Physics 441/541 Spring 2022: Problem Set #6 Solutions

1. *Consider a binary star system with masses M_1 and M_2 (and $M_1 > M_2$) in a circular orbit with separation a. We can write the total orbital kinetic energy as $K = M_1 v_1^2/2 + M_2 v_2^2/2 = \mu v^2/2$, where $\mu = M_1 M_2/(M_1 + M_2)$ is the reduced mass, and $v = v_1 + v_2$ is the relative orbital speed. We can also write the potential energy as $U = -GM_1M_2/a = -GM\mu/a$ where $M = M_1 + M_2$ is the total mass. The virial theorem applies here ($E_{\rm tot} = -K = U/2$); use it to solve for v in terms of the masses and separation.*

   The virial theorem says that $2\langle K\rangle = -\langle U\rangle$, and for a circular orbit the speeds and separation are constant, so we have

$$2\left(\frac{\mu v^2}{2}\right) = \frac{GM\mu}{a} \quad\Rightarrow\quad v^2 = \frac{GM}{a} \quad\Rightarrow\quad v = \left(\frac{GM}{a}\right)^{1/2}$$

   Now imagine that M_1 instantaneously (and spherically symmetrically) loses ΔM of its mass (e.g., in a supernova explosion), so that the relative velocity v and separation a are the same before and immediately after the explosion. For what values of ΔM does the system become unbound? Hint: *recall that a bound system has $E_{\rm tot} < 0$.*

   Recall that bound systems have negative total energy. So the system would become unbound if the total energy $E = K + U \geq 0$. The new reduced mass and new kinetic energy are

$$\mu' = \frac{(M_1 - \Delta M)M_2}{M_1 - \Delta M + M_2} \qquad K' = \frac{\mu' v^2}{2} = \frac{(M_1 - \Delta M)M_2}{M_1 - \Delta M + M_2}\left[\frac{G(M_1 + M_2)}{2a}\right]$$

   The new potential energy is

$$U' = -\frac{G(M_1 - \Delta M)M_2}{a}$$

   The new total energy is

$$\begin{aligned} E' = K' + U' &= \frac{(M_1 - \Delta M)M_2}{M_1 - \Delta M + M_2}\left[\frac{G(M_1 + M_2)}{2a}\right] - \frac{G(M_1 - \Delta M)M_2}{a} \\ &= \frac{G(M_1 - \Delta M)M_2}{a}\left[\frac{M_1 + M_2}{2(M_1 - \Delta M + M_2)} - 1\right] \end{aligned}$$

   This will be positive if the term in brackets is non-negative; we can write this in terms of the total original mass $M = M_1 + M_2$. So we need

$$\frac{M_1 + M_2}{2(M_1 - \Delta M + M_2)} = \frac{M}{2(M - \Delta M)} \geq 1 \;\Rightarrow\; M/2 \geq M - \Delta M \;\Rightarrow\; \Delta M \geq M/2$$

   In other words, the system will become unbound if the ejected mass is equal to or more than half of the original total mass of the system.

2. *(Lamers & Levesque problem 12.2)*

   (a) *Estimate the radii at the beginning and end of the Hayashi contraction phase and at the beginning and end of the pre-main-sequence contraction for stars of 0.1, 0.3, 1.0, 3, 10, 30, and 100* $M_\odot$. Hint: *review L&L sections 12.6, 12.7, and 12.8.*

   At the beginning of the Hayashi contraction phase, the star has $R/R_\odot \approx 100M/M_\odot$ (L&L eqn. 12.13). L&L (pg. 12-8,9) further describe that during the Hayashi phase, the internal temperature increases from $\sim 7 \times 10^4$ K to $\sim 3 \times 10^6$, an increase of a factor 50, that requires a decrease in the radius of a factor of 50. Thus at the end of Hayashi phase, the star has $R/R_\odot \approx 2M/M_\odot$.

   The end of the Hayashi phase is the start of the pre-main-sequence phase, which ends on the main sequence. Then the star has a radius given by L&L eqn. 12.16, $R/R_\odot \approx (M/M_\odot)^{0.7}$. So that gives

Mass $M/M_\odot$	Hayashi start $R/R_\odot \approx 100\ M/M_\odot$	Hayashi end/PMS start $R/R_\odot \approx 2\ M/M_\odot$	PMS end/MS start $R/R_\odot \approx (M/M_\odot)^{0.7}$
0.1	10	0.2	0.2
0.3	30	0.6	0.4
1	100	2	1.0
3	300	6	2.2
10	1000	20	5.0
30	3000	60	11
100	10000	200	25

   (b) *Estimate the duration of the Hayashi contraction phase and of the pre-main-sequence contraction for stars of 0.1, 0.3, 1.0, 3, 10, 30, and 100* $M_\odot$. Hint: *see L&L page 12-10, including eqn. 12.17.*

   L&L derive that the Hayashi contraction phase takes $\sim 10^6$ years for a 1 $M_\odot$ star, and that $\tau_{\text{Hayashi}} \propto 1/M$. So we can write $\tau_{\text{Hayashi}} = 10^6\ (M_\odot/M)$ yr. The duration of the pre-main-sequence phase is given by L&L eqn. 12.17, $\tau_{\text{PMS}} \approx 6 \times 10^7\ (M/M_\odot)^{-2.5}$ yr. So we get

Mass $M/M_\odot$	time on Hayashi track $\tau_{\text{Hayashi}} = 10^6\ (M_\odot/M)$ yr	time on pre-main-sequence $\tau_{\text{PMS}} \approx 6 \times 10^7\ (M/M_\odot)^{-2.5}$ yr
0.1	10^7	1.9×10^{10}
0.3	3.3×10^6	1.2×10^9
1	10^6	6×10^7
3	3.3×10^5	3.8×10^6
10	10^5	1.9×10^5
30	3.3×10^4	1.2×10^4
100	10^4	600

3. **This problem is longer than the others so it will count for double the points.** *Type Ia supernovae are the thermonuclear explosions of accreting white dwarfs that approach the Chandrasekhar limit. In this problem, consider an exploding white dwarf of mass* 1.4 $M_\odot$ *and radius* 10^4 km.

   (a) *The explosive fusion occurs in several steps, but the ultimate result is that carbon and oxygen are fused into nickel:* $2\ {}^{12}_{6}\mathrm{C} + 2\ {}^{16}_{8}\mathrm{O} \rightarrow {}^{56}_{28}\mathrm{Ni}$. *The atomic masses are 12 amu (for carbon-12), 15.994915 amu (for oxygen-16) and 55.942132 amu (for nickel-56). Assuming that the entire white dwarf is half carbon and half oxygen that fuses to nickel, how much total energy is released in the explosion?*

   The energy released in one reaction results from the mass difference between the inputs and the outputs

$$\begin{aligned}
\Delta m &= (2m_C + 2m_O) - (m_{Ni}) \\
&= 2(12.000000) + 2(15.994915) - 55.942132 = 0.047698 \text{ amu} \\
E &= \Delta m c^2 = (0.047698 \times 1.660539 \times 10^{-24} \text{ g})(3.0 \times 10^{10} \text{ cm sec}^{-1})^2 \\
&= 7.13 \times 10^{-5} \text{ erg}
\end{aligned}$$

   Then we just need to figure out how many times this reaction occurs. We divide up the total mass by the mass of the inputs

$$\begin{aligned}
N_{\text{reactions}} &= \frac{M}{2m_C + 2m_O} = \frac{1.38 \times 1.99 \times 10^{33} \text{ g}}{(2(12.000000) + 2(15.994915))(1.660539 \times 10^{-24} \text{ g})} \\
&= 2.95 \times 10^{55}
\end{aligned}$$

   That's a lot of fusion! Note that unlike for the energy yield (when we were subtracting masses) it doesn't much matter for this part to get the masses exactly right; we could have used $56m_p$ and been very close to the same answer.

   The total energy released is the energy per reaction times the number of reactions,

$$E_{\text{total}} = (7.13 \times 10^{-5} \text{ erg})(2.95 \times 10^{55}) = 2.1 \times 10^{51} \text{ erg}$$

   Another way to do this problem is in terms of the efficiency of the fusion

$$\varepsilon = \frac{m_{\text{start}} - m_{\text{end}}}{m_{\text{start}}} = \frac{2(12.000000) + 2(15.994915) - 55.942132}{2(12.000000) + 2(15.994915)} = 0.00085$$

   Recall that the efficiency of hydrogen fusion to helium was $\varepsilon = 0.007$, more than 8 times higher! Because of the shape of the nuclear binding energy curve, you don't get as much bang for the buck fusing carbon and oxygen as you do fusing hydrogen into helium.

   Finally if M is the total mass fused, the total energy released is just

$$E_{\text{total}} = \varepsilon M c^2 = (0.00085)(1.38 \times 1.99 \times 10^{33} \text{ g})(3.0 \times 10^{10} \text{ cm sec}^{-1})^2 = 2.1 \times 10^{51} \text{ erg}$$

(b) *What is the gravitational binding energy of the white dwarf? Assume a polytrope model with a relativistic equation of state (and look back through our class notes for a relatively simple formula for the gravitational potential energy of a polytrope with index n). Compare the explosive fusion energy with the binding energy.*

For an ultrarelativistic degenerate equation of state, we have $\gamma = 4/3$ and a polytropic index of $n = 1/(\gamma - 1) = 3$. The potential energy of a polytrope is given by (see Lecture 3, slide 15)

$$\begin{aligned}\Omega = U = E_{\rm pot} &= -\frac{3}{5-n}\frac{GM^2}{R} = -\frac{3}{2}\frac{GM^2}{R} \\ &= -\frac{3}{2}\frac{(6.67\times10^{-8}\ {\rm cm^3\ g^{-1}\,sec^{-2}})(1.4\times1.99\times10^{33}\ {\rm g})^2}{10^9\ {\rm cm}} \\ &= -7.8\times10^{50}\ {\rm erg}\end{aligned}$$

We see the explosive fusion energy is $\sim$3 times more than the binding energy, so we expect to unbind the star in the explosion.

(c) *The optical light we see from a type Ia supernova is produced mainly by the radioactive decay of* ^{56}Ni, *first to* ^{56}Co, *and then to* ^{56}Fe. *Let's examine the first step, from nickel to cobalt. This reaction releases energy, because the atomic mass of cobalt-56 is 55.939839 amu. Determine the total amount of energy,* $E_{\rm decay}$, *released from the radioactive decay of all the nickel to cobalt.*

We can do this the same way as for part (a). Let's use the efficiency approach, as it is quicker. The efficiency of energy release in the decay of ^{56}Ni to ^{56}Co is

$$\varepsilon = \frac{m(^{56}{\rm Ni}) - m(^{56}{\rm Co})}{m(^{56}{\rm Ni})} = \frac{55.942132\ {\rm u} - 55.939839\ {\rm u}}{55.942132\ {\rm u}} = 4.1\times10^{-5}$$

Thus the total radioactive energy released is

$$E_{\rm decay} = \varepsilon Mc^2 = (4.1\times10^{-5})(1.4\times1.99\times10^{33}\ {\rm g})(3.0\times10^{10}\ {\rm cm\,sec^{-1}})^2 = 1.0\times10^{50}\ {\rm erg}$$

(d) *The radioactive energy is not released all at once, because the half-life of this decay is* $t_{1/2} = 6.075$ *days. We can write the number of radioactive nickel atoms remaining at any time as* $N(t) = N_0\, e^{-t/\tau}$, *where* N_0 *is the number of radioactive nickel atoms at* $t = 0$. *The half-life is defined as* $N(t_{1/2}) = N_0/2$. *Use the half-life to determine* τ.

By definition of the half-life, at $t = t_{1/2}$, the number of radioactive atoms is $N_0/2$. So we have

$$N_0/2 = N_0 e^{-t_{1/2}/\tau} \quad \Rightarrow \quad 1/2 = e^{-t_{1/2}/\tau} \quad \Rightarrow \quad t_{1/2}/\tau = \ln 2 \quad \Rightarrow \quad \tau = t_{1/2}/(\ln 2)$$

Thus for $^{56}{\rm Ni} \to {}^{56}{\rm Co}$, with $t_{1/2} = 6.075$ d, we find $\tau = t_{1/2}/(\ln 2) = 8.764$ days.

(e) *The rate of radioactive energy release is the bolometric luminosity, with $L(t) = L_0\, e^{-t/\tau}$. Use your results from parts (b) and (c), and the fact that $E_{\rm decay} = \int_0^\infty L(t)\, dt$ to calculate the initial bolometric luminosity L_0 in units of $L_\odot$. What is the initial bolometric absolute magnitude?*

Integrating the luminosity (an exponential decay), we find

$$\begin{aligned} E_{\rm decay} &= \int_0^\infty L_0\, e^{-t/\tau}\, dt = L_0\, \tau \int_0^\infty e^{-u}\, du = L_0\, \tau \left[-e^{-u}\right]_0^\infty = L_0\, \tau\, [0-(-1)] \\ &= L_0\, \tau \end{aligned}$$

where I've made the substitution $u = t/\tau$, so $t = \tau\, u$ and thus $dt = \tau\, du$, and the limits of integration did not change (because $u = 0$ when $t = 0$, and $u \to \infty$ when $t \to \infty$). So we find that $E_{\rm decay} = L_0\, \tau \;\Rightarrow\; L_0 = E_{\rm decay}/\tau$. Plugging in the numbers from parts (b) and (c), we derive

$$L_0 = \frac{E_{\rm decay}}{\tau} = \frac{1.0 \times 10^{50}\ \rm erg}{8.764 \times 24 \times 3600\, \rm sec} = 1.3 \times 10^{44}\ \rm erg\, sec^{-1} = 3.4 \times 10^{10}\ L_\odot$$

So a Type Ia supernova can be initially as bright as 30 billion Suns! This corresponds to a bolometric absolute magnitude of (e.g., L&L eqn. 2.7)

$$M_{\rm bol} = -2.5 \log\left(L/L_\odot\right) + 4.74 = -2.5 \log\left(3.4 \times 10^{10}\right) + 4.74 = -21.5$$

It turns out in reality SN Ia only reach a bit over a tenth of this brightness (peak $M_{\rm bol} \approx -19.3$), because the fusion energy release is less than what we calculated (not all of mass fuses to nickel), and the energy takes time to leave the star (see the next subquestion).

(f) *According to this simplistic model, the type Ia supernova should be brightest at the time of explosion, but in actuality the supernova light curve rises for a few weeks because of its* opacity. *Review our class discussion of radiative diffusion to provide an explanation of the delayed peak.*

If the supernova is opaque (high opacity), then the radioactive energy that is released has a diffusion time before it escapes the surface. In other words the energy is trapped as photons travel many mean-free-paths before diffusing to the surface to escape.

For supernovae, it turns out that as the explosion debris expands, it becomes lower density and so the mean free path for photons increases and eventually the radioactive energy escapes more easily and promptly.

(g) *Compare the energy released in a Type Ia supernova to the energy released in a core-collapse supernova (with a $\sim 1.4\ M_\odot$ iron core collapsing down to a neutron* *star with $R = 12$ km). Given your result, briefly explain how a typical Type Ia supernova can have a higher luminosity than a typical core-collapse supernova.*

The core-collapse energy release is given by the change in potential energy, which we will continue to assume is given by a polytrope with $n = 3$,

$$E_{\text{released}} = -U_{\text{final}} - (-U_{\text{init}}) = \frac{3}{2}\frac{GM^2}{R_{\text{final}}} - \frac{3}{2}\frac{GM^2}{R_{\text{init}}} = \frac{3GM^2}{2}\left(\frac{1}{R_{\text{final}}} - \frac{1}{R_{\text{init}}}\right)$$

For $R_{\text{init}} \approx 10^4$ km $\gg R_{\text{final}} = 12$ km we can ignore the initial radius and find

$$\begin{aligned} E_{\text{released}} &\approx \frac{3GM^2}{2R_{\text{final}}} = \frac{3(6.67 \times 10^{-8}\ \text{cm}^3\ \text{g}^{-1}\,\text{sec}^{-2})(1.4 \times 1.99 \times 10^{33}\ \text{g})^2}{2(12 \times 10^5\ \text{cm})} \\ &\approx 6.5 \times 10^{53}\ \text{erg} \end{aligned}$$

This is much larger than the energy released in a Type Ia supernova; however the reason SN Ia can be brighter is that most of the energy released by a core-collapse supernova goes into neutrinos. This is not the case for Type Ia supernovae.

4. **(Required for 541; extra credit for 441)** *On our Canvas site, under Files → Problem Set Resources, there is a handout called homology.pdf from Prof. Joe Shields at Ohio University. "Homology" is an approach to stellar structure where we approximate the relevant variables as power laws of density, temperature, etc., and grossly simplify all the derivatives, allowing us to derive interesting (though very approximate) scaling relations between stellar masses, luminosities, radii, etc. Read through this document before proceeding with this question.*

   (a) *To apply this methodology, we need to parameterize the nuclear energy generation rate (per unit mass) as $\epsilon \propto \rho^\alpha T^\beta$. List α and β for*

   i. *hydrogen fusion via the p-p chain (see Lamers & Levesque, eqn. 8.13),*

   $\varepsilon_{\text{pp}} \propto \rho T^4$, so $\alpha = 1$ and $\beta = 4$.

   ii. *hydrogen fusion via the CNO cycle (see L&L, eqn. 8.14), and*

   $\varepsilon_{\text{CNO}} \propto \rho T^{18} \ \Rightarrow\ \alpha = 1,\ \beta = 18.$

   iii. *helium fusion via the triple-α process (see L&L, eqn. 8.18a).*

   $\varepsilon_{3\alpha} \propto \rho^2 T^{40} \ \Rightarrow\ \alpha = 2,\ \beta = 40.$

   For the two-body interactions (in hydrogen fusion), we would expect the energy generation rate to go as ρ^2; this means the energy generation rate per unit mass ε should go as ρ (one less power of the density because it is per unit mass), as it does. Similarly, for the three-body interaction in triple-alpha fusion, we have $\varepsilon \propto \rho^2$. The temperature scalings depend on the Coulomb barrier and are higher for fusion of higher atomic number nuclei.

(b) *Now, set up a matrix equation following equation 32 in the document, for each of the three cases above, and solve for α_R, α_ρ, α_L, and α_T for each case.*

Equation 32 from the handout has

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 \\ 0 & \alpha & -1 & \beta \\ 4 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} \alpha_R \\ \alpha_\rho \\ \alpha_L \\ \alpha_T \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

Solving this for the $p-p$ chain: $\alpha = 1$, $\beta = 4$ by, for example Wolfram alpha: , we get $\alpha_R = 3/7$, $\alpha_\rho = -2/7$, $\alpha_L = 3$, and $\alpha_T = 4/7$.

For the CNO cycle, $\alpha = 1$, $\beta = 18$, we similarly get $\alpha_R = 17/21$, $\alpha_\rho = -10/7$, $\alpha_L = 3$, and $\alpha_T = 4/21$.

For triple-alpha, $\alpha = 2$, $\beta = 40$, we similarly get $\alpha_R = 20/23$, $\alpha_\rho = -37/23$, $\alpha_L = 3$, and $\alpha_T = 3/23$.

(c) *This matrix approach seems a bit like magic, so let's double check things are making sense. We expect the interior temperature T to scale as M/R (recall our order of magnitude estimate for the central temperature was $T_c \approx GM\mu m_p/kR$). What relation does that imply between α_R and α_T? Is that relation satisfied by your results?*

We have $T \propto M^{\alpha_T}$ and $R \propto M^{\alpha_R}$, so if $T \propto M/R$ that implies $\alpha_T = 1 - \alpha_R$ or $\alpha_R + \alpha_T = 1$. From our results, that relation is satisfied in all three cases.

(d) *We cannot directly observe the interior temperature, T. Rather, we observe the surface effective temperature T_{eff}. Use the scaling implied by the Stefan-Boltzmann law and your homology results to derive the scaling between effective temperature and mass for these three cases.*

From the Stefan-Boltzmann law we have

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4 \;\Rightarrow\; T_{\text{eff}} \propto \left(\frac{L}{R^2}\right)^{1/4} = \frac{L^{1/4}}{R^{1/2}} \;\Rightarrow\; T_{\text{eff}} \propto M^{(\alpha_L/4)-(\alpha_R/2)}$$

Thus for the $p-p$ chain we get $T_{\text{eff}} \propto M^{(3/4)-(3/14)} \propto M^{15/28}$.

For the CNO cycle we get $T_{\text{eff}} \propto M^{(3/4)-(17/42)} \propto M^{29/84}$.

For triple-alpha, $T_{\text{eff}} \propto M^{(3/4)-(10/23)} \propto M^{29/92}$.

(e) *Compare your results for the* mass-luminosity *relationship (given by α_L) with real observations of stars on the main sequence (e.g., L&L Figure 2.3). How well do your homology results work? Comment specifically on very low-mass stars, medium mass stars ($\sim$1 $M_\odot$), and higher-mass stars.*

Our homology results give $\alpha_L = 3$ for all three cases, meaning $L \propto M^3$. This best matches the high-mass stars (over 6 $M_\odot$ which have $\alpha_L \approx 2.9$ on the L&L figure). The luminosity dependence on mass is steeper for solar-type stars ($\alpha_L \approx 4.0$ from the observations), and shallower ($\alpha_L \approx 2.4$) for the lowest-mass stars.