252 ELEMENTARY DIFFERENTIAL EQUATIONS: HW4 SOLUTION

(1) Let $Y_1(t), Y_2(t)$ be two solutions to the linear system

$$Y' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} Y.$$

Let $D(t) = \det \begin{pmatrix} Y_1(t) & Y_2(t) \end{pmatrix}$.

(a) Show that D satisfies the differential equation

$$D' = \operatorname{tr} A \cdot D, \quad \text{where } \operatorname{tr} A = a + d.$$

<u>Solution:</u> We write

$$Y_1(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}, \quad Y_2(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}.$$

Then

$$\begin{aligned} D &= \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1 \\ \Rightarrow D' &= x_1' y_2 + x_1 y_2' - x_2' y_1 - x_2 y_1'. \end{aligned}$$

Next we use the linear systems $Y_1' = AY, Y_2' = AY$, so

$$\begin{aligned} \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ &= \begin{pmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{pmatrix} \end{aligned}$$

Similarly

$$\begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{pmatrix}$$

Thus we have

$$\begin{aligned} D' &= (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1) \\ &= ax_1y_2 + dx_1y_2 - ax_2y_1 - dx_2y_1 \\ &= (a+d)(x_1y_2 - x_2y_1) \\ &= (a+d)D. \end{aligned}$$

(b) Conclude that if $Y_1(0), Y_2(0)$ is linearly independent, then $Y_1(t), Y_2(t)$ is linearly independent for all t.

<u>Solution:</u> Solving the differetial equation $D' = (a+d)D$ gives

$$D(t) = D(0)e^{(a+d)t}.$$

Since $e^{(a+d)t}$ is never zero, $D(t)$ is nonzero if and only if $D(0)$ is nonzero. Thus $\{Y_1(0), Y_2(0)\}$ is linearly independent if and only if $\{Y_1(t), Y_2(t)\}$ is linearly independent.

(2) Solve the IVP:

$$\frac{dY}{dt} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} Y, \quad Y(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Solution: The characteristic polynomial is given by

$$\begin{vmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (\lambda+2)(\lambda-1) - 4 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2).$$

Thus A has two distinct real eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$. Since

$$A - \lambda_1 I = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix},$$

$V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of A with respect to λ_1. Similarly one find the other eigenvector, which is $V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Thus the general solution is given by

$$Y(t) = k_1 e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

where k_1, k_2 are any real numbers. Now we use the initial condition to set

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = k_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow k_1 = \frac{4}{5}, \; k_2 = -\frac{3}{5}.$$

Thus the solution to the IVP is

$$Y(t) = \frac{4}{5} e^{-3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{3}{5} e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

(3) Find the general solution to the following system:

$$\frac{dY}{dt} = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix} Y, \quad .$$

Solution: The characteristic polynomial is given by

$$\begin{vmatrix} -3-\lambda & -5 \\ 3 & 1-\lambda \end{vmatrix} = (\lambda+3)(\lambda-1) + 15 = \lambda^2 + 2\lambda + 12 = (\lambda+1)^2 + 11.$$

Thus A has two distinct imaginary eigenvalues $\lambda = -1 + \sqrt{11}i$ (and $\bar{\lambda}$). Note that $\vec{v} = \begin{pmatrix} 5 \\ -2 - \sqrt{11}i \end{pmatrix}$ is an eigenvector of A with respect to λ. Thus $e^{\lambda t}\vec{v}$ is a (complex) solution to the system. Note

$$\begin{aligned}
e^{\lambda t}\vec{v} &= e^{(-1+\sqrt{11}i)t}\begin{pmatrix} 5 \\ -2-\sqrt{11}i \end{pmatrix} \\
&= e^{-t}(\cos(\sqrt{11}t) + i\sin(\sqrt{11}t))\left(\begin{pmatrix} 5 \\ -2 \end{pmatrix} + i\begin{pmatrix} 0 \\ -\sqrt{11} \end{pmatrix}\right) \\
&= \begin{pmatrix} 5e^{-t}\cos(\sqrt{11}t) \\ e^{-t}(-2\cos(\sqrt{11}t) + \sqrt{11}\sin(\sqrt{11}t)) \end{pmatrix} \\
&\quad + i\begin{pmatrix} 5e^{-t}\sin(\sqrt{11}t) \\ e^{-t}(-2\sin(\sqrt{11}t) - \sqrt{11}\cos(\sqrt{11}t)) \end{pmatrix} \\
&= Y_{re} + iY_{im}.
\end{aligned}$$

Thus the general solution is given by $k_1Y_{re} + k_2Y_{im}$.

(4) Solve the IVP:

$$\frac{dY}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} Y, \quad Y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solution: The characteristic polynomial is given by

$$\begin{vmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{vmatrix} = (\lambda+2)(\lambda+4) + 1 = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2.$$

Thus A has repeated eigenvalues $\lambda = -3$. Note that for any W_0,

$$Y(t) = e^{\lambda t}W_0 + te^{\lambda t}W_1, \quad \text{where } W_1 = (A - \lambda I)W_0,$$

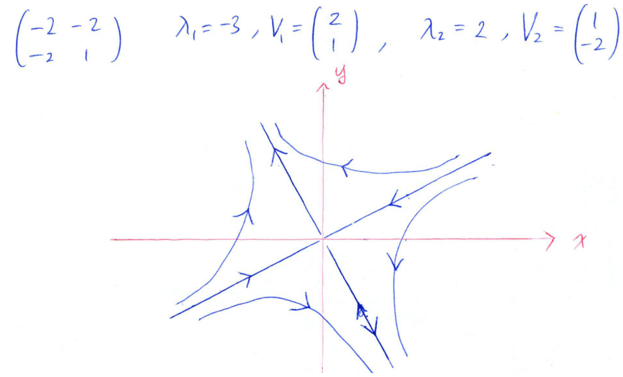
is the solution to the system with $Y(0) = W_0$. In our situation,

$$W_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

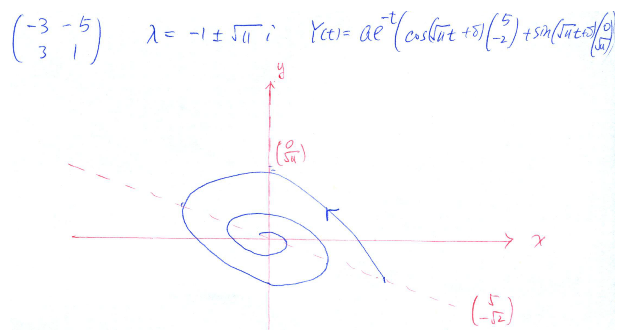
thus the solution to the IVP is

$$Y(t) = e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-3t} + te^{-3t} \\ te^{-3t} \end{pmatrix}.$$

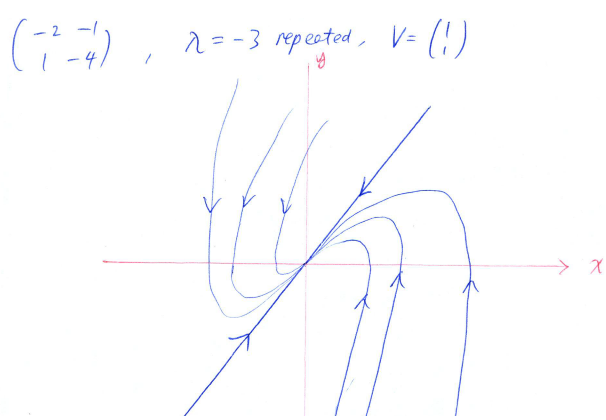
(5) Sketch the phase portraits of the system given in Q2, 3, and 4.
- Phase portrait for question 2:



- Phase portrait for question 3:



- Phase portrait for question 4:



(6) Let B be a matrix with a repeated zero eigenvalues. Then show that $B^2 = 0$ (the 2×2 zero matrix). Use this to show: if A has a repeated eigenvalue λ_0, then $(A - \lambda_0 I)^2 = 0$.

Solution: Write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the characteristic polynomial is given by

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= \lambda^2 - (a+d)\lambda + ad - bc. \end{aligned}$$

If B has repeated zero eigenvalue, then the characteristic polynomial has to be λ^2. Compare with the above, we conclude

$$a + d = 0, \quad ad - bc = 0.$$

This is sufficient to show that B^2 is the zero matrix: indeed,

$$B^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

and

$$\begin{aligned} a^2 + bc &= a^2 + ad = a(a+d) = 0, \\ ab + bd &= b(a+d) = 0, \\ ac + cd &= c(a+d) = 0, \\ bc + d^2 &= ad + d^2 = d(a+d) = 0. \end{aligned}$$

Now if A has repeated eigenvalue λ_0, then $A - \lambda_0 I$ has repeated eigenvalue zero and thus $(A - \lambda_0 I)^2 = 0$.

(7) Let A be a 2×2 matrix. Assume that

$$Y_1(t) = \begin{pmatrix} e^t \\ -2e^t \end{pmatrix}, \quad Y_2(t) = \begin{pmatrix} 3e^{-2t} \\ e^{-2t} \end{pmatrix}$$

and both solutions to the system $Y' = AY$. Then solve the IVP

$$Y' = AY, \quad Y(0) = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Solution: By the linearity principle, since Y_1 and Y_2 are solutions,

$$Y(t) = k_1 Y_1(t) + k_2 Y_2(t)$$

are also solutions for all constants k_1, k_2. To solve the IVP, set

$$k_1 Y_1(0) + k_2 Y_2(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \begin{cases} k_1 + 3k_2 = 1 \\ -2k_1 + k_2 = 5 \end{cases}.$$

Solving the systems of linear equations gives $k_1 = -2, k_2 = 1$. Thus

$$-2Y_1(t) + Y_2(t)$$

solves the IVP.