MATH252 VARIATION OF PARAMETERS

In this short note, we are interested in solving the linear system

$$Y' = AY + \vec{F}(t), \tag{0.1}$$

where A is a 2×2 matrix with constant coefficients and $\vec{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$ is a vector.

We can compare (0.1) with the first order linear differential equation

$$y' = ay + f(t),$$

which can be solved by integration factor:

$$\begin{aligned} y' - ay &= f(t), \\ e^{-at}(y' - ay) &= e^{-at} f(t) \\ (e^{-at} y)' &= e^{-at} f(t) \\ e^{-at} y - y(0) &= \int_0^t e^{-au} f(u) du \\ \Rightarrow y &= \int_0^t e^{a(t-u)} f(u) du + e^{at} y_0. \end{aligned}$$

The same argument works for (0.1), only that we need to use the matrix exponential:

Theorem 0.1 (Variation of parameters)**.** *The general solution to (0.1) is given by*

$$Y(t) = \int_0^t e^{(t-u)A} \vec{F}(u) du + e^{At} Y_0. \tag{0.2}$$

Note that in (0.2), integration refers to component-wise integration.

1. Second order linear differential equation

Next we study the second order linear differential equation with constant coefficients with a forcing term

$$y'' + py' + qy = f(t). \tag{1.1}$$

The above is equivalent to the following first order system:

$$\begin{aligned} y' &= v, \\ v' &= -qy - pv + f(t), \end{aligned}$$

which can be written as a first order linear systems

$$\begin{pmatrix} y \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}. \tag{1.2}$$

That is,

$$A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}, \quad \vec{F} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

We will then use (0.2) to obtain $Y = \begin{pmatrix} y \\ v \end{pmatrix}$ and in particular y. First we calculate explicitly e^{sA}. We consider only the cases where A has distinct (real or complex) eigenvalues.

Let λ_1, λ_2 be the eigenvalues of A. Then the corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}.$$

Thus

$$\begin{aligned} sA &= \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} s\lambda_1 & 0 \\ 0 & s\lambda_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1} \\ \Rightarrow e^{sA} &= \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} e^{s\lambda_1} & 0 \\ 0 & e^{s\lambda_2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} e^{s\lambda_1} & e^{s\lambda_2} \\ \lambda_1 e^{s\lambda_1} & \lambda_2 e^{s\lambda_2} \end{pmatrix} \left[\frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \right] \\ &= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{s\lambda_1} - \lambda_1 e^{s\lambda_2} & -e^{s\lambda_1} + e^{s\lambda_2} \\ \lambda_1\lambda_2(e^{s\lambda_1} - e^{s\lambda_2}) & -\lambda_1 e^{s\lambda_1} + \lambda_2 e^{s\lambda_2} \end{pmatrix} \end{aligned}$$

Next we choose $s = t - u$ and apply (0.2). Note that we need only a particular solution, so we choose $Y_0 = \vec{0}$. Also, we will only calculate the y-component (i.e., the first component) in (0.2) since we want only y. From the above calculation of e^{sA}, the first component of $e^{(t-u)A}\vec{F}$ is

$$\frac{e^{(t-u)\lambda_2} - e^{(t-u)\lambda_1}}{\lambda_2 - \lambda_1} f(u),$$

thus we have the formula for the particular solution:

$$y_p(t) = \frac{1}{\lambda_2 - \lambda_1} \int_0^t (e^{(t-u)\lambda_2} - e^{(t-u)\lambda_1}) f(u) du. \tag{1.3}$$

Example 1.1. We use (1.3) to find the general solution to the differential equation

$$y'' - 3y' + 2y = te^t.$$

First we find y_h. Setting $s^2 - 3s + 2 = 0$ we obtain $s = 1, 2$. Thus $y_h = k_1 e^t + k_2 e^{2t}$. Next we let $\lambda_1 = 1, \lambda_2 = 2$ and use (1.3) to get

$$\begin{aligned} y_p(t) &= \int_0^t (e^{2(t-u)} - e^{(t-u)}) u e^u du \\ &= e^{2t} \int_0^t u e^{-u} du - e^t \int_0^t u du \\ &= e^{2t} \left[-u e^{-u} \Big|_0^t + \int_0^t e^{-u} du \right] - \frac{1}{2} t^2 e^t \\ &= e^{2t}(-t e^{-t} - e^{-t} + 1) - \frac{1}{2} t^2 e^t \\ &= -(t^2/2 + t) e^t + e^{2t} - e^t. \end{aligned}$$

(Note that since $e^{2t} - e^t$ satisfies the corresponding homogeneous equation, one can also use $-(t^2/2 + t)e^t$ as a particular solution).

Note that (1.3) holds even when the eigenvalues are complex $\lambda = \alpha \pm \beta i$. We can also represent (0.2) using α and β: let $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, then

$$\lambda_2 - \lambda_1 = -2\beta i$$

and by Euler's formula

$$\begin{aligned} & e^{(t-u)\lambda_2} - e^{(t-u)\lambda_1} \\ &= e^{(t-u)(\alpha - \beta i)} - e^{(t-u)(\alpha + \beta i)} \\ &= e^{\alpha(t-u)} (e^{-\beta(t-u)i} - e^{\beta(t-u)i}) \\ &= e^{\alpha(t-u)} \big(\cos(-\beta(t-u)) + i \sin(-\beta(t-u)) - \cos(\beta(t-u)) - i \sin(\beta(t-u)) \big) \\ &= -2i e^{\alpha(t-u)} \sin(\beta(t-u)). \end{aligned}$$

Together with (1.3), we have

$$y_p(t) = \frac{1}{\beta} \int_0^t e^{\alpha(t-u)} \sin(\beta(t-u)) f(u) du. \tag{1.4}$$

Lastly we deal with the situation for repeated eigenvalues. Let λ be the repeated eigenvalues. Then

$$y_p(t) = \int_0^t (t-u) e^{\lambda(t-u)} f(u) du \tag{1.5}$$

is a particular solution to (1.1). This can be checked directly, but one can also give the following intuitive argument: for any $\beta \neq 0$,

$$y_p(t) = \int_0^t e^{\lambda(t-u)} \frac{\sin(\beta(t-u))}{\beta} f(u) du$$

is a particular solution when (1.1) has complex eigenvalues $\lambda \pm \beta i$. The expression (1.5) can be found by taking $\beta \to 0$: when $\beta \to 0$, we have

$$\frac{\sin(\beta(t-u))}{\beta} = (t-u)\frac{\sin(\beta(t-u))}{\beta(t-u)} \to (t-u).$$

2. Exercises

In the following, use either (1.3), (1.4) or (1.5) to find the general solutions.

(1) $y'' + 2y' - 3y = te^{2t}$.
(2) $y'' + 2y' - 8y = e^{-2t}\sin t$.
(3) $y'' + 2y' + 2y = t^2$.
(4) $y'' - 3y' - 15y = t^2 e^t$.
(5) $y'' - 2y' + y = \frac{e^t}{1+t^2}$.