Linearization: Sketching phase protrait around an equilibrium.

Given a nonlinear system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \quad \cdots \quad (1)$$

Goal: Understang the phase portrait of (1) around an equilibrium.

Recall: (x_0, y_0) is called equilibrium of the system (1) if $f(x_0,y_0) = 0 = g(x_0,y_0)$.

e.g.

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -y - \sin x \end{cases} \quad \cdots \quad (2)$$

has equilibria $(0,0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, ...

To understand the behavior of (2) around (e.g.) $(0,0)$, consider the taylor expansion of $\sin x$ at $x=0$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

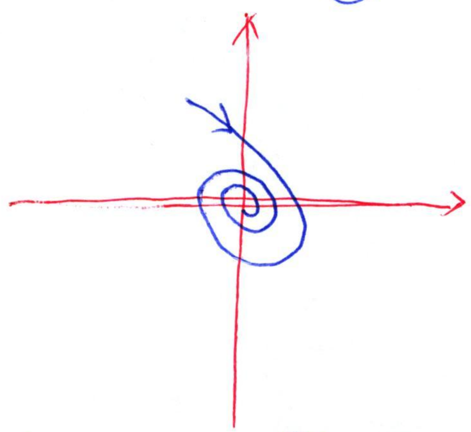
$$= x + \text{higher order terms.}$$

$\Longrightarrow$ $\sin x \approx x$ when x is small.

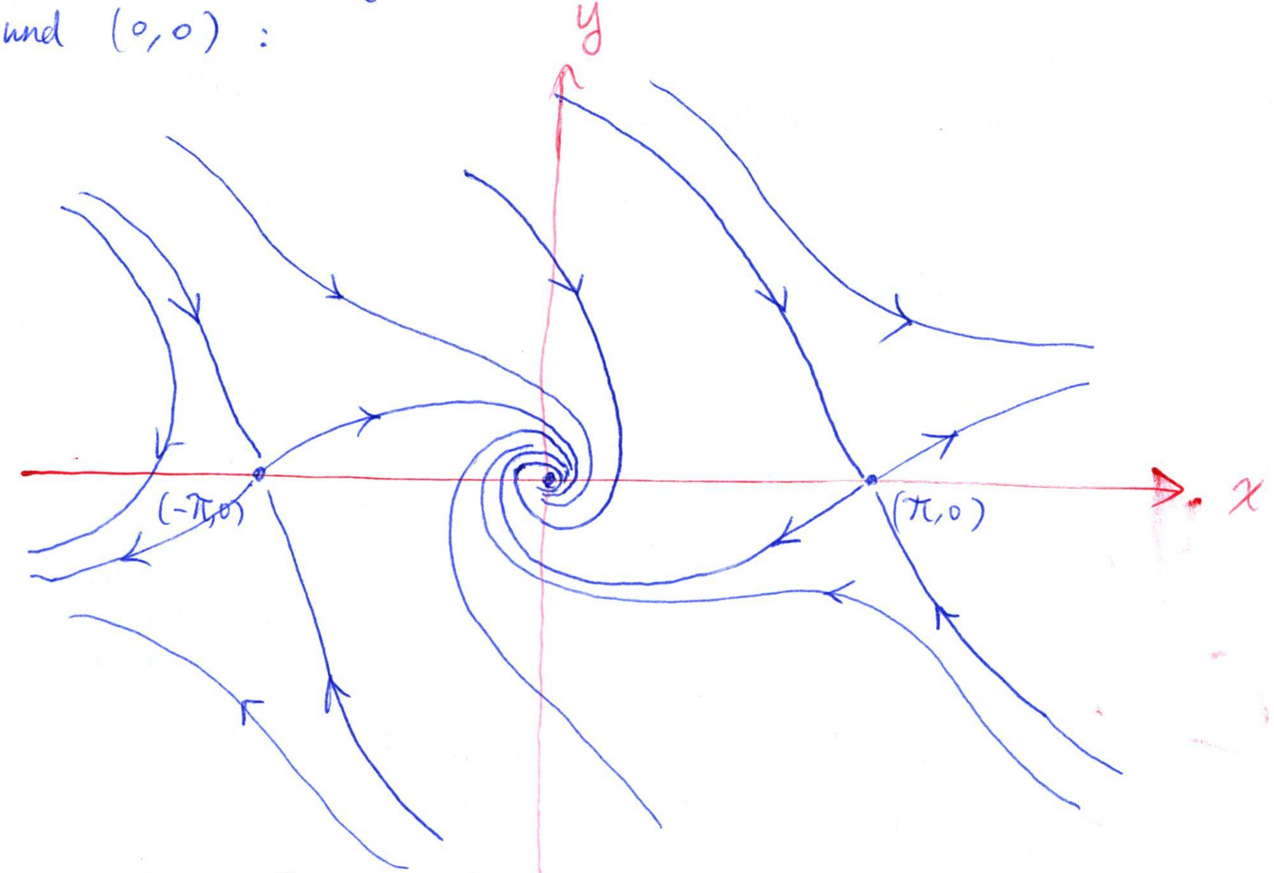
Consider the following "linear approximation" of system ② at $(0,0)$:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -y - x \end{cases} \qquad \cdots \quad (2')$$

It's linear, with matrix $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. The matrix has $T = 0 + (-1) = -1$, $D = 1 \Rightarrow$ (2') is a spiral sink:



Indeed, the above system (2') approximates system ② well around $(0,0)$:



Phase portrait of system ②.

In general, we have the following linear approximation:

$$f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + \text{h.o.t.}$$

$$g(x,y) = g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial g}{\partial y}(x_0,y_0)(y-y_0) + \text{h.o.t.}$$

So at an equilibrium,

$$f(x,y) \approx \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$g(x,y) \approx \frac{\partial g}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial g}{\partial y}(x_0,y_0)(y-y_0)$$

and system ① can be approximated (around (x_0,y_0)) by

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \dfrac{\partial f}{\partial y}(x_0,y_0)(y-y_0) \\ \dfrac{dy}{dt} = \dfrac{\partial g}{\partial x}(x_0,y_0)(x-x_0) + \dfrac{\partial g}{\partial y}(x_0,y_0)(y-y_0), \end{cases}$$

or

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0,y_0) & \dfrac{\partial f}{\partial y}(x_0,y_0) \\ \dfrac{\partial g}{\partial x}(x_0,y_0) & \dfrac{\partial g}{\partial y}(x_0,y_0) \end{pmatrix} \begin{pmatrix} x-x_0 \\ y-\cancel{x_0}\, y_0 \end{pmatrix} \quad \cdots \quad ①$$

The above matrix is called the linearization of the system ① at (x_0,y_0), and is denoted $D\mathcal{F}_{(x_0,y_0)}$.

e.g. For system ②, we have $f(x,y) = y$, $g(x,y) = -y - \sin x$.

$$\Rightarrow D\mathcal{F}_{(x,y)} = \begin{pmatrix} 0 & 1 \\ -\cos x & -1 \end{pmatrix} \quad \text{and thus}$$

$$D\mathcal{F}_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad D\mathcal{F}_{(\pi,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{etc.}$$

Note that

- $DF_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ is the matrix corresponding to system (2').

- $DF_{(\pi,0)} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ has $T=-1$, $D=-1$. Thus the linearized system

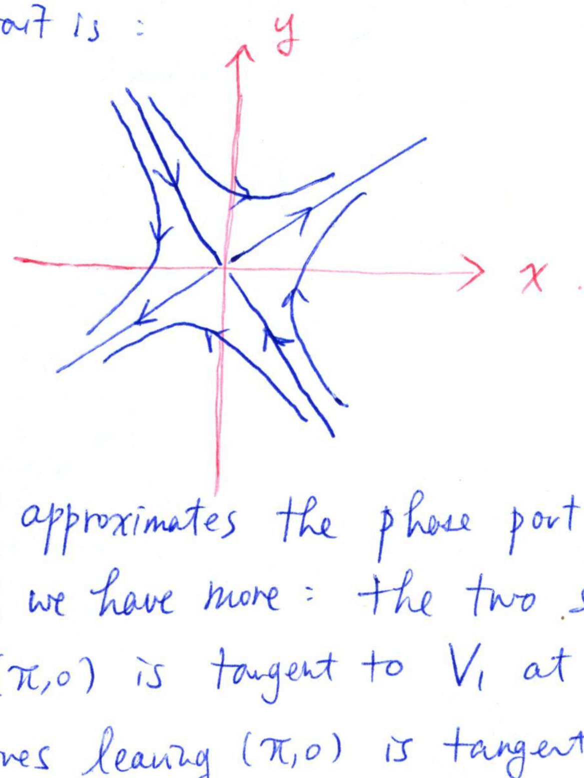
at $(\pi, 0)$ is a saddle. It has eigenvalues

$$\lambda_1 = -\frac{1}{2} - \frac{\sqrt{5}}{2}, \qquad \lambda_2 = \frac{\sqrt{5}-1}{2}$$

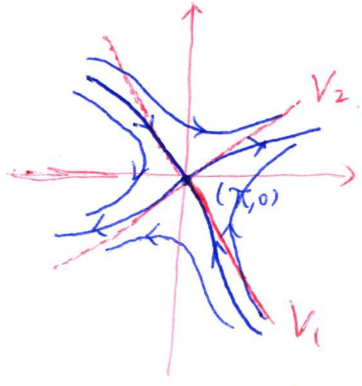
with corresponding eigenvectors

$$V_1 = \begin{pmatrix} -1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}.$$

The phase portrait is :



Again it well approximates the phase portrait of (2) around $(\pi, 0)$. Indeed we have more : the two solution curves which tend towards $(\pi,0)$ is tangent to V_1 at $(\pi, 0)$, and the two solution curves leaving $(\pi,0)$ is tangent to V_2.



(Phase portrait of (2) locally at $(\pi,0)$)

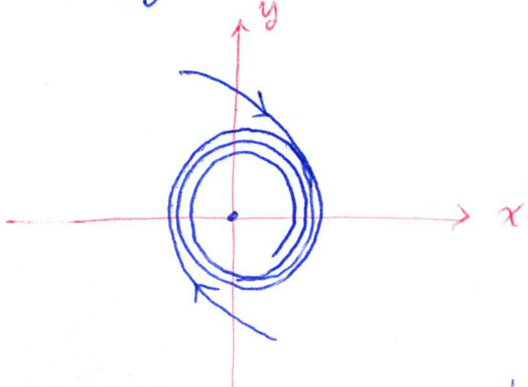
In general, ~~ha~~ moreover, linearization does not work when

- the linearization has purely imaginary eigenvalues, and
- the linearization has zero eigenvalues.

e.g.

$$\begin{cases} \dfrac{dx}{dt} = y - (x^2+y^2)\,x \\ \dfrac{dy}{dt} = -x - (x^2+y^2)\,y \end{cases} \quad \cdots \quad (3)$$

The origin $(0,0)$ is an equilibrium of (3), with linearization $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The matrix has purely imaginary eigenvalues and thus is a center, ~~not~~ but system (3) has no periodic solutions:



phase portrait of (3).

Note the above two situations are the borderline case. ~~and~~ Other than those two cases, the linearized system at an equilibrium well-approximates the ~~ordinary~~ original nonlinear system.