Decoupled System:

General system:

$$\begin{cases} \frac{dx}{dt} = f(x,y), \\ \frac{dy}{dt} = g(x,y). \end{cases}$$

It's "coupled": the ~~solution~~ function y in $f(x,y)$ affects $\frac{dx}{dt}$, and vice versa. The solution $x(t)$, $y(t)$ cannot be found explicitly in general.

Two examples where solutions can be found explicitly:

(I) Completely decoupled system:

$$\begin{cases} \frac{dx}{dt} = f(x). & \leftarrow \text{Nothing to do with } y \\ \frac{dy}{dt} = g(y) & \leftarrow \text{Nothing to do with } x. \end{cases}$$

e.g. $$\begin{cases} \frac{dx}{dt} = x^2+1 \\ \frac{dy}{dt} = 2y \end{cases}$$

$\Rightarrow$ Systems can be solved by solving the equations one by one.

e.g. $\frac{dx}{dt} = x^2+1 \Rightarrow \frac{1}{x^2+1}\frac{dx}{dt} = 1 \Rightarrow \arctan x = t + C_1$

$\Rightarrow x = \tan(t+C_1)$

Note C_1, C_2 are different constants.

$\frac{dy}{dt} = 2y \Rightarrow y = C_2 e^{2t}$

$$\Rightarrow Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \tan(t+C_1) \\ C_2 e^{2t} \end{pmatrix},$$

where C_1, C_2 are ~~two~~ any constants.

(II) Partially decoupled system

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(y) \end{cases} \quad \text{or} \quad \begin{cases} \frac{dx}{dt} = f(x) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

One of the equation is independent of the other.

e.g. $\begin{cases} \frac{dx}{dt} = 2x - y^2 \\ \frac{dy}{dt} = 3y. \end{cases}$ ← Nothing to do with x.

Partially decoupled system can be solved explicitly, ~~they~~

e.g. $\begin{cases} \frac{dx}{dt} = 2x - y^2 \\ \frac{dy}{dt} = 3y \end{cases}$

For the second equation, one finds

$$y(t) = C_1 e^{3t}.$$

Plug this into the the first equation,

$$\frac{dx}{dt} = 2x - (C_1 e^{3t})^2$$
$$= 2x - C_1^2 e^{6t}.$$

It's a first order linear D.E., constant ~~coe~~ linear coefficients

$\Rightarrow$ Guessing: $x_p = Ae^{6t}$.

$\Rightarrow x_p' = 6Ae^{6t}$.

$\Rightarrow x_p' - 2x_p = 4Ae^{6t}$

$\Rightarrow$ Set $4Ae^{6t} = -C_1^2 e^{6t} \Rightarrow A = -\frac{C_1^2}{4}$.

$\Rightarrow x_p = -\frac{C_1^2}{4}e^{6t}$.

$\Rightarrow$ the general solution for $\frac{dx}{dt} = 2x - C_1^2 e^{6t}$ is

$$x = x_h + x_p$$

$$= C_2 e^{2t} - \frac{C_1^2}{4} e^{6t}.$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_2 e^{2t} - \frac{C_1^2}{4} e^{6t} \\ C_1 e^{3t} \end{pmatrix}.$$