3.1 Linear system (Basic property)

Definition: A system of 1st order linear D.E.s is called linear if it's of the form

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$$

where a, b, c, d are constants.

In matrix form:

$$Y = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad Y' = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$\Rightarrow$ A linear system ~~is~~ can be written as

$$Y' = AY$$

↑ matrix multiplication.

Special cases:

① $b = c = 0$.

$\Rightarrow \begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dy}{dt} = dy. \end{cases}$ (Completely decoupled) (or $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is diagonal)

② $b = 0$ or $c = 0$.

$\begin{cases} \dfrac{dx}{dt} = ax \\ \dfrac{dy}{dt} = cx + dy \end{cases}$ or $\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = dy \end{cases}$ (Partially decoupled)

(or A is upper/lower triangular)

Basic properties:

(I) Linearity Principle:

If $Y_1(t)$, $Y_2(t)$ are both solutions to a linear system, then so is

$$Y(t) = k_1 Y_1(t) + k_2 Y_2(t).$$

for any constants k_1, k_2.

Reason:

$$\begin{aligned} Y'(t) &= \left(k_1 Y_1(t) + k_2 Y_2(t)\right)' \\ &= k_1 Y_1'(t) + k_2 Y_2'(t) \quad (\because \text{differentiation is linear}) \\ &= k_1 A Y_1(t) + k_2 A Y_2(t). \quad (\because Y_1, Y_2 \text{ satisfy } Y' = AY) \\ &= A(k_1 Y_1(t) + k_2 Y_2(t)) \quad (\because \text{properties of matrix multiplication}) \\ &= A Y(t). \end{aligned}$$

$\Longrightarrow$ $Y(t)$ is also a solution

Indeed, if $Y_1(t)$, $Y_2(t)$ are not "the same" (one of them is not ~~the~~ a constant multiple of the other), then $Y(t) = k_1 Y_1 + k_2 Y_2$ is all possible solutions:

(II) If in particular that $Y_1(0)$, $Y_2(0)$ are linearly independent, then the general solution to the system is

$$Y(t) = k_1 Y_1(t) + k_2 Y_2(t),$$

where k_1, k_2 are any constants.

e.g. Solve the IVP

$$\begin{cases} Y' = \begin{pmatrix} 2 & 3 \\ 0 & -4 \end{pmatrix} Y, & Y(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \end{cases}$$

~~wf~~ given that $Y_1(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$, $Y_2(t) = \begin{pmatrix} e^{-4t} \\ -2e^{-4t} \end{pmatrix}$ are solutions to the system.

Let $Y(t) = k_1 Y_1(t) + k_2 Y_2(t) = k_1 \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} e^{-4t} \\ -2e^{-4t} \end{pmatrix}$

we find k_1, k_2 so that $Y(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is satisfied. so we set

$$Y(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

$$\Rightarrow \quad k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

or $\begin{cases} k_1 + k_2 = 2 \\ -2k_2 = -3 \end{cases} \Rightarrow k_2 = 3/2, \ k_1 = \frac{1}{2}.$

$$\Rightarrow \quad Y(t) = \frac{1}{2} \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} e^{-4t} \\ -2e^{-4t} \end{pmatrix}$$

solves the IVP.