SPRING 20 ELEMENTARY DIFFERENTIAL EQUATIONS: SAMPLE FINAL EXAMINATION SOLUTION

1. (10 marks) Solve the following initial value problem:
$$x' = x^2(t+1), \quad x(1) = 1.$$

Solution:
$$\begin{aligned} \frac{x'}{x^2} &= t+1 \\ \int \frac{x'}{x^2} dt &= \int (t+1) dt \\ \int \frac{1}{x^2} dx &= \int (t+1) dt \\ -\frac{1}{x} &= \frac{1}{2}t^2 + t + C \\ x &= -\frac{1}{\frac{1}{2}t^2 + t + C}. \end{aligned}$$

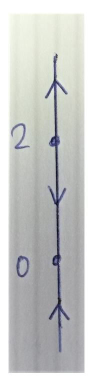
Since $x(1) = 1$,
$$1 = -\frac{1}{1/2 + 1 + C} \Rightarrow C = -5/2.$$

Thus $x(t) = -(\frac{1}{2}t^2 + t - \frac{5}{2})^{-1}$ solves the IVP.

2. (10 marks) Sketch the phase lines of the following differential equation:
$$y' = y(y-2).$$

If y satisfies the differential equation and $y(2) = 1$, what can we say about $y(t)$ as $t \to +\infty$?

Solution:



Since in the interval $0 < y < 2$ we have $y(y-2) < 0$, $y(t) \to 0$ as $t \to +\infty$.

3. (10 marks) Find the general solution to the following differential equation:

$$y' + \frac{3}{t+1}y = t - 1.$$

Solution: Write $g(t) = \frac{3}{t+1}$, then

$$\mu = e^{\int g(t)dt} = e^{\int \frac{3}{t+1}dt} = (t+1)^3.$$

Multiplying μ on both sides gives

$$\begin{aligned}
(t+1)^3y' + 3(t+1)^2y &= (t+1)^3(t-1) \\
((t+1)^3y)' &= (t+1)^4 - 2(t+1)^3 \\
(t+1)^3y &= \frac{1}{5}(t+1)^5 - \frac{1}{2}(t+1)^4 + C \\
y &= \frac{1}{5}(t+1)^2 - \frac{1}{2}(t+1) + \frac{C}{(t+1)^3}.
\end{aligned}$$

4. (10 marks) Let $y(t)$ be a function which satisfies the differential equation

$$y' = (t+1)(y-2)^2 - t$$

for all t and $y(0) = 1$. Show that $y(t) < 2$ for all $t \geq 0$.

Solution: We argue by contradiction. Suppose that $y(t) \geq 2$ for some $t > 0$, then since $y(0) = 1 < 2$, there is a time $t_0 > 0$ so that $y(t_0) = 2$ and $y(t) < 2$ for all $t < t_0$. In particular, one has $y'(t_0) \geq 0$. However, using the differential equation

$$y'(t_0) = (t_0+1)(y(t_0)-2)^2 - t_0 = -t_0 < 0.$$

But this is not impossible. Thus $y(t) < 2$ for all $t > 0$.

5. (12 marks) Find the general solution to the following first order system:

$$\begin{aligned}
x' &= x + 1, \\
y' &= 3y + x^2.
\end{aligned}$$

Solving the first equation:

$$\begin{aligned}
\frac{x'}{x+1} &= 1 \\
\int \frac{1}{x+1}dx &= t + C \\
x + 1 &= C_1e^t \\
\Rightarrow x &= C_1e^t - 1.
\end{aligned}$$

Plug into the second gives

$$y' = 3y + C_1^2e^{2t} - 2C_1e^t + 1.$$

The general solution is $y = y_h + y_p = C_2 e^{3t} + y_p$. To find y_p, try

$$y_p = Ae^{2t} + Be^t + D.$$

Then

$$\begin{aligned} y_p' &= 2Ae^{2t} + Be^{2t}, \\ \Rightarrow y_p' - 3y_p &= -Ae^{2t} - 2Be^t - 3D. \end{aligned}$$

So we choose $A = -C_1^2$, $B = C_1$ and $D = -1/3$. Thus

$$\begin{aligned} x &= C_1 e^t - 1, \\ y &= C_2 e^{3t} - C_1^2 e^{2t} + C_1 e^t - 1/3. \end{aligned}$$

5. (8 marks) Calculate e^A, where A is the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Solution: The characteristic polynomial is

$$(3-\lambda)(3-\lambda) - 1 = (\lambda - 2)(\lambda - 4).$$

Thus the eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 2$. To find the eigenvectors,

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ A - \lambda_2 I &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \end{aligned}$$

Thus

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ e^A &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^4 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \end{aligned}$$

6. (10 marks) Sketch the phase portrait of the following linear system:

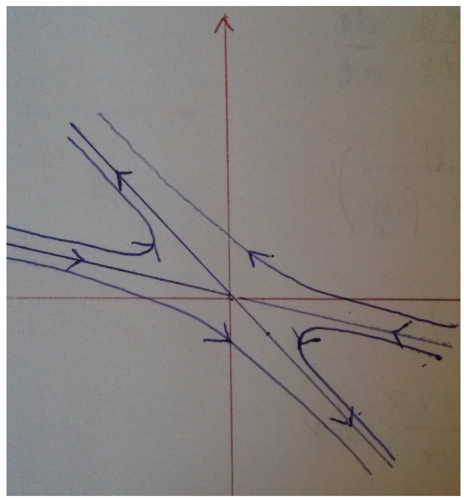
$$Y' = \begin{pmatrix} -3 & -6 \\ 1 & 4 \end{pmatrix} Y$$

Solution: The characteristic polynomial is

$$(-3-\lambda)(4-\lambda) + 6 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Thus the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 3$. The phase portrait is a saddle. To find the eigenvectors,

$$A - \lambda_1 I = \begin{pmatrix} -1 & -6 \\ 1 & 6 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$
$$A - \lambda_2 I = \begin{pmatrix} -6 & -6 \\ 1 & 1 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$



7. (8 marks) Solve the following initial value problem:

$$Y' = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} Y, \quad Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution: The characteristic polynomial is

$$(2-\lambda)(6-\lambda) + 4 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

Thus the eigenvalues are $\lambda = 4$ (repeated). We have

$$A - \lambda I = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$$

and

$$Y(t) = e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + te^{4t}\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

solves the IVP.

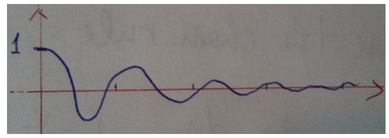
8. (12 marks) The harmonic oscillator is governed by the following differential equation

$$my'' + by' + ky = 0.$$

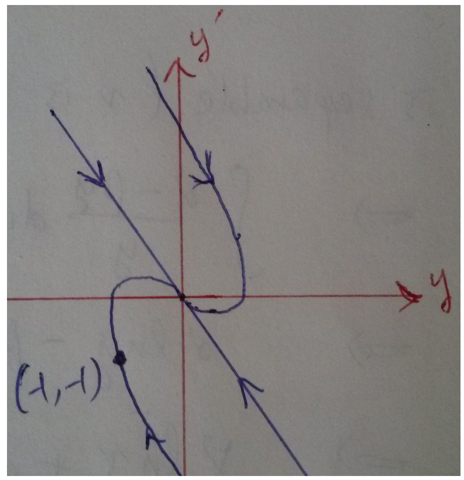
In the following three situations, sketch the function $y(t)$.

(a) $m = 1$, $b = 2$, $k = 3$ with $y(0) = 1, y'(0) = 0$.

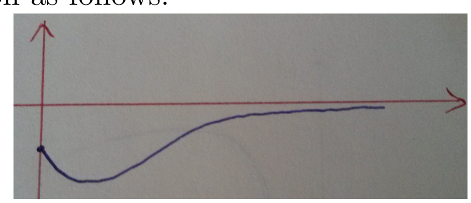
Solution: We have $b^2 = 4 < 12 = 4km$. It's underdamped.



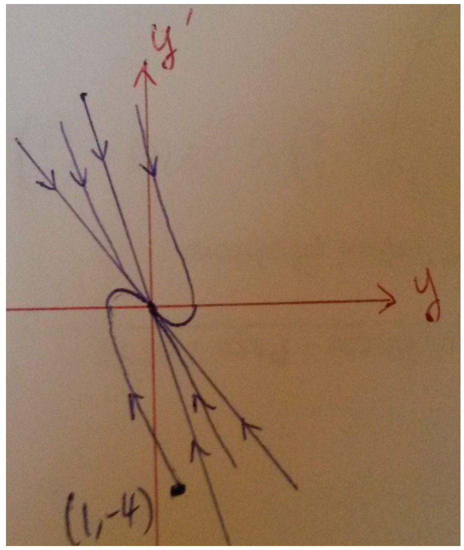
(b) $m = 1$, $b = 4$, $k = 4$ with $y(0) = -1$, $y'(0) = -1$.
Solution: $b^2 = 16 = 4km$. It's critically damped. The corresponding linear systems has a repeated eigenvalue -2 with eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, the phase portrait is



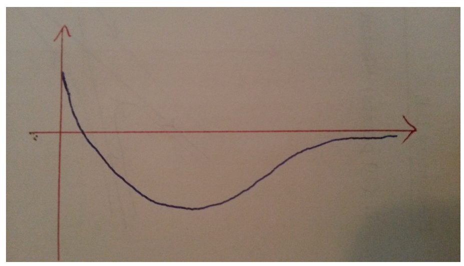
which gives the graph as follows:



(c) $m = 1$, $b = 5$, $k = 6$ with $y(0) = 1$, $y'(0) = -4$.
Solution: $b^2 = 25 > 24 = 4km$. It's overdamped. The phase portrait of the corresponding first order linear system is,



which gives the graph as follows:



9. (12 marks) Find the general solution to the following differential equation:

$$y'' + 3y' + 2y = 2\sin 2t.$$

Solution: By the linearity principle, $y = y_h + y_p$. To find y_h, we solve

$$s^2 + 3s + 2 = (s+2)(s+1) = 0.$$

Thus $s = -1, -2$ and $y_h = k_1 e^{-2t} + k_2 e^{-t}$. To find y_p, we guess

$$y_p = A\sin 2t + B\cos 2t.$$

Thus

$$\begin{aligned} y_p' &= 2A\cos 2t - 2B\sin 2t, \\ y_p'' &= -4A\sin 2t - 4B\cos 2t. \end{aligned}$$

Then

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= -4A\sin 2t - 4B\cos 2t + 3(2A\cos 2t - 2B\sin 2t) \\ &\quad + 2(A\sin 2t + B\cos 2t) \\ &= (-2A - 6B)\sin 2t + (6A - 2B)\cos 2t. \end{aligned}$$

Thus we choose

$$\begin{aligned} -2A - 6B &= 2, \\ 6A - 2B &= 0. \end{aligned}$$

Which give $A = -1/10$, $B = -3/10$. Thus the general solution is

$$y = k_1 e^{-2t} + k_2 e^{-t} - \frac{1}{10}\sin 2t - \frac{3}{10}\cos 2t.$$

10. (12 marks) Find the general solution to the following differential equation:

$$y'' + 3y + 2y = te^{-t}.$$

Solution Again by the linearity principle, $y = y_h + y_p$. Since the LHS of the equation is the same as in the previous question, we have $y_h = k_1 e^{-2t} + k_2 e^{-t}$. Write $\lambda_2 = -1$, $\lambda_1 = -2$, then

$$y_p(t) = \int_0^t \frac{e^{\lambda_2(t-u)} - e^{\lambda_1(t-u)}}{\lambda_2 - \lambda_1}(ue^{-u})du$$

is a particular solution. Calculating the integral gives

$$\begin{aligned} y_p(t) &= \int_0^t (e^{u-t} - e^{2u-2t})ue^{-u}du \\ &= e^{-t}\int_0^t udu - e^{-2t}\int_0^t ue^u du \\ &= \frac{t^2}{2}e^{-t} - e^{-2t}\left(te^t - \int_0^t e^u du\right) \\ &= \frac{t^2}{2}e^{-t} - e^{-2t}\left(te^t - e^t + 1\right) \\ &= (t^2/2 - t + 1)e^{-t} - e^{-2t}. \end{aligned}$$

Thus the general solution is given by

$$y = k_1 e^{-2t} + k_2 e^{-t} + (t^2/2 - t + 1)e^{-t}.$$

11. (20 marks) In this question we restrict our attention to the first quadrant ($x, y \geq 0$). Given the following system:

$$\begin{aligned} x' &= x(2 - x - y) \\ y' &= y(y - x). \end{aligned}$$

Note that $(0,0)$, $(1,1)$, $(2,0)$ are equilibria.

(a) Calculate the linearization at the equilibria $(1,1)$ and $(2,0)$. State if it is a sink, a source or a saddle.
Solution The Jacobian matrix is

$$\begin{pmatrix} 2-2x-y & -x \\ -y & 2y-x \end{pmatrix}.$$

At $(1,1)$, the linearization

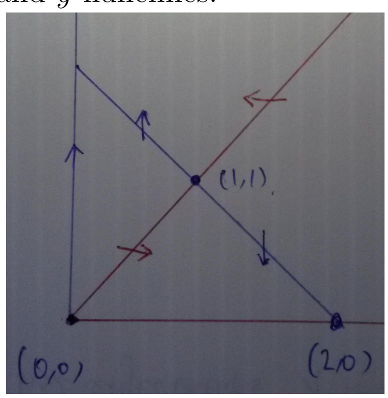
$$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$$

has eigenvalues $\pm\sqrt{2}$. Thus it is a saddle.
At $(2,0)$, the linearization

$$\begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix}$$

has eigenvalues -2 (repeated). This is a sink.

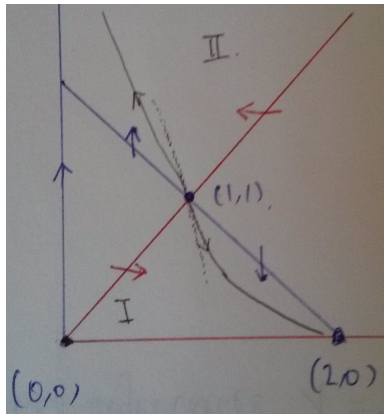
(b) Sketch the x-nullclines and y-nullclines.



(c) Argue that there is exactly one solution curve $Y(t)$ (up to time translation) so that

$$Y(t) \to (1,1) \text{ as } t \to -\infty,$$
$$Y(t) \to (2,0) \text{ as } t \to +\infty.$$

Solution: From (a), the linearization at $(1,1)$ is a saddle. Then there are two solution curves $Y(t)$ (up to time translation) so that $Y(t) \to (1,1)$ as $t \to -\infty$ which corresponds to the straight line solution of the linearization with eigenvalues $-\sqrt{2}$. Those two solution curves are tangential to the eigenvector, which is $\begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix}$.



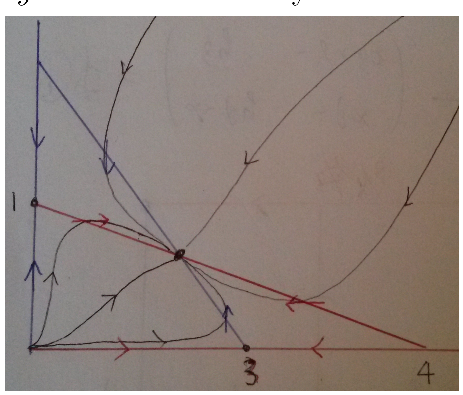
One of these two is in the stable region I, and it converges to $(2,0)$ as $t \to +\infty$. The other one is in the region II, which is moving in the top left direction. In particular, this does not converge to $(2,0)$. Thus there is exactly one such solution curve.

12. (16 marks) Consider the following model for two competitive species X, Y sharing the same habitat:

$$x' = x(3-x) - bxy,$$
$$y' = 4y(1-y) - xy,$$

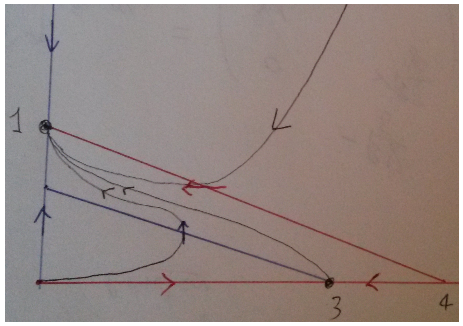
here $x(t)$ and $y(t)$ are the population of the species X and Y at time t respectively. b is a unknown positive parameter. The term $-bxy$ measures the (negative) effect to the growth rate of X due to the presence of Y. Find the critical value b_0, so that whenever $b > b_0$, the species Y dominates the competition and the species X becomes extinct eventually. Please explain your answer.

Solution: The x- and y- intercepts of the line $3 - x - by = 0$ are 3 and $3/b$ respectively, while the x- and y- intercepts of $4-4y-x = 0$ are 4 and 1 respectively, thus these two lines intersect in the first quardrant if and only if $3/b < 1$, or $b > 3$. When $b < 3$, the x- and y- nullclines of the system are as follows:



which implies that all solution curves converges to the intersection of $3-x-by = 0$ and $4-4y-x = 0$, which is an equilibria. That is, the two species co-exists.

When $b > 3$, the x- and y- nullclines of the system are as follows:



all solutions to the system converges to $(0, 1)$ as $t \to +\infty$. That is, the species X becomes extincts eventually (as $t \to +\infty$). Thus the critical value is $b_0 = 3$.

13. (10 marks) Check if the following system is Hamiltonian. If so, find a Hamiltonian H:

$$x' = 2y + e^x \sin y,$$
$$y' = -2x + e^x \cos y.$$

Solution: Write

$$f = 2y + e^x \sin y,$$
$$g = -2x + e^x \cos y.$$

Then

$$\frac{\partial f}{\partial x} = e^x \sin y,$$
$$\frac{\partial g}{\partial y} = -e^x \sin y.$$

Hence $\frac{\partial f}{\partial x} - \frac{\partial g}{\partial y}$ and the system is Hamiltonian. To find the Hamiltonian function, let

$$\frac{\partial H}{\partial y} = 2y + e^x \sin y,$$

integrating with respect to y gives

$$H = y^2 - e^x \cos y + C(x).$$

Then

$$\frac{\partial H}{\partial x} = -e^x \cos y + C'(x)$$

since $\frac{\partial H}{\partial x} = -g = 2x - e^x \cos y$, we obtain

$$C'(x) = 2x$$

and we choose $C(x) = x^2$. Thus

$$H = x^2 + y^2 - e^x \cos y.$$