Modelling: Lotka Volterra Equation:

① Competitive species:

$x(t)$, $y(t)$: No. of animals of ~~a~~ two different species.

Assumption: When the two species live in different habitats, they both satisfy~~s~~ the Logistic growth model

$$x' = k_1 x\left(1 - \frac{x}{N_1}\right), \quad y' = k_2 y\left(1 - \frac{x}{N_2}\right).$$

If the two species share the same habitat, they compete for natural resources. (e.g. Alligator and Python in Everglade). Thus the presence of one species has a negative effect on the growth of the other species:

$$\begin{cases} x' = k_1 x\left(1 - \frac{x}{N_1}\right) - C_1 xy \\ y' = k_2 x\left(1 - \frac{x}{N_2}\right) - C_2 xy \end{cases}$$

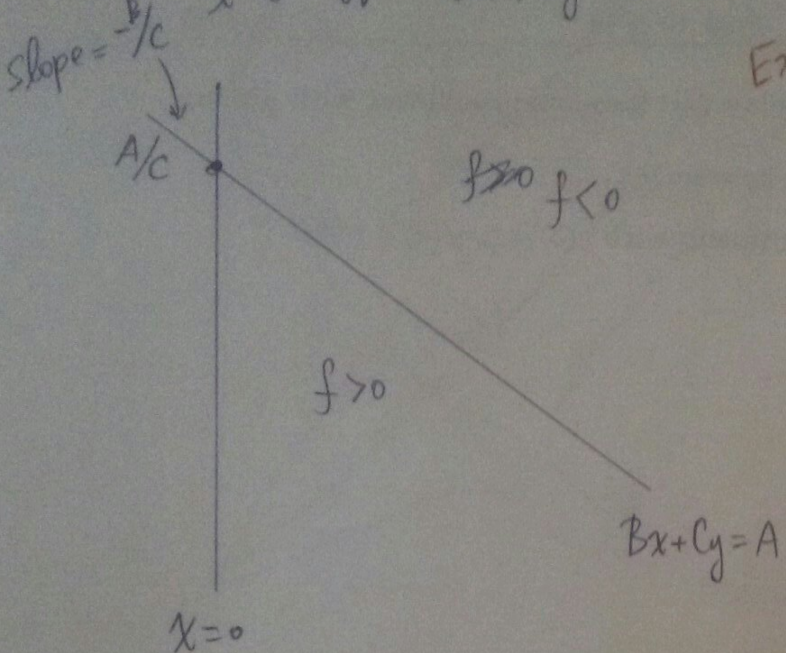
Negative effect.

which can be written as

$$\begin{cases} x' = x(A - Bx - Cy) \\ y' = y(D - Ex - Fy) \end{cases} \qquad (A, B, C, D, E, F > 0)$$

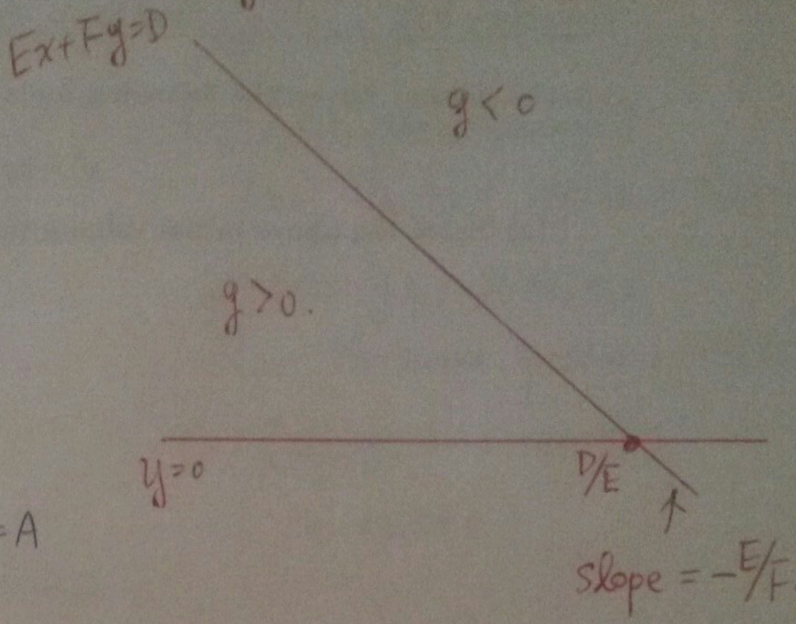
x-nullclines:

$x=0$ or $Bx+Cy=A$



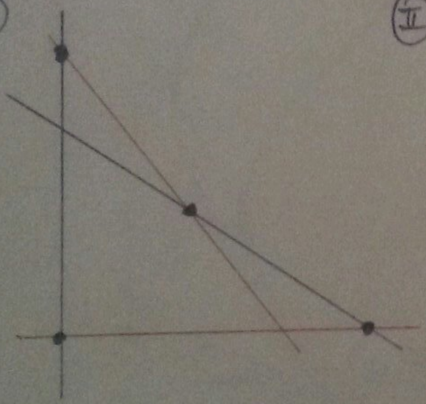
y-nullclines

$y=0$ or $Ex+Fy=D$.

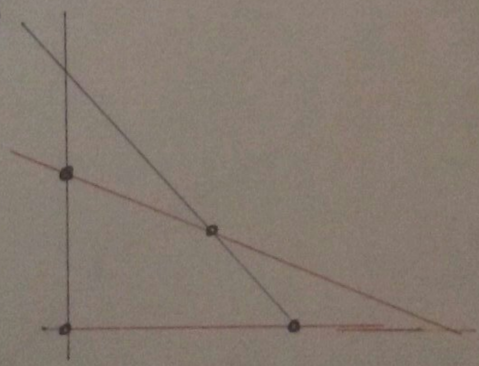


$\Rightarrow$ There are 4 possibilities (consider only $x \geq 0$, $y \geq 0$):

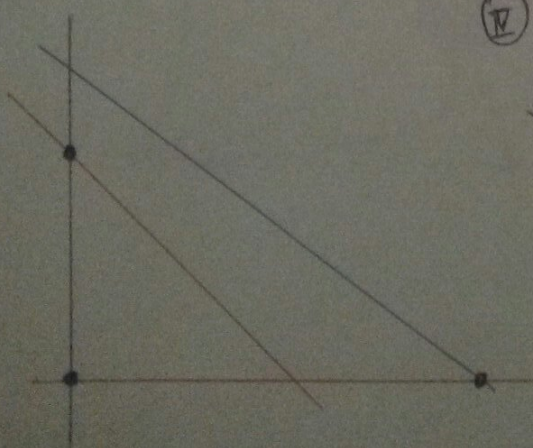
I



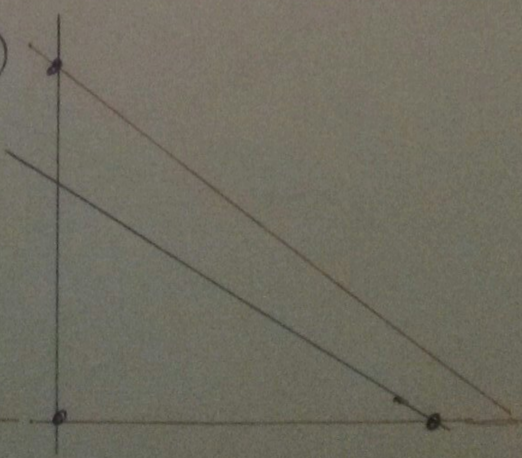
II



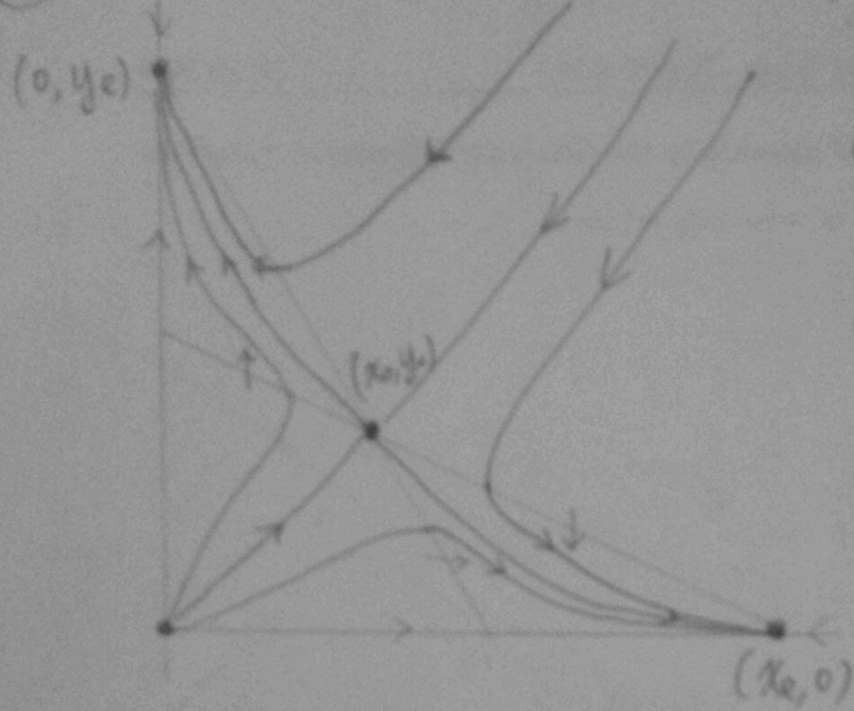
III



IV



(I)



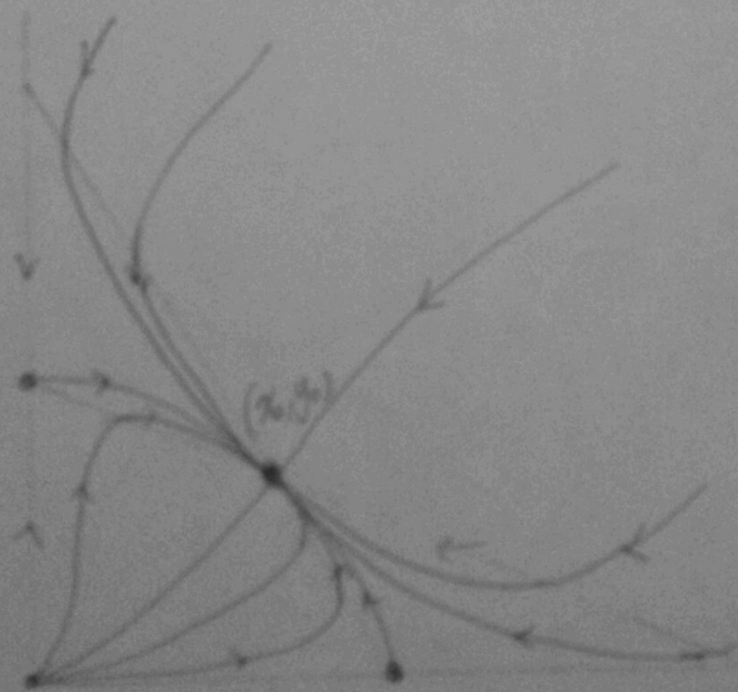
Depending on initial condition, as $t \to +\infty$, either

(i) $(x(t), y(t)) \to (0, y_c)$
(i.e. the first species extincts)

(ii) $(x(t), y(t)) \to (x_e, 0)$
(i.e. the second species extincts)

(iii) $(x(t), y(t)) \to (x_0, y_0)$
(co-exists)

(II)

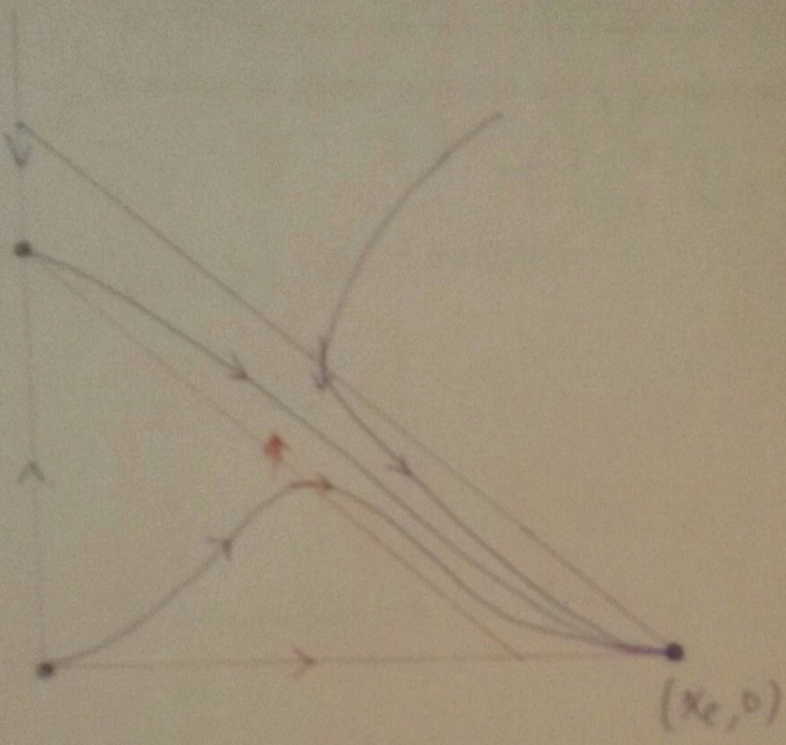


for all initial conditions

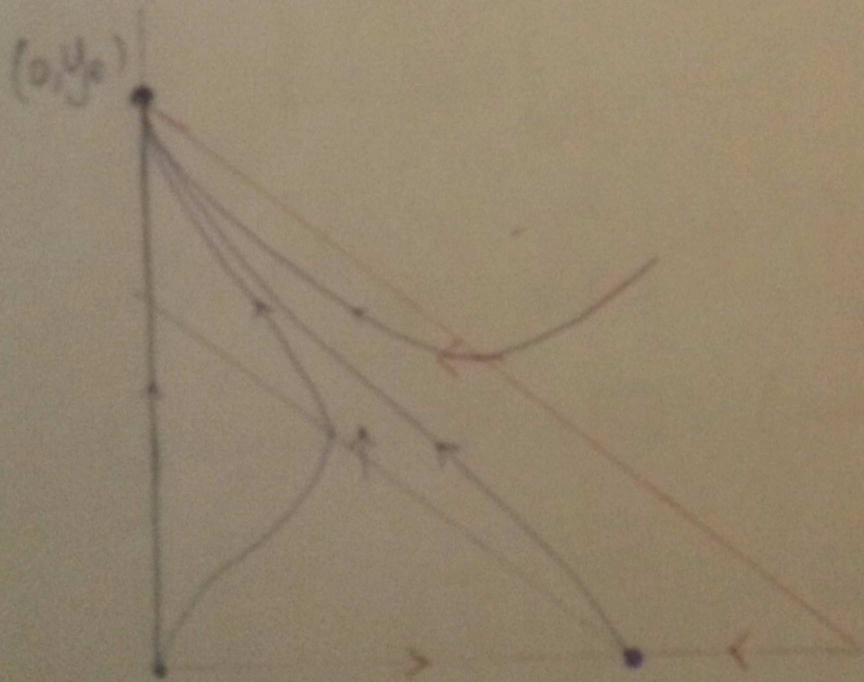
$(x(t), y(t)) \to (x_0, y_0)$.

i.e. Two species co-exists with each other.

III, IV : Dominance of one species



all converges to $(x_e, 0)$
as $t \to +\infty$
(i.e. 2nd species extincts)

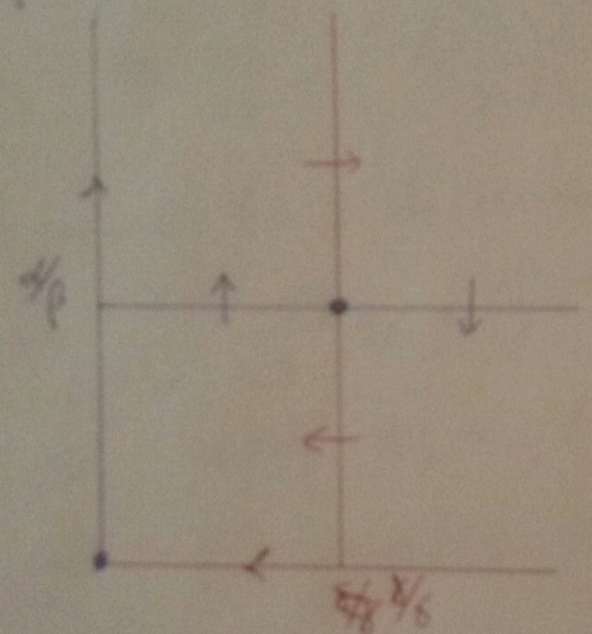


all converges to $(0, y_e)$
as $t \to +\infty$
(i.e. 1st species extincts)

② Predator - Prey system: x: prey y: predator.

$$\begin{cases} x' = \alpha x - \beta x y = x(\alpha - \beta y) \\ y' = -\gamma y + \delta x y = y(-\gamma + \delta x) \end{cases}$$

$\alpha, \beta, \gamma, \delta > 0$ are constants.



- Spiralling around the equilibrium $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$
- Can't tell if it's a spiral sink, spiral source, or a center

$$DF = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{pmatrix} \Rightarrow DF_{(\gamma/\delta, \alpha/\beta)} = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\delta\alpha}{\beta} & 0 \end{pmatrix}$$

has eigenvalues $\lambda = \pm\sqrt{\alpha\gamma}\, i$ (zero real part, center)

$\Downarrow$

inconclusive again.

Instead we divide one equation from the other:

$$\cancel{\frac{dx}{dt}} \quad \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y(-\gamma+\delta x)}{x(\alpha-\beta y)}.$$

and use the chain rule: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

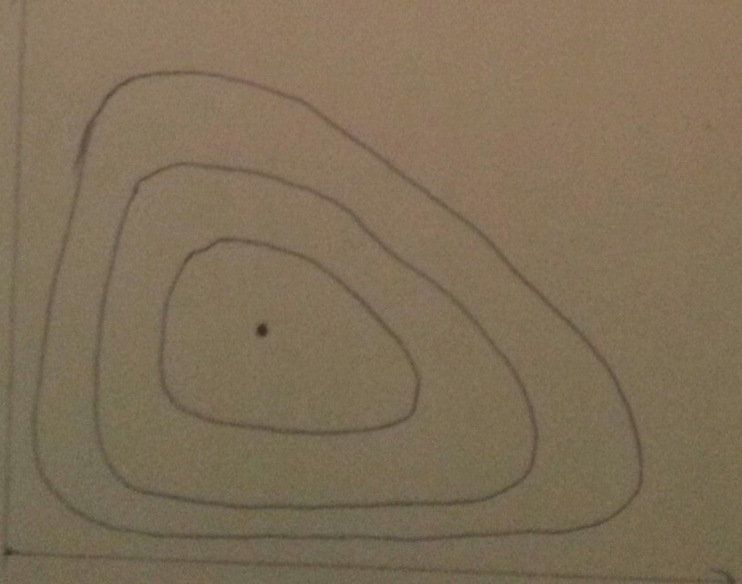
$$\Rightarrow \quad \cancel{dy} \ \frac{dy}{dx} = \left(\frac{-\gamma+\delta x}{x}\right)\left(\frac{y}{\alpha-\beta y}\right).$$

This is separable (x is the variable).

$$\Rightarrow \quad \int \frac{\alpha-\beta y}{y}\, dy = \int \frac{-\gamma+\delta x}{x}\, dx$$

$$\Rightarrow \quad \alpha \ln y - \beta y = -\gamma \ln x + \delta x + C.$$

$$\Rightarrow \quad \gamma \ln x + \alpha \ln y - \beta y - \delta x = C.$$



In particular, the equilibrium is a <u>center</u>.