Separation of variables; initial value problems:

In some situation, it's easy to find all explicit solutions of a D.E.:

$$\boxed{\text{e.g.} \quad y' = t^2+1. \qquad (f(t,y) = t^2+1)}.$$

$$\xRightarrow{\int \cdot\, dt} \quad \int y'(t)\,dt = \int (t^2+1)\,dt = \frac{1}{3}t^3+t+C.$$

FTC: $\quad y(t) = \int y'(t)\,dt.$

$$\rightarrow \quad \boxed{y(t) = \frac{1}{3}t^3+t+C}$$

Not working if there's "y" on the RHS:

$$\boxed{\text{e.g.} \qquad y' = t+y^2}$$

$$\xRightarrow{\int \cdot\, dt} \quad y(t) = \int y'(t)\,dt$$

$$= \int (t+y^2(t))\,dt$$

$$= \int t\,dt + \boxed{\int y^2(t)\,dt.} \leftarrow \text{Dunno what to do.}$$

The separation of variables is a method to solve D.E. by

- Moves all "y" terms to the L.H.S., and
- $\int \cdot\, dt$ on ~~the~~ both sides.

e.g. $y' = t(y^2+1)$.

Dividing (y^2+1) : $\frac{y'}{y^2+1} = t$.

$\int \cdot \, dt$: $\int \frac{y'}{y^2+1} dt = \int t \, dt$

without the term y, this can be calculated.

$= \frac{1}{2}t^2 + C$.

Substitution

$y'dt = dy$: $\int \frac{1}{y^2+1} dy = \frac{1}{2}t^2 + C$.

Integrate L.H.S and isolate y : $\arctan y = \frac{1}{2}t^2 + C$.

$\Rightarrow \quad y = \tan\left(\frac{1}{2}t^2 + C\right)$ #.

In general, the above works if the D.E. is of the form:

$$y' = g(t)\, h(y).$$

- Dividing by $h(y)$: $\frac{y'}{h(y)} = g(t)$.
- $\int \cdot \, dt$: $\int \frac{y'}{h(y)} dt = \int g(t)\, dt$.
- $y'dt = dy$: $\int \frac{dy}{h(y)} = \int g(t)\, dt$.
- Evaluate the integrals and isolate y. $\Rightarrow$ Done!

e.g. $y' = \dfrac{2t}{y - t^2 y}$

$\because \dfrac{2t}{y-t^2y} = \left(\dfrac{1}{y}\right)\cdot\left(\dfrac{2t}{1-t^2}\right) \Rightarrow$ Separable!

$\Longrightarrow \quad y y' = \dfrac{2t}{1-t^2}.$

$\xrightarrow{\int \cdot \, dt} \quad \int y y' \, dt = \int \dfrac{2t}{1-t^2} \, dt$

$\int y \, dy = \int \dfrac{-1}{u} \, du \quad (u = 1-t^2)$

$\Rightarrow \quad \frac{1}{2} y^2 = -\ln|1-t^2| + C.$

$\Rightarrow \quad y = \pm\sqrt{2(C - \ln|1-t^2|)}.$

Initial Value Problem (IVP): Find solution to a given D.E. with an extra conditions on $y(t_0)$.

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0. \end{cases}$$

Roughly speaking, the extra initial condition $y(t_0) = y_0$ corresponds to fixing/specifying the integration constant C.

e.g. Solve the IVP

$$\begin{cases} y' = t(y^2+1), \\ y(0) = \sqrt{3}. \end{cases}$$

Solution: We calculated already that

$$y(t) = \tan\left(\tfrac{1}{2}t^2 + C\right) \quad \cdots \quad (*).$$

if $y(0) = \sqrt{3}$.

set $t=0$ in $(*)$ $\Rightarrow$ $y(0) = \tan C$.

$\Rightarrow$ $\sqrt{3} = \tan C$.

i.e. $C = \frac{\pi}{3}$.

$\Longrightarrow$ $y(t) = \tan\left(\frac{1}{2}t^2 + \frac{\pi}{3}\right)$ solves the IVP.

Caution: Missing solutions.

e.g. Applying separation of variables to $y' = y$. :

$\Rightarrow \frac{y'}{y} = 1 \Rightarrow \int \frac{y'}{y}\,dt = t + C \Rightarrow \ln|y| = t + C$

$\Rightarrow |y| = Ce^t \ (C>0) \Rightarrow y = \pm C e^t$.

But it misses the trivial solution $y_0(t) \equiv 0$.

Reason: In writing $\frac{y'}{y}$, we implicitly assumed that $y(t) \neq 0$ for all t.

Caution 2: To apply the method, the D.E. Must be of the form

$$y' = g(t)\,h(y)$$

e.g. $\quad y' = t + y.$

$$\rightsquigarrow \quad y' - y = t$$

$$\int \cdot\, dt \rightsquigarrow \quad \int (y' - y)\,dt = \int t\,dt$$

$$\rightsquigarrow \quad \underset{\substack{\parallel \\ y}}{\int y'\,dt} - \boxed{\int y\,dt} = \tfrac{1}{2}t^2 + C$$

Dunno what to do: it's ~~dy~~ dt, not dy.