SPRING 20 ELEMENTARY DIFFERENTIAL EQUATIONS: TAKE HOME MIDTERM 2 SOLUTION

1. (6 marks) Find the equilibria for the following systems of differential equations:
$$x' = (x-1)(2x+y-2),$$
$$y' = y(x+y-3).$$

   Solution: Set
$$(x-1)(2x+y-2) = 0,$$
$$y(x+y-3) = 0.$$

   Which implies
$$x = 1 \text{ or } 2x+y-2 = 0,$$
$$y = 0 \text{ or } x+y-3 = 0.$$

   So there are four cases:
   - $x = 1$ and $y = 0 \Rightarrow (1,0)$ is an equilibrium.
   - $x = 1$ and $x+y-3 = 0 \Rightarrow (1,2)$ is an equilibrium.
   - $2x+y-2 = 0$ and $y = 0 \Rightarrow (1,0)$ is an equilibrium.
   - $2x+y-2 = 0$ and $x+y-3 = 0 \Rightarrow (-1,4)$ is an equilibrium.

   Thus there are three equilibria: $(1,0),(1,2)$ and $(-1,4)$.

2. (4 marks) Transform the following second order differential equation into a systems of two differential equations:
$$y'' + y^2y' + y'\sin t + t^2 = 0.$$

   Solution: Set $v = y'$, so
$$v' = y'' = -y^2y' - y'\sin t - t^2 = -y^2v - v\sin t - t^2.$$

   Thus the corresponding first order system is
$$y' = v,$$
$$v' = -y^2v - v\sin t - t^2.$$

3. (8 marks) Find the general solution to the following system of differential equations
$$x' = 2x,$$
$$y' = 4y + 2x^2.$$

Solution: Solving the first equation gives

$$x = C_1 e^{2t}.$$

Plugging into the second equation gives

$$y' = 4y + 2C_1^2 e^{4t}.$$

We use the guessing method: write

$$y = y_h + y_p,$$

where $y_h = C_2 e^{4t}$ and $y_p = Ate^{4t}$. Then

$$y_p' = Ae^{4t} + 4Ate^{4t}.$$

So

$$\begin{aligned} y_p' - 4y_p &= Ae^{4t} + 4Ate^{4t} - 4Ate^{4t} \\ &= Ae^{4t}. \end{aligned}$$

Thus we choose $A = 2C_1^2$. Thus the general solution is

$$\begin{aligned} x &= C_1 e^{2t}, \\ y &= C_2 e^{4t} + 2C_1^2 te^{4t}. \end{aligned}$$

4. (6 marks) Let $Y_1(t), Y_2(t)$ be two solutions to an autonomous system $Y' = F(Y)$, where F and the partial derivatives of F are both continuous. If $Y_2(0) = Y_1(2)$, what is the relationship between Y_1 and Y_2. Please explain your answer.

   Solution: Let $Y_3(t) = Y_1(t+2)$. When $Y_3(t)$ is also a solution to $Y' = F(Y)$ since the system is autonomous. Also,

$$Y_3(0) = Y_1(0+2) = Y_1(2) = Y_2(0).$$

   Thus Y_3 and Y_2 agree when $t = 0$. Since F and the first derivatives of F are continuous, one can apply the uniqueness theorem for system to conclude that $Y_3(t) = Y_2(t)$ for all t. Thus

$$Y_2(t) = Y_1(t+2).$$

5. (6 marks) Calculate e^A, where A is the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix}.$$

   Solution: The characteristic polynomials is

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 - 5\lambda + 5$$

thus by the quadratic formula we have

$$\lambda_1 = \frac{1}{2}(5 - \sqrt{5}), \quad \lambda_2 = \frac{1}{2}(5 + \sqrt{5}).$$

To find the corresponding eigenvectors v_1, v_2,

$$A - \lambda_1 I = \begin{pmatrix} -\frac{1}{2}(3+\sqrt{5}) & 1 \\ -1 & \frac{1}{2}(3-\sqrt{5}) \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ \frac{1}{2}(3+\sqrt{5}) \end{pmatrix},$$
$$A - \lambda_2 I = \begin{pmatrix} -\frac{1}{2}(3-\sqrt{5}) & 1 \\ -1 & \frac{1}{2}(3+\sqrt{5}) \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ \frac{1}{2}(3-\sqrt{5}) \end{pmatrix}.$$

Thus

$$\begin{aligned} A &= S\Lambda S^{-1} \\ &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2}(5-\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(5+\sqrt{5}) \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1} \\ \Rightarrow e^A &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}(5-\sqrt{5})} & 0 \\ 0 & e^{\frac{1}{2}(5+\sqrt{5})} \end{pmatrix} \begin{pmatrix} v_1 & v_2 \end{pmatrix}^{-1}. \end{aligned}$$

6. (6 marks) Calculate e^{tB}, where

$$B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Solution: Write I as the identity matrix and

$$U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then

$$B = 2I + U.$$

Then since $U^2 = 0$ (the zero matrix) and $UB = 2U$ (direct calculation),

$$\begin{aligned} B^2 &= (2I + U)(2I + U) \\ &= 4I + 2IU + 2UI + U^2 \\ &= 4I + 4U = 4(I + U). \end{aligned}$$

and

$$\begin{aligned} B^3 &= B^2 B = 4(I + U)(2I + U) \\ &= 4(2I + 2UI + IU) = 8I + 4 \cdot 3U. \end{aligned}$$

and similarly

$$\begin{aligned} B^4 &= B^3 B = (8I + 4 \cdot 3U)(2I + U) \\ &= 16I + 8 \cdot 4U \end{aligned}$$

in general if $B^n = 2^n I + 2^{n-1} n U$ for some n, then .

$$\begin{aligned} B^{n+1} &= B^n B \\ &= (2^n I + 2^{n-1} n U)(2I + U) \\ &= 2^{n+1} I + 2^n I U + 2^{n-1} n U (2I) + U^2 \\ &= 2^{n+1} I + 2^n (n+1) U. \end{aligned}$$

Thus we have $B^n = 2^n I + 2^{n-1} n U$ for all n. Then

$$\begin{aligned} e^{tB} &= I + tB + \frac{t^2 B^2}{2} + \cdots + \frac{t^n B^n}{n!} + \cdots \\ &= I + t(2I + U) + \frac{t^2(4I + 4U)}{2} + \cdots + \frac{t^n(2^n I + 2^{n-1} n U)}{n!} + \cdots \\ &= \left(1 + 2t + \frac{(2t)^2}{2} + \cdots + \frac{(2t)^n}{n!} + \cdots\right) I \\ &\quad + \left(t + 2t^2 + \cdots + \frac{2^{n-1} t^n}{(n-1)!} + \cdots\right) U \\ &= e^{2t} I + t\left(1 + (2t) + \cdots + \frac{(2t)^{n-1}}{(n-1)!} + \cdots\right) U \\ &= e^{2t} I + t e^{2t} U \\ &= \begin{pmatrix} e^{2t} & t e^{2t} \\ 0 & e^{2t} \end{pmatrix}. \end{aligned}$$

7. (6 marks) Let

$$Y_1(t) = \begin{pmatrix} 2e^t - e^{-2t} \\ e^t + e^{-2t} \end{pmatrix}, \quad Y_2(t) = \begin{pmatrix} -2e^t + 2e^{-2t} \\ -e^t - 2e^{-2t} \end{pmatrix}$$

be two solutions to a linear system $Y' = AY$. Solve the initial value problem

$$\begin{cases} Y' = AY, \\ Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{cases}$$

Solution: Let $Y(t) = k_1 Y_1(t) + k_2 Y_2(t)$. If we plug in the initial condition, we have

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ -3 \end{pmatrix},$$

which gives the equations

$$1 = k_1, 1 = 2k_1 - 3k_2$$

and thus $k_1 = 1, k_2 = 1/3$. Then

$$\begin{pmatrix} 2e^t - e^{-2t} \\ e^t + e^{-2t} \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -2e^t + 2e^{-2t} \\ -e^t - 2e^{-2t} \end{pmatrix}$$

solves the IVP.

8. (6 marks) Let v be a nonzero vector so that $e^{\lambda t}v$ is a solution to the system $Y' = AY$. Show that λ is an eigenvalue of A with eigenvector v.

Solution: It is given that $e^{\lambda t}v$ satisfies $Y' = AY$. Thus

$$\begin{aligned}(e^{\lambda t}v)' &= A(e^{\lambda t}v)\\ \lambda e^{\lambda t}v &= e^{\lambda t}Av\\ e^{\lambda t}\lambda v &= e^{\lambda t}Av.\end{aligned}$$

In particular, $Av = \lambda v$. Since v is a non-zero eigenvector, λ is an eigenvalue of A with eigenvector v.

9. (8 marks) Solve the initial value problem

$$Y' = \begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} Y, \quad Y(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Solution: First we find the eigenvalues:

$$\begin{aligned}p(\lambda) &= \det \begin{pmatrix} 1-\lambda & -2 \\ 4 & 3-\lambda \end{pmatrix}\\ &= \lambda^2 - 4\lambda + 11\\ &= (\lambda - 2)^2 + 7.\end{aligned}$$

Thus A has complex eigenvalues $2 \pm \sqrt{7}i$. Let $\lambda = 2 + \sqrt{7}i$, then

$$A - \lambda I = \begin{pmatrix} -1-\sqrt{7}i & -2 \\ 4 & 1-\sqrt{7}i \end{pmatrix}$$

Thus

$$v = \begin{pmatrix} -2 \\ 1+\sqrt{7}i \end{pmatrix}$$

is an eigenvector. Then

$$\begin{aligned}e^{\lambda t}v &= e^{(2+\sqrt{7}i)t} \begin{pmatrix} -2 \\ 1+\sqrt{7}i \end{pmatrix}\\ &= e^{2t}(\cos\sqrt{7}t + i\sin\sqrt{7}t) \begin{pmatrix} -2 \\ 1+\sqrt{7}i \end{pmatrix}\\ &= \begin{pmatrix} -2e^{2t}\cos\sqrt{7}t \\ e^{2t}(\cos\sqrt{7}t - \sqrt{7}\sin\sqrt{7}t) \end{pmatrix}\\ &\quad + i \begin{pmatrix} -2e^{2t}\sin\sqrt{7}t \\ e^{2t}(\sin\sqrt{7}t + \sqrt{7}\cos\sqrt{7}t) \end{pmatrix}\\ &= Y_{re}(t) + iY_{im}(t),\end{aligned}$$

and the general solution is $k_1 Y_{re}(t) + k_2 Y_{im}(t)$. To solve the IVP, set

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ \sqrt{7} \end{pmatrix}$$

which gives $k_1 = -1$, $k_2 = 4/\sqrt{7}$.

10. (8 marks) Sketch the phase portrait of the following system:

$$Y' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Y.$$

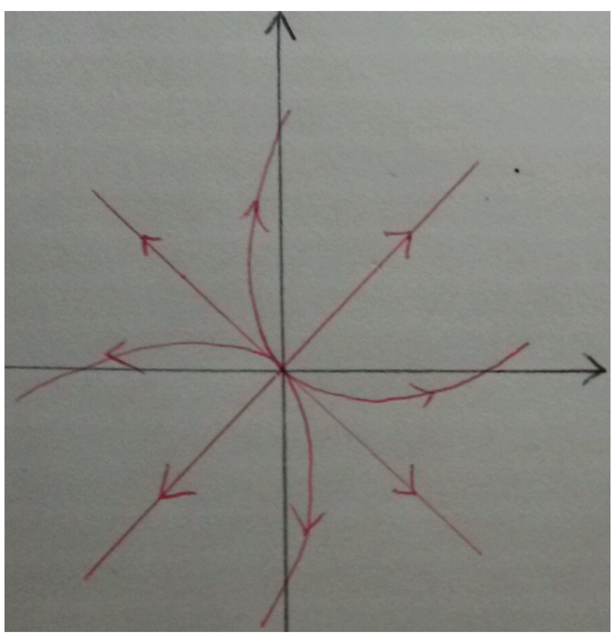
Solution: First we calculate the eigenvalues:

$$\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

Thus $\lambda_1 = 3$ and $\lambda_2 = 1$ are the eigenvalues. To find the corresponding eigenvectors,

$$A - \lambda_1 I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$



11. (8 marks) Sketch the phase portrait of the following system:

$$Y' = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} Y.$$

Solution: First we calculate the eigenvalues:

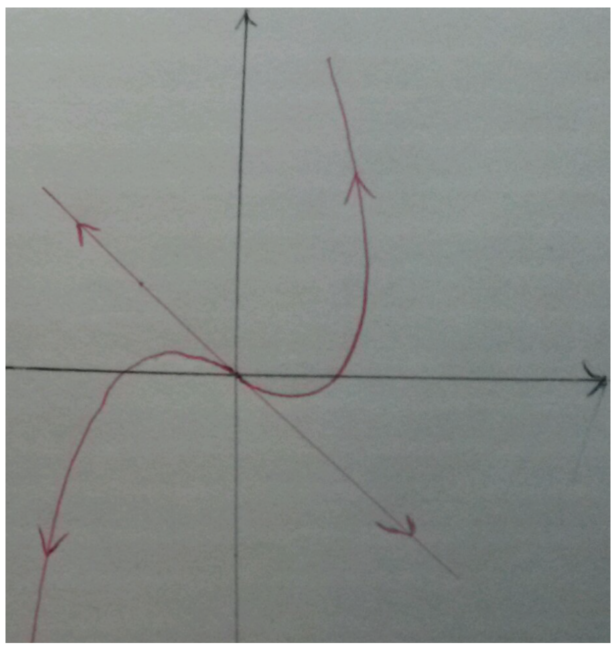
$$\det(A - \lambda I) = \begin{pmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2.$$

Thus A has a repeated eigenvalues $\lambda = 3$. To find the corresponding eigenvectors,

$$A - \lambda_1 I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

To check the orientation, consider a point away from the straight line solution, say, $Y = (1,1)$. Then

$$AY = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$



12. (8 marks) Let A be a 2×2 matrix with eigenvalues λ_1, λ_2. What are the conditions on λ_1, λ_2, so that all solutions to $Y' = AY$ converges to $(0,0)$ as t goes to $+\infty$? Please explain your answer.

Solution: First of all, λ_i must be nonzero: if one of λ_i is zero, then $Av = 0$ for some nonzero vector v. Then $Y(t) = v$ is a constant solutions (equilirium) to $Y' = AY$, which does not converge to the origin.

When both of the eigenvalues are non-zero, we split into three cases:

- When λ_1, λ_2 are distinct, real: then the general solution is

$$Y(t) = k_1 e^{\lambda_1 t} v_1 + k_2 e^{\lambda_2 t} v_2$$

and all such $Y(t)$ converges to the origin $(0,0)$ as $t \to +\infty$ if and only if both λ_1, λ_2 are negative.

- When $\lambda = \lambda_1 = \lambda_2$ (i.e. repeated eigenvalues), then the general solution is
$$Y(t) = e^{\lambda t} W_0 + t e^{\lambda t} W_1, \quad W_1 = (A - \lambda I) W_0.$$
In this cases $Y(t)$ converges to the origin $(0, 0)$ as $t \to +\infty$ if and only if λ is negative, since $te^{\lambda t}$ also converge to 0 as $t \to +\infty$.
- Lastly, when the eigenvalues are complex, write $\lambda = \alpha + \beta i$ (the other eigenvalues is $\alpha - \beta i$), then the general solution is
$$Y(t) = k_1 Y_{re}(t) + k_2 Y_{im}(t).$$
Where $e^{\lambda t} v = Y_{re}(t) + i Y_{im}(t)$. Since both $Y_{re}(t), Y_i m(t)$ are of the form $e^{\alpha t} Y_p(t)$, where Y_p is some periodic functions, we see that $Y(t)$ converges to the origin $(0, 0)$ as $t \to +\infty$ if and only if α is negative.

We can sum up and say that all solutions converge to $(0, 0)$ as $t \to +\infty$ if and only if the real parts of the eigenvalues are negative.