Repeated eigenvalues:

Consider $Y' = AY$,

where A has repeated eigenvalues

e.g. $A = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix}$ $\quad p(\lambda) = \det(A - \lambda I)$

$$= (-5-\lambda)(-3-\lambda) + 1$$

$$= \lambda^2 + 8\lambda + 16 = (\lambda+4)^2$$

$\lambda = -4$ is a double root.

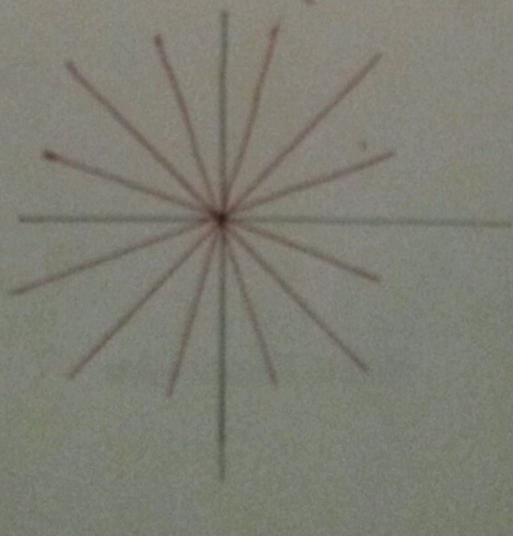
Simplest case:

$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

- $Av = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} v = \lambda v$ for all ~~eigenve~~ vectors v.

$\Longrightarrow$ All nonzero vectors are eigenvectors.

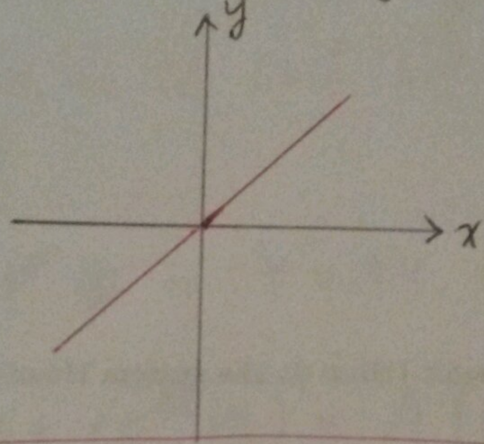
$\Longrightarrow$ All solutions to $Y' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} Y$ are straight line solutions.



In general, when λ is repeated, one might $\boxed{\text{not}}$ have "2" eigenvectors

e.g. $A = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix}$ has repeated eigenvalues $\lambda = -4$,

but just 1 direction of eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or $\begin{pmatrix} c \\ c \end{pmatrix}$)



(Only "one" straight line solution)

General solution:

Start with a simple example:

$$Y' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} Y. \iff \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2 is a repeated eigenvalue.

$$\iff \begin{aligned} x' &= 2x + y \\ y' &= 2y \end{aligned}$$ (decoupled!)

Solving $y' = 2y$ gives

$y(t) = C_1 e^{2t}$.

$$\Rightarrow x' = \boxed{2}x + C_1 e^{\boxed{2}t} \Rightarrow x' = \underbrace{C_2 e^{2t}}_{x_h} + C \underbrace{te^{2t}}_{x_p}$$

the same.

te^{2t} shows up!

It suggests the following "guessing": the general solution to $Y' = AY$ when A has repeated eigenvalues λ is

$$Y(t) = e^{\lambda t} W_0 + t e^{\lambda t} W_1 \quad \cdots \quad (*)$$

for some choices of W_0, W_1.

Q: Conditions on W_0, W_1 such that $Y(t)$ in $(*)$ satisfies $Y' = AY$?

A: Plug in $(*)$ to $Y' = AY$:

$$(e^{\lambda t} W_0 + t e^{\lambda t} W_1)' = A(e^{\lambda t} W_0 + t e^{\lambda t} W_1)$$

$$\Rightarrow \quad \lambda \cancel{e^{\lambda t}} W_0 + (\cancel{e^{\lambda t}} + \lambda t \cancel{e^{\lambda t}}) W_1 = \cancel{e^{\lambda t}} A W_0 + t \cancel{e^{\lambda t}} A W_1$$

$$\Rightarrow \quad \lambda W_0 + W_1 + \lambda t W_1 = A W_0 + t A W_1.$$

Comparing constant and t-coefficients:

$$\begin{cases} \lambda W_0 + W_1 = A W_0, \\ \lambda W_1 = A W_1 \end{cases}$$

or

$$\begin{cases} W_1 = (A - \lambda I) W_0, \\ 0 = (A - \lambda I) W_1. \end{cases}$$

Note $\quad$ As $(A-\lambda I)W_1 = (A-\lambda I)(A-\lambda I)W_0$

$$= (A-\lambda I)^2 W_0.$$

Using $(A-\lambda I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (Exercise!),

$(A-\lambda I)W_1 = 0$ is always satisfied.

~~i.e.~~

Theorem: Let W_0 be any vector, then

$$Y(t) = e^{\lambda t} W_0 + t e^{\lambda t} W_1$$

with $W_1 = (A-\lambda I)W_0$ satisfies $Y' = AY$,

i.e. - The general solution is

$$Y(t) = e^{\lambda t} W_0 + t e^{\lambda t} W_1$$

with W_0 arbitrary and $W_1 = (A-\lambda I)W_0$.

- Indeed, since

$$Y(0) = e^0 W_0 + 0 e^0 W_1 = W_0,$$

W_0 serves as the initial condition for the IVP.

$$\begin{cases} Y' = AY \\ Y(0) = W_0. \end{cases}$$

Phase portrait: We have

$$Y(t) = e^{\lambda t} W_0 + t e^{\lambda t} W_1$$

We split into 2 cases:

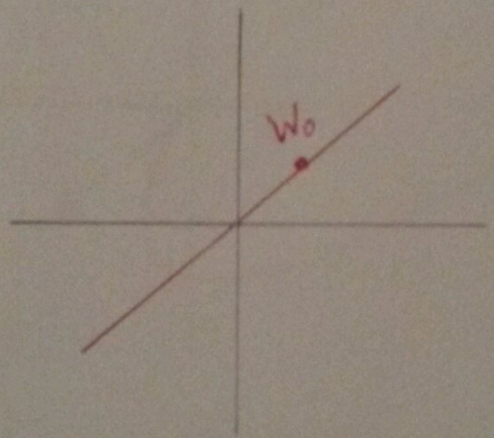
① $W_1 = 0$ (zero vector).

$\Longleftrightarrow (A - \lambda I) W_0 = 0 \Longleftrightarrow A W_0 = \lambda W_0$

So $W_1 = 0 \Longleftrightarrow W_0$ is an eigenvector. and in this case,

$$Y(t) = e^{\lambda t} W_0$$

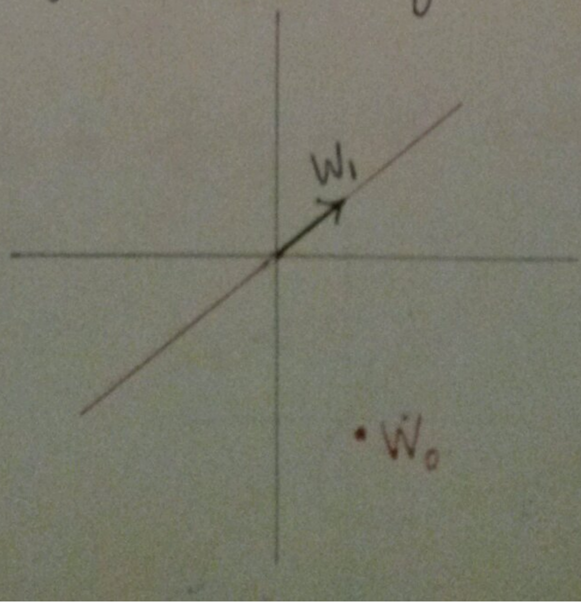
is a straight line solution.



② $W_1 \neq 0$.

In this case W_0 is not an eigenvector.

Note $A W_1 = \lambda W_1$, so W_1 is an eigenvector.



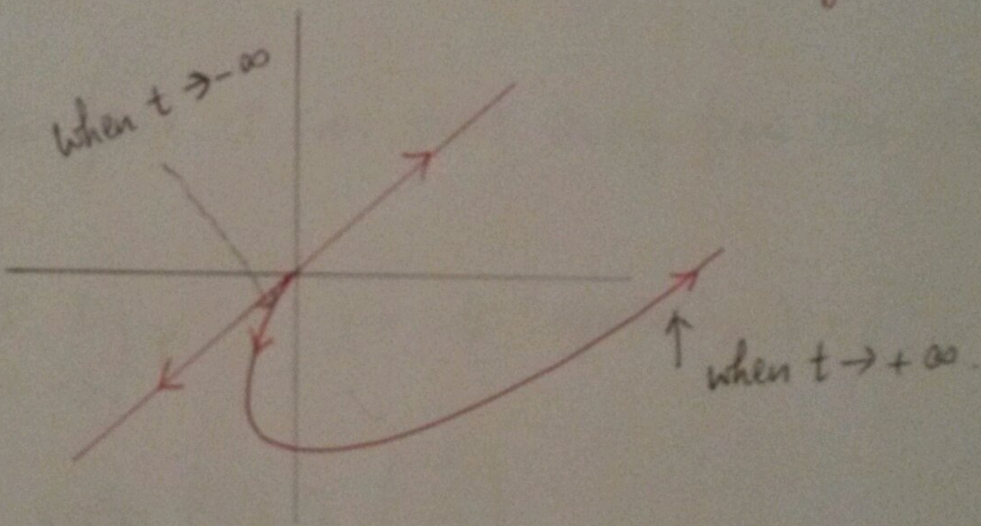
Write

$$Y(t) = e^{\lambda t} W_0 + t e^{\lambda t} W_1 = e^{\lambda t}(W_0 + t W_1).$$

$\Rightarrow$ as $t \to \pm\infty$, $Y(t) \approx e^{\lambda t}(0 + t W_1) = t e^{\lambda t} W_1$.

If $\lambda > 0$, then

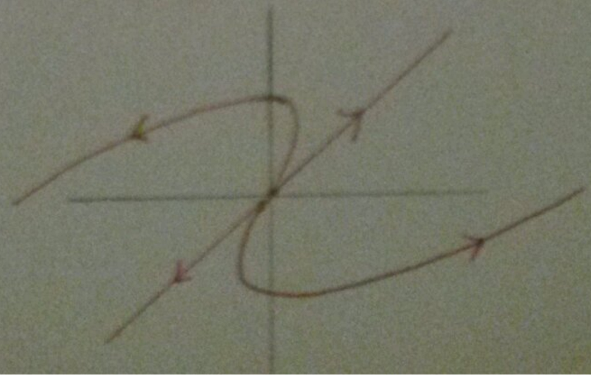
- $t e^{\lambda t} \longrightarrow +\infty$ as $t \to +\infty$
- $t e^{\lambda t} \longrightarrow 0$ as $t \to -\infty$ (checked by L'Hospital Rule)



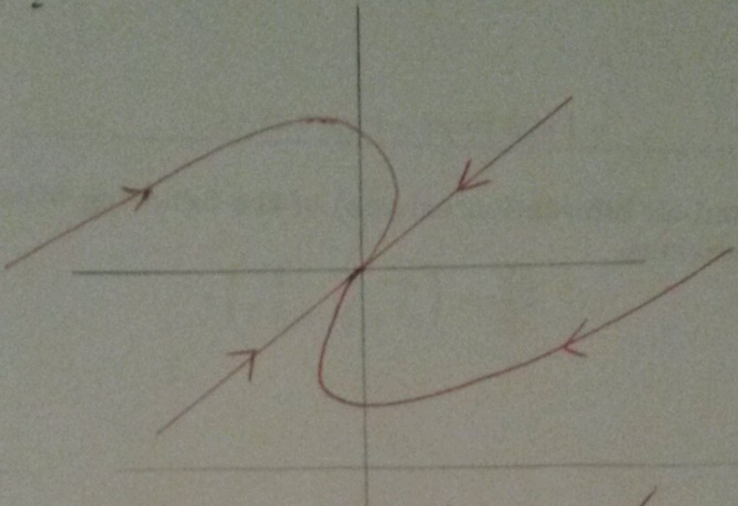
- $Y(t)$
  - tends to $(0,0)$ in the direction of the straight line solution as $t \to -\infty$,
  - Makes a "U" turn.
  - then tends to infinity in the direction of the straight line solution.

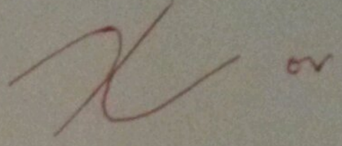
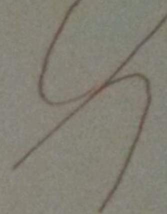
"Source"

Phase portrait:



$\lambda < 0$: similar : "sink".



Orientation : how to tell if it's  or  ?

- Check the vector field at some point away from the straight line solution :

e.g. $Y' = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix} Y$. $\lambda = -4$ (repeated), $V = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

at $(1,0)$, the vector field is $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$