Bifurcation:

One parameter family of D.E.:

$$y' = f_\mu(y).$$

(different $\mu \Rightarrow$ different $f_\mu(y) \Rightarrow$ different D.E.).

e.g. $\quad y' = y^2 - 2y + \mu$

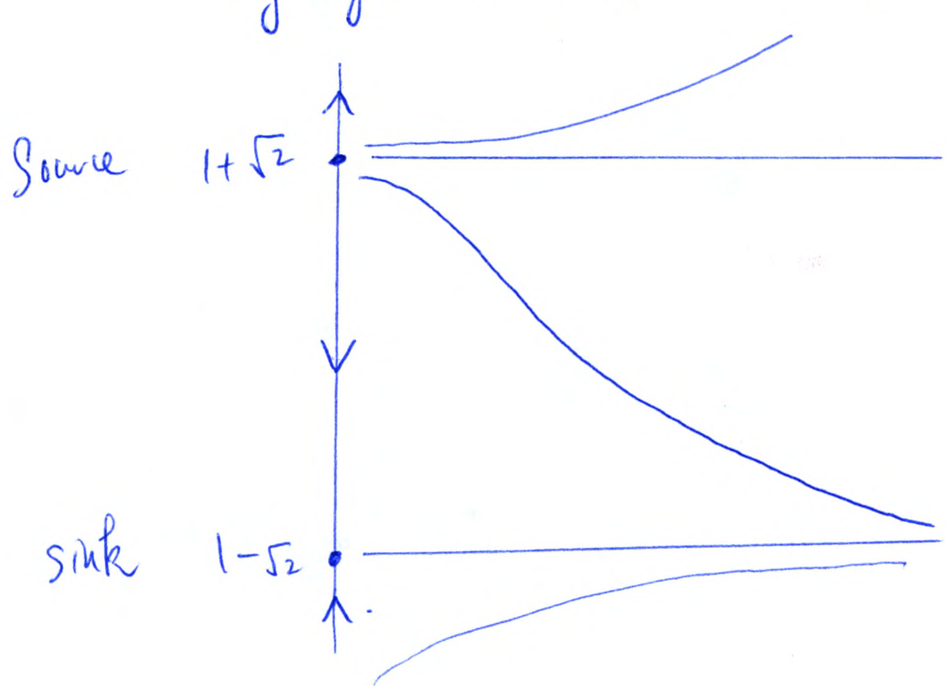
$\mu = 0$: $\quad y' = y^2 - 2y$

$\mu = 1$: $\quad y' = y^2 - 2y + 1$

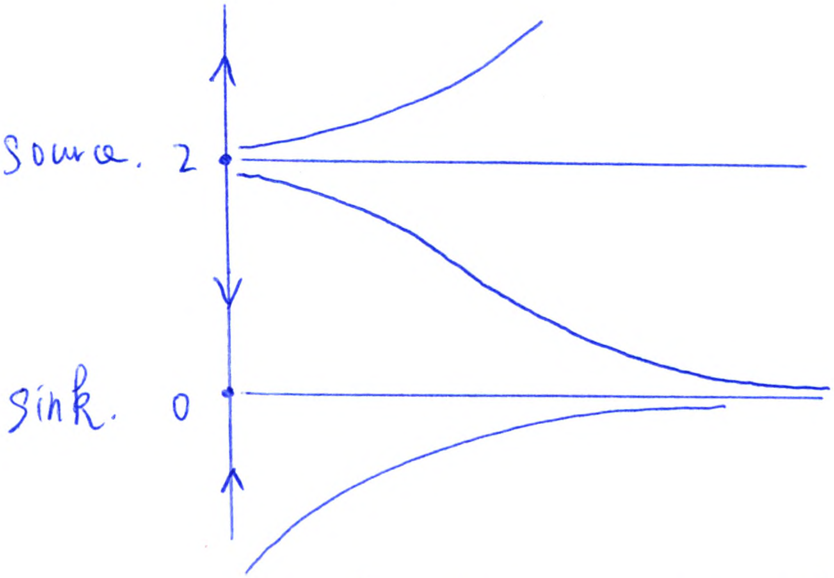
$\mu = 2$: $\quad y' = y^2 - 2y + 2.$

$\mu = -1$ $\quad y' = y^2 - 2y - 1$

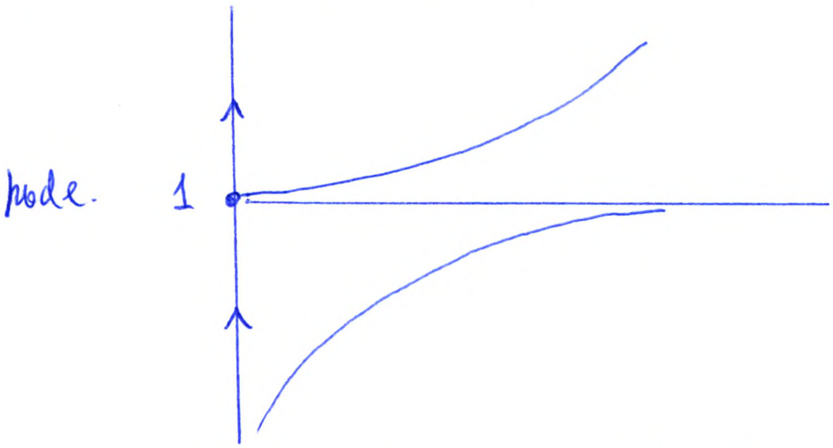
$$y^2 - 2y - 1 = 0 \Rightarrow (y-1)^2 = 2 \Rightarrow y = 1 \pm \sqrt{2}.$$



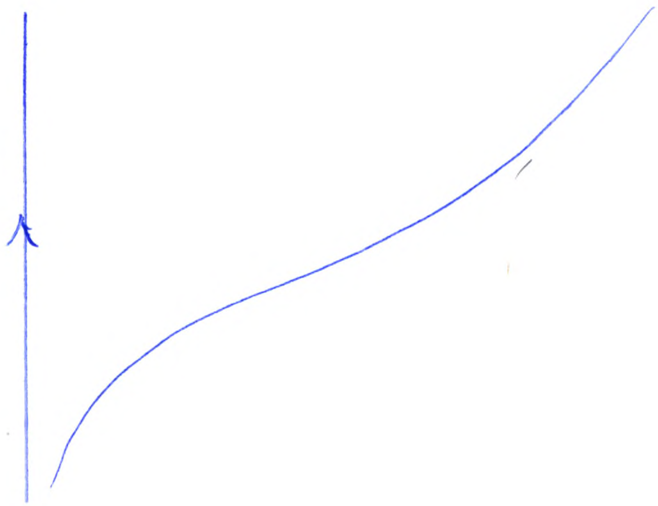
$\mu=0$. $y' = y^2 - 2y = y(y-2)$



$\mu = 1$: $y' = y^2 - 2y + 1 = (y-1)^2$.

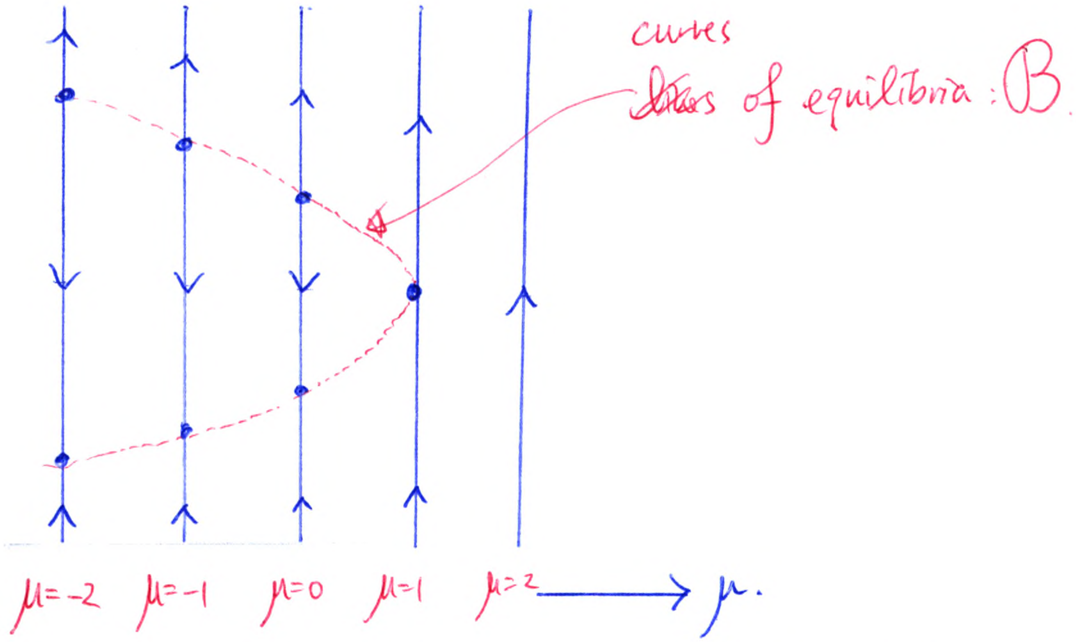


$\mu=2$. $y' = y^2 - 2y + 2 = (y-1)^2 + 1$. (No equilibrium).



Note:
"$\mu=-2$" similar to "$\mu=0$": 2 equilibria, one source, one sink.
Which is different from
$\mu=1$ (only one node)
$\mu=2$ (No equilibrium)

Put the phase line together



From the above picture, the behavior of $y' = y^2 - 2y + \mu$ are similar when $\mu < 1$, but change when $\mu = 1$.

Call $\mu = 1$ the $\boxed{\text{bifurcation value}}$ of the family $y' = y^2 - 2y + \mu$.

To find the dotted red line : for each fixed μ, solve for $f_\mu(y) = 0$. (i.e. equilibria of $y' = f_\mu(y)$).

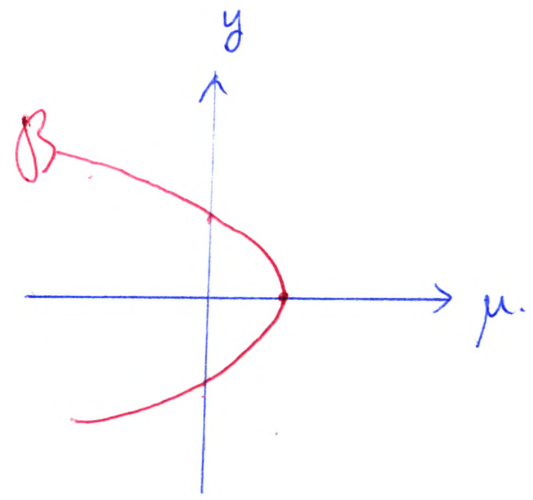
i.e. The curve of equilibria for $y' = f_\mu(y)$ is

$$\mathcal{B} = \{ (\mu, y) : f_\mu(y) = 0 \}.$$

e.g. $y' = y^2 - 2y + \mu$.

Set $y^2 - 2y + \mu = 0$, or $\mu = 2y - y^2$.

$= -(y-1)^2 + 1$.

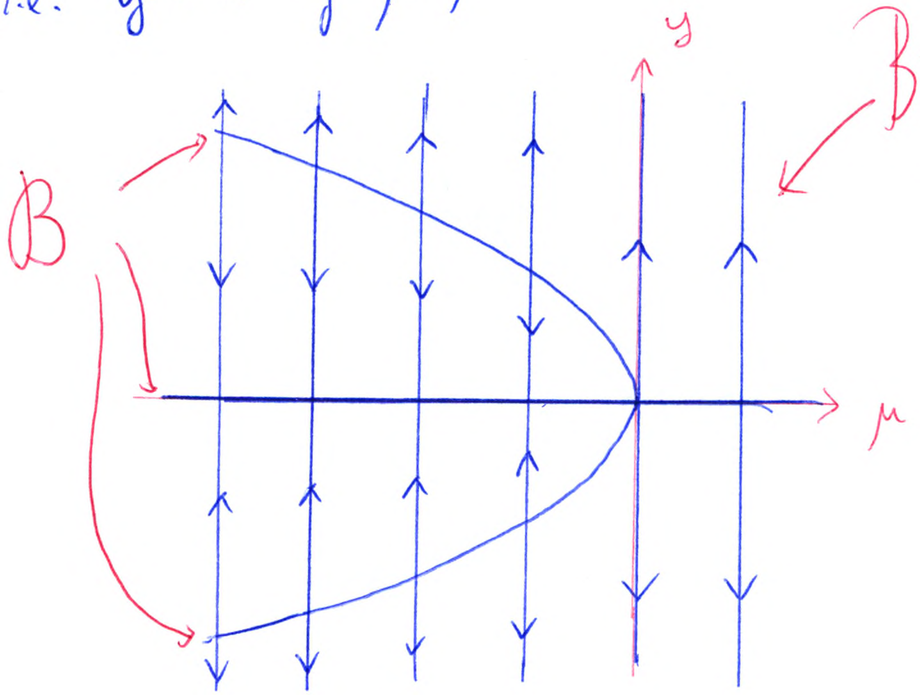


Bifurcation Diagram: (Just bundle of phase lines)

e.g. $y' = y^3 - \mu y = y(y^2+\mu)$.

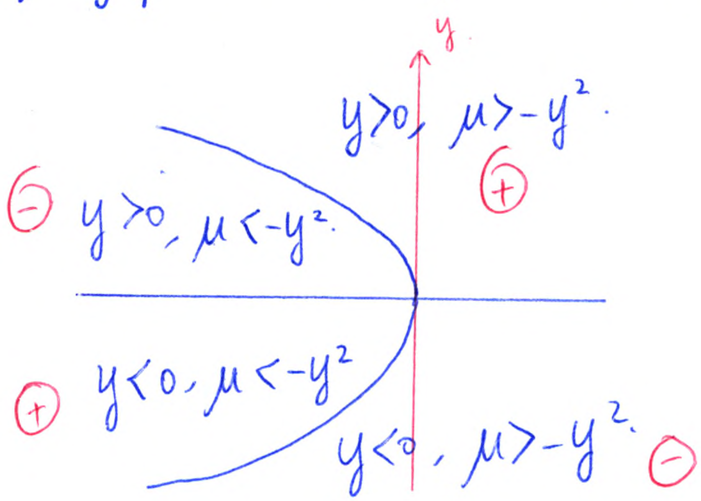
$\mathcal{B} = \{(\mu, y) : y(y^2+\mu) = 0\}$

(i.e. $y=0$ or $-y^2 = \mu$.)



Bifurcation diagram of $y' = y^3 - \mu y$.

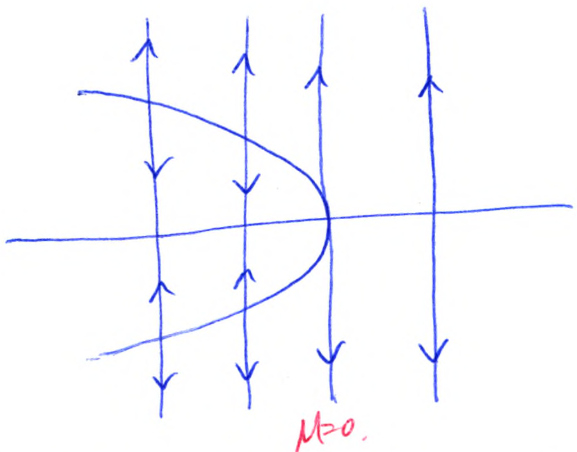
$\mathcal{B}$ decomposes μ-y plane into 4 regions. Need to check $\uparrow$, $\downarrow$ one by one:



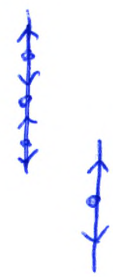
Bifurcation Diagram is useful in identifying bifurcation value:

(i.e. μ so that the behavior of $y' = f_\mu(y)$ changes)

e.g. $y' = y^3 - \mu y$.

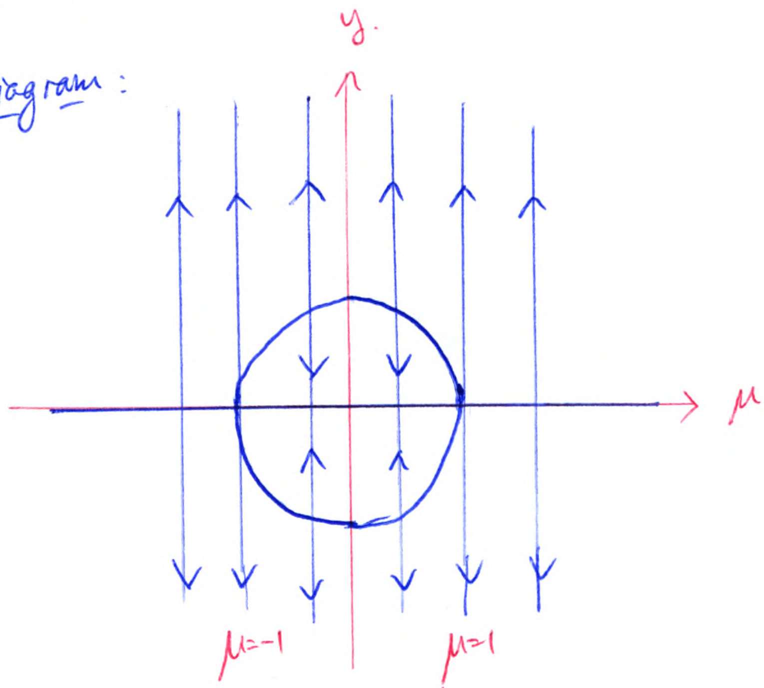


$\mu = 0$ is the bifurcation value: $\mu < 0 \rightarrow$ 3 equilibria.
$\mu > 0 \rightarrow$ 1 equilibrium



e.g. $y' = y(y^2 + \mu^2 - 1)$

Bifurcation Diagram:



$\mu = \pm 1$ are the bifurcation value.

eg. <u>Harvesting</u>:

$P(t)$ = population of fish in a ~~pond~~ lake (at time t)

Logistic model: $P' = 2P\left(1 - \frac{P}{100}\right)$. $P'(0) = 100$.

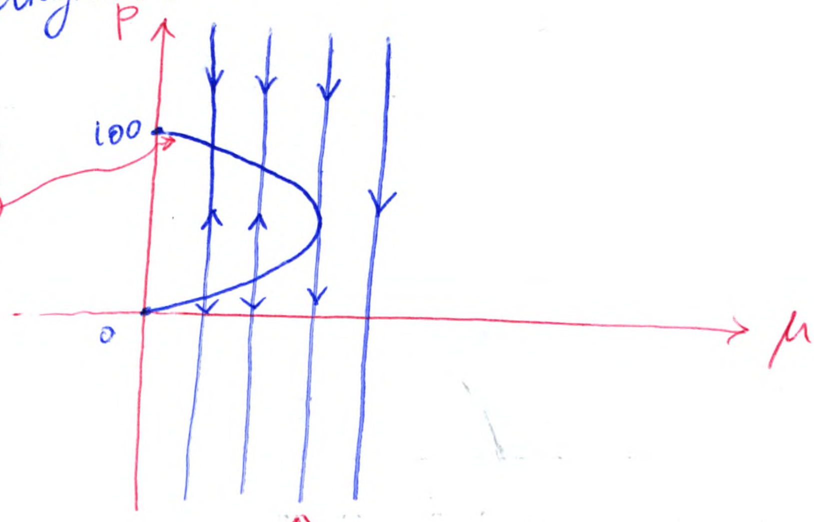
If the city council decided to issue fishing license, where one can fish 1 fish per day. How many license can they issue, so that the population won't die to zero?

μ = # of license issued.

$$P' = 2P\left(1 - \frac{P}{100}\right) - \mu.$$

Bifurcation diagram:

$B = \left\{\mu = 2P\left(1 - \frac{P}{100}\right)\right\}$



$\mu = ?$

when $P = 50$ (i.e. $\mu = 2(50)\left(1 - \frac{50}{100}\right) = 50$)

i.e. When $\mu < 50$. When $\mu = 50$. When $\mu > 50$

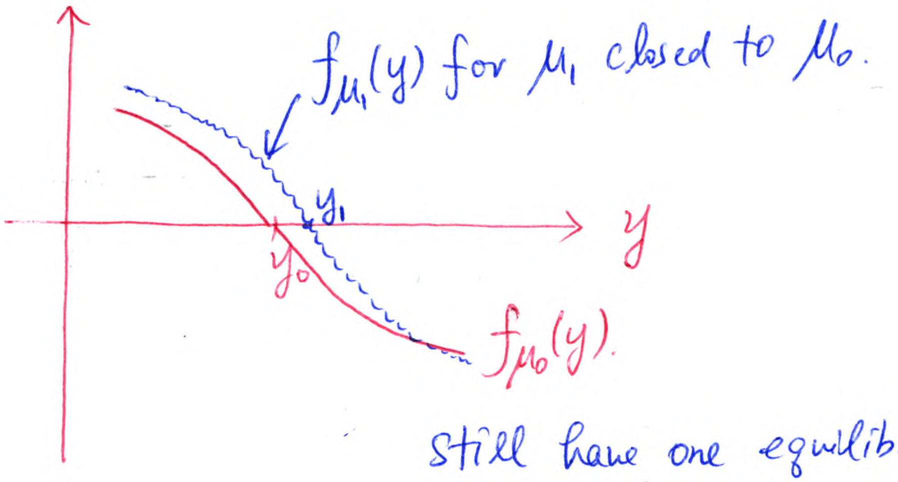
To determine if μ is not a bifurcation value:

Thm: If μ_0 has the property:

$$f'_{\mu_0}(y_0) \neq 0 \quad \text{for all equilibria of } y' = f_{\mu_0}(y),$$

then μ_0 is not a bifurcation value.

Reason:



still have one equilibrium y_1 around y_0.