Physics 441/541: Stars and Star Formation
Spring 2022 Final Exam Solutions

Brief calculation/derivation [12 pts total]

1. *Consider the following nuclear reaction:* ${}^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} \rightarrow {}^{28}_{14}\mathrm{Si} + {}^{4}_{2}\mathrm{He}$

   (a) [2 pts] *Write an expression for the energy released by this reaction in terms of the nuclear masses m(O), m(Si), m(He), and any needed fundamental constants. You don't have to calculate a numerical answer; just write an equation.*

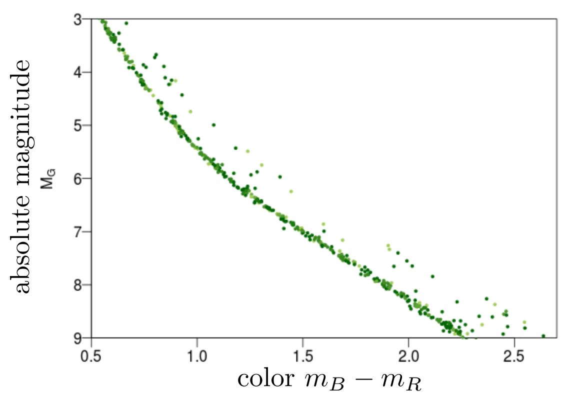
   We know that $E = \Delta mc^2$, so the energy released is given by the change in mass between the reactants and the products (note: energy is *released* if the mass of the products is *less* than the mass of the reactants)

   $$E = \Delta mc^2 = [m(\mathrm{O}) + m(\mathrm{O}) - m(\mathrm{Si}) - m(\mathrm{He})]\, c^2 = [2m(\mathrm{O}) - m(\mathrm{Si}) - m(\mathrm{He})]\, c^2$$

   (b) [1 pts] *In what kind of stars would you expect this reaction to occur?*

   This is oxygen burning (oxygen fusion), and it only occurs in high-mass stars ($M \gtrsim 8\ M_\odot$); lower-mass stars do not reach high enough temperatures in the core for tunneling through the high Coloumb barrier to fuse oxygen this way.

2. [3 pts] *The* Gaia *mission released this color-magnitude diagram for the Hyades star cluster. They said that in addition to many stars on the main sequence, there are also "a scattering of double stars up to 0.75 mag above the main sequence." Explain what that means.* Hint: *what flux or luminosity ratio does a difference of 0.75 mag correspond to? Why are those points above the others on the diagram?*



As implied by "double stars" these are binary star systems, where there are two stars rather than just one, so we would expect them to be brighter (lower absolute magnitude) than single stars on the main sequence. A lower magnitude of -0.75 mag corresponds to a flux ratio of:

$$m = -2.5 \log(f/f_0)\ \Rightarrow\ f/f_0 = 10^{-0.4m} = 10^{[(-0.4)(-0.75)]} \approx 2$$

suggesting some of these binary stars are up to twice as bright as a single star, as you might expect.

3. *On Problem Set 5 we showed that the evolution of the orbital separation of a binary star system with mass transfer can be written*

$$\frac{d\ln a}{dt} = 2\frac{d\ln J}{dt} + \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_1}\right) + \dot{M}_2\left(\frac{1}{M_1+M_2} - \frac{2}{M_2}\right)$$

(a) [2 pts] *Simplify this expression for* conservative *mass transfer.*

In conservative mass transfer, mass flows from one star to the other and none is lost from the system. That means the rate of change of mass from one star is equal and opposite to the rate of change of mass of the other star: $\dot{M}_1 = -\dot{M}_2$.

Additionally, for conservative mass transfer, angular momentum is conserved, so $dJ/dt = 0$ or $d\ln J/dt = 0$.

Thus we can simplify our expression

$$\begin{aligned}
\frac{d\ln a}{dt} &= 2\frac{d\ln J}{dt} + \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_1}\right) + \dot{M}_2\left(\frac{1}{M_1+M_2} - \frac{2}{M_2}\right) \\
\frac{d\ln a}{dt} &= 0 + \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_1}\right) - \dot{M}_1\left(\frac{1}{M_1+M_2} - \frac{2}{M_2}\right) \\
\frac{d\ln a}{dt} &= \dot{M}_1\left(-\frac{2}{M_1} + \frac{2}{M_2}\right) = 2\dot{M}_1\left(\frac{1}{M_2} - \frac{1}{M_1}\right)
\end{aligned}$$

(b) [2 pts] *Show that in the case of conservative mass transfer, the orbital separation is minimum when the two stellar masses are equal, $M_1 = M_2$.*

The orbital separation is minimized when $da/dt = 0$ or equivalently, when $d\ln a/dt = 0$. Based on our expression above, this would imply that either $\dot{M}_1 = 0$ (but that would mean no mass transfer) or $(1/M_2 - 1/M_1) = 0 \Rightarrow M_2 = M_1$.

To show this is really a minimum (not a maximum), we could show the second time derivative is positive. More simply, if $\dot{M}_1$ is positive, then a little time after $M_1 = M_2$ we would have $M_1 > M_2$ and thus $(1/M_2 - 1/M_1)$ would be positive and so $d\ln a/dt$ would be positive and the separation would increase. Conversely, if $\dot{M}_1$ is negative, then a little time after $M_1 = M_2$ we would have $M_1 < M_2$ and thus $(1/M_2 - 1/M_1)$ would be *negative* and so $d\ln a/dt$ would *still be positive* and the separation would still increase. That means the separation is a *minimum* when $M_1 = M_2$.

(c) [2 pts] *Show that for conservative mass transfer, the quantity $M_1^2M_2^2a$ is constant.* Hint: *check the front page formula sheet.*

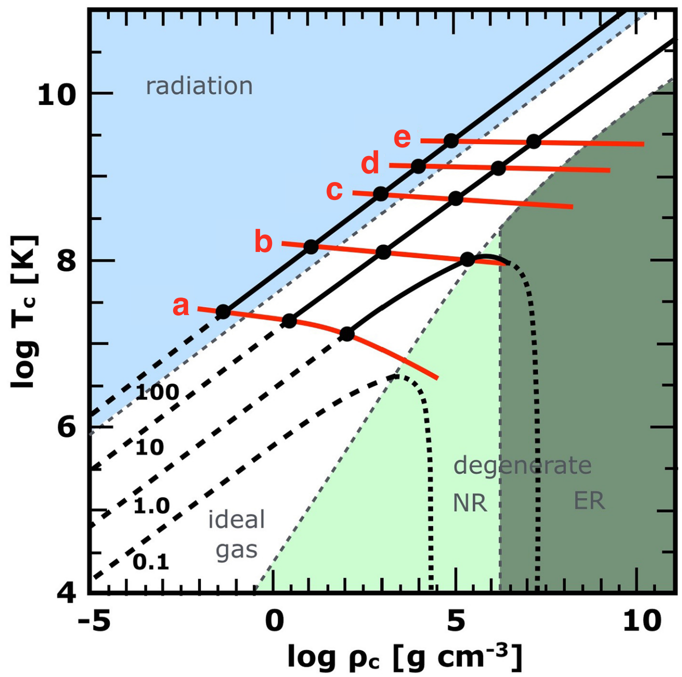
The formula sheet gives $J^2 = GaM_1^2M_2^2/(M_1+M_2)$. We can rearrange this as $M_1^2M_2^2a = J^2(M_1+M_2)/G$. The right hand side has only constants for conservative mass transfer: the total angular momentum J, the total mass M_1+M_2, and the universal constant G. Thus the left hand side must also be constant.

Short answer [8 pts total]

4. [3 pts] *How do the main sequence lifetimes of stars depend on their mass? Why?*

   More massive stars have shorter main sequence lifetimes. This is because even though they have more hydrogen fuel to fuse, their higher mass requires more thermal pressure support and thus a higher core temperature. The hydrogen fusion rate strongly depends on temperature, so higher mass stars fuse at a much higher rate than lower mass stars, leading to a much higher luminosity and a *shorter* main sequence lifetime before all the hydrogen in the core is used up.

5. [5 pts] *Briefly explain this diagram (adapted from L&L). What are the mostly-horizontal lines labeled a,b,c,d,e? What do the labels 100, 10, 1.0, and 0.1 represent?*



   The shaded regions show the dominant source of a star's pressure support as a function of the central density (along the x-axis) or central temperature (along the y-axis): radiation pressure (blue), ideal gas thermal pressure (white), or degeneracy pressure (non-relativistic: light green; extreme relativistic: dark green).

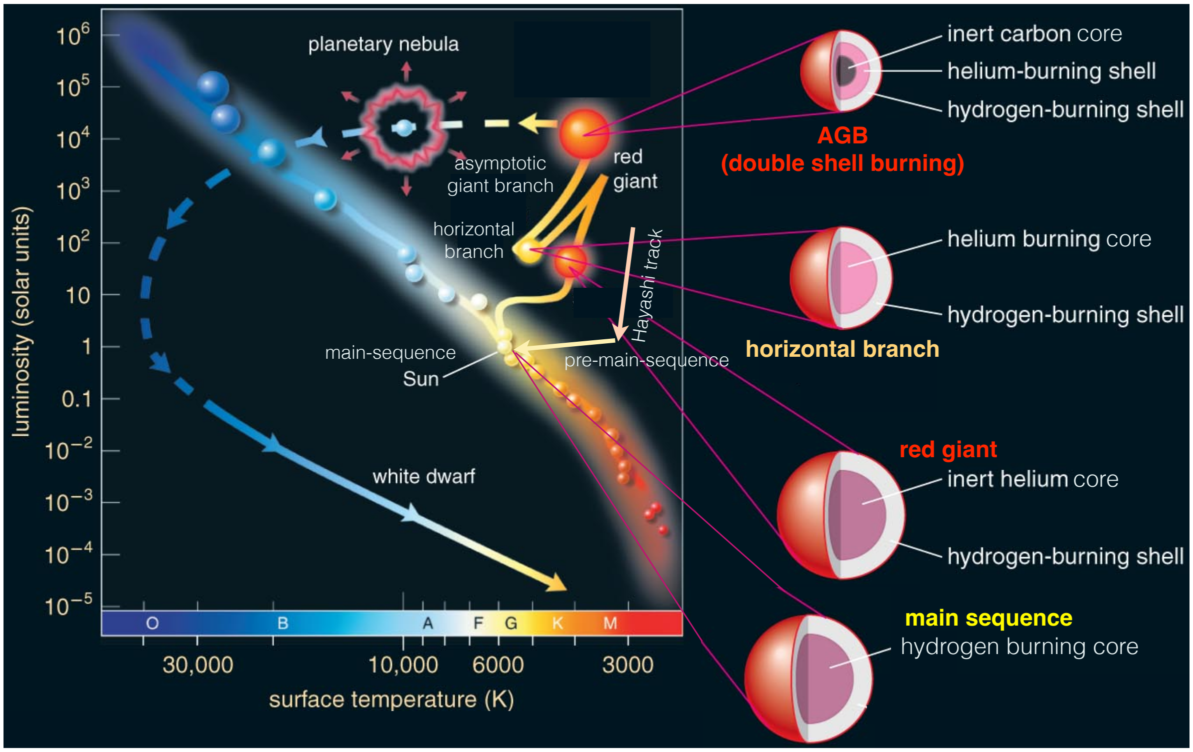
   The red lines are thresholds for different kinds of fusion: above line a, a star is hot enough in its core to fuse hydrogen (to helium). Line b marks where helium fusion (to carbon and oxygen) starts. Lines c, d, and e mark later stages of fusion (carbon-fusion, oxygen-fusion, and silicon-fusion to iron) that occur only in massive stars.

   The labels 0.1, 1.0, 10, and 100 refer to the mass of a star in solar masses. The two high mass stars (10 and 100 $M_\odot$) go through all fusion stages, while the lower mass stars end up as degenerate white dwarfs. The stars are "born" when they reach hydrogen fusion, line (a), so that's why the track turns solid at that point.

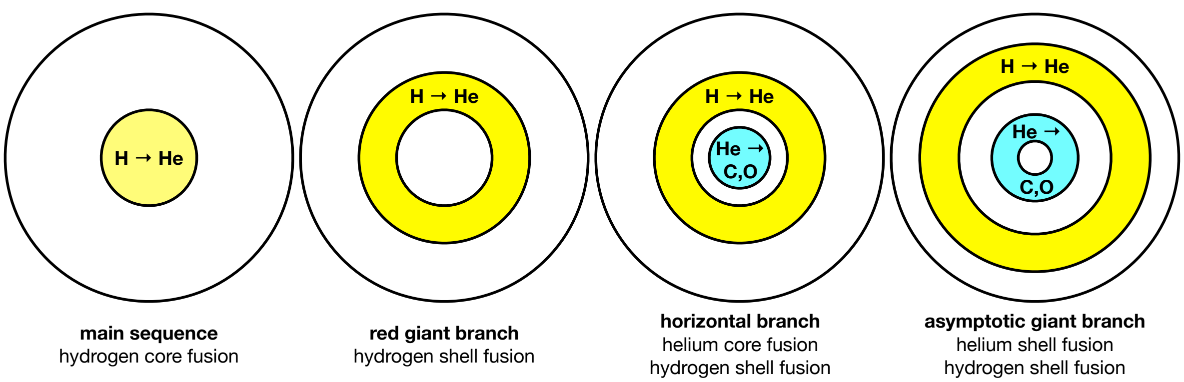
Slightly longer answer [21 pts total]

6. [6 pts] *Sketch the evolution of the Sun on the Hertzsprung-Russell diagram. Label your axes, and mark the Sun's position on the main sequence. Sketch and label these phases of the evolution: (1) the Hayashi track/pre-main-sequence, (2) the main sequence, (3) the red giant branch, (4) the horizontal branch, (5) the asymptotic giant branch, (6) the planetary nebula phase, and (7) the end state for the Sun (what is it?).*

   The figure below is adapted from Lecture 14, slide 4.



7. [4 pts] *The Sun will go through four distinct phases of fusion during this evolution. For each of these four phases, sketch the Sun as a circle and label where and what kind of fusion is going on. An example sketch is given below. Copy this one and create three more. What phases from the previous question do each of your four diagrams correspond to?*



8. [5 pts] *Explain conceptually the Chandrasekhar mass limit for a white dwarf and how it is derived. What provides the pressure support and what kind of equation of state is relevant for white dwarfs with $M \ll M_{Ch}$? How does the radius depend on the mass for such white dwarfs? What is different for white dwarfs with mass closer to M_{Ch}?*

   The Chandrasekhar mass limit is the maximum mass for a white dwarf. We derived it using a polytrope model for a star supported by electron degeneracy pressure. For $M \ll M_{Ch}$ the equation of state is given by non-relativistic electron degeneracy pressure and leads to $R \propto M^{-1/3}$. In other words, more massive white dwarfs are smaller in size. This is because the electrons need to be squeezed to higher density to provide the extra pressure to balance against the higher gravity of more mass. As the mass approaches M_{Ch} the squeezed electrons move faster and faster and become ultra-relativistic.

9. (a) [3 pts] *What is the energy source for a core-collapse supernova? What kind of stars explode as core-collapse supernovae? What kind of particle carries away most of the energy in a core-collapse supernova?*

   The energy for a core-collapse supernova comes from the release of gravitational potential energy as the iron core collapses from the size of a white dwarf ($\sim 10^4$ km) down to the size of a neutron star (~ 10 km). Only massive stars ($M \gtrsim 8\ M_\odot$) undergo this. Neutrinos carry away the vast majority of the energy from a core-collapse supernova explosion.

   (b) [2 pts] *What is the energy source for a thermonuclear (type Ia) supernova?* Hint: *recall Problem Set 6. What kind of stars explode as thermonuclear supernovae?*

   A thermonuclear (type Ia) supernova is powered by the explosive nuclear fusion of carbon and oxygen to elements like iron and nickel. White dwarf stars (possibly ones approaching the Chandrasekhar mass limit) explode as thermonuclear supernovae.

   (c) [1 pt] *A typical core-collapse supernova occurs when an iron core of $\sim 1.4\ M_\odot$ is formed. A typical thermonuclear (type Ia) supernova produces $\sim 0.8\ M_\odot$ of iron. Core-collapse supernovae occur more frequently than thermonuclear supernovae, and yet most of the iron in the Universe (and on the Earth and in your blood) came from thermonuclear supernovae. Explain why.*

   The iron core of a core-collapse supernova gets crushed into becoming a neutron star or a black hole and thus that iron is not available to be released out into the Universe.

Group projects short answer [9 points total]

10–12. [3 pts each] Answer any ***three*** of these questions. Your choice, but **only 3**.

a. *Describe the stellar initial mass function. Explain why young star clusters (with all stars on the main sequence) look blue, referencing the stellar initial mass function in your answer.*

The stellar initial mass function describes how many stars are born of a given mass. There are many more low-mass stars compared to high-mass stars. Nonetheless, high-mass stars are much more luminous, so in a young star cluster, the bright, blue, high-mass stars outshine the other stars, making the cluster look blue.

b. *What is a brown dwarf?*

A brown dwarf is a "failed" star that does not have enough mass to sustainably fuse hydrogen into helium. Brown dwarfs may fuse deuterium or lithium, but this minimal fusion does not stabilize their luminosity. Brown dwarfs have masses from about 13 to 80 Jupiter masses ($\approx 0.08\ M_\odot$).

c. *In what part of the electromagnetic spectrum was the light from the first stars energetic enough to reionize hydrogen? Because of the expansion of the Universe, in what part of the electromagnetic spectrum would we observe that light today?*

Ultraviolet (UV) light is needed to ionize hydrogen. The expansion of the Universe causes this light to be redshifted (stretched) to the infrared (IR) today. Such light is a prime target for JWST.

d. *How is the fraction of stars with exoplanets related to the heavy element abundance (metallicity) of those stars?*

The exoplanet occurrence frequency goes up as metallicity increases. Perhaps the higher metal content in the accretion disk around a protostar leads to more efficient formation of planets.

e. *What is the study of the oscillation modes of the Sun called? It allows us to understand the density and pressure in the solar interior. What is it called when we study the oscillation of other stars?*

In the Sun it is helioseismology. For other stars it is asteroseismology.

f. *What is an astronomical standard candle? Describe what this term means and give one example.*

An astronomical standard candle is an object with known luminosity, for which a measurement of its apparent brightness gives an estimate of its distance. Examples include Cepheid variable stars and type Ia supernovae.

g. *A stellar evolution code like MESA uses what differential equation to relate the pressure, density, and gravitational acceleration through a star? Give either the equation or its name.*

   This is the equation of hydrostatic equilibrium: $dP(r)/dr = -\rho(r)g(r)$.

h. *What is a magnetar?*

   A magnetar is a neutron star with an extremely strong magnetic field.

i. *What does LIGO detect? What kind of instrument is it?* Hint: *the answers to both of these questions are in the expansion of the acronym "LIGO".*

   LIGO detects gravitational waves; it is an interferometer. LIGO stands for Laser Interferometer Gravitational-wave Observatory.