Physics 441/541 Spring 2022: Problem Set #5
due Friday April 15 at 11:00 am in PDF format on Canvas

You are encouraged to work in groups on these problems, but you must write up the solutions individually. In your writeup, list your collaborators and cite any external sources you used. You **may not** consult previous solution sets for this class or other similar classes.

1. In general relativity, photons emitted from a gravitational potential well lose energy as they travel out, leading to a *gravitational redshift* (see Phillips eq. 6.87):

$$z_g = \frac{\Delta\lambda}{\lambda} = \frac{\lambda_{\rm obs} - \lambda_{\rm emitted}}{\lambda_{\rm emitted}} = \left(1 - \frac{R_S}{R}\right)^{-1/2} - 1$$

where $R_S = 2GM/c^2$ is the Schwarzschild radius.

(a) Simplify the expression for the gravitational redshift z_g in the "weak"-gravity limit where $R \gg R_S$. *Hint:* recall that $(1-x)^\alpha \approx 1 - \alpha x$ for $x \ll 1$.

(b) The nearby white dwarf Sirius B has $M = 1.02\ M_\odot$ and $R = 0.0084\ R_\odot$. What is the gravitational redshift from its surface? At what wavelength $\lambda_{\rm obs}$ would a distant observer measure the $H\alpha$ line from Sirius B? For $H\alpha$, take $\lambda_{\rm emitted} =$ 656.2801 nm and assume no relative motion between the observer and the star.

(c) Repeat part (b) for the white dwarf Procyon B ($M = 0.60\ M_\odot$, $R = 0.012\ R_\odot$).

(d) Does the lower mass white dwarf have a higher or lower gravitational redshift? Why? How does the gravitational redshift of a fully-degenerate, non-relativistic white dwarf scale with the mass?

(e) Are you justified in using the weak-field expression for the gravitational redshift for these two white dwarfs? What about for a neutron star with $M = 1.4\ M_\odot$ and $R = 12$ km? Compare the exact and weak-field approximation values for the gravitational redshift for such a neutron star.

2. The Kamiokande-II detector observed 12 neutrinos[1] from Supernova (SN) 1987A in our neighbor galaxy the Large Magellanic Cloud (LMC) at a distance of 50 kpc. The average energy of the detected neutrinos was ∼10 MeV and the total energy emitted in neutrinos by the supernova was $\sim 10^{53}$ erg.

The planned Hyper-Kamiokande detector (scheduled to be online in 2025) will have about 320 times the volume of Kamiokande-II.

[1] These were actually anti-neutrinos, but the distinction is not so important for this problem. At the relevant (high) temperatures, neutrinos and anti-neutrinos are made in roughly even numbers via $e^+e^- \to \nu\bar{\nu}$.

(a) How many neutrinos would Hyper-K see from a supernova like SN 1987A if it happened in our Galaxy at a distance of 10 kpc?

(b) During the $\sim$1 day before core-collapse, a massive star will be undergoing silicon burning in the core, with a neutrino luminosity of $\sim 10^{45}$ erg sec^{-1} and a typical neutrino energy of 2 MeV. How close would such a star have to be for Hyper-K to see at least 100 neutrinos from silicon burning in the day before explosion? How many neutrinos would be seen from the subsequent supernova?

3. The mass-luminosity relation for massive stars on the main sequence is approximately

$$\log\left(\frac{L}{L_\odot}\right) \approx 0.781 + 2.760 \log\left(\frac{M_i}{M_\odot}\right)$$

where M_i is the initial mass. The mass loss rate ($\dot{M} = dM/dt$) of massive stars can roughly be approximated by

$$\log\left(\frac{\dot{M}}{M_\odot \text{ yr}^{-1}}\right) \approx -12.76 + 1.3 \log\left(\frac{L}{L_\odot}\right)$$

The main-sequence lifetime $t_{\rm MS}$ of these stars is approximately

$$\log\left(\frac{t_{\rm MS}}{\rm yr}\right) \approx 7.719 - 0.655 \log\left(\frac{M_i}{M_\odot}\right)$$

(a) Calculate the fraction of mass that is lost by massive stars with $M_i = 25$, 40, 60, 85, and 120 $M_\odot$ during the main sequence phase. Make a plot of the fraction of mass lost versus initial mass. Note that the luminosity depends on the *initial* mass and stays roughly constant during this phase.

(b) A star with an initial mass of 85 $M_\odot$ on the zero-age main sequence has a convective core that contains 83% of the mass. Calculate the time at which products of nuclear burning will appear at the surface of the star.

(c) Wolf-Rayet stars are massive stars that have lost nearly all of their hydrogen rich envelope. They can be classified according to their surface abundances:

i. WC: no H; high abundances of He, C, and O.

ii. WNE: no H; N/He abundance ratio consistent with CNO cycle equilibrium

iii. WNL: some H; N/He abundance ratio consistent with CNO cycle equilibrium

Put these classifications in "chronological" order, i.e., as a massive star evolves in what order would we see these stages? What class is the Wolf-Rayet star described in part (b)?

4. In class (Lecture 18, slide 21; L & L eqn. 28.5) we saw that the total angular momentum of a circular binary orbit with separation a was given by

$$J^2 = Ga\frac{M_1^2 M_2^2}{M_1 + M_2}$$

   (a) Show that with mass or angular momentum loss or gain we can write

$$\frac{d\ln a}{dt} = 2\frac{d\ln J}{dt} + \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_1}\right) + \dot{M}_2\left(\frac{1}{M_1+M_2} - \frac{2}{M_2}\right)$$

   Hint: first solve for a, then take the logarithm, and then take a time derivative.

   (b) Simplify the expression above for *conservative* mass transfer.

5. (adapted from L&L problem 28.1) Consider a close binary consisting of star 1 with a mass of 25 $M_\odot$ and a He core of 8 $M_\odot$ at the end of the main-sequence (MS) phase, and star 2 with a mass of 10 $M_\odot$ still on the main sequence. The orbits are circular and the initial orbital period is 5 days.

   (a) Calculate the orbital separation (in AU and $R_\odot$).

   (b) Show that star 1 fills it Roche lobe shortly after the terminal-age main sequence (TAMS). See L & L Appendix D, and assume solar metallicity.

   Assume conservative mass transfer from star 1 to star 2 begins and continues until star 1 has lost its entire envelope, leaving only its He core.

   (c) Calculate the minimum orbital period and minimum orbital separation for the binary. *Hint:* look at L&L section 28.4 and equation 28.17.

   (d) Calculate the final orbital period and final orbital separation.

   (e) *Extra credit:* Assuming (unrealistically) a constant mass transfer rate $\dot{M}_2 = 10^{-5}\ M_\odot\ \mathrm{yr}^{-1}$, make plots of the stellar masses ($M_1(t)$ and $M_2(t)$ on the same plot), the orbital separation $a(t)$, and the orbital period $P(t)$ versus time.

6. **(Required for 541; extra credit for 441)** Free neutrons spontaneously decay via the reaction $n \to p + e^- + \bar{\nu}_e$. In principle, this could happen even inside neutron stars.

   (a) What is the maximum kinetic energy of the emitted electron, in MeV? Is such an electron non-relativistic, mildly relativistic, or ultra relativistic?

(b) Degeneracy pressure will be important for the neutrons, protons, and electrons. Recall that the Fermi momentum for a degenerate particle is given by

$$p_F = \left(\frac{3h^3 n}{8\pi}\right)^{1/3}$$

Write expressions for the Fermi energy E_F for neutrons, protons, and electrons, in terms of n_n, n_p, and n_e, respectively, and any necessary constants (don't forget to include the rest mass energy as part of E_F). At nuclear densities, you can take the neutrons and protons to be non-relativistic (so then $E_F \approx mc^2 + p_F^2/2m$), but for the electrons you will need to use the relativistic energy-momentum formula. What is the relation between n_p and n_e?

(c) If we start with pure neutrons, with $n_n = \rho/m_n$, and no protons or electrons, neutron decay is energetically favorable, with $E_{F,n} \geq E_{F,p} + E_{F,e}$. But as some of the neutrons decay, this no longer becomes the case. Let's call the proton to neutron fraction $f = n_p/n_n$, so that $n_p = fn_n$. Show that we can then write the neutron number density as

$$n_n = \frac{\rho}{m_n + fm_p}$$

(d) Take the mass density $\rho = 2 \times 10^{14}$ g cm^{-3}. On the same plot graph $E_{F,n} - E_{F,p}$ and $E_{F,e}$ (both in units of MeV) versus $f = n_p/n_n$ ranging from 0 to 0.01.

(e) You should see the curves cross at the equilibrium value where $E_{F,n} = E_{F,p} + E_{F,e}$ and neutron decay no longer occurs. At what value of $f = n_p/n_n$ does this happen? Is it still reasonable to call this a neutron star?

Group Project Assignment

The Group Project assignment has been uploaded to Canvas. Presentations will start Friday April 22; see the posted schedule. **A draft of your presentation is due a week before your presentation date.**

Lecture Date	Group	Members: Topic
Apr 22 (Fri)	1	Haonan Cheng, Yoon Choi, Matthew Wang: Stellar Initial Mass Function
Apr 22 (Fri)	2	Frank Genty, Anthony Pizzarelli, Khovesh Ramdin: Brown Dwarfs
Apr 22 (Fri)	3	Barbara Benda, Avery Kiihne, Harshill Patel: First Stars and Reionization
Apr 26 (Tue)	4	George Kharchilava, Geet Purohit, Anish Seth: Exoplanet Host Stars
Apr 26 (Tue)	5	Ava Marie Friedrich, Seung Hee Sung: Helioseismology
Apr 26 (Tue)	6	Rujuta Mokal, Michael Wozniak, Orion Yeung: Standard Candles
Apr 29 (Fri)	7	Aidan Boyce, Kailash Raman: MESA code
Apr 29 (Fri)	8	Arya Lakshmanan, Ina Park, Brandon Shane: Magnetars
Apr 29 (Fri)	9	Bradley Butler, Christine Carvajal, Connor Lane: LIGO Black Holes