# MATH252 NOTES ON EIGENVALUES AND EIGENVECTORS

In this short note we recall the definition of eigenvalues and eigenvectors.

## 1. Definition of eigenvalues and eigenvectors

**Definition 1.1.** Let $A$ be a $n \times n$ matrix. Then a number $\lambda$ is called an **eigenvalue** of $A$ if there is an non-zero vectors $\vec{v}$ so that $A\vec{v} = \lambda\vec{v}$. In this case we call $\vec{v}$ an **eigenvector** of $A$ corresponding to $\lambda$.

**Example 1.1.** Let $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$. Then $\lambda = 3$, and $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector since

$$\begin{aligned} A\vec{v} &= \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot 2 + 4 \cdot 1 \\ 1 \cdot 2 + 1 \cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \\ &= 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\vec{v}. \end{aligned}$$

*Remark* 1. For a fixed eigenvalues, the eigenvectors are not unique: $\vec{v}$ is an eigenvector, then so is $c\vec{v}$ for any non-zero number $c$.

**Example 1.2.** We will allow $\lambda$ to be complex numbers and $\vec{v}$ to be complex vectors (that is, the entries might be complex). For example, if

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$

then $\lambda = 2 + i$ is an eigenvalue of $A$ with corresponding eigenvector $\vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$, since

$$\begin{aligned} A\vec{v} &= \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2i - 1 \\ i + 2 \end{pmatrix} \\ &= \begin{pmatrix} 2i + i^2 \\ 2 + i \end{pmatrix} \qquad \text{(note that } i^2 = -1\text{)} \\ &= (2 + i) \begin{pmatrix} i \\ 1 \end{pmatrix} = (2 + i)\vec{v}. \end{aligned}$$

## 2. To find eigenvalues, eigenvectors

Given an $n \times n$ matrix $A$, the **characteristic polynomial** of $A$ is given by

$$p(\lambda) = \det(A - \lambda I), \tag{2.1}$$

where $I$ is the identity matrix and det denote the determinant: we will recall the case for $n = 2$: given a $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its determinant is given by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc. \tag{2.2}$$

One important properties of determinant is that an $n \times n$ matrix $B$ is invertible if and only if $\det B \neq 0$. Indeed we have a formula for the inverse $B^{-1}$ in terms of $B$ and $\det B$. We recall only the formula for $n = 2$, which is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{2.3}$$

**Example 2.1.** If $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$, then its characteristic polynomial is

$$\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \det \left( \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&= \det \begin{pmatrix} 1 - \lambda & 4 \\ 1 & 1 - \lambda \end{pmatrix} \\
&= (1 - \lambda)(1 - \lambda) - 4 \\
&= \lambda^2 - 2\lambda - 3.
\end{aligned}$$

If $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $\vec{v}$, then $A\vec{v} = \lambda\vec{v}$, or

$$A\vec{v} - \lambda\vec{v} = 0.$$

Writing $\vec{v} = I\vec{v}$, where $I$ is the identity matrix, we have

$$(A - \lambda I)\vec{v} = 0. \tag{2.4}$$

In particular, we have

**Theorem 2.1.** *A number $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is a root of the characteristic polynomial.*

*Proof.* If $\lambda$ is an eigenvalue, then (2.4) is satisfied. In particular, $\det(A - \lambda I)$ must be zero: if not, then $(A - \lambda I)^{-1}$ exists. Multiplying $(A - \lambda I)^{-1}$ to both sides of (2.4) gives $\vec{v} = 0$, which is not true by the definition of an eigenvector. Thus $\det(A - \lambda I)$ has to be zero. $\square$

This gives us a convenient way to find all possible eigenvalues.

**Example 2.2.** In example 2.1 we calculated the characteristic polynomial of $\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$, which is $\lambda^2 - 2\lambda - 3$. Either using factorization

$$\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

or the quadratic formula, we found by Theorem 2.1 that 3 and $-1$ are the only eigenvalues of the matrix.

Next we indicate how to find an eigenvector corresponding to an eigenvalue. In general, when an eigenvalue $\lambda$ is given, one can think of (2.4) as $n$ linear equations with $n$-unknowns (given by the entries of $\vec{v}$), so that (2.4) can be solved by (e.g.) row reduction. We remark that in the simple situation $n = 2$, one can read off $\vec{v}$ directly from the matrix $A - \lambda I$, using the following observations:

**Lemma 2.1.** *Let $B$ be a $2 \times 2$ matrix with $\det B = 0$, then either*

$$B = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} ka & kb \\ a & b \end{pmatrix},$$

*that is, the two row vectors are proportional to each other.*

*Proof.* Write $B = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix}$. Since $\det B = 0$, we have $ad = bc$, or $d/b = c/a$. Write $k = d/b$ $(= c/a)$. Then

$$c = (d/b)a = ka, \quad d = (c/a)b = kb.$$

□

Using the above lemma, we have:

**Proposition 2.1.** *If $\lambda$ is an eigenvalue of $A$ and*

$$A - \lambda I = \begin{pmatrix} a & b \\ ka & kb \end{pmatrix},$$

*then $\vec{v} = \begin{pmatrix} b \\ -a \end{pmatrix}$ is an eigenvector of $A$ corresponding to $\lambda$.*

*Proof.* To find the eigenvector, write $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Then (2.4) becomes

$$\begin{pmatrix} a & b \\ ka & kb \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is the same as

$$\begin{cases} ax + by = 0, \\ kax + kby = 0. \end{cases}$$

Note that the second equation is redundant since $kax + kby = k(ax + by)$. Thus it suffices to find $x, y$ so that $ax + by = 0$. This is satisfied when $x = b$, $y = -a$. Thus $\vec{v} = \begin{pmatrix} b \\ -a \end{pmatrix}$ is an eigenvector of $A$ corresponding to $\lambda$. □

**Example 2.3.** Using the same example $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$, we calculated that the eigenvalues are 3 and $-1$. Next we find the corresponding eigenvectors:

- When $\lambda = 3$,
$$A - 3I = \begin{pmatrix} 1-3 & 4 \\ 1 & 1-3 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix},$$
then $\vec{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ is an eigenvector.
- When $\lambda = -1$,
$$A - (-1)I = \begin{pmatrix} 1-(-1) & 4 \\ 1 & 1-(-1) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix},$$
then $\vec{v} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ is an eigenvector.