344 Recitation

Θ · $=$
o $<$
O $\le$

$f_1 = o(f_2)$
$f_1 = O(f_2)$
$f_1 = \Theta(f_2)$

(1) $\sqrt{n}$ n^2

$\sqrt{n} \overset{<}{=} o(n^2)$ $\sqrt{n} \overset{\le}{=} O(n^2)$ $\sqrt{n} \ne \Theta(n^2)$

(2) $\log n$ $\log \sqrt{n}$

$\log(\sqrt{n}) = \frac{1}{2} \log n$

$\log n \ne o(\log \sqrt{n})$ $\log n = \Theta(\log \sqrt{n})$

(3) 2^n 2^{n-1}

2^n $2^{n-1} = \frac{2^n}{2}$

$\lim \frac{2^n}{2^{n-1}} = 2$

$2^n = \Theta(2^{n-1})$

(4) 2^n 3^n

$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

$2^n = o(3^n)$

(5) $n!$ $\left(\frac{n}{\ldots}\right)^{n/3}$ $\ldots^{n/3}$ $\ldots(\ldots)$

$(5) \quad \binom{n}{3} \qquad \binom{n}{3} = O(n^3)$

$$(6) \quad \log(n!) = \Theta(n \log n)$$

$$n! = O(n^n)$$

Increasing order of asymptotic notation

n^3

$\log(n^2)$

$\log \log n^4$

$2^{\log n}$

n^n

$2^{n^2 \log n}$

Prove by induction : $\displaystyle\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$

$\displaystyle\rightarrow \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

A^* $\quad f_1 = \log n$

$f_2 = \log\log n$

$f_2 = O(f_1)$

$f_2 < f_1$

$f_2 = o(f_1)$

$$\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = +\infty$$

$$\lim_{n\to\infty} \frac{\log n}{\log\log n} = \lim_{t\to\infty} \frac{2^t}{t} = +\infty$$

$t = \log \log n$

B $\quad f_1 = n \qquad f_2 = \sqrt{n}$

$$f_2 = O(f_1)$$

$$\lim_{n \to \infty} \frac{f_1(n)}{f_2(n)} = \lim_{n \to \infty} \sqrt{n} = +\infty$$

C $\quad f_1 = 10 \log n$

$f_2 = \sqrt{n}$

$f_3 = \frac{1}{2} n^2 \qquad\qquad f_1 = o(f_2)$

$f_4 = 2^n \qquad\qquad f_1 < f_2$

f_1 vs f_2

$$\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = \lim_{n\to\infty} \left(\frac{10\log n}{\sqrt{n}}\right) = 0$$

f_2 vs f_3

From class $\leftarrow$ $n^c = O(n^{c+1})$

$\sqrt{n}$ $\quad n^2$

$n^{1/2} = O(n^{\frac{1}{2}+\frac{3}{2}})$

f_3 vs f_4

$\frac{1}{2}n^2$ vs 2^n

$$\lim_{n\to\infty} \frac{\frac{1}{2}n^2}{2^n} = \lim_{n\to\infty} \frac{n^2}{2^{n+1}} = 0$$

$$\lim_{n\to\infty} \frac{n^c}{2^n} = 0$$

for any constant c.

C ✓

$f_? = O(f_1)$

D $f_1 = 2^n$ ✗ $f_2 = \log n$

$f_1 \neq O(f_2)$

$f_1 = \omega(f_2)$

$$\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = \lim_{n\to\infty} \frac{2^n}{\log n} = +\infty$$

$f_3 = 1000n$ $f_4 = \sqrt{n}$

$$\lim_{n\to\infty} \frac{2^n}{\log n} = +\infty$$

↑ function $f(n) \leq f(n+1)$

L'Hopital Rule

$$\lim_{n\to\infty} \frac{2^n}{\frac{1}{n}} \; c = \lim_{n\to\infty} 2^n n \to +\infty$$

$< C$ - little - o and big O not θ

$f_1 = \theta(f_2)$

$\lim = C$

$\lim \frac{f_1(n)}{...} \leq C \to O$

$$\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} = C \quad C \neq 0 \quad = \Theta$$

Correct — $\lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} \geq C \rightarrow \Omega$

$$\left.\begin{aligned} \lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} &\leq C_1 \\ \lim_{n\to\infty} \frac{f_1(n)}{f_2(n)} &\geq C_2 \end{aligned}\right] \rightarrow \Theta \quad \neq o \text{ little} \cdot o$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f(n) = o(g(n)) \\ c \quad c > 0 & f(n) = \Theta(g(n)) \\ \infty & f(n) = \omega(g(n)) \end{cases}$$

$f(n) = O(g(n))$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f(n) = O(g(n)) \\ C & f(n) = O(g(n)) \\ \infty & \end{cases}$$

Q1 quiz

odd digit - letter A

5 - B

A → even or odd

Behind an odd digit → A
does not mean behind A there is an odd digit.

F
odd → A at the back.

statement $\rightarrow$

$D \rightarrow 3$ - counter example

$\overline{\overline{A}}$

$4 \rightarrow A$ doesn't matter.

$4 \rightarrow$ not A

2^n vs 3^n

$$\lim_{n \to \infty} \frac{2^n}{3^n} = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0$$

$2^n = o(3^n)$ $\quad <$ $\quad \leq$ also.

$\left(\frac{n}{3}\right)^{\frac{n}{3}}$ $\quad n!$

$$\lim_{n \to \infty} \frac{\left(\frac{n}{3}\right)^{\frac{n}{3}}}{n!}$$

$$\lim_{n \to \infty} \frac{\left(\frac{n}{3}\right)\left(\frac{n}{3}\right)\left(\cdots\right)\cdots\left(\frac{n}{3}\right)}{(n)(n-1)(n-2)\cdots\left(n-\frac{n}{3}\right)} \quad \frac{1}{\left(\frac{2n}{3}-1\right)\cdots}$$

$\frac{2n}{3}$

$c \geq 0$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$$

$f(n) = O(g(n))$

$g(n) = \Omega(f(n)$

$$\leq \lim_{n \to \infty} \frac{1}{\left(\frac{2n}{3}-1\right)!}$$

$$\lim_{n \to \infty} \frac{\left(\frac{n}{3}\right)^{\frac{n}{3}}}{n!} = 0 \longrightarrow \left(\frac{n}{3}\right)^{n/3} = o(n!)$$

$(n \quad {}^{n/3} \quad \quad n!$

$\left(\frac{n}{3}\right)^{\frac{n}{3}}$ / $n!$

$$\lim_{n\to\infty} \frac{\overbrace{\frac{n}{3} \times \frac{n}{3} \times \cdots \times \frac{n}{3}}^{\frac{n}{3}\text{ times}}}{\underbrace{n \times (n-1) \times n-2 \cdots}_{\frac{n}{3}\text{ terms}} \underbrace{\left(\frac{2n}{3}-1\right)\left(\frac{2n}{3}-2\right)\cdots \times 1}_{\frac{2n}{3}\text{ terms aside.}}}$$

$$= \lim_{n\to\infty} \frac{\frac{n}{3}}{n} \times \frac{\frac{n}{3}}{n-1} \cdots \frac{\frac{n}{3}}{\frac{2n}{3}} \times \frac{1}{\left(\frac{2n}{3}-1\right)!}$$

Remaining $\frac{2n}{3}$ terms

$$\leq \lim_{n\to\infty} 1 \times 1 \cdots \times 1 \times \frac{1}{\left(\frac{2n}{3}-1\right)!}$$

$$= \underline{\underline{0}}$$

Try to prove by induction

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

$$\frac{1}{i(i+1)} = \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$i=1$ $\quad 1 - \cancel{\frac{1}{2}}$

$i=2$ $\quad \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$i=3$ $\quad \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$ $\quad = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

$\vdots$

$i=n$ $\quad \cancel{\frac{1}{n}} - \frac{1}{n+1}$

Base case :

$$i=1 \qquad \frac{1}{1\times 2} = \frac{1}{2}$$

Induction Hypothesis

Assume $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$.

Want to prove $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$ using the hypothesis.

$$P(1) \to P(2) \to P(3) \ldots\ldots P(n)$$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \underbrace{\sum_{i=1}^{k} \frac{1}{i(i+1)}}_{\frac{k}{k+1}} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{1}{k+1}\left[k + \frac{1}{k+2} \right]$$

$$= \frac{1}{k+1}\left[\frac{k^2+2k+1}{k+2} \right]$$

$$= \frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)} = \frac{(k+1)}{(k+2)}$$

$$\log(n!) = \Theta(n \log n)$$

$$\log(n!) = \underline{O}(n \log n)$$

$$\log(n!) = \log(1 \times 2 \times \dots \times n)$$

$$\leq \log(n \times n \times \dots \times n)$$

$$= \log(n^n)$$

$= n \log n.$

$\log(n!) \geq \frac{n}{3} \log\left(\frac{n}{3}\right)$

$n! \geq \left(\frac{n}{3}\right)^{n/3}$ $n \log n$

$\log(n!) \geq \frac{n}{3} \log\left(\frac{n}{3}\right)$

$= \frac{1}{3}\left(n \log n - n \log 3\right)$

$\geq \frac{1}{4} n \log n$

$\log(n!) = \Omega(n \log n)$

$\log(n!) = \Theta(n \log n)$

2^n 3^n

2 2

Tuesday 11am - 12pm