CS 344: Design and Analysis of Computer Algorithms

(Spring 2022 — Sections 5,6,7,8)

Lecture 19:
Prim's Algorithm for MST, Shortest Path Algorithms

The Minimum Spanning Tree Problem

- **Input:**
  - An undirected connected graph $G = (V, E)$
  - Positive weights on edges of G: edge e has weight $w_e > 0$
- **Output:**
  - A spanning tree T in G with minimum weight
    - Weight of T = $\sum_{e \in T} w_e$

A Generic "Algorithm" for MST

A Generic Meta-Algorithm

- Let $F = \varnothing$ be an empty forest initially
- For i = 1 to n-1 steps:
  - **Find a safe edge e for the current forest F**
  - Update F = F + e
- Output the final F as an MST

- This is NOT really an algorithm

An Approach for Finding Safe Edges

- **Theorem:**
  - Suppose F is MST-good but not a tree yet
  - Let (S,V-S) be any cut with no cut edge in F
  - Then edge e in G-F with minimum weight among cut edges of (S,V-S) is safe for F

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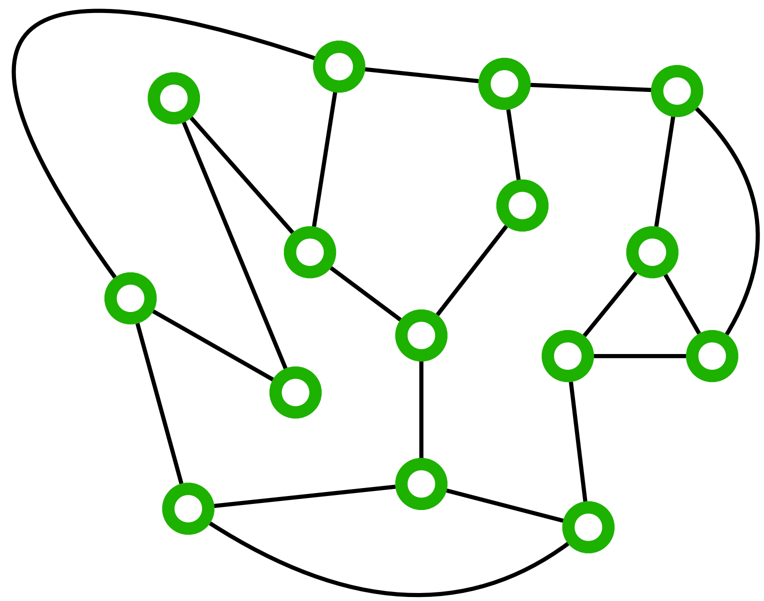

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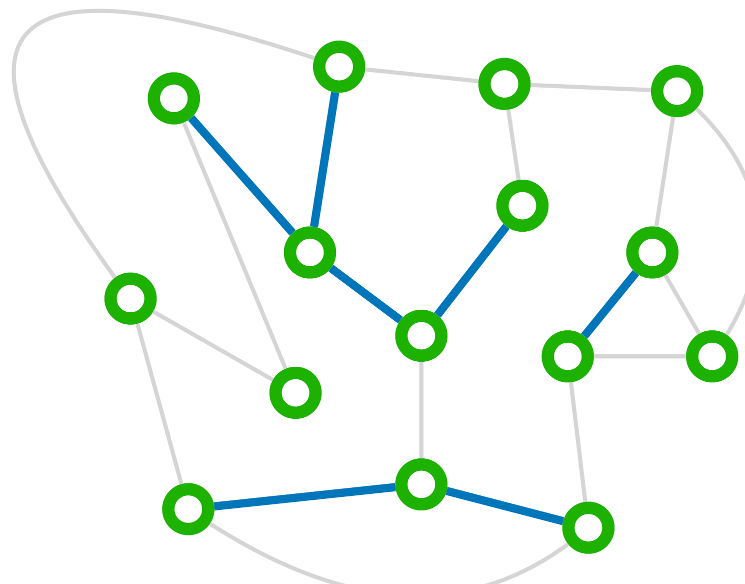

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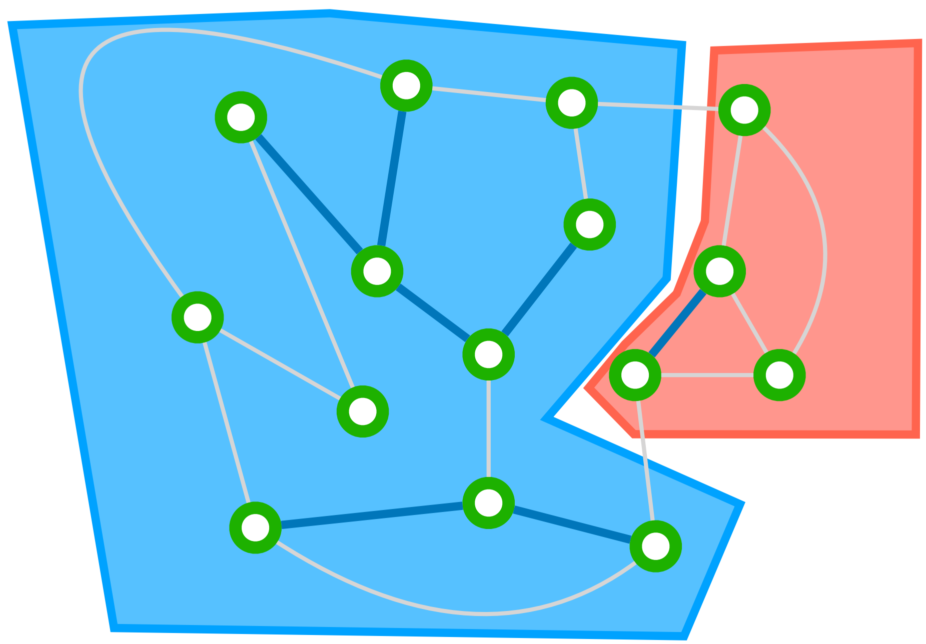

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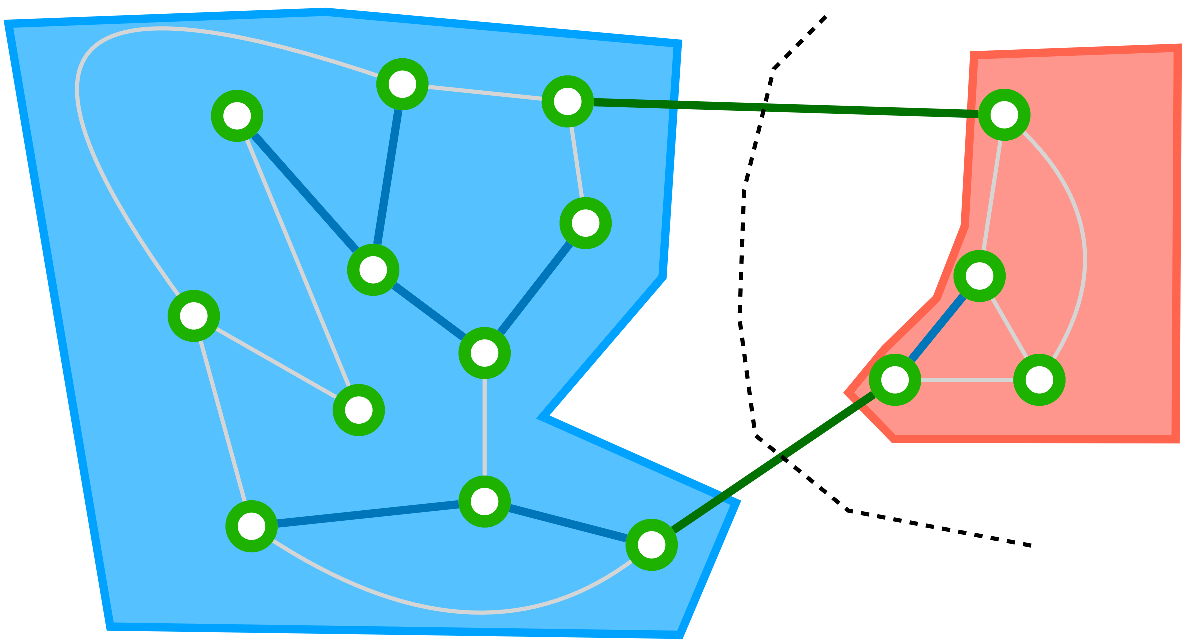
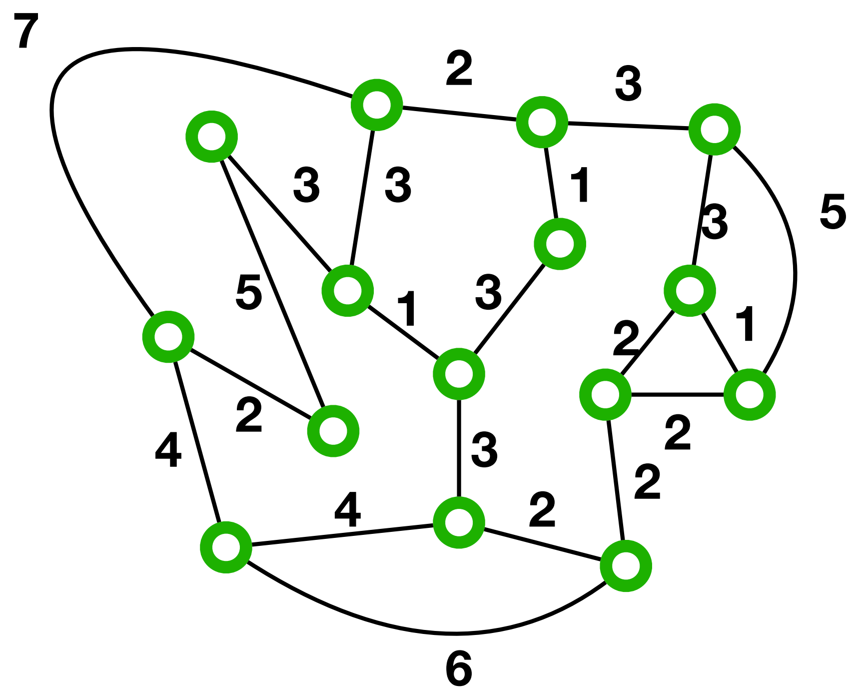


Prim’s Algorithm

Prim's Algorithm

- Let mark[1:n] = false for all vertices and s be any arbitrary vertex
- Let F = ∅, mark[s] = true and H be the set of edges incident on s
- While H is not empty:
  - Remove the minimum weight edge e=(u,v) from H
  - If mark[u]=mark[v] = true, ignore this edge and go to the next iteration of the while-loop
  - Otherwise, let us assume by symmetry mark[u] = true only
  - Add the edge (u,v) to F and all edges incident on v to H; set mark[v] = true.
- Return F as an MST of the input graph

Example



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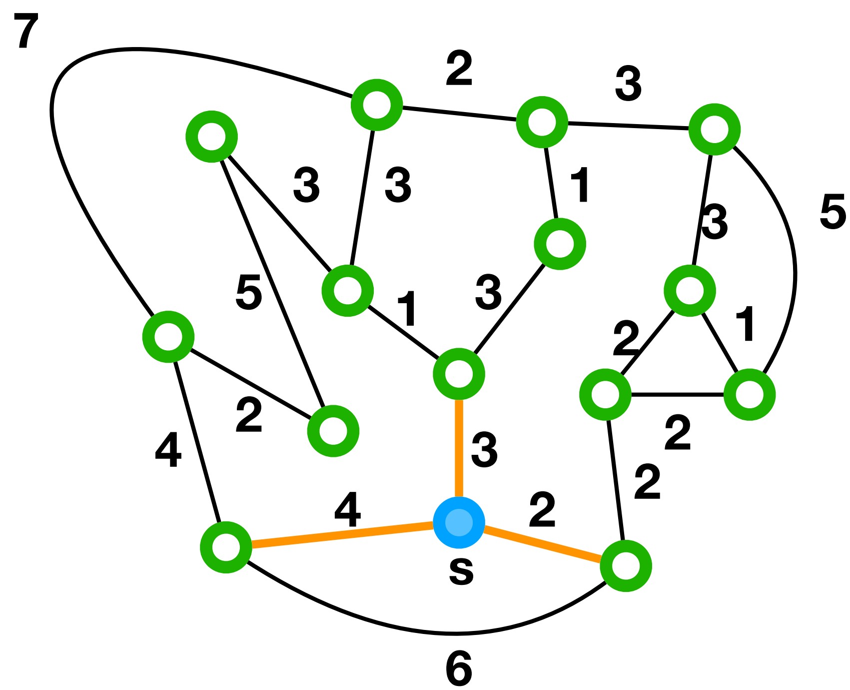

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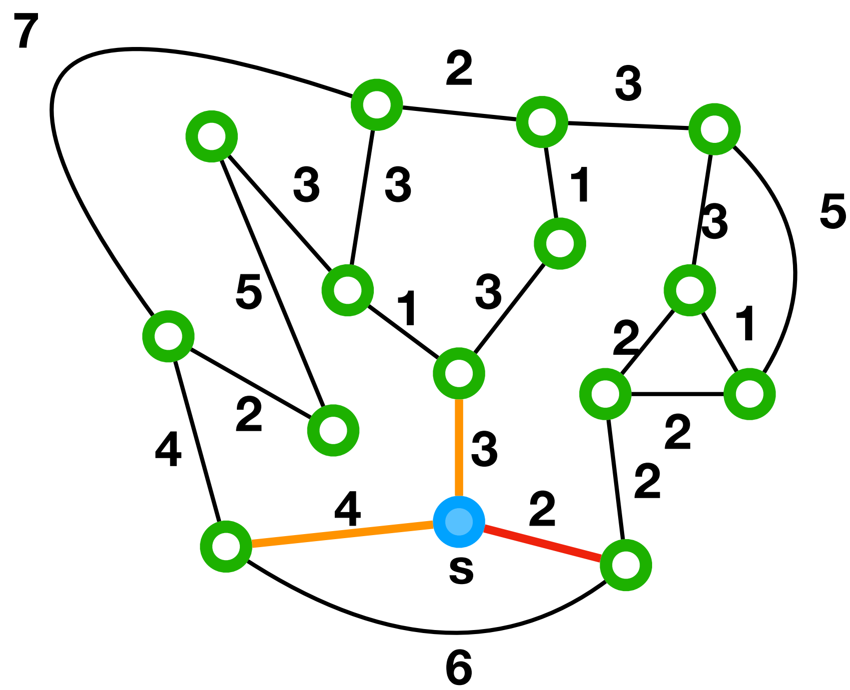

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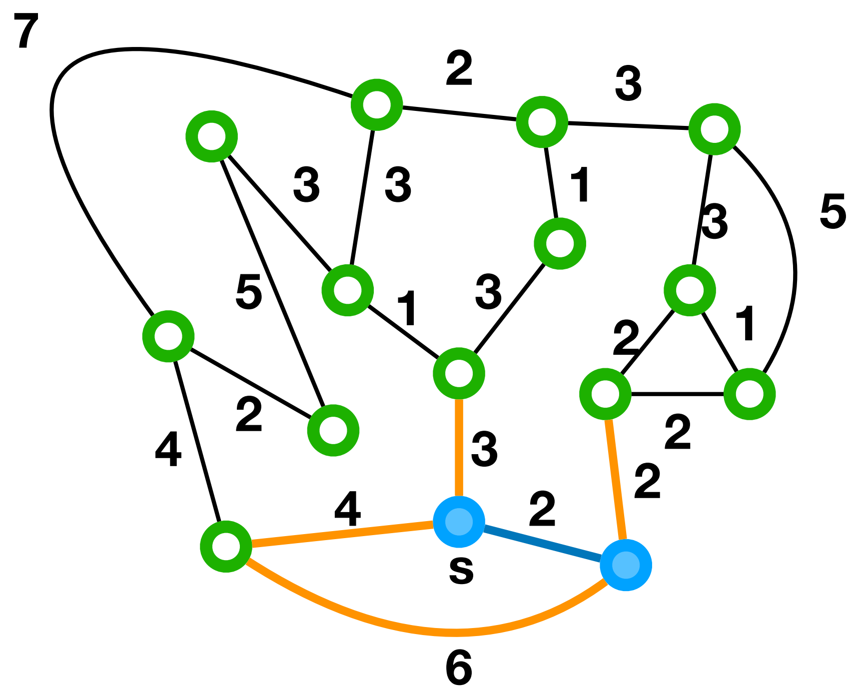

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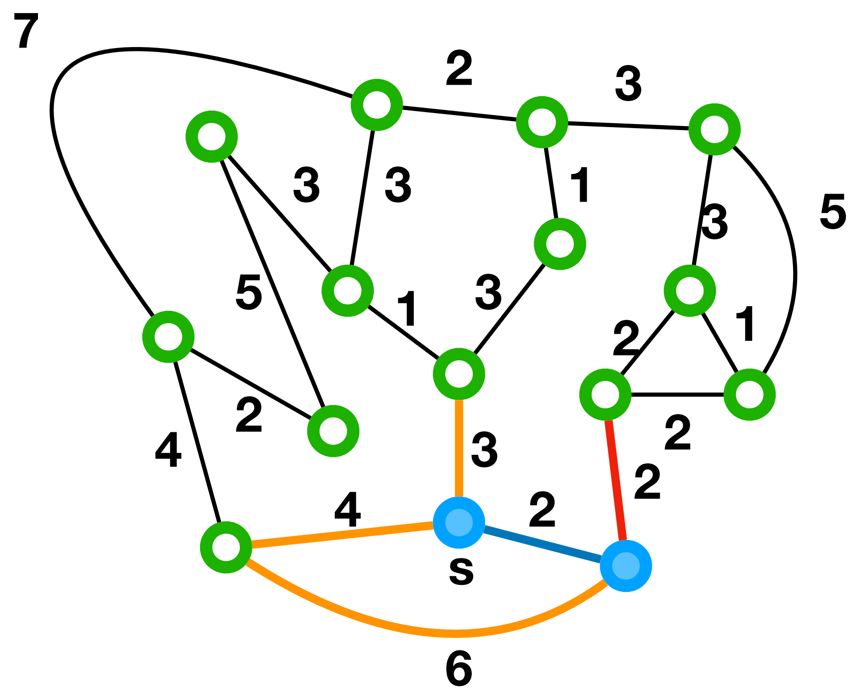

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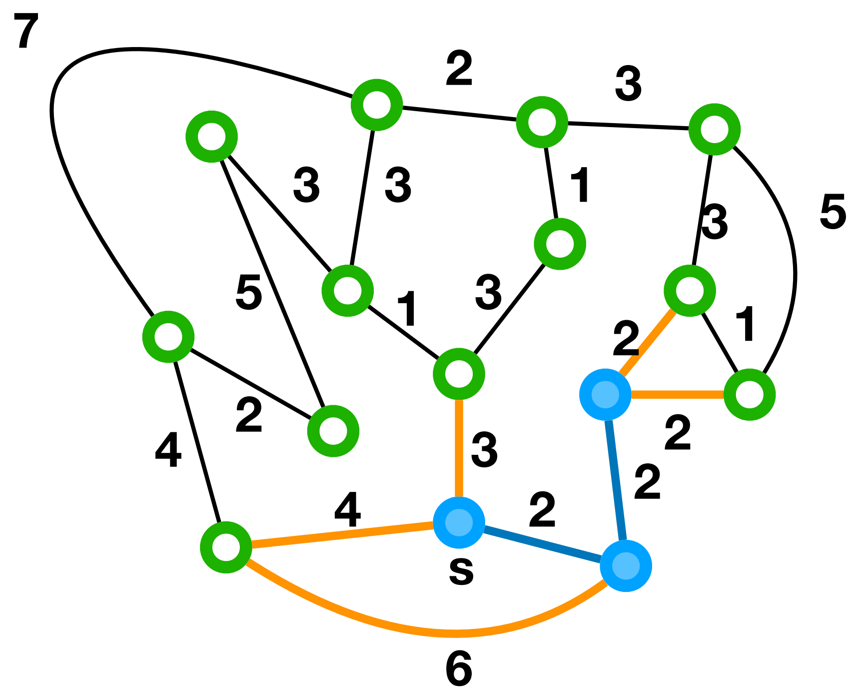

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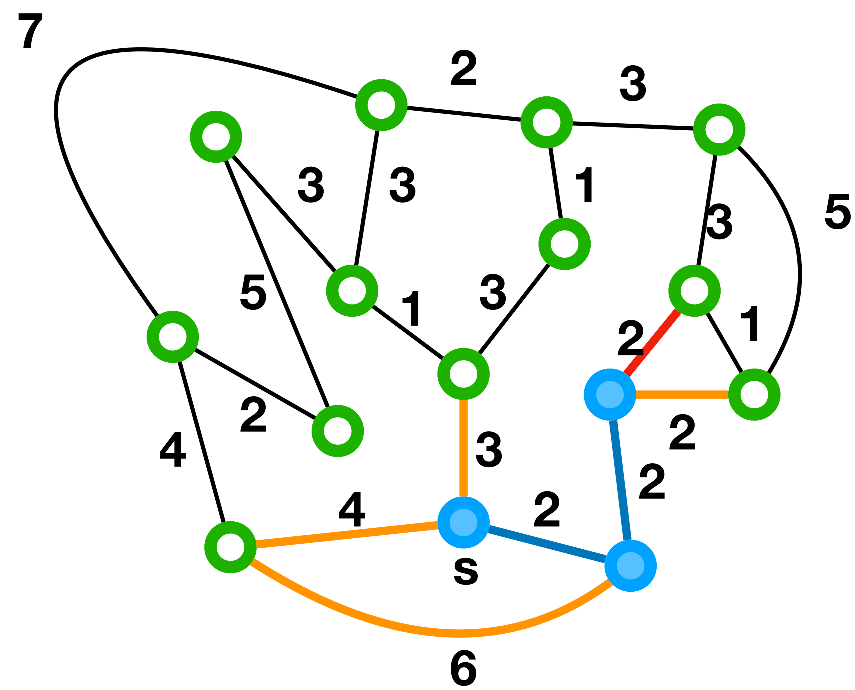

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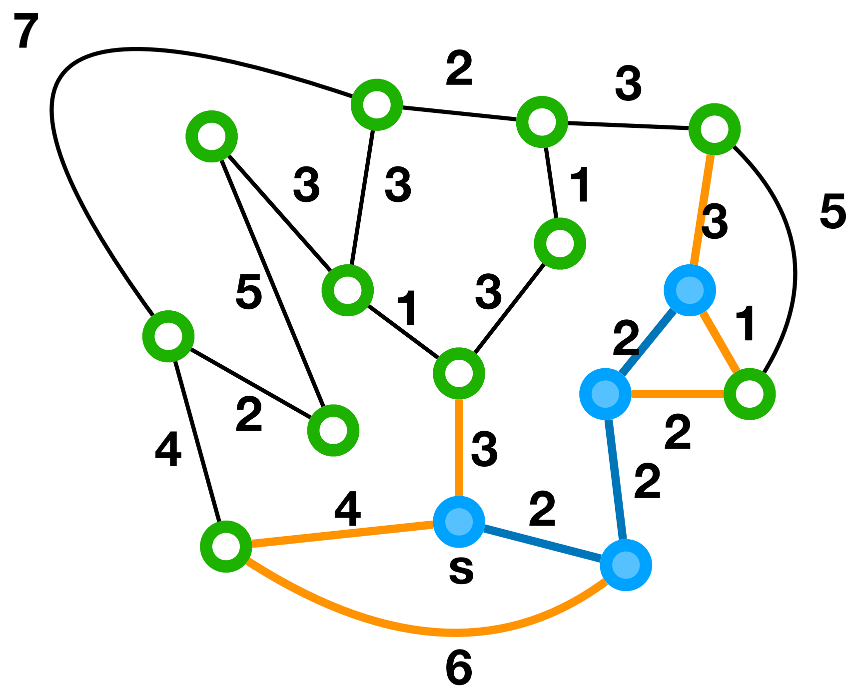

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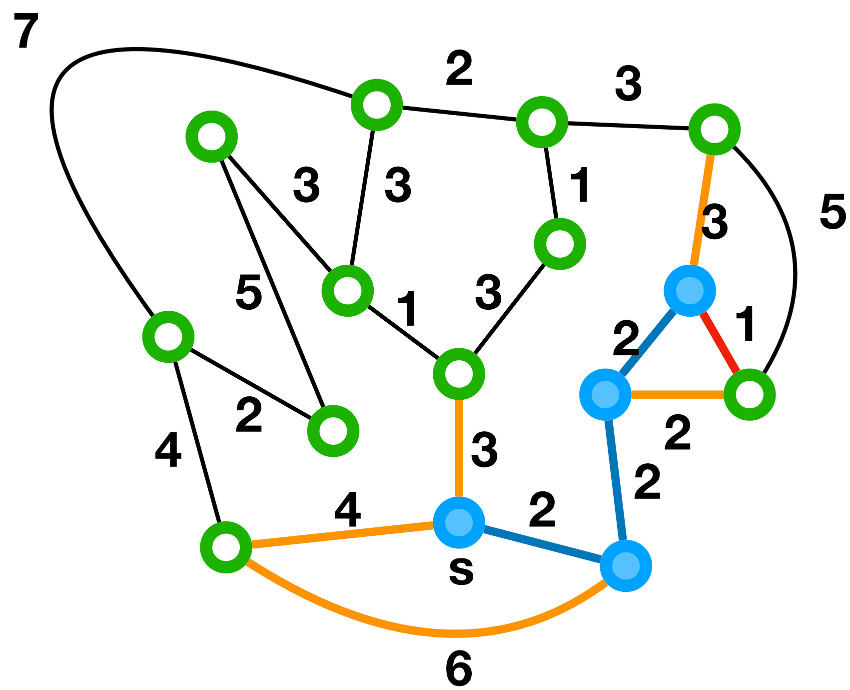

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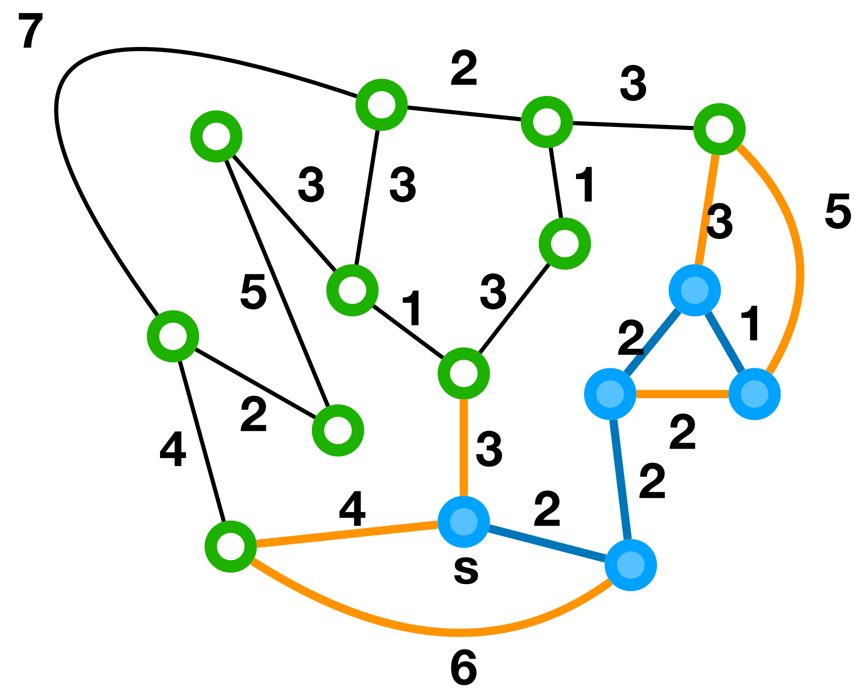

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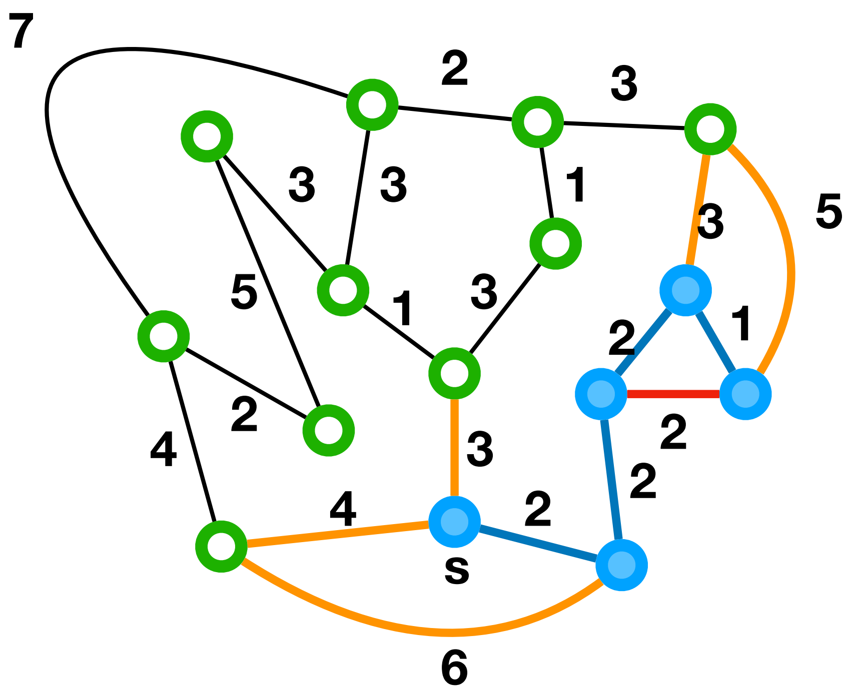

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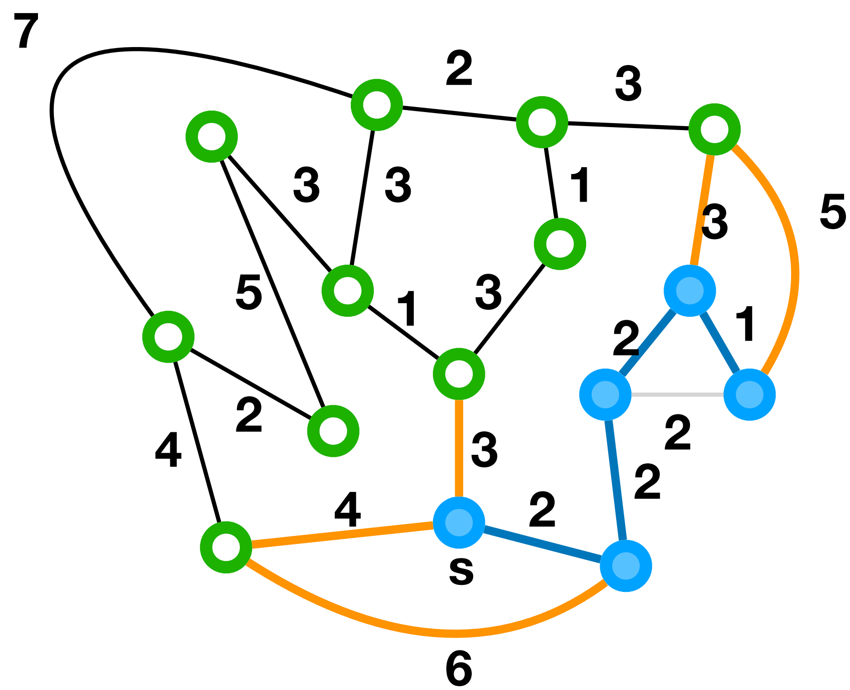

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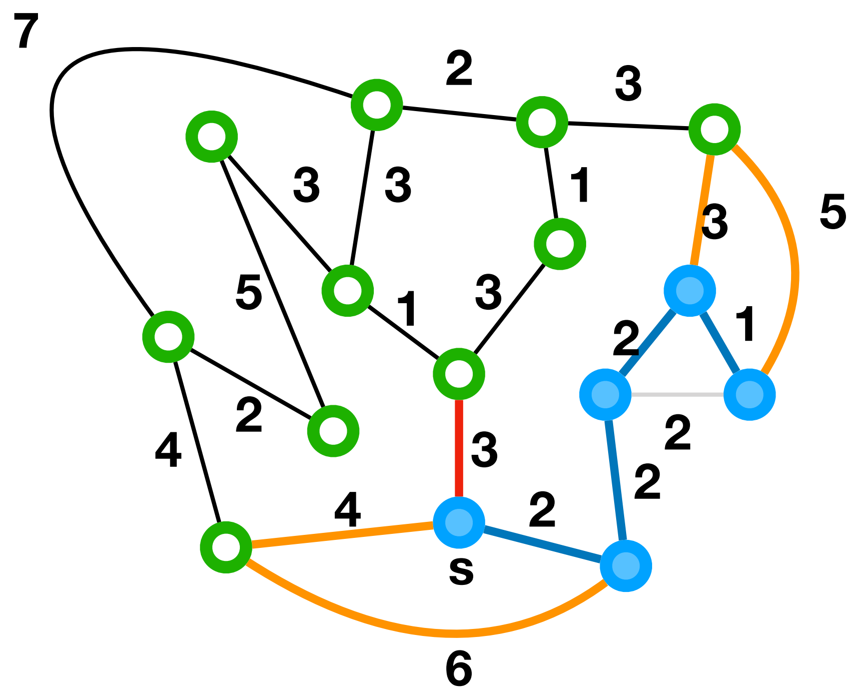

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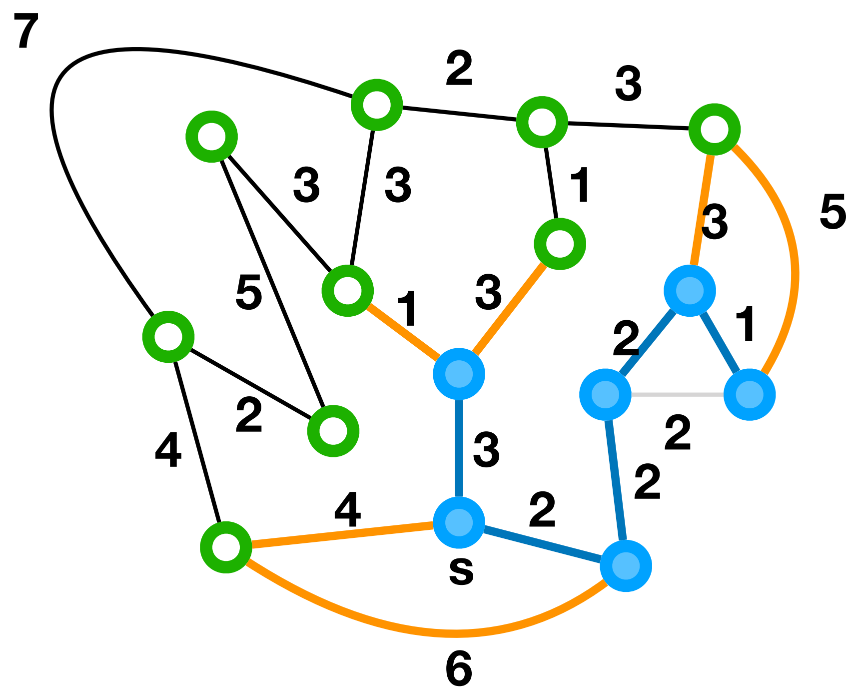

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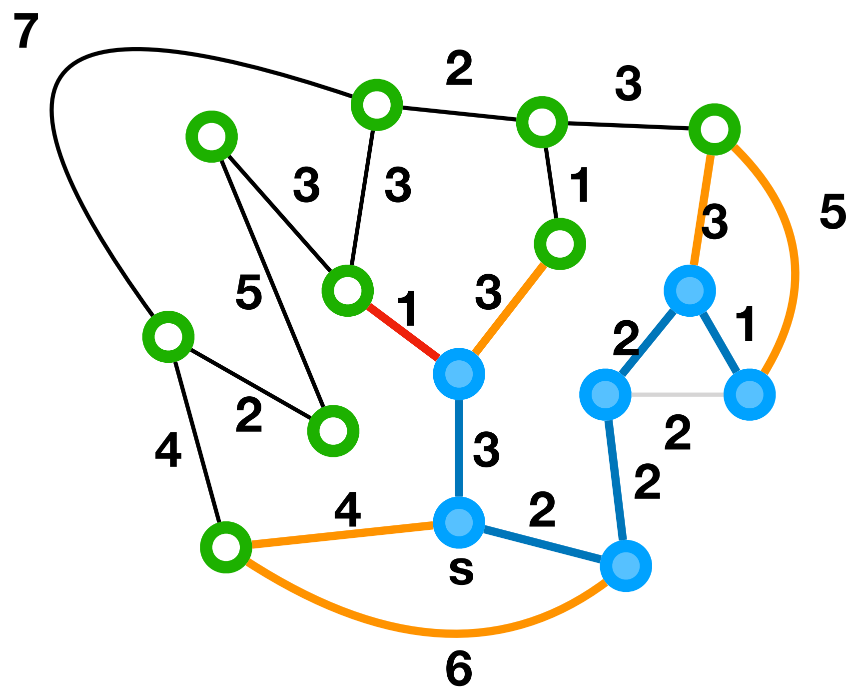

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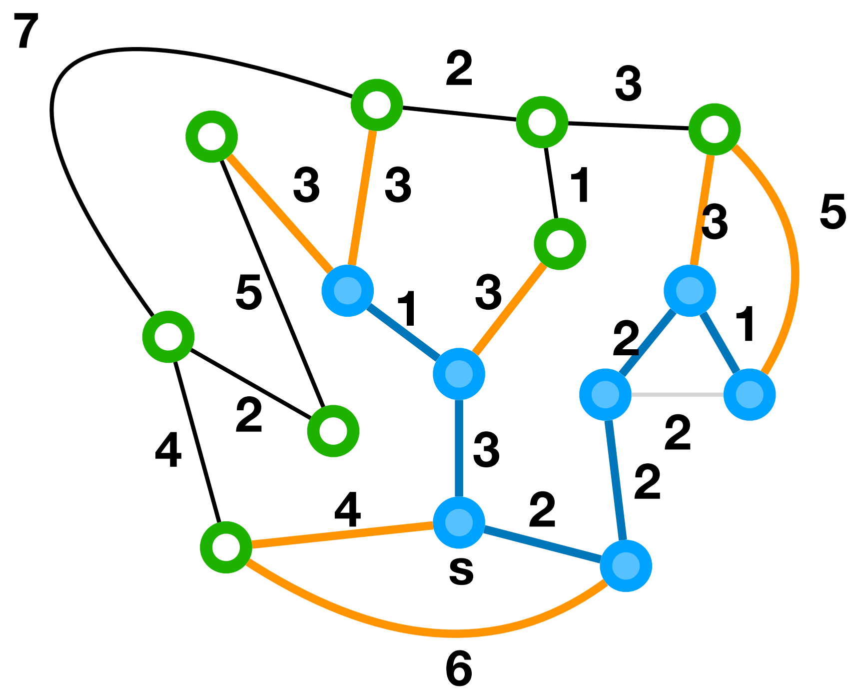

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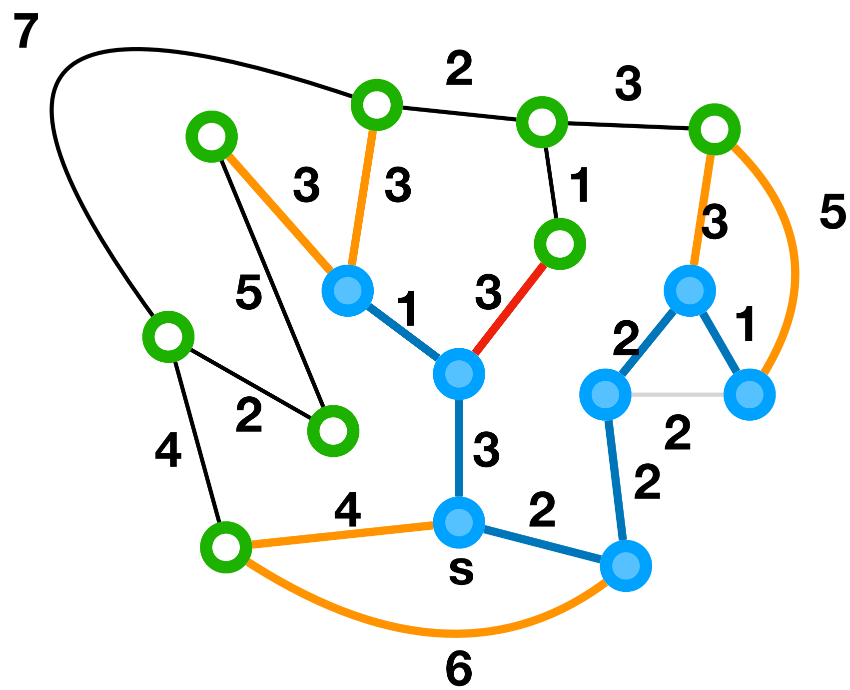

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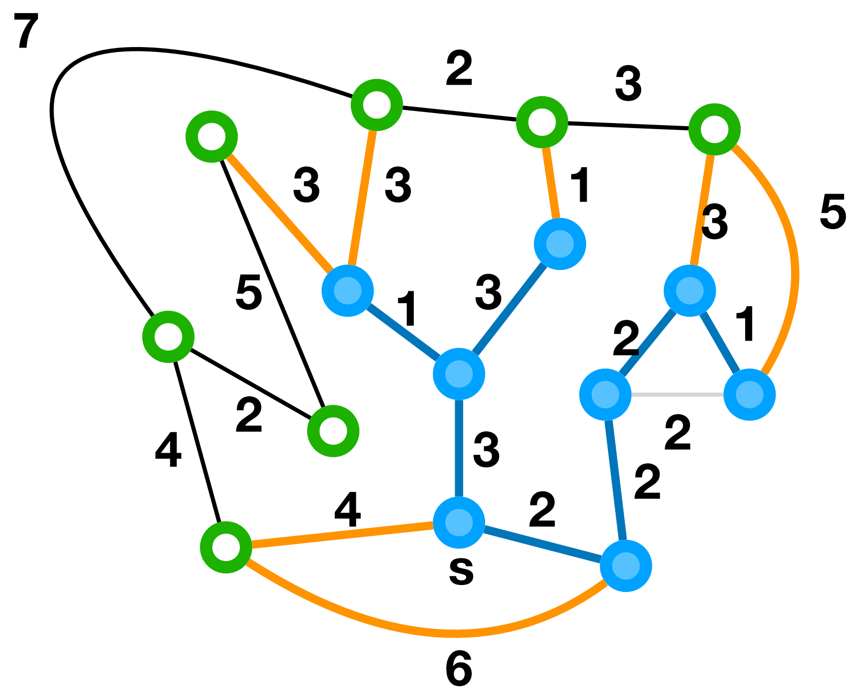

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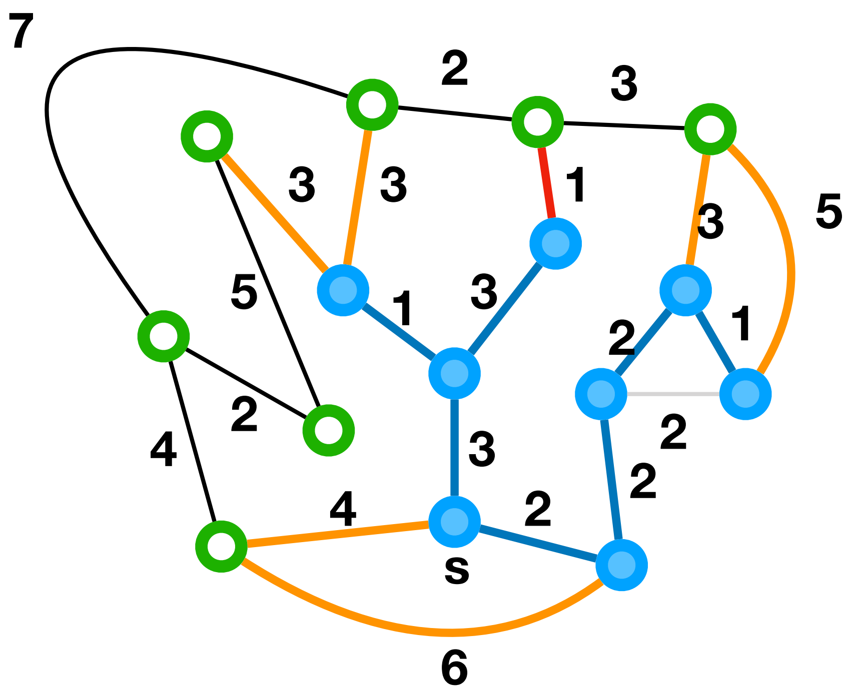

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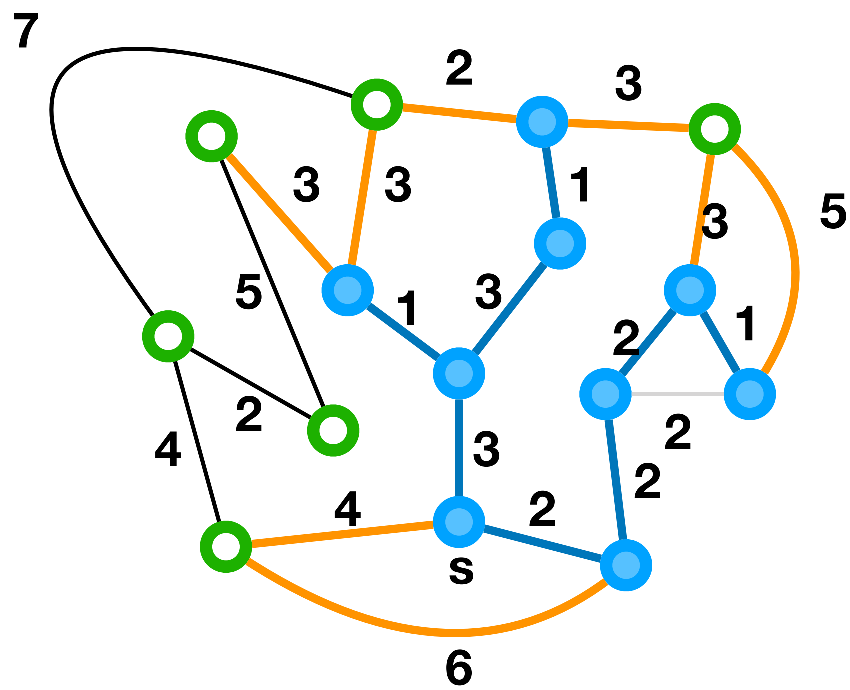

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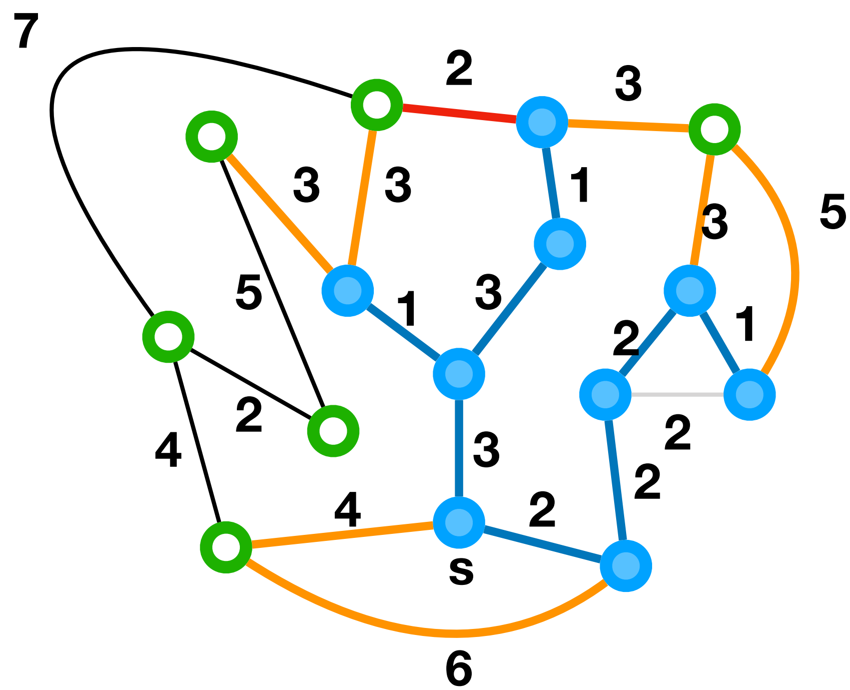

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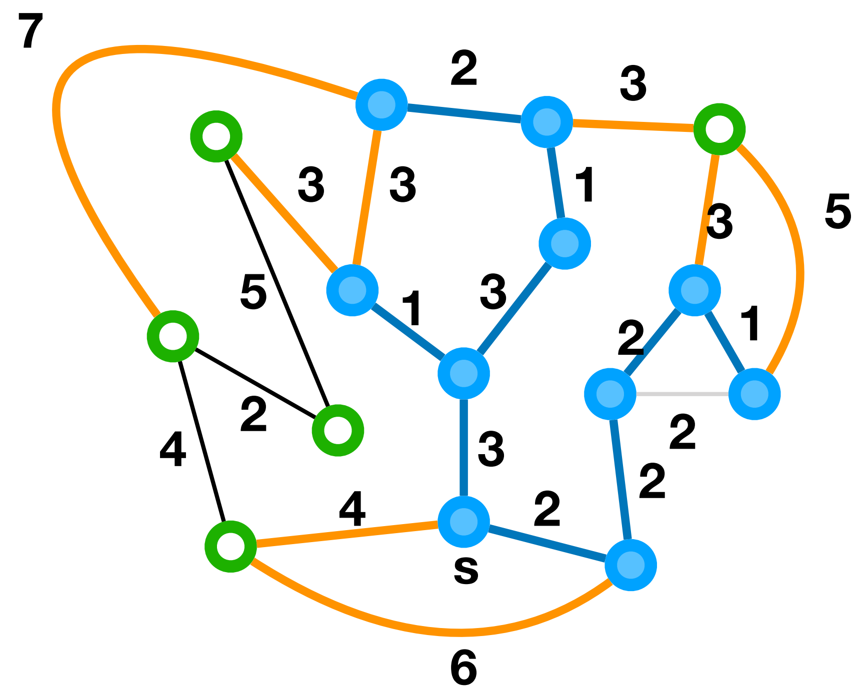

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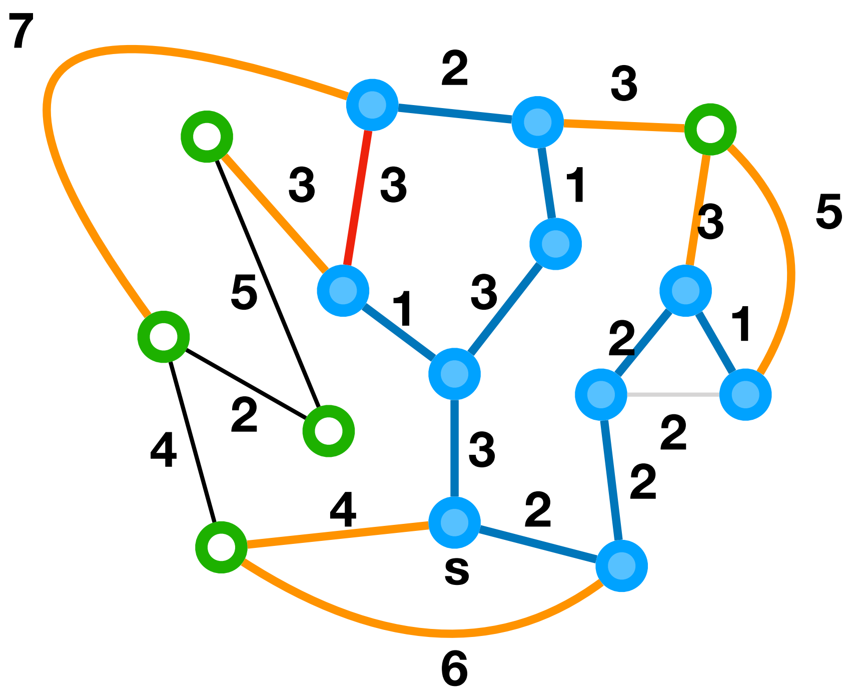

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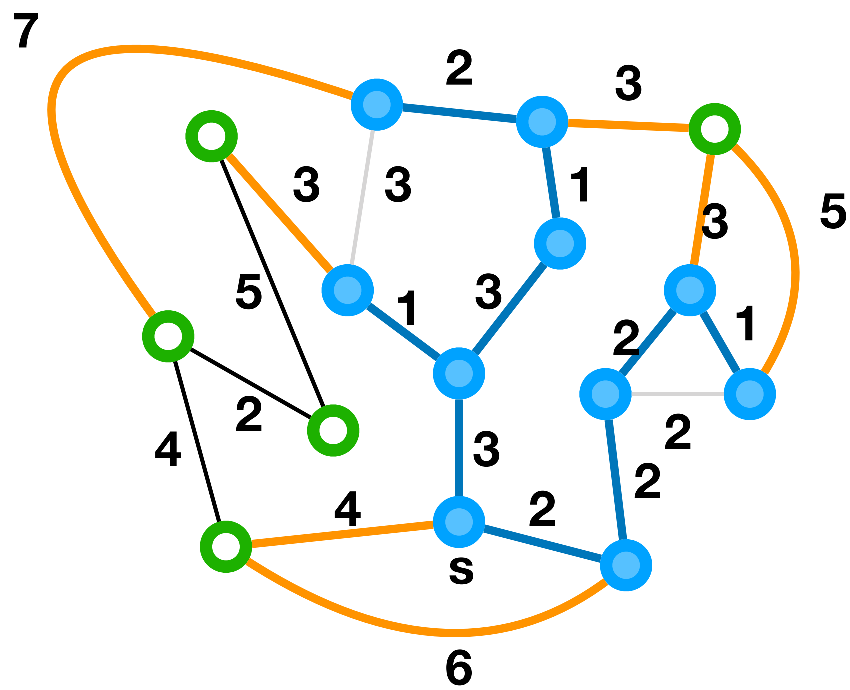

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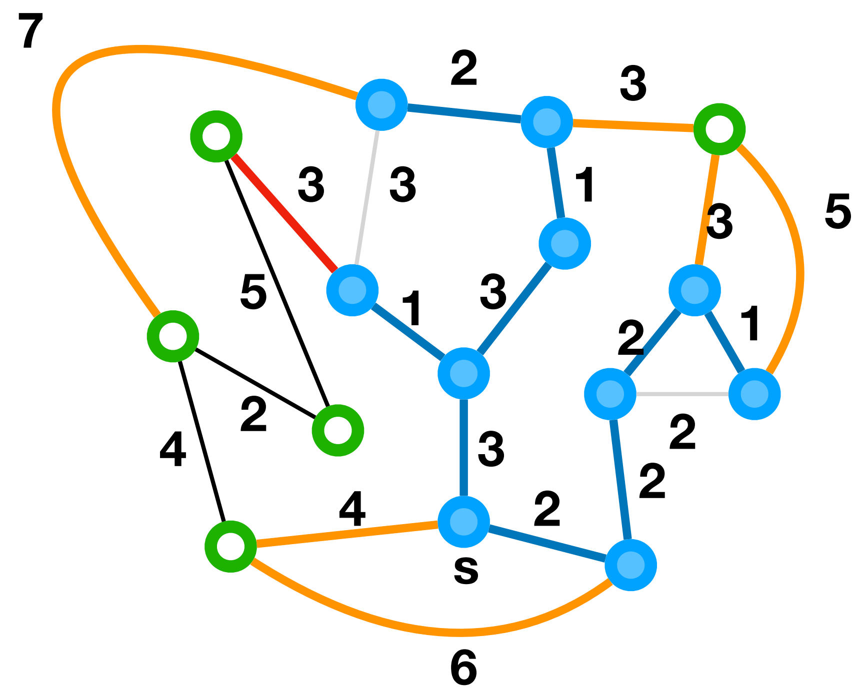

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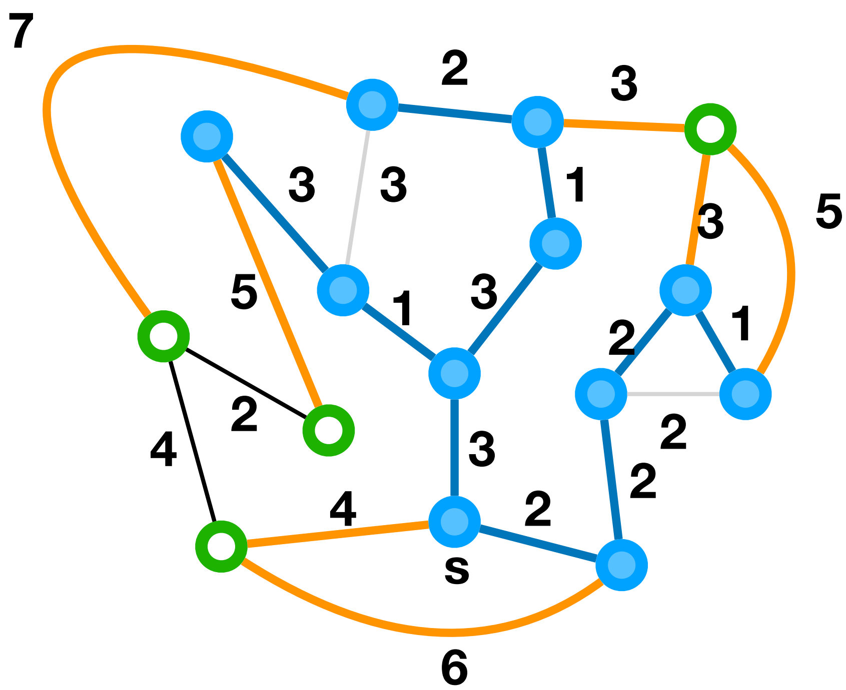

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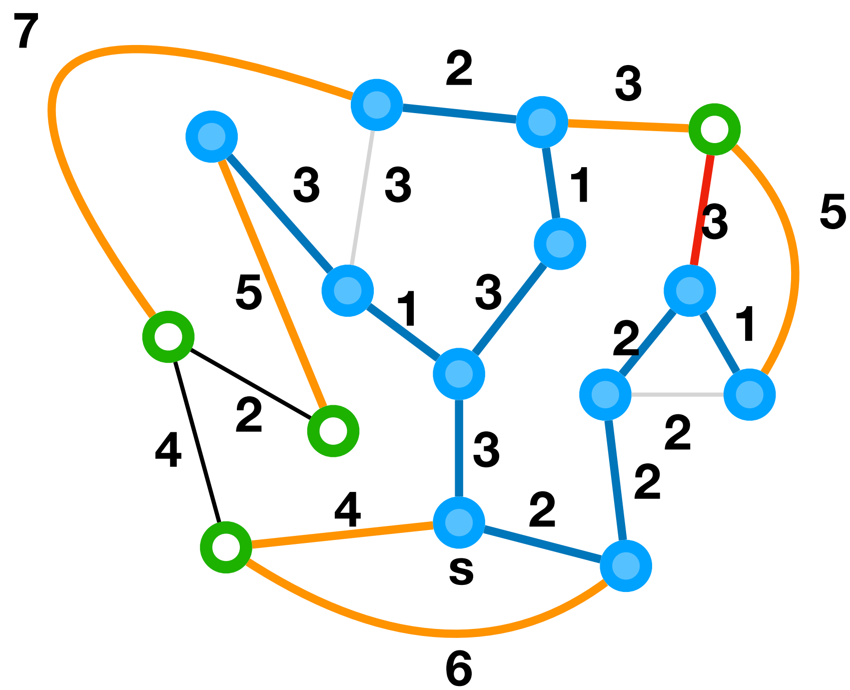

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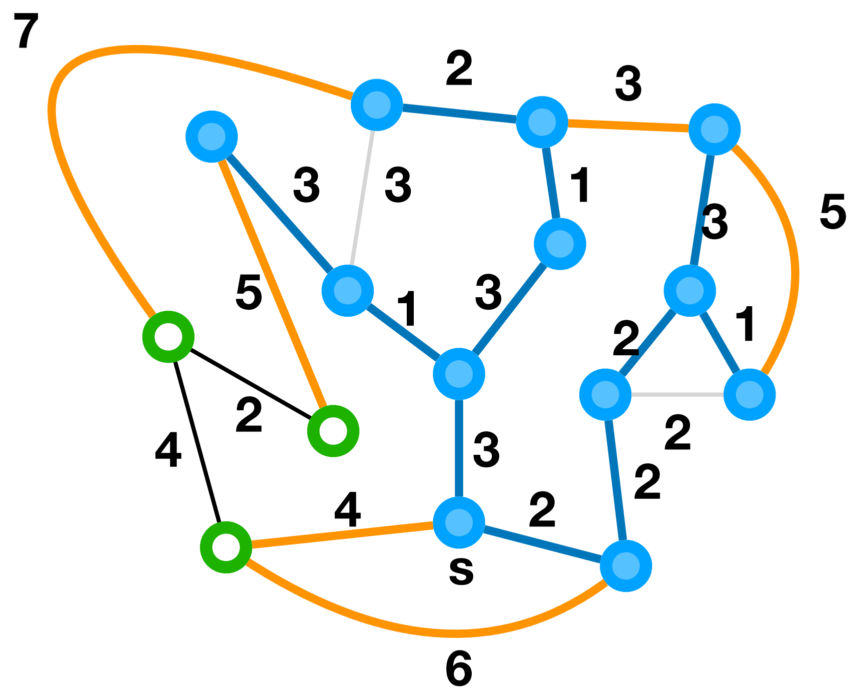

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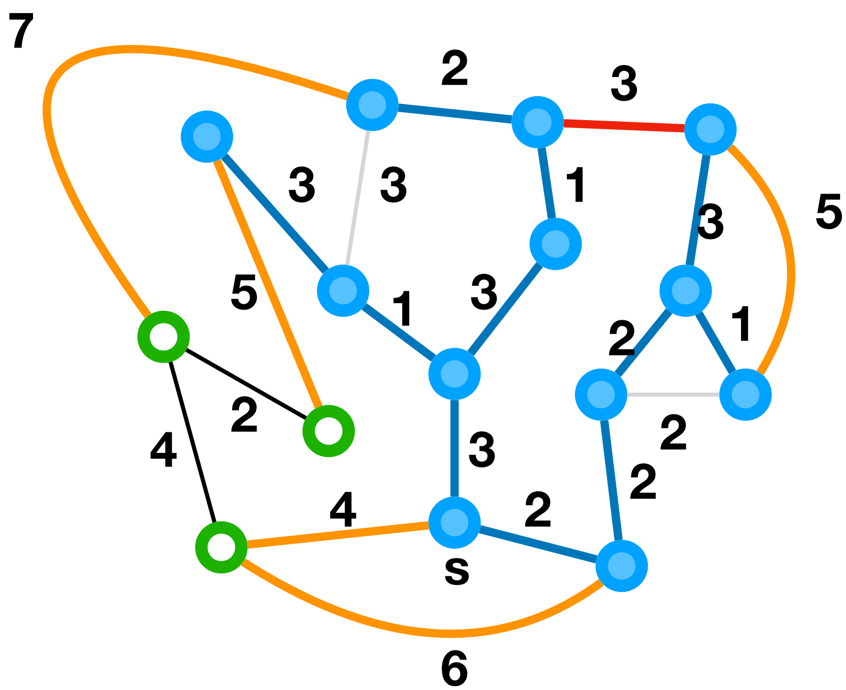

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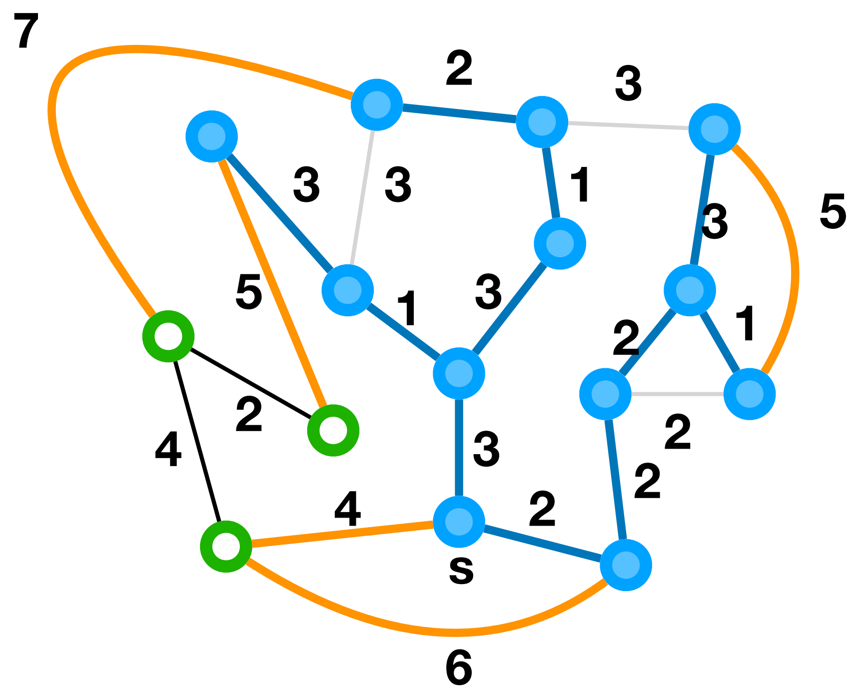

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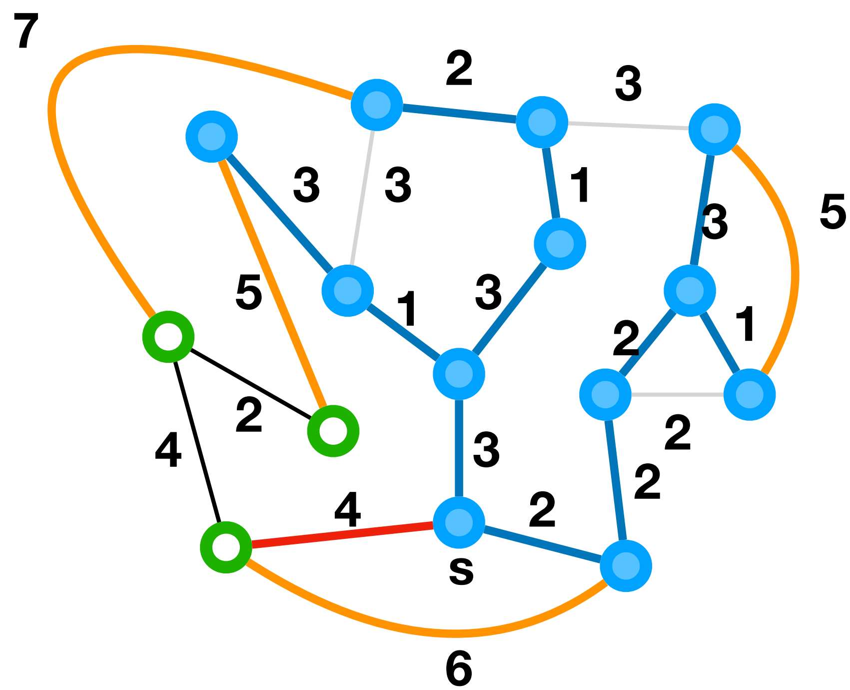

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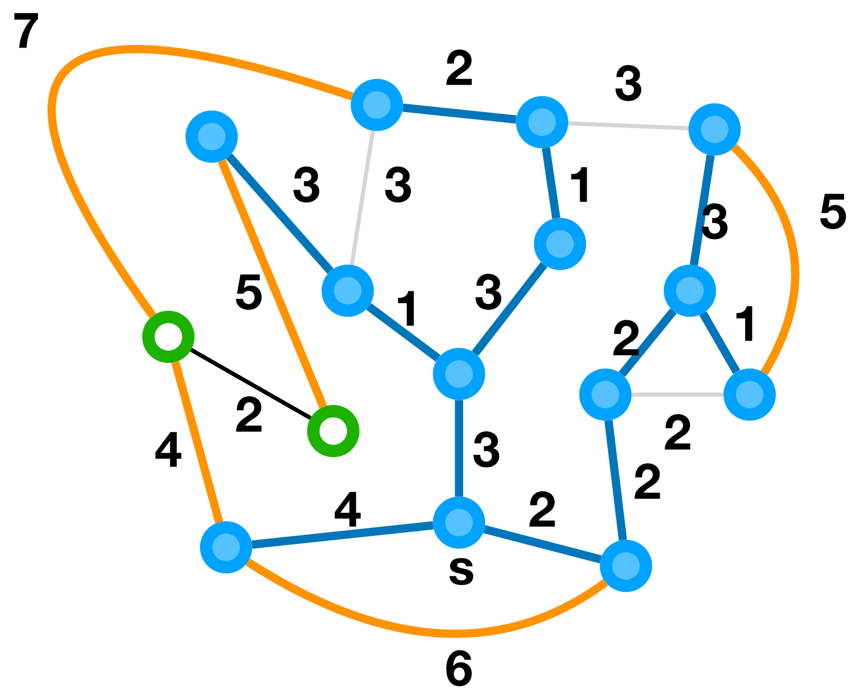

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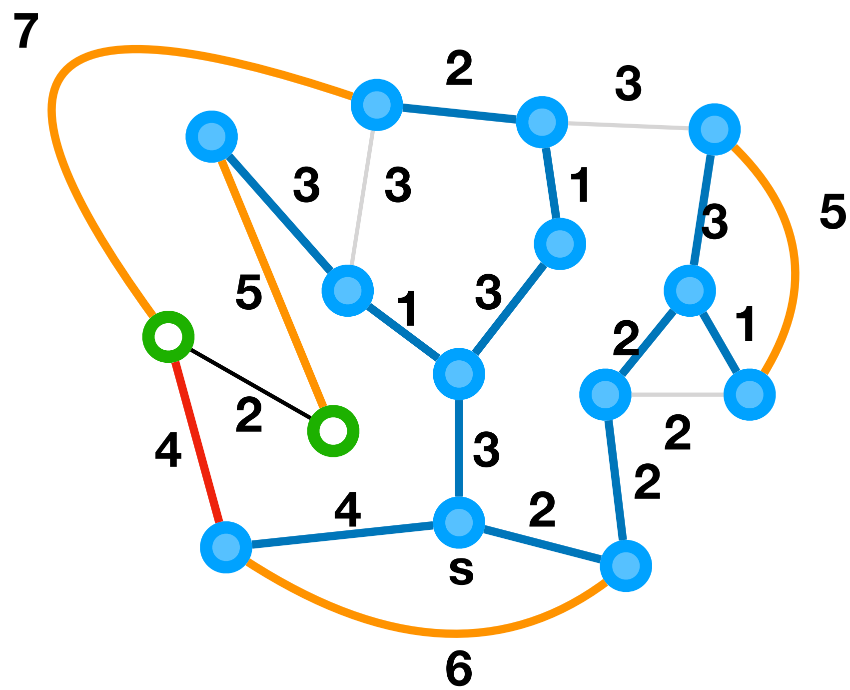

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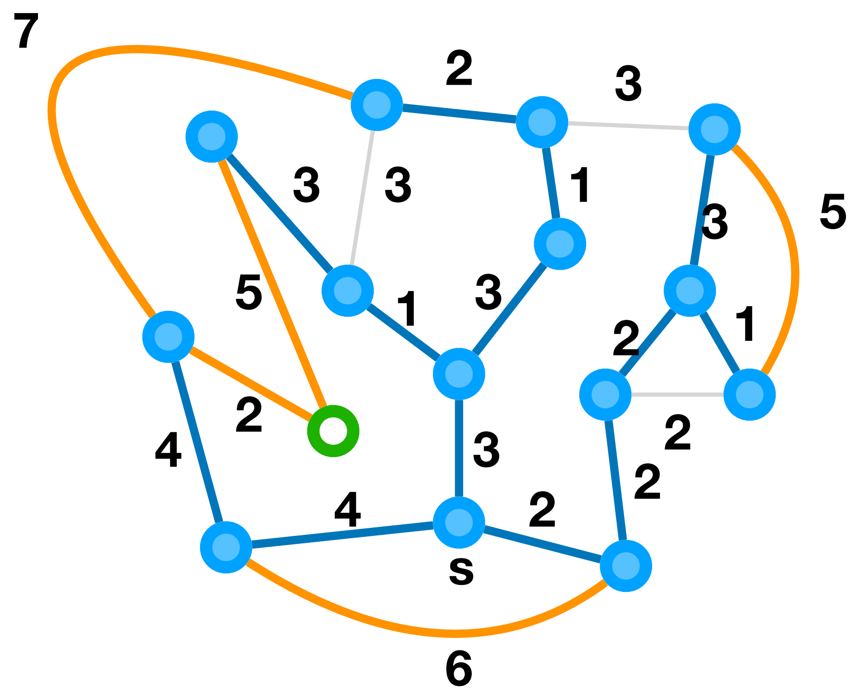

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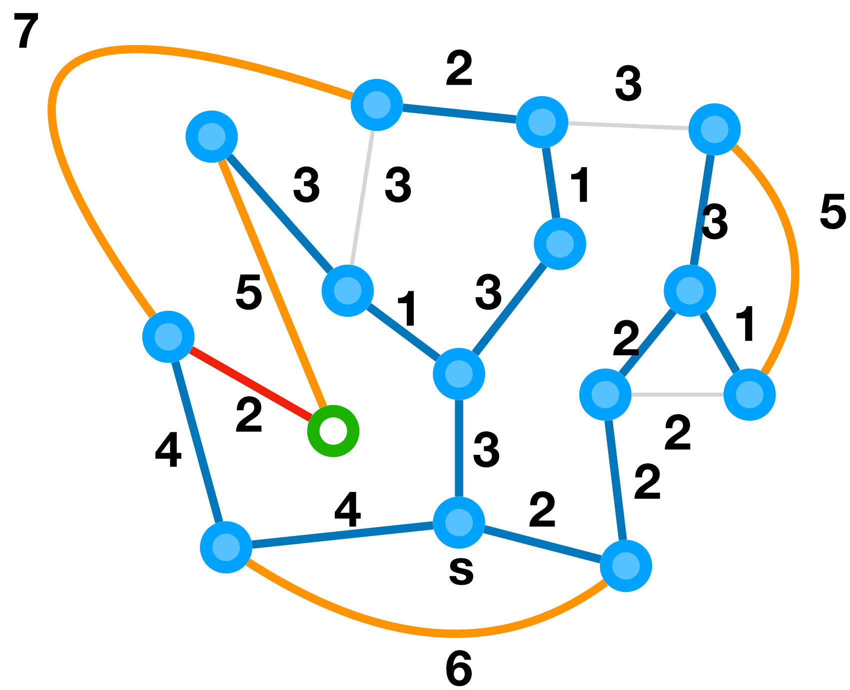

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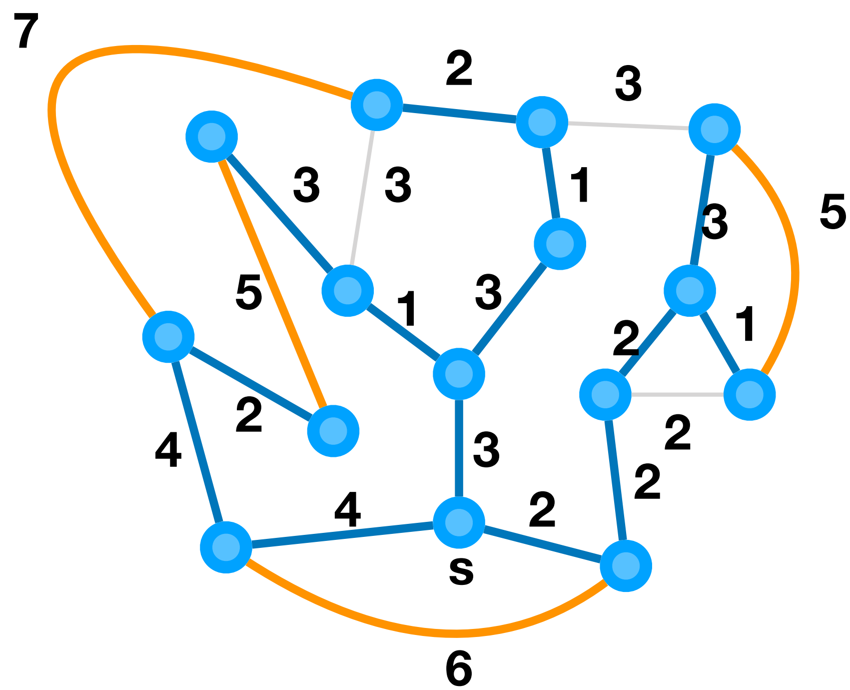

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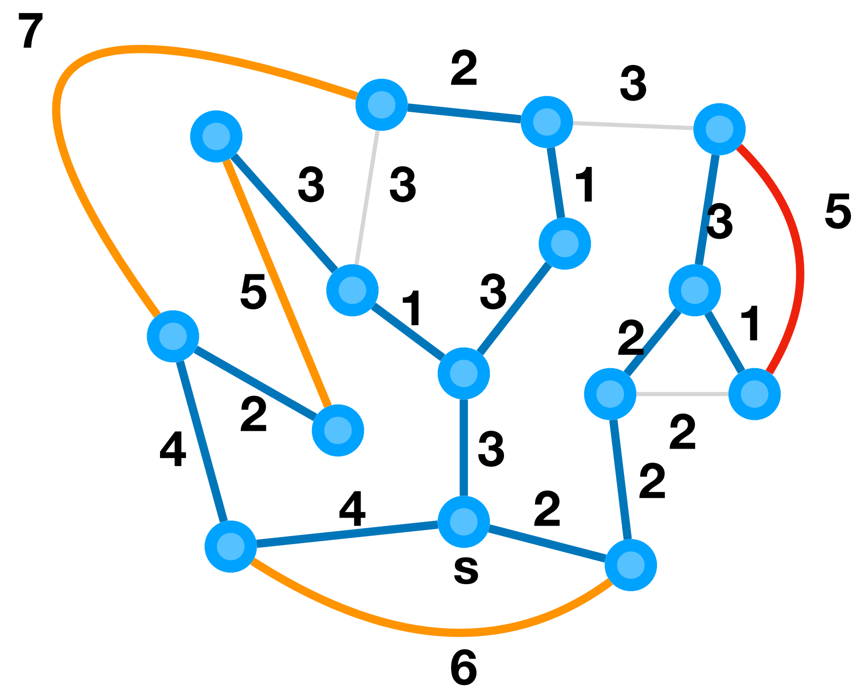

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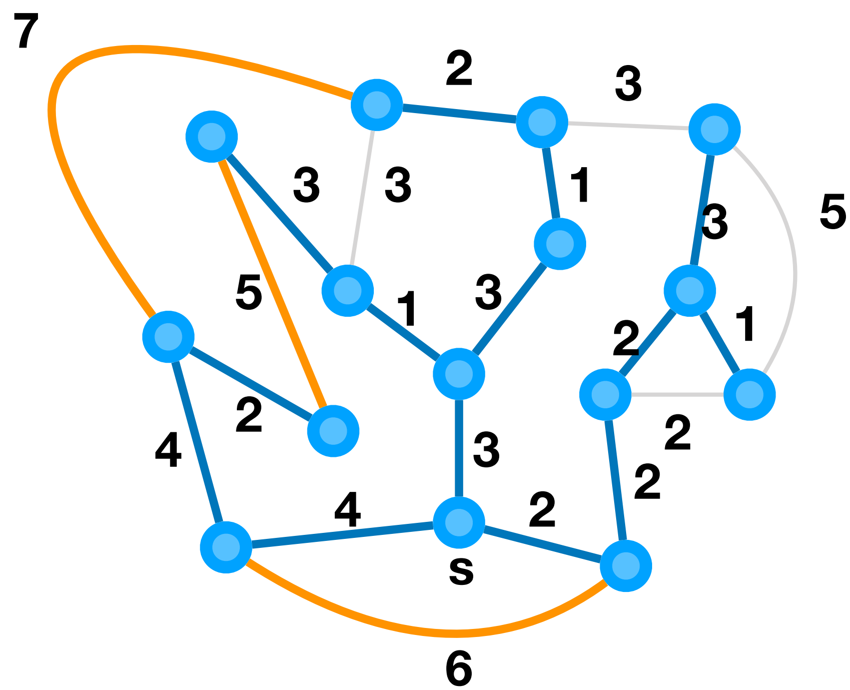

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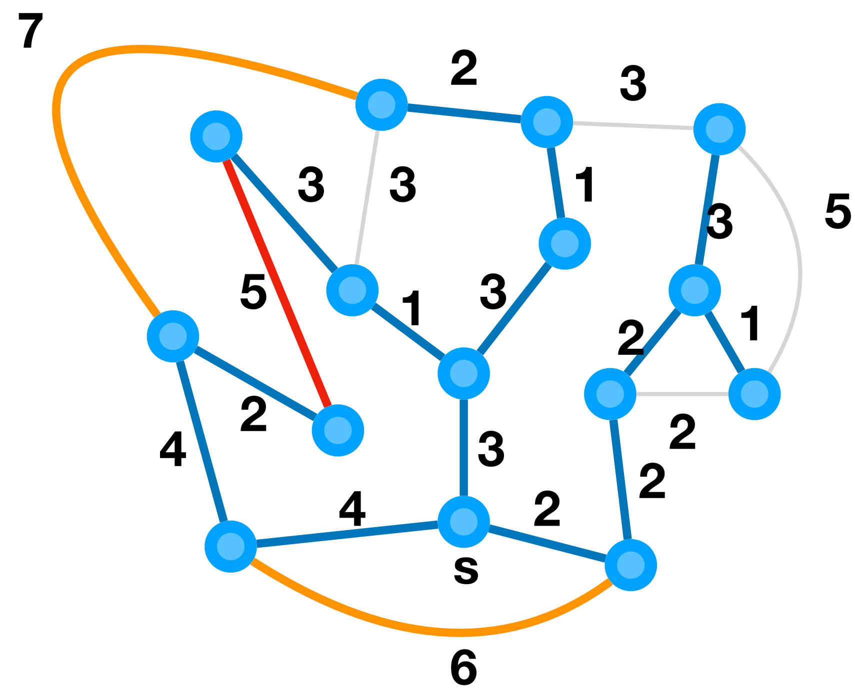

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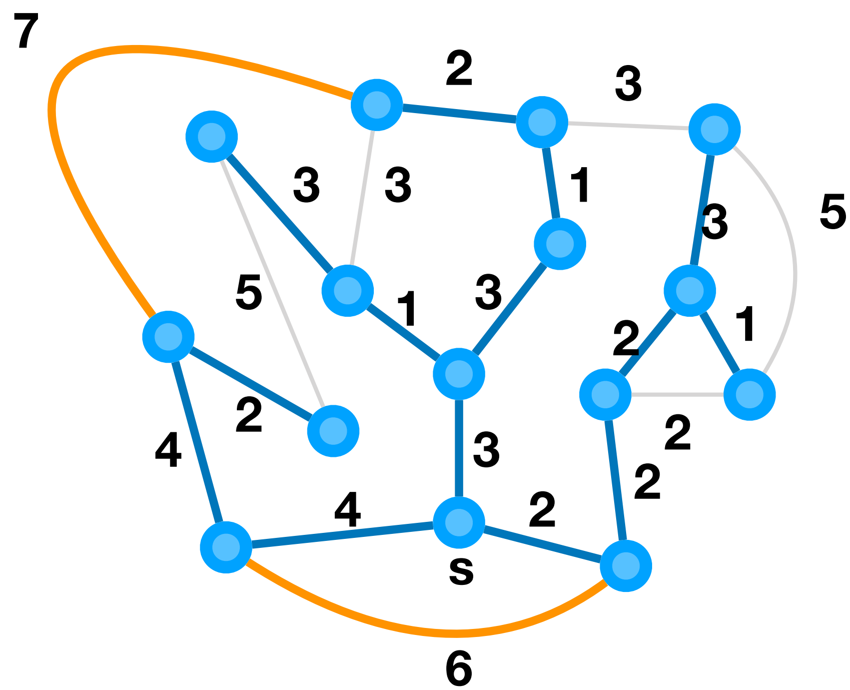

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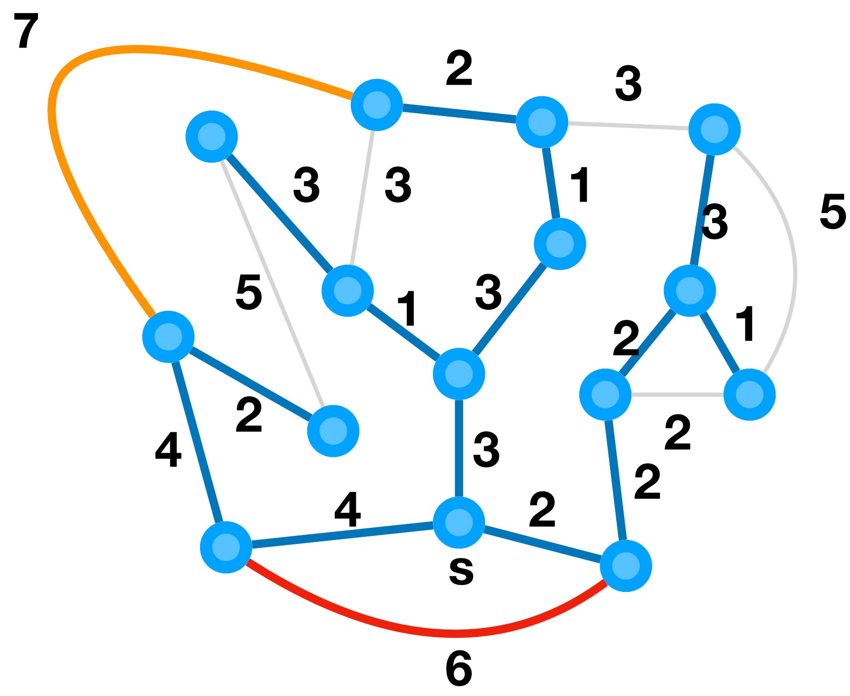

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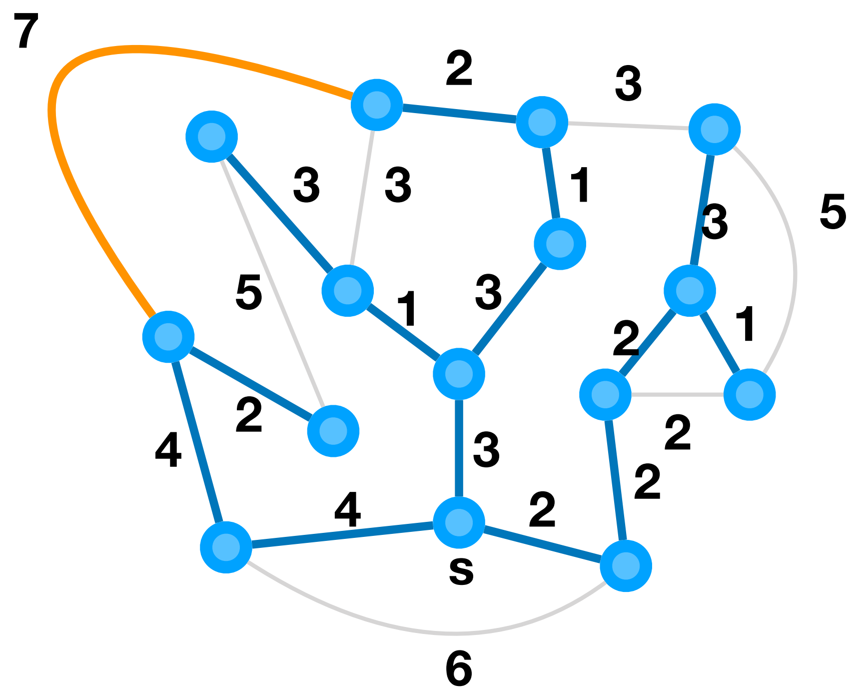

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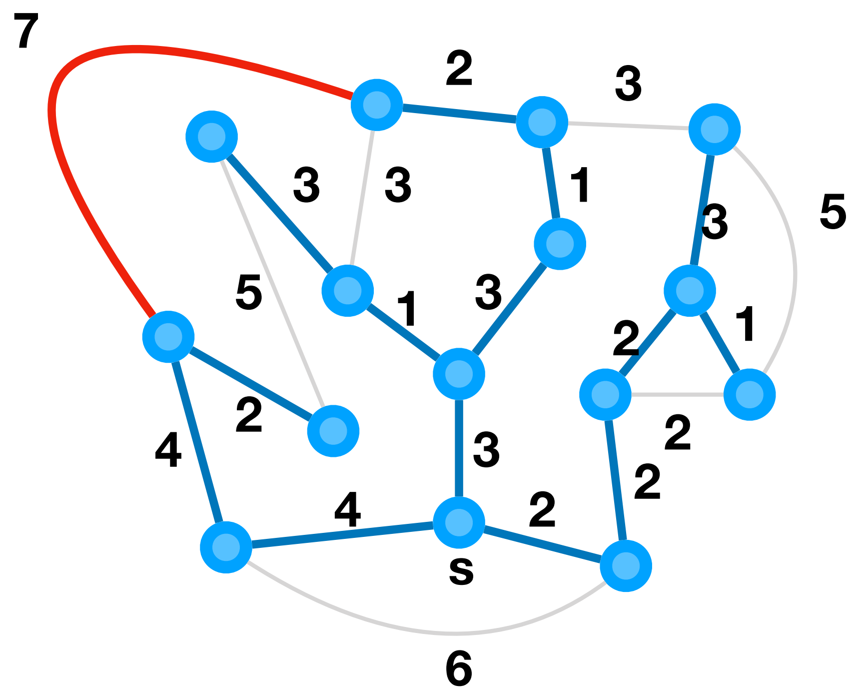

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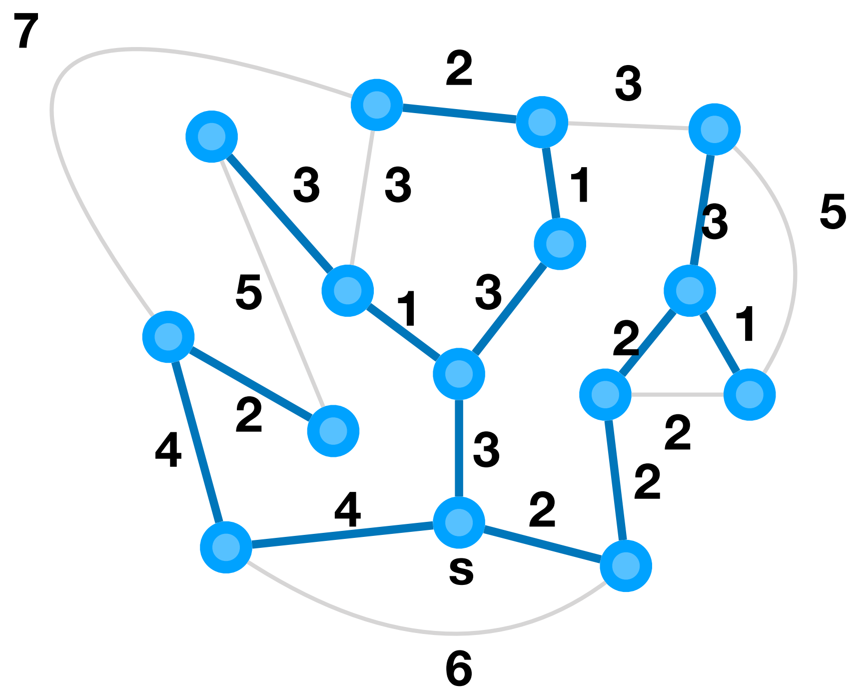

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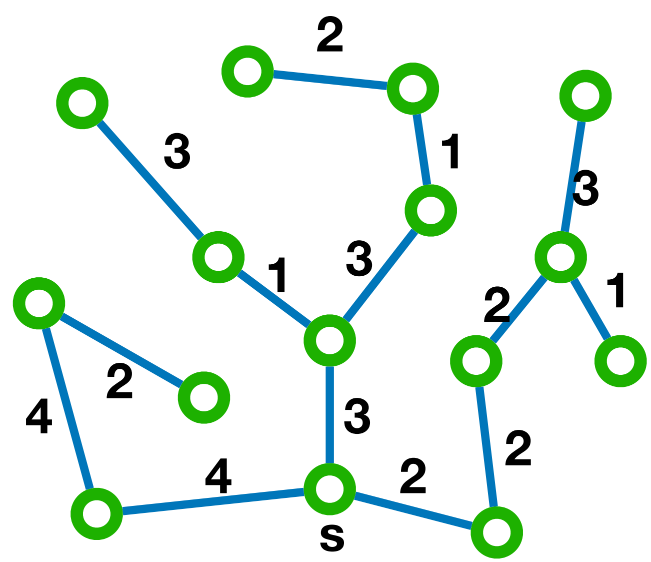

Proof of Correctness

- Let mark[1:n] = false for all vertices and s be any arbitrary vertex
- Let F = ∅, mark[s] = true and H be the set of edges incident on s
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  - Remove minimum weight edge (u,v) from H
  - If mark[u]=mark[v] = true, ignore the edge and go to the next iteration of while-loop
  - Otherwise, let us assume by symmetry mark[u] = true only
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- MST problem: finding a spanning tree with minimum weight
- We saw two different algorithms for finding MST
  - Kruskal: based on sorting edges first
  - Prim: based on a graph search + min-heap
- They are both different implementation of a generic meta-algorithm based on safe edges and minimum weight edge of cuts
- Both algorithms take O(m log m) time

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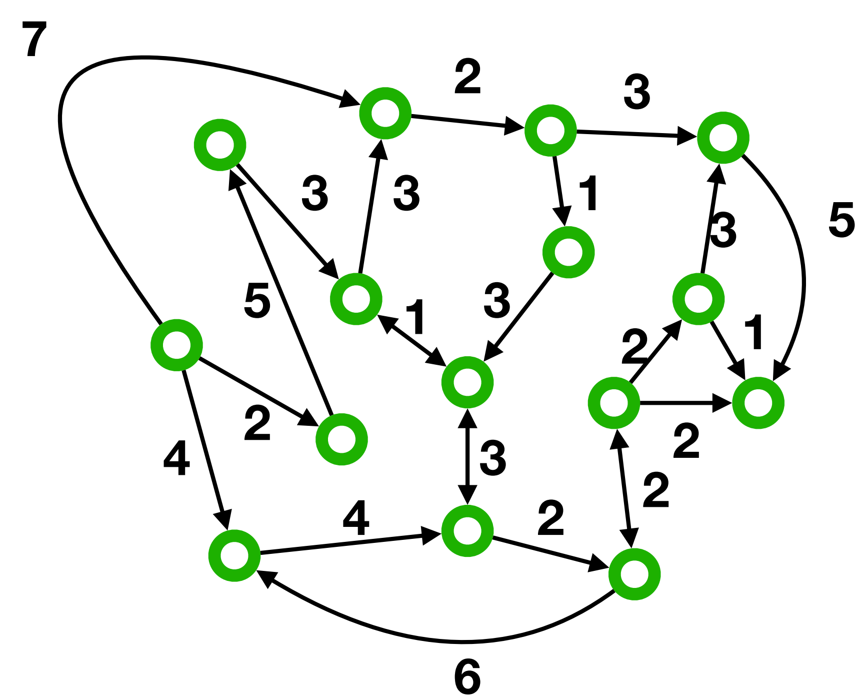
- They are both different implementation of a generic meta-algorithm based on safe edges and minimum weight edge of cuts
- Both algorithms take $O(m \log m)$ time
- There are even faster algorithms for this problem but they are way beyond the scope of our course

The Single-Source Shortest Path Problem

Shortest Paths

- For a graph G=(V,E) (directed or undirected) with weights w_e over each edge e



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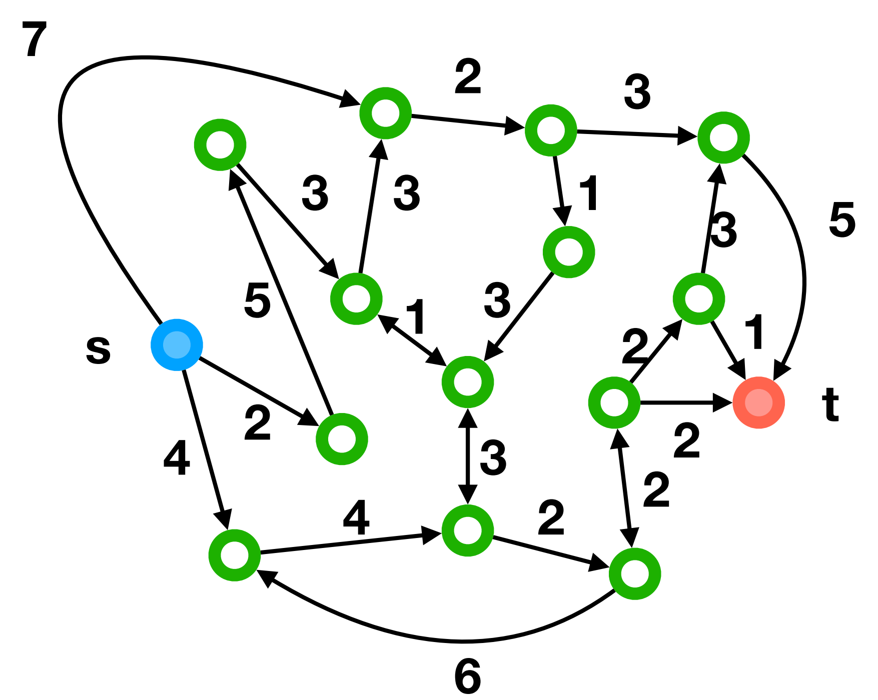

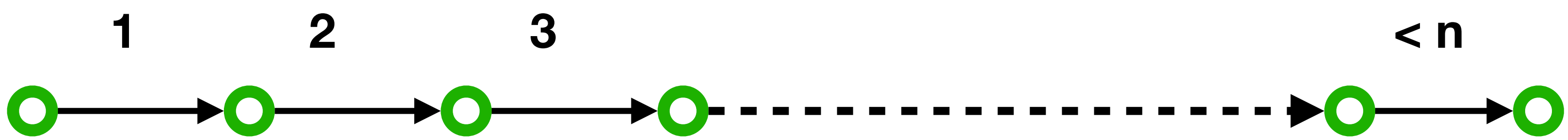
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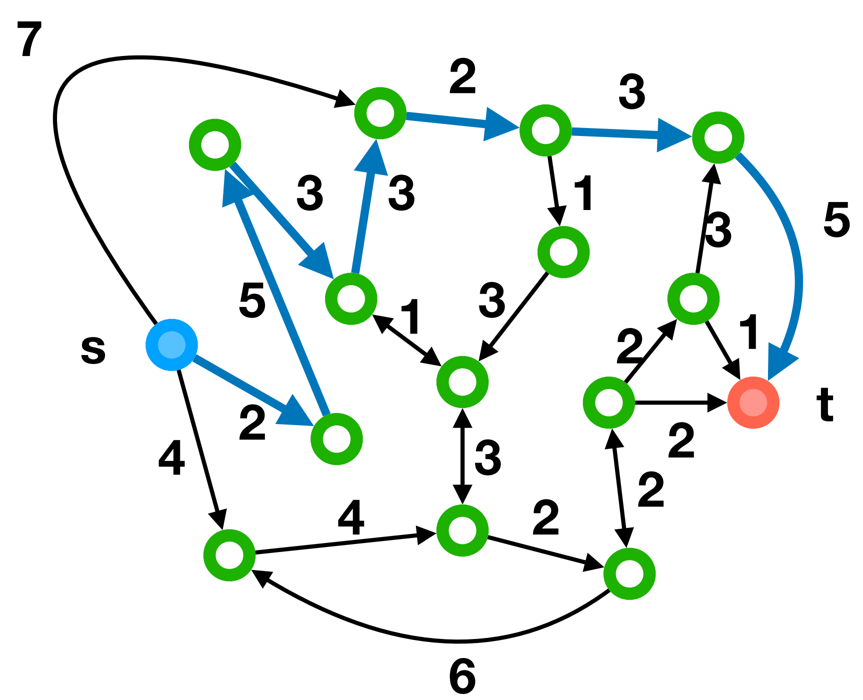

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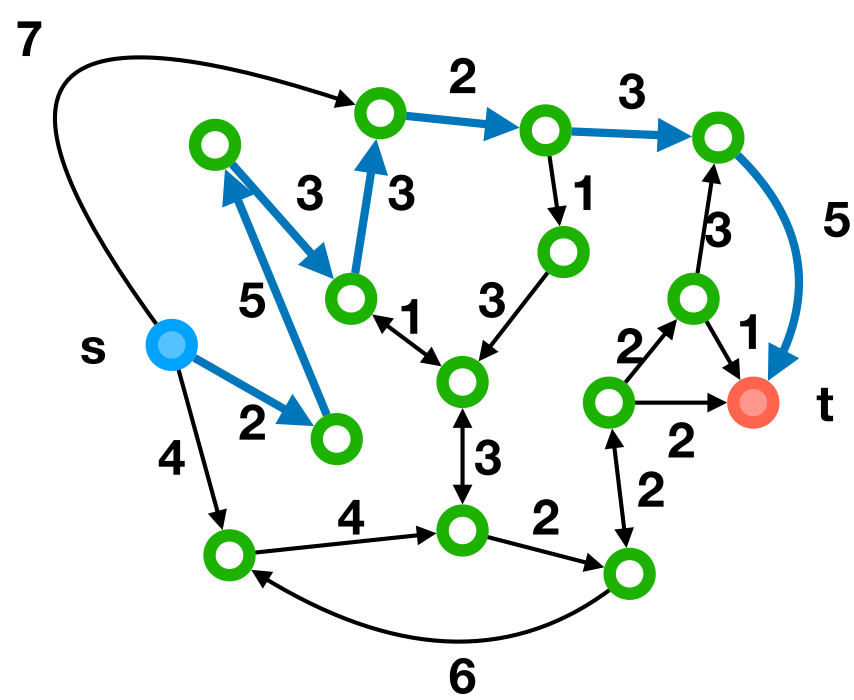
- Weight of the path:

  - $$w(P_{st}) = \sum_{e \in P} w_e$$



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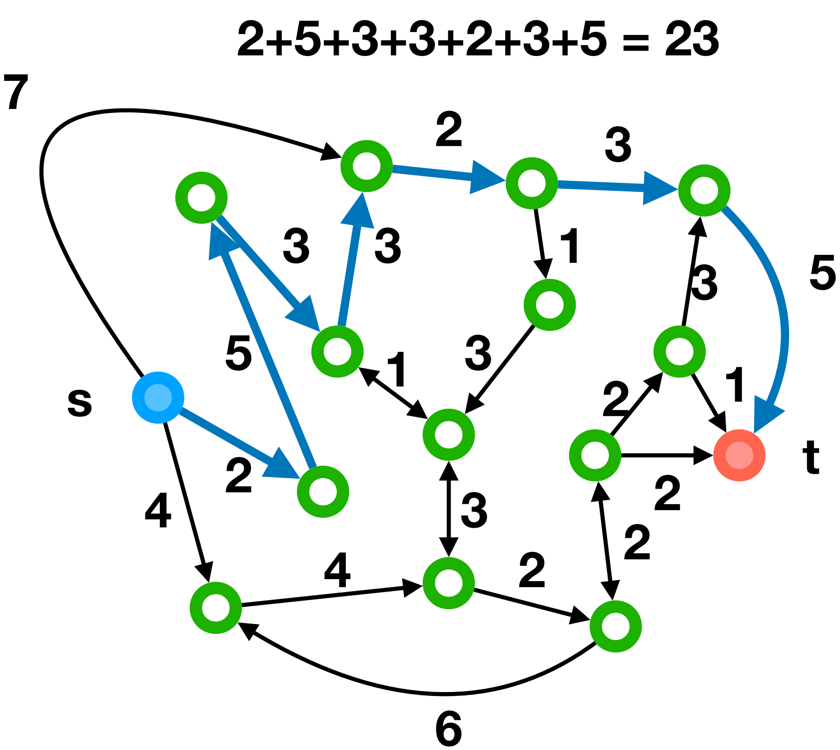

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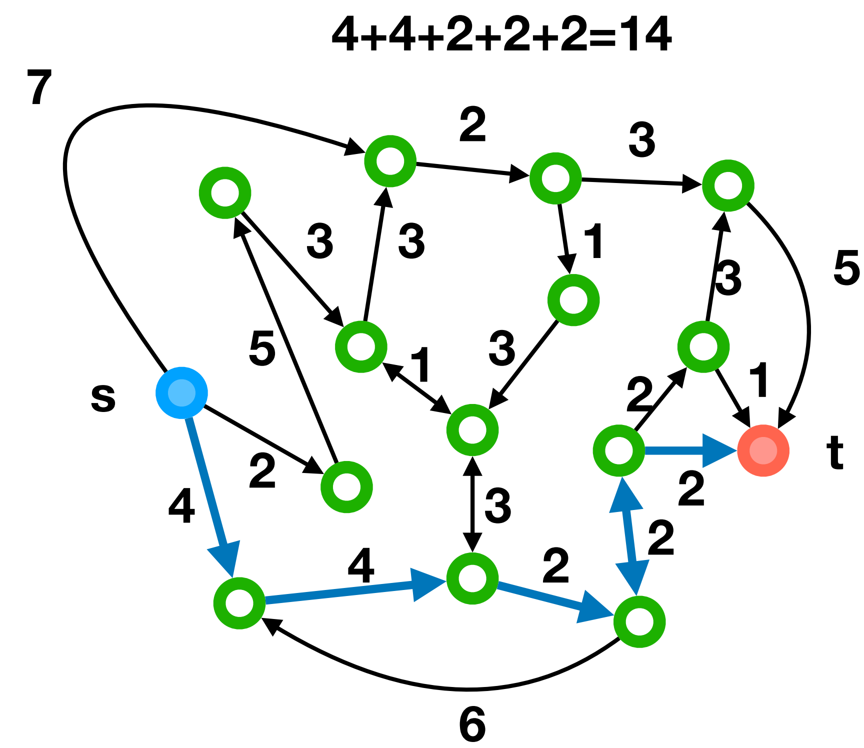

Shortest Paths

- Shortest s-t Path:
  - The path with minimum weight
  - P_{st} that minimizes $w(P_{st})$
- Distance of s to t, $dist(s, t)$:
  - Weight of shortest path from s to t

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- Application? Any navigation app/method you ever use

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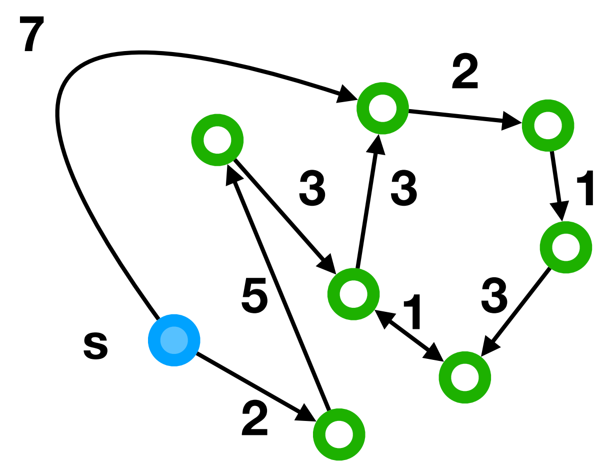

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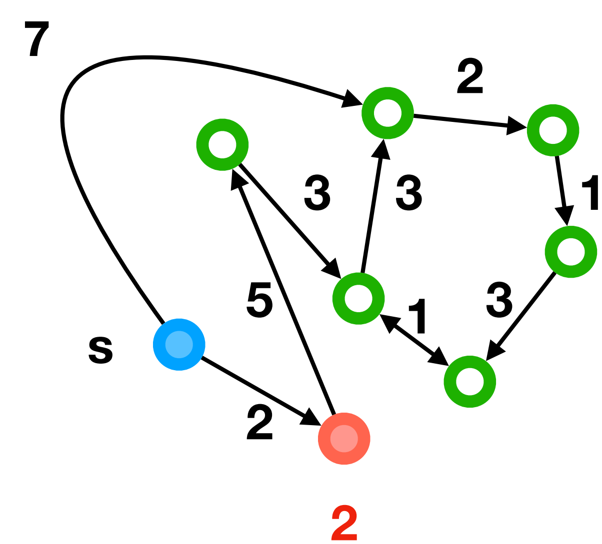

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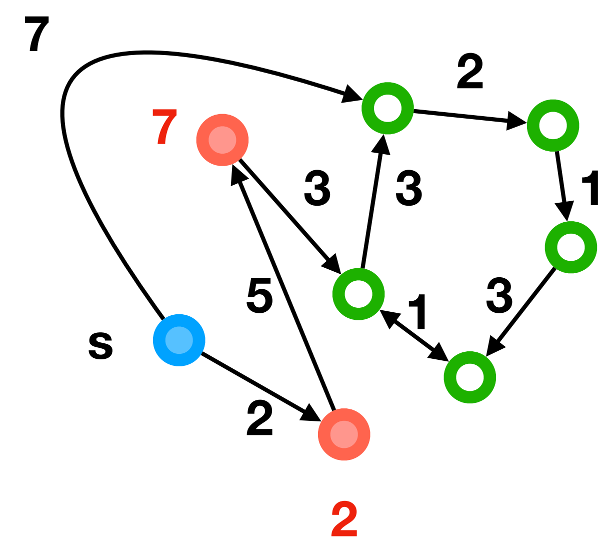

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- Shortest s-t Path:
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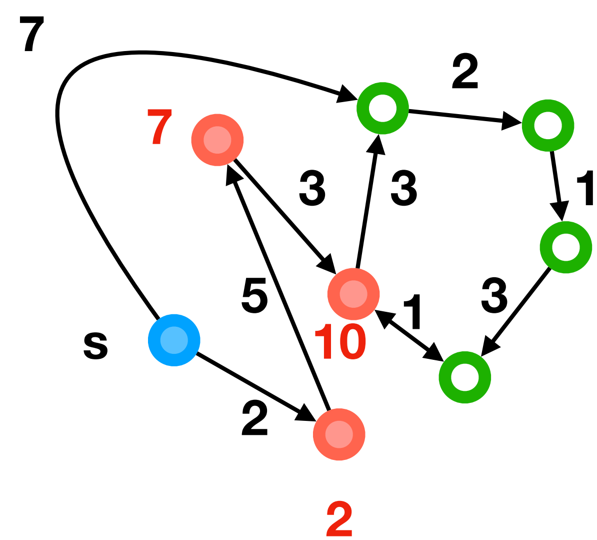

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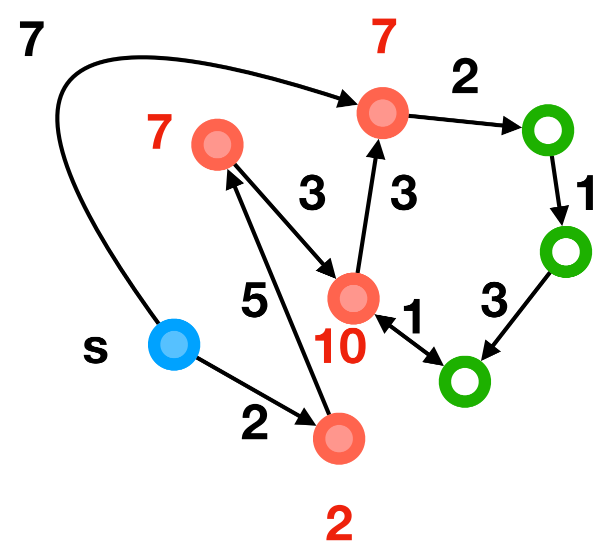

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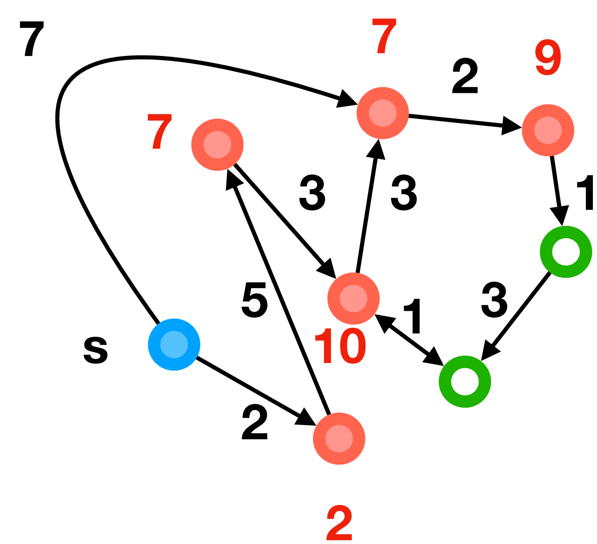

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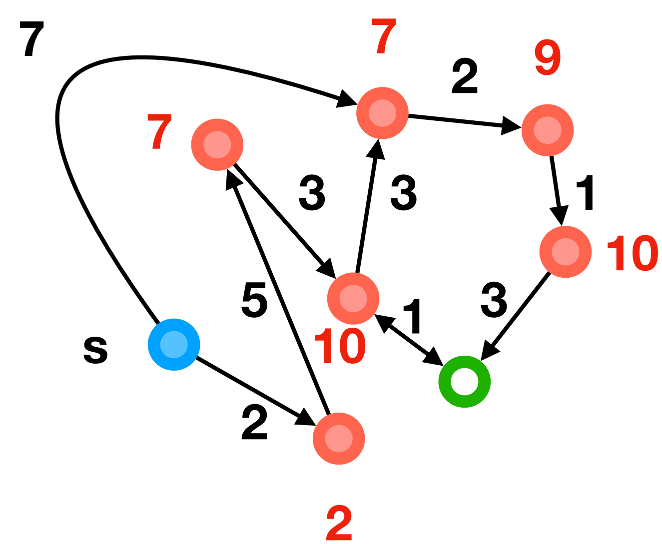

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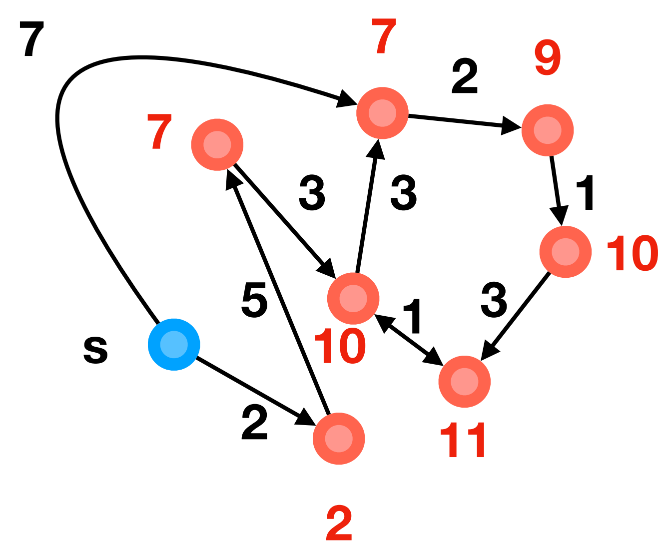


Single-Source Shortest Path Problem

- **Input:**
  - A graph G=(V,E) (undirected or directed)
  - Weights w_e on each edge e
  - A single vertex s called source
- **Output:**
  - The distance of s to all other vertices: $dist(s, v)$ for all $v \in V$

From SSSP to Finding Shortest Paths

SSSP and Distances

- The SSSP problem we defined only outputs the distances
- What if we want to find a shortest path from s to some vertex t?

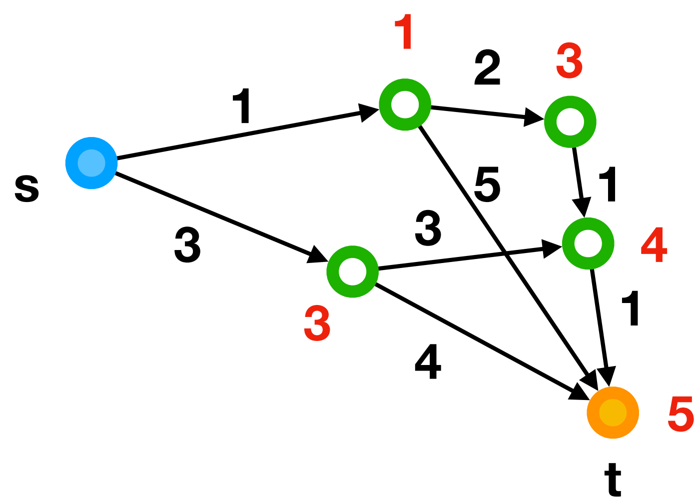
SSSP and Distances

- The SSSP problem we defined only outputs the distances
- What if we want to find a shortest path from s to some vertex t?
- We can build the path given the distances easily

Finding the Path

- Let v=t and $P_{st} = \varnothing$
- While $v \neq s$
  - Find the in-neighbor u of v such that $dist(s, v) = dist(s, u) + w_{uv}$
  - Add the edge (u,v) to the beginning of P_{st}
  - Let $v \leftarrow u$
- Output P_{st}

Example



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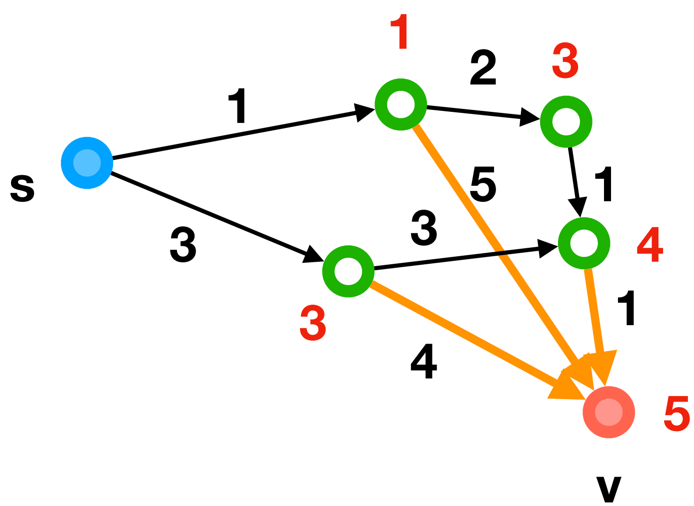

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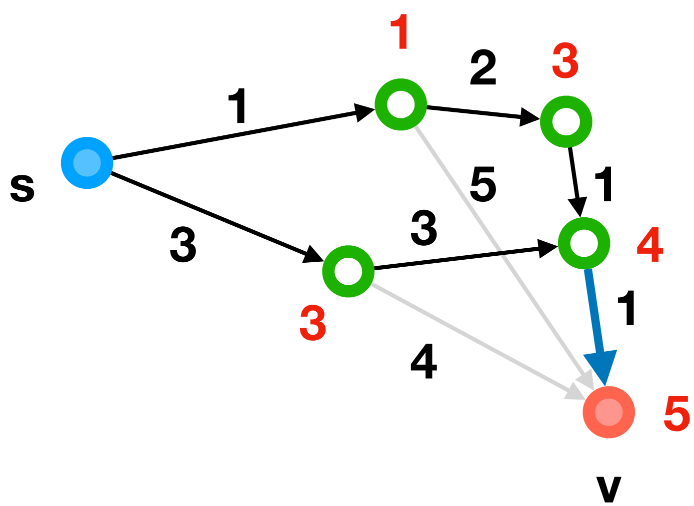

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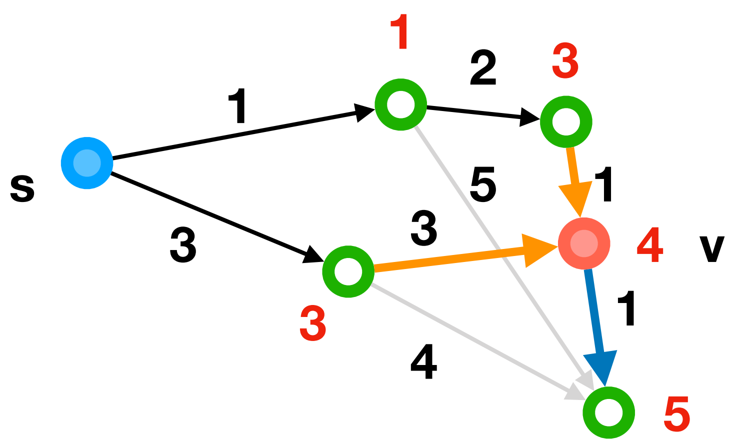

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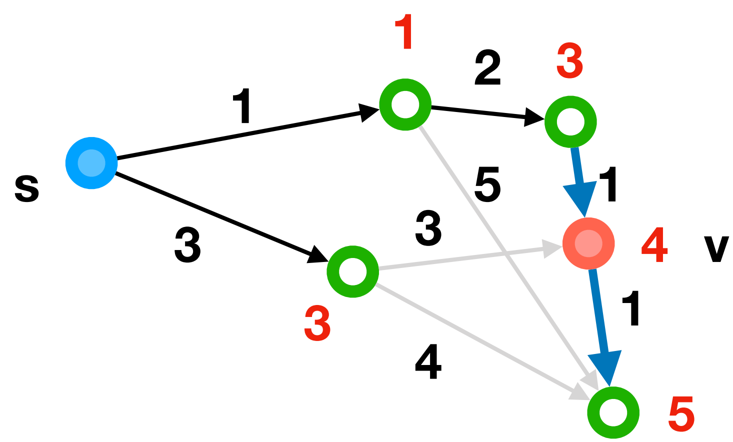

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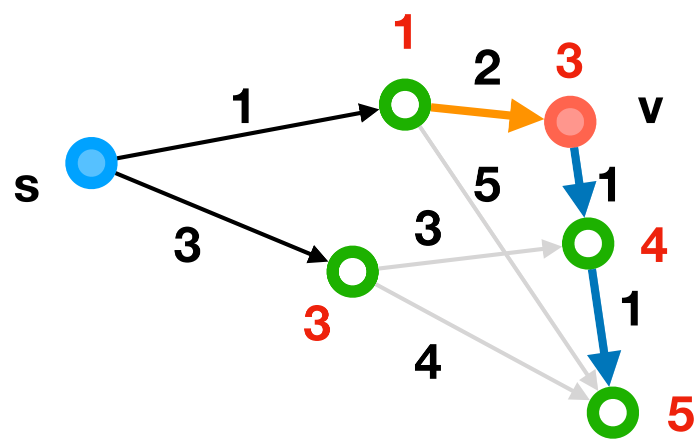

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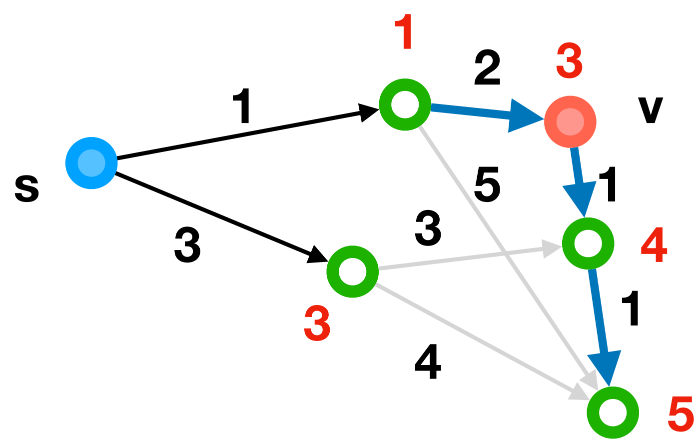

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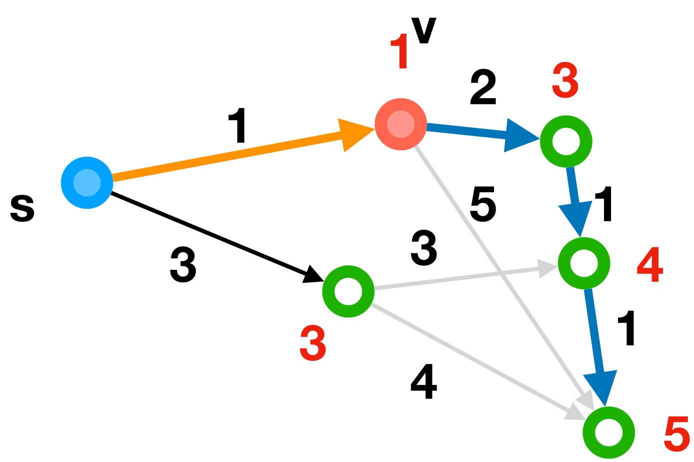

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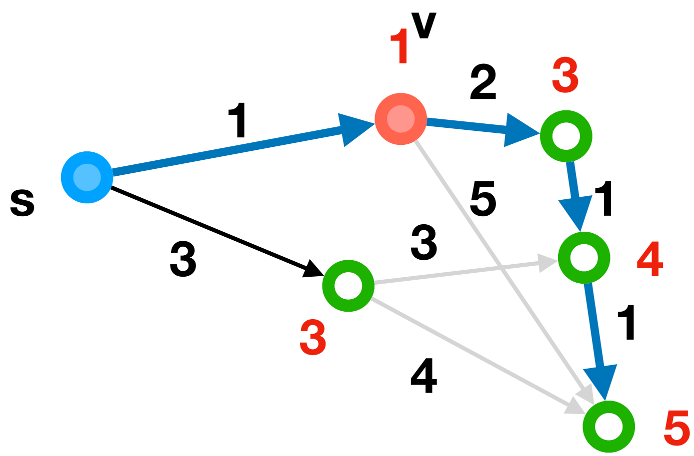

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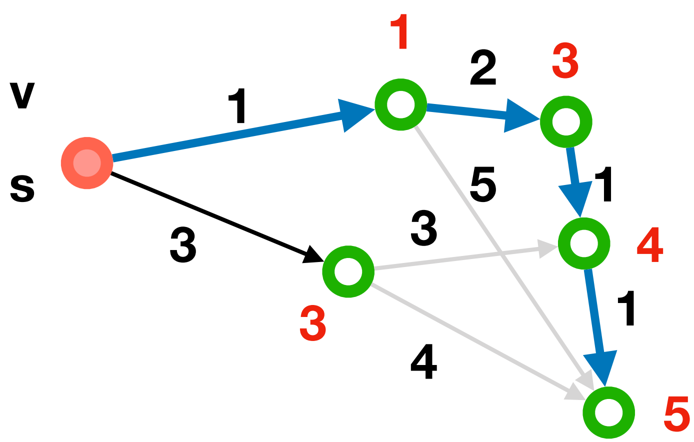

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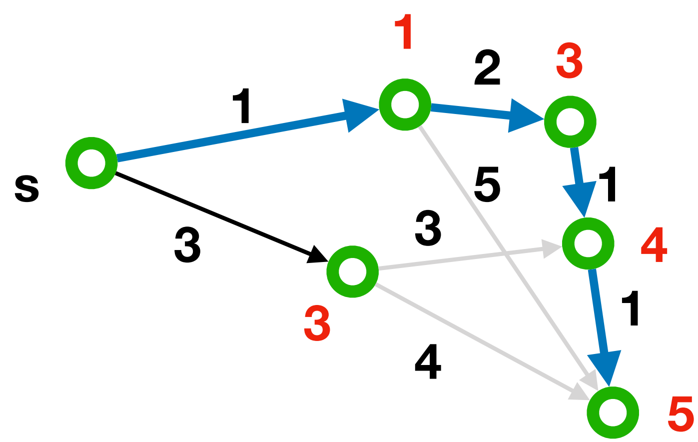

Proof of Correctness

- Let v=t and $P_{st} = \emptyset$
- While $v \neq s$
  - Find the in-neighbor u of v such that $dist(s, v) = dist(s, u) + w_{uv}$
  - Add the edge (u,v) to the beginning of P_{st}
  - Let $v \leftarrow u$
- Output P_{st}

- The weight of the path is equal to $dist(s, t)$
- So it is a shortest path from s to t
- We will see more on this later in the lecture

Runtime

- Let v=t and $P_{st} = \emptyset$
- While $v \neq s$
  - Find the in-neighbor u of v such that $dist(s,u) = dist(s,v) + w_{uv}$
  - Add the edge (u,v) to the beginning of P_{st}
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- Output P_{st}

- No edge or vertex is visited more than once
- So O(n+m) time at most

Bellman-Ford Algorithm

Bellman-Ford Algorithm

- An extremely simple algorithm for SSSP
- General idea:
- We start with some value d[v] for every vertex v
- We make sure that d[v] is always equal to weight of some s-v path
- We would like to eventually have d[v] = dist(s,v) but originally d[v] can be much larger
- We update values like this:
  - for any edge (u,v), if $d[v] > d[u] + w_{uv}$ set $d[v] \leftarrow d[u] + w_{uv}$

Bellman-Ford Algorithm

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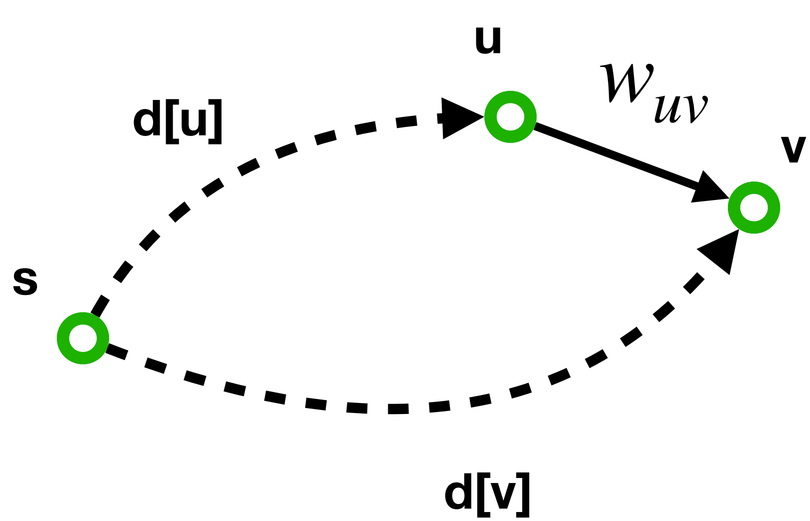

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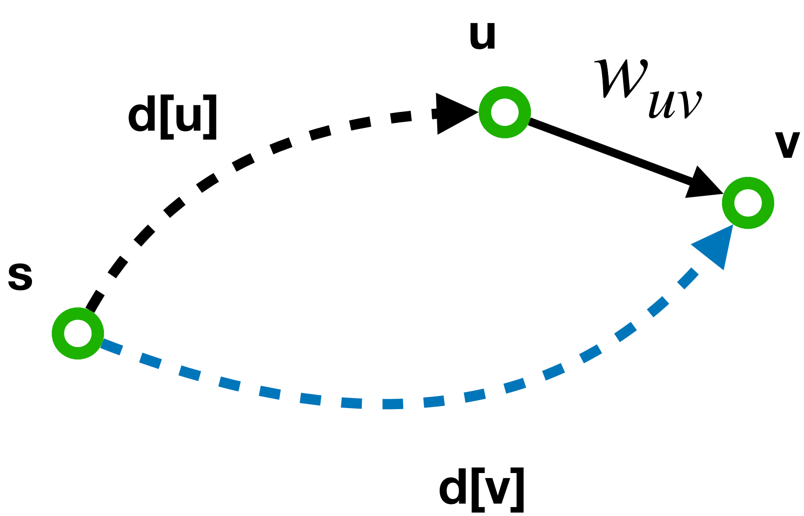

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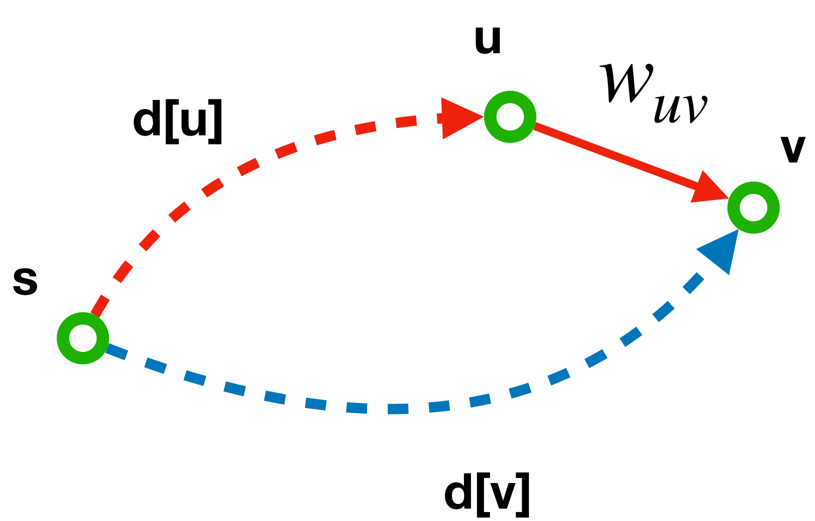

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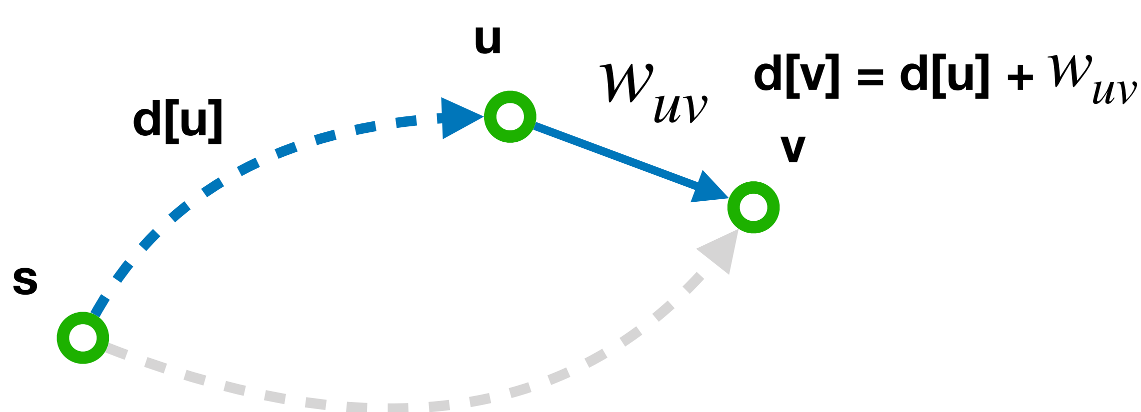

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- Let d[s] = 0 and d[v] = $+\infty$ for $v \in V - \{s\}$
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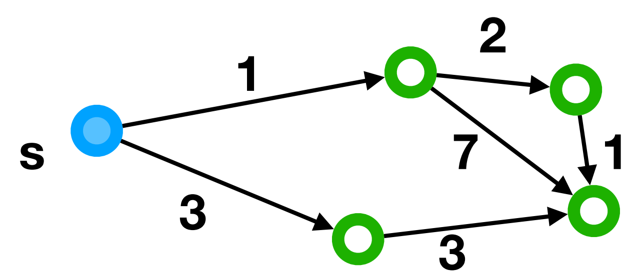

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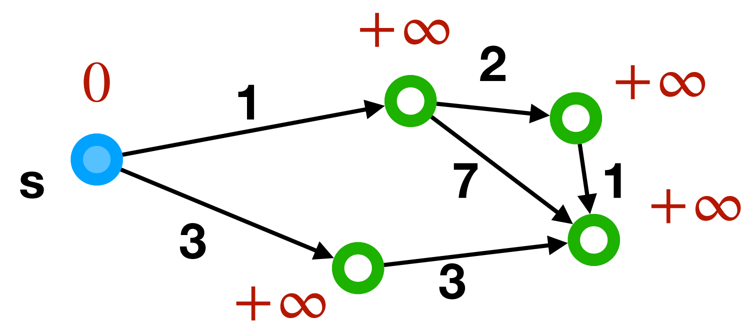

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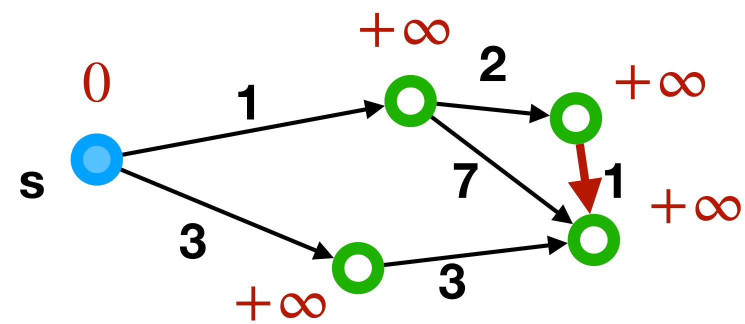

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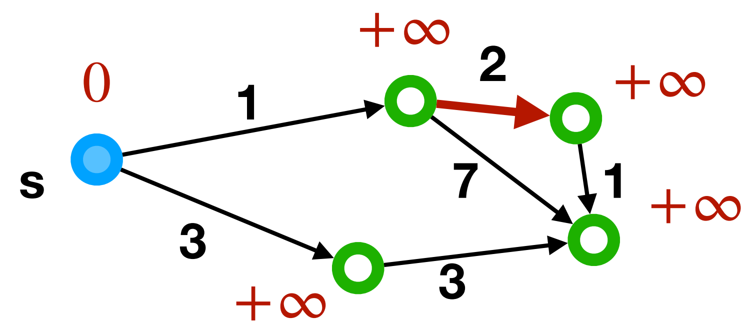

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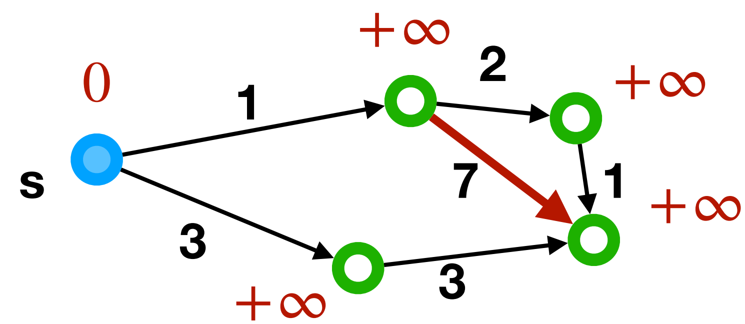

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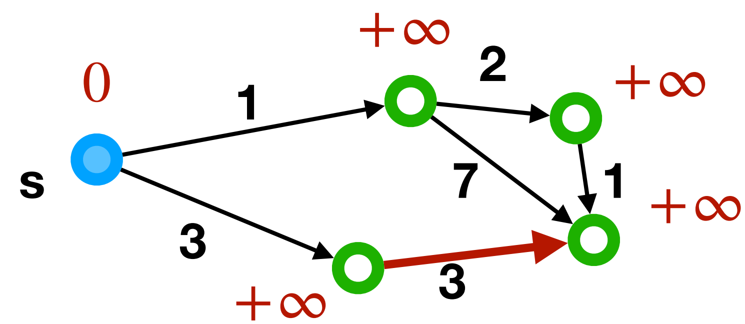

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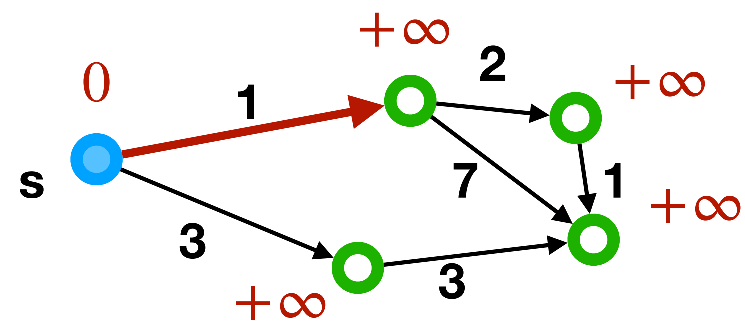

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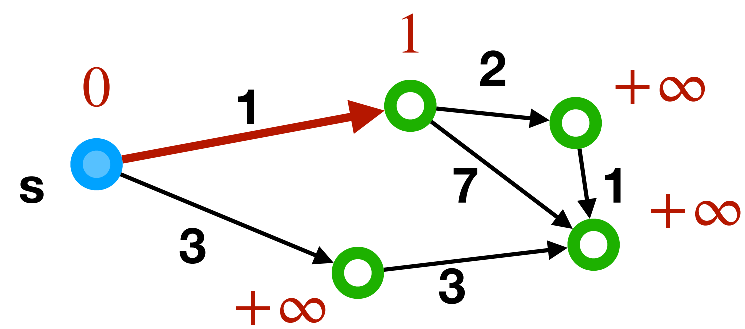

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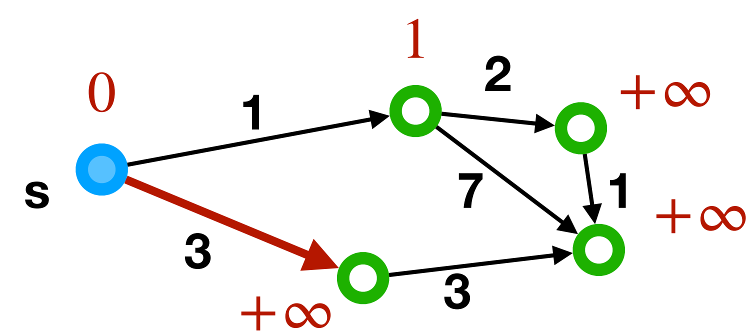

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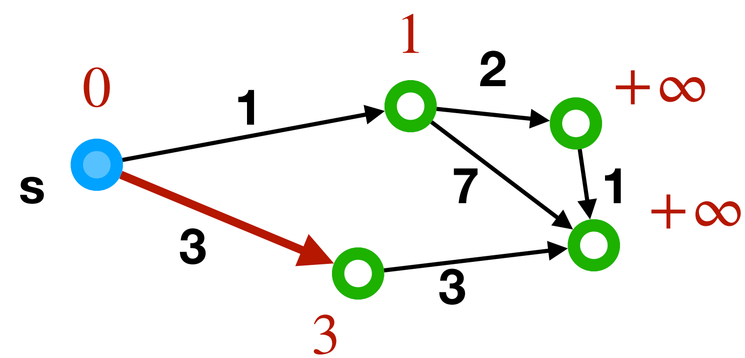

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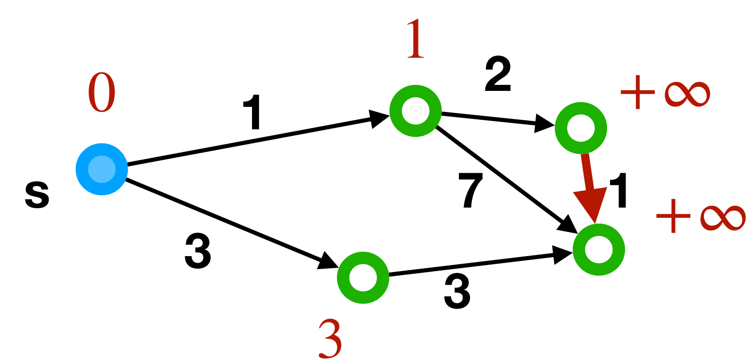

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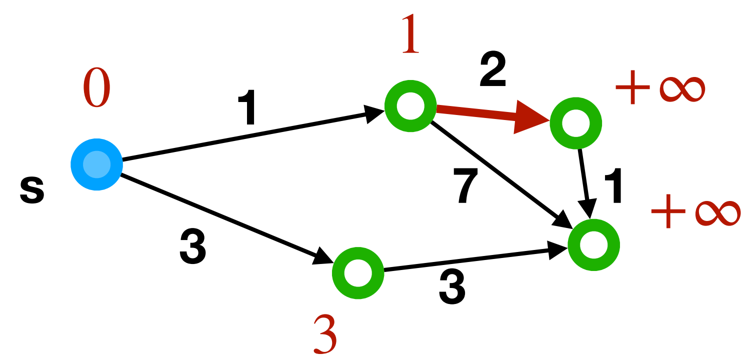

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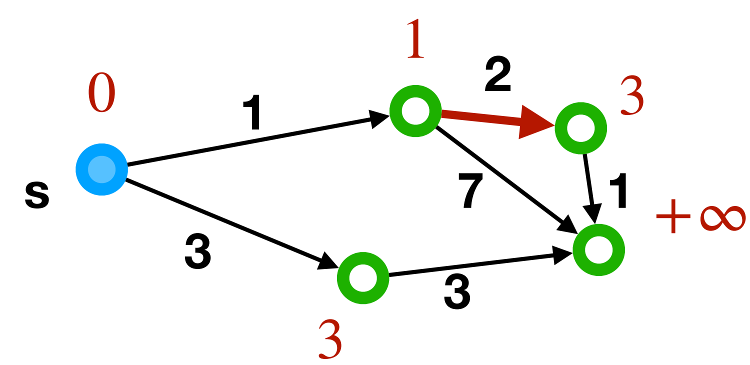

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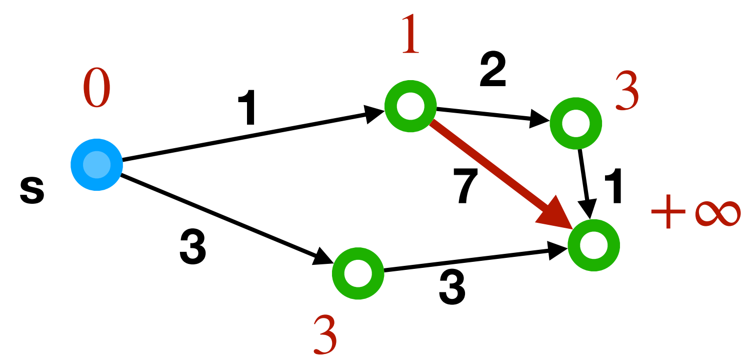

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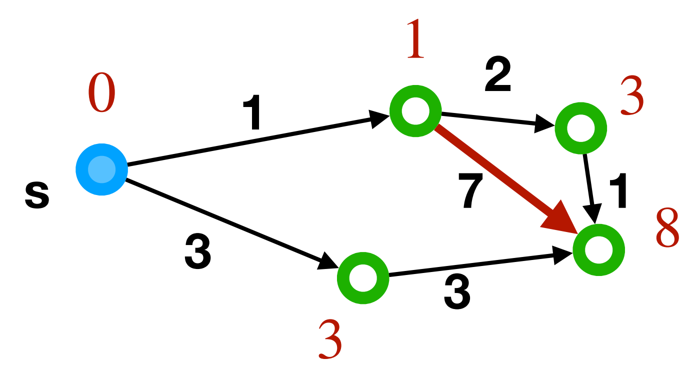

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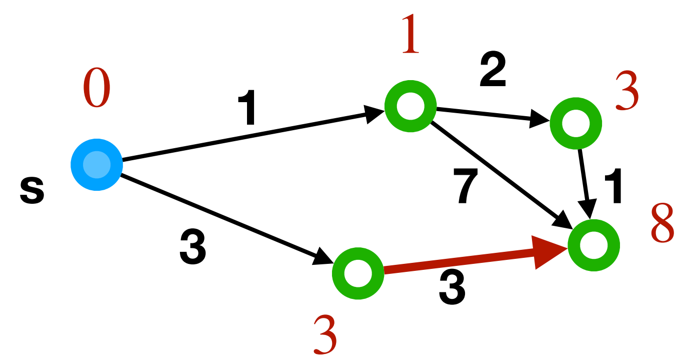

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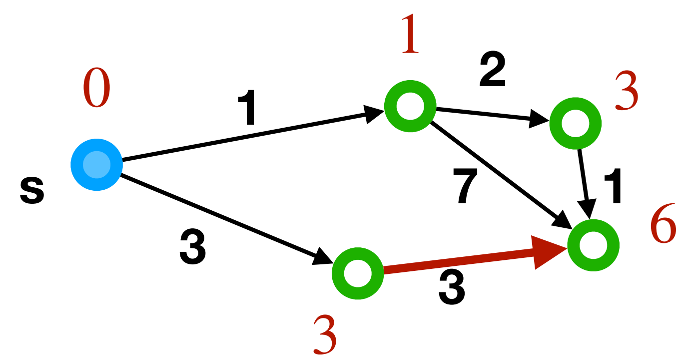

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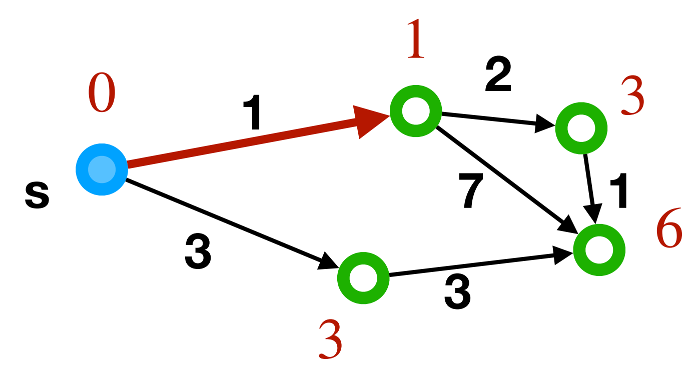

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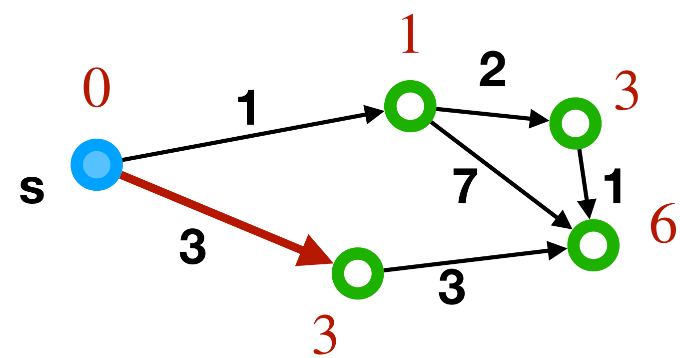

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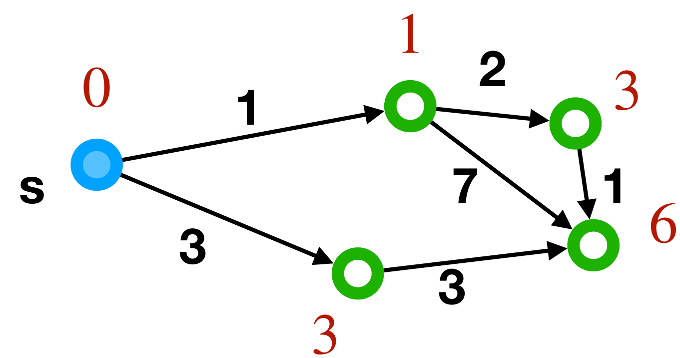

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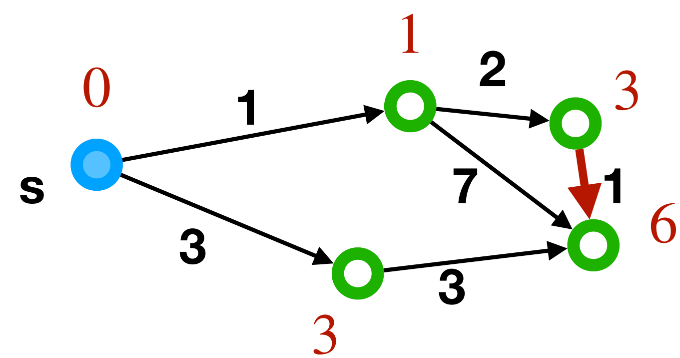

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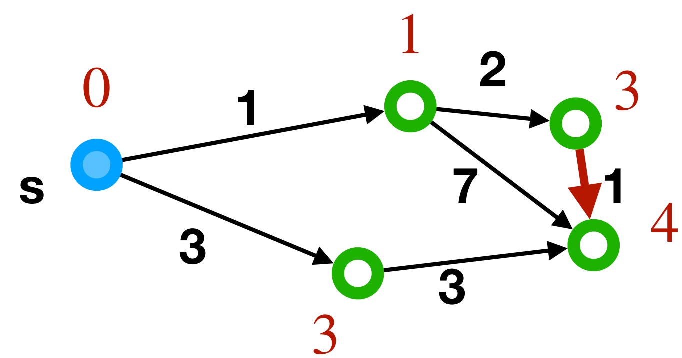

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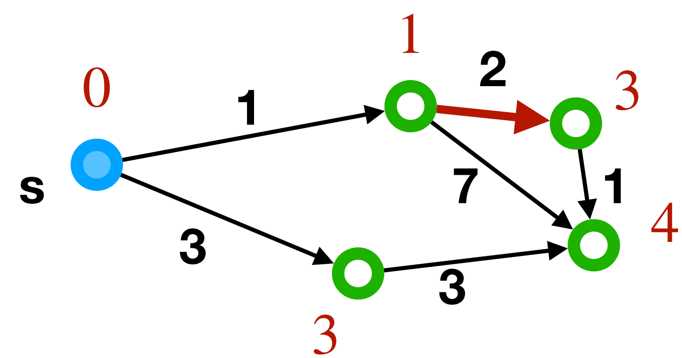

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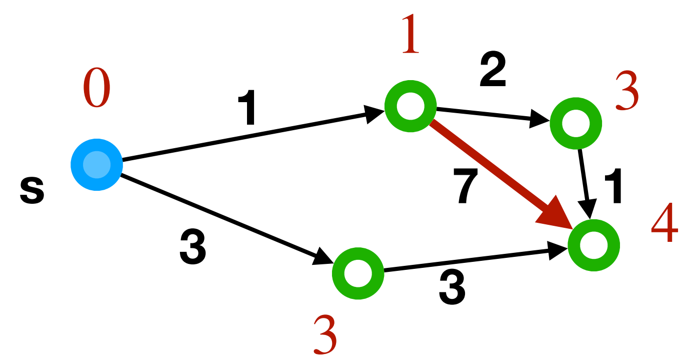

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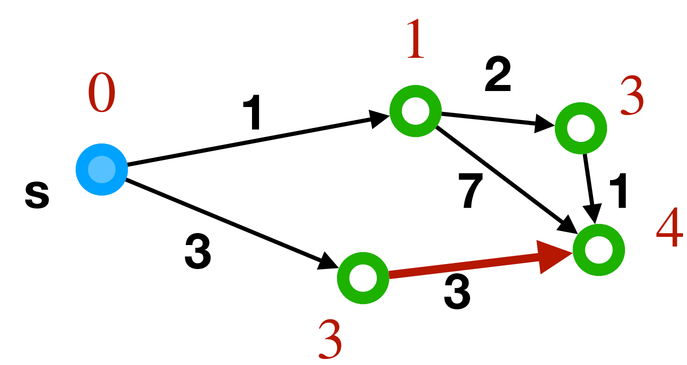

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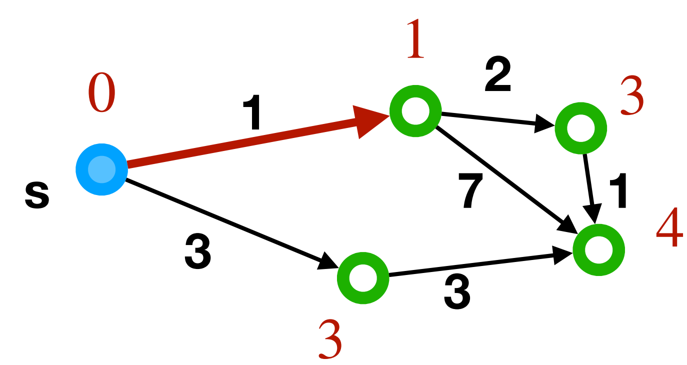

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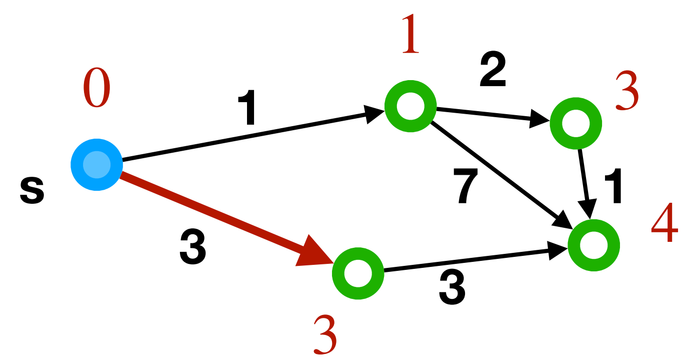

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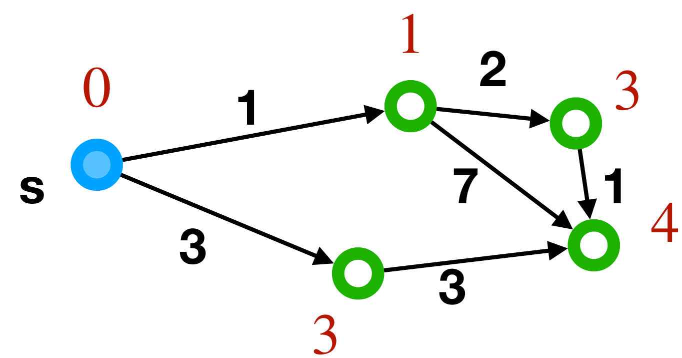

Proof of Correctness

- Let d[s] = 0 and d[v] = $+\infty$ for $v \in V - \{s\}$

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- Define $dist_i(s, v)$ as the weight of shortest path from s to v using at most i edges

- **Inductive statement:** For any i, after i-th run of the for-loop entirely, $d[v] \leq dist_i(s, v)$

Runtime Analysis

- Let d[s] = 0 and d[v] = $+\infty$ for $v \in V - \{s\}$

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- If no update happened in the for-loop terminate, otherwise run the for-loop again.

- We can have at most n iterations of the for-loop

- (By the proof of correctness)

- Each iteration takes O(m) time

- So total runtime is O(mn)

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- Is extremely agile and can be used in different settings:
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- But its runtime is too slow
- We will see another algorithm with much faster runtime

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- Btw, Bellman-Ford is a dynamic programming algorithm — in fact, perhaps the first serious one ever invented!