Existence and Uniqueness for system:

Given an IVP:

$$\begin{cases} Y' = \vec{F}(Y), \\ Y(t_0) = Y_0 \end{cases} \quad \left(\text{or} \begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \\ x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases} \right)$$

Then:

Existence and Uniqueness theorem:

If $\vec{F}$ and the first derivatives of $\vec{F}$ are both continuous, then for ~~and~~ any t_0 and Y_0, the IVP has an unique solution $Y(t)$ ~~dep~~ ~~de~~, where $Y(t)$ is defined in an interval $t_0 - \varepsilon < t < t_0 + \varepsilon$

Remark: By first derivatives of $\vec{F}$ we mean the four terms:

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$$

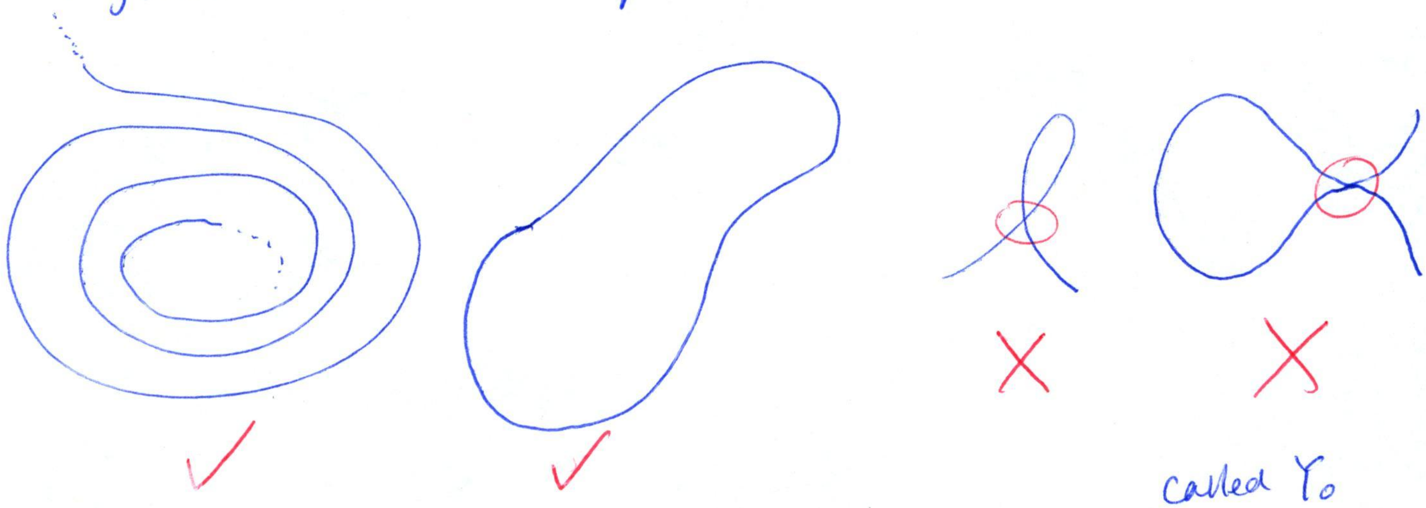
Remark: The same conclusion holds for non-autonomous system:

$$\begin{cases} Y' = \vec{F}(t, Y_0), \\ Y(t_0) = Y_0 \end{cases}.$$

Remark: As an the case for one D.E., one has no control on $\varepsilon > 0$. It could be small even if $\vec{F}$ is everywhere continuous and its 1st derivatives.

Consequence of the uniqueness theorem (for autonomous system)

① Solution curves (i) do not intersect itself, (ii) unless it forms a smooth loop :



called Y_0

Reason : If it does have an intersection : $Y(t_0) = Y(t_1)$ for $t_1 \neq t_0$.

then
$$\begin{cases} Y' = \vec{F}(Y) \\ Y(t_0) = Y_0 \end{cases}$$

has two solutions $Y(t)$ and $Y(t+t_1-t_0)$

Uniqueness $\Rightarrow$ $Y(t) = Y(t+t_1-t_0)$

$\Rightarrow$ $Y(t)$ is periodic $\Rightarrow$ it forms a smooth loop.

Note : To see why $Y(t+t_1-t_0)$ is a solution :

$$\cancel{Y'(t+t} \; \frac{d}{dt}\left(Y(t+t_1-t_0)\right) = Y'(t+t_1-t_0)\cdot \frac{d}{dt}(t+t_1-t_0) \quad \text{(Chain rule)}$$
$$= Y'(t+t_1-t_0)$$
$$= \vec{F}\left(Y(t+t_1-t_0)\right)$$

It's essential that $\vec{F}$ is autonomous here.

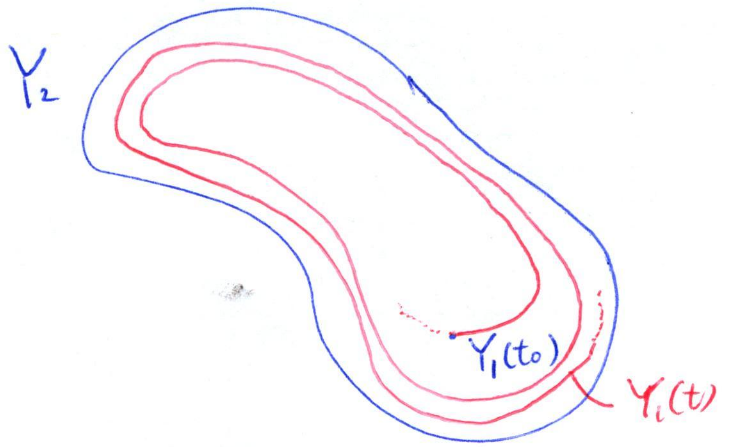
② Comparison Principle:

If Y_1, Y_2 are two solutions to the same autonomous system and

(i) $Y_2(t)$ forms a smooth loop,

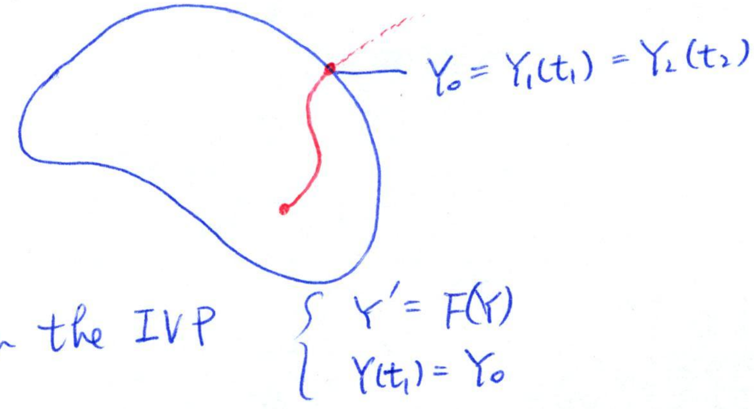
(ii) $Y_1(t_0)$ is inside of Y_2.

Then $Y_1(t)$ stays inside of Y_2 for all t.



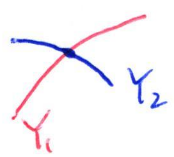
$Y_1(t)$ always stay inside of Y_2.

Reason: If $\nexists$ Y_1 does cross Y_2 at $Y_0 = Y_1(t_1) = Y_2(t_2)$,



Then the IVP $\begin{cases} Y' = F(Y) \\ Y(t_1) = Y_0 \end{cases}$

has two distinct solutions $Y_1(t)$ and $Y_2(t + t_2 - t_1)$, which contradicts the uniqueness theorem.



One application of the comparison principle:

Given the following system

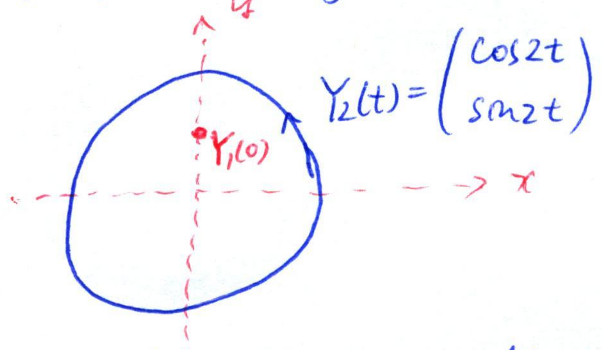
$$\begin{cases} \dfrac{dx}{dt} = -y(1+x^2+y^2). \\ \dfrac{dy}{dt} = 2x \end{cases}$$

(i) Note that $Y_2(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$ is a solution (check!)

(ii) So if $Y(t)$ is also a solution, ~~then~~ and $Y_1(0) = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$

then $x^2(t) + y^2(t) < 1$ for all t (here $Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$).

Reason: $Y_1(0) = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$ is inside of the unit circle, AND the unit circle is the trace of Y_2:



Then since $-y(1+x^2+y^2)$, $2x$ are continuous and have continuous 1st derivatives, comparison principle is applicable and thus $Y_1(t)$ stay inside of Y_2 (= the unit circle)

Remark: The comparison principle also work when $Y_1(t_0)$ is outside of Y_2: in this case $Y_1(t)$ always stays outside of Y_2.