Linear system: Complex eigenvalues:

e.g. $Y' = AY$, $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

characteristic polynomial: $p(\lambda) = \det(A - \lambda I)$

$$= \det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix}$$

$$= (2-\lambda)^2 + 1.$$

$\Rightarrow$ A has eigenvalues $\lambda = 2 \pm i$. ($i = \sqrt{-1}$)

Remark: Let v be ~~a~~ an eigenvector of λ, then

$$e^{\lambda t} v$$

is a "solution" to $Y' = AY$, even when λ is complex

- But $e^{\lambda t} v$ cannot be sketch in the x-y plane: $\underline{e^{\lambda t}}$ and $\underline{v}$ are both complex.

e.g. $A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ has $\lambda = 2 \pm i$. For $\lambda = 2 + i$

$A - \lambda I = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \Rightarrow v = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector.

Basic property:

① If λ is an eigenvalue, with eigenvector v, then $\bar{\lambda}$ is also an eigenvalue, with eigenvector $\bar{v}$.

(Recall: if $\lambda = \alpha + \beta i$, then $\underset{\uparrow}{\bar{\lambda}} = \alpha - \beta i$), if $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then

"conjugate" of λ

$\bar{v} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}$.

Reason for ①: If $Av = \lambda v$,

$$\Rightarrow \overline{Av} = \overline{\lambda v} \Rightarrow \bar{A}\bar{v} = \bar{\lambda}\bar{v}.$$

$$\Rightarrow \underline{A\bar{v} = \bar{\lambda}\bar{v}}$$ (we used $\bar{A} = A$, since the entries of A are real).

so $\bar{\lambda}$ is an eigenvalue, with eigenvector $\bar{v}$.

② : If $e^{\lambda t}v = \underbrace{Y_{re}(t)}_{\text{real part}} + i\underbrace{Y_{im}(t)}_{\text{imaginary part}}$, where λ, v are eigenvalues, eigenvector respectively,

$\Longrightarrow Y_{re}, Y_{im}$ both satisfy $Y' = AY$.

Reason for ②: Since $e^{\lambda t}v$ satisfies $Y' = AY$

$$\Longrightarrow (Y_{re} + iY_{im})' = A(Y_{re} + iY_{im})$$

$$\Longrightarrow Y_{re}' + iY_{im}' = AY_{re} + iAY_{im}.$$

real parts

imaginary parts

$$\Longrightarrow Y_{re}' = AY_{re}, \quad Y_{im}' = AY_{im}.$$

By ② and Linearity Principle:

- Let λ be a complex eigenvalue of A and v is an eigenvector. Write $e^{\lambda t} v = Y_{re}(t) + i\, Y_{im}(t)$.

Then the general solution to $Y' = AY$ is given by.

$$\boxed{Y(t) = k_1 Y_{re}(t) + k_2 Y_{im}(t).}$$

Q: How to explicitly write $e^{\lambda t}$ as $Y_{re}(t) + i\, Y_{im}(t)$??

Recall: Euler's formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

e.g. Back to $Y' = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} Y$. $\lambda = 2+i$, $v = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

$$\Rightarrow e^{\lambda t} v = e^{(2+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} = e^{2t+it}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= e^{2t} e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} \overset{\text{Euler's}}{=} e^{2t}(\cancel{\cos t}\cos t + i\sin t)\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{pmatrix}$$

$(\because i^2 = -1)$

$$= e^{2t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\, e^{2t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

$Y_{re}(t)$ $\qquad$ $Y_{im}(t)$.

$\Rightarrow$ General Solution to $Y' = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} Y$ is $k_1 e^{2t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

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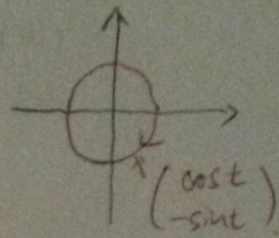
Phase portrait:

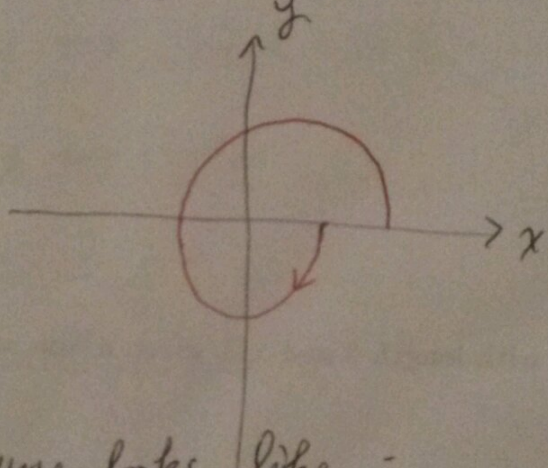
e.g. How to sketch $e^{2t}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$?

Note: • $\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$ periodic, circle around the origin

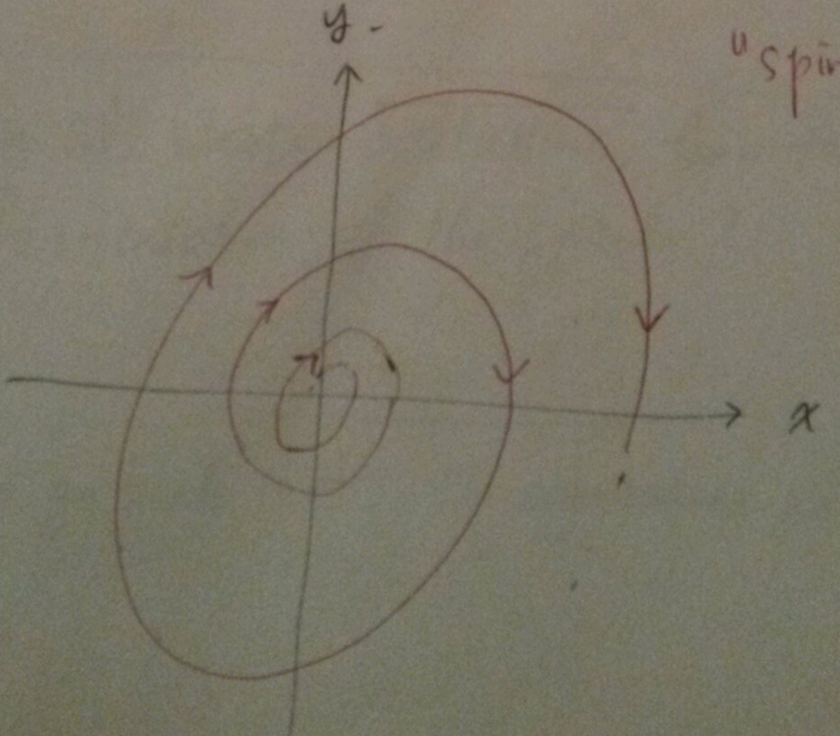


• e^{2t} increasing in t.

i.e. After a complete term, $e^{2t}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}$ is ~~far~~ further away from the origin, so we have



Thus the solution curve looks like:



In general, let $\lambda = \alpha + \beta i$, then

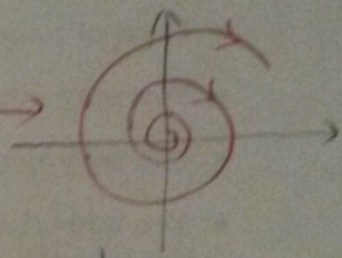
$$e^{\lambda t} v = e^{(\alpha+\beta i)t} v = e^{\alpha t} e^{i\beta t} v$$

$$= e^{\alpha t} (\cos\beta t + i \sin\beta t) v .$$

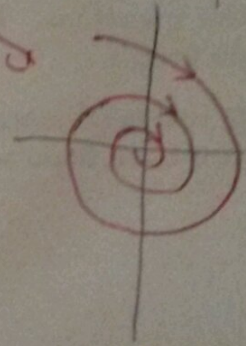
$$= e^{\alpha t} (\text{something periodic, circling around the origin}).$$

Thus we have the following 3 cases:

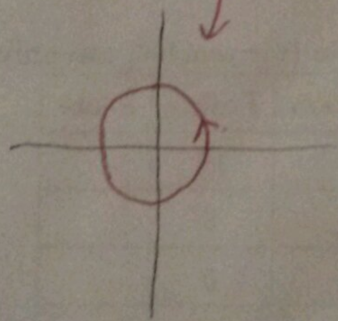
$\alpha > 0 \Rightarrow e^{\alpha t}$ increasing $\Rightarrow$ Spiral source



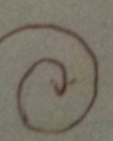
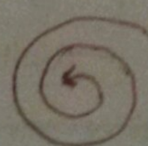
$\alpha < 0 \Rightarrow e^{\alpha t}$ decreasing $\Rightarrow$ Spiral sink.



$\alpha = 0 \Rightarrow e^{0t} = 1$ is constant $\Rightarrow$ Center



Orientation: In all ~~situation~~ 3 situations, how to find out the orientation of the spirals?

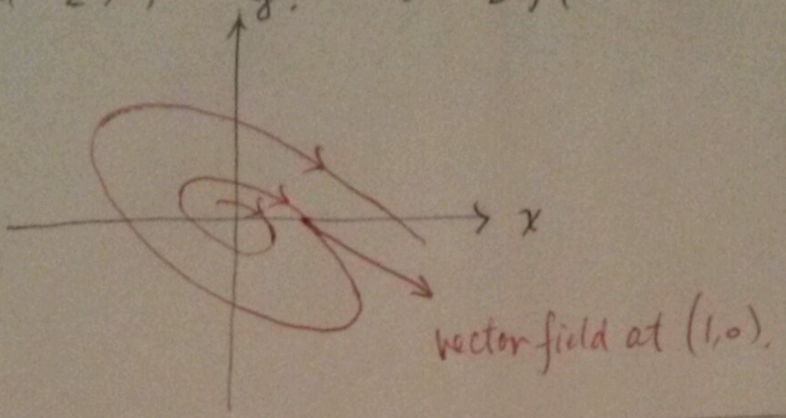
 or  ?

Rmk: Need to go back to A: the eigenvalues alone cannot tell.

To find the orientation, pick the point $(1,0)$ and consider the vector field at that point.

e.g. $A=\begin{pmatrix}2&1\\-1&2\end{pmatrix}$, then $\begin{pmatrix}2&1\\-1&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\-1\end{pmatrix}$.

$\Rightarrow$



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e.g. $Y'=\begin{pmatrix}2&-1\\1&2\end{pmatrix}Y$. has $\lambda=\underbrace{2}_{>0}\pm i$. $\Rightarrow$ spiral source.

At $(1,0)$. $\begin{pmatrix}2&-1\\1&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$.