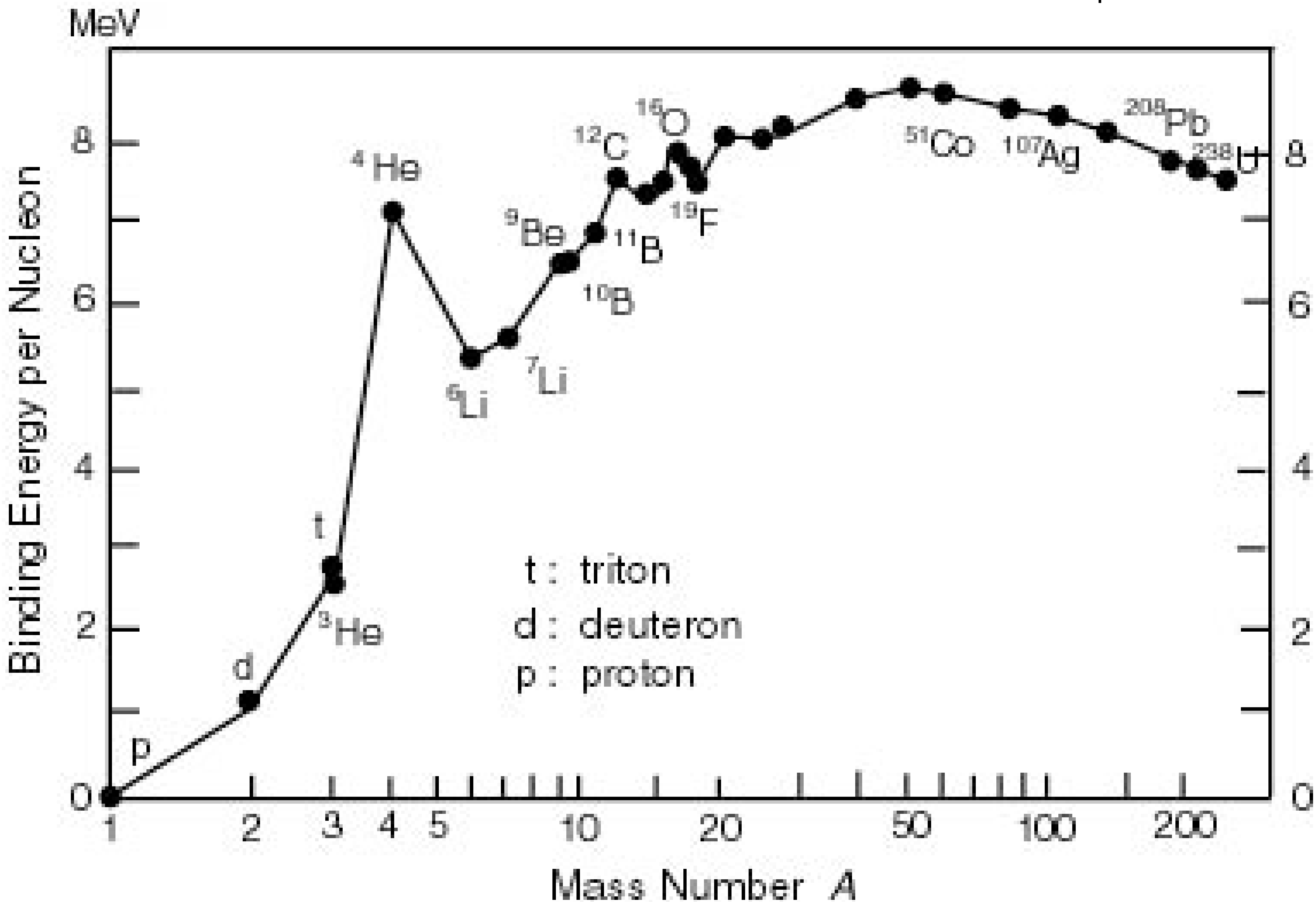
# Lecture 8: Nuclear Energy Generation

Phillips Ch. 4



# radiation (radiative diffusion = photon conduction)

thermal conductivity

$$j(x) = -K\frac{dT}{dx} \quad \text{with} \quad K \approx \frac{1}{3}\bar{v}\ell C$$

$$\bar{v} = c \qquad u = aT^4 \quad C = \frac{du}{dT} = 4aT^3$$

radiation constant
(see Lecture 1, slide 17)

$$a = \frac{8\pi^5 k^4}{15h^3c^3}$$

The heat flux density due to radiation is

$$j(x) = -K_r\frac{dT}{dx} \qquad \text{with} \qquad K_r \approx \frac{4}{3}claT^3$$

in general we write the photon mean free path in terms of the *opacity* $\kappa$

$$\boxed{j(x) = -\frac{4ac}{3}\frac{T^3}{\rho\kappa}\frac{dT}{dx} \qquad \text{where} \qquad l = \frac{1}{\rho\kappa}}$$

# Sources of Opacity

## 1) Electron scattering

$$\kappa_{es} = \frac{\sigma_T}{m_H} \cdot \frac{1}{\mu_e} = \frac{0.40}{\mu_e} \simeq 0.40\,\frac{1+X}{2} \approx 0.2(1+\mathrm{X})\ \mathrm{cm}^2/\mathrm{g}$$

## 2) Free-free absorption

$$\kappa_{\mathrm{ff}} = 7.5 \cdot 10^{22} \left(\frac{1+X}{2}\right) \left\langle \frac{Z^2}{A} \right\rangle \rho T^{-7/2}\ \mathrm{cm}^2/\mathrm{g}$$

$\rho$ in $\mathrm{g/cm}^3$, $T$ in K

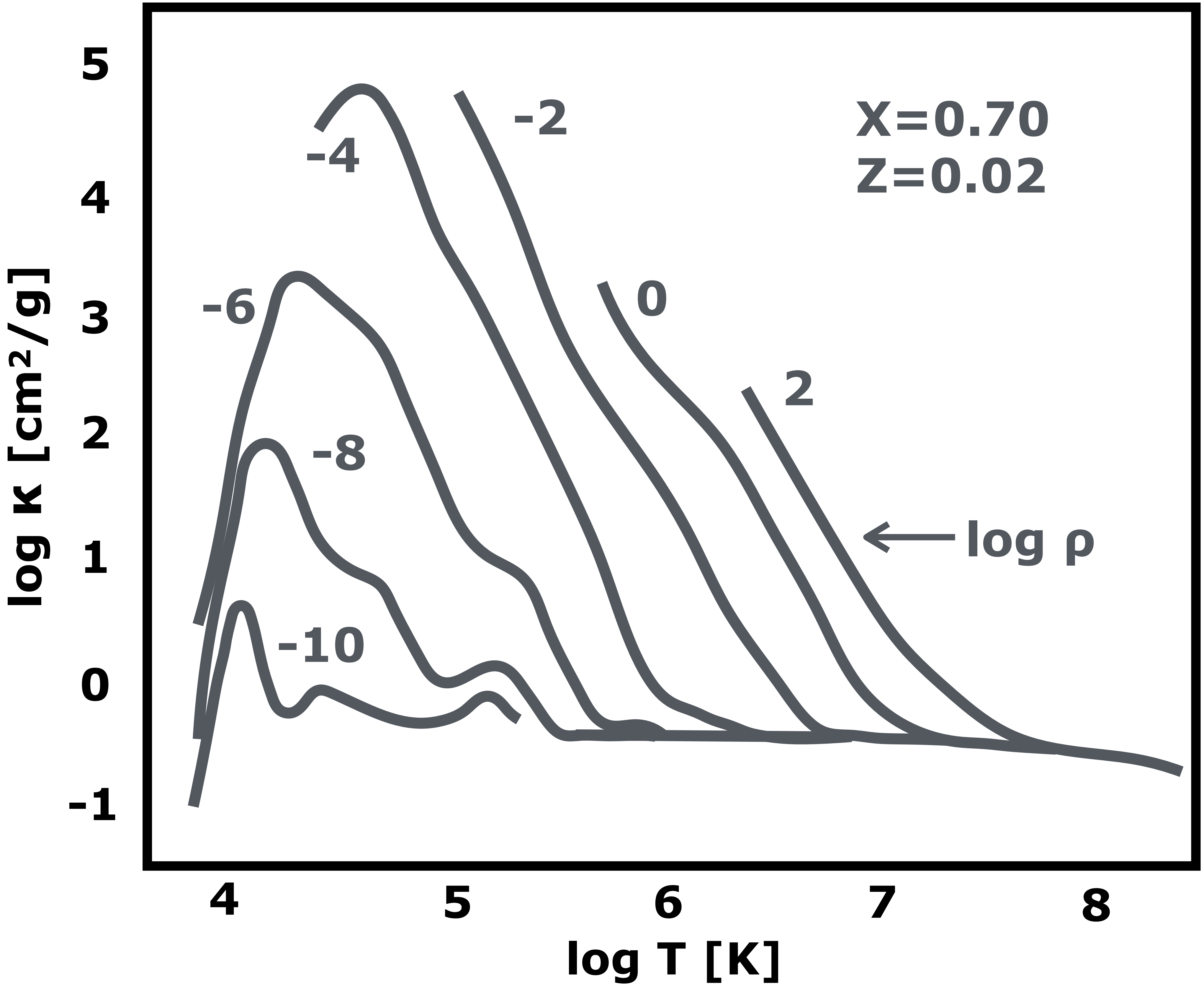
## 3) Bound-free absorption

$$\kappa_{\mathrm{bf}} = 4.3 \times 10^{25}\ (1+\mathrm{X})\ \mathrm{Z} \cdot \rho \cdot \mathrm{T}^{-7/2}\ \left(\mathrm{cm}^2/\mathrm{g}\right)$$

## 4) Bound-bound absorption

$\kappa_{\mathrm{bb}} =$ unpleasant

# Sources of Opacity



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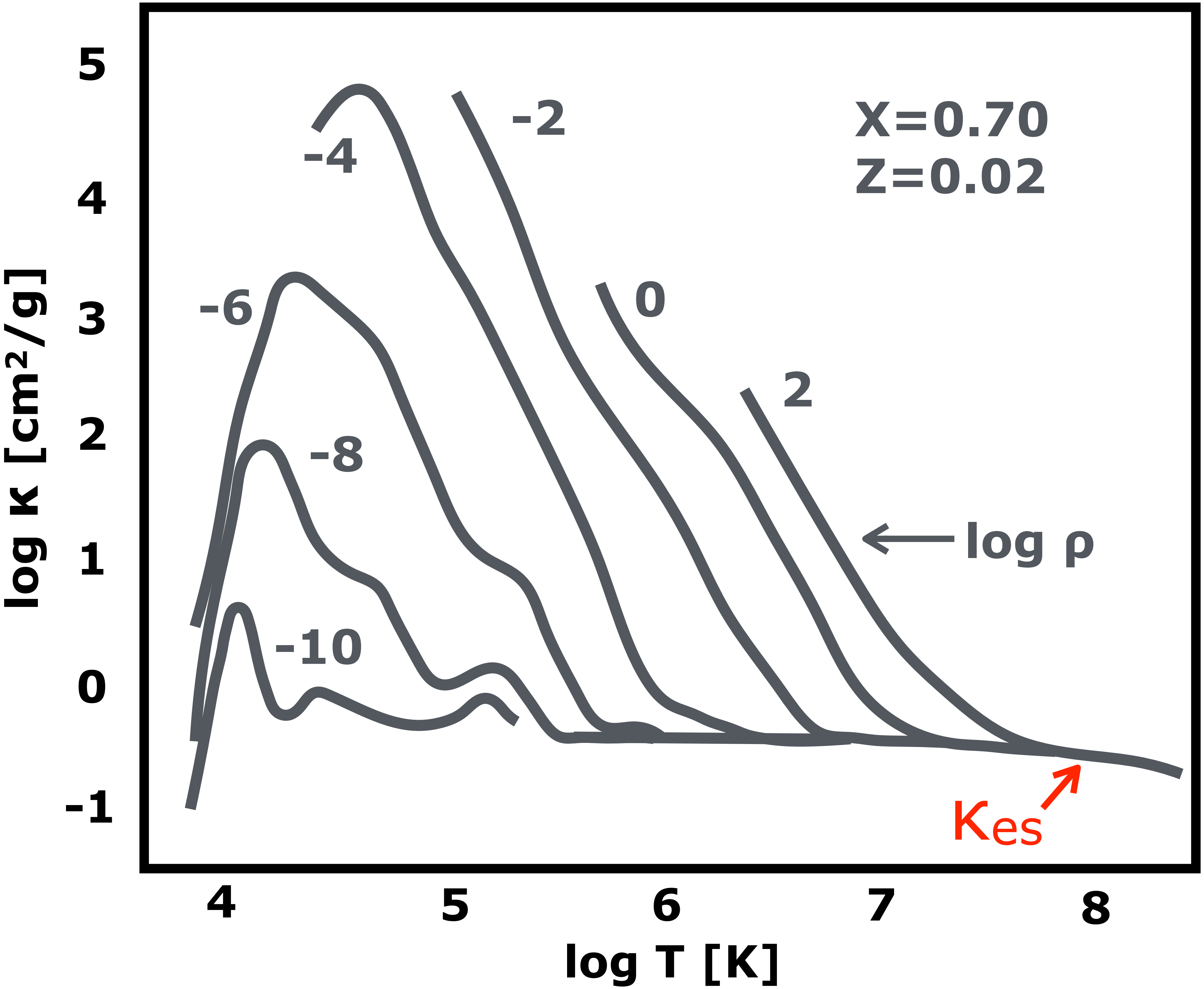

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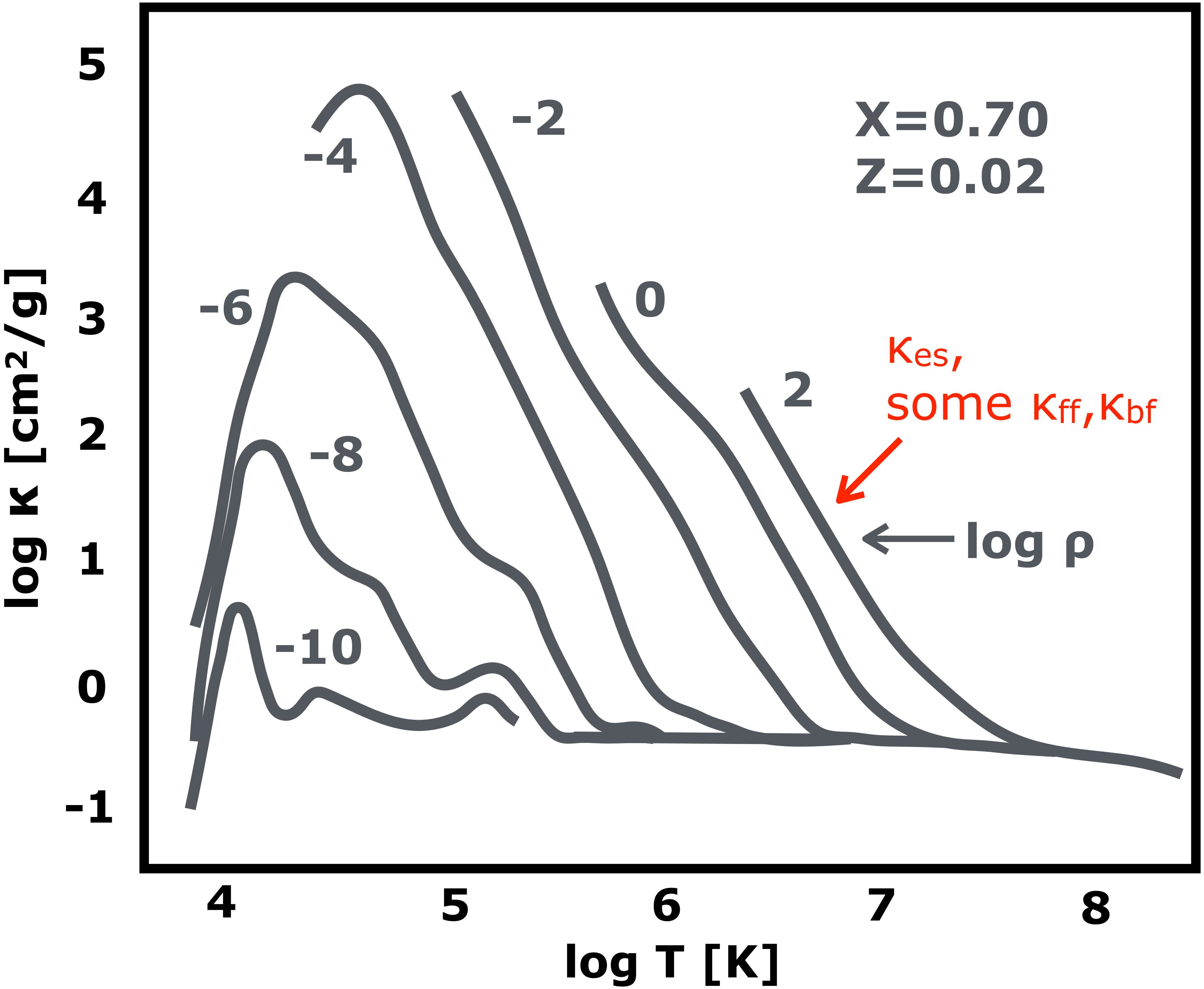

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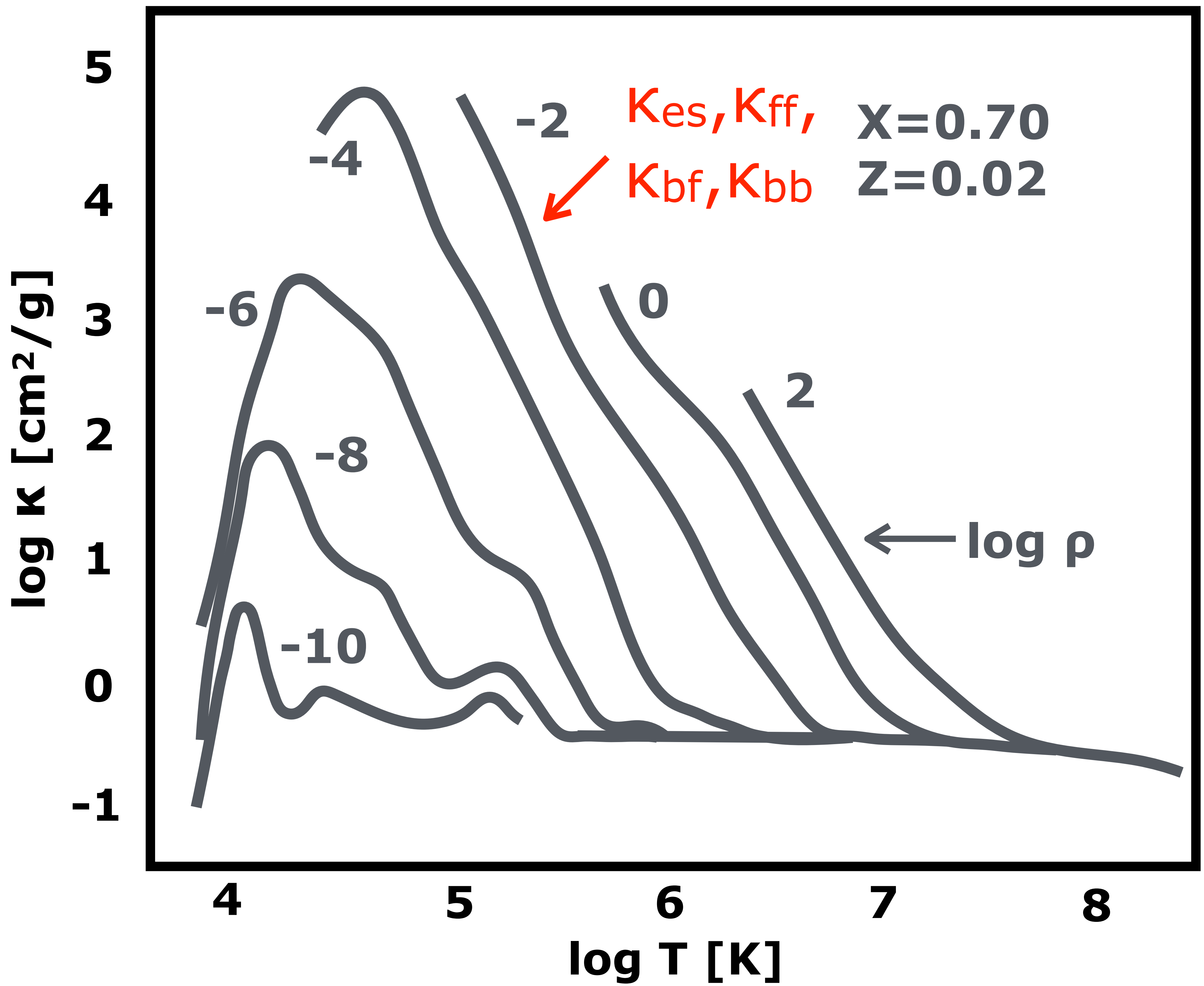

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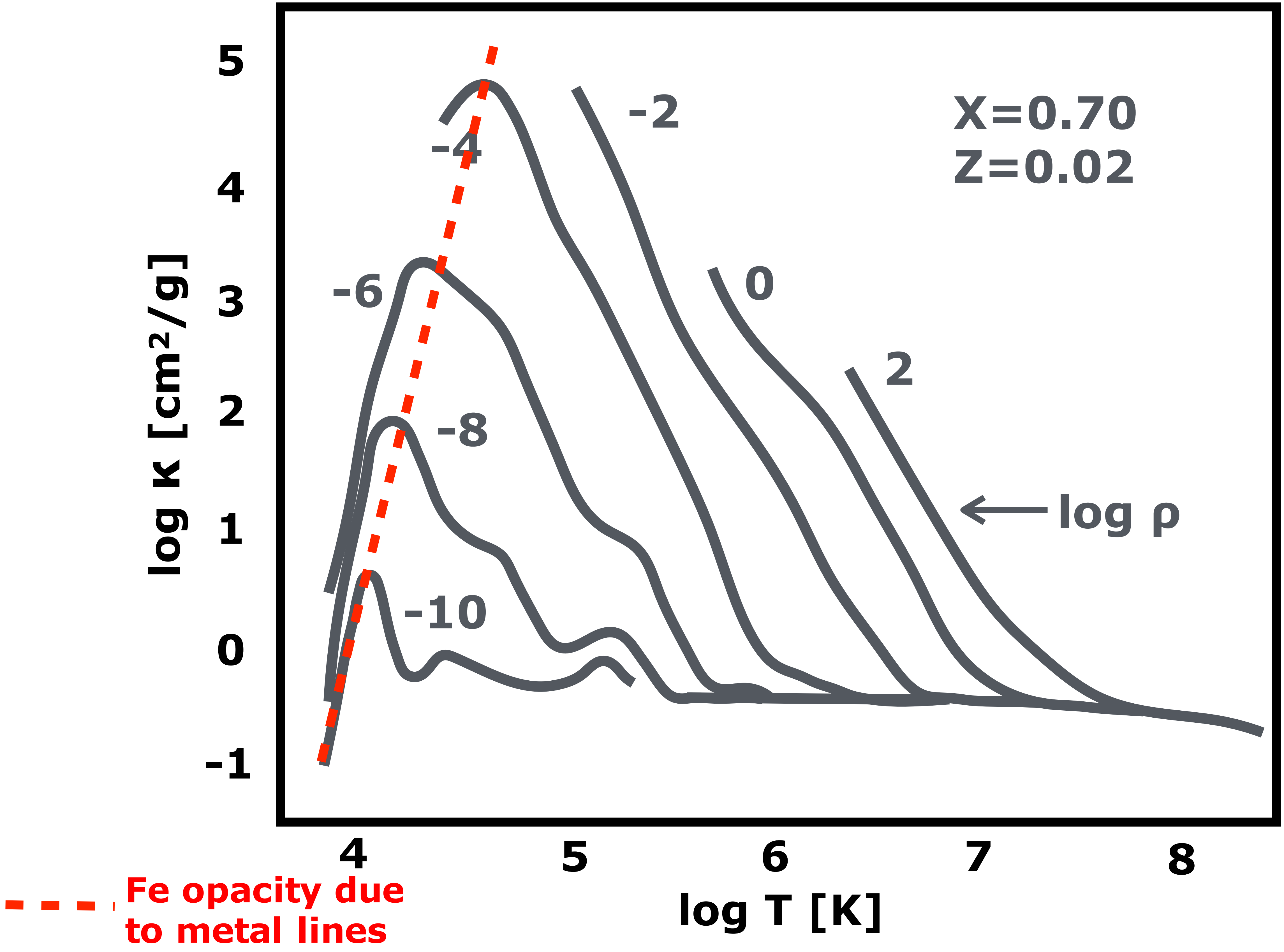


# radiative temperature gradient

**Temperature in a Star**

First, we need to talk about luminosity. We will call $L(r)$ the amount of energy that flows through a spherical surface of radius $r$ for a star. For stars powered by nuclear fusion, $L(r)$ increases outward until we reach the radius where nuclear fusion is no longer occurring. We will call $\epsilon(r)$ the nuclear power generated per unit volume at radius $r$. Then the power generated in a shell between $r$ and $r+dr$ is

$$\epsilon(r)4\pi r^2 dr.$$

Because this power generated adds to the overall energy flow, we have

$$dL = 4\pi r^2 \epsilon(r) dr$$
$$\frac{dL}{dr} = 4\pi r^2 \epsilon(r).$$

After the radius where nuclear energy stops,

$$\epsilon(r) = 0,$$

and

$$\frac{dL}{dr} = 0.$$

After this radius, the $L(r)$ is a constant, and is equivalent to the surface luminosity.

If radiative diffusion is the dominant energy transport, then

$$L(r) = 4\pi r^2 j(r).$$

Therefore,

$$\frac{L(r)}{4\pi r^2} = -\frac{4ac}{3}\frac{T^3(r)}{\rho(r)\kappa(r)}\frac{dT}{dr},$$

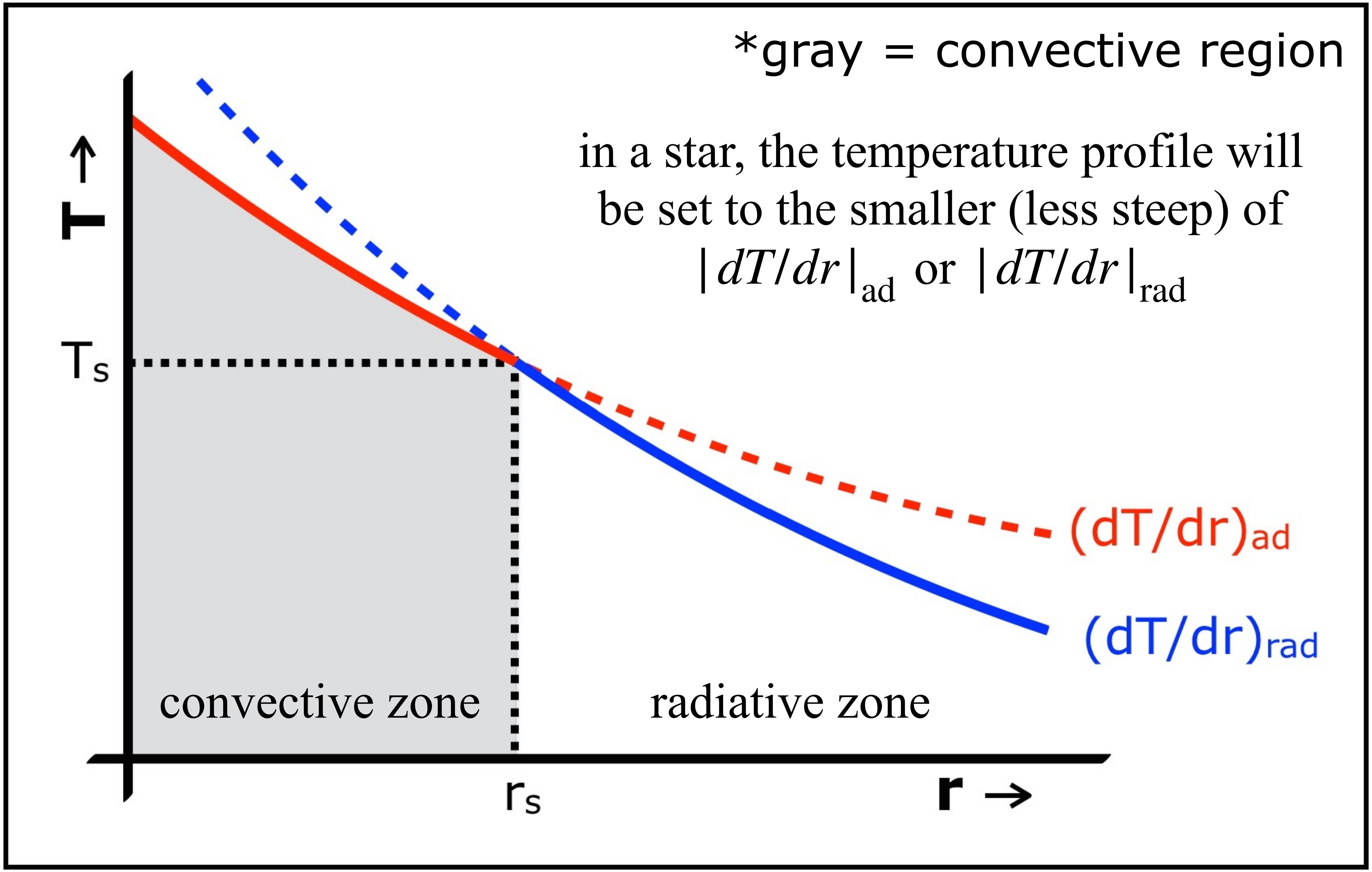
and

$$\boxed{\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa(r)\rho(r)}{T^3(r)}\frac{L(r)}{4\pi r^2}} \qquad \left|\frac{dT}{dr}\right|_{\mathrm{rad}}$$

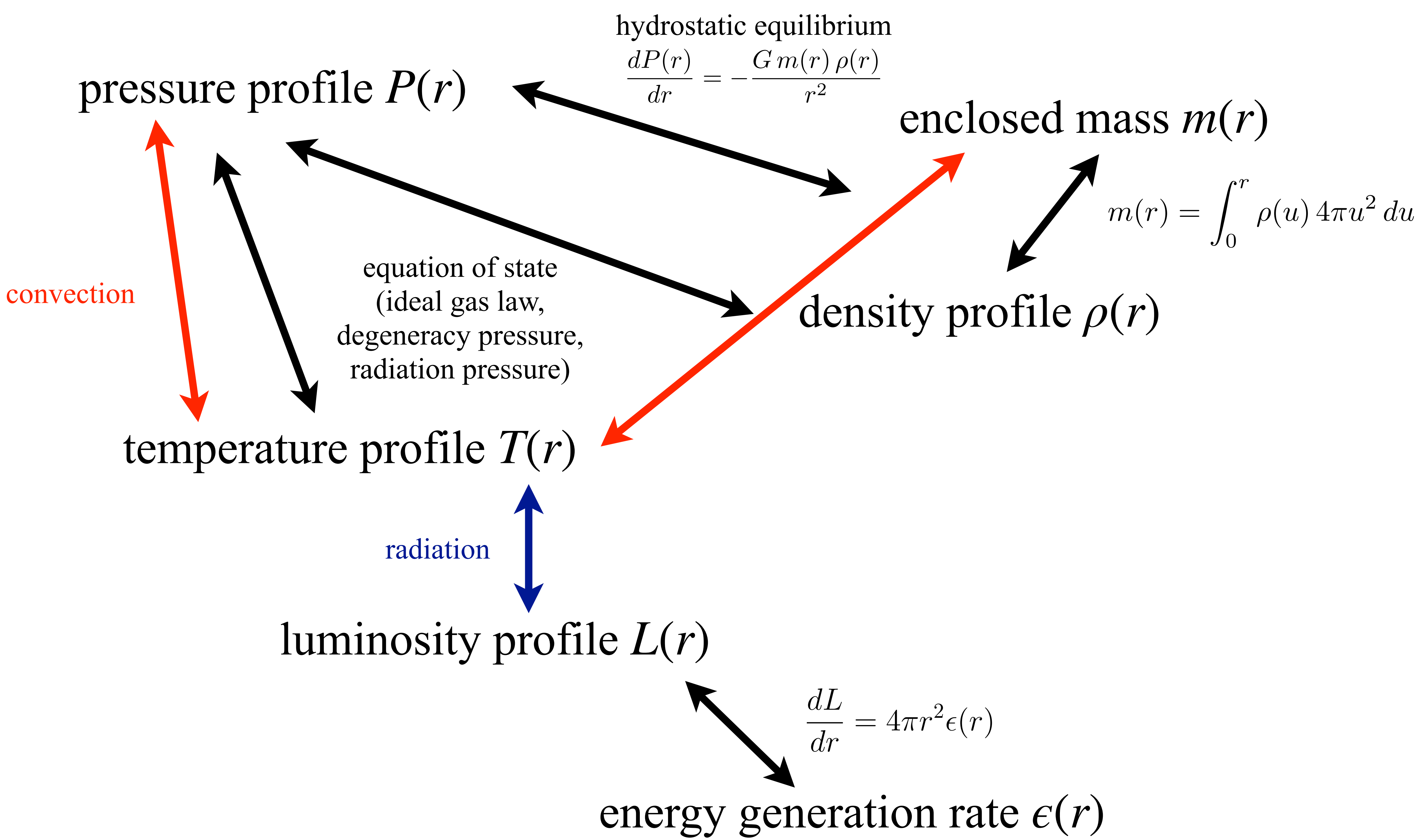
radiative temperature gradient

# Radiation vs. Convection

**Schwarzschild criterion** $|dT/dr|_{ad} < |dT/dr|_{rad}$



# stellar structure



# Coulomb barrier

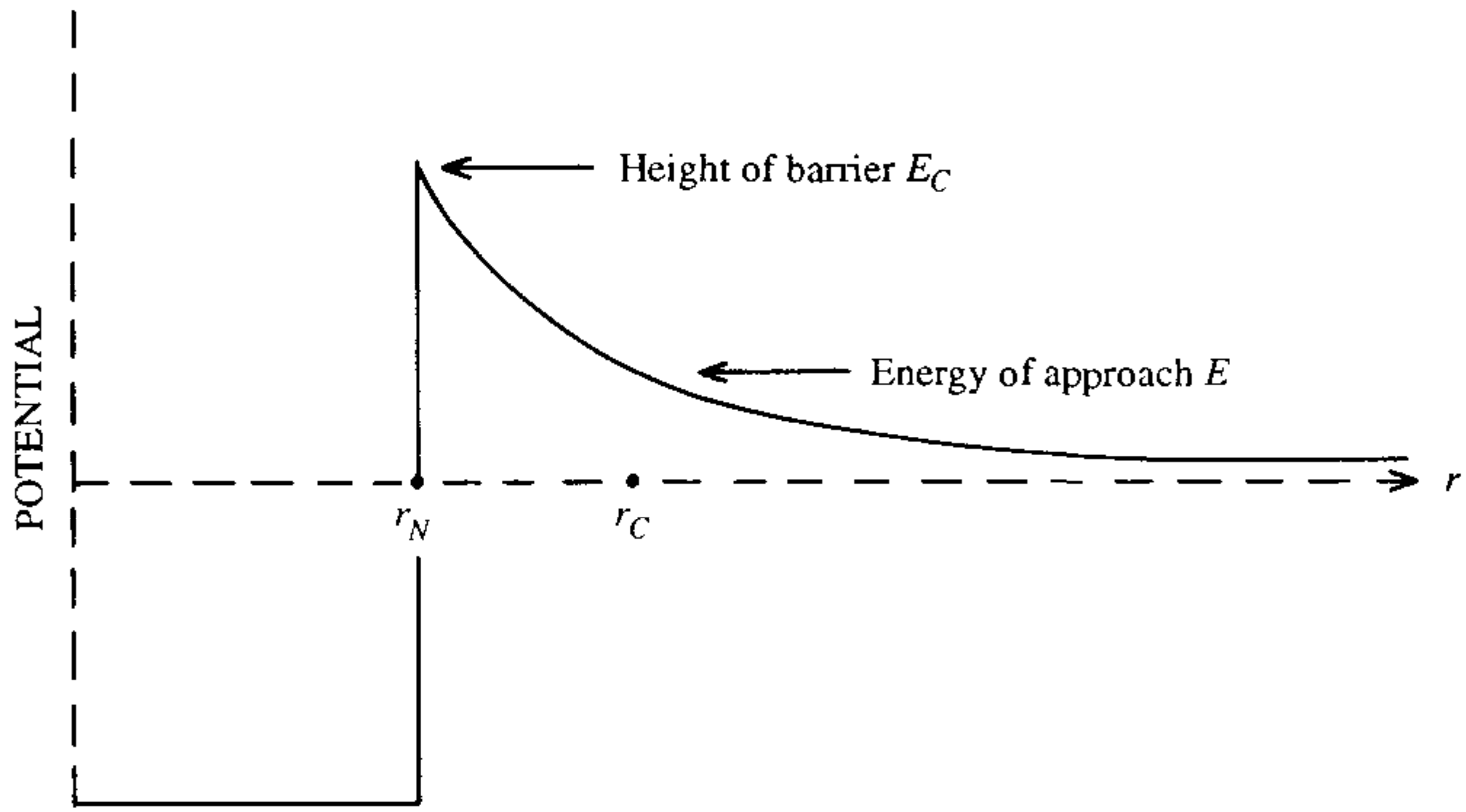
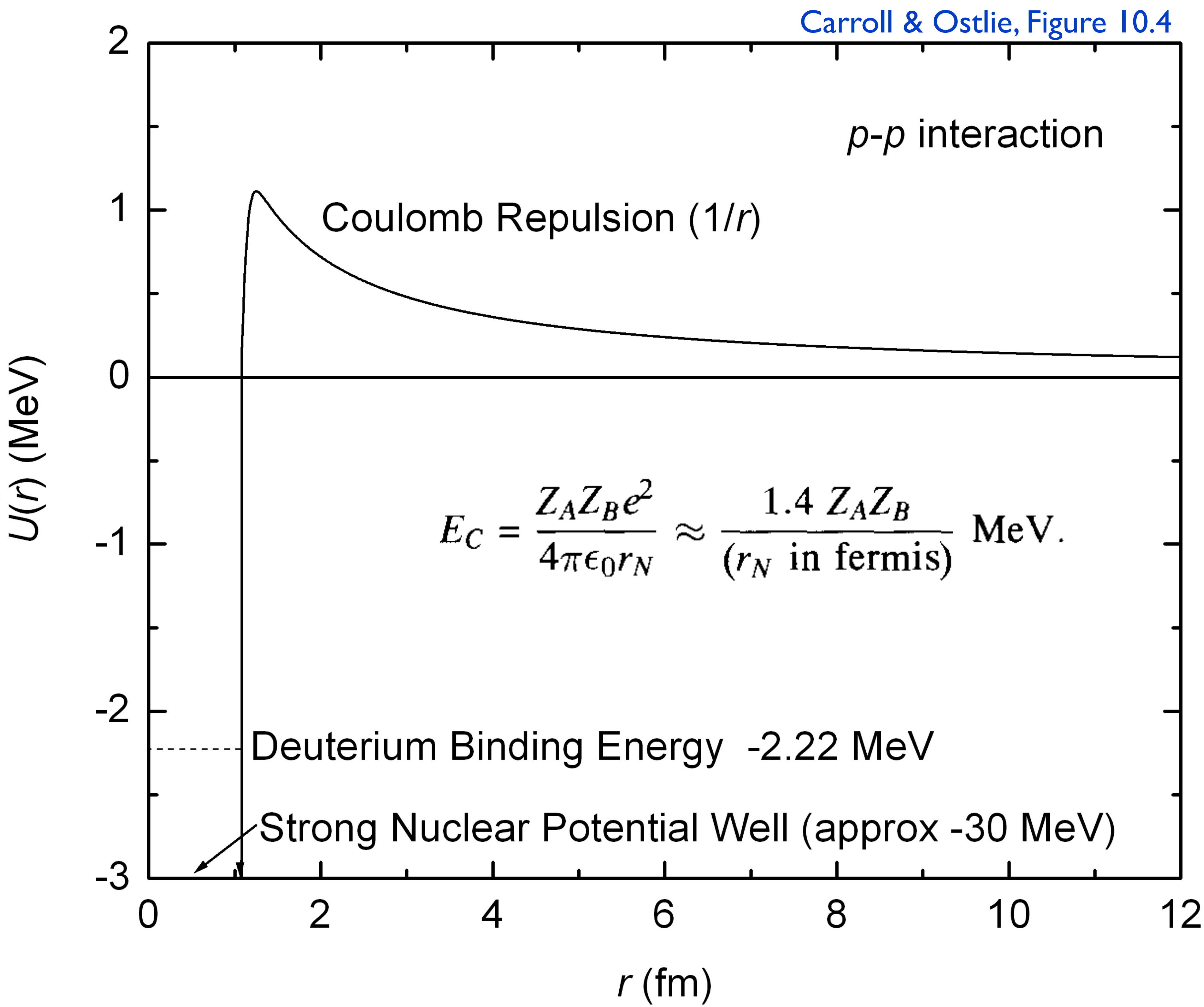



Fig. 4.1 A representation of the Coulomb and nuclear potentials between two nuclei of charge $Z_A$ and $Z_B$. The distance $r_C$ is the classical distance of closest approach for nuclei with an energy of approach equal to $E$. The distance $r_N$ represents the range of short-range nuclear forces. $E_C$ is the height of the Coulomb barrier keeping the nuclei apart.

Phillips Fig. 4.1

# Coulomb barrier




1 fm = 1 femtometer = 1 fermi = $10^{-15}$ m

# Coulomb barrier

near the center of the Sun, $T \approx 10^7$ K $\Rightarrow kT \approx 1$ keV, so the Coulomb barrier of ~1 MeV is about 1000 times larger than the typical proton energy

what fraction of protons have enough energy to overcome Coulomb barrier?

Boltzmann factor: $p \approx e^{-E/kT} = e^{-\frac{1\ \mathrm{MeV}}{1\ \mathrm{keV}}} = e^{-1000} \approx 10^{-436} \approx 0$

Phillips: *Thus at first sight, Coulomb repulsion presents an insurmountable barrier to fusion in stars.*

# Coulomb barrier: tunneling

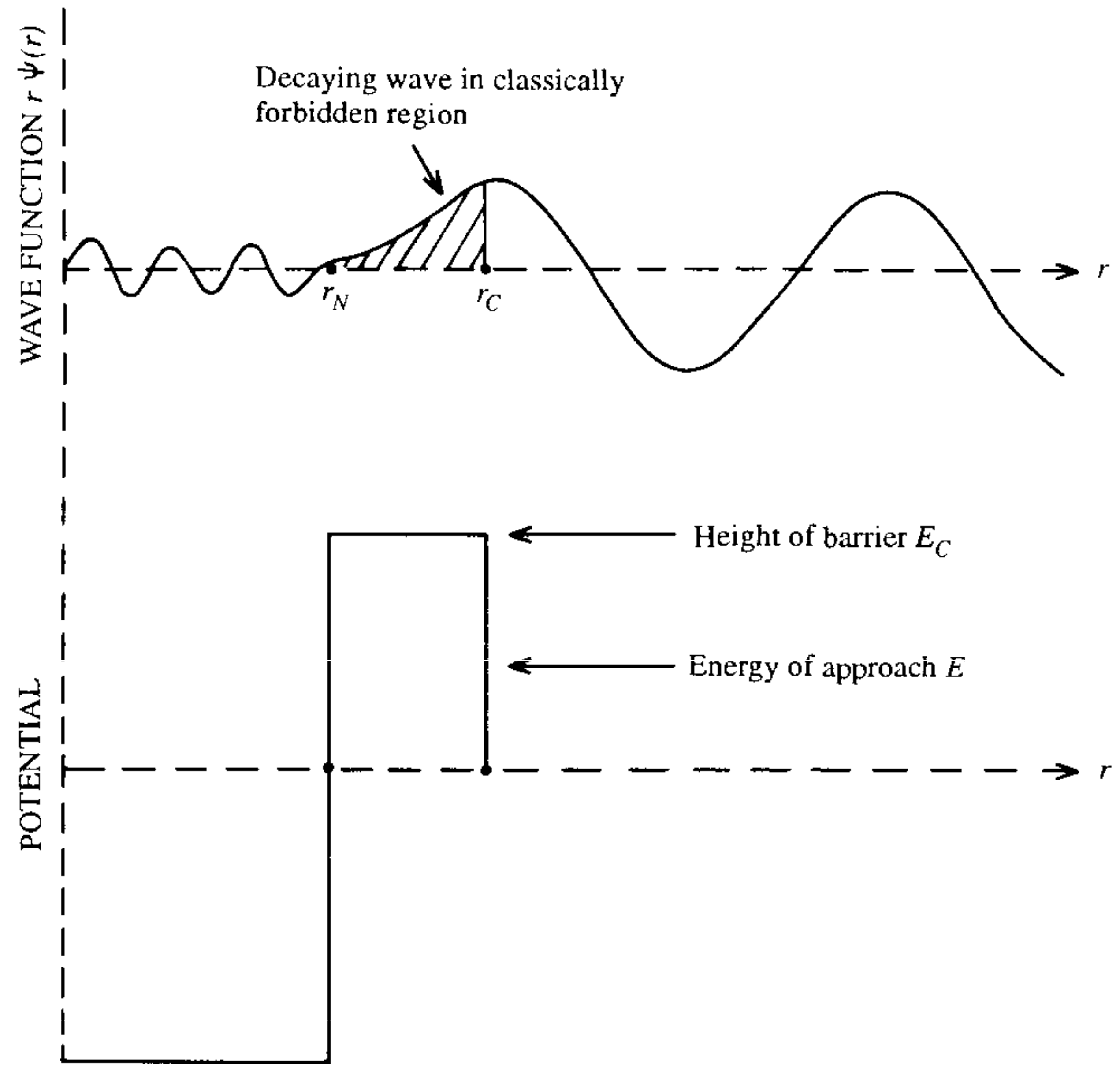



Fig. 4.2 The wave function representing the penetration of a barrier of constant height $E_C$ by particles whose energy of approach $E$ is below the barrier. The wave function, $r\psi(r)$, oscillates sinusoidally in the outer and inner classically allowed regions. It decays exponentially in the intervening classically forbidden region. In stellar thermonuclear fusion the wavelength for the relative motion of the nuclei in the outer classically allowed region is very long compared with the range of nuclear forces $r_N$.

Phillips Fig. 4.2

# Coulomb barrier: tunneling

Gamow energy, defined by

see Phillips for derivation

$$E_G = (\pi \alpha Z_A Z_B)^2 2 m_r c^2, \tag{4.10}$$

where $\alpha$ is the dimensionless fine structure constant,

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}. \tag{4.11}$$

Equation (4.9) then leads to

$$\text{Probability of tunneling} \approx \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right]. \tag{4.12}$$

Thus, the Coulomb barrier keeping charged nuclei apart need not be overcome in order to give the nuclei a chance to fuse. In practice, stars evolve slowly by adjusting their temperature so that the average thermal energy of nuclei is well below the Coulomb barrier. Fusion then proceeds at a rate proportional to the probability of penetration of the barrier. Because this probability is very low, fusion proceeds at a slow pace and the nuclear fuel lasts for an astronomically long time. We note that the penetrability of the barrier is completely described by its Gamow energy, Eq. (4.10). For the fusion of two protons $E_G$ is 493 keV. If the temperature is about $10^7$ K, the typical thermal energy, $kT$, is about 1 keV, and the penetration probability for two protons with this typical energy is $\exp[-(E_G/kT)^{1/2}] \approx \exp[-22]$. There are, of course, protons present with higher kinetic energy which will have a better chance of penetrating the Coulomb barrier.

$$p \approx e^{-22} \approx 10^{-10}$$

# Nuclear reactions: particles

let’s list some subatomic particles:

# Nuclear reactions: nuclei

mass number (protons + neutrons)

atomic number (protons; optional)

$$^{A}_{Z}\mathrm{Element}$$

example: isotopes of carbon $^{12}_{6}\mathrm{C}$ $^{13}_{6}\mathrm{C}$ $^{14}_{6}\mathrm{C}$

or just $^{12}\mathrm{C}$ $^{13}\mathrm{C}$ $^{14}\mathrm{C}$

# Nuclear reactions: conservation laws

1. electric charge
   positive (proton, positron)    negative (electron)    zero (neutron)

2. lepton number
   +1 (electron, neutrino)    –1 (positron, antineutrino)

3. nucleon number (protons + neutrons)

for example, here is the first reaction in the *p-p* chain that eventually fuses hydrogen into helium:

$$ {}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \;\rightarrow\; {}^{2}_{1}\mathrm{H} + e^{+} + \nu_e $$

for the *strong nuclear force* protons stay as protons, neutrons stay as neutrons
the *weak nuclear force* can convert between a neutron ↔ proton

# Nuclear reaction energies

Nuclear fusion involves the conversion of mass to energy via $E = mc^2$. It works because, starting from hydrogen and moving toward iron, progressively heavier nuclei are actually *less than* the sum of their parts. It is common for masses to be quoted in terms of equivalent energy, $m = E/c^2$.

$$1 \text{ u} = 1.660539 \times 10^{-24} \text{ g} = 931.494 \text{ MeV}/c^2$$

This is 1/12th the mass of a carbon-12 nucleus, and slightly smaller than the mass of a proton or neutron:

$$\begin{aligned}
m_p &= 938.272 \text{ MeV}/c^2 = 1.672622 \times 10^{-24} \text{ g} = 1.00727647 \text{ u} \\
m_n &= 939.563 \text{ MeV}/c^2 = 1.674927 \times 10^{-24} \text{ g} = 1.00866492 \text{ u} \\
m_e &= 0.511 \text{ MeV}/c^2 = 9.109382 \times 10^{-28} \text{ g}
\end{aligned}$$

# Nuclear reaction energies

We will see that the nuclear reaction powering the Sun is the fusion of 4 hydrogen atoms into 1 helium atom. What is the mass involved?

$$\begin{aligned}
\text{4 hydrogen :} \qquad m_{4\mathrm{H}} &= 3755.13\ \mathrm{MeV}/c^2 = 4.031300\ \mathrm{u} \\
\text{helium :} \qquad m_{\mathrm{He}} &= 3728.40\ \mathrm{MeV}/c^2 = 4.002603\ \mathrm{u} \\
\Rightarrow \Delta m &= 26.73\ \mathrm{MeV}/c^2 = 0.007\ m_{4\mathrm{H}} = 0.028697\ \mathrm{u} \\
E &= \Delta m\ c^2 = 26.73\ \mathrm{MeV}
\end{aligned}$$

In other words, each hydrogen atom gives up 0.7% of its mass. This quantity is sometimes called the **efficiency** $\epsilon$ of a nuclear reaction

$$\epsilon = \frac{m_{\mathrm{start}} - m_{\mathrm{end}}}{m_{\mathrm{start}}} = \frac{\Delta m}{m_{\mathrm{start}}}$$

so that the total energy released starting with nuclear "fuel" of mass $M$ is just

$$E = \Delta m\, c^2 = \epsilon M c^2$$

For hydrogen fusion into helium, the efficiency is $\epsilon = 0.007$.

# Nuclear energy in the Sun

How much fusion energy is available? A first estimate is:

$$
\begin{aligned}
E_{\rm nuclear} &\sim 0.007 \times M_\odot\, c^2 = 1.3 \times 10^{52}\ \mathrm{erg} \\
t_{\rm nuclear} &\sim \frac{E_{\rm nuclear}}{L_\odot} \sim 10^{11}\ \mathrm{yr}
\end{aligned}
$$

Roughly 100 billion years. In fact, only about 10% of the Sun's mass is available for fusion (i.e., in a region where the temperature is high enough for fusion to occur). So the actual energy and lifetime are about a factor of 10 smaller (10 billion years). Nevertheless, that is still plenty to explain the Sun.

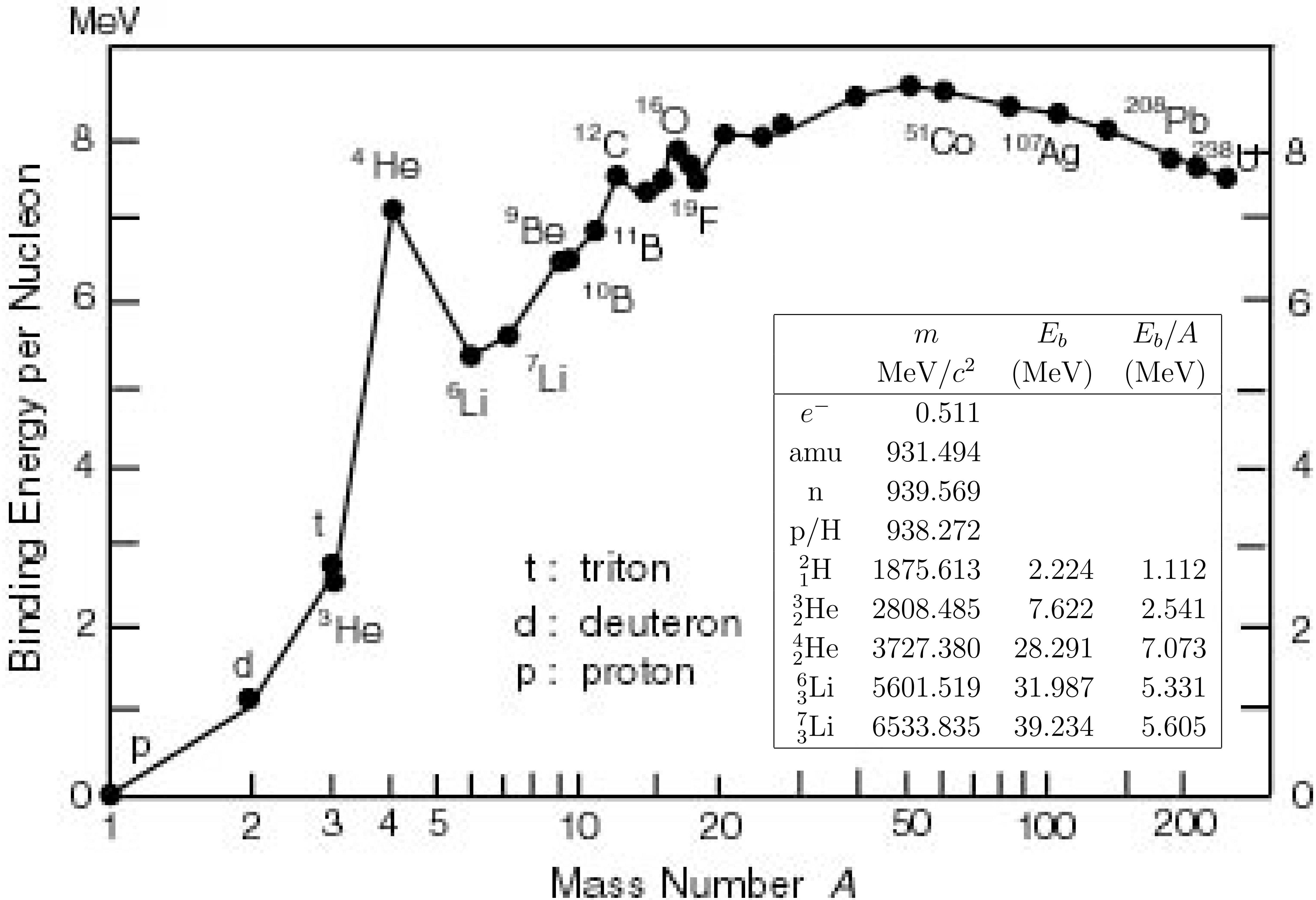
# Nuclear binding energy

To understand how much energy can be released in fusion, let's tabulate some masses and energies. Define the **binding energy** to be the difference between the actual mass of a nucleus and the mass of the same number of isolated protons and neutrons:

$$E_b = \left[ Z\, m_p + (A - Z)\, m_n - m_{\text{nuc}} \right] c^2$$

| | $m$ MeV/$c^2$ | $E_b$ (MeV) | $E_b/A$ (MeV) |
|---|---|---|---|
| $e^-$ | 0.511 | | |
| n | 939.569 | | |
| p/H | 938.272 | | |
| ${}^2_1$H | 1875.613 | 2.224 | 1.112 |
| ${}^3_2$He | 2808.485 | 7.622 | 2.541 |
| ${}^4_2$He | 3727.380 | 28.291 | 7.073 |
| ${}^6_3$Li | 5601.519 | 31.987 | 5.331 |
| ${}^7_3$Li | 6533.835 | 39.234 | 5.605 |

# Nuclear binding energy per nucleon




| | $m$ MeV/$c^2$ | $E_b$ (MeV) | $E_b/A$ (MeV) |
|---|---|---|---|
| $e^-$ | 0.511 | | |
| amu | 931.494 | | |
| n | 939.569 | | |
| p/H | 938.272 | | |
| $^2_1$H | 1875.613 | 2.224 | 1.112 |
| $^3_2$He | 2808.485 | 7.622 | 2.541 |
| $^4_2$He | 3727.380 | 28.291 | 7.073 |
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from http://www.kutl.kyushu-u.ac.jp/seminar/MicroWorld3_E/3Part2_E/3P23_E/nuclear_mass_E.htm