Systems of D.E.s.

1st order systems of D.E.s.

$$(*)\begin{cases} x' = f(t,x,y) \\ y' = g(t,x,y). \end{cases}$$

Where t is the variable, $f(t,x,y)$, $g(t,x,y)$ are given functions, and ~~$x(t)$~~ $x = x(t)$, $y = y(t)$ are to be determined.

$(x(t), y(t))$ is called a solution to $(*)$, if

$$\begin{cases} x'(t) = f(t, x(t), y(t)) \\ y'(t) = g(t, x(t), y(t)) \end{cases}$$

are both satisfied for all t (in the domain of $x(t)$, $y(t)$).

e.g.
$$\begin{cases} x' = x^2 + y^2 + \sin(tx) \\ y' = x^3 - y + e^t + 1. \end{cases}$$

e.g. Predator-Prey system

$$\begin{cases} R' = 2R - 1.2RF, \\ F' = -F + 0.9RF. \end{cases}$$

Equivalence between Higher order D.E.s and 1st order system:

Given a ~~higher~~ second order D.E.

$$y'' = f(t, y, y'),$$

one can defined an extra variable

$$v = y',$$

and write

$$\begin{cases} y' = v \\ v' = f(t, y, v) \end{cases}$$

$$(\because v' = (y')' = y'' = f(t, y, y') = f(t, y, v))$$

e.g. The equation

$$y'' = ky$$

is equivalent to the system

$$\begin{cases} y' = v \\ v' = ky. \end{cases}$$

Checking solution:

$$\begin{cases} x' = -x+y \\ y' = -3x-5y. \end{cases}$$

Check: $(x(t), y(t)) = (e^{-4t} - 3e^{-2t}, -3e^{-4t} + 3e^{-2t})$

is a solution:

$$x' = -4e^{-4t} + 6e^{-2t}.$$

$$-x+y = (-e^{-4t} + 3e^{-2t}) + (-3e^{-4t} + 3e^{-2t})$$

$$= -4e^{-4t} + 6e^{-2t}.$$

$$\Longrightarrow \quad x' = -x+y.$$

(Similarly $y' = -3x - 5y$.

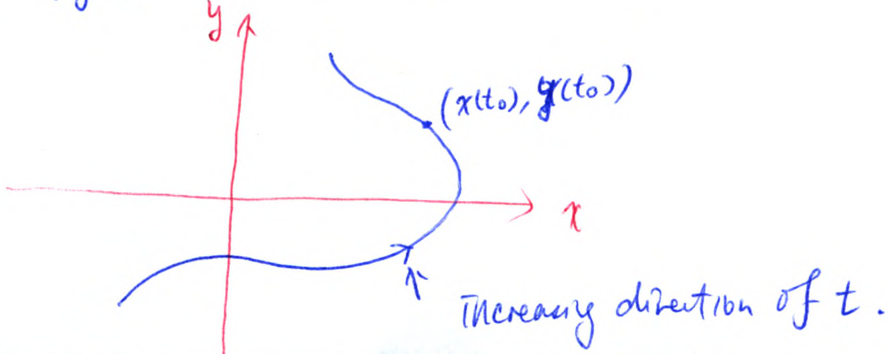
Geometric Interpretations:

Write $Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $F(Y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$, $Y'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$.

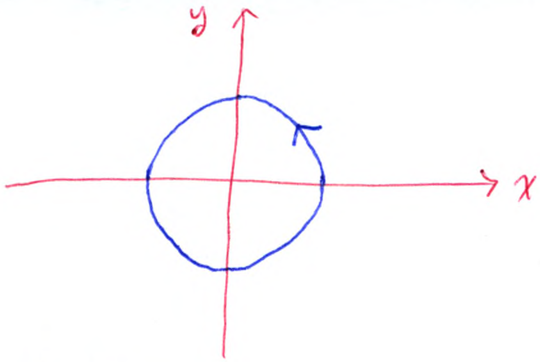
Q: How to interpret

$$Y'(t) = \vec{F}(Y) \quad ?$$

① $Y(t)$: Think of $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ as a curve in the x-y plane:

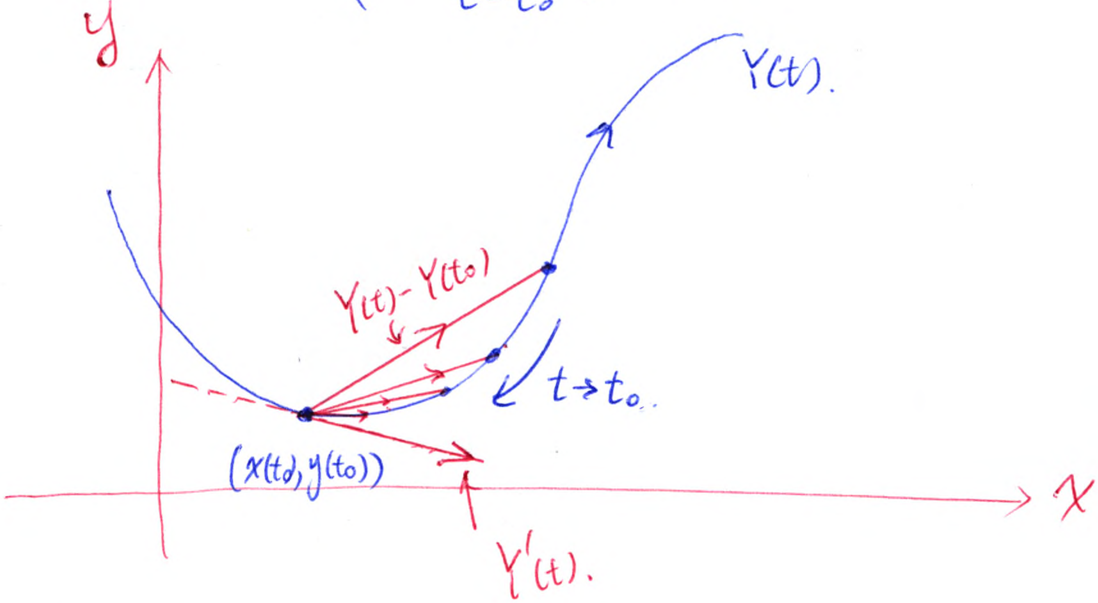


e.g. $\quad Y(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

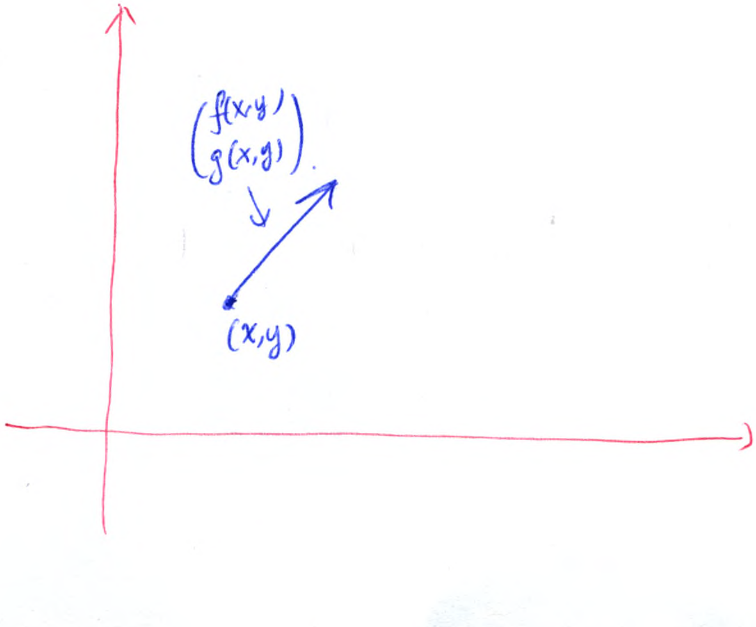


② $Y'(t)$: tangent vectors at $Y(t)$.

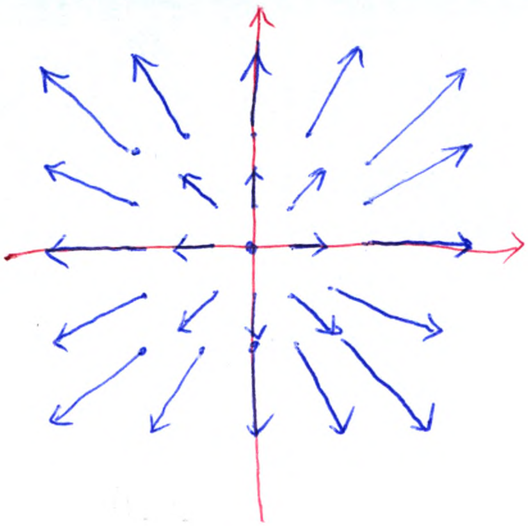
Reason: $Y'(t_0) = \lim_{t \to t_0} \begin{pmatrix} \frac{x(t) - x(t_0)}{t - t_0} \\ \frac{y(t) - y(t_0)}{t - t_0} \end{pmatrix} = \lim_{t \to t_0} \frac{1}{t - t_0} \left(\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} \right)$.



③ $F(Y)$: Vector fields : at each $Y = \begin{pmatrix} x \\ y \end{pmatrix}$, attain to it the vector $\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$.

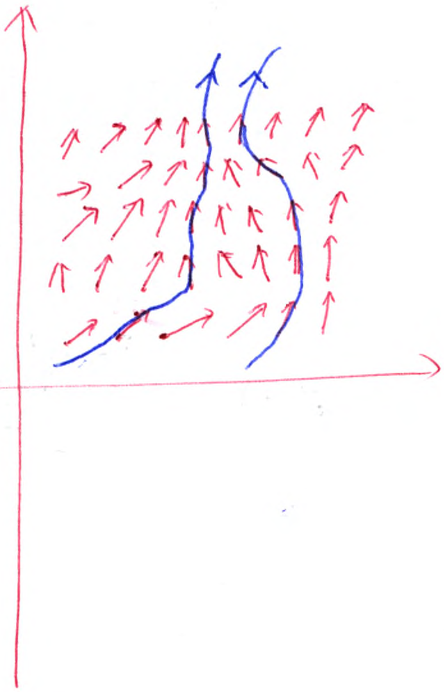


e.g. $\vec{F}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$.



$Y'(t) = \vec{F}(Y)$:

$Y'(t)$ tangent vectors.



i.e. the curve $Y(t)$ moves along the vector field.