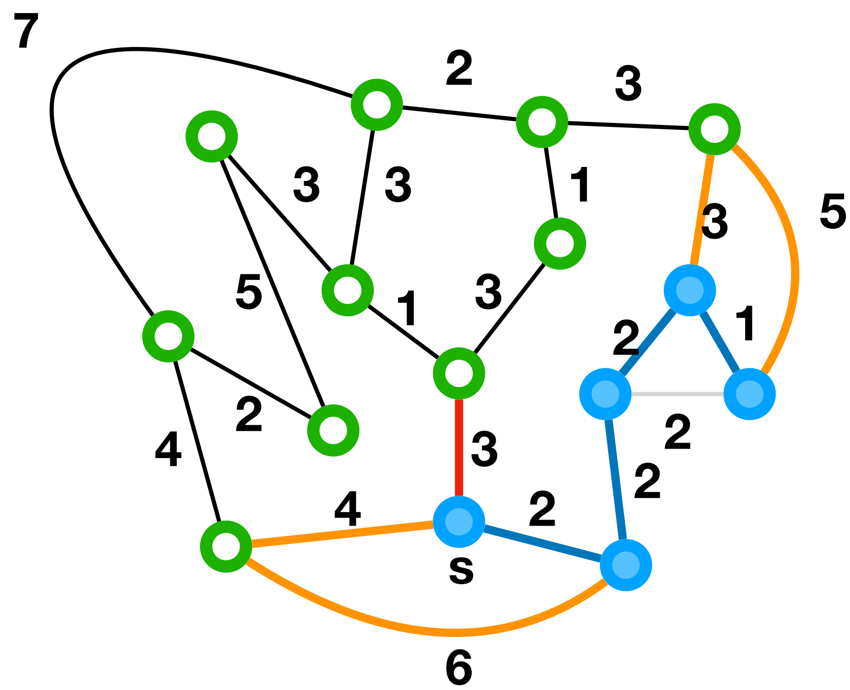

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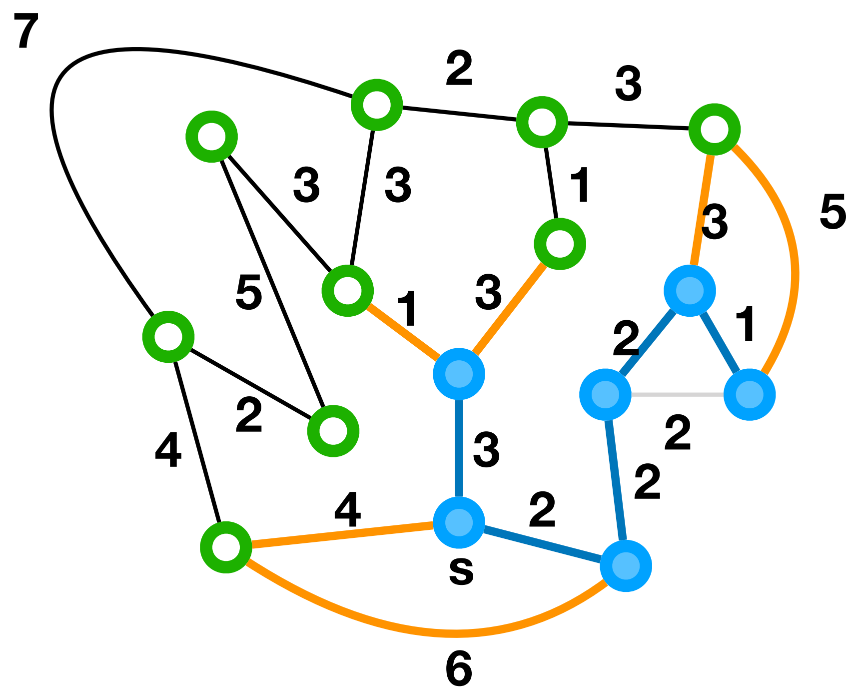

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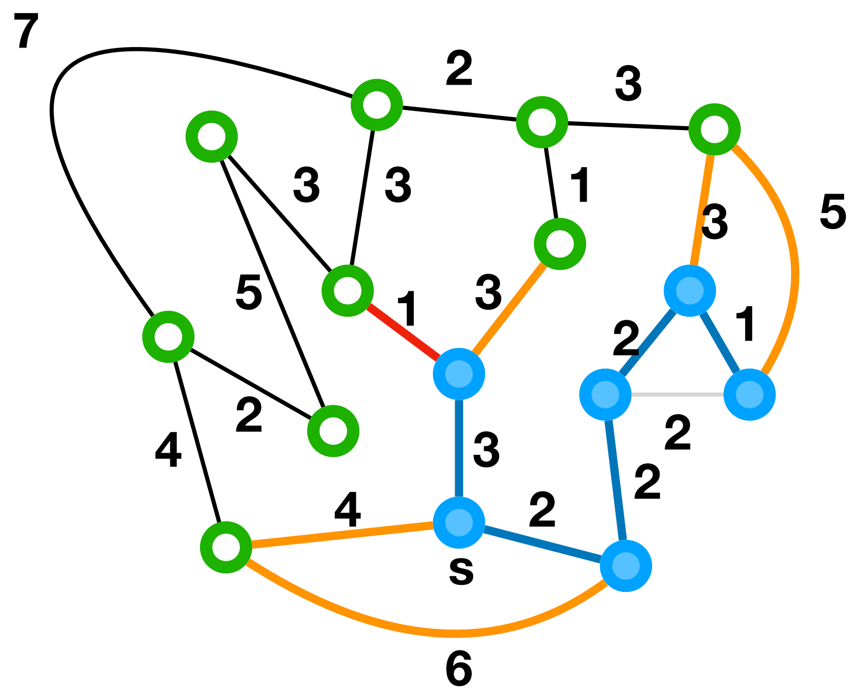

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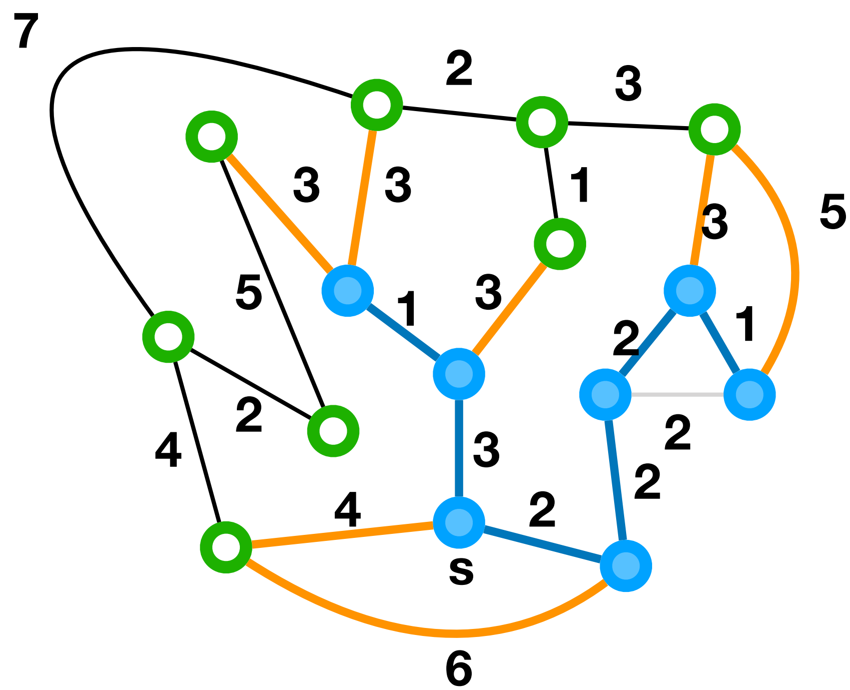

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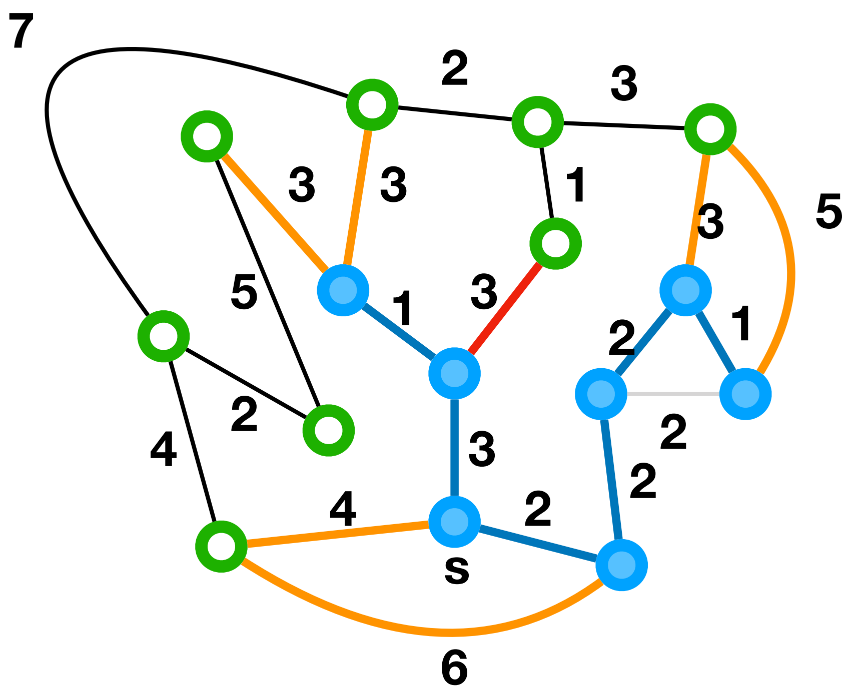

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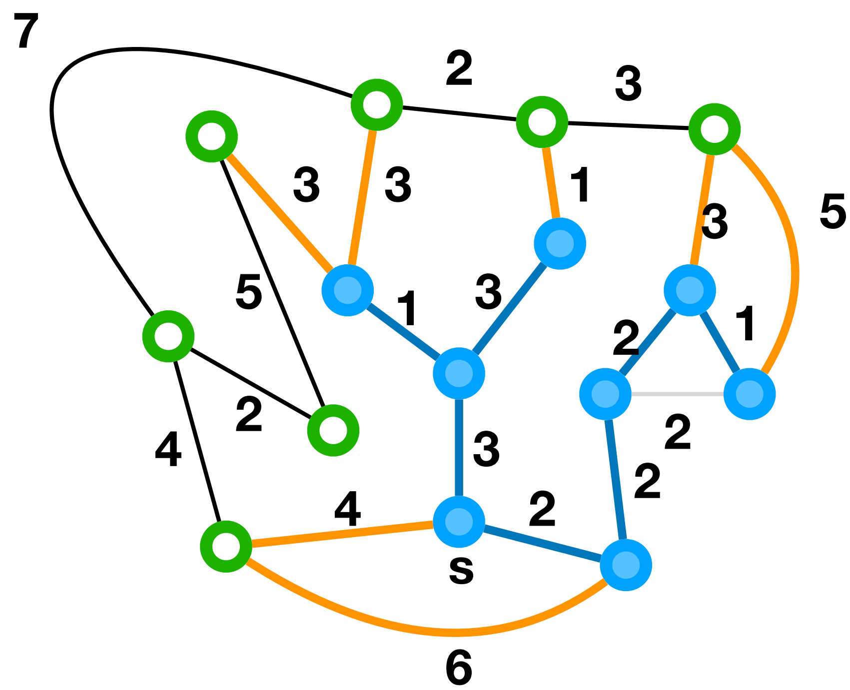

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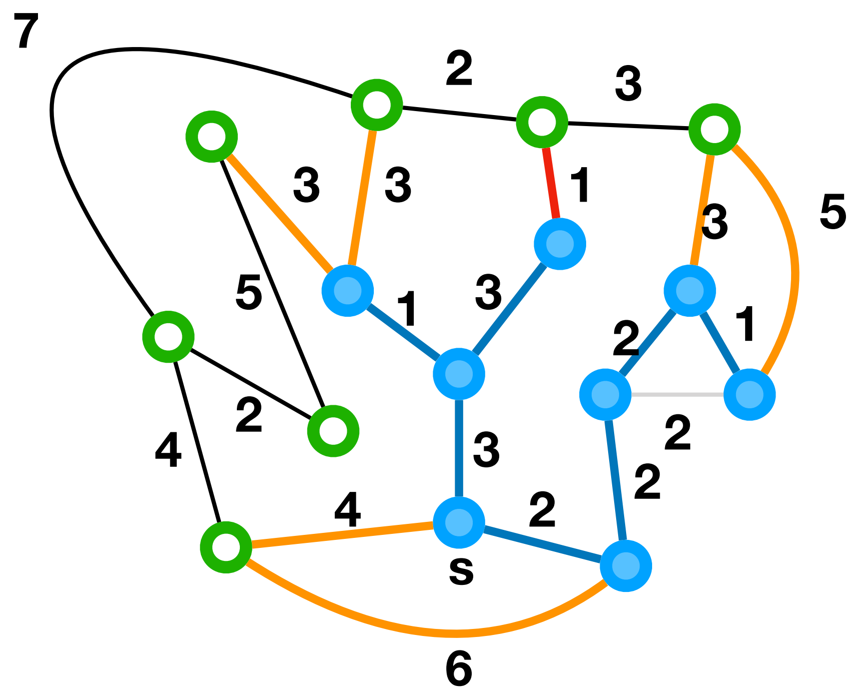

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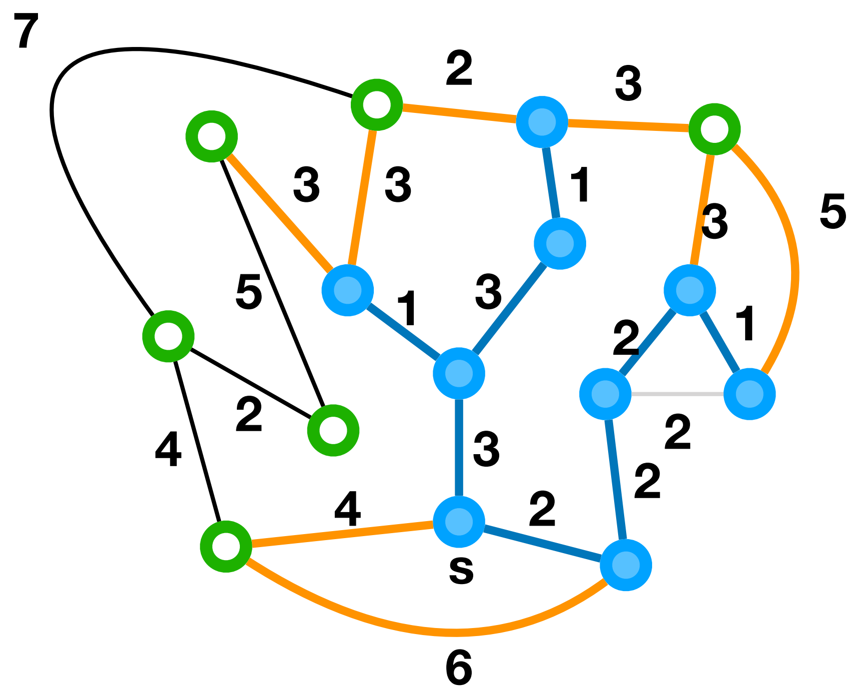

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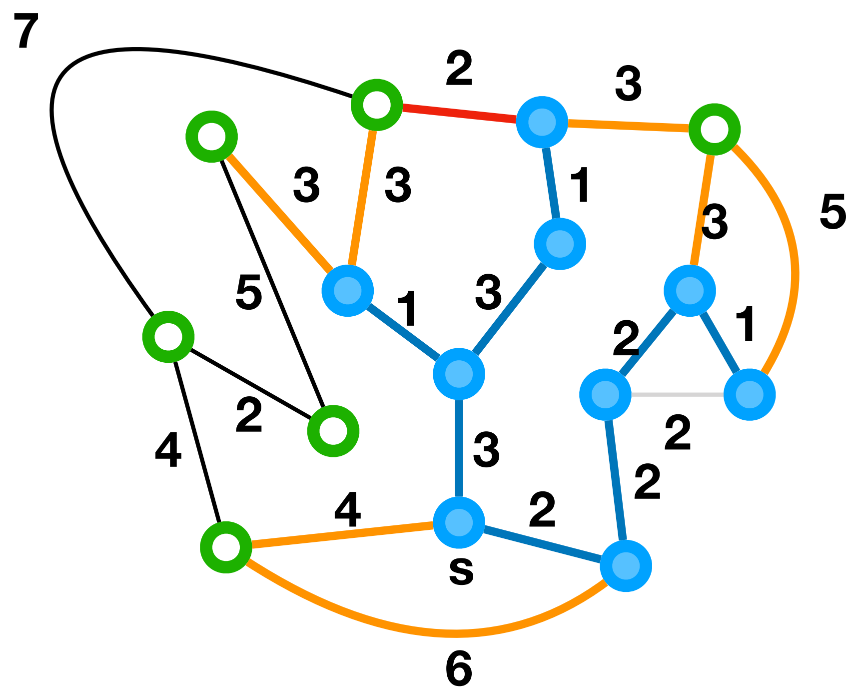

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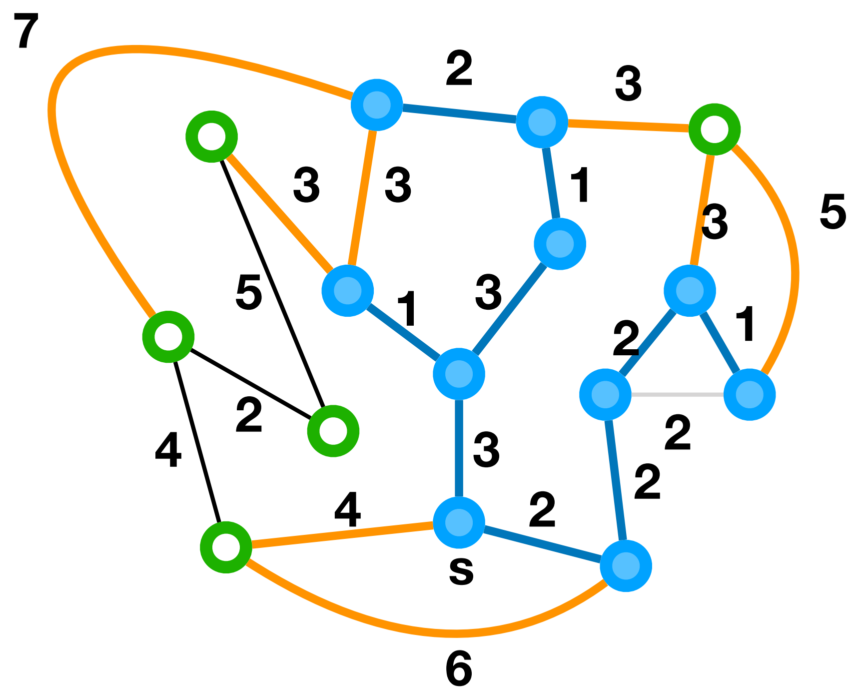

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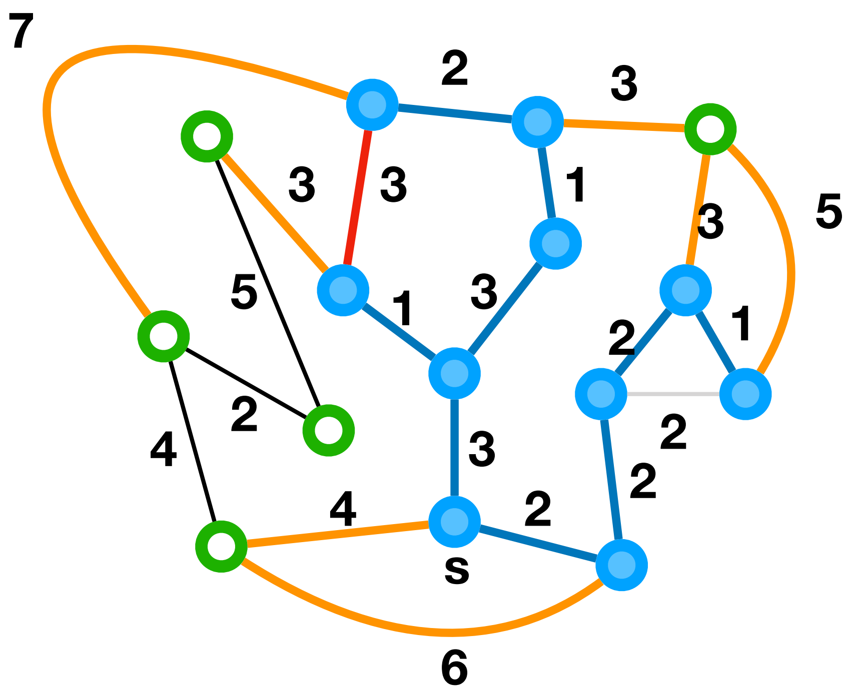

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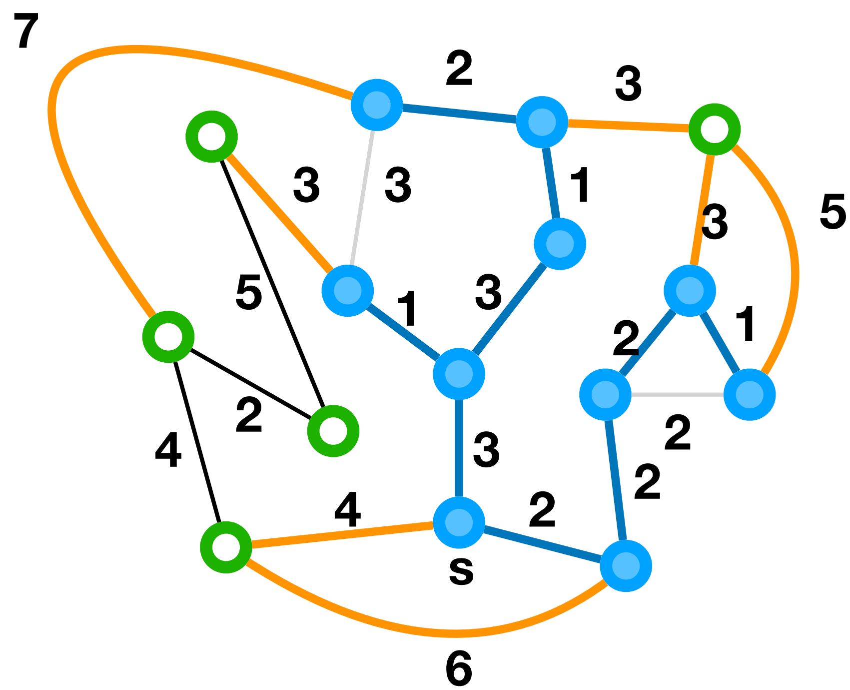

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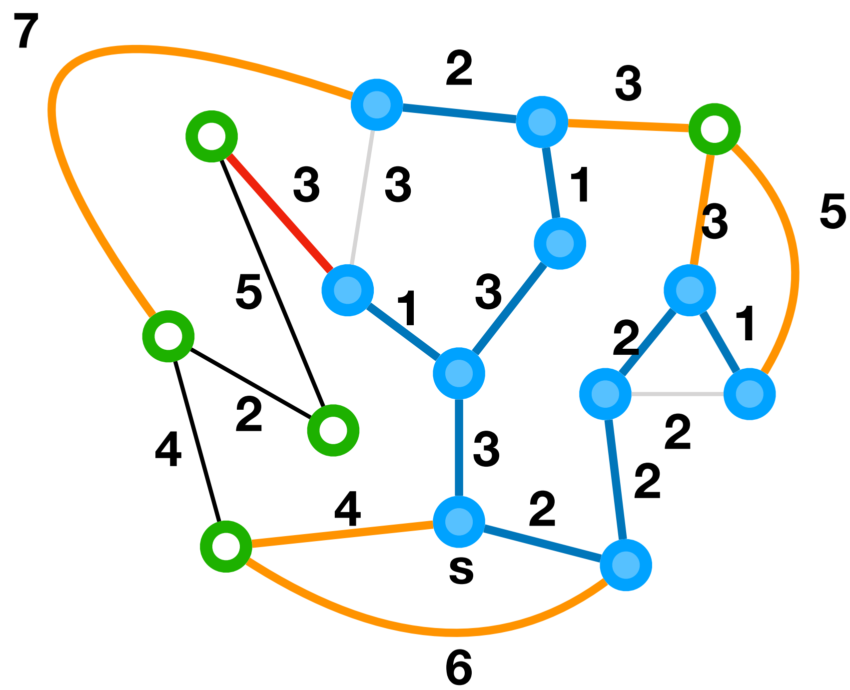

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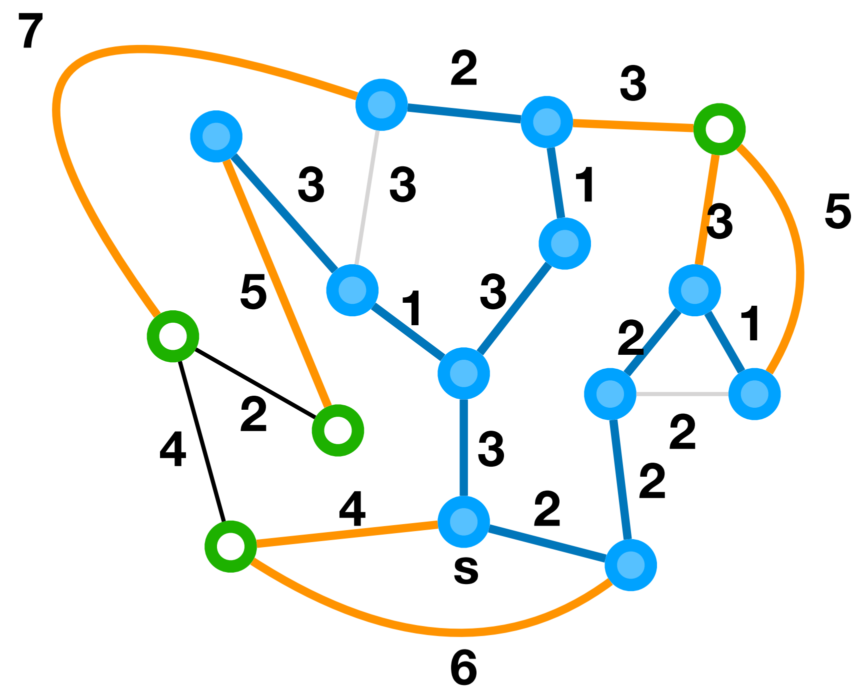

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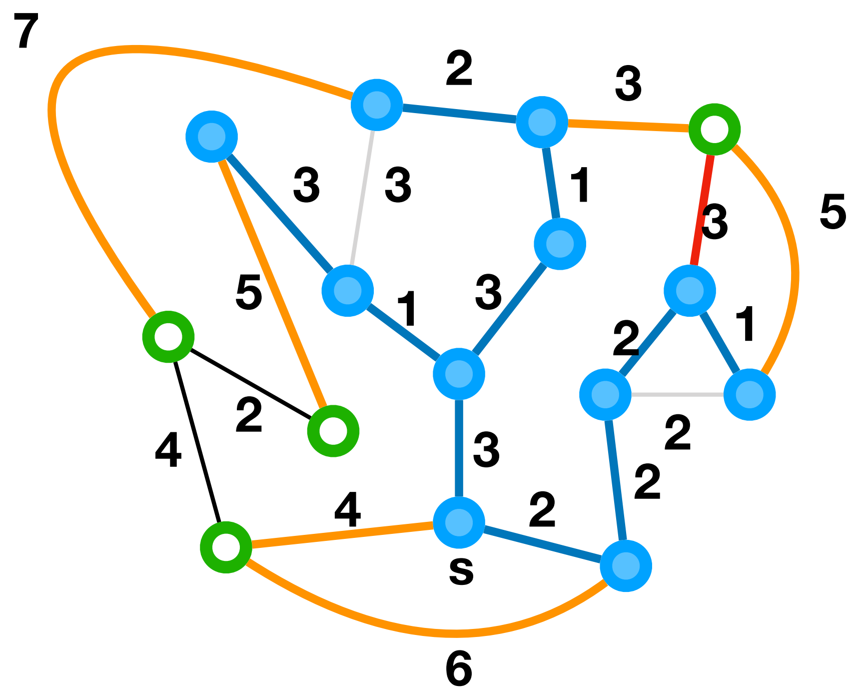

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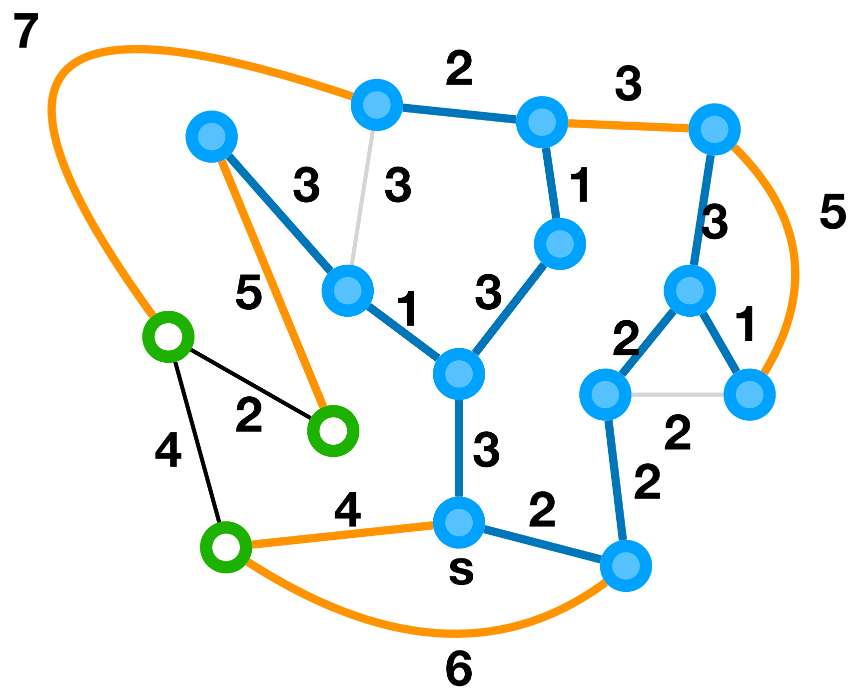

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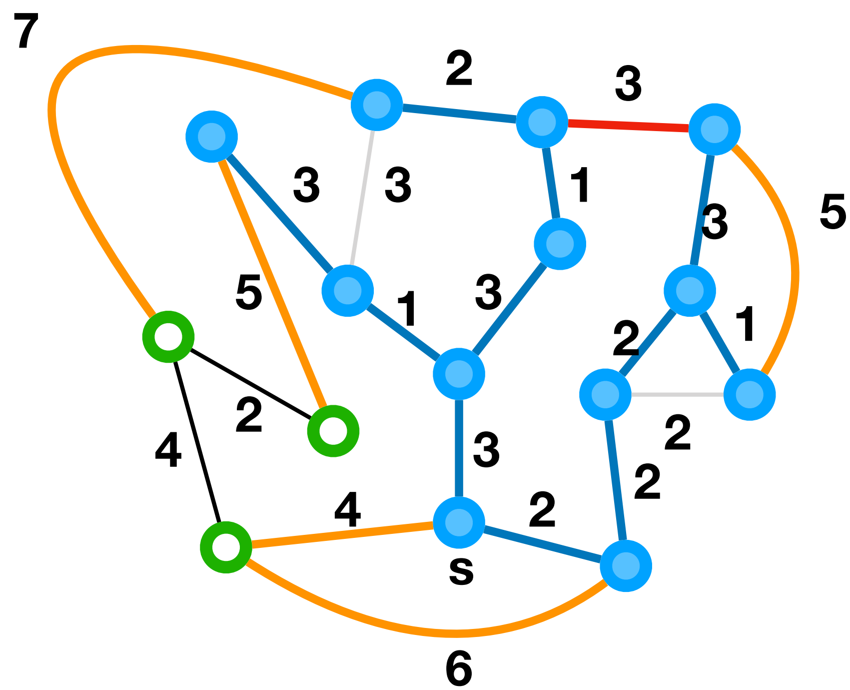

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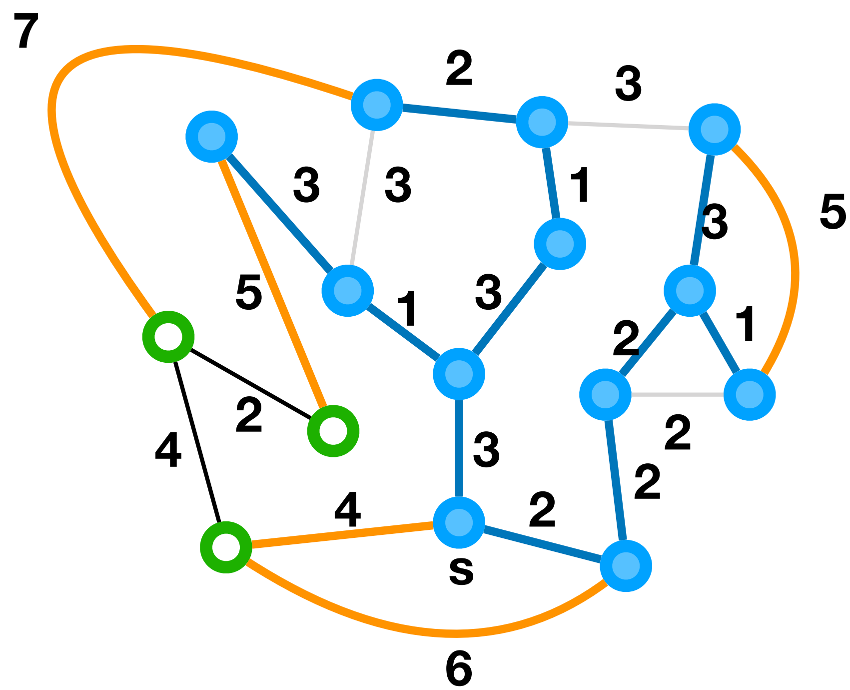

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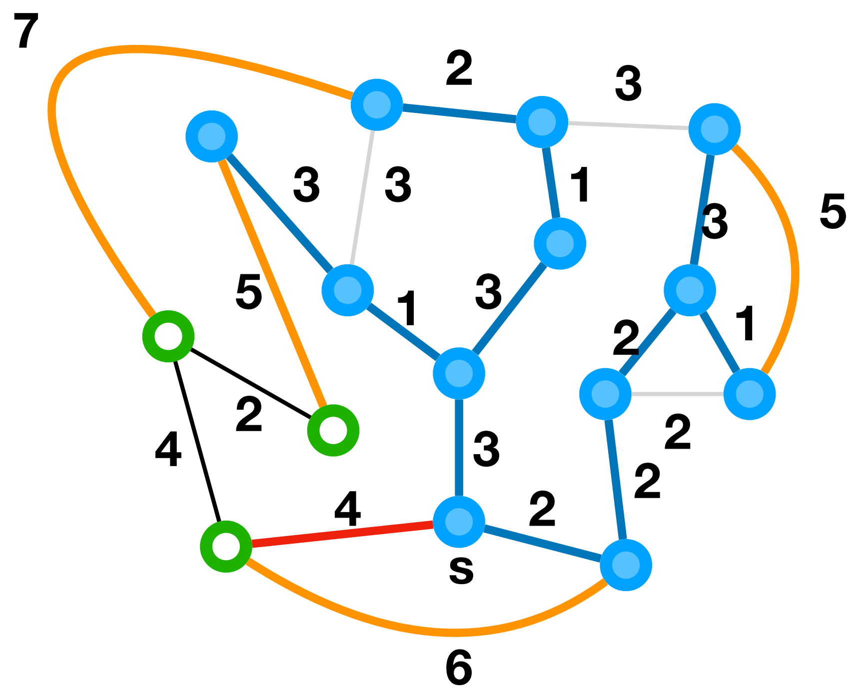

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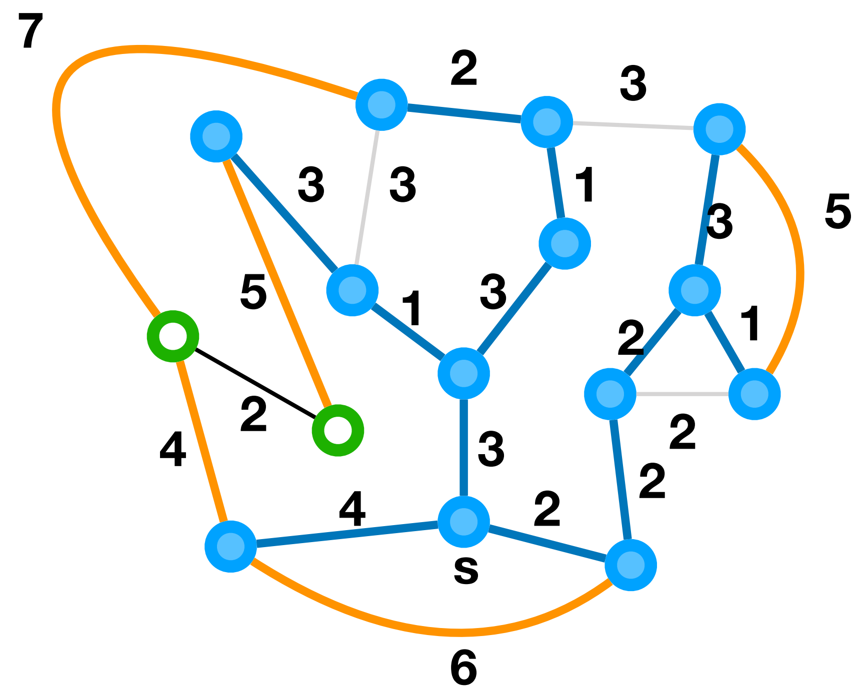

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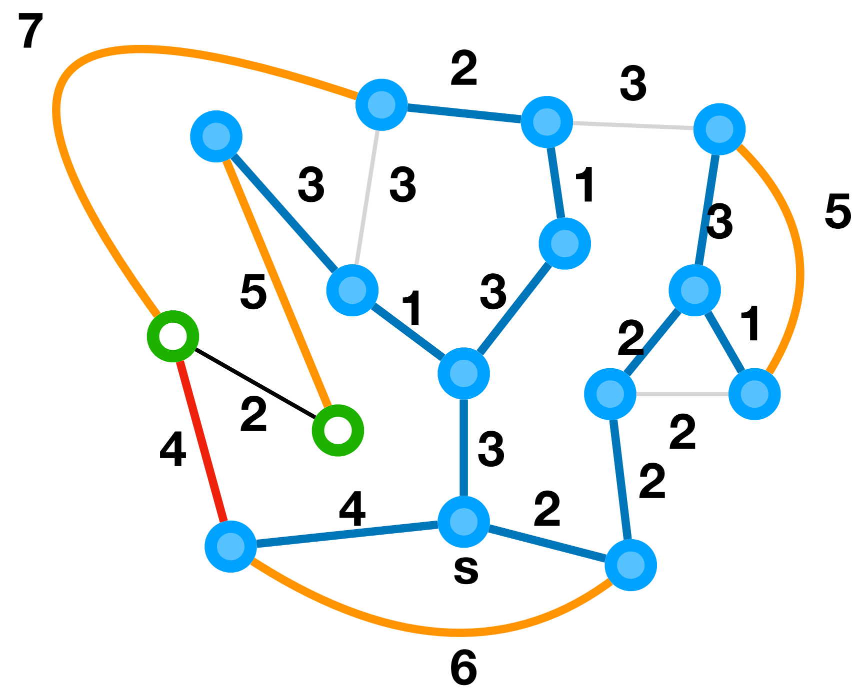

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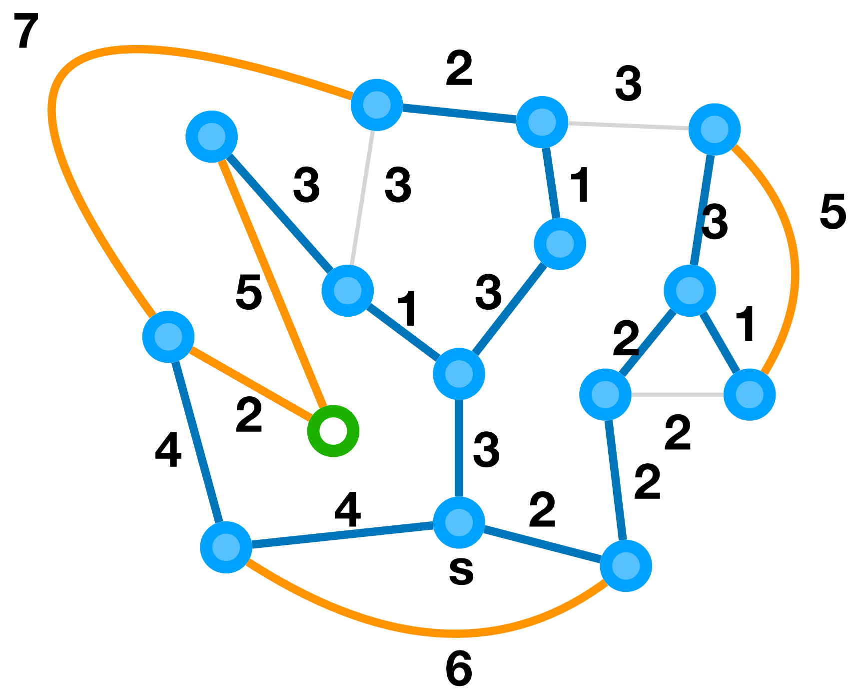

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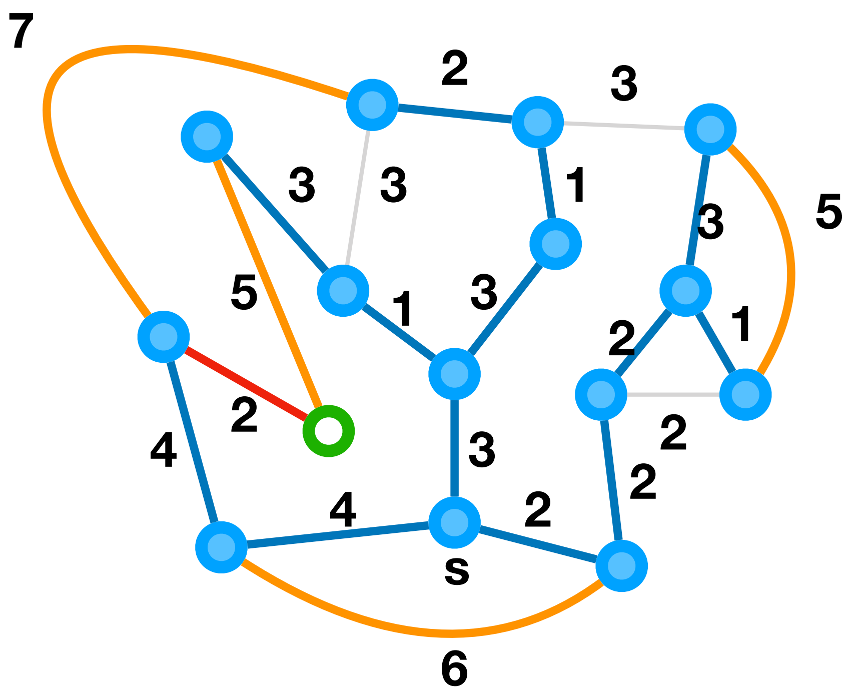

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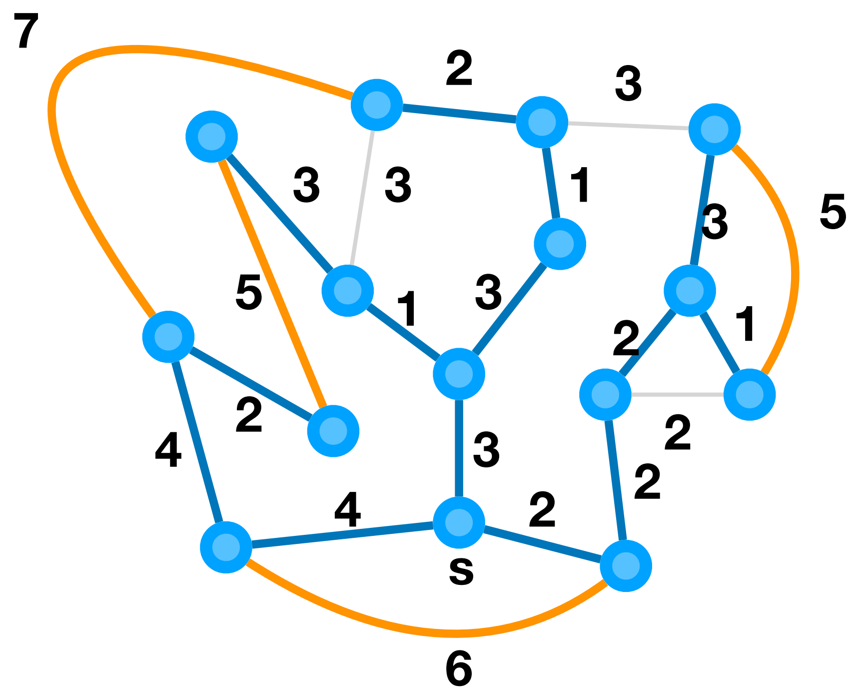

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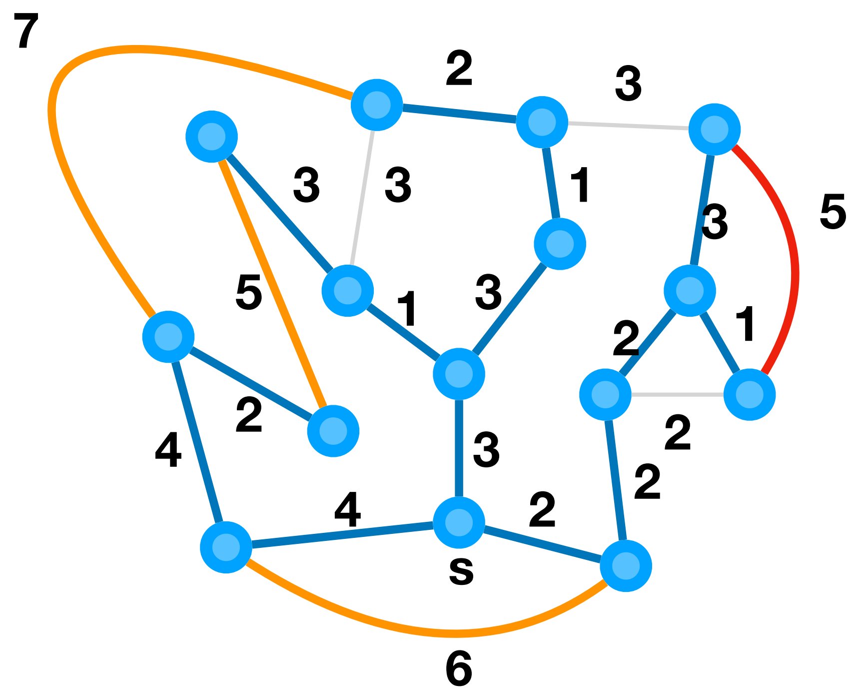

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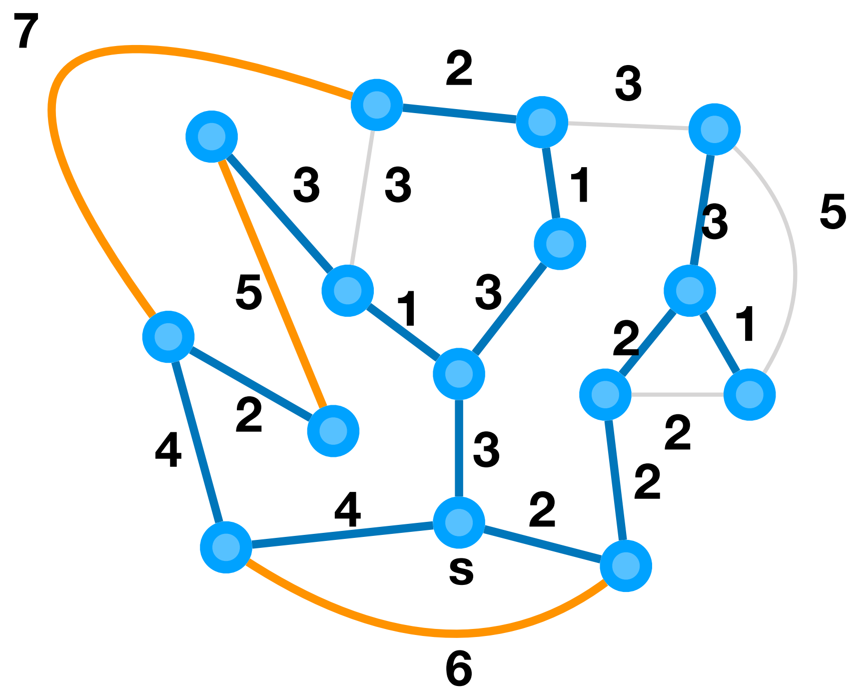

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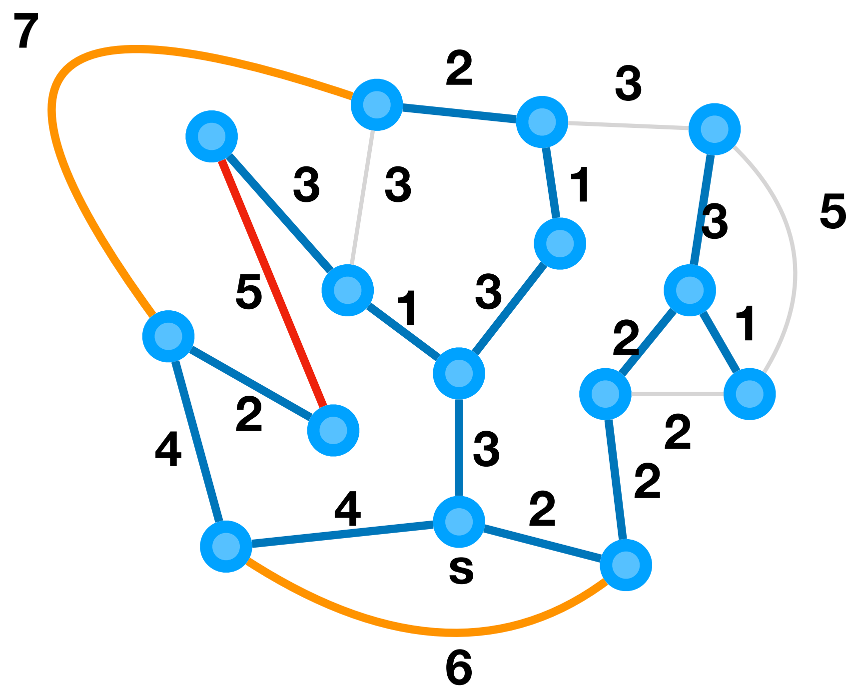

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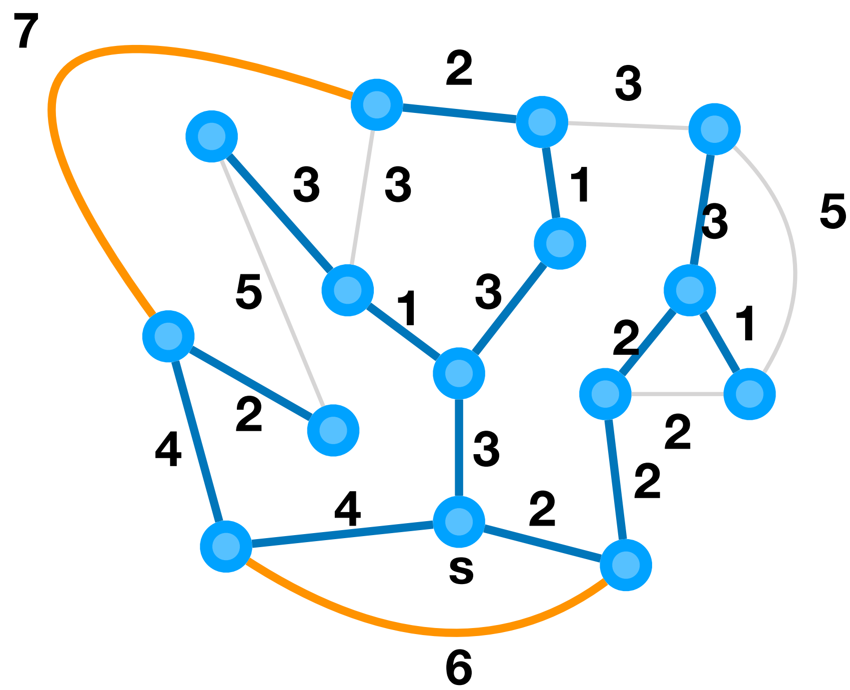

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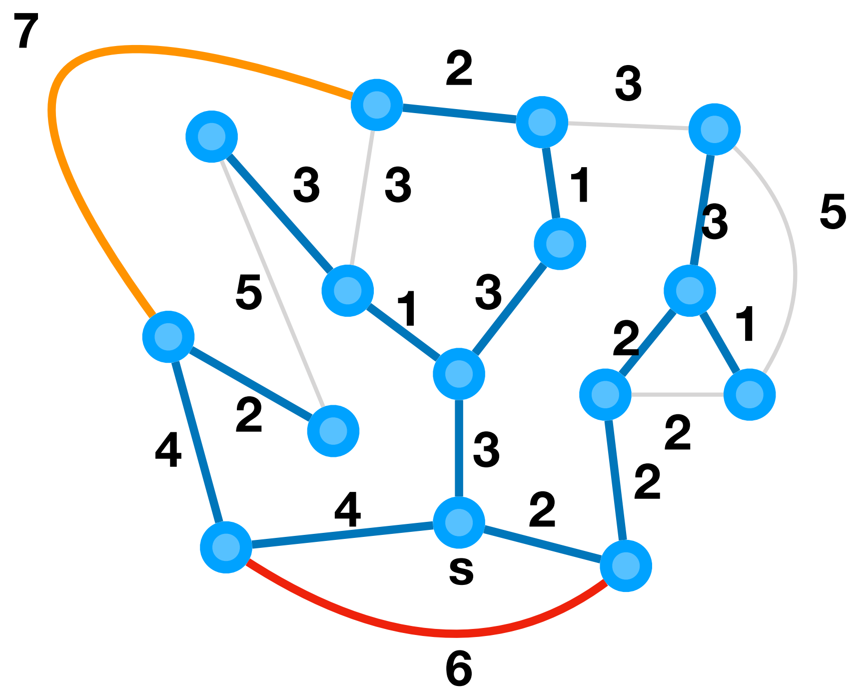

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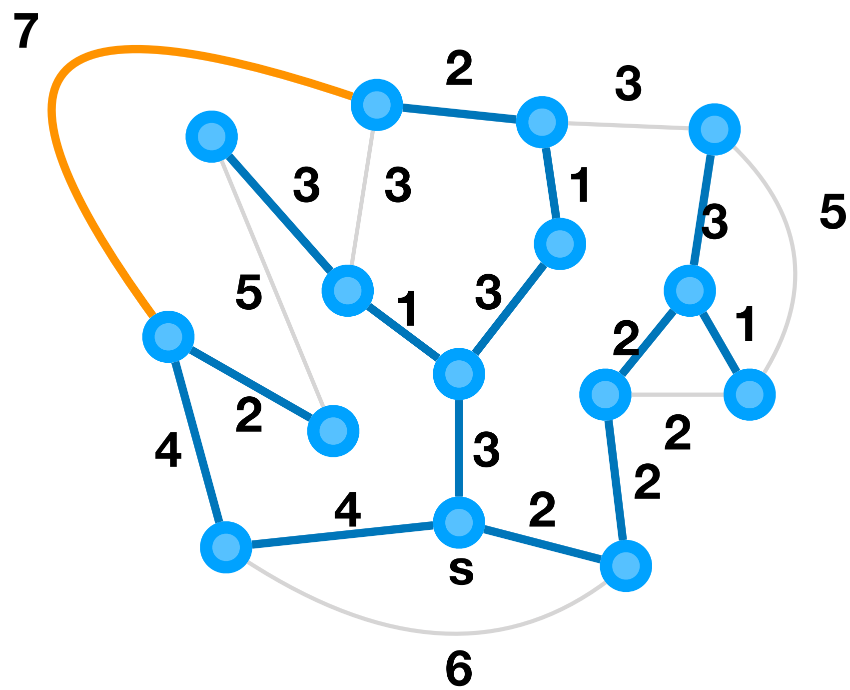

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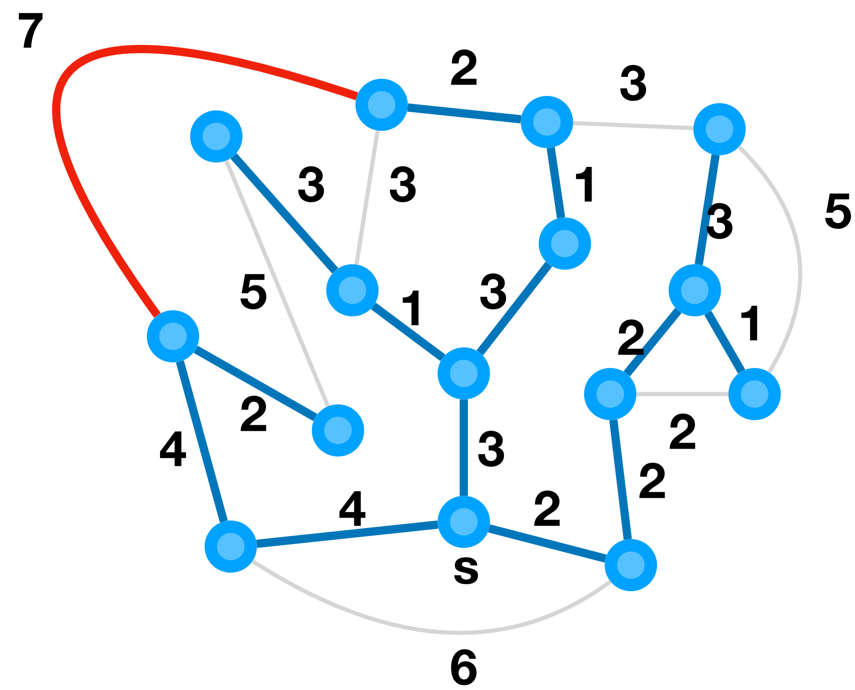

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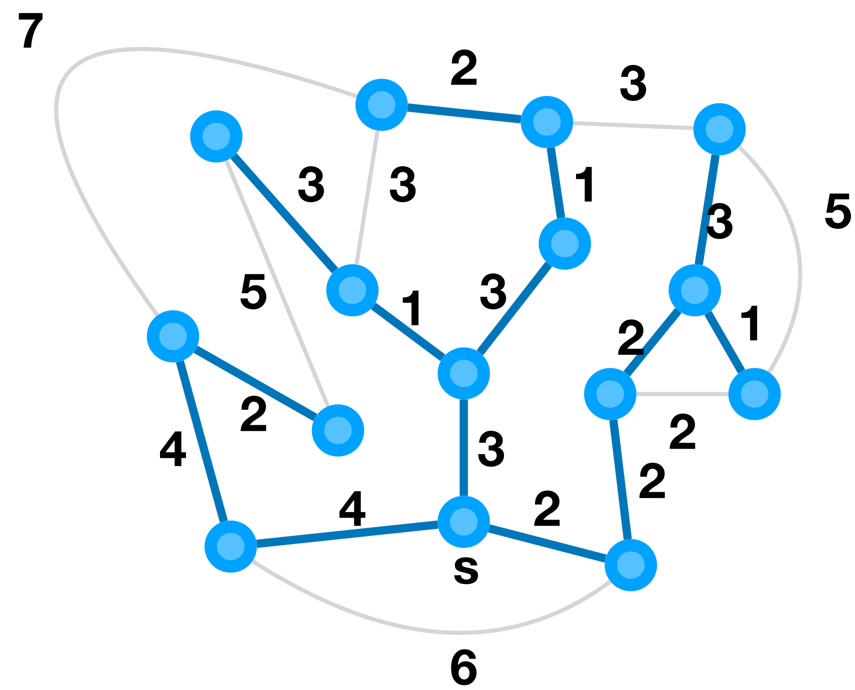

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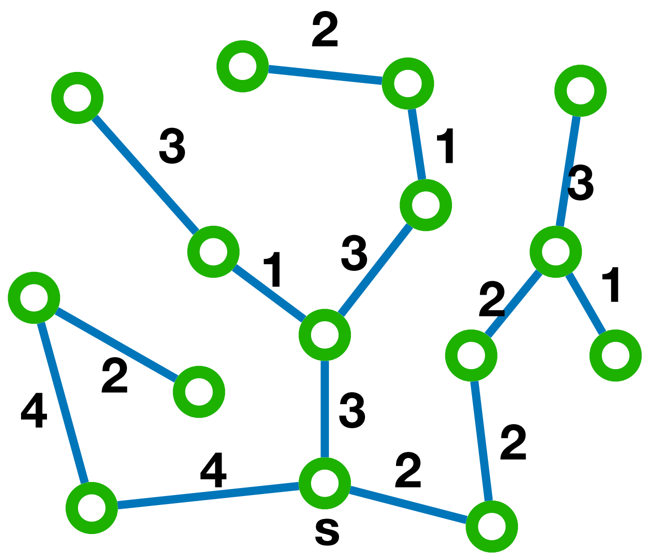

Proof of Correctness

- Let mark[1:n] = false for all vertices and s be any arbitrary vertex
- Let F = ∅, mark[s] = true and H be the set of edges incident on s
- While H is not empty:
  - Remove minimum weight edge (u,v) from H
  - If mark[u]=mark[v] = true, ignore the edge and go to the next iteration of while-loop
  - Otherwise, let us assume by symmetry mark[u] = true only
  - Add (u,v) to F and all edges incident on v to H; set mark[v] = true.

- **Theorem:**
  - Suppose F is MST-good but not a tree yet
  - Let (S,V-S) be any cut with no cut edge in F
  - Then edge e in G-F with minimum weight among cut edges of (S,V-S) is safe for F

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- Consider edge (u,v) inserted to F
- Let S be the connected component of s in F before inserting (u,v)
- By the same argument as DFS/BFS, S is the set of all marked vertices
- So here u belongs to S and v does not
- We claim (u,v) is the minimum weight edge of the cut (S, V-S)
- So by our theorem, (u,v) is safe

Runtime Analysis

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- If we store H in a linked list and do a linear search in every step:
- O(m) time to find minimum weight edge in H
- O(m) iterations in the while-loop
- So $O(m^2)$ time in total

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- Again, too slow

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- We should store H as a min-heap
- Insertion takes O(log m) time
- Deletion takes O(log m) time
- Each vertex is marked once and takes $O(\deg(v) \cdot \log m)$ time to insert its edges to H
- We do at most m deletions from the min-heap
- So O(m log m) time in total

Summary

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- MST problem: finding a spanning tree with minimum weight
- We saw two different algorithms for finding MST
  - Kruskal: based on sorting edges first
  - Prim: based on a graph search + min-heap
- They are both different implementation of a generic meta-algorithm based on safe edges and minimum weight edge of cuts
- Both algorithms take $O(m \log m)$ time

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- They are both different implementation of a generic meta-algorithm based on safe edges and minimum weight edge of cuts
- Both algorithms take O(m log m) time
- There are even faster algorithms for this problem but they are way beyond the scope of our course