Linear System - matrix with real, distinct, nonzero eigenvalues.

Given a linear system

$$Y' = AY \quad , \quad A: 2\times 2 \text{ matrix}.$$

Goal: To sketch the phase protrait using eigenvalues eigenvectors.

~~In this~~

First observation:

$\boxed{\text{If } \underset{\text{(real)}}{\lambda} \text{ is an eigenvalue with eigenvector } V\text{, then } Y(t) = e^{\lambda t} V \text{ is a solution to the linear system.}}$

Direct checking: write $Y(t) = e^{\lambda t} V$.

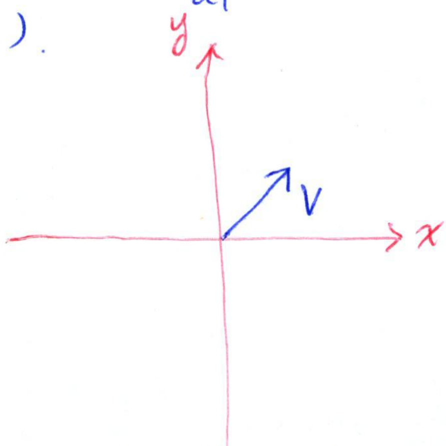
$$\Longrightarrow \quad Y'(t) = (e^{\lambda t} V)' = (e^{\lambda t})' V = \lambda e^{\lambda t} V = \lambda Y(t)$$

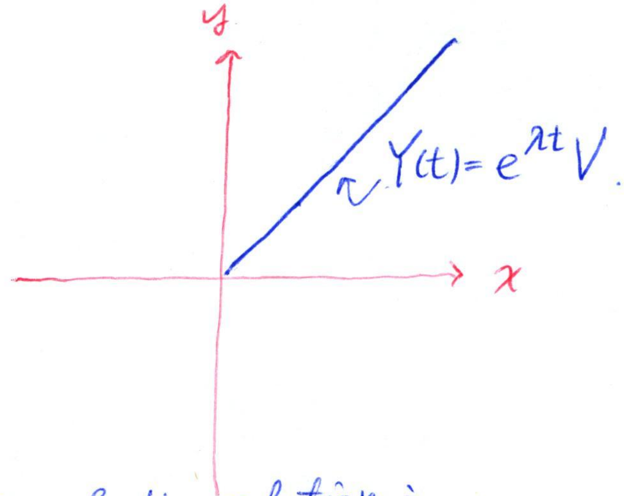
and

$$AY(t) = A(e^{\lambda t} V) = e^{\lambda t} AV \underset{\substack{\uparrow \\ AV = \lambda V}}{=} e^{\lambda t} \lambda V = \lambda Y(t)$$

$$\Longrightarrow \quad Y'(t) = AY(t) \qquad \square$$

Solutions of the form $Y(t) = e^{\lambda t} V$, with $\lambda \neq 0$, are called straight line solutions: $e^{\lambda t} V$ is always parallel to V, and is always pointing at the ~~d~~ same direction as V (since $e^{\lambda t} > 0$ for all t).

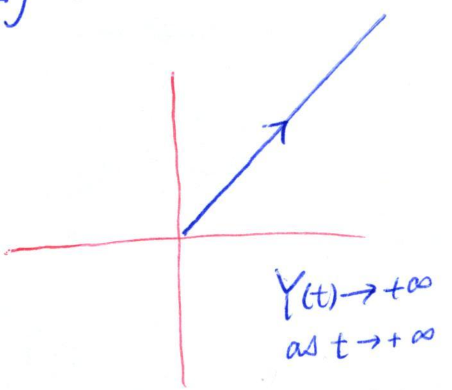




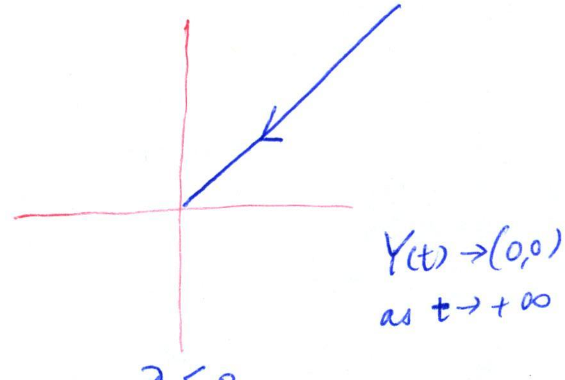
We can also find the orientation of the solution:

(i) If $\lambda > 0$, $e^{\lambda t}$ is increasing $\Rightarrow$ $e^{\lambda t} V$ moves away from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(ii) If $\lambda < 0$, $e^{\lambda t}$ is decreasing $\Rightarrow$ $e^{\lambda t} V$ moves towards $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

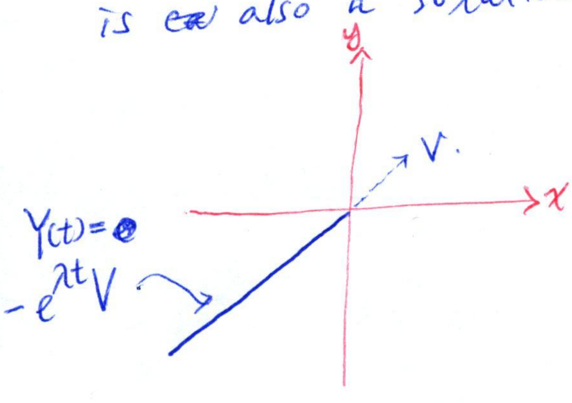


$\lambda > 0$

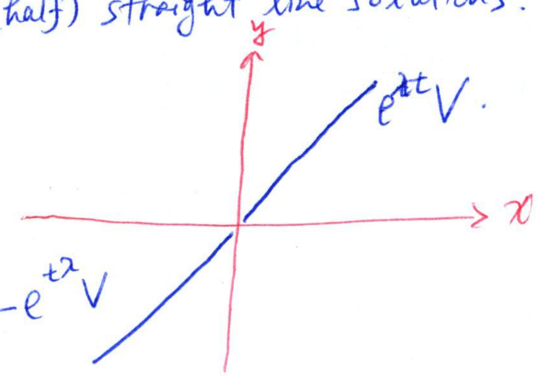


$\lambda < 0$

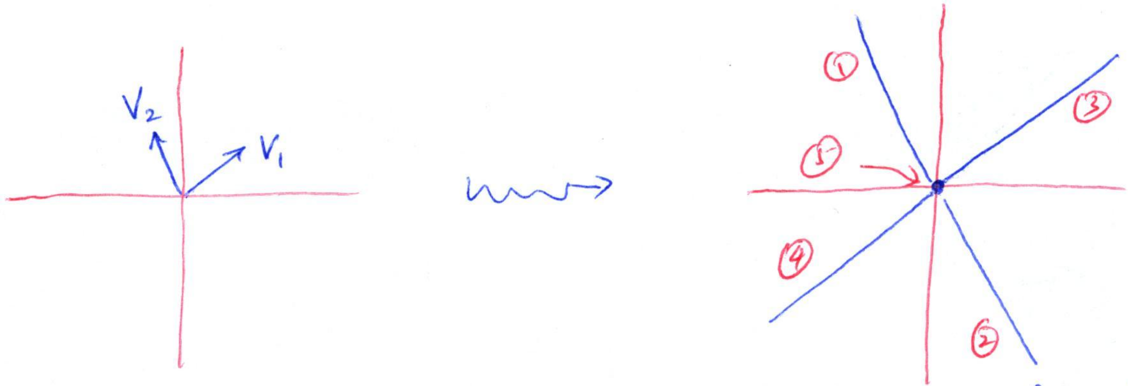
Note that the same eigenvalue λ gives also another solution: $-V$ is also an eigenvector, so $Y(t) = e^{\lambda t}(-V) = -e^{\lambda t} V$ is ~~ex~~ also a solution:



Thus the information λ, V gives us two (half) straight line solutions.



Thus if we have 2 real eigenvalues with linearly independent eigenvectors, then there are 4 (half) straight line solutions. Together with the equilibrium $Y(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all t, we have already 5 solutions.



(arrows on the lines depend on sign of λ_1, λ_2).

Remark: If λ_1, λ_2 are distinct, then V_1, V_2 are linearly independent.

Remark: If V_1, V_2 are linearly independent, then the general solution is given by

$$Y(t) = k_1 e^{\lambda_1 t} V_1 + k_2 e^{\lambda_2 t} V_2. \qquad (*)$$

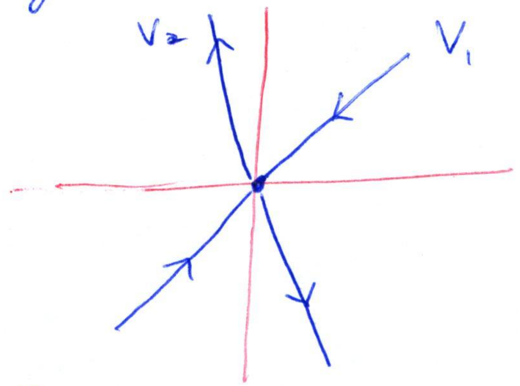
for any k_1, k_2.

Now we use $(*)$ to sketch the phase protrait. We divide into 3 cases.

1. $\lambda_1 < 0 < \lambda_2$.
2. $0 < \lambda_2 < \lambda_1$
3. $\lambda_1 < \lambda_2 < 0$

① $\lambda_1 < 0 < \lambda_2$

We have already 5 solutions



to fill in the others, we use

$$Y(t) = k_1 e^{\lambda_1 t} V_1 + k_2 e^{\lambda_2 t} V_2 .$$

Note that as $t \to +\infty$, $e^{\lambda_1 t} \to 0$ (since $\lambda_1 < 0$). Thus

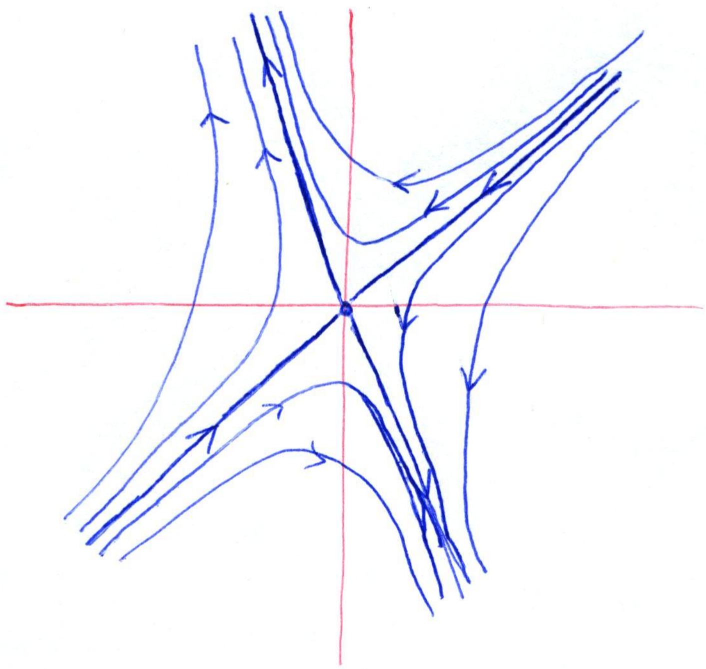
$$Y(t) \approx k_2 e^{\lambda_2 t} V$$

when t is large.

Similarly, as $t \to -\infty$, $\cancel{k_2} e^{\lambda_2 t} \to 0$ (since $\lambda_2 > 0$). Thus

$$Y(t) \approx k_1 e^{\lambda_1 t} V_1$$

when $t \to -\infty$. (or, very negative). So we have :



"Saddle"

all solutions comes from ~~and~~ infinity, get closed to the origin, but then go back to infinity.

" "

② $0 < \lambda_2 < \lambda_1$

Note that in this case,

- when $t \to +\infty$, $e^{\lambda_1 t}, e^{\lambda_2 t} \to +\infty$ as $\lambda_1, \lambda_2 > 0$.
- when $t \to -\infty$, $e^{\lambda_1 t}, e^{\lambda_2 t} \to 0$ as $\lambda_1, \lambda_2 > 0$.

Thus <u>all</u> solution $Y(t) = k_1 e^{\lambda_1 t} V_1 + k_2 e^{\lambda_2 t} V_2$

- tends to infinity as $t \to +\infty$,
- tends to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to -\infty$.

Also,

$\boxed{\text{As } t \to -\infty,\ Y(t) \text{ converges to the origin "in the direction of } V_2\text{".}}$

Precisely, it means $Y(t)$ is almost parallel to V_2 when t is very negative.

Reason: If $Y(t) = k_1 e^{\lambda_1 t} V_1 + k_2 e^{\lambda_2 t} V_2$.

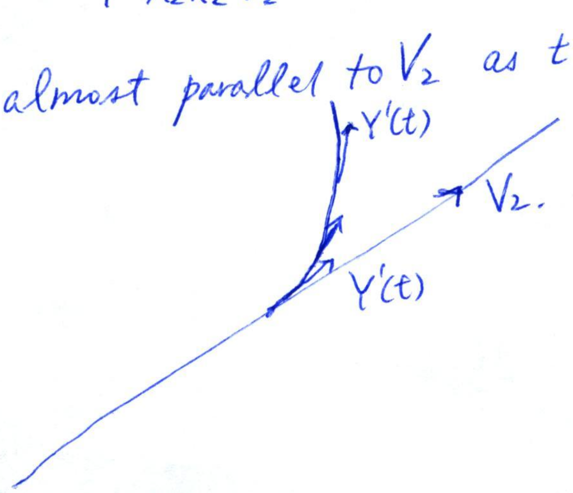
$$\Rightarrow Y'(t) = k_1 \lambda_1 e^{\lambda_1 t} V_1 + k_2 \lambda_2 e^{\lambda_2 t} V_2$$

$$= e^{\lambda_2 t} \left(k_1 \lambda_1 e^{(\lambda_1 - \lambda_2)t} V_1 + k_2 \lambda_2 V_2 \right)$$

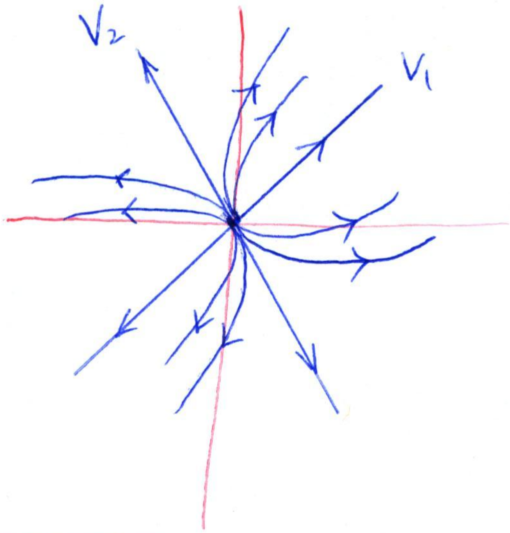
thus $Y'(t)$ is parallel to $k_1 \lambda_1 e^{(\lambda_1 - \lambda_2)t} V_1 + \lambda_2 k_2 V_2$. As $t \to -\infty$, $e^{(\lambda_1 - \lambda_2)t} \to 0$ since $\lambda_1 - \lambda_2 > 0$ (this is where we use the convention $0 < \lambda_2 < \lambda_1$)

$$\Rightarrow k_1 \lambda_1 e^{(\lambda_1 - \lambda_2)t} + \lambda_2 k_2 V_2 \approx \lambda_2 k_2 V_2 \text{ if } t \text{ is very negative.}$$

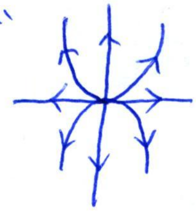
$\Rightarrow Y(t)$ is almost parallel to V_2 as $t \to -\infty$.



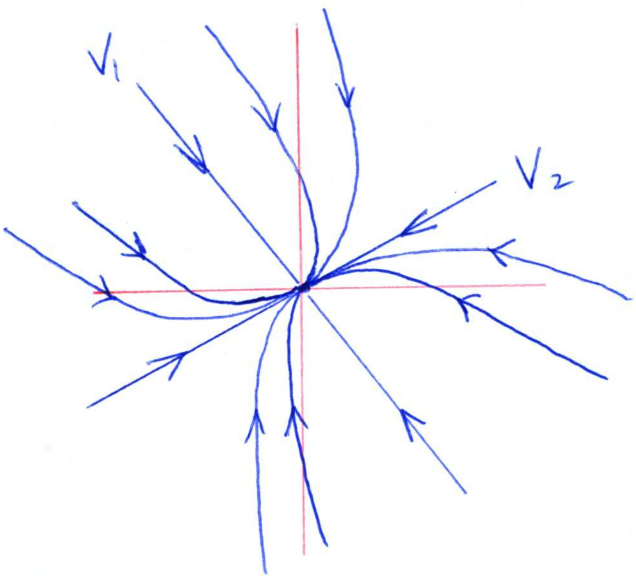
To sum up, when $0 < \lambda_2 < \lambda_1$,



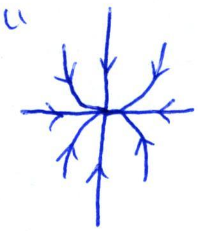
"Source".
ALL solutions go to infinity as $t \to +\infty$, go to $\binom{0}{0}$ as $t \to -\infty$ in the direction of V_2 (except those 2 solution given by V_1)

"  "

③ $\lambda_1 < \lambda_2 < 0$. This is similar to ②. We have



"Sink"
ALL solutions tends to $\binom{0}{0}$ as $t \to +\infty$, in the direction of V_2 (except those 2 given by V_1). ALL solution go to infinity as $t \to -\infty$.

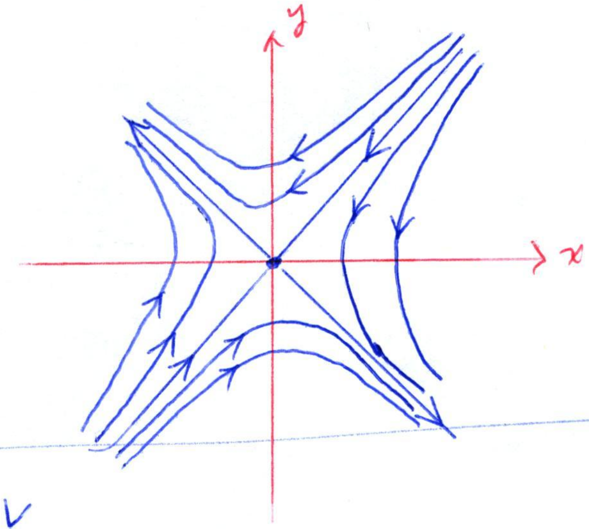
"  "

e.g. $Y' = \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix} Y$

$A = \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}$ has eigenvalues $-5, 1 \Rightarrow \underbrace{\lambda_1}_{-5} < 0 < \underbrace{\lambda_2}_{1}$

(saddle)

One can check $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are the eigenvectors with respect to λ_1, λ_2 respectively. So



e.g. $Y' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} Y$.

(source)

$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ has eigenvalues $1, 4 \Rightarrow$ Call $\lambda_1 = 4$, $\lambda_2 = 1$.

(Note the convention: we choose λ_1, λ_2 so that λ_2 is closer to 0)

Corresponding eigenvectors: $V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$