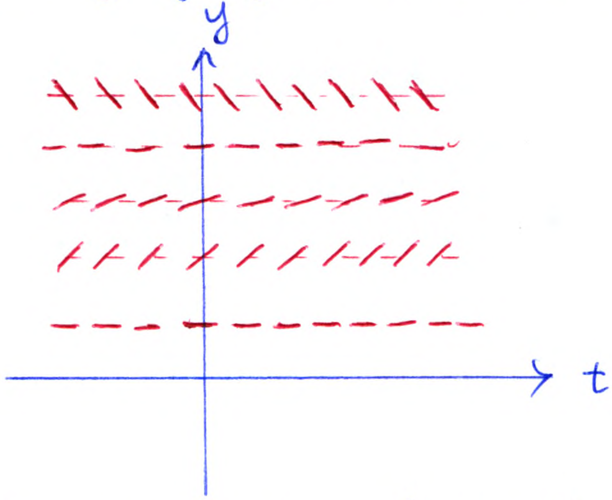
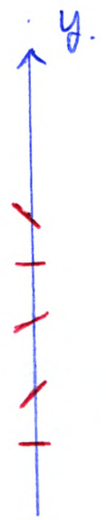
Phase line :

Work for $y' = f(y)$ (i.e. autonomous D.E.) AND that $f, \frac{\partial f}{\partial y}$ are continuous (i.e. Uniqueness theorem, comparison principle holds)

Recall: Slope field of $y' = f(y)$ looks the same horizontally



So it suffices to sketch on (e.g.) the y-axis



To simplify more, we use

- • to denote where $f(y)=0$ —
- ↑ to denote intervals with $f(y)>0$ /
- ↓ to denote intervals with $f(y)<0$ \

↖ Phase line of $y'=f(y)$.

e.g. $y' = y(y-1)(y-2)$.

- $y(y-1)(y-2)=0$ when $y=0,1,2$.
- $y>2 \Rightarrow f(y)>0$
- $2>y>1 \Rightarrow f(y)<0$
- $1>y>0 \Rightarrow f(y)>0$
- $0>y \Rightarrow f(y)<0$.

↖ Phase line of $y'=y(y-1)(y-2)$.

Interpretation:

① At y_0 so that $f(y_0)=0$.

- Called Equilibrium point of $y'=f(y)$
- $y(t) \equiv y_0$ is a solution to the system.

② If $y(t)$ lies in region with $f(y)>0$ $\uparrow$.

- $y'=f(y)>0 \Rightarrow y(t)$ is increasing.

③ If $y(t_0)$ lies between two equilibria y_0, y_1, then the same is true for all t. (Comparison Principle), and $y(t)$ is defined for all $-\infty < t < +\infty$.

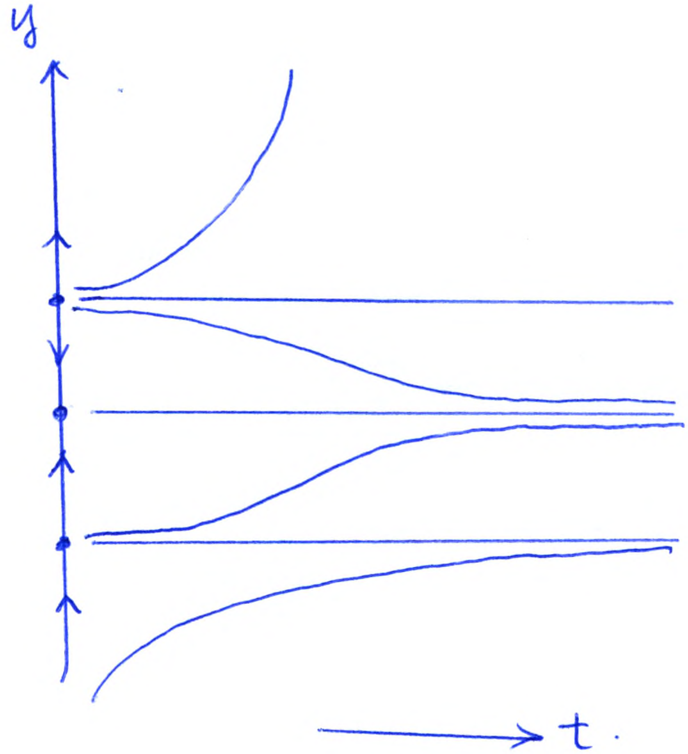
④ As in ③, ~~we have~~ if $y_0 < y(t_0) < y_1$ and $f(y)>0$ in (y_0, y_1),

$$\lim_{t\to+\infty} y(t) = y_1 \ , \ \lim_{t\to-\infty} y(t) = y_0 .$$

Reason: $y(t)$ increasing in (y_0, y_1). then $y' \to 0$ as $t \to +\infty$.
$\Rightarrow f(y(t)) \to 0$ as $t\to+\infty$. i.e. $y(t)$ tends to some y's. with $f(y')=0 \Rightarrow y' = y_1$.

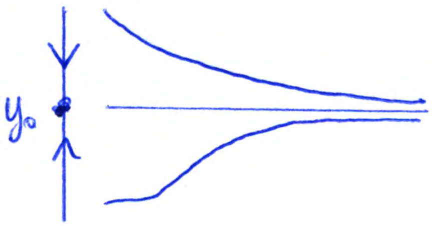
⑤ Similar result of ③, ④ in intervals where $f(y)<0$.

⑥ If $f(y)>0$ in (a,∞). then y is increasing, ~~but~~ and tends to $+\infty$ in either finite or infinite time.



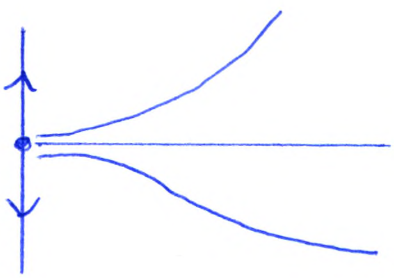
Classification of Equilibria:

- Sink:



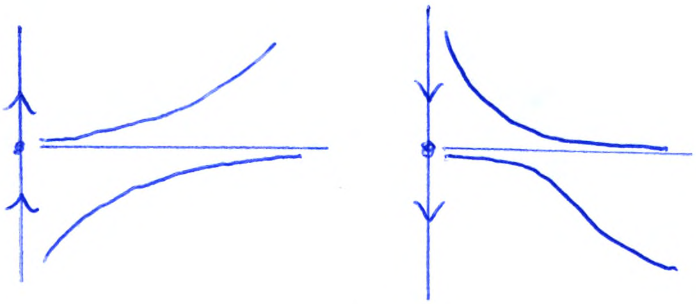
- All solutions $y(t)$ with $y(t_0)$ closed to y_0 tends to y_0 as $\boxed{t \to +\infty}$.

- Source



- All solutions $y(t)$ with $y(t_0)$ closed to y_0 tends to y_0 as $\boxed{t \to -\infty}$

- Node

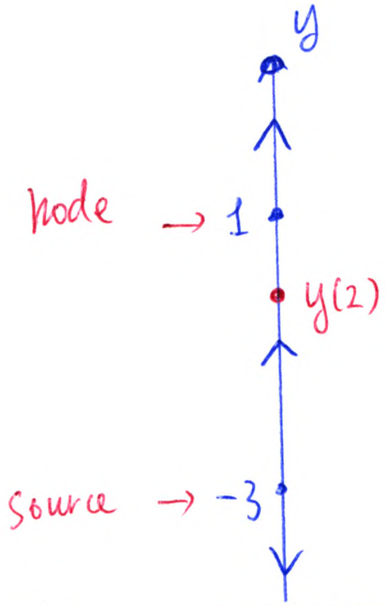


Not much can be said about $\lim_{t \to \pm\infty} y(t)$.

e.g. Sketch the phase line ~~of~~ and classify the equilibria of.

$$y' = (y-1)^2(y+3). \quad \text{--} \quad (*)$$

Describe the behavior of $y(t)$ as $t \to +\infty$, Here $y(t)$ is a solution to $(*)$ with $y(2) = 0$.



$\Rightarrow$ $y(t) \to 1$ as $t \to +\infty$.

(Remark: "2" in $y(2)$ plays no role : $y' = f(y)$ is autonomous)

~~e.g.~~ To ~~f~~ classify ~~p~~ an equilibrium without ~~do~~ sketching the whole phase line :

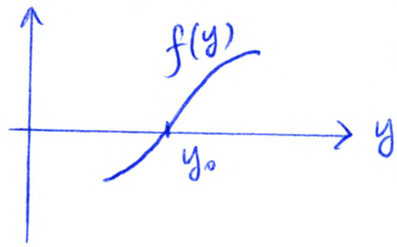
<u>Thm</u>: If y_0 is an equilibrium ~~po~~ of $y' = f(y)$ AND

$$f'(y_0) > 0 \quad \text{(resp. } f'(y_0) < 0 \text{)},$$

then y_0 is a source (resp. sink).

"pf" $f(y_0) = 0$, $f'(y_0) > 0$

$\rightsquigarrow$ f locally looks like.



$\leftarrow$ $f(y) > 0$ when $y > y_0$, $f(y) < 0$ when $y < y_0$ $\rightsquigarrow$

$\rightsquigarrow$ Source!