MATH252 NOTES ON EXISTENCE AND UNIQUENESS OF SOLUTIONS TO IVP

1. Existence and Uniqueness: The Statements

Given a first order differential equation

$$y' = f(t, y), \tag{1.1}$$

we study the initial value problem (IVP): that is, we look for function(s) $y(t)$ which satisfy

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0. \end{cases} \tag{1.2}$$

We have the following existence and uniqueness theorem: We will assume that R is a rectangle $R = \{a < t < b, c < y < d\}$ in the $t - y$ plane and (t_0, y_0) is a point in R.

Theorem 1.1 (Existence theorem)**.** *Assume that $f(t, y)$ is continuous in the rectangle R. Then there is an $\epsilon > 0$ and a solution $y(t)$ defined on $(t_0 - \epsilon, t_0 + \epsilon)$ which solve the IVP.*

One might wonder why there is a term $\epsilon > 0$ in the statement, and how ϵ is related to (e.g.) the size of R. The following example demonstrates that ϵ cannot be large even when R is as large as one wish.

Example 1.1. Consider the IVP:

$$\begin{cases} y' = y^2 + 1, \\ y(0) = 0. \end{cases}$$

Then by separation of variable:

$$\begin{aligned} &\frac{y'}{y^2+1} = 1 \\ \Rightarrow &\int \frac{1}{y^2+1} dy = \int 1 dt = t + C \\ \Rightarrow &\arctan y = t + C \\ \Rightarrow &y = \tan(t + C). \end{aligned}$$

The initial condition implies that $\tan C = 0$, thus one can choose $C = 0$. Thus $y(t) = \tan t$ solves the IVP. Although $f(t, y) = y^2 + 1$ is continuous with continuous derivative everywhere (i.e. R can be chosen to be as large as one wish), the solution $\tan t$ is only defined in $(-\pi/2, \pi/2)$. That is, $\epsilon = \pi/2$.

Next we discuss the uniqueness theorem. First we give an example where non-uniqueness does happen.

Example 1.2. Consider the IVP

$$\begin{cases} y' = 3y^{2/3}, \\ y(0) = 0. \end{cases}$$

First one observe that the constant function $y_1(t) = 0$ solves the IVP. On the other hand, by separation of variables,

$$\begin{aligned} \frac{y'}{3y^{2/3}} &= 1 \\ \Rightarrow y^{1/3} &= t + C \\ \Rightarrow y &= (t + C)^3 \end{aligned}$$

and the initial condition $y(0) = 0$ implies that $C = 0$. Thus $y_2(t) = t^3$ also solves the IVP.

Note that in the above example, $f(t, y) = 3y^{2/3}$ is continuous. Thus continuity of f (i.e. the assumption in the Existence Theorem) alone is not enough to guarantee uniqueness of solution. Note also that in the above example,

$$\frac{\partial f}{\partial y} = \frac{1}{2} y^{-1/3}$$

is not continuous at $y = 0$.

Theorem 1.2 (Uniqueness theorem)**.** *Assume that both $f(t, y)$ and $f_y(t, y)$ are continuous in the rectangle R and (t_0, y_0) is in R. If both $y_1(t), y_2(t)$ are solutions (in R) to the IVP, then $y_1(t) = y_2(t)$ for all t in $(t_0 - \epsilon, t_0 + \epsilon)$.*

Go back to Example 1.2. As pointed out, $\frac{\partial f}{\partial y} = 2y^{-1/3}$ is not continuous when $y = 0$. Thus there is no way to find a rectangle R which contains $(t_0, y_0) = (0, 0)$ and that $\frac{\partial f}{\partial y}$ is continuous on R. Thus the Uniqueness Theorem is not applicable to Example 1.2.

Next we state the following important consequence of the Uniqueness Theorem.

Theorem 1.3 (Comparison Principle)**.** *Given a differential equation $y' = f(t, y)$, where f satisfies the condition in the Uniqueness Theorem. Let $y_1(t), y_2(t)$ be two solutions the differential equation and*

$$y_1(t_0) < y_2(t_0)$$

for some time t_0. Then

$$y_1(t) < y_2(t)$$

for **all** *time t.*

Proof. If instead $y_1(t) < y_2(t)$ does not hold, then the graph of y_1 crosses that of y_2. Let t_1 be the first instance where $y_1(t_1) = y_2(t_1)$. Then the IVP

$$\begin{cases} y' = f(t, y), \\ y(t_1) = y_1(t_1) \end{cases}$$

admits two different solutions $y_1(t)$, $y_2(t)$ (they are different since t_1 is the first instance where y_1, y_2 touch each other: so $y_1(t) < y_2(t)$ if $t < t_1$). $\square$

The Comparison Principle is useful since it provides qualitative information about a solution y_1 (which might not be explicitly found) in terms of another solution y_2 (which one might find easily).

Example 1.3. Consider the differential equation $y' = e^{y^2}(y-1)$. First, $f(t, y) = e^{y^2}(y-1)$ and its partial derivatives are both continuous everywhere, so Comparison Principle holds. Second, $y_2(t) = 1$ is a particular solution to the equation.

Then if $y(t)$ is a solution with $y(0) < 1 = y_2(0)$, then by Comparison Principle, we have

$$y(t) < y_2(t) = 1$$

for all t.

2. Proof of the uniqueness theorem

To start the proof we first change the differential equation to an integral one: note that we have

$$y'(t) = f(t, y(t)).$$

Thus by the Fundamental Theorem of Calculus,

$$\begin{aligned} y(t) - y(t_0) &= \int_{t_0}^{t} y'(s)ds \\ &= \int_{t_0}^{t} f(s, y(s))ds. \end{aligned}$$

That is,

$$y(t) = \int_{t_0}^{t} f(s, y(s))ds + y(t_0). \tag{2.1}$$

Now assume that $y_1(t), y_2(t)$ are both solutions to the same initial value problem (that is,

$$y_1(t_0) = y_2(t_0) = y_0. \tag{2.2}$$

Putting $y = y_1$ or y_2 into (2.1) and use (2.2), we have

$$\begin{aligned} y_1(t) &= \int_{t_0}^{t} f(s, y_1(s))ds + y_0, \\ y_2(t) &= \int_{t_0}^{t} f(s, y_2(s))ds + y_0. \end{aligned}$$

Substracting one equation from the other gives

$$y_1(t) - y_2(t) = \int_{t_0}^{t} \Big(f(s, y_1(s)) - f(s, y_2(s)) \Big) ds. \tag{2.3}$$

Now consider the integrand on the right hand side. Using the Mean Value Theorem (and thinking of $f(s,y)$ as a function of y with s fixed), there is τ between $y_1(s)$ and $y_2(s)$ so that

$$f(s,y_1(s)) - f(s,y_2(s)) = \frac{\partial f}{\partial y}(s,\tau)(y_1(s) - y_2(s)).$$

From now on, we make the assumption that $\left|\frac{\partial f}{\partial y}\right| \le M$ for some positive constant M (This is possible since in the assumption of the Uniqueness Theorem, $\frac{\partial f}{\partial y}$ is continuous, so locally it is bounded). Then we have

$$|f(s,y_1(s)) - f(s,y_2(s)| \le M|y_1(s) - y_2(s)|.$$

Putting this into (2.3) we obtain that

$$|y_1(t) - y_2(t)| \le M\int_{t_0}^{t} |y_1(s) - y_2(s)|ds.$$

The above inequality is all we need to show that $y_1(t) = y_2(t)$ whenever t is closed to t_0: to be precise, assume that $y_1(t), y_2(t)$ are both defined in the interval $\left[t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}\right]$. Let the function $|y_1(t) - y_2(t)|$ attains its maximum at some point $\tilde{t}$. Then

$$0 \le |y_1(s) - y_2(s)| \le |y_1(\tilde{t}) - y_2(\tilde{t})|$$

for all s in $\left[t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}\right]$. Thus

$$\begin{aligned}
|y_1(\tilde{t}) - y_2(\tilde{t})| &\le M\int_{t_0}^{\tilde{t}} |y_1(s) - y_2(s)|ds \\
&\le M\int_{t_0}^{\tilde{t}} |y_1(\tilde{t}) - y_2(\tilde{t})|ds \\
&= M|\tilde{t} - t_0||y_1(\tilde{t}) - y_2(\tilde{t})| \\
&\le \frac{1}{2}|y_1(\tilde{t}) - y_2(\tilde{t})|.
\end{aligned}$$

In the last inequaltiy we used $|\tilde{t} - t_0| \le \frac{1}{2M}$. Thus

$$|y_1(\tilde{t}) - y_2(\tilde{t})| \le 0,$$

or $y_1(t) = y_2(t)$ for all t in $\left[t_0 - \frac{1}{2M}, t_0 + \frac{1}{2M}\right]$.