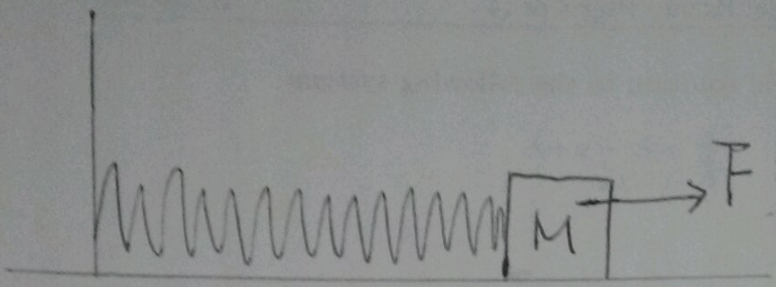
Forced harmonic oscillator.



$$my'' = -ky - by + f(t), \text{ or}$$

$$my'' + by' + ky = f(t)$$

(1) General solution to $ay'' + by' + cy = f(t)$. ← 2nd order linear equation.

Linearity Principle $\Rightarrow$ general solution $= y = y_h + y_p$.

y_h : general solution to the homogeneous equation

$$ay'' + by' + cy = 0.$$

y_p : Any particular solution to the original equation.

Guessing Method :

e.g. $y'' + 2y' - 3y = e^{-t}$.

$y = y_h + y_p$.

(i) to find y_h, find s such that $s^2 + 2s - 3 = 0 \Leftrightarrow (s+3)(s-1) = 0$

$\Leftrightarrow s = -3$ or 1.

$\Rightarrow y_h = k_1 e^{-3t} + k_2 e^t$.

(ii) to find y_p, guess $y_p = Ce^{-t}$.

$\Rightarrow y_p'' + 2y_p' - 3y_p = (Ce^{-t}) + 2(-Ce^{-t}) - 3(Ce^{-t})$
$= -4Ce^{-t}$.

$\Rightarrow$ choose $-4C = 1$. $\Rightarrow y_p = -\frac{1}{4}e^{-t}$.

(i)+(ii) $\Rightarrow y = k_1 e^{-3t} + k_2 e^t - \frac{1}{4}e^{-t}$

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e.g. $y'' + 2y' - 3y = \underline{\underline{e^t}}$

$y = y_h + y_p \Rightarrow$, $y_h = k_1 e^{-3t} + k_2 \underline{\underline{e^t}}$.

forcing term is also "in y_h"
$\Downarrow$
Simplest guessing $y_p = Ce^t$ does not work.

Try: $y_p = Cte^t$.

$\vdots$

Another e.g. $y'' + 2y' + 5y = \sin 2t$.

$y = y_h + y_p$, to find y_h: Consider $s^2 + 2s + 5 = 0 \Rightarrow s = -1 \pm 2i$

$\Rightarrow y_h = k_1 e^{-t}\sin 2t + k_2 e^{-t}\cos 2t$.

to find y_p, set $y_p = A\sin 2t + B\cos 2t$.

$\Rightarrow y_p'' + 2y_p' + 5y_p = (-4A\sin 2t - 4B\cos 2t) + 2(2A\cos 2t - 2B\sin 2t)$
$+ 5(A\sin 2t + B\cos 2t)$
$= (-4A - 4B + 5A)\sin 2t + (-4B + 4A + 5B)\cos 2t$.
$= (A - 4B)\sin 2t + (4A + B)\cos 2t$.

$\Rightarrow$ Set $\begin{cases} A - 4B = 1 \\ 4A + B = 0 \end{cases} \Rightarrow A = \frac{1}{17}, B = -\frac{4}{17} \Rightarrow y_p = \frac{1}{17}\sin 2t - \frac{4}{17}\cos 2t$.

(I) Periodic forcing of underdamped harmonic oscillator:

e.g. $y'' + 2y' + 2y = \sin t$.

General solution: $y = y_h + y_p$.

(i) to find y_h, solve $s^2 + 2s + 2 = 0 \Rightarrow s = -1 \pm i$.

$\Rightarrow y_h = k_1 e^{-t} \sin t + k_2 e^{-t} \cos t$

(ii) to find y_p, guess $y_p = C_1 \sin t + C_2 \cos t$,

$y_p'' + 2y_p' + 2y_p = (C_1 - 2C_1) \sin t + (2C_1 + C_2) \cos t$.

$\Rightarrow \begin{cases} C_1 - 2C_2 = 1 \\ 2C_1 + C_2 = 0 \end{cases} \Rightarrow C_1 = \frac{1}{5},\ C_2 = -\frac{2}{5}$.

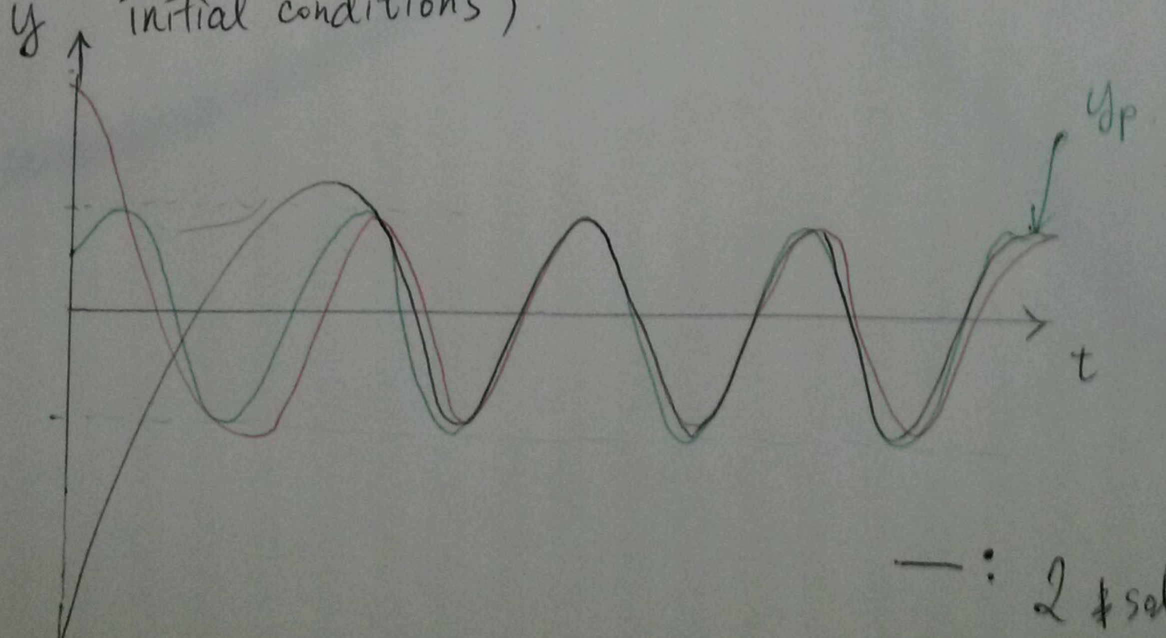
(i)+(ii)

$\Longrightarrow \quad y = k_1 e^{-t} \sin t + k_2 e^{-t} \cos t + \frac{1}{5} \sin t - \frac{2}{5} \cos t$.

Observation:

① $y(t) \longrightarrow y_p = \frac{1}{5} \sin t - \frac{2}{5} \cos t$. as $t \to +\infty$ since $e^{-t} \to 0$ as $t \to +\infty$

(For all choices of k_1, k_2, i.e. for all choices of initial conditions)

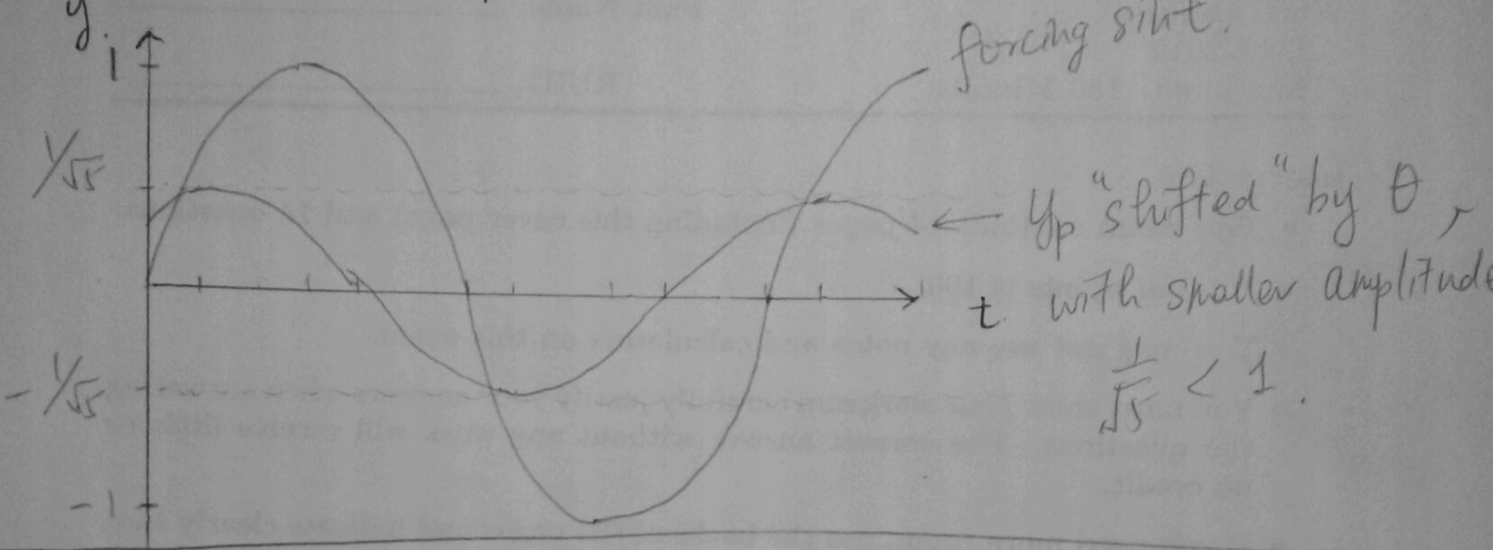


—: 2 ≠ solutions:

② $y_p = \frac{1}{5}\sin t - \frac{2}{5}\cos t \underset{\uparrow}{=} \underbrace{\frac{1}{\sqrt{5}}}\sin(t+\theta)$

Exercise Amplitude. $\theta \approx 63°$.

i.e. "Delayed" response from the forcing $\sin t$.



(II) Periodic forcing of undamped harmonic oscillator:

e.g.
$$y'' + qy = \cos \omega t.$$

↑ forcing frequency.

$$y_h = k_1 \sin\sqrt{q}\,t + k_2\cos\sqrt{q}\,t.$$

$\sqrt{q}$ = Natural frequency.

(a) When $\sqrt{q} \neq \omega$: choose

$$y_p = A\sin\omega t + B\cos\omega t.$$

$$\Rightarrow y_p'' + q y_p = -\omega^2 A\sin\omega t - \omega^2 B\cos\omega t + q(A\sin\omega t + B\cos\omega t)$$
$$= (q-\omega^2)A\sin\omega t + (q-\omega^2)B\cos\omega t.$$

$\Rightarrow$ Choose $A=0$, $B=\frac{1}{q-\omega^2} \Rightarrow y_p = \frac{1}{q-\omega^2}\cos\omega t.$

$$\Longrightarrow y(t) = k_1 \sin\sqrt{q}\,t + k_2 \cos\sqrt{q}\,t + \frac{1}{q-\omega^2}\cos\omega t.$$

Special situation: $y(0)=0$, $y'(0)=0$.

$\Downarrow$ Check!

$$k_2 = \frac{-1}{q-\omega^2},\quad k_1 = 0.$$

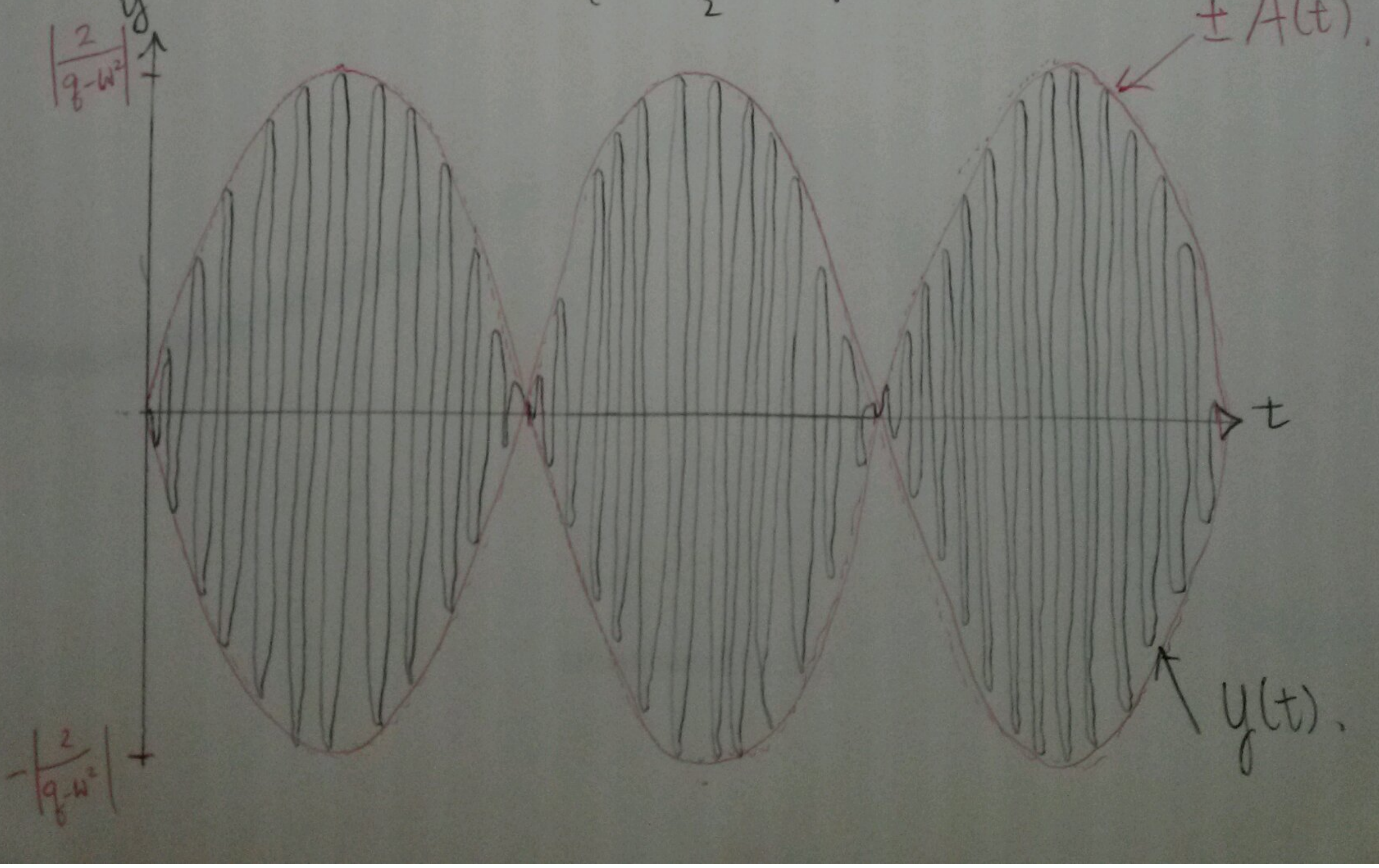
$$\Longrightarrow y(t) = \frac{1}{q-\omega^2}\left(\cos\omega t - \cos\sqrt{q}\,t\right).$$

(Exercises)

$$= \underbrace{\frac{-2}{q-\omega^2}\left(\sin\left(\frac{\omega-\sqrt{q}}{2}\right)t\right)}_{A(t)}\left(\sin\left(\frac{\omega+\sqrt{q}}{2}\right)t\right).$$

Think of $A(t)$ as "amplitude".

When $\omega \approx \sqrt{q}$, $\Longrightarrow \begin{cases} \frac{\omega-\sqrt{q}}{2} \approx 0 \\ \frac{\omega+\sqrt{q}}{2} \approx \sqrt{q}. \end{cases}$



(b) $\sqrt{q} = w$: Resonance :

$$y_p = At\cos \underset{\overset{\shortparallel}{\sqrt{q}}}{\omega} t + Bt\sin \underset{\overset{\shortparallel}{\sqrt{q}}}{\omega} t.$$

$$y_p'' + qy_p = 2(-A\sqrt{q}\sin\sqrt{q}t + B\sqrt{q}\cos\sqrt{q}t)$$

$$\Longrightarrow A = 0,\quad B = \cancel{\frac{t}{2\sqrt{q}}\sin\sqrt{q}t}.\ \frac{1}{2\sqrt{q}}.$$

$$\Longrightarrow y = k_1 \sin\sqrt{q} + k_2\cos\sqrt{q}t + \frac{t}{2\sqrt{q}}\sin\sqrt{q}t.$$

Special situation: $y(0) = 0,\ y'(0) = 0.$

$$\Downarrow$$

$$k_2 = 0,\ k_1 = 0. \qquad \Longrightarrow y(t) = \frac{t}{2\sqrt{q}}\sin\sqrt{q}t.$$