SPRING 20 ELEMENTARY DIFFERENTIAL EQUATIONS: TAKE HOME MIDTERM 2

1. Instructions

- The midterm 2 consists of 12 questions. Please finishes all question.
- The full mark is 80.
- This is a take home examination.
- You must show your work and carefully justify your answers when answering the questions. The correct answer without any work will receive little or no credit.
- You are not allowed to discuss with anyone concerning the midterm 2. Answers copied directly elsewhere will receive no credit.
- Upload your solution on Canvas as an assignment before 10pm on April 9th (Local time).
- Only pdf file are allowed.

2. Questions

1. (6 marks) Find the equilibria for the following systems of differential equations:

$$\begin{aligned} x' &= (x-1)(2x+y-2), \\ y' &= y(x+y-3). \end{aligned}$$

2. (4 marks) Transform the following second order differential equation into a systems of two differential equations:

$$y'' + y^2 y' + y' \sin t + t^2 = 0.$$

3. (8 marks) Find the general solution to the following system of differential equations

$$\begin{aligned} x' &= 2x, \\ y' &= 4y + 2x^2. \end{aligned}$$

4. (6 marks) Let $Y_1(t), Y_2(t)$ be two solutions to an autonomous system $Y' = F(Y)$, where F and the partial derivatives of F are both continuous. If $Y_2(0) = Y_1(2)$, what is the relationship between Y_1 and Y_2. Please explain your answer.

5. (6 marks) Calculate e^A, where A is the matrix
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}.$$

6. (6 marks) Calculate e^{tB}, where
$$B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$
Hint: Write
$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
and calculate B^2, B^3 and so on.

7. (6 marks) Let
$$Y_1(t) = \begin{pmatrix} 2e^t - e^{-2t} \\ e^t + e^{-2t} \end{pmatrix}, \quad Y_2(t) = \begin{pmatrix} -2e^t + 2e^{-2t} \\ -e^t - 2e^{-2t} \end{pmatrix}$$
be two solutions to a linear system $Y' = AY$. Solve the initial value problem
$$\begin{cases} Y' = AY, \\ Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{cases}$$

8. (6 marks) Let v be a nonzero vector so that $e^{\lambda t}v$ is a solution to the system $Y' = AY$. Show that λ is an eigenvalue of A with eigenvector v.

9. (8 marks) Solve the initial value problem
$$Y' = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} Y, \quad Y(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

10. (8 marks) Sketch the phase portrait of the following system:
$$Y' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y.$$

11. (8 marks) Sketch the phase portrait of the following system:
$$Y' = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} Y.$$

12. (8 marks) Let A be a 2×2 matrix with eigenvalues λ_1, λ_2. What are the conditions on λ_1, λ_2, so that all solutions to $Y' = AY$ converges to $(0, 0)$ as t goes to $+\infty$? Please explain your answer.