01:640:252 ELEMENTARY DIFFERENTIAL EQUATIONS: HW6 SOLUTION

(1) For the following system, find and classify all equilibrium points.

$$\begin{cases} \frac{dx}{dt} = x(-x-3y+150), \\ \frac{dy}{dt} = y(-2x-y+100). \end{cases}$$

Solution To find the equilibria, set

$$x(-x-3y+150) = 0,$$
$$y(-2x-y+100) = 0.$$

The first equation implies $x = 0$ or $-x-3y+150 = 0$. If $x = 0$, then the second equation becomes

$$y(-y+100) = 0 \Rightarrow y = 0 \text{ or } y = 100$$

and thus $(0,0)$, $(0,100)$ are two equilibria. If $-x-3y+150 = 0$, then $x = -3y+150$ and so

$$y(-2(-3y+150) - y + 100) = 0 \Rightarrow y = 0 \text{ or } 40.$$

Putting it back to the first equations give another two equilibria $(150,0)$, $(30,40)$.

The linearization is

$$D\mathcal{F}_{(x,y)} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -2x-3y+150 & -3x \\ -2y & -2x-2y+100 \end{pmatrix}$$

Next we calculate the trace and determinand at each equilibrium to find the type:

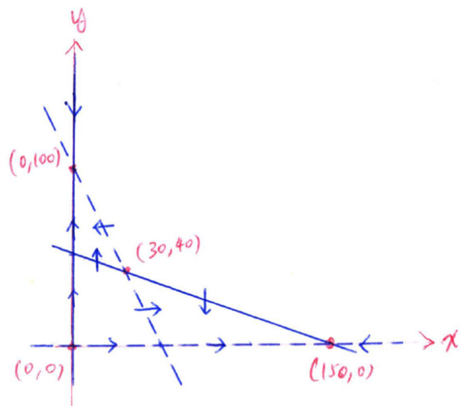
$$D\mathcal{F}_{(0,0)} = \begin{pmatrix} 150 & 0 \\ 0 & 100 \end{pmatrix} \Rightarrow \lambda = 150, 100 \Rightarrow \text{Source},$$

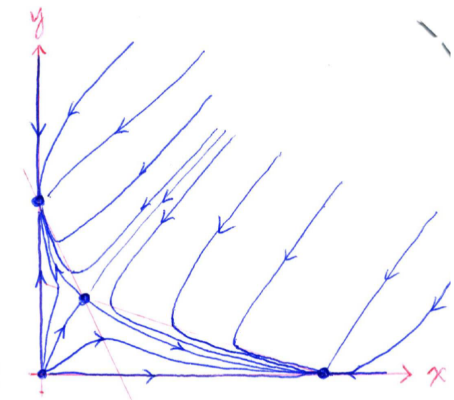
$$D\mathcal{F}_{(0,100)} = \begin{pmatrix} -150 & 0 \\ -200 & -100 \end{pmatrix} \Rightarrow \lambda = -150, -100 \Rightarrow \text{Sink},$$

$$D\mathcal{F}_{(150,0)} = \begin{pmatrix} -150 & -450 \\ 0 & -200 \end{pmatrix} \Rightarrow \lambda = -150, -450 \Rightarrow \text{Sink},$$

$$D\mathcal{F}_{(30,40)} = \begin{pmatrix} -30 & -90 \\ -80 & -40 \end{pmatrix} \Rightarrow \lambda = 120, -50 \Rightarrow \text{Saddle}.$$

(2) Sketch the x-nullcline and y-nullcline of the system in the previous question. Sketch the phase portrait.





(3) For the following system, find all equilibria.

$$\begin{cases} \frac{dx}{dt} = x(2-x-y), \\ \frac{dy}{dt} = y(y-x^2). \end{cases}$$

<u>Solution</u> To find the equilibria, set

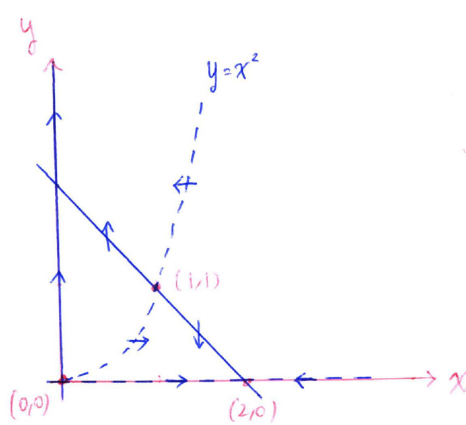
$$x(2-x-y) = 0,$$
$$y(y-x^2) = 0.$$

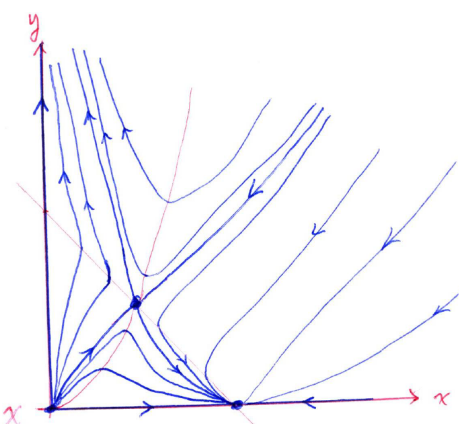
The first equation implies $x = 0$ or $2 - x - y = 0$. If $x = 0$, then the second equation becomes $y^2 = 0$, thus $(0,0)$ is one of the equilibrium. If $2 - x - y = 0$, then $y = 2 - x$ and so

$$(2-x)(2-x-x^2) = 0 \Rightarrow x = 2 \text{ or } 1.$$

Putting it back to the first equations give another two equilibria $(2,0)$, $(1,1)$.

(4) Sketch the x-nullcline and y-nullcline of the system in the previous question. Sketch the phase portrait.





(5) For the following systems, check if it is Hamiltonian. If yes, find a Hamiltonian function.

(a) $\begin{cases} x' = 3xy^2 + e^x + 1 \\ y' = -y^3 - e^x \end{cases}$

Solution: Write

$$f = 3xy^2 + e^x + 1,$$
$$g = -y^3 - ye^x.$$

Then

$$\frac{\partial f}{\partial x} = 3y^2 + e^x,$$
$$\frac{\partial g}{\partial y} = -3y^2 - e^x.$$

Hence $\frac{\partial f}{\partial x} \neq -\frac{\partial g}{\partial y}$ and the system is not Hamiltonian. To find the Hamiltonian function, let

$$\frac{\partial H}{\partial y} = 3xy^2 + e^x + 1,$$

integrating with respect to y gives

$$H = xy^3 + ye^x + y + C(x).$$

Then

$$\frac{\partial H}{\partial x} = y^3 + ye^x + C'(x)$$

since $\frac{\partial H}{\partial x} = -g = y^3 + ye^x$, we obtain $C'(x) = 0$ and we choose $C(x) = 0$. Thus

$$H = xy^3 + ye^x + y.$$

is a Hamiltonian.

(b) $\begin{cases} x' = 2y\cos(y^2) + x^2e^y \\ y' = -2xe^y \end{cases}$,

Solution: Again this is Hamiltonian since

$$\frac{\partial f}{\partial x} = 2xe^y = -\frac{\partial g}{\partial y}$$

To find the Hamiltonian function, let

$$\frac{\partial H}{\partial y} = 2y\cos(y^2) + x^2e^y,$$

integrating with respect to y gives

$$H = \sin(y^2) + x^2e^y + C(x).$$

Then

$$\frac{\partial H}{\partial x} = 2xe^y + C'(x)$$

since $\frac{\partial H}{\partial x} = -g = 2xe^y$, we obtain $C'(x) = 0$ and we choose $C(x) = 0$. Thus

$$H = \sin(y^2) + x^2 e^y.$$