# 252 ELEMENTARY DIFFERENTIAL EQUATIONS: HW3 SOLUTION

(1) Solve the following intial value problem

$$\begin{cases} \frac{dx}{dt} = 3x, \\ \frac{dy}{dt} = 4y - x^2 \end{cases}$$

and $(x(0), y(0)) = (1, 2)$.

**Solution:** The first equation gives

$$x = C_1 e^{3t},$$

where $C_1$ is any constant. Thus

$$\frac{dy}{dt} = 4y - C_1^2 e^{6t}.$$

Next we use guessing. Let $y_p = Ae^{6t}$. Then $y_p' = A6e^{6t}$. Putting $y = y_p$ into the second equation,

$$6Ae^{6t} = 4Ae^{6t} - C_1^2 e^{6t} \Rightarrow A = -C_1^2/2.$$

Thus $y_p = -\frac{1}{2}C_1^2 e^{6t}$ is a particular solution and the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_2 e^{4t} - \frac{1}{2}C_1^2 e^{6t}. \end{aligned}$$

Thus the general solution is to the system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{3t} \\ C_2 e^{4t} - \frac{1}{2}C_1^2 e^{6t} \end{pmatrix}.$$

Next we use the initial value $(x(0), y(0)) = (1, 2)$ to obtain

$$\begin{aligned} 1 &= C_1 e^0, \\ 2 &= C_2 e^0 - \frac{1}{2}C_1^2 e^0. \end{aligned}$$

Which implies $C_1 = 1$ and $C_2 = 5/2$. Thus

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{3t} \\ \frac{5}{2}e^{4t} - \frac{1}{2}e^{6t} \end{pmatrix}$$

solves the IVP.

(2) Consider the system

$$\begin{cases} x' = x^2 + y \\ y' = x^2 y^2 \end{cases}$$

Show that, for the solution $(x(t), y(t))$ with intial condition $(x(0), y(0)) = (0, 1)$, there is a time $t_*$ such that $x(t) \to +\infty$ as $t \to t_*$. In other words the solution blows up in finite time.

**Solution:** Since $y' = x^2 y^2$, we have $y' \geq 0$ for all $t$. Thus $y(t)$ is a non-decreasing function, and thus $y(t) \geq y(0)$ whenever $t > 0$. Using the initial condition, $y(t) \geq 1$ for all $t > 0$. Put this into the equation for $x'$, we obtain

$$x' \geq x^2 + 1, \quad \text{for all } t > 0.$$

Then we use the same technique for separation of variable:

$$\begin{aligned} \frac{x'}{x^2+1} &\geq 1 \\ \Rightarrow \int_0^t \frac{x'}{x^2+1} dt &\geq \int_0^t 1 dt = t, \\ \Rightarrow \int_{x(0)}^{x(t)} \frac{1}{x^2+1} dx &\geq t \\ \Rightarrow \arctan x(t) - \arctan x(0) &\geq t \\ \Rightarrow \arctan x(t) &\geq t \quad \text{(since } x(0) = 0) \\ \Rightarrow x(t) &\geq \tan t. \end{aligned}$$

Note that at the last step we apply tan to both sides. This is possible since tan is non-decreasing. Since $\tan t$ blows up to $+\infty$ in finite time (as $t \to \pi/2$), the inequality $x(t) \geq \tan t$ implies that $x(t)$ also blows up at finite time.

(3) Rewrite the following system of differetial equations in matrix form:

$$\begin{aligned} \frac{dp}{dt} &= 2p - q + 6r, \\ \frac{dq}{dt} &= -p + 3r, \\ \frac{dr}{dt} &= 7q + 2r. \end{aligned}$$

**Solution:**

$$\begin{pmatrix} p' \\ q' \\ r' \end{pmatrix} = \begin{pmatrix} 2 & -1 & 6 \\ -1 & 0 & 3 \\ 0 & 7 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

(4) Find the equilibria of the following systems of differential equations:

$$\begin{cases} x' = -3y(1-x-y) \\ y' = x(3-2x-y) \end{cases}$$

**Solution:** To find the equilibria, we set the RHS of the system to zero:

$$-3y(1-x-y) = 0,$$
$$x(3-2x-y) = 0.$$

The first equation gives $y = 0$ or $1 - x - y = 0$. We split into two cases:

- When $y = 0$, the second equation gives $x(3-2x) = 0$, thus $x = 0$ or $x = 3/2$. Thus $(0,0)$ and $(3/2, 0)$ are both equilibria.
- When $1 - x - y = 0$, write $y = 1 - x$ and plug into the second equation. Thus
$$x(3 - 2x - (1-x)) = 0,$$
which implies $x = 0$ or $x = 2$. Using $y = 1-x$, we find that $(0,1)$ and $(2,-1)$ are also equilibria.

To sum up, the system has four equilibria

$$(0,0), (3/2,0), (0,1) \text{ and } (2,-1).$$

(5) Consider the following system of differential equations:

$$\begin{cases} \frac{dx}{dt} = -3y(1+x^2+y^2) \\ \frac{dy}{dt} = 2x(1+2x^2+2y^2) \end{cases}$$

(a) Show that $(\cos 6t, \sin 6t)$ is one of the solution.
**Solution:** Direct checking:

$$\begin{aligned} -3y(1+x^2+y^2) &= -3\sin 6t(1+\cos^2 6t + \sin^2 6t) \\ &= -6\sin 6t \\ &= \frac{dx}{dt}. \end{aligned}$$

The checking for the second is similar and is skipped.

(b) Show that if $(x(t), y(t))$ is another solution with $(x(1), y(1)) = (0.5, 0.5)$, then $x(t)^2 + y(t)^2 < 1$ for all $t$.
**Solution:** Since $Y_2(t) = (\cos 6t, \sin 6t)$ is a solution by (a), and $(x(1), y(1))$ is inside of the unit circle, which is the trace of $Y_2$. Thus by the comparison principle for autonomous systems, $(x(t), y(t))$ must stay inside of $Y_2$ for all $t$ and thus $x(t)^2 + y(t)^2 < 1$.

(6) In each of the following, factor the matrix $A$ into a product $S\Lambda S^{-1}$, with $\Lambda$ a diagonal matrix.

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

**Solution** The eigenvalues is determined by the characteristic polynomial:

$$0 = \det(A - \lambda I) = (1-\lambda)(-\lambda)$$

which gives $\lambda = 0, 1$. The corresponding eigenvectors are

$$V_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

(b) $A = \begin{pmatrix} 5 & 6 \\ -1 & -2 \end{pmatrix}$.

**Solution** The characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= (5-\lambda)(-2-\lambda) + 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda - 4)(\lambda + 1). \end{aligned}$$

Thus $\lambda = -1, 4$ are the eigenvalues.
The corresponding eigenvalues are

$$V_{-1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}.$$

Then we have

$$\begin{aligned} A &= \begin{pmatrix} 1 & 6 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 6 \\ -1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 6 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -1/5 & -6/5 \\ -1/5 & 1/5 \end{pmatrix} \end{aligned}$$

(7) Calculate $A^4$:

**Solution:** We use the fact that $A^4 = S\Lambda^4 S^{-1}$. Thus for (a)

$$\begin{aligned} A^4 &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^4 \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \\ &= A. \end{aligned}$$

and for (b)

$$\begin{aligned} A^4 &= \begin{pmatrix} 1 & 6 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}^4 \begin{pmatrix} -1/5 & -6/5 \\ -1/5 & 1/5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 6 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 256 \end{pmatrix} \begin{pmatrix} -1/5 & -6/5 \\ -1/5 & 1/5 \end{pmatrix}. \end{aligned}$$

(8) Calculate $e^{At}$.
**Solution** We use the fact that $e^{At} = Se^{\lambda t}S^{-1}$. Thus for (a)

$$\begin{aligned} e^{At} &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} e^t & e^t \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

and for (b)

$$e^{At} = \begin{pmatrix} 1 & 6 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{4t} \end{pmatrix} \begin{pmatrix} -1/5 & -6/5 \\ -1/5 & 1/5 \end{pmatrix}$$