1st order Linear D.E.s

$$y' = \underbrace{a(t)\,y + b(t)}_{\text{linear in } y}.$$

1st order

linear in y: i.e. No $y^2, y^3, \sin y, \ldots$

e.g. $y' = (\sin t)\,y + \cos t$

$y' = e^t y + 1.$

Special case:

(i) $b(t) = 0$ $\quad y' = a(t)\,y.$

(ii) $a(t), b(t)$ constants $\quad y' = \cancel{a(t)}\ ay + b.$

both separable.

(i) $y' = a(t)\,y \Rightarrow \dfrac{y'}{y} = a(t).$

$$\Rightarrow \int \frac{dy}{y} = \int a(t)\,dt.$$

$$\Rightarrow \ln|y| = \int a(t)\,dt + C.$$

$$\Rightarrow \boxed{y = Ce^{\int a(t)\,dt}}$$

Linearity Principle:

Given a 1st order linear D.E:

$$y' = a(t)y + b(t) \qquad (1).$$

let the corresponding homogeneous equation be.

$$y' = a(t)y. \qquad (2)$$

Then if y_p is a particular solution to (1), then

$$y = y_h + y_p$$

is ~~ge~~ the general solution to (1), where y_h is the general solution to (2). (i.e. $y_h = Ce^{\int a(t)dt}$).

Reason: Let y be any solution to (1), then $y - y_p$ satisfies.

$$\begin{aligned}(y-y_p)' &= y' - y_p' \\ &= a(t)y + \cancel{b(t)} - (a(t)y_p + \cancel{b(t)}) \qquad (y, y_p \text{ satisfies } 1) \\ &= a(t)y - a(t)y_p \\ &= a(t)(y - y_p).\end{aligned}$$

i.e. $y - y_p$ satisfies 2 $\Rightarrow y - y_p = y_h$. or $y = y_p + y_h$.

$\square$

e.g. $y' = (\cos t) y + \frac{1}{5}(1 - t\cos t)$. (*)

Note $y_p(t) = t/5$ satisfies (*), and (*) is linear.

Direct checking.

$\Rightarrow$ the general solution to (*) is given by

$$y = y_h + y_p,$$

where $y_h = Ce^{\int \cos t\, dt} = Ce^{\sin t}$.

The main question: how to find y_p??

Method 1: Guessing

Works only for $a(t) = a$ (constant) and $b(t)$ special.

$$y' = ay + b(t).$$

e.g. $y' = 2y + e^{3t}$.

Idea: Want y_p s.t. $y_p' - 2y_p = e^{3t}$.

Try $y_p = Ce^{3t}$. (since derivative of e^{3t} is $3e^{3t}$.

$y_p = Ce^{3t}, \quad y_p' = 3Ce^{3t}.$

$\Rightarrow \quad y_p' - 2y_p = 3Ce^{3t} - 2Ce^{3t} = Ce^{3t}.$

i.e. choose $C=1 \Rightarrow y_p = e^{3t}$ satisfies $y' = 2y + e^{3t}$.

$\Rightarrow$ General solution to $y' = 2y + e^{3t}$ is

$$y = y_h + y_p$$
$$= Ce^{2t} + e^{3t} \quad \#.$$

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e.g. 2 $\quad y' = 2y + \cos t.$

Cannot use $y_p = \cos t$, since $y_p' = -\sin t$ (Not $\cos t$).

Try: $\quad y_p = A\sin t + B\cos t.$

$\Rightarrow \quad y_p' = A\cos t - B\sin t.$

$\Rightarrow \quad y_p' - 2y_p = A\cos t - B\sin t - 2(A\sin t + B\cos t).$
$= (-B - 2A)\sin t + (A - 2B)\cos t.$

i.e. We want

$$\begin{cases} -2A - B = 0 \\ A - 2B = 1 \end{cases} \Rightarrow A = \frac{1}{5}, \quad B = \frac{-2}{5}.$$

$\longrightarrow \quad y_p = \frac{1}{5}\sin t - \frac{2}{5}\cos t.$

e.g. $y' = 2y + 3e^{2t}$.

the same.

$y_p = Ce^{2t}$ does not work, since

$$y_p' - 2y_p = 2Ce^{2t} - 2Ce^{2t} = 0.$$

Instead, try $y_p = Cte^{2t}$.

$$\Rightarrow y_p' = Ce^{2t} + 2Cte^{2t}.$$

$$\Rightarrow y_p' - 2y_p = Ce^{2t} + 2Cte^{2t} - 2Cte^{2t}$$
$$= Ce^{2t}.$$

i.e. choose $C = 3 \Rightarrow y_p = 3te^{2t}$ is a particular solution.

$y' = ay + b(t)$.	
$b(t)$	Guess: y_p
$Ae^{\alpha t}$, $\alpha \neq a$	$Ce^{\alpha t}$.
Ae^{at}	Cte^{at}.
$\cos Bt$, $\sin Bt$.	$A \sin Bt + C\cos Bt$.
$p(t)$ polynomial.	$q(t)$ (with the same degree

Integrating factor

- Another method to find explicit functions to

$$y' = a(t)y + b(t).$$

- Essentially no restriction on $a(t)$, $b(t)$,
- Strictly stronger than the guessing method.
- Harder to implement: Need to evaluate integrals.

For our convenience, we instead consider

$$\boxed{y' + g(t)y = b(t)} \quad \cdots \quad (*)$$

(i.e. $g(t) = -a(t)$)

Idea: Cannot just integrate $(*)$, because of the term $g(t)y$.
So try to multiply $(*)$ by a function $\mu(t)$, so that

$$\mu(t)y' + g(t)\mu(t)y = (B(t)y)' \cdots (**)$$

- We want this since then we can integrate:

$$(B(t)y)' = \mu(t)b(t)$$

$$\Rightarrow \quad B(t)y = \int \mu(t)b(t)dt. \Rightarrow y = \frac{1}{B(t)}\int \mu(t)b(t)dt.$$

- By product rule:

$$(B(t)y)' = B'(t)y + B(t)y' \quad (\text{similar to } \mu(t)y' + g(t)\mu(t)y)$$

to this end, we set

$$\mu(t) y' + g(t)\mu(t) y = (B(t) y)'.$$

(where $g(t)$ is given, and we want μ, B).

① Product rule:

$(B(t)y)' = B(t)y' + B'(t)y \Rightarrow B(t) = \mu(t)$. (comparing y' coeff.)

i.e. $(\mu(t)y)' = \mu(t)y' + \mu'(t)y$.

$$\Rightarrow \mu'(t) = g(t)\mu(t) \quad \text{(comparing } y\text{- coeff.)}$$

This is separable! Solutions are

(***) ... $\mu(t) = e^{\int g(t)dt}$. (the general solution is $Ce^{\int g(t)dt}$, we choose $C=1$ for simplicity)

i.e. $\mu(t)y' + \mu(t)g(t)y = \mu(t)b(t)$

$\Rightarrow (\mu(t)y)' = \mu(t)b(t)$ (assume μ satisfies (***))

$\Rightarrow \mu(t)y = \int \mu(t)b(t)dt$

$$\Rightarrow \boxed{y = \frac{1}{\mu(t)}\int \mu(t)b(t)dt}$$

e.g. $y' + \frac{2}{t}y = t-1.$

$g(t) = \frac{2}{t} \Rightarrow \cancel{\mu(t)=} \int \frac{2}{t}dt = 2\ln t.$

$\Rightarrow \mu(t) = e^{\int g(t)dt} = e^{2\ln t} = e^{\ln t^2} = t^2.$

$\times t^2$

$t^2y' + 2ty = t^2(t-1).$

$\Rightarrow (t^2y)' = t^3 - t^2.$

$\xrightarrow{\int \cdot dt} t^2y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + C.$

$\Rightarrow y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{C}{t^2}$ ※.

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e.g. $y' = ay + e^{at}$ (In guessing method, we'll need $y_p = Cte^{at}$).

$\Leftrightarrow y' - ay = e^{at}.$

$g(t) = -a \Rightarrow \mu(t) = e^{\int g(t)dt} = e^{-at}.$

$\Rightarrow e^{-at}y' - ae^{-at}y = 1.$

$\Rightarrow (e^{-at}y)' = 1$

$\xrightarrow{\int \cdot dt} e^{-at}y = t + C.$

$\Rightarrow y = \underbrace{te^{at}}_{y_p} + \underbrace{Ce^{at}}_{y_h}.$