SPRING 20 ELEMENTARY DIFFERENTIAL EQUATIONS: SAMPLE FINAL EXAMINATION

This is NOT the official final examination!!!

1. Instructions

- The (sample) final examination consists of 13 questions. Please finishes all question.
- The full mark is 160.
- This is an open book, open note examination.
- You must show your work and carefully justify your answers when answering the questions. The correct answer without any work will receive little or no credit.
- You are not allowed to discuss the questions with anyone. Answers copied directly elsewhere will receive no credit.
- You may upload your solution on Canvas as an assignment before 11am May 10th (Local time).
- Only pdf file are allowed.

2. Questions

1. (10 marks) Solve the following initial value problem:
$$x' = x^2(t+1), \quad x(1) = 1.$$

2. (10 marks) Sketch the phase lines of the following differential equation:
$$y' = y(y-2).$$
If y satisfies the differential equation and $y(2) = 1$, what can we say about $y(t)$ as $t \to +\infty$?

3. (10 marks) Find the general solution to the following differential equation:
$$y' + \frac{3}{t+1} y = t - 1.$$

4. (10 marks) Let $y(t)$ be a function which satisfies the differential equation $y' = (t+1)(y-2)^2 - t$ for all t and $y(0) = 1$. Show that $y(t) < 2$ for all $t \geq 0$.

5. (12 marks) Find the general solution to the following first order system:
$$x' = x + 1,$$
$$y' = 3y + x^2.$$

5. (8 marks) Calculate e^A, where A is the matrix
$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

6. (10 marks) Sketch the phase portrait of the following linear system:
$$Y' = \begin{pmatrix} -3 & -6 \\ 1 & 4 \end{pmatrix} Y$$

7. (8 marks) Solve the following initial value problem:
$$Y' = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} Y, \qquad Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

8. (12 marks) The harmonic oscillator is governed by the following differential equation
$$my'' + by' + ky = 0.$$
In the following three situations, sketch the function $y(t)$.
(a) $m = 1$, $b = 2$, $k = 3$ with $y(0) = 1, y'(0) = 0$.
(b) $m = 1$, $b = 4$, $k = 4$ with $y(0) = -1$, $y'(0) = -1$.
(c) $m = 1$, $b = 5$, $k = 6$ with $y(0) = 1$, $y'(0) = -4$.

9. (12 marks) Find the general solution to the following differential equation:
$$y'' + 3y' + 2y = 2\sin 2t.$$

10. (12 marks) Find the general solution to the following differential equation:
$$y'' + 3y + 2y = te^{-t}.$$

11. (20 marks) In this question we restrict our attention to the first quadrant ($x, y \geq 0$). Given the following system:
$$x' = x(2 - x - y)$$
$$y' = y(y - x).$$
Note that $(0,0)$, $(1,1)$, $(2,0)$ are the equilibria.
(a) Calculate the linearization at the equilibria $(1,1)$ and $(2,0)$. State if it is a sink, a source or a saddle.
(b) Sketch the x-nullclines and y-nullclines.
(c) Argue that there is exactly one solution curve $Y(t)$ (up to time translation) so that
$$Y(t) \to (1,1) \text{ as } t \to -\infty,$$
$$Y(t) \to (2,0) \text{ as } t \to +\infty.$$

12. (16 marks) Consider the following model for two competitive species X, Y sharing the same habitat:

$$\begin{aligned} x' &= x(3-x) - bxy, \\ y' &= 4y(1-y) - xy, \end{aligned}$$

here $x(t)$ and $y(t)$ are the population of the species X and Y at time t respectively. b is a unknown positive parameter. The term $-bxy$ measures the (negative) effect to the growth rate of X due to the presence of Y. Find the critical value b_0, so that whenever $b > b_0$, the species Y dominates the competition and the species X becomes extinct eventually. Please explain your answer.

13. (10 marks) Check if the following system is Hamiltonian. If so, find a Hamiltonian H:

$$\begin{aligned} x' &= 2y + e^x \sin y, \\ y' &= -2x + e^x \cos y. \end{aligned}$$

End of Sample Final Examination