Hamiltonian System

Def. Given a system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases} \quad \cdots \text{①}$$

A function $H(x,y)$ is called a conserved quantity of ①, if for any solution curve $(x(t), y(t))$ of ①,

$$H(x(t), y(t))$$

is constant.

e.g.

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x - x^2 \end{cases} \quad \cdots \text{②}.$$

$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$. is a conserved quantity. Indeed,

$$\begin{aligned} \frac{d}{dt}\left(H(x(t), y(t))\right) &= \frac{d}{dt}\left(\frac{1}{2}y(t)^2 - \frac{1}{2}x(t)^2 + \frac{1}{3}x(t)^3\right) \\ &= y(t)\,y'(t) - x(t)\,x'(t) + x^2(t)\,x'(t) \\ &\overset{②}{=} y(x - x^2) - x(y) + x^2(y) \\ &= 0. \end{aligned}$$

$\Rightarrow$ $H(x(t), y(t))$ is constant.

Remark:

(i) All systems have conserved quantity: the constant function $H(x,y)=c$ is a conserved quantity.

(ii) If $(x(t), y(t))$ is a solution curve, then

$$0 = \frac{d}{dt}\left(H(x(t), y(t))\right) = \frac{\partial H}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt} + \frac{\partial H}{\partial y}(x(t), y(t)) \frac{dy}{dt}$$

$$= \frac{\partial H}{\partial x} \cdot f + \frac{\partial H}{\partial y} \cdot g.$$

$$= \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} f \\ g \end{pmatrix}.$$

↑ dot product

i.e. H is a conserved quantity iff the gradient vector $\nabla H = \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix}$ is everywhere perdendicular to the vector fields $\begin{pmatrix} f \\ g \end{pmatrix}$

(iii) If H is a conserved quantity of ①, then it's also a conserved quantity of

$$\begin{cases} \frac{dx}{dt} = \Phi(x,y)\, f(x,y) \\ \frac{dy}{dt} = \Phi(x,y)\, g(x,y). \end{cases}$$

(Reason: $\begin{pmatrix} \Phi f \\ \Phi g \end{pmatrix}$ is parallel to $\begin{pmatrix} f \\ g \end{pmatrix}$, so by (ii), $\begin{pmatrix} \Phi f \\ \Phi g \end{pmatrix}$ is perdendicular to ∇H).

From (i), (iii), ~~a~~ the "conserved quantity" is a very flexible concept, which unfortunately means that it's hard to find a nontrivial (i.e non-constant) conserved quantity of a system.

Def: A system is called Hamiltonian if

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H}{\partial y}, \\ \frac{dy}{dt} = -\frac{\partial H}{\partial x}. \end{cases} \quad \cdots \quad (3)$$

for some function $H(x,y)$.

Remark:

(iv) Note that $\begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix}$ is always perpendicular to $\begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix}$. Thus H is a conserved quantity of the Hamiltonian system. The function H is called the Hamiltonian of (3).

It turns out that a Hamiltonian system is more rigid than merely having a conserved quantity. In particular, we can answer the following question:

Q: Given a system (1),

- how to check if it's Hamiltonian?
- If it's Hamiltonian, how to find H?

First of all, if (1) is Hamiltonian, then

$$\begin{cases} f = \frac{\partial H}{\partial y} \\ g = -\frac{\partial H}{\partial x} \end{cases} \quad \cdots \quad (4)$$

for some function H.

$$\Rightarrow \frac{\partial}{\partial x} f = \frac{\partial}{\partial x}\frac{\partial H}{\partial y} \Rightarrow \frac{\partial f}{\partial x} = \frac{\partial^2 H}{\partial x \partial y}$$

and $$\frac{\partial}{\partial y} g = \frac{\partial}{\partial y}\left(-\frac{\partial H}{\partial x}\right) \Rightarrow \frac{\partial g}{\partial y} = -\frac{\partial^2 H}{\partial y \partial x}$$

Since partial derivatives commutes.

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x} \Rightarrow$$

$$\Rightarrow \frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} \quad \cdots \quad (5).$$

Indeed, we have also the converse:

$$\boxed{\text{If (5) holds, then (1) is a Hamiltonian system:}}$$

To see why, we integrate the 1st equation in (4) with respect to y:

$$H = \int f(x,y)\,dy + C(x)$$

"integration constant" with respect to y: a function of x.

Differentiate the above with respect to x:

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\int f(x,y)\,dy + C'(x).$$

and use the 2nd equation of (4):

$$\Rightarrow \quad C'(x) = -\frac{\partial}{\partial x}\int f(x,y)\,dy - g(x,y).$$

thus one can integrate (with respect to x) to find $C(x)$.

e.g. $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = x - x^2 \end{cases}$ ... ②

here $\frac{\partial f}{\partial x} = 0$, $\frac{\partial g}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$ and thus ② is Hamiltonian. To find H, set

$$\begin{cases} \frac{\partial H}{\partial y} = y \\ -\frac{\partial H}{\partial x} = x - x^2 \end{cases}$$

$$\Rightarrow H = \frac{1}{2}y^2 + C(x)$$

$$\Rightarrow \frac{\partial H}{\partial x} = 0 + C'(x)$$

So $C'(x) = x^2 - x \Rightarrow C(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2$ (choose the integration constant $= 0$)

$\Longrightarrow H = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$ is the Hamiltonian.

e.g. $\begin{cases} \frac{dx}{dt} = -x\sin y + 2y \\ \frac{dy}{dt} = -\cos y \end{cases}$

$\Rightarrow \frac{\partial f}{\partial x} = -\sin y$, $\frac{\partial g}{\partial y} = \sin y \Rightarrow \frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y} \Rightarrow$ Hamiltonian.

To find H, set

$$\frac{\partial H}{\partial y} = -x\sin y + 2y$$

$$-\frac{\partial H}{\partial x} = -\cos y$$

$$\Rightarrow H = x\cos y + y^2 + C(x)$$

$$\Rightarrow \frac{\partial H}{\partial x} = \cos y + C'(x)$$

$$\Rightarrow \cos y = \cos y + C'(x)$$

$$\Rightarrow C'(x) = 0 \Rightarrow C(x) = 0$$

$\Longrightarrow H = x\cos y + y^2$ is the Hamiltonian.

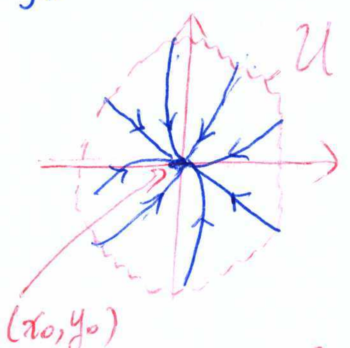
A property of concerning phase portrait of Hamiltonian System:

~~If a~~ The phase portrait of a Hamiltonian System

AT

- Has no sinks, sources:
- Has only saddle and center.

Reason: If it does have a sinks (the case for source is similar)



local phase portrait of the system around (x_0, y_0)

Then for all (x,y) in U, the solution curve $(x(t), y(t))$ passing through (x,y) tends to (x_0, y_0) as $t \to +\infty$ (since it's a sink)

Since H is conserved,

$$H(x,y) = H(x(t), y(t)) \quad \text{for all } t.$$

Take $t \to +\infty$.

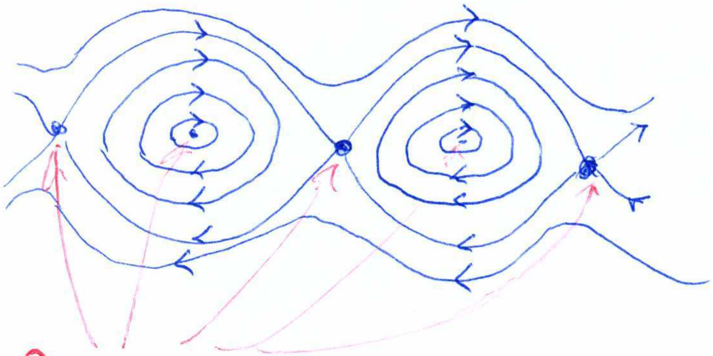
$$\Rightarrow H(x,y) = \lim_{t \to +\infty} H\big(x(t), y(t)\big)$$

$$= H(x_0, y_0) \quad (\because (x(t), y(t)) \to (x_0, y_0)).$$

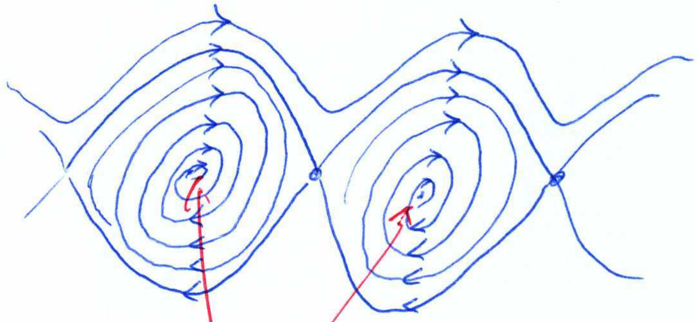
$$\Rightarrow H(x,y) = H(x_0, y_0).$$

But (x,y) in U is arbitrary $\Rightarrow H(x,y) = H(x_0, y_0)$ for all (x,y) in U. Thus H is constant in U and $\frac{\partial H}{\partial y} = \frac{\partial H}{\partial x} = 0$ in U. $\Rightarrow$ impossible since the system is not locally constant.

e.g.



Only saddle and center:
Could be Hamiltonian.



Spiral sink => Not Hamiltonian.