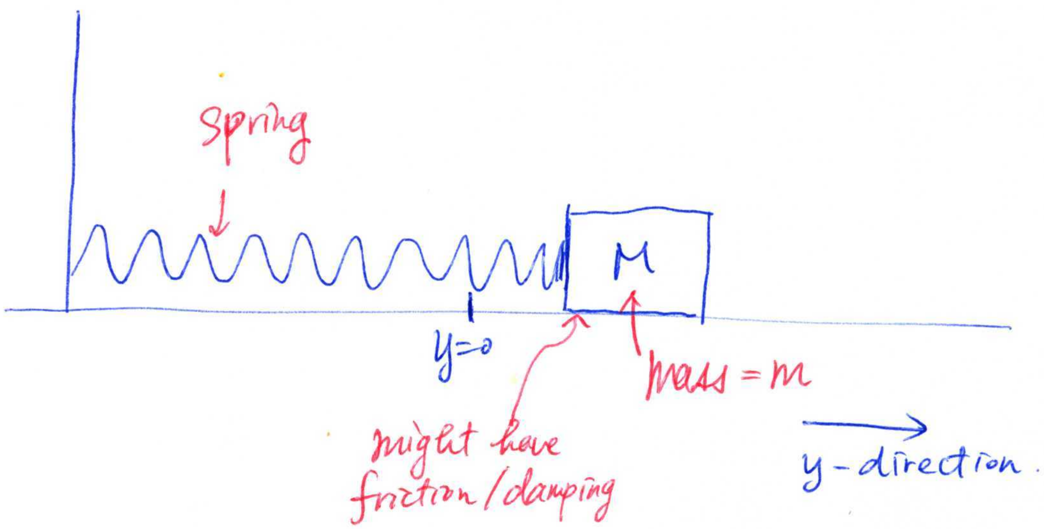
Second order linear D.E. with constant coefficients

$$ay'' + by' + cy = 0, \quad a, b, c \text{ constants.}$$

Example / Motivation: Harmonic Oscillator:



Let $y(t)$ be the position of the mass M at time t. Here $y=0$ is the stationary/rest position. (i.e. No force asserted to M from the spring.

$\Rightarrow \quad y'(t) = $ velocity

$\quad\quad y''(t) = $ acceleration.

Newton's 2nd Law:

$$F = my''$$

F: force ~~a~~ experienced by M.

To calculate the force, we use

- the Hooke's Law: the force asserted to M from the spring is proportional to the position y from the rest position: $F_1 = -\underset{\sim}{k} y$

  Hooke's constant, or spring constant. $k>0$.

- the assumption that the friction / damping is proportional to the velocity: $F_2 = -by'$.

$$\Longrightarrow \quad my'' = F = F_1 + F_2 = -ky - by', \text{ or}$$

$$\boxed{my'' + by' + ky = 0}$$

← Equation for harmonic oscillator.

To solve the 2nd order linear D.E., we use the guessing

$$y = e^{st},$$

where s is a some constants.

$$\Rightarrow \quad y' = se^{st}, \quad y'' = s^2 e^{st}.$$

So $ay'' + by' + cy = 0$ is the same as

$$a(s^2 e^{st}) + b(se^{st}) + ce^{st} = 0$$

$$\Rightarrow \quad e^{st}(as^2 + bs + c) = 0.$$

$$\Rightarrow \quad as^2 + bs + c = 0.$$

Thus e^{st} solves the 2nd order D.E. if and only if

$$as^2+bs+c=0$$

The general solutions can be found as follows:

(I): If $as^2+bs+c=0$ has 2 distinct real roots λ_1, λ_2, then

$$\boxed{Y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t}} \quad k_1, k_2 \text{ any constants,}$$

is the general solution.

(II): If $as^2+bs+c=0$ has imaginary roots $\lambda = \alpha \pm \beta i$. Then

$$\begin{aligned} e^{\lambda t} &= e^{(\alpha+\beta i)t} \qquad (\text{choose } \lambda = \alpha+\beta i) \\ &= e^{\alpha t} e^{i\beta t} \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t) \qquad (\text{Euler's formula}) \\ &= e^{\alpha t}\cos\beta t + i\, e^{\alpha t}\sin\beta t. \end{aligned}$$

$$\Rightarrow \boxed{Y(t) = k_1 e^{\alpha t}\cos\beta t + k_2 e^{\alpha t}\sin\beta t}, \quad k_1, k_2 \text{ any constants.}$$

is the general solution.

(III): If $as^2+bs+c=0$ has repeated root λ. Then

$$\boxed{Y(t) = k_1 e^{\lambda t} + k_2 t e^{\lambda t}}, \quad k_1, k_2 \text{ any constants,}$$

is the general solution.

Transformation to 1st order linear system:

Introduce a new variable $v = y'$, thus

$$\begin{aligned} v' &= y'' \\ &= -\frac{b}{a}y' - \frac{c}{a}y \qquad (\because ay''+by'+cy=0) \\ &= -\frac{c}{a}y - \frac{b}{a}v. \end{aligned}$$

Thus the 2nd order D.E. is equivalent to the first order system

$$\begin{cases} y' = v \\ v' = -\frac{c}{a}y - \frac{b}{a}v, \end{cases}$$

or

$$Y' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} Y, \quad Y = \begin{pmatrix} y \\ v \end{pmatrix}, \; Y' = \begin{pmatrix} y' \\ v' \end{pmatrix}.$$

Simple observations:

① The characteristic polynomial of $\begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$ is

$$\begin{aligned} \det\begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a}-\lambda \end{pmatrix} &= \lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a} \\ &= \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} \\ &= \frac{1}{a}\left(a\lambda^2 + b\lambda + c\right). \end{aligned}$$

$\Longrightarrow$ Roots of $as^2+bs+c=0$ is the eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$.

② If λ is an eigenvalue of $\begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}$, then

$$\begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a}-\lambda \end{pmatrix}.$$

$\Rightarrow$ the corresponding eigenvector is $V_\lambda = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$.

Understanding the harmonic oscillator using the phase portrait:

The equation for harmonic oscillator

$$my'' + by' + ky = 0$$

is equivalent to

$$Y' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} Y.$$

the eigenvalues satisfies

$$m\lambda^2 + \cancel{k} b\lambda + k = 0,$$

so

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4km}}{2m}.$$

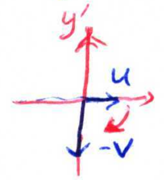
Thus the phase portrait depend on the eigenvalue λ. We split into several case according to the size of b, the friction.

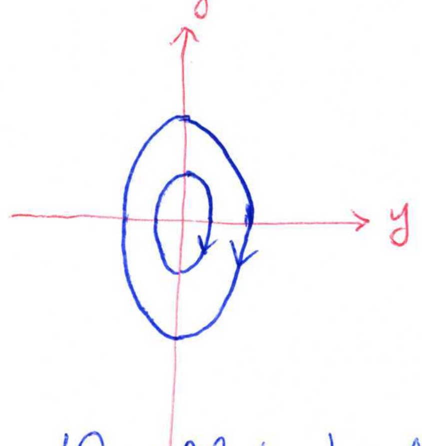
(i) $b=0$ (No friction, undamped)

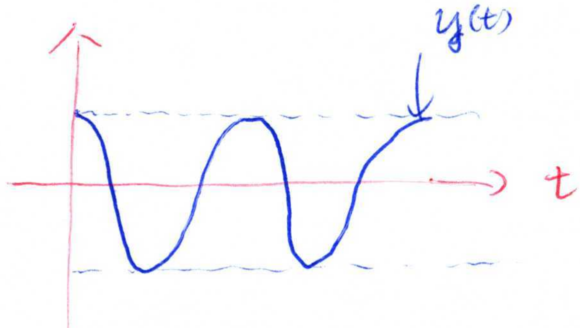
$$\Rightarrow \lambda = \frac{\pm\sqrt{-4km}}{2m} = \pm\sqrt{\frac{k}{m}}\, i$$

$\Rightarrow$ Imaginary eigenvalues with zero real parts $\Rightarrow$ (Center)

$$V_\lambda = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{\frac{k}{m}}\, i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{u} + i \underbrace{\begin{pmatrix} 0 \\ \sqrt{\frac{k}{m}} \end{pmatrix}}_{v}.$$

(so )





- Oscillate back and forth, periodic solution with period $T = 2\pi\sqrt{\frac{m}{k}}$.
- No "energy" loss.

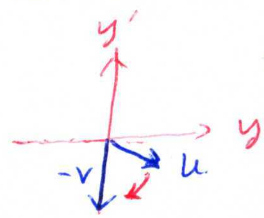
(ii) $b>0$, $b^2 < 4km$ (Underdamped).

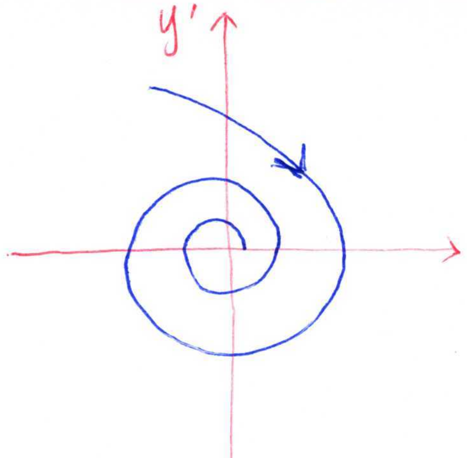
$$\lambda = \frac{-b}{2m} \pm \frac{\sqrt{b^2-4km}}{2m} = \frac{-b}{2m} \pm \frac{\sqrt{4km-b^2}}{2m}\, i$$

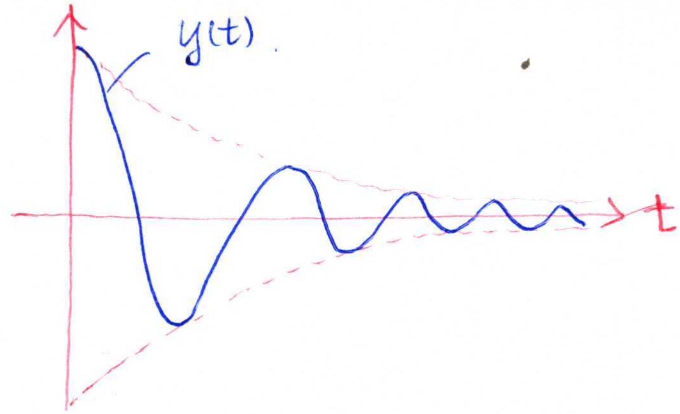
$\Rightarrow$ Complex eigenvalues with negative real part $\frac{-b}{2m}$ ($\because b>0, m>0$)

$\rightsquigarrow$ spiral sink.

$$V_\lambda = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{b}{2m} \end{pmatrix} + i \begin{pmatrix} 0 \\ \frac{\sqrt{4km-b^2}}{2m} \end{pmatrix} = u + iv$$

(so )





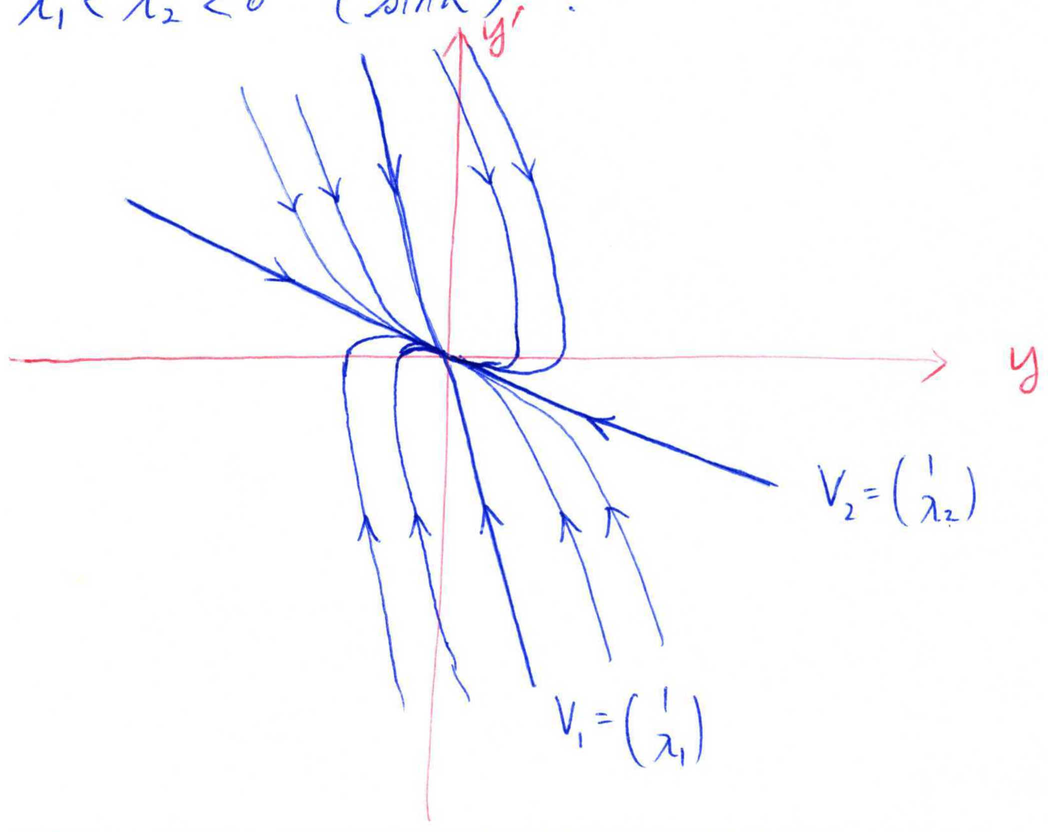
- The mass oscillate back and forth.
- The magnitude decreases and tends to zero. ~~but~~

(iii) $b>0$, $b^2>4km$ (Overdamped).

$$\lambda = \frac{-b \pm \sqrt{b^2-4km}}{2m}.$$

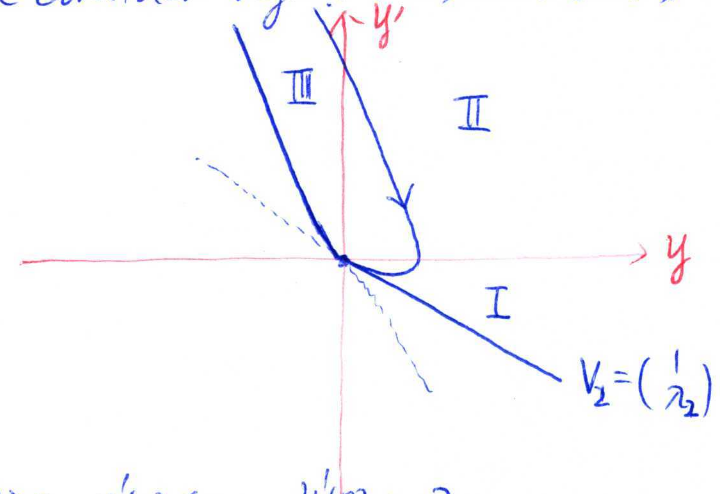
Note $b^2-4km < b^2$ (as $k,m>0$), so
$\sqrt{b^2-4km} < b$.

$\Longrightarrow$ $-b+\sqrt{b^2-4km} < 0$. And $-b-\sqrt{b^2-4km}<0$ obviously, thus the system has 2 distinct negative eigenvalues $\lambda_1 < \lambda_2 < 0$ (sink).



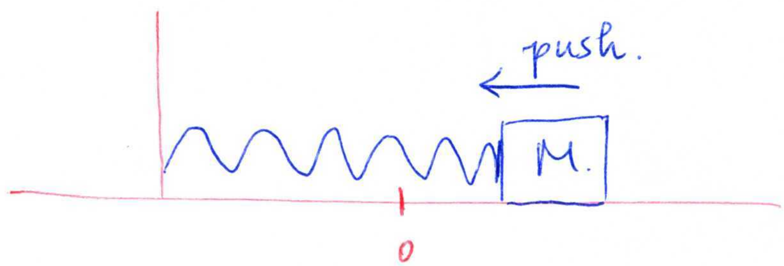
- All solutions converges to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, but the precise motion depends on initial conditions $y(0)$, $y'(0)$:

e.g. We consider region (I), (II), (III):

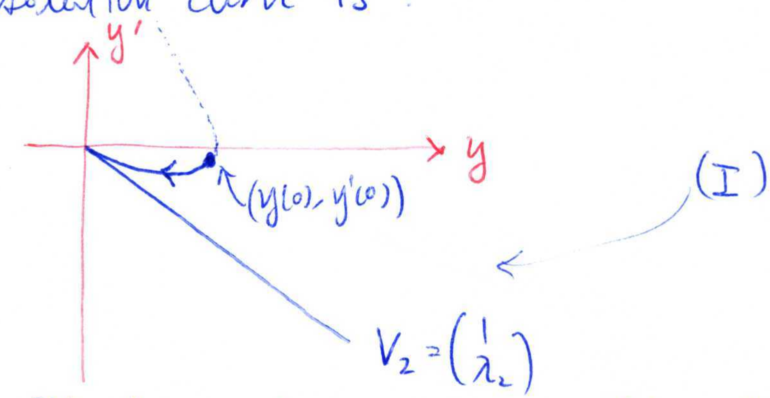


(I) $y(0) > 0$, $y'(0) < 0$, $\frac{y'(0)}{y(0)} > \lambda_2$.

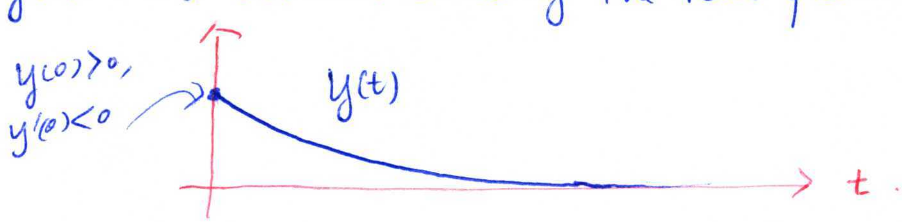
(Interpretation : M initially is on the right $y(0) > 0$, pushed to the left $y'(0) < 0$, while the push is small $\frac{|y'(0)|}{y(0)} < |\lambda_2|$.)



Then the solution curve is :

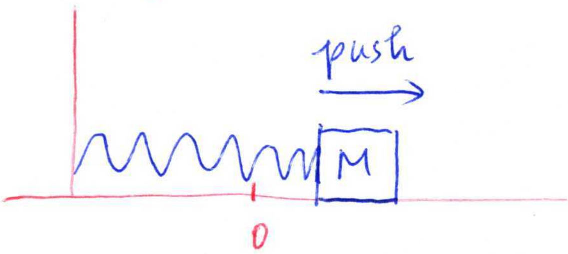


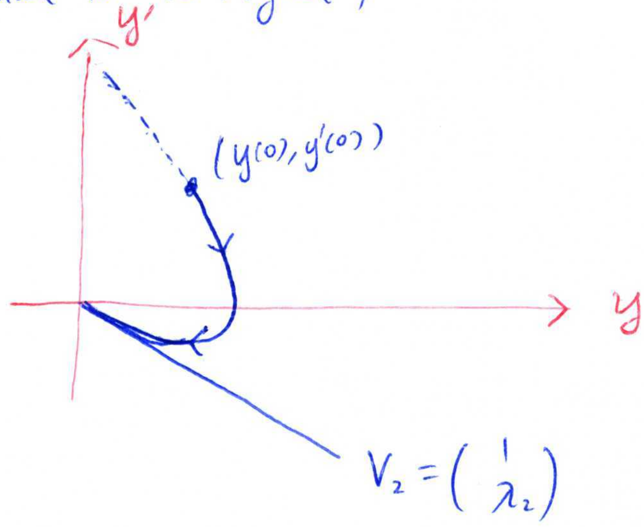
$y(t) \to 0$, without crossing the rest position $y = 0$:



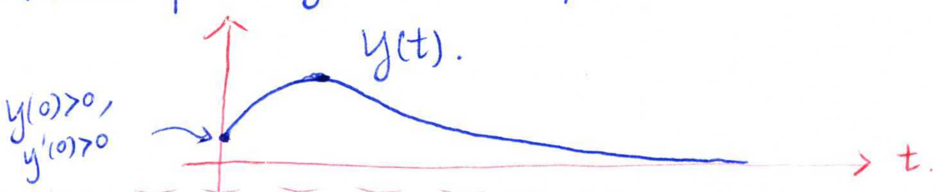
(II): $y(0)>0$, $y'(0)>0$

(Initially on the right, then pushed to the right)

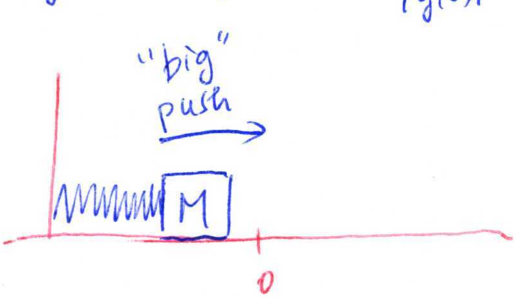


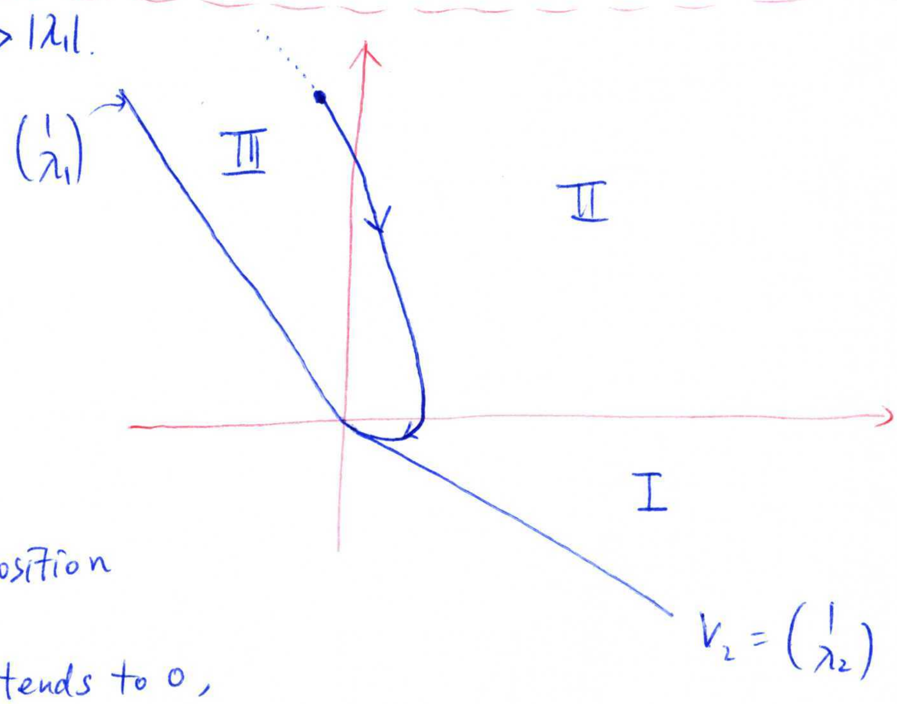


- M moves to the right,
- then stop, and
- changes its direction and tends to 0,
- without passing the rest position

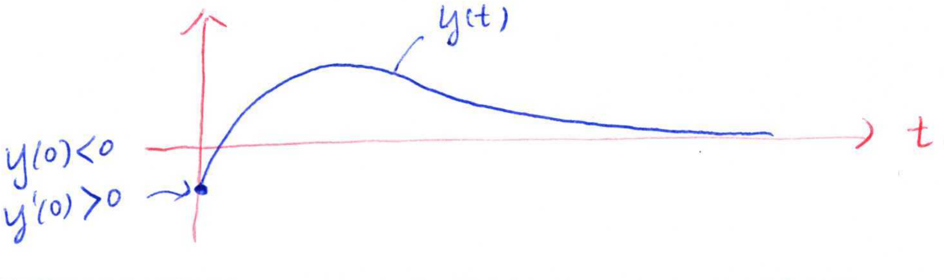


(III): $y(0)<0$, $y'(0)>0$, $\frac{y'(0)}{|y(0)|} > |\lambda_1|$.





- M moves to the right,
- passes through the rest position
- then stop,
- changes its direction and tends to 0,
- without passing the rest position (again).

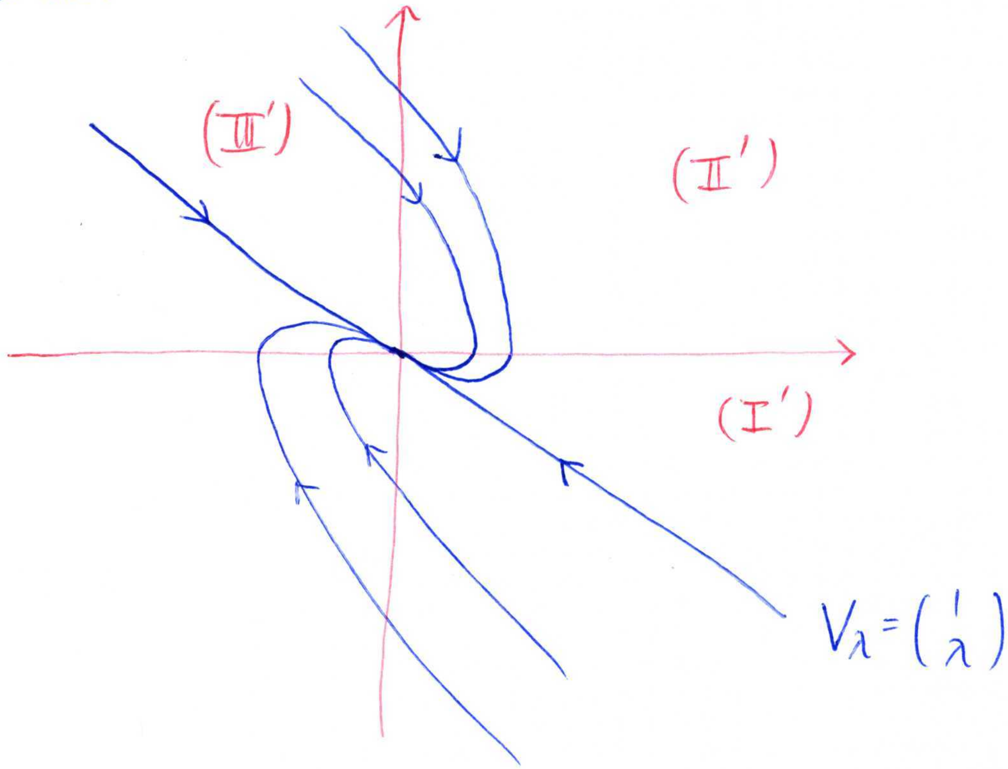


Lastly:

(iv) $b > 0$, $b^2 = 4km$ (critically damped).

$$\lambda = \frac{-b}{m} \text{ (repeated negative root).}$$

phase portrait:



- All solutions tend to 0.
- Interpretation of (I'), (II'), (III') similar to (I), (II), (III) in the overdamped case.