Physics 441/541 Spring 2022: Problem Set #4
due Friday April 1 at 11:00 am in PDF format on Canvas

You are encouraged to work in groups on these problems, but you must write up the solutions individually. In your writeup, list your collaborators and cite any external sources you used. You **may not** consult previous solution sets for this class or other similar classes.

1. On Problem Set #2, we looked at stellar models from Ekström et al. (2012, A&A, 537, 146), and we will use these again here. You can access the paper via `https://ui.adsabs.harvard.edu/abs/2012A%26A...537A.146E/abstract`. A direct link to the PDF is `https://www.aanda.org/articles/aa/pdf/2012/01/aa17751-11.pdf`).

   (a) I encourage you to skim through the whole paper! For now we need to grab one important piece of information: what is the initial hydrogen mass fraction X for the stellar models? (See the beginning of section 2 in the article.)

   From this paper, I have downloaded data for the (non-rotating) stellar models at two points in each star's evolution: the **zero-age main sequence** (ZAMS) when hydrogen fusion starts in the core and the **terminal-age main sequence** (TAMS, also called the "turn-off") when the star runs out of hydrogen to fuse in the center. You can grab the data either at this Google sheet or on Canvas under Files → Problem Set Resources as zams-tams.xlsx or zams-tams.csv. The data give the initial mass for each star (in solar masses) and the time (in years after formation), the $\log_{10}$ of the luminosity in solar units: $\log_{10}(L/L_\odot)$, and the $\log_{10}$ of the effective temperature in Kelvin, for both the ZAMS (start of the main-sequence) and TAMS (end of the main sequence).

   (b) Using this data, make a plot of the stars' main-sequence lifetime (between ZAMS and TAMS) as a function of their initial mass. Use logarithmic x and y axes.

   (c) Make an H-R diagram of this data with log Te on the x-axis (increasing to the left) and log L on the y-axis. Plot the ZAMS data with one set of symbols/color and the TAMS data with another set of symbols/color, all on the same plot.

   (d) Let's look at what fraction of the star's hydrogen is fused on the main sequence. First calculate the average luminosity of each star during this time, as

   $$(\log L)_{\text{avg}} = \left[\frac{(\log L)_{\text{ZAMS}} \ + \ (\log L)_{\text{TAMS}}}{2}\right] \qquad L_{\text{avg}} = 10^{(\log L)_{\text{avg}}}$$

   Then calculate the total energy released during the main sequence for each star, by multiplying this average luminosity by the main-sequence lifetime from part (b). Next, use this to estimate the mass of hydrogen fused on the main sequence by each star. Finally divide this mass by the total starting hydrogen mass of the star, making use of the initial mass and hydrogen mass fraction from part (a). Make a plot of the fraction of each star's hydrogen that is fused on the main sequence versus stellar mass (use a logarithmic x-axis). Comment on your results.

2. The data below come from two locations in the interior of a single main-sequence stellar model, tabulating the radius, enclosed mass, enclosed luminosity, temperature, density, and opacity at the specified locations:

r	$m(r)$	$L(r)$	$T(r)$	$\rho(r)$	$\kappa(r)$
$0.242\ R_\odot$	$0.199\ M_\odot$	$340\ L_\odot$	2.52×10^7 K	1.88×10^4 kg m^{-3}	0.044 m^2 kg^{-1}
$0.670\ R_\odot$	$2.487\ M_\odot$	$528\ L_\odot$	1.47×10^7 K	6.91×10^3 kg m^{-3}	0.059 m^2 kg^{-1}

   (a) Is energy transport at each of these two locations in the star radiative or convective? You may assume that radiation pressure is negligible, so that the gas behaves like a monoatomic ideal gas. Take the mean molecular weight to be $\mu = 0.7$.

   (b) Based on your results, what kind of star is this?

3. Let's examine the structure of the Sun after the main sequence when it moves onto the red giant branch (RGB) of the H-R diagram. As a red giant, the Sun will have a dense, degenerate helium core surrounded by a low density envelope.

   (a) The gravitational potential energy of a sphere with constant density is given by $\Omega = U = E_{\rm pot} = -\frac{3}{5}GM^2/R$. Calculate this value for the Sun today, assuming a constant density. Also calculate it for when the Sun expands as a red giant to $R \approx 100\ R_\odot$ (again assuming a constant density, and ignoring any mass loss).

   (b) The virial theorem says that in equilibrium, the total energy of the star is given by $E_{\rm tot} = U/2$. How much does the total energy change between the Sun today and the red-giant Sun based on your calculation above? Does the Sun's total energy really change by this much or have we made a major error in our assumptions? Explain.

4. The radius of a non-relativistic, fully degenerate star is given by

$$R = 2.7 \times 10^4\ \mu_e^{-5/3} \left(\frac{M}{M_\odot}\right)^{-1/3}\ \text{km} \tag{1}$$

   where μ_e is the mean molecular weight per free electron, $\mu_e = \rho/(n_e m_p)$.

   (a) Show that for fully ionized material, we can approximate $\mu_e = 2/(1 + X)$, where X is the hydrogen mass fraction.

   (b) Assume that the red giant Sun has a pure helium core with a mass $M = 0.25\ M_\odot$. What is the radius of the core? What is its average density? Compare your result to the average density of the Sun today ($\bar{\rho}_\odot = 1.4$ g cm^{-3}) and the central density of the Sun today ($\rho_c \approx 150$ g cm^{-3}).

   (c) What is the average density of the envelope (everything outside the helium core) of the red giant Sun? Compare this to the typical density of air at Earth's surface.

5. The equation of state for an ultra-relativistic white dwarf is

$$P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3}$$

   (a) Replace the electron number density to rewrite the pressure in terms of the mass density ρ and mean molecular weight per free electron μ_e. The result is a polytrope, with $P = K\rho^\gamma$. What are K, γ, and the polytropic index n?

   (b) Recalling our previous results about polytropes, show that there is a unique value of the mass of such a star, with

$$M \approx 2.02\left(\frac{\sqrt{3\pi}}{2}\right)\left(\frac{1}{\mu_e m_p}\right)^2\left(\frac{\hbar c}{G}\right)^{3/2}$$

   Hint: look at Lecture 3, slide 15. Derive α and plug in to the expression for M.

   (c) Plug in constants to derive the Chandrasekhar limit for a carbon-oxygen white dwarf, $M_{\mathrm{Ch}} \approx 1.44\ M_\odot$.

6. (Required for 541; extra credit for 441) Use the procedure from problem 5, except this time for a non-relativistic, fully-degenerate equation of state

$$P = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} n_e^{5/3}$$

to determine the mass-radius relation for non-relativistic white dwarfs. As in that problem, derive your result with symbols first. Then at the end plug in the constants to get a numerical value that you can compare to equation (1) of problem 4.

Group Project Assignment

The Group Project assignment has been uploaded to Canvas. Presentations will start Friday April 22; see the posted schedule. A draft of your presentation is due a week before your presentation date.