252 Elementary differential equations
Fall 2019
Midterm 1
10/10/2019
Time Limit: 80 Minutes

Last Name: ______________________

First Name: ______________________

RUID: ______________________

Instruction

- **This exam contains 7 pages (including this cover page) and 11 questions.**
- **Total of points is 80.**
- **You may not use any notes on this exam.**
- **You may not use calculators on this exam.**
- **You must show your work and carefully justify your answers when answering the questions. The correct answer without any work will receive little or no credit.**
- **If you need more room, use the backs of the pages and indicate clearly that you have done so.**

Grade Table (for teacher use only)

Question	Points	Score
1	8	
2	6	
3	6	
4	8	
5	6	
6	8	
7	8	
8	6	
9	6	
10	10	
11	8	
Total:	80	

1. (8 points) Solve the initial value problem

$$y' = 2ty^2 + 3t^2y^2, \quad y(0) = -1.$$

Solution:

$$\begin{aligned} y' &= y^2(2t + 3t^2) \\ \frac{y'}{y^2} &= 2t + 3t^2 \\ -\frac{1}{y} &= t^2 + t^3 + C, \\ y(t) &= \frac{-1}{t^2 + t^3 + C}. \end{aligned}$$

$y(0) = -1$ **implies** $-1 = \frac{-1}{0+C}$**. So** $C = 1$ **and** $y(t) = \frac{-1}{t^2+t^3+1}$ **solves the IVP.**

2. (6 points) Given the differential equation

$$y' = \sin y \; (\cos^2(\pi y^2) - e^{-y^3+1}).$$

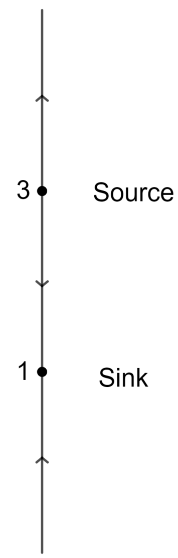
Note that $y = 0$ **is an equilibrium value. Classify if this value is a sink, a source of a node.**

Solution: Write $f(y) = \sin y \; (\cos^2(\pi y^2) - e^{-y^3+1})$**, then**

$$\begin{aligned} f'(y) &= \cos y(\cos^2(\pi y^2) - e^{-y^3+1}) + \sin y(\cos^2(\pi y^2) - e^{-y^3+1})' \\ \Rightarrow f'(0) &= \cos 0(\cos^2 0 - e^{0+1}) + 0 \\ &= 1 - e < 0. \end{aligned}$$

Thus $y = 0$ **is a sink.**

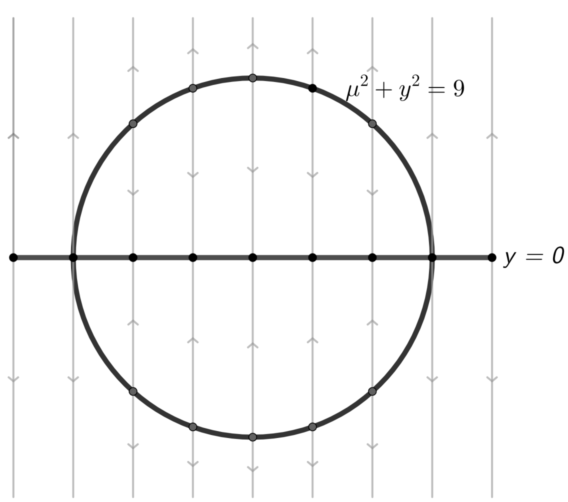
3. (6 points) Sketch the phase lines of the differential equation $y' = (y-1)(y-3)$. Locate all equilibrium values. For each equilibrium values, states if it is a sink, a source or a node.



4. (8 points) Sketch the bifurcation diagram of the following one parameter family of differential equations

$$y' = y(y^2 + \mu^2 - 9).$$

Find all bifurcation values.



The bifurcation values are $\mu = 3, -3$.

5. **(6 points) Find two different solutions $y_1(t), y_2(t)$ to the initial value problem**

$$\begin{cases} y' = 2y^{1/2} \\ y(0) = 0. \end{cases}$$

Why does it not contradicting the comparison principle?

Solution: First $y_1(t) = 0$ is obviously a solution. By separation of variables,

$$\frac{1}{2y^{1/2}} y' = 1 \Rightarrow y^{1/2} = t + C \Rightarrow y = (t+C)^2.$$

With the initial condition $y(0) = 0$, then $C = 0$ and $y_2(t) = t^2$ is another solution.

Note that the differential equation $y' = 2y^{1/2}$ has $f(y) = 2y^{1/2}$. Since $f'(y) = y^{-1/2}$ is NOT continuous at $y = 0$, one cannot apply comparison principle to this differential equation.

6. **(8 points) Find the general solution to the differential equation**

$$y' = -\frac{y}{1+t} + 2.$$

Solution: Write $y' + \frac{y}{1+t} = 2$. The integration factor is

$$\mu = e^{\int \frac{dt}{1+t}} = e^{ln(1+t)} = 1 + t$$

So

$$\begin{aligned} (1+t)y' + y &= 2(1+t) \\ ((1+t)y)' &= 2 + 2t \\ (1+t)y &= 2t + t^2 + C \\ y(t) &= \frac{2t + t^2 + C}{1+t}. \end{aligned}$$

7. (8 points) Find the general solutions to the differential equation

$$y' = 3y + \sin 2t.$$

Solution: The general solution is given by $y = y_h + y_p$, where $y_h = Ce^{3t}$. To find y_p, let

$$y_p = A\sin 2t + B\cos 2t \Rightarrow y_p' = 2A\cos 2t - 2B\sin 2t.$$

Thus

$$\begin{aligned} y_p' - 3y_p &= 2A\cos 2t - 2B\sin 2t - 3(A\sin 2t + B\cos 2t) \\ &= (-3A - 2B)\sin 2t + (2A - 3B)\cos 2t. \end{aligned}$$

Then we choose

$$\begin{cases} -3A - 2B = 1, \\ 2A - 3B = 0. \end{cases} \quad \Rightarrow A = -\frac{3}{13}, B = -\frac{2}{13}.$$

Thus the general solution is $y(t)Ce^{3t} - \frac{3}{13}\sin 2t - \frac{2}{13}\cos 2t$.

8. (6 points) Sketch the phase plane of the following systems

$$\begin{cases} \frac{dx}{dt} = -2y(1 + x^2 + y^2) \\ \frac{dy}{dt} = 2x(1 + x^2 + y^2). \end{cases}$$

Solution: Using Chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = -\frac{x}{y} \Rightarrow y\frac{dy}{dx} = -x.$$

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, or $x^2 + y^2 = C$. Thus the phase plane is a union of circles.

9. (6 points) Use the Euler method with $\Delta t = 1$ to approximate $y(3)$, where $y(t)$ satisfies $y' = t + y^2$ and $y(0) = 1$.

Solution: $y_{k+1} = y_k + \Delta t f(t_k, y_k)$. **With** $\Delta t = 1$, $t_0 = 0$, $y_0 = 1$,

$$\begin{aligned} y_1 &= 1 + 1(0 + 1^2) = 2, \\ y_2 &= 2 + 1(1 + 2^2) = 7, \\ y_3 &= 7 + 1(2 + 7^2) = 58. \end{aligned}$$

Thus $y(3)$ **is approximated by** 58.

10. (10 points) Find the general solution to the following system of differential equations

$$\begin{cases} x' &= 2x \\ y' &= xy. \end{cases}$$

Solution: First, solving $x' = 2x$ **gives** $x = C_1 e^{2t}$. **Put this into the second equation. Thus**

$$\begin{aligned} y' &= C_1 e^{2t} y \\ \Rightarrow y &= C_2 e^{C_1 e^{2t}}. \end{aligned}$$

11. (8 points) The Florida Fish and Wildlife Conversation Commission (FWC) plans to hire hunters to kill the Burmese python, an invasive species in Everglades, South Florida. It is estimated that there are 100,000 pythons in Everglades and their population is governed by

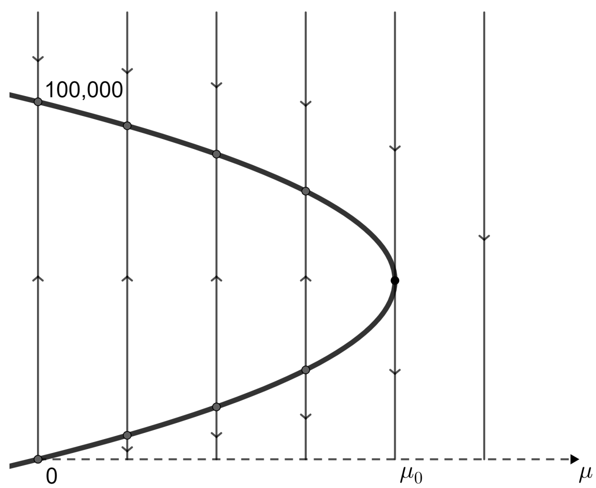
$$P' = \frac{P}{10,000}(100,000 - P).$$

Assume that each hunter kills 10 pythons everyday. At least how many hunters should FWC hire in order to completely remove the python from the area?

Solution: Let μ be the number of hunters hired. Then the equation becomes

$$P' = \frac{P}{10,000}(100,000 - P) - 10\mu.$$

Treat μ as the parameter, we have the following bifurcation diagram



From the diagram, one observe that if $\mu \geq \mu_0$, then the phase line has no equilibrium and thus P decrease to 0 in finite time. To find μ_0, note that the maximum of the parabola is attained when $P = 50,000$. Then

$$10\mu_0 = \frac{50,000}{10,000}(100,000 - 50,000) \Rightarrow \mu_0 = 25,000.$$

Thus at least $25,000$ hunters must be hired to kill all the pythons.

End of Midterm