Nullclines

Given a system

$$\begin{cases} \frac{dx}{dt} = f(x,y), & \cdots \quad ① \\ \frac{dy}{dt} = g(x,y). \end{cases}$$

Definition: The x-nullcline (resp. y-nullcline) of ① is the curve which satisfies $f(x,y)=0$ (resp. $g(x,y)=0$).

e.g.

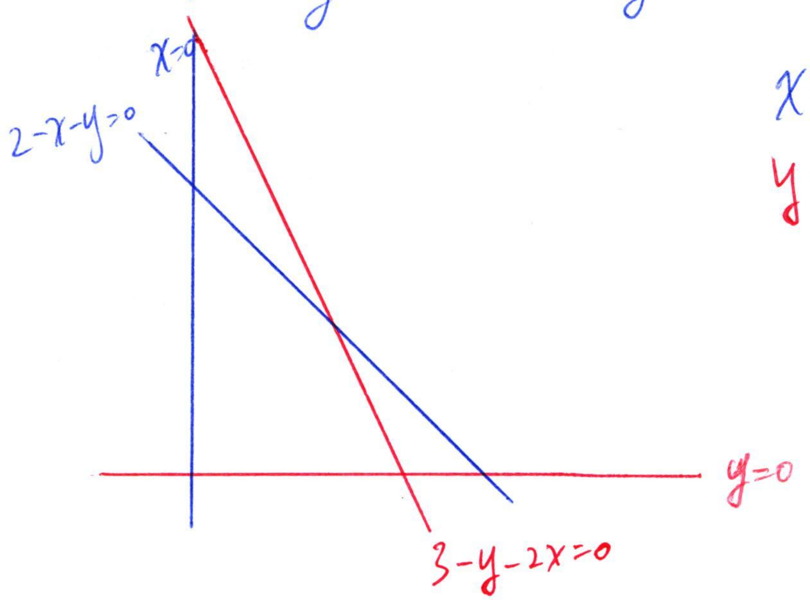
$$\begin{cases} \frac{dx}{dt} = x(2-x-y) & \cdots \quad ② \\ \frac{dy}{dt} = y(3-y-2x) \end{cases}$$

x-nullclines: $x(2-x-y)=0$

$\Rightarrow$ $x=0$ or $y=0$.

y-nullclines: $y(3-y-2x)=0$

$\Rightarrow$ $y=0$ or $3-y-2x=0$.



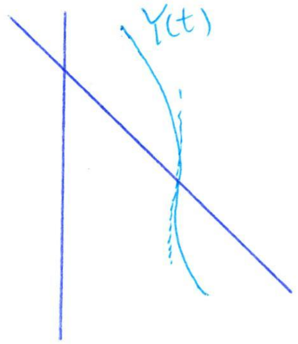
x- nullclines

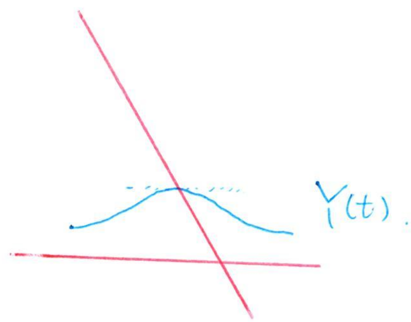
y-nullclines.

Remark:

- Equilibria = intersection of x-nullcline and y-nullclines
  ($f(x,y)=0$) ($g(x,y)=0$).

- Solution curve $Y(t)$ passes through the x-nullcline vertically, and the y-nullclines horizontally.





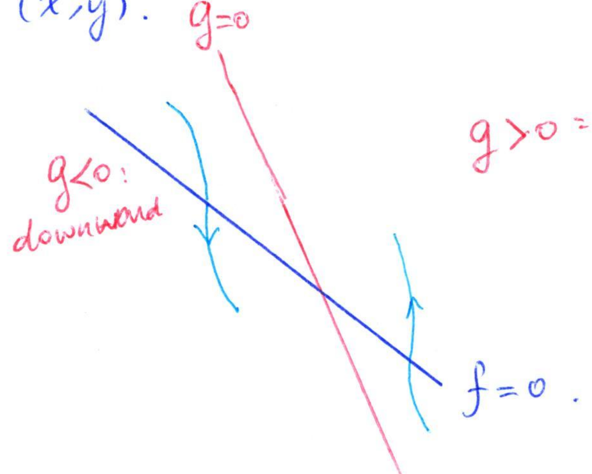
Reason: On the x-nullcline, $f(x,y)=0$ and so

$$Y'(t) = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 \\ g(x,y) \end{pmatrix} \leftarrow \text{vertical vector.}$$

Similar for y-nullclines.

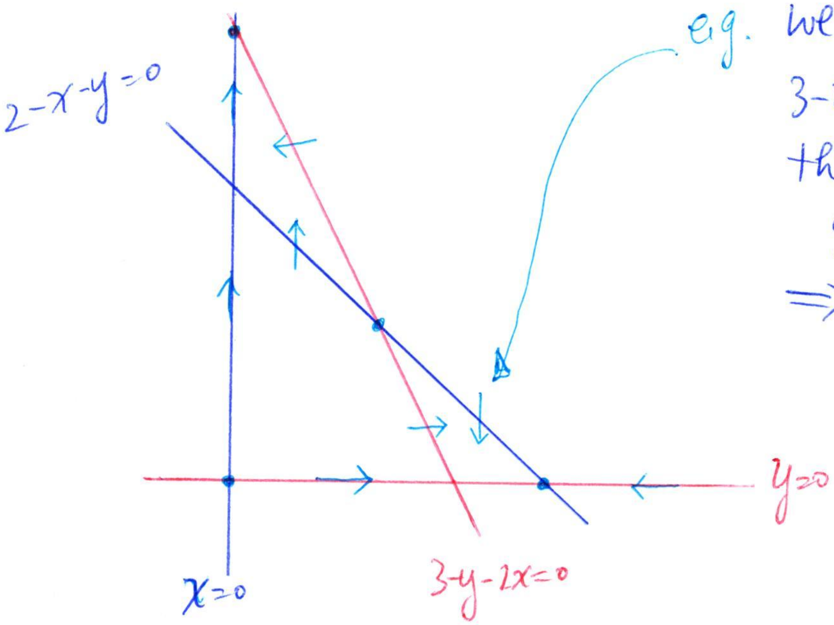
Indeed we can say more: If $Y(t)$ passes through the x-nullcline at (x,y) and $g(x,y)>0$, then $Y(t)$ is going upward at (x,y).

e.g.



We use uparrow $\uparrow$ to indicate $g>0$, downarrow $\downarrow$ to indicate $g<0$ on the x-nullcline.

e.g. Back to system ②, we have.

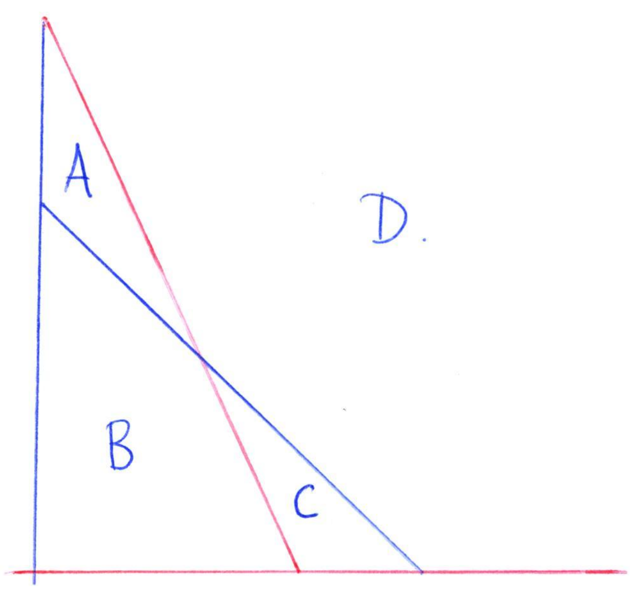


e.g. we have $y>0$ and $3-y-2x<0$ at this point, thus

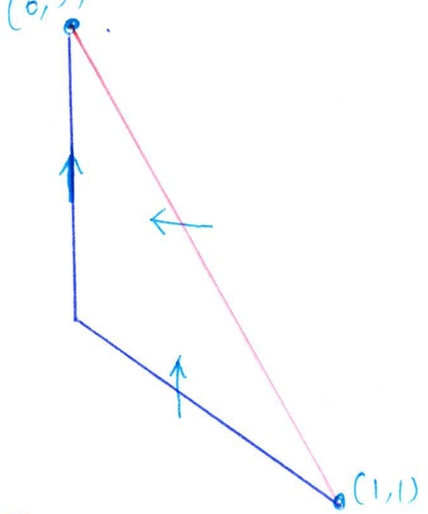
$$g(x,y) = y(3-y-2x)<0.$$

$\Rightarrow$ downarrow.

Now we are ready to sketch the phase portrait. We split into the following four regions (consider $x, y \geq 0$ for simplicity)

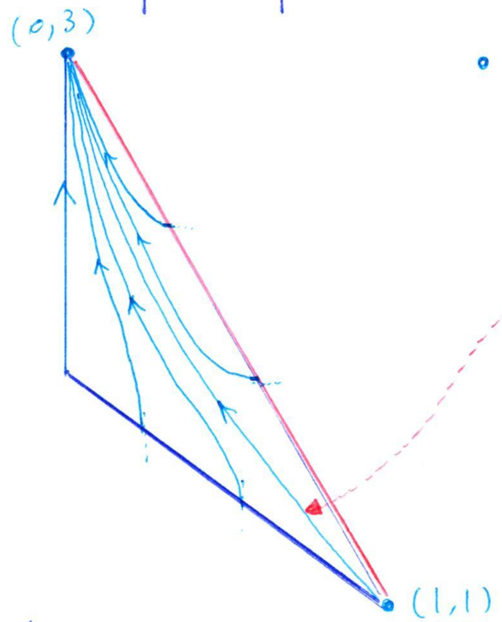


A :



- ***Stable*** ***region***: $Y(t)$ stays in the region for all $t \geq t_0$ if $Y(t_0)$ is in the region
  (Since all arrows on the boundary of A are **NOT** pointing outward)
- All $Y(t)$ moves in the direction $\uparrow + \leftarrow = \nwarrow$, thus
- All $Y(t)$ tends to $(0,3)$ as $t \to +\infty$.

Thus the phase portrait in A looks like:

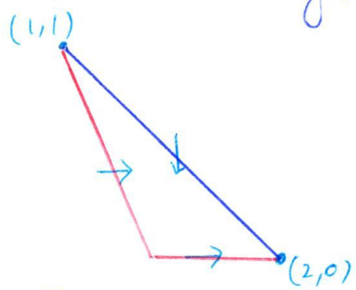


- ~~All~~ ~~sol~~ Solutions ~~enter~~ either
  - Enter A from the x-nullcline vertically,
  - Enter A from the y-nullcline horizontally,
  - there one special $Y(t)$ "connecting" $(1,1)$ and $(0,3)$, in the sense that

$$\lim_{t \to +\infty} Y(t) = (a,3),$$

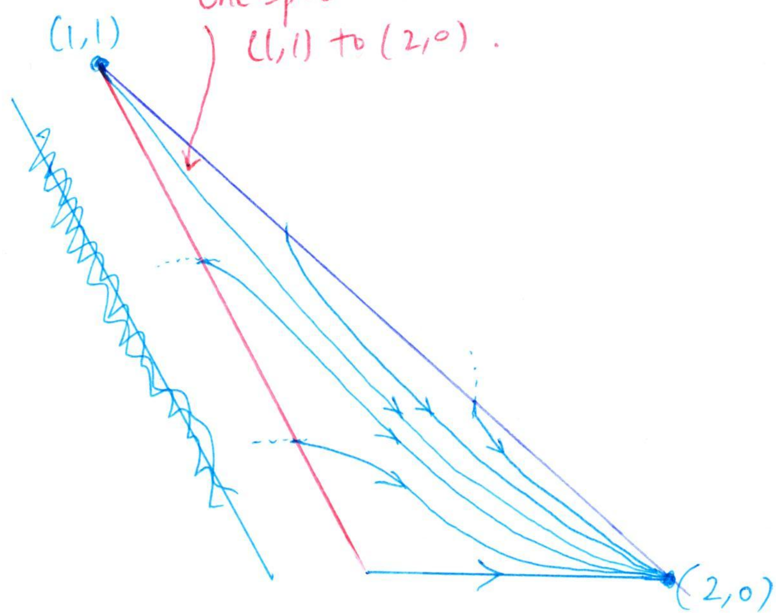
$$\lim_{t \to -\infty} Y(t) = (1,1).$$

C ~~B~~ : Similar as region A:

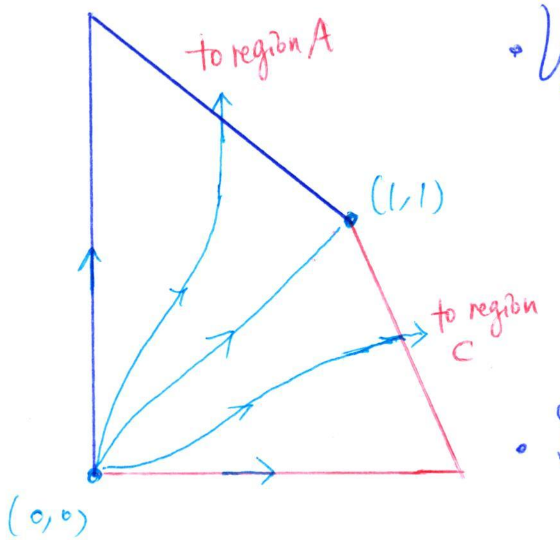


- Stable region,
- Direction : $\downarrow + \rightarrow = \searrow$
- All $Y(t)$ tend to $(2,0)$ as $t \to +\infty$.

One special solution "connecting" $(1,1)$ to $(2,0)$.

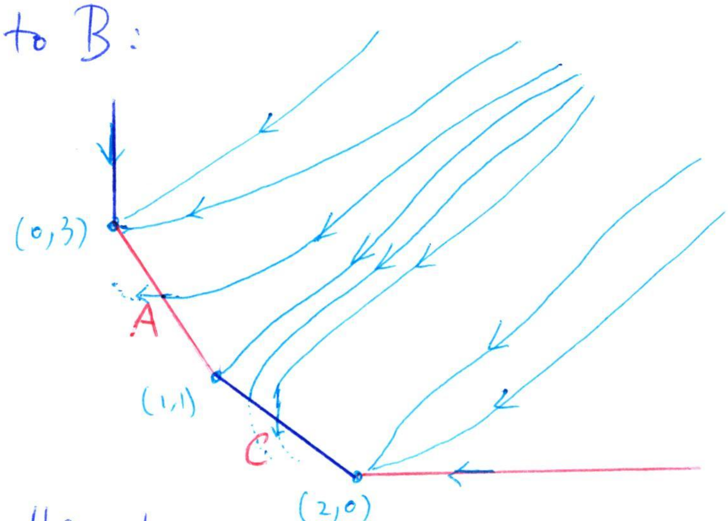


B:

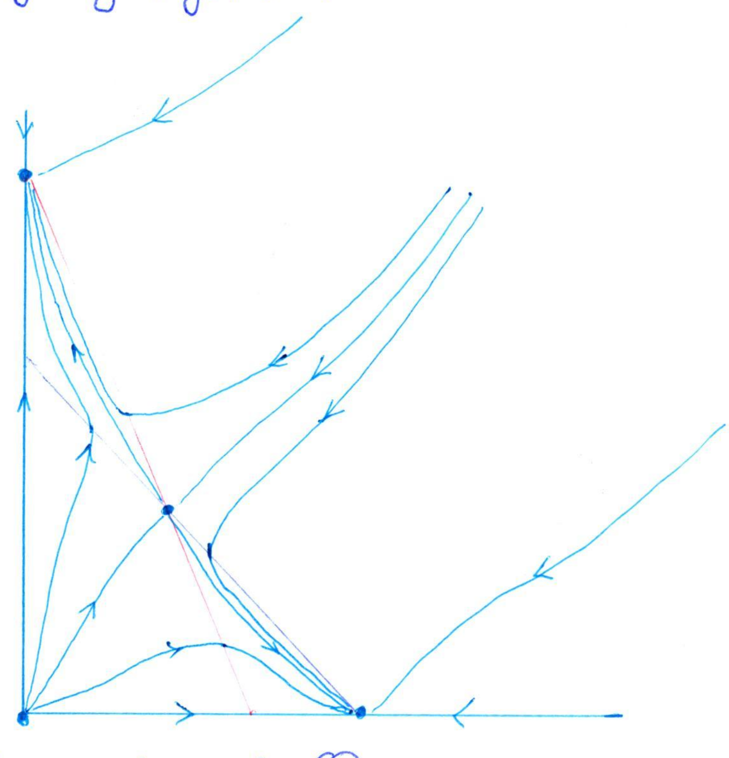


- **Unstable** region: $Y(t)$ might either
  - leave B via the x-nullcline,
  - leave B via the y-nullcline, or
  - $Y(t) \to (1,1)$ as $t \to +\infty$.
- Direction: $\uparrow + \rightarrow = \nearrow$

D: Similar to B:



Putting everything together:



phase portrait of 2.

Remark: $(1,1)$ looks like a saddle: indeed this is true and can be checked using linearization:

$$DF_{(x,y)} = \begin{pmatrix} 2-2x-y & -x \\ -2y & 3-2y-2x \end{pmatrix} \Rightarrow DF_{(1,1)} = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}.$$

has $T = -2$ and $D = -1 < 0 \Rightarrow$ saddle.

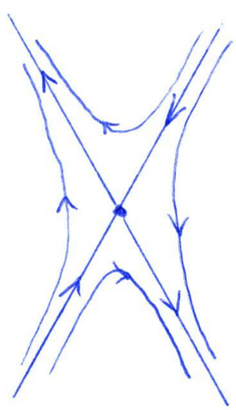
Moreover, we can improve the sketch of ② around $(1,1)$ by finding the phase portrait of the linear system $DF_{(1,1)}$:

It has eigenvalues

$$\lambda_1 = -1-\sqrt{2} < 0 \quad , \quad \lambda_2 = -1+\sqrt{2} > 0.$$

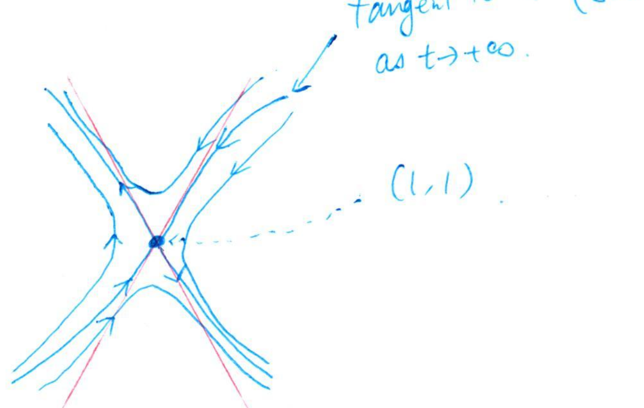
with corresponding eigenvector

$$V_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$



phase portrait of

$Y' = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix} Y$



local phase portrait of ② around $(1,1)$.

Another example: Given

$$\begin{cases} \dfrac{dx}{dt} = x(x-1) \\ \dfrac{dy}{dt} = x^2 - y. \end{cases}$$

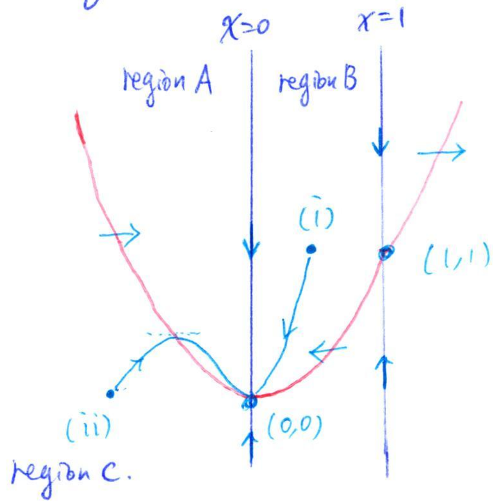
(a) Find the limit of $Y(t)$ as $t \to +\infty$ if

(i) $Y(0) = (0.5, 1)$,

(ii) $Y(0) = (-1, 0)$.

(b) Sketch the phase portrait.

Solution: x: nullclines : $x=0$ or $x=1$, y-nullclines : $y = x^2$.



(i) $(0.5, 1)$ is in B (stable), direction $\swarrow$

$\Rightarrow Y(t) \to (0,0)$ as $t \to +\infty$.

(ii) $(-1, 0)$ in C (unstable), direction $\nearrow$

$\Rightarrow Y(t) \to (0,0)$ as $t \to +\infty$.

(b) Phase portrait: