# 01:640:252 ELEMENTARY DIFFERENTIAL EQUATIONS: HW5 SOLUTION

(1) Find the general solution to the following differential equations:

$$y'' - y' - 6y = e^{4t}.$$

**Solution** By the linearity principle, the general solution is given by $y = y_h + y_p$, where $y_h$ is the general solution to the homogeneous equation and $y_p$ is a particular solution. To find $y_h$, set $y = e^{st}$ for some constants $s$, then

$$0 = y'' - y' - 6y = e^{st}(s^2 - s - 6) = e^{st}(s-3)(s+2).$$

Thus $s = 3$ or $-2$. Then

$$y_h = k_1 e^{3t} + k_2 e^{-2t}.$$

To find $y_p$, try $y_p = Ce^{4t}$, where $C$ is some constants to be found. Then

$$y_p'' - y_p - 6y_p = 16Ce^{4t} - 4Ce^{4t} - 6Ce^{4t} = 6Ce^{4t},$$

thus we set $C = 1/6$. Then the general solution is given by $y = k_1 e^{3t} + k_2 e^{-2t} + \frac{1}{6}e^{4t}$.

(2) Find the solution to the following initial value problem:

$$y'' + 4y' + 20y = -3\sin 2t, \text{ with } y(0) = y'(0) = 0.$$

**Solution** Again $y = y_h + y_p$. To find $y_h$, set $y = e^{st}$ for some constants $s$, then

$$0 = y'' + 4y' + 20y = e^{st}(s^2 + 4s + 20) \Rightarrow s = -2 \pm 4i$$

and $e^{(-2+4i)t}$ satisfies the homogeneous equation. Using Euler's formula,

$$e^{(-2+4i)t} = e^{-2t}e^{i4t} = e^{-2t}\cos 4t + ie^{-2t}\sin 4t.$$

Thus $y_h = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t$. To find $y_p$, try $y_p = A\cos 2t + B\sin 2t$. Then

$$\begin{aligned}
y_p'' + 4y_p' + 20y_p &= (-4A\cos 2t - 4B\sin 2t) + 4(-2A\sin 2t + 2B\cos 2t) \\
&\quad + 20(A\cos 2t + B\sin 2t) \\
&= (-4A + 8B + 20A)\cos 2t + (-4B - 8A + 20B) \\
&= (16A + 8B)\cos 2t + (-8A + 16B)\sin 2t.
\end{aligned}$$

So we set

$$\begin{aligned}
16A + 8B &= 0 \\
-8A + 16B &= -3.
\end{aligned}$$

Thus $A = \frac{3}{40}, B = \frac{-3}{20}$. Thus the general solution is given by

$$y = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t + \frac{3}{40}\cos 2t - \frac{3}{20}\sin 2t.$$

Lastly we find $k_1$ and $k_2$ using the initial conditions. Since $y(0) = y'(0) = 0$,

$$0 = k_1 + \frac{3}{40},$$
$$0 = -2k_1 + 4k_2 + \frac{3}{10}.$$

Thus $k_1 = -\frac{3}{40}$, $k_2 = -\frac{3}{80}$.

(3) Use the angle sum formula

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$

to write

$$A \sin\theta + B\cos\theta = C\sin(\theta + \phi).$$

Write $C, \phi$ in terms of $A, B$.
**Solution** Using the formula, we write

$$C\sin(\theta + \phi) = C\sin\theta\cos\phi + C\cos\theta\sin\phi$$

In order that the right hand side equals $A\sin\theta + B\cos\theta$, we choose

$$C\cos\phi = A,$$
$$C\sin\phi = B.$$

Thus

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{A}{B} \Rightarrow \phi = \tan^{-1}\frac{B}{A}.$$

On the other hand, using $\sin^2\phi + \cos^2\phi = 1$,

$$C^2 = C^2\sin^2\phi + C^2\cos^2\phi = A^2 + B^2.$$

Thus $C = \sqrt{A^2 + B^2}$.

(4) Use the angle sum formulae

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2,$$
$$\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$$

to show

$$\cos\phi_1 - \cos\phi_2 = -2\sin\left(\frac{\phi_1 + \phi_2}{2}\right)\sin\left(\frac{\phi_1 - \phi_2}{2}\right).$$

**Solution** Choose

$$\theta_1 = \frac{\phi_1 + \phi_2}{2}, \quad \theta_2 = \frac{\phi_1 - \phi_2}{2},$$

then

$$\theta_1 + \theta_2 = \phi_1, \quad \theta_1 - \theta_2 = \phi_2$$

and we have

$$\begin{aligned}\cos\phi_1 - \cos\phi_2 &= \cos(\theta_1+\theta_2) - \cos(\theta_1-\theta_2)\\ &= -2\sin\theta_1\sin\theta_2\\ &= -2\sin\left(\frac{\phi_1+\phi_2}{2}\right)\sin\left(\frac{\phi_1-\phi_2}{2}\right).\end{aligned}$$

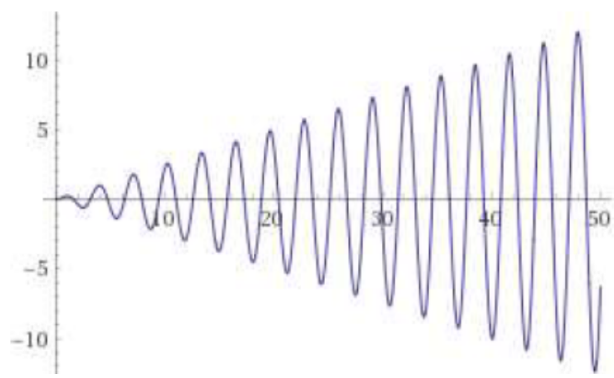
(5) Solve the following initial value problem and sketch the solution.

$$y'' + 4y = \cos 2t, \quad y(0) = y'(0) = 0.$$

**Solution** This is the case when $\omega^2 = q = 4$. The calculations in the note (Last page in "forced harmonic oscillators") gives

$$y(t) = k_1\cos 2t + k_2\sin 2t + \frac{t}{4}\sin 2t.$$

Using the intial conditions, we find $k_1 = k_2 = 0$ and thus $y(t) = \frac{t}{4}\sin 2t$:




(6) Find the general solution to the following first order (non-homogeneous) linear system

$$Y' = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} Y + \begin{pmatrix}e^{2t}\\ e^t\end{pmatrix}.$$

**Solution:** Write

$$A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}, \quad \vec{F} = \begin{pmatrix}e^{2t}\\ e^t\end{pmatrix}$$

We use (0.2) in note 4.0, which is

$$Y(t) = \int_0^t e^{(t-u)A}\vec{F}(u)du + e^{tA}Y(0).$$

We need to calculate the matrix exponential. $A$ has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$ with corresponding eigenvectors

$$v_1 = \begin{pmatrix}1\\ -1\end{pmatrix}, \quad v_2 = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

Thus

$$\begin{aligned} e^{sA} &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^s & 0 \\ 0 & e^{3s}1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} e^s & e^{3s} \\ -e^s & e^{3s} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^s + e^{3s} & -e^s + e^{3s} \\ -e^s + e^{3s} & e^s + e^{3s} \end{pmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} e^{(t-u)A}\vec{F}(u) &= \frac{1}{2} \begin{pmatrix} e^{t-u} + e^{3(t-u)} & -e^{t-u} + e^{3(t-u)} \\ -e^{t-u} + e^{3(t-u)} & e^{t-u} + e^{3(t-u)} \end{pmatrix} \begin{pmatrix} e^{2u} \\ e^u \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{t+u} + e^{3t-u} - e^t + e^{3t-2u} \\ -e^{t+u} + e^{3t-u} + e^t + e^{3t-2u} \end{pmatrix} \\ \Rightarrow \int_0^t e^{(t-u)A}\vec{F}(u)du &= \frac{1}{2} \begin{pmatrix} \int_0^t \left(e^{t+u} + e^{3t-u} - e^t + e^{3t-2u}\right)du \\ \int_0^t \left(-e^{t+u} + e^{3t-u} + e^t + e^{3t-2u}\right)du \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^t(e^t-1) - e^{3t}(e^{-t}-1) - te^t - 1/2e^{3t}(e^{-2t}-1) \\ -e^t(e^t-1) - e^{3t}(e^{-t}-1) + te^t - 1/2e^{3t}(e^{-2t}-1) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -e^t + e^{3t} - te^t - \frac{1}{2}e^t + \frac{1}{2}e^{3t} \\ -e^{2t} + e^t - e^{2t} + e^{3t} + te^t - \frac{1}{2}e^t + \frac{1}{2}e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{4}e^t + \frac{3}{4}e^{3t} - \frac{1}{2}te^t \\ \frac{1}{4}e^t + \frac{3}{4}e^{3t} + \frac{1}{2}te^t - e^{2t} \end{pmatrix}. \end{aligned}$$

Thus the general solution is given by

$$Y(t) = \begin{pmatrix} -\frac{3}{4}e^t + \frac{3}{4}e^{3t} - \frac{1}{2}te^t \\ \frac{1}{4}e^t + \frac{3}{4}e^{3t} + \frac{1}{2}te^t - e^{2t} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} e^s + e^{3s} & -e^s + e^{3s} \\ -e^s + e^{3s} & e^s + e^{3s} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Here $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = Y(0)$ is the initial condtion, which is arbitrary.

Remark: We can also interpret the above general solution using the Linearity Principle: write

$$\begin{aligned} Y_p(t) &= \int_0^t e^{(t-u)A}\vec{F}(u)du \\ &= \begin{pmatrix} -\frac{3}{4}e^t + \frac{3}{4}e^{3t} - \frac{1}{2}te^t \\ \frac{1}{4}e^t + \frac{3}{4}e^{3t} + \frac{1}{2}te^t - e^{2t} \end{pmatrix}, \\ Y_h(t) = e^{tA}Y(0) &= \frac{1}{2} \begin{pmatrix} e^s + e^{3s} & -e^s + e^{3s} \\ -e^s + e^{3s} & e^s + e^{3s} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}. \end{aligned}$$

$Y_p$ is particular solution and $Y_h$ is the general solution to the homogeneous system $Y' = AY$. One can also write $Y_h$ as

$$c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(7) Find the general solution to the following second order differential equation:

$$y'' - 2y' - 3y = te^{-t}.$$

**Solution:** set $y(t) = e^{st}$, then

$$y'' - 2y' - 3y = e^{st}(s^2 - 2s - 3) = e^{st}(s-3)(s+1),$$

thus the general solution to the homogeneous equation is $k_1 e^{-t} + k_2 e^{3t}$. To find a particular solution, we use (1.3) in Note 4.0, which is

$$y_p(t) = \int_0^t \frac{e^{(t-u)\lambda_2} - e^{(t-u)\lambda_1}}{\lambda_2 - \lambda_1} f(u)du.$$

In our case we choose $\lambda_1 = -1$, $\lambda_2 = 3$ and $f(u) = ue^{-u}$, then

$$\begin{aligned}
y_p(t) &= \int_0^t \frac{e^{3(t-u)} - e^{u-t}}{4} ue^{-u} du \\
&= \frac{1}{4}\left(e^{3t}\int_0^t ue^{-4u}du - e^{-t}\int_0^t udu\right) \\
&= \frac{1}{4}\left(e^{3t}\left(-\frac{1}{4}te^{-4t} + \frac{1}{4}\int_0^t e^{-4u}du\right) - e^{-t}\frac{1}{2}t^2\right) \\
&= \frac{1}{4}\left(e^{3t}\left(-\frac{1}{4}te^{-4t} - \frac{1}{16}(e^{-4t} - 1)\right) - \frac{1}{2}t^2e^{-t}\right) \\
&= \frac{1}{4}\left(-\frac{1}{4}te^{-t} - \frac{1}{16}e^{-t} + \frac{1}{16}e^{3t} - \frac{1}{2}t^2e^{-t}\right) \\
&= -\frac{1}{64}e^{-t} + \frac{1}{64}e^{3t} - \frac{1}{16}te^{-t} - \frac{1}{8}t^2e^{-t}.
\end{aligned}$$

Thus the general solution is given by

$$y = k_1e^{-t} + k_2e^{3t} - \frac{1}{16}te^{-t} - \frac{1}{8}t^2e^{-t}$$

(The term $-\frac{1}{64}e^{-t} + \frac{1}{64}e^{3t}$ is absorbed into $k_1e^{-t} + k_2e^{3t}$).

(8) Show directly that

$$y_p(t) = \int_0^t (t-u)e^{t-u}f(u)du$$

is a particular solution to

$$y'' - 2y' + y = f(t).$$

**Solution:** Write

$$\begin{aligned} y_p(t) &= \int_0^t (t-u)e^{t-u} f(u)du \\ &= \int_0^t (te^{t-u} - ue^{t-u}) f(u)du \\ &= te^t \int_0^t e^{-u} f(u)du - e^t \int_0^t ue^{-u} f(u)du. \end{aligned}$$

Then by the product rule and the fundamental theorem of calculus,

$$\begin{aligned} y_p(t)' &= (te^t)' \int_0^t e^{-u} f(u)du + te^t(e^{-t} f(t)) - e^t \int_0^t ue^{-u} f(u)du - e^t(te^{-t} f(t)) \\ &= (te^t)' \int_0^t e^{-u} f(u)du - e^t \int_0^t ue^{-u} f(u)du. \end{aligned}$$

differentiating again,

$$\begin{aligned} y_p(t)'' &= (te^t)'' \int_0^t e^{-u} f(u)du + (te^t)' e^{-t} f(t) - e^t \int_0^t ue^{-u} f(u)du - e^t(te^{-t} f(t)) \\ &= (te^t)'' \int_0^t e^{-u} f(u)du - e^t \int_0^t ue^{-u} f(u)du + f(t), \end{aligned}$$

note we used $(te^t)' = te^t + e^t$. Then we have

$$y_p'' - 2y_p' + y_p = ((te^t)'' - 2(te^t)' + te^t) \int_0^t e^{-u} f(u)du + f(t) = f(t)$$

since $(te^t)'' - 2(te^t)' + te^t = 0$.