Physics 441/541 Spring 2022: Problem Set #5 Solutions

1. *In general relativity, photons emitted from a gravitational potential well lose energy as they travel out, leading to a* gravitational redshift *(see Phillips eq. 6.87):*

$$z_g = \frac{\Delta\lambda}{\lambda} = \frac{\lambda_{\rm obs} - \lambda_{\rm emitted}}{\lambda_{\rm emitted}} = \left(1 - \frac{R_S}{R}\right)^{-1/2} - 1$$

where $R_S = 2GM/c^2$ is the Schwarzschild radius.

(a) *Simplify the expression for the gravitational redshift z_g in the "weak"-gravity limit where $R \gg R_S$.* Hint: *recall that $(1-x)^\alpha \approx 1 - \alpha x$ for $x \ll 1$.*

For $R \gg R_S$, we have $R_S/R \ll 1$ and thus following the hint, we can approximate $\left(1 - \frac{R_S}{R}\right)^{-1/2} \approx 1 - (-1/2)(R_S/R) = 1 + R_S/2R$. That gives

$$z_g = \left(1 - \frac{R_S}{R}\right)^{-1/2} - 1 \approx \frac{R_S}{2R} \approx \frac{GM}{c^2 R}$$

(b) *The nearby white dwarf Sirius B has $M = 1.02\ M_\odot$ and $R = 0.0084\ R_\odot$. What is the gravitational redshift from its surface? At what wavelength $\lambda_{\rm obs}$ would a distant observer measure the Hα line from Sirius B? For Hα, take $\lambda_{\rm emitted} = 656.2801$ nm and assume no relative motion between the observer and the star.*

Note that $R = 0.0084 \times 6.96 \times 10^{10}$ cm ≈ 5800 km is much greater than $r_S \approx 3$ km for a solar mass object, so we can use the approximate formula to find

$$z = \frac{GM}{c^2 R} = \frac{(6.67 \times 10^{-8}\ {\rm cm^3\ g^{-1}\,sec^{-2}})(1.02 \times 1.99 \times 10^{33}\ {\rm g})}{(3.0 \times 10^{10}\ {\rm cm\,sec^{-1}})^2 (0.0084 \times 6.96 \times 10^{10}\ {\rm cm})} = 2.6 \times 10^{-4}$$

For the observed wavelength, we have

$$\begin{aligned} z &= \frac{\Delta\lambda}{\lambda} = \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}} = \frac{\lambda_{\rm obs}}{\lambda_{\rm em}} - 1 \quad \Rightarrow \quad \lambda_{\rm obs} = \lambda_{\rm em}(1+z) \\ \lambda_{\rm obs} &= (656.2801\ {\rm nm})(1 + 2.6 \times 10^{-4}) = 656.45\ {\rm nm} \end{aligned}$$

(c) *Repeat part (b) for the white dwarf Procyon B ($M = 0.60\ M_\odot$, $R = 0.012\ R_\odot$).*

We get

$$\begin{aligned} z &= \frac{(6.67 \times 10^{-8}\ {\rm cm^3\ g^{-1}\,sec^{-2}})(0.60 \times 1.99 \times 10^{33}\ {\rm g})}{(3.0 \times 10^{10}\ {\rm cm\,sec^{-1}})^2 (0.012 \times 6.96 \times 10^{10}\ {\rm cm})} = 1.1 \times 10^{-5} \\ \lambda_{\rm obs} &= (656.2801\ {\rm nm})(1 + 1.1 \times 10^{-5}) = 656.287\ {\rm nm} \end{aligned}$$

(d) *Does the lower mass white dwarf have a higher or lower gravitational redshift? Why? How does the gravitational redshift of a fully-degenerate, non-relativistic white dwarf scale with the mass?*

The lower mass white dwarf has a lower gravitational redshift, because *both* the mass is lower and the radius is larger. From the mass-radius relationship for non-relativistic degenerate objects, $R \propto M^{-1/3}$. Thus we find $z \propto M^{4/3}$:

$$z \approx \frac{GM}{c^2 R} \propto \frac{M}{R} \propto \frac{M}{M^{-1/3}} \propto M^{4/3}$$

(e) *Are you justified in using the weak-field expression for the gravitational redshift for these two white dwarfs? What about for a neutron star with $M = 1.4\ M_\odot$ and $R = 12$ km? Compare the exact and weak-field approximation values for the gravitational redshift for such a neutron star.*

We saw that for the white dwarfs $r_S \ll R$, or equivalently $z \ll 1$, so the weak-field approximation is fine. For a neutron star with $M = 1.4\ M_\odot$, however,

$$R_S = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-8}\ \mathrm{cm^3\ g^{-1}\,sec^{-2}})(1.4 \times 1.99 \times 10^{33}\ \mathrm{g})}{(3.0 \times 10^{10}\ \mathrm{cm\,sec^{-1}})^2} = 4.1\ \mathrm{km}$$

so $R = 12$ km is not much larger and we should probably use the exact formula. Comparing the approximate and exact formulas for the neutron star we find

$$\begin{aligned} z_{\mathrm{exact}} &= \left(1 - \frac{R_S}{R}\right)^{-1/2} - 1 = \left(1 - \frac{4.1\ \mathrm{km}}{12\ \mathrm{km}}\right)^{-1/2} - 1 = 0.23 \\ z_{\mathrm{approx}} &= \frac{GM}{c^2 R} = \frac{r_S}{2R} = \frac{4.1\ \mathrm{km}}{2 \times 12\ \mathrm{km}} = 0.17 \end{aligned}$$

2. *The Kamiokande-II detector observed 12 neutrinos*[1] *from Supernova (SN) 1987A in our neighbor galaxy the Large Magellanic Cloud (LMC) at a distance of 50 kpc. The average energy of the detected neutrinos was ~10 MeV and the total energy emitted in neutrinos by the supernova was ~10^{53}* erg.

The planned Hyper-Kamiokande detector (scheduled to be online in 2025) will have about 320 times the volume of Kamiokande-II.

(a) *How many neutrinos would Hyper-K see from a supernova like SN 1987A if it happened in our Galaxy at a distance of 10 kpc?*

The neutrino flux will go as $1/(\mathrm{distance})^2$, so for an object $(50/10) = 5$ times closer, that means $(50/10)^2 = 25$ times the neutrino flux. So for a detector with 320 times the volume we would get $N \approx 12 \times 25 \times 320 \approx 96000$ neutrinos.

[1] These were actually anti-neutrinos, but the distinction is not so important for this problem. At the relevant (high) temperatures, neutrinos and anti-neutrinos are made in roughly even numbers via $e^+e^- \to \nu\bar{\nu}$.

(b) *During the ∼1 day before core-collapse, a massive star will be undergoing silicon burning in the core, with a neutrino luminosity of ∼10^{45} erg sec^{-1} and a typical neutrino energy of 2 MeV. How close would such a star have to be for Hyper-K to see at least 100 neutrinos from silicon burning in the day before explosion? How many neutrinos would be seen from the subsequent supernova?*

The energy released over that ∼1 day is $E = 10^{45}$ erg sec^{-1} $\times 24 \times 3600$ sec $=$ 8.64×10^{49} erg. This is about three orders of magnitude less energy than in the SN expolosion, but note that each SN neutrino has 5 times higher energy on average, so the ratio of Si-burning neutrinos to SN neutrinos is

$$\frac{N_{\nu,\mathrm{Si}}}{N_{\nu,\mathrm{SN}}} = \frac{8.64 \times 10^{49}\ \mathrm{erg}\ /\ 2\ \mathrm{MeV}}{10^{53}\ \mathrm{erg}\ /\ 10\ \mathrm{MeV}} = 0.0043$$

So for example, at $d = 10$ kpc, where we saw $N_{\nu,\mathrm{SN}} = 96000$, we would have $N_{\nu,\mathrm{Si}} \approx 410$. That means we would see at least 100 Si-burning neutrinos in a day for a star out to a bit farther than that distance: $d \approx 10 \times \sqrt{410/100} \approx 20$ kpc. At that distance we would see approximately $100/0.0043 = 23000$ neutrinos from the subsequent supernova.

3. *The mass-luminosity relation for massive stars on the main sequence is approximately*

$$\log\left(\frac{L}{L_\odot}\right) \approx 0.781 + 2.760 \log\left(\frac{M_i}{M_\odot}\right)$$

where M_i is the initial mass. The mass loss rate ($\dot{M} = dM/dt$) of massive stars can roughly be approximated by

$$\log\left(\frac{\dot{M}}{M_\odot\ \mathrm{yr}^{-1}}\right) \approx -12.76 + 1.3 \log\left(\frac{L}{L_\odot}\right)$$

The main-sequence lifetime t_{MS} of these stars is approximately

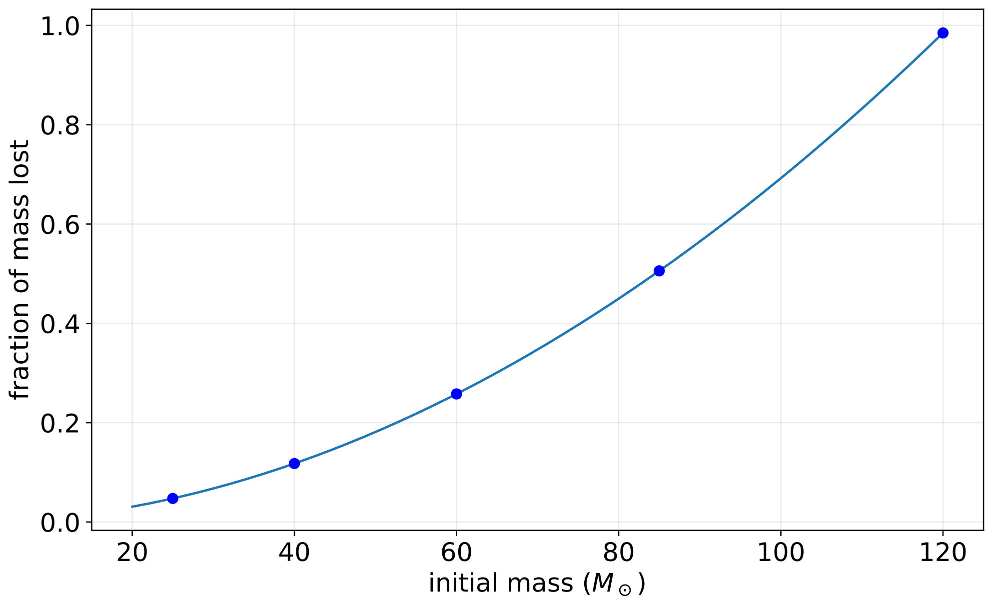
$$\log\left(\frac{t_{\mathrm{MS}}}{\mathrm{yr}}\right) \approx 7.719 - 0.655 \log\left(\frac{M_i}{M_\odot}\right)$$

(a) *Calculate the fraction of mass that is lost by massive stars with M_i = 25, 40, 60, 85, and 120 $M_\odot$ during the main sequence phase. Make a plot of the fraction of mass lost versus initial mass. Note that the luminosity depends on the* initial *mass and stays roughly constant during this phase.*

I've coded this up at http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps05q3.ipynb.

The total mass lost on the main sequence is the mass loss rate times the main-sequence lifetime, $M_{\mathrm{lost}} = \dot{M} \times t_{\mathrm{MS}}$ and the fraction of mass lost is just M_{lost}/M_i. Plugging in the relations gives

$M_i/M_\odot$	25	40	60	85	120
$M_{\rm lost}/M_i$	0.047	0.118	0.258	0.506	0.985



(b) *A star with an initial mass of* 85 $M_\odot$ *on the zero-age main sequence has a convective core that contains 83% of the mass. Calculate the time at which products of nuclear burning will appear at the surface of the star.*

The star needs to lose $100\% - 83\% = 17\%$ of its mass, $0.17 \times 85\ M_\odot = 14.45\ M_\odot$ before the convective core is exposed. Plugging the mass into our formulas gives a mass-loss rate of $\dot{M} = 1.51 \times 10^{-5}\ M_\odot\ \mathrm{yr}^{-1}$. The time needed is thus

$$t = M/\dot{M} = 9.6 \times 10^5\ \mathrm{yr}$$

That's about one-third of its main-sequence lifetime.

(c) *Wolf-Rayet stars are massive stars that have lost nearly all of their hydrogen rich envelope. They can be classified according to their surface abundances:*

i. *WC: no H; high abundances of He, C, and O.*
ii. *WNE: no H; N/He abundance ratio consistent with CNO cycle equilibrium*
iii. *WNL: some H; N/He abundance ratio consistent with CNO cycle equilibrium*

Put these classifications in "chronological" order, i.e., as a massive star evolves in what order would we see these stages? What class is the Wolf-Rayet star described in part (b)?

The mass loss occurs from the surface. Before the star is a Wolf-Rayet star, we don't see any evidence of fusion at the surface (because the convection zone is limited to the core). But as mass is lost, we can begin to see the convective core.

First we will see the zone which has convection but not much fusion, so there will still be some hydrogen. The helium abundance will be enhanced as a product of hydrogen fusion; in massive stars this fusion proceeds by the CNO cycle, so we will see enhanced nitrogen (which is the "slow" step of the CNO cycle, causing a buildup of nitrogen nuclei). Thus we will first see the star as a WNL type Wolf-Rayet star.

As we continue to lose mass, the surface will eventually reach a region where no hydrogen is left, but the products of hydrogen fusion will still be there; thus it would be a WNE star.

Finally as we lose more mass we will expose the more central parts of the star where helium fusion products can be seen; these include the triple-alpha production of carbon, and carbon+helium fusion to oxygen. This would be a WC star.

The star in part (b) is still on the main sequence (so it is still fusing hydrogen in the core and thus can't be a WC star). When we first see the convective core, there will still be hydrogen mixing to the surface, and so we would expect it to be a WNL star.

4. *In class (Lecture 18, slide 21; L & L eqn. 28.5) we saw that the total angular momentum of a circular binary orbit with separation a was given by*

$$J^2 = Ga\frac{M_1^2 M_2^2}{M_1 + M_2}$$

   (a) *Show that with mass or angular momentum loss or gain we can write*

$$\frac{d\ln a}{dt} = 2\frac{d\ln J}{dt} + \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_1}\right) + \dot{M}_2\left(\frac{1}{M_1+M_2} - \frac{2}{M_2}\right)$$

Hint: *first solve for a, then take the logarithm, and then take a time derivative.*

Solving for a we find

$$a = \frac{J^2\left(M_1+M_2\right)}{GM_1^2M_2^2}$$

and then taking a natural log of both sides we get

$$\ln a = 2\ln J - \ln G + \ln(M_1+M_2) - 2\ln M_1 - 2\ln M_2$$

Differentiating with respect to time, and noting that G is constant, we get

$$\begin{aligned}\frac{d\ln a}{dt} &= 2\frac{d\ln J}{dt} + \frac{1}{M_1+M_2}(\dot{M}_1+\dot{M}_2) - \frac{2}{M_1}\dot{M}_1 - \frac{2}{M_2}\dot{M}_2 \\ &= 2\frac{d\ln J}{dt} + \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_1}\right) + \dot{M}_2\left(\frac{1}{M_1+M_2} - \frac{2}{M_2}\right)\end{aligned}$$

as desired.

(b) *Simplify the expression above for* conservative *mass transfer.*

For *conservative* mass transfer (see Lecture 18, slide 54; L & L eqn. 28.13), the angular momentum and total mass are conserved, so $dJ/dt = 0$ and $\dot{M}_1 = -\dot{M}_2$. Plugging in these we get

$$\frac{d\ln a}{dt} = \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_1}\right) - \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_2}\right) = 2\dot{M}_1\left(\frac{1}{M_2} - \frac{1}{M_1}\right)$$

5. *(adapted from L&L problem 28.1) Consider a close binary consisting of star 1 with a mass of* 25 $M_\odot$ *and a He core of* 8 $M_\odot$ *at the end of the main-sequence (MS) phase, and star 2 with a mass of* 10 $M_\odot$ *still on the main sequence. The orbits are circular and the initial orbital period is 5 days.*

(a) *Calculate the orbital separation (in AU and $R_\odot$).*

We can use Kepler's 3rd law

$$a^3 = \frac{G(M_1+M_2)}{4\pi^2}P^2$$

In solar units (separation in AU, masses in $M_\odot$, period in years), this simplifies to $a^3 = (M_1 + M_2)P^2$ (think about why that is!), so we can determine

$$a^3 = (35)(5/365.24)^2 \text{ AU}^3 = 0.00656 \text{ AU}^3 \quad \Rightarrow \quad a = 0.19 \text{ AU}$$

Converting to solar radii, we also have $a = 0.19$ AU $= 41\ R_\odot$.

(b) *Show that star 1 fills it Roche lobe shortly after the terminal-age main sequence (TAMS). See L & L Appendix D, and assume solar metallicity.*

L&L Appendix D gives a table of stellar parameters. For $M_1 = 25\ M_\odot$, using solar metallicity, we find from columns 6 and 7, $\log(L/L_\odot) = 5.19$ and $\log(T_{\rm eff}\text{ in K}) = 4.43$. From the Stefan-Boltzmann law, $L = 4\pi R^2\sigma T_{\rm eff}^4$, we can calculate the radius

$$\begin{aligned} R &= \left(\frac{L}{4\pi\sigma T_{\rm eff}^4}\right)^{1/2} = \left(\frac{10^{5.19}\times 3.83\times 10^{33}\ \text{erg sec}^{-1}}{4\pi(5.67\times10^{-5}\ \text{erg sec}^{-1}\ \text{cm}^{-2}\ \text{K}^{-4})(10^{4.43}\ \text{K})^4}\right)^{1/2} \\ &= 1.26\times10^{12}\ \text{cm} = 18\ R_\odot \end{aligned}$$

The Roche lobe radius for star 1 is given by L&L eqn. 28.11. For a mass ratio $x = M_1/M_2 = 2.5$, we have

$$\frac{R_{L1}}{a} \approx \frac{0.49\,x^{2/3}}{0.6\,x^{2/3} + \ln(1+x^{1/3})} \approx 0.46$$

and thus the Roche lobe radius is $R_{L1} \approx 0.46\,a \approx 18.9\ R_\odot$. This is just slightly larger than the radius of the star, so soon after star 1 leaves the main sequence and expands, it will overflow its Roche lobe.

Assume conservative mass transfer from star 1 to star 2 begins and continues until star 1 has lost its entire envelope, leaving only its He core.

(c) *Calculate the minimum orbital period and minimum orbital separation for the binary.* Hint: *look at L&L section 28.4 and equation 28.17.*

L&L eqn. 28.17 gives expressions for the separation and period as a function of the masses and the initial values, in the case of conservative mass transfer:

$$\frac{a}{a_i} = \left(\frac{M_{1i}}{M_1}\frac{M_{2i}}{M_2}\right)^2 \quad \text{and} \quad \frac{P}{P_i} = \left(\frac{M_{1i}}{M_1}\frac{M_{2i}}{M_2}\right)^3$$

The minimum separation and period are when the masses are equal, in this case $M_1 = M_2 = 17.5\ M_\odot$. So we have

$$\frac{a}{a_i} = \left(\frac{25}{17.5}\frac{10}{17.5}\right)^2 = 0.67 \quad \text{and} \quad \frac{P}{P_i} = \left(\frac{25}{17.5}\frac{10}{17.5}\right)^3 = 0.54$$

and thus $a_{\min} = 0.67\,a_i = 0.67(0.19\ \text{AU}) = 0.13\ \text{AU}$ and $P_{\min} = 0.54\,P_i = 0.54(5\ \text{day}) = 2.7\ \text{day}$.

(d) *Calculate the final orbital period and final orbital separation.*

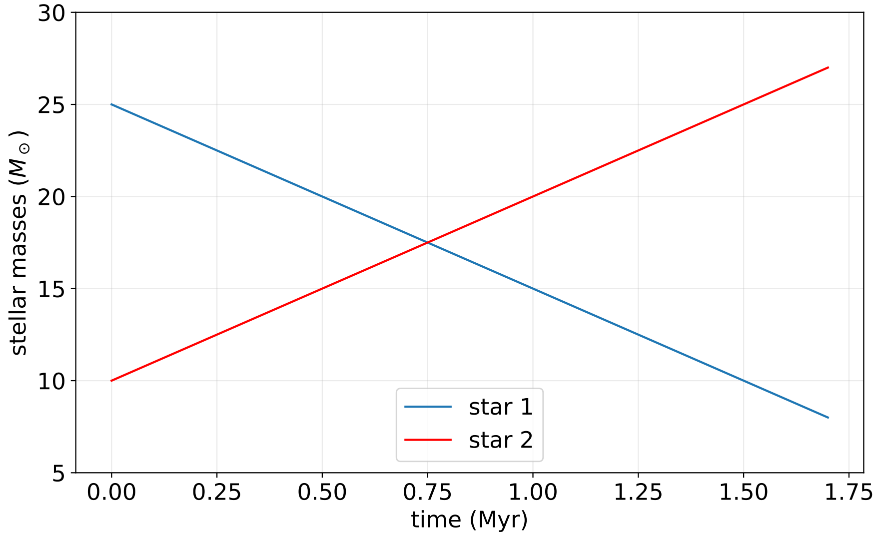
The final separation and period occur when $M_1 = 8\ M_\odot$ and $M_2 = 27\ M_\odot$. So

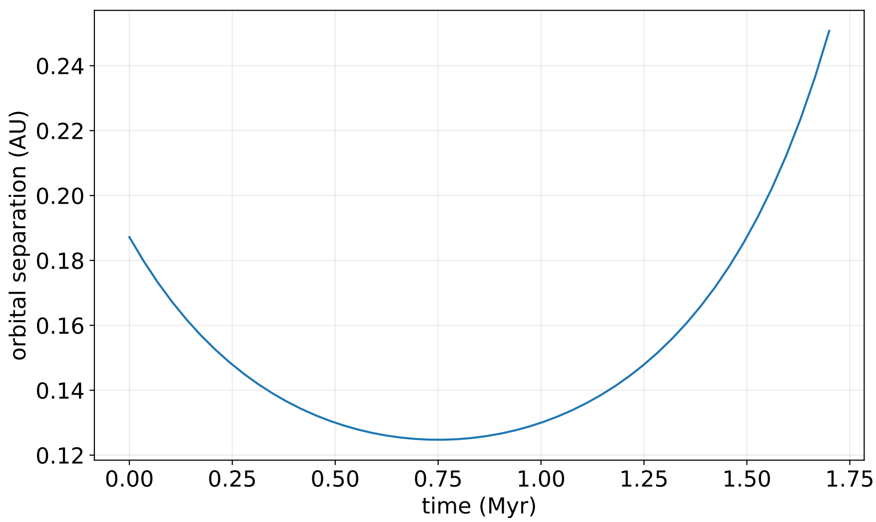
$$\frac{a}{a_i} = \left(\frac{25}{8}\frac{10}{27}\right)^2 = 1.34 \quad \text{and} \quad \frac{P}{P_i} = \left(\frac{25}{8}\frac{10}{27}\right)^3 = 1.55$$

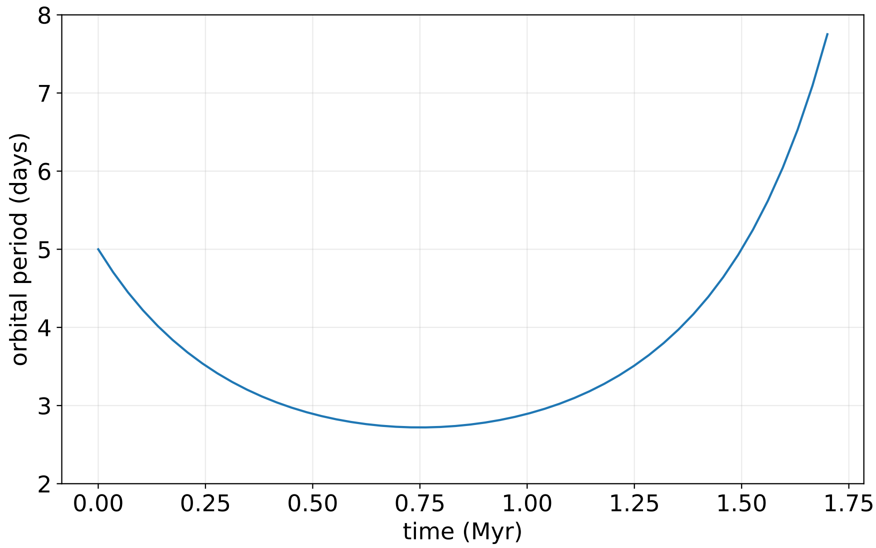
and thus $a_f = 1.34\,a_i = 1.34(0.19\ \text{AU}) = 0.25\ \text{AU}$ and $P_f = 1.55\,P_i = 1.55(5\ \text{day}) = 7.8\ \text{day}$.

(e) Extra credit: *Assuming (unrealistically) a constant mass transfer rate* $\dot{M}_2 = 10^{-5}\ M_\odot\ \text{yr}^{-1}$*, make plots of the stellar masses ($M_1(t)$ and $M_2(t)$ on the same plot), the orbital separation $a(t)$, and the orbital period $P(t)$ versus time.*

See `http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps05q5e.ipynb` for the notebook to create these plots.


stellar masses ($M_\odot$)
time (Myr)
star 1
star 2


orbital separation (AU)
time (Myr)


orbital period (days)
time (Myr)

6. (Required for 541; extra credit for 441) *Free neutrons spontaneously decay via the reaction $n \to p + e^- + \bar{\nu}_e$. In principle, this could happen even inside neutron stars.*

(a) *What is the maximum kinetic energy of the emitted electron, in MeV? Is such an electron non-relativistic, mildly relativistic, or ultra relativistic?*

The rest mass energy of the neutron, proton, and electron are $m_n = 939.565$ MeV, $m_p = 938.272$ MeV, and $m_e = 0.511$ MeV. Assuming zero kinetic energy for the neutrino (whose mass is at the $\sim$eV scale, i.e. 10^{-6} MeV, so negligible), the excess energy available is $939.565 - (938.272 + 0.511) = 0.782$ MeV. This is maximum kinetic energy for the electron.

This is a bit more than the electron's rest mass energy, so this is mildly relativistic. The electron's total energy would be $0.511 + 0.782 = 1.293$ MeV, corresponding to a Lorentz factor of $\gamma = E/mc^2 = 1.293/0.511 = 2.5$.

(b) *Degeneracy pressure will be important for the neutrons, protons, and electrons. Recall that the Fermi momentum for a degenerate particle is given by*

$$p_F = \left(\frac{3h^3 n}{8\pi}\right)^{1/3}$$

Write expressions for the Fermi energy E_F for neutrons, protons, and electrons, in terms of n_n, n_p, and n_e, respectively, and any necessary constants (don't forget to include the rest mass energy as part of E_F). At nuclear densities, you can take the neutrons and protons to be non-relativistic (so then $E_F \approx mc^2 + p_F^2/2m$), but for the electrons you will need to use the relativistic energy-momentum formula. What is the relation between n_p and n_e?

The Fermi momenta for the three species are given by p_F above; we just have to set the number density to n_n, n_p, or n_e as appropriate. The Fermi energy is related to the Fermi momentum; to do this properly, we should use the relativistic relationship between total energy and momentum, $E = [(mc^2)^2 + (pc)^2]^{1/2}$.

For the neutrons and protons, however, we can simplify to the non-relativistic limit, $E \approx mc^2 + p^2/2m$, where we see the rest energy and the non-relativisitic kinetic energy. So for neutrons and protons, we have

$$E_{Fn} = m_n c^2 + \frac{1}{2m_n}\left(\frac{3h^3 n_n}{8\pi}\right)^{2/3} \quad \text{and} \quad E_{Fp} = m_p c^2 + \frac{1}{2m_p}\left(\frac{3h^3 n_p}{8\pi}\right)^{2/3}$$

For the electrons, we have to use the relativistic formula, which gives

$$E_{Fe} = \left[(m_e c^2)^2 + \left(\frac{3h^3 c^3 n_e}{8\pi}\right)^{2/3}\right]^{1/2}$$

The material is purely neutral, so as neutrons decay we get equal numbers of protons and electrons, with $n_p = n_e$.

(c) *If we start with pure neutrons, with $n_n = \rho/m_n$, and no protons or electrons, neutron decay is energetically favorable, with $E_{F,n} \geq E_{F,p} + E_{F,e}$. But as some of the neutrons decay, this no longer becomes the case. Let's call the proton to neutron fraction $f = n_p/n_n$, so that $n_p = f n_n$. Show that we can then write the neutron number density as*

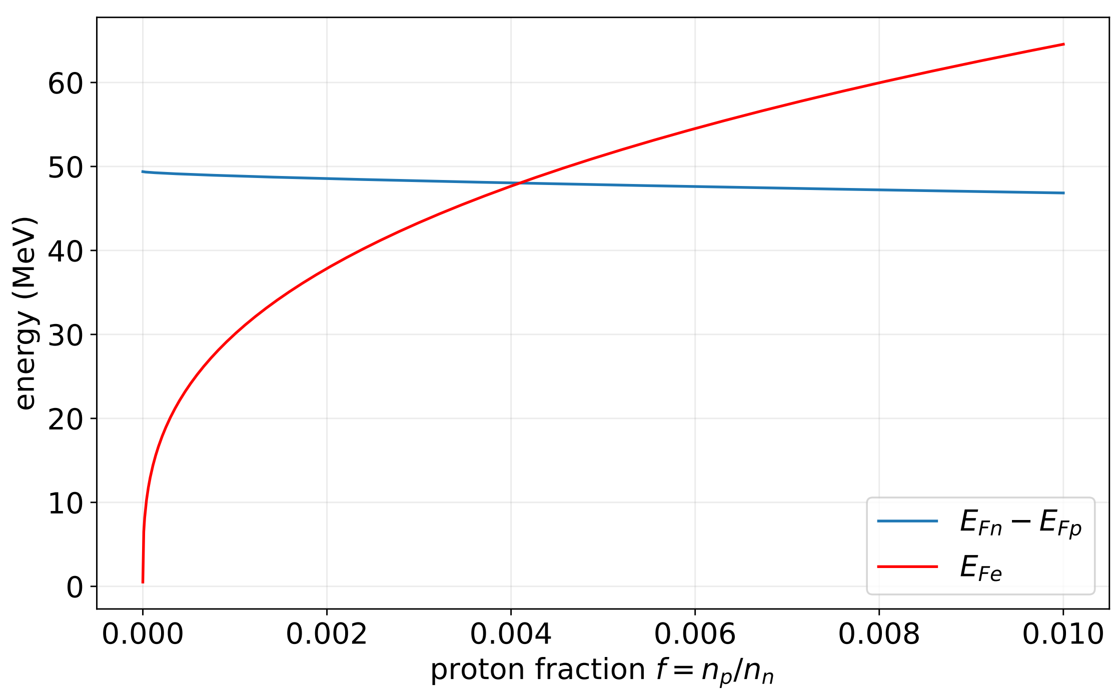
$$n_n = \frac{\rho}{m_n + f m_p}$$

When we have neutrons, protons, and electrons, we can write the total density as $\rho = n_n m_n + n_p m_p + n_e m_e$. As above, we know that $n_p = n_e$ because the material is neutral, and given that $m_e \ll m_p$, we can ignore the $n_e m_e$ contribution to the density. Thus we have

$$\rho \approx n_n m_n + n_p m_p = n_n \left(m_n + \frac{n_p}{n_n} m_p \right) \;\Rightarrow\; n_n \approx \frac{\rho}{m_n + m_p n_p/n_n} = \frac{\rho}{m_n + f m_p}$$

(d) *Take the mass density $\rho = 2 \times 10^{14}$ g cm^{-3}. On the same plot graph $E_{F,n} - E_{F,p}$ and $E_{F,e}$ (both in units of MeV) versus $f = n_p/n_n$ ranging from 0 to 0.01.*

See http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps05q6d.ipynb for the notebook to create the following plot.



(e) *You should see the curves cross at the equilibrium value where* $E_{F,n} = E_{F,p} + E_{F,e}$ *and neutron decay no longer occurs. At what value of* $f = n_p/n_n$ *does this happen? Is it still reasonable to call this a neutron star?*

We find that the curves cross at an equilibrium value that corresponds to a proton fraction $n_p/n_e \approx 0.004$. The reason for this is that as the electron density increases, the electron Fermi energy also increases. Eventually the electron produced by the neutron decay needs to be such high energy that it is no longer energetically favorable to decay.

For a proton fraction $n_p/n_n \approx 0.004 = 0.4\%$, we see that the star is still nearly completely dominated by neutrons, and so calling it a neutron star is appropriate.