Matrix Exponential.

Introduction: Consider the ~~D.E.~~ IVP

$$\begin{cases} y' = ay, \\ y(0) = y_0. \end{cases}$$

It has solution $y(t) = e^{ta} y_0$. (e.g. solved by separation of variables)

Want to define a matrix e^{tA}, so that the solution to the IVP

$$\begin{cases} Y' = AY \\ Y(0) = Y_0 \end{cases}$$

(here $Y = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)

is given by $Y(t) = e^{tA} \cdot Y_0$.

↑ matrix multiplication.

Since the exponential function e^x admits a Taylor expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \qquad (n! = n(n-1)(n-2)\cdots 2 \cdot 1)$$

we define:

Def: Given a matrix B, define e^B as

$$\boxed{e^B = I + B + \frac{B^2}{2!} + \cdots + \frac{B^n}{n!} + \cdots}$$

identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

e.g. $B^2 = B \cdot B$ (matrix multiplications)

Key property: (*) $\boxed{\frac{d}{dt} e^{tA} = A \cdot e^{tA}}$

component-wise differentiation: i.e. if $C(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$,

then $\frac{d}{dt} C(t) = \begin{pmatrix} a'(t) & b'(t) \\ c'(t) & d'(t) \end{pmatrix}$.

Proof of (*):

$$\frac{d}{dt} e^{tA} = \frac{d}{dt}\left(I + tA + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots \right)$$

$$= \frac{d}{dt} I + \frac{d}{dt}(tA) + \frac{d}{dt}\left(\frac{t^2A^2}{2!}\right) + \cdots + \frac{d}{dt}\left(\frac{t^nA^n}{n!}\right) + \cdots$$

(Power rule)

$$= 0 + A + (2t)\frac{A^2}{2!} + \cdots (nt^{n-1})\frac{A^n}{n!} + \cdots$$

$$= A + tA^2 + \cdots + \frac{t^{n-1}A^n}{(n-1)!} + \cdots \quad \left(\text{Note } \frac{n}{n!} = \frac{1}{(n-1)!}\right)$$

$$= A \cdot I + A \cdot (tA) + \cdots + A \cdot \left(\frac{t^{n-1}A^{n-1}}{(n-1)!}\right) + \cdots$$

$$= A\left(I + tA + \cdots + \frac{t^{n-1}A^{n-1}}{(n-1)!} + \cdots\right)$$

$$= A \cdot e^{tA} \quad \square$$

From (*) we can conclude: $Y(t) = e^{tA} \cdot Y_0$ satisfies

$$Y' = AY.$$

Proof: $\frac{d}{dt} Y = \frac{d}{dt}\left(e^{tA} \cdot Y_0\right) = \left(\frac{d}{dt} e^{tA}\right) \cdot Y_0$ (Y_0 is independent of t)

$$\overset{(*)}{=} A e^{tA} \cdot Y_0$$

$$= AY \quad \blacksquare$$

Together with the fact $(t=0)$

$$\cancel{Y(0)=} \; e^{0A} = I + (0A) + \frac{(0A)^2}{2!} + \cdots + \frac{(0A)^n}{n!} + \cdots$$

$$= I.$$

$$\Rightarrow Y(0) = e^{0A} \cdot Y_0 = I \cdot Y_0 = Y_0,$$

we have shown that $Y(t) = e^{tA} \cdot Y_0$ solves the IVP

$$\begin{cases} Y' = AY, \\ Y(0) = Y_0. \end{cases}$$

Question: how to calculate e^{tA} explicitly? Note that it's hard to compute e^B of a matrix directly from the definition: e.g. if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$B^2 = B \cdot B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$$

is already quite complicated.

To calculate e^B effectively, we use two observations:

(I) If $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is diagonal, then

$$e^{\Lambda} = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}.$$

Proof of (I): Note that $\Lambda^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$= \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

and in general $\Lambda^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$

$$\Rightarrow e^{\Lambda} = I + \Lambda + \frac{\Lambda^2}{2!} + \cdots + \frac{\Lambda^n}{n!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix} + \cdots + \frac{1}{n!}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1+\lambda_1+\frac{\lambda_1^2}{2}+\cdots+\frac{\lambda_1^n}{n!}+\cdots & 0 \\ 0 & 1+\lambda_2+\frac{\lambda_2^2}{2!}+\cdots+\frac{\lambda_2^n}{n!}+\cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix} \quad \blacksquare$$

(II) If $A = S\Lambda S^{-1}$, then $e^A = S e^{\Lambda} S^{-1}$.

(Here S^{-1} is the inverse matrix of S)

Proof of (II): Note that if $A = S\Lambda S^{-1}$, then

$$A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1})$$

$$= S\Lambda(S^{-1}S)\Lambda S^{-1}$$

$$= S\Lambda\Lambda S^{-1}$$

$$= S\Lambda^2 S^{-1}.$$

Similarly $$A^n = S\Lambda^n S^{-1}$$

$$\Rightarrow e^A = I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots$$
$$= SIS^{-1} + S\Lambda S^{-1} + S\frac{\Lambda^2}{2!}S^{-1} + \cdots + S\frac{\Lambda^n}{n!}S^{-1} + \cdots$$
$$= S\left(I + \Lambda + \frac{\Lambda^2}{2!} + \cdots + \frac{\Lambda^n}{n!} + \cdots\right)S^{-1}$$
$$= Se^{\Lambda}S^{-1} \quad \square$$

Thus, using (I), (II), if one can find S and $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, so that $A = S\Lambda S^{-1}$, then

$$e^A \underset{\text{(II)}}{=} Se^{\Lambda}S^{-1} \underset{\text{(I)}}{=} S\begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}S^{-1}$$

can be computed.

To find S and Λ, we recall some linear algebra:

Def (Eigenvalue, eigenvector) A number λ is called an eigenvalue of a matrix A, if there is a non-zero vector V so that

$$AV = \lambda V.$$

In this case V is called an eigenvector with respect to λ.

If λ_1, λ_2 are two eigenvalues of A, so that the corresponding eigenvectors V_1, V_2 is linearly independent, then

$$A = S \Lambda S^{-1}$$

with $S = \begin{pmatrix} V_1 & V_2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Putting 2 column vectors together to form the 2×2 matrix S.

Reason: $AV_1 = \lambda_1 V_1$, $AV_2 = \lambda_2 V_2$

$$\Longrightarrow \begin{pmatrix} AV_1 & AV_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 V_1 & \lambda_2 V_2 \end{pmatrix}$$

$$\Longrightarrow A\begin{pmatrix} V_1 & V_2 \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\Longrightarrow AS = S\Lambda$$

(S invertible)

$$\Longrightarrow A = S\Lambda S^{-1}$$

<u>To sum up</u>: To calculate e^A,

(i) Find its eigenvalues. λ_1, λ_2

(ii) Find the corresponding eigenvectors, V_1, V_2.

(iii) $\Longrightarrow e^A = \begin{pmatrix} V_1 & V_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}\begin{pmatrix} V_1 & V_2 \end{pmatrix}^{-1}$

Example: Calculate e^A for $A = \begin{pmatrix} 5 & 6 \\ -1 & -2 \end{pmatrix}$

To find the eigenvalues, set

$$0 = \det(A - \lambda I) \leftarrow \text{characteristic polynomial.}$$

$$= \det\begin{pmatrix} 5-\lambda & 6 \\ -1 & -2-\lambda \end{pmatrix}$$

$$= (5-\lambda)(-2-\lambda) + 6$$

$$= \lambda^2 - 3\lambda - 10 + 6$$

$$= \lambda^2 - 3\lambda - 4$$

$$= (\lambda+1)(\lambda-4).$$

$$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = 4.$$

To find the eigenvectors,

$\underline{\lambda_1 = -1}$: $\quad A - \lambda_1 I = \begin{pmatrix} 5 & 6 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ -1 & -1 \end{pmatrix}$

$\Rightarrow \quad V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is one eigenvectors.

$\underline{\lambda_2 = 4}$: $\quad A - \lambda_2 I = \begin{pmatrix} 1 & 6 \\ -1 & -6 \end{pmatrix} \Rightarrow V_2 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}.$

$$\Rightarrow S = (V_1 \; V_2) = \begin{pmatrix} 1 & 6 \\ -1 & -1 \end{pmatrix} \Rightarrow S^{-1} = \frac{1}{-1+6}\begin{pmatrix} -1 & -6 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{5} & -\frac{6}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}.$$

$$\Longrightarrow e^A = \begin{pmatrix} 1 & 6 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} e^{-1} & 0 \\ 0 & e^4 \end{pmatrix}\begin{pmatrix} -\frac{1}{5} & -\frac{6}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix} \qquad \square$$