Lecture 15: White Dwarfs & Chandrasekhar Limit

Phillips 6.1
Lamers & Levesque Ch. 20

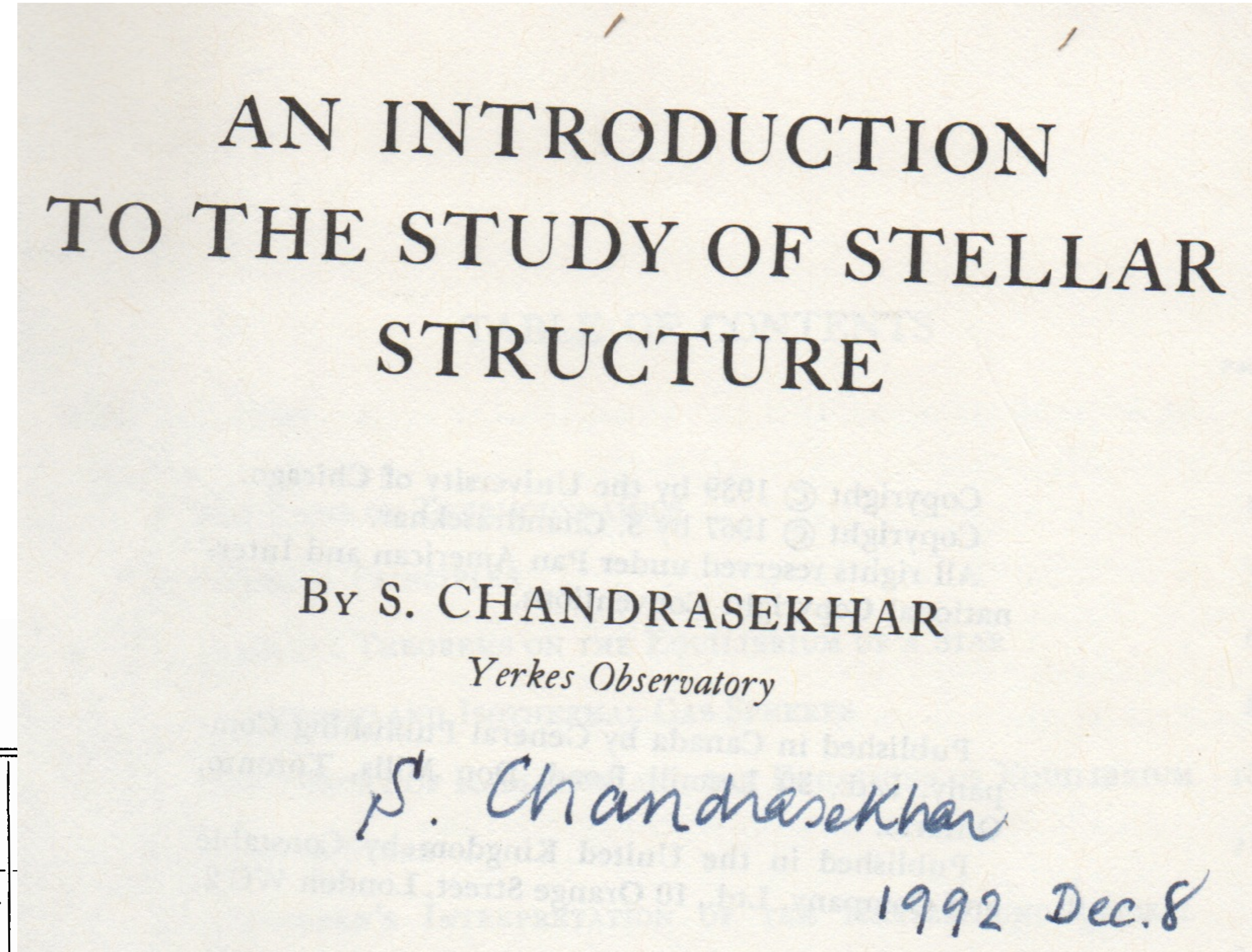


TABLE 4

THE CONSTANTS OF THE LANE-EMDEN FUNCTIONS*

n	ξ_1	$-\xi_1^2\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$	$\omega_n = -\xi_1^{\frac{n+1}{n-1}}\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	N_n		
0	2.4494	4.8988	1.0000	0.33333	....		
0.5	2.7528	3.7871	1.8361	0.02156	2.270		
1.0	3.14159	3.14159	3.28987	....	0.63662		
1.5	3.65375	2.71406	5.99071	132.3843	0.42422	0.770140	0.53849
2.0	4.35287	2.41105	11.40254	10.4950	0.36475	1.63818	0.60180
2.5	5.35528	2.18720	23.40646	3.82662	0.35150	3.90906	0.69956
3.0	6.89685	2.01824	54.1825	2.01824	0.36394	11.05066	0.85432
3.25	8.01894	1.94980	88.153	1.54716	0.37898	20.365	0.96769
3.5	9.53581	1.89056	152.884	1.20426	0.40104	40.9098	1.12087
4.0	14.97155	1.79723	622.408	0.729202	0.47720	247.558	1.66606
4.5	31.83646	1.73780	6189.47	0.394356	0.65798	4922.125	3.33100
4.9	169.47	1.7355	934800	0.14239	1.340	3.693×10^6	16.550
5.0	∞	1.73205	∞	0	∞	∞	∞

* The values for $n = 0.5$ and 4.9 are computed from Emden's integrations of θ_n; for $n = 3.25$ an unpublished integration by Chandrasekhar has been used. $n = 5$ corresponds to the Schuster-Emden integral. For the other values of n the *British Association Tables*, Vol. II, has been used.

Midterm & Assignments

midterm solutions posted on Canvas

average score: 43 out 50 – nice job!

Problem Set 4 available on Canvas
due Friday April 1

group project assignment document
also posted on Canvas

Group project

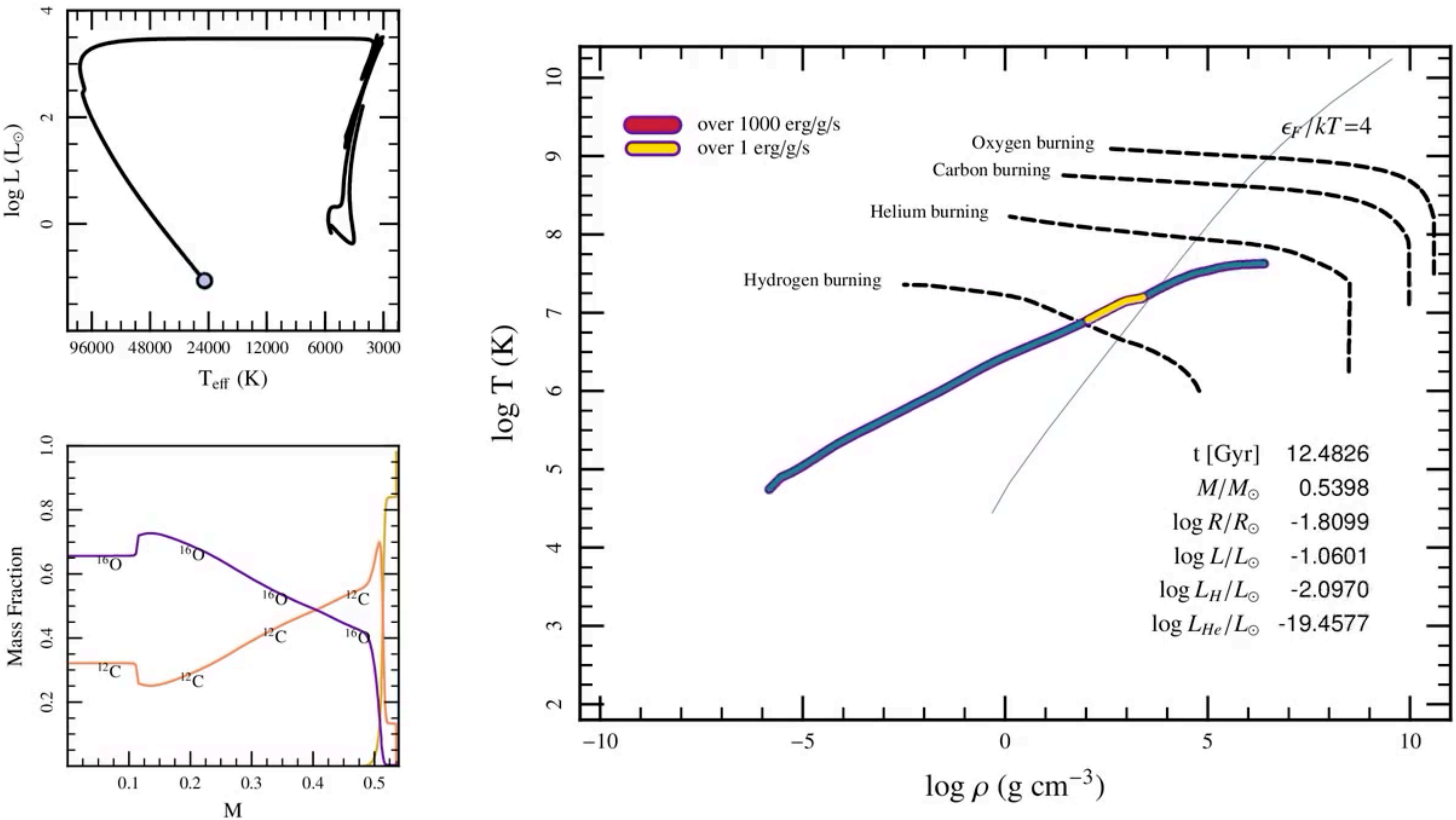
presentation, like our lecture: basic physics of the topic, relevant equations/derivations, examples, etc.

3-person groups: 15–20 minutes total
2-person groups: 10–15 minutes total

rough draft of slides due one week in advance

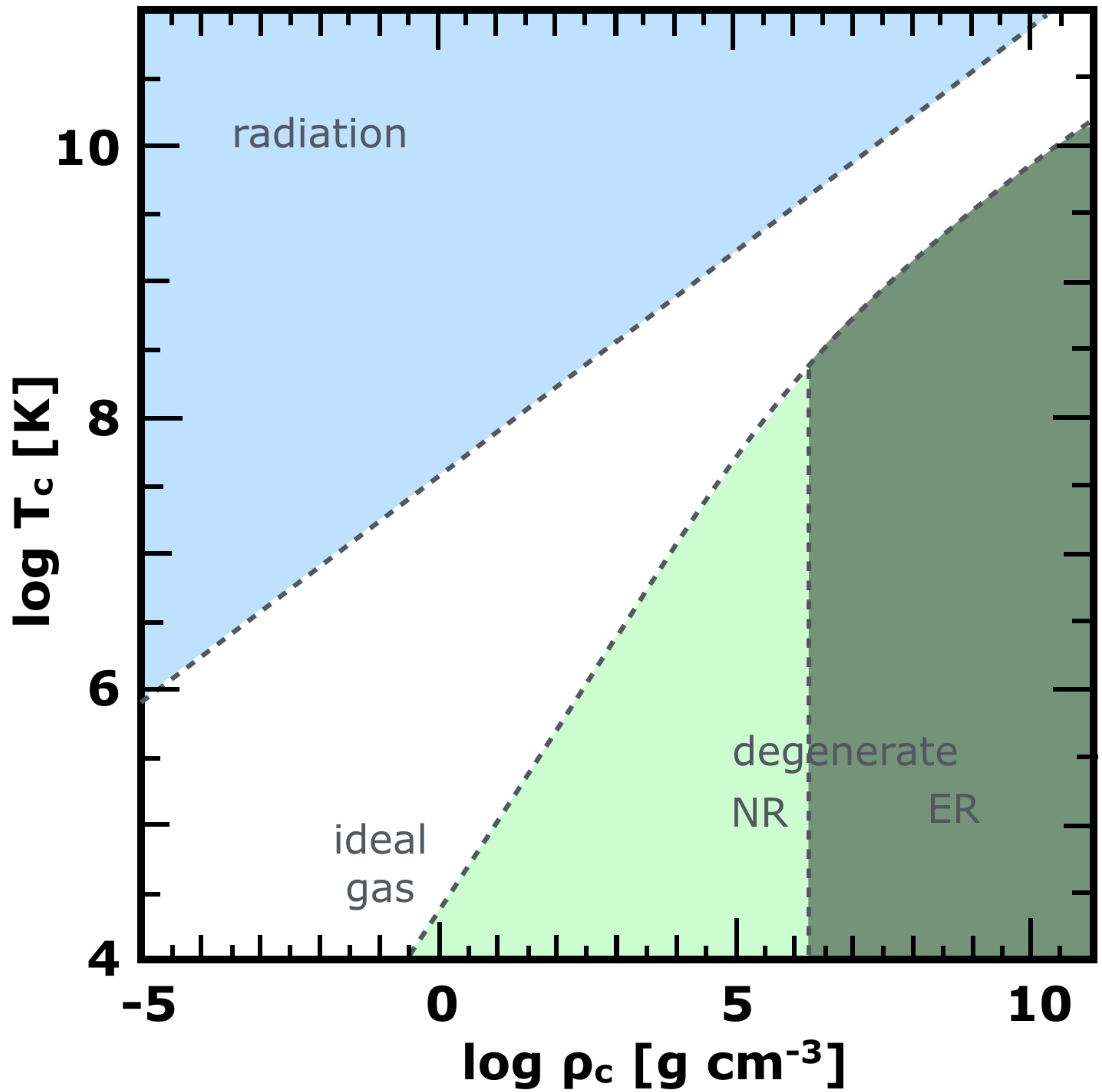
Lecture Date	Group	Members: Topic
Apr 22 (Fri)	1	Haonan Cheng, Yoon Choi, Matthew Wang: Stellar Initial Mass Function
Apr 22 (Fri)	2	Frank Genty, Anthony Pizzarelli, Khovesh Ramdin: Brown Dwarfs
Apr 22 (Fri)	3	Barbara Benda, Avery Kiihne, Harshill Patel: First Stars and Reionization
Apr 26 (Tue)	4	George Kharchilava, Geet Purohit, Anish Seth: Exoplanet Host Stars
Apr 26 (Tue)	5	Ava Marie Friedrich, Seung Hee Sung: Helioseismology
Apr 26 (Tue)	6	Rujuta Mokal, Michael Wozniak, Orion Yeung: Standard Candles
Apr 29 (Fri)	7	Aidan Boyce, Kailash Raman: MESA code
Apr 29 (Fri)	8	Arya Lakshmanan, Ina Park, Brandon Shane: Magnetars
Apr 28 (Fri)	9	Bradley Butler, Christine Carvajal, Connor Lane: LIGO Black Holes

MESA: open stellar modeling code



from Josiah Schwab, MESA model of a 1 solar mass star
https://www.youtube.com/watch?v=oZY3TtA63sE

Fusion and Stellar Evolution



Adapted from Onno Pols, 2011.

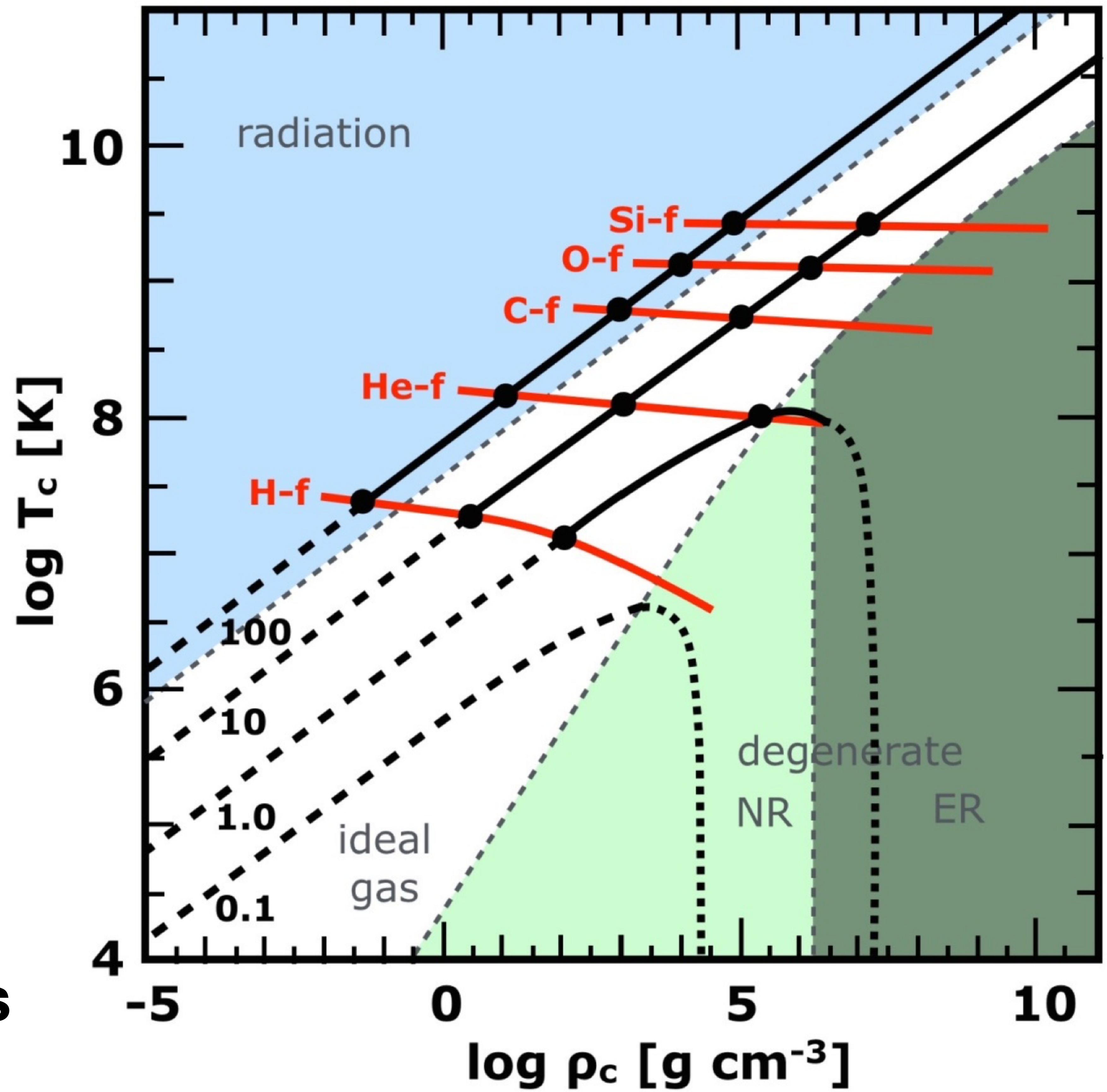
Fusion and Stellar Evolution

stars evolve up and to the right (hotter & denser center) until WD

lowest mass stars only fuse H→He

"low-mass stars" fuse H→He→C,O

high mass (> 8 $M_\odot$) stars fuse to iron & do not make WDs



Adapted from Onno Pols, 2011.

Degenerate Stars

White Dwarfs

All stars with $M_i < 8 M_\odot$ end their lives as white dwarfs (WD), degenerate stars with no more nuclear energy source.

WD can be split into three spectral types:

DA: spectra dominated by H lines
DB: spectra dominated by He lines
DC: continuum spectra without absorption lines

Internal composition can be quite different; due to high surface gravity ($\sim 10^7$ cm s^{-2}) and stable atmospheres WD spectral types are driven by gravitational diffusion, with heavier elements sinking to the bottom of the photosphere and lighter elements appearing at the top.

for white dwarfs, what you see is **not** what you get

Degenerate Stars

White Dwarfs

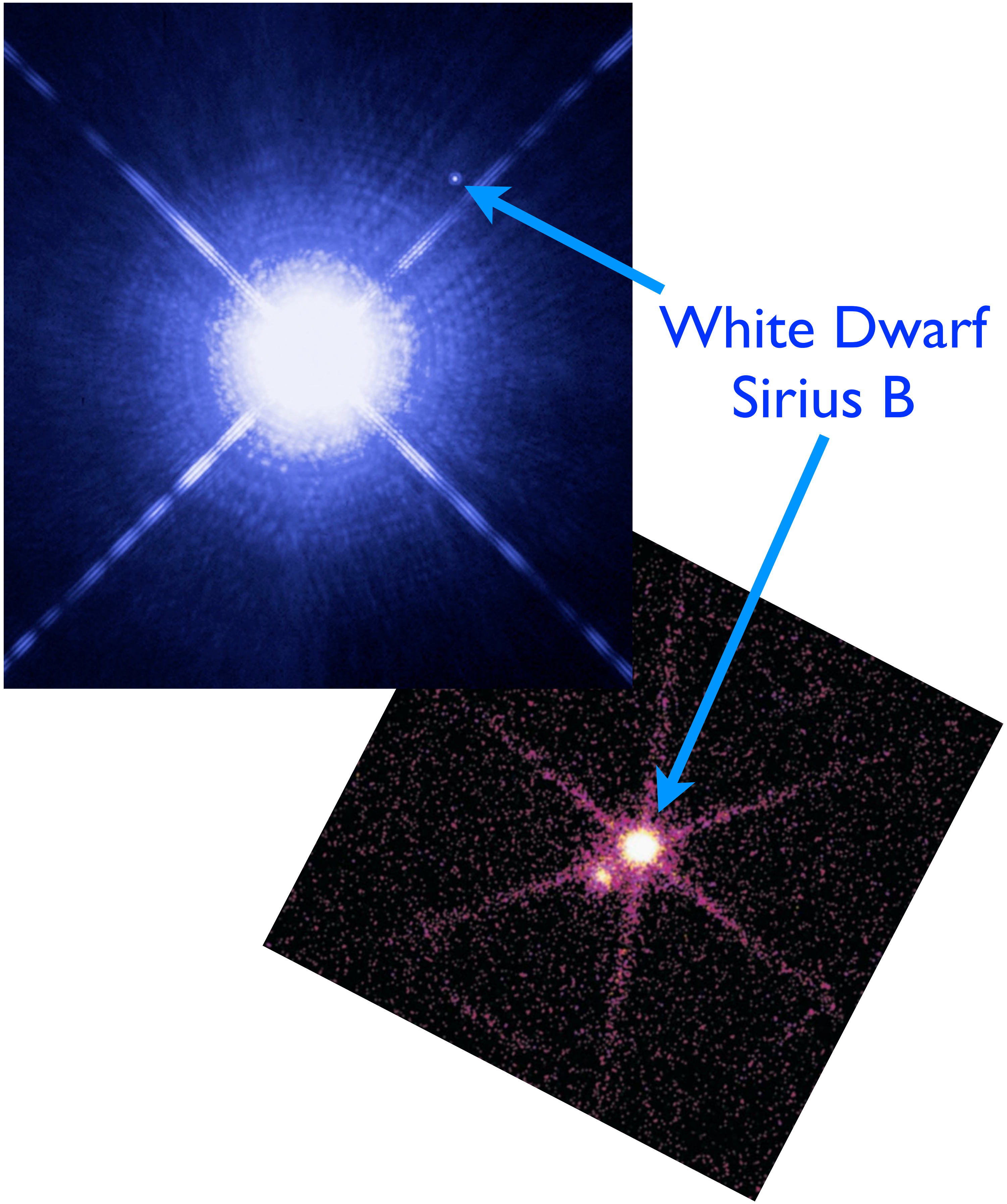
$0.8M_\odot < M_i < 8M_\odot$: stars go through core H and core He fusion, ending as C- and O-rich degenerate stars, **CO WDs**

$0.5M_\odot < M_i < 0.8M_\odot$: stars reach core H fusion but are not massive enough to fuse He. End lives as **He-rich WDs**.
—single-star formation channel would be longer than the age of the universe; existing He-rich WDs are presumed to have formed via stripping of an initially more massive star by a close companion

—binary star companion stripping of even lower mass stars can result in the formation of **H-rich WDs**.

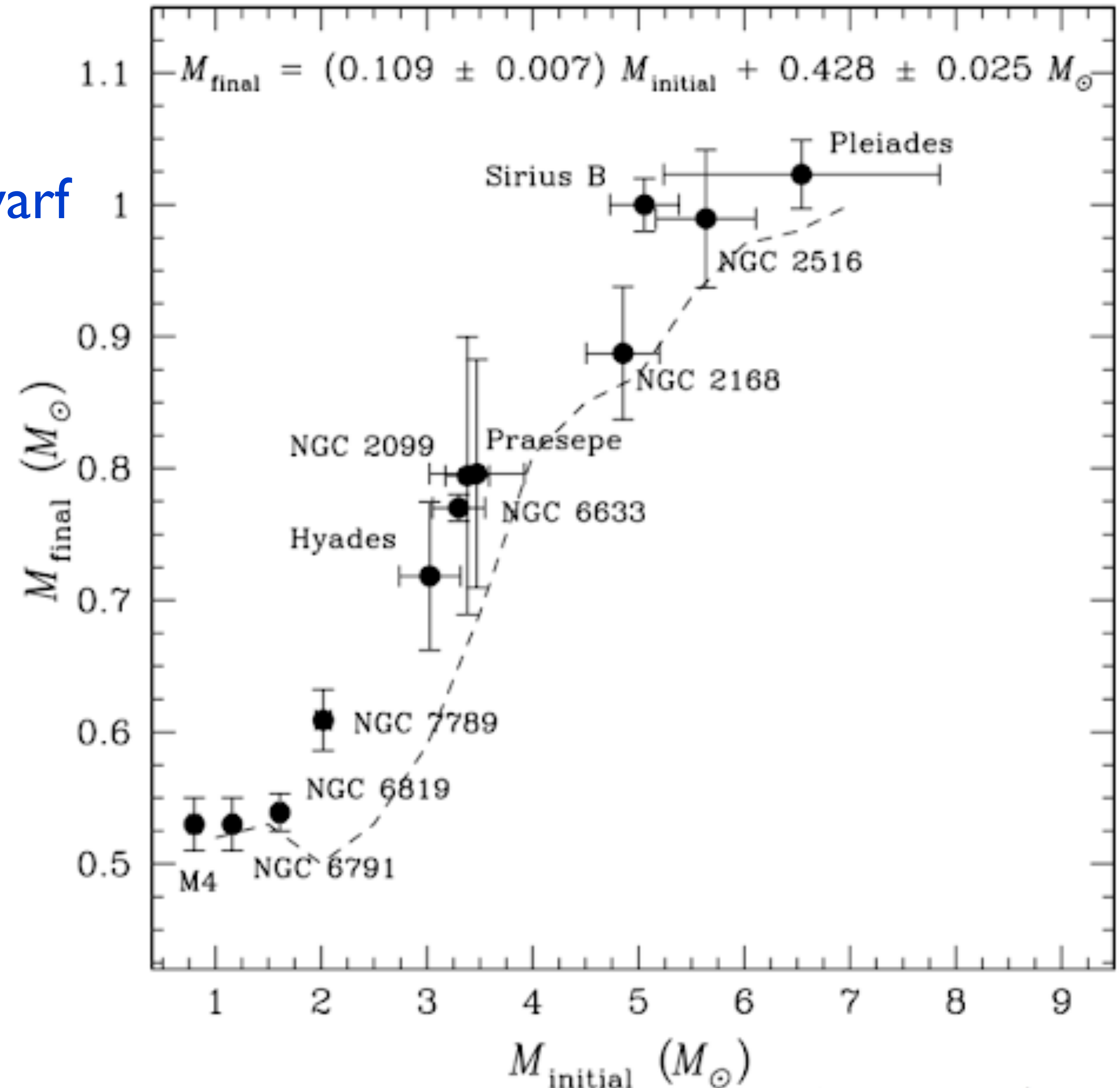
White Dwarfs

- White dwarfs are the remaining cores of dead low-mass stars ($M < 8\ M_{Sun}$).
- Electron degeneracy pressure supports them against the crush of gravity.



Initial-final mass relation

white dwarf mass



ZAMS stellar mass

Kalirai et al. (2008, ApJ, 676, 594)
Kalirai et al. (2009, ApJ, 705, 408)

Initial-final mass relation

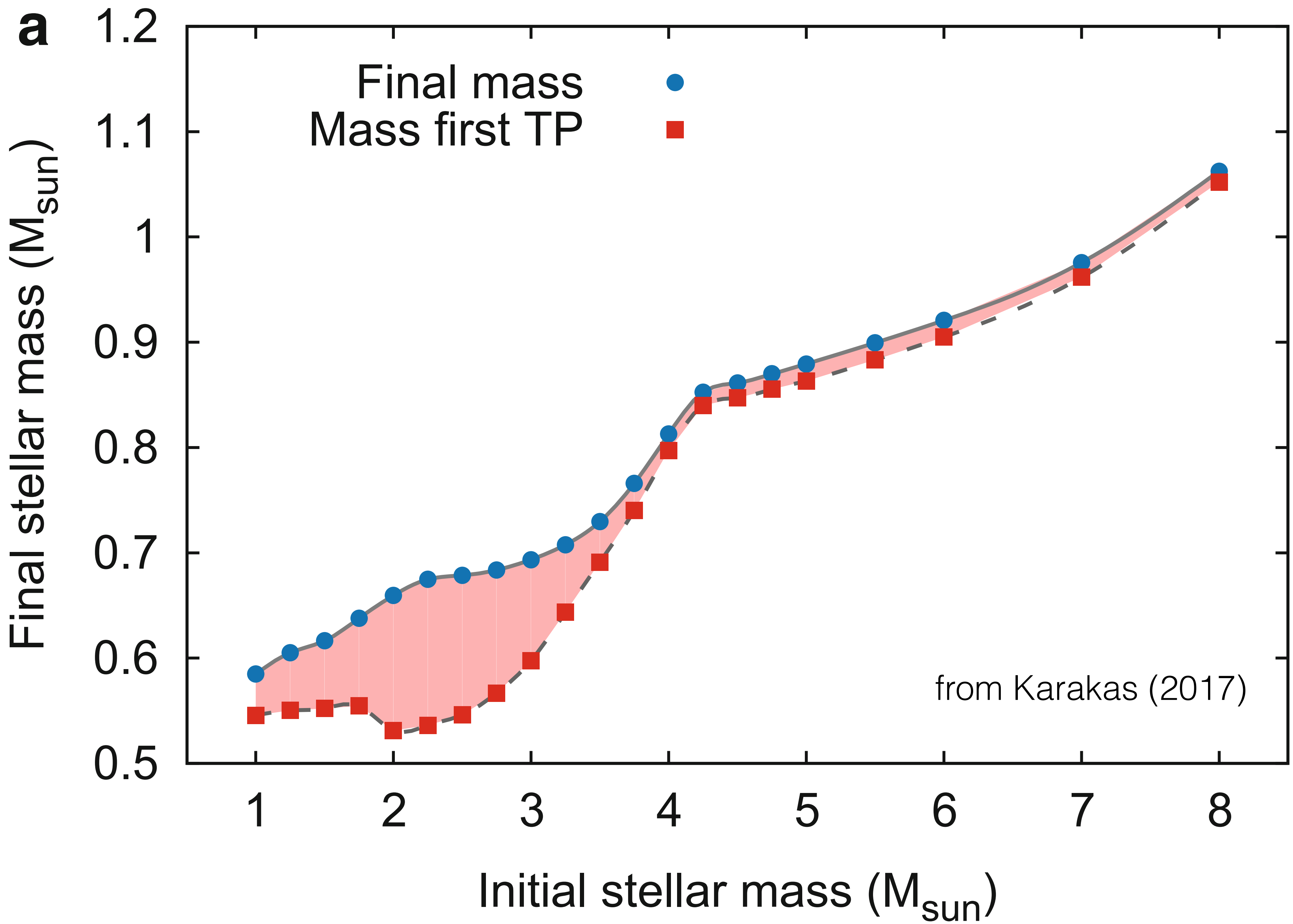


Fig. 8 Initial-final mass relation for models of solar metallicity $Z = 0.014$ and for metal-poor models of $Z = 0.001$. The metal poor models use data from Fishlock et al. (2014) and the solar metallicity models use data from Karakas (2014). The core mass at the first thermal pulse is also shown, while the *shaded region* shows the core growth on the AGB

Review: degeneracy pressure

see lectures 4 & 5! and PS #2, problem 3

pressure integral $$P = \frac{1}{3}\int_0^\infty p\, v(p)\, n(p)\, dp$$

$$p_F = (3h^3 n_e/8\pi)^{1/3} \qquad P = \frac{8\pi}{3h^3}\int_0^{p_F} \frac{p^4 c^2}{(p^2c^2 + m_e^2 c^4)^{1/2}}\, dp$$

complete degeneracy: 2 electrons in every h³ of phase space volume

non-relativistic

$$P = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} n^{5/3}$$

ultra-relativistic

$$P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}$$

these limits correspond to polytropes

$$\gamma = 5/3 \quad n = 3/2$$

$$\gamma = 4/3 \quad n = 3$$

$$R \propto M^{\frac{1-n}{3-n}} \qquad R \propto M^{-1/3}$$

slide from Lecture 3

polytropes: summary

see also Phillips section 5.2
Lamers & Levesque chapter 11

Assume pressure depends only on density:

$$P = K\rho^{\gamma} \quad P = K\rho^{1+1/n} \qquad \gamma = 1 + 1/n \;\Leftrightarrow\; n = 1/(\gamma - 1)$$

Define dimensionless variables θ (1 at center, 0 at surface) and ξ (like radius, 0 at center), and α (not dimensionless: constant with units of length):

$$\rho = \rho_c \theta^n, \qquad P = P_c \theta^{n+1}, \qquad r = \alpha\xi, \qquad \alpha^2 = \frac{K\,(n+1)\,\rho_c^{\frac{1-n}{n}}}{4\pi G},$$

Then combined equations for hydrostatic equilibrium and enclosed mass can be rewritten:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \quad \Longrightarrow \quad \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

Lane-Emden equation

Use boundary conditions at center: $\theta(\xi = 0) = 1$ and $\left.\dfrac{d\theta}{d\xi}\right|_{\xi=0} = 0$

Then can solve equations (typically numerically) for a given value of polytropic index *n.*
Mathematical solutions of Lane-Emden equation just depend on *n* (and boundary conditions).

Physical solutions start at $\xi = 0$ (center) and extend to ξ_1 where $\theta(\xi_1) = 0$ (surface)
To get **physical units**, we **require two more parameters**. Choose **any two** of the following: total mass *M*, radius *R*, central density ρ_c, eqn of state const *K*, central pressure P_c, etc.
Once you specify two of those, all the others can be solved in terms of the rest, e.g.:

$$R = \alpha\xi_1 = \left[\frac{K}{G}\frac{n+1}{4\pi}\right]^{1/2} \rho_c^{\frac{1-n}{2n}} \underbrace{\xi_1} \qquad M = 4\pi\alpha^3\rho_c \underbrace{\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}} \qquad \Omega = E_{\rm pot} = -\frac{3}{5-n}\frac{GM^2}{R}$$

look these up in tables, for a given value of *n*

Review: degeneracy pressure

non-relativistic	ultra-relativistic
$P = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} n^{5/3}$	$P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}$

these limits correspond to polytropes

$$\gamma = 5/3 \quad n = 3/2 \qquad\qquad \gamma = 4/3 \quad n = 3$$

$$R \propto M^{\frac{1-n}{3-n}} \qquad R \propto M^{-1/3}$$

Qualitatively, the degeneracy pressure gets stronger as the density increases. So for degeneracy pressure to support a larger mass against gravity, the density needs to be higher, and the white dwarf needs to have a physically smaller radius.

complete degeneracy: $P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}\, I(x)$ see PS #2, problem 6

partial degeneracy is harder (temperature dependent); see L&L § 4.5.3 and Lecture 5 slides 5 & 6

where $x = p_F/(m_e c)$ *and*

$$I(x) = \frac{3}{2x^4}\left\{x\left(1+x^2\right)^{1/2}\left(\frac{2x^2}{3}-1\right) + \ln\left[x + \left(1+x^2\right)^{1/2}\right]\right\}.$$

In the ultra-relativistic limit, we take $x \gg 1$.

$$\lim_{x\to 0} I(x) = 4x/5 \qquad\qquad \lim_{x\to\infty} I(x) = 1$$

slide from Lecture 3

from Lamers & Levesque, Ch. 11

The radius of the star is defined by $\rho = 0$, so $\theta(\xi_1) = 0$, and ξ_1 is the value of the ordinate where $\theta = 0$. This means that

$$R = \alpha\xi_1 \equiv \alpha R_{\mathrm{n}}. \tag{11.8}$$

The mass of the star is given by

typo in L&L eqn 11.9 fixed here

$$M = 4\pi \int_0^R \rho r^2 \, dr = 4\pi\alpha^3 \rho_c \int_0^{\xi 1} \xi^2 \theta^n \, d\xi. \tag{11.9}$$

Using Equation (11.7) results in

$$M = -4\pi\alpha^3 \rho_c \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi 1} \equiv 4\pi\alpha^3 \rho_c M_n, \tag{11.10a}$$

with

$$M_n = -\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi 1}. \tag{11.10b}$$

Table 11.1. Physical Parameters of Polytrope Models.

n	γ	$R_n = \xi_1$	$M_n = -\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$
1.00	2.00	3.14	3.14
1.50	5/3	3.65	2.71
3.00	4/3	6.90	2.02

from Lamers & Levesque, Ch. 11

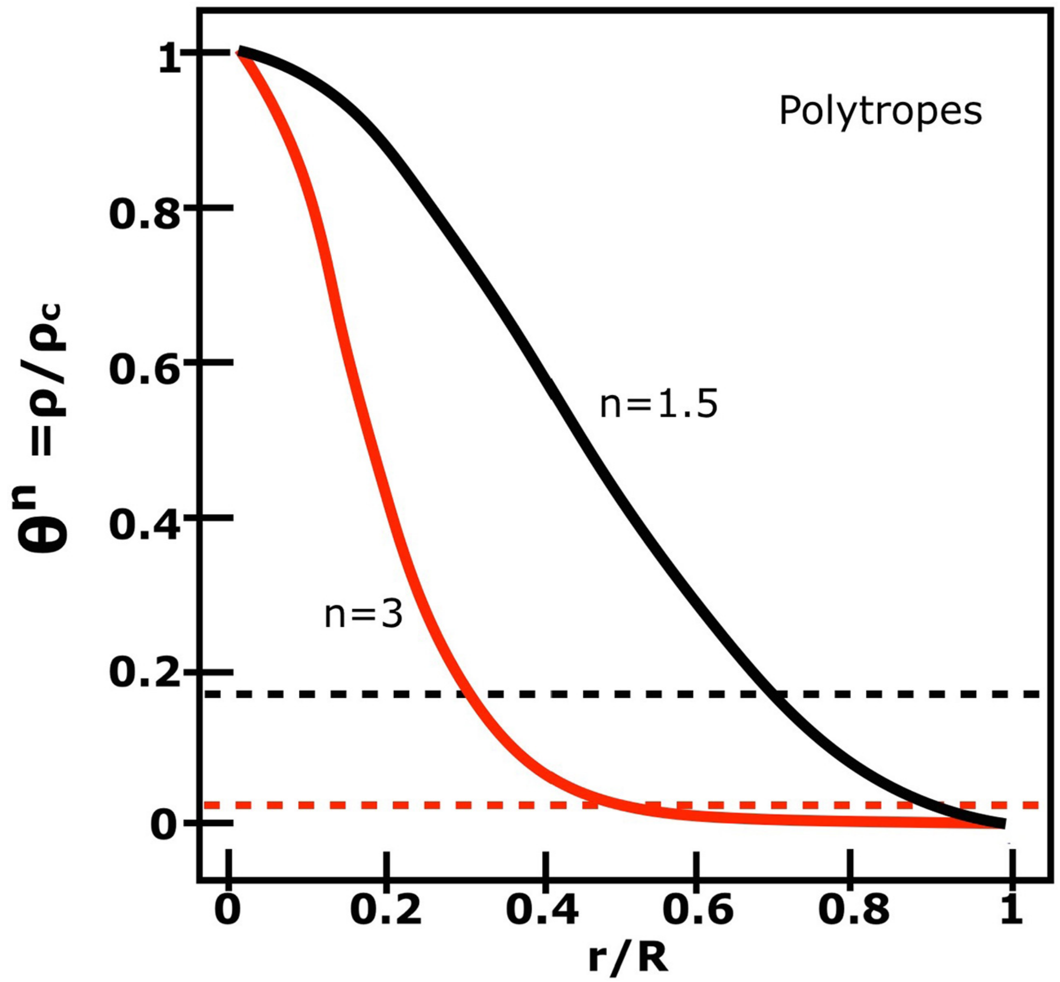


Figure 11.1. Density structure of polytropes for $n = 3$ ($\gamma = 4/3$) (red) and $n = 1.5$ ($\gamma = 5/3$) (black). The dashed lines show the mean density of the two models.

Table 11.2. Properties of Polytropic Stars

n	γ	Property	M–R Relation	Consequence
0	∞	Incompressible	$R \sim M^{1/3}$	Constant ρ
1	2	$P \sim \rho^2 \rightarrow T \sim \rho$	$R \sim M^{0}$	Constant R
1.5	5/3	Nonrelat. degeneracy	$R \sim M^{-1/3}$	Volume $\sim 1/M$
		Fully convective (ideal gas)		
3.0	4/3	Relativ. degeneracy	$R \sim M^{\infty}$	$M \sim R^{0}$
		Constant P_{gas}/P_{rad} (ideal gas)		
∞	1	Isothermal: $P \sim \rho$	$R \rightarrow \infty$	Infinite radius

Degenerate Stars

White Dwarfs

Because they are non-relativistic electron degenerate, the WD equation of state can be expressed as $P \sim \rho^{5/3}$, making them polytropes with $\gamma = 5/3$, $n = 1.5$

We know that there is a strict M-R relation for degenerate stars; solving the polytrope model (see PS #4) gives:

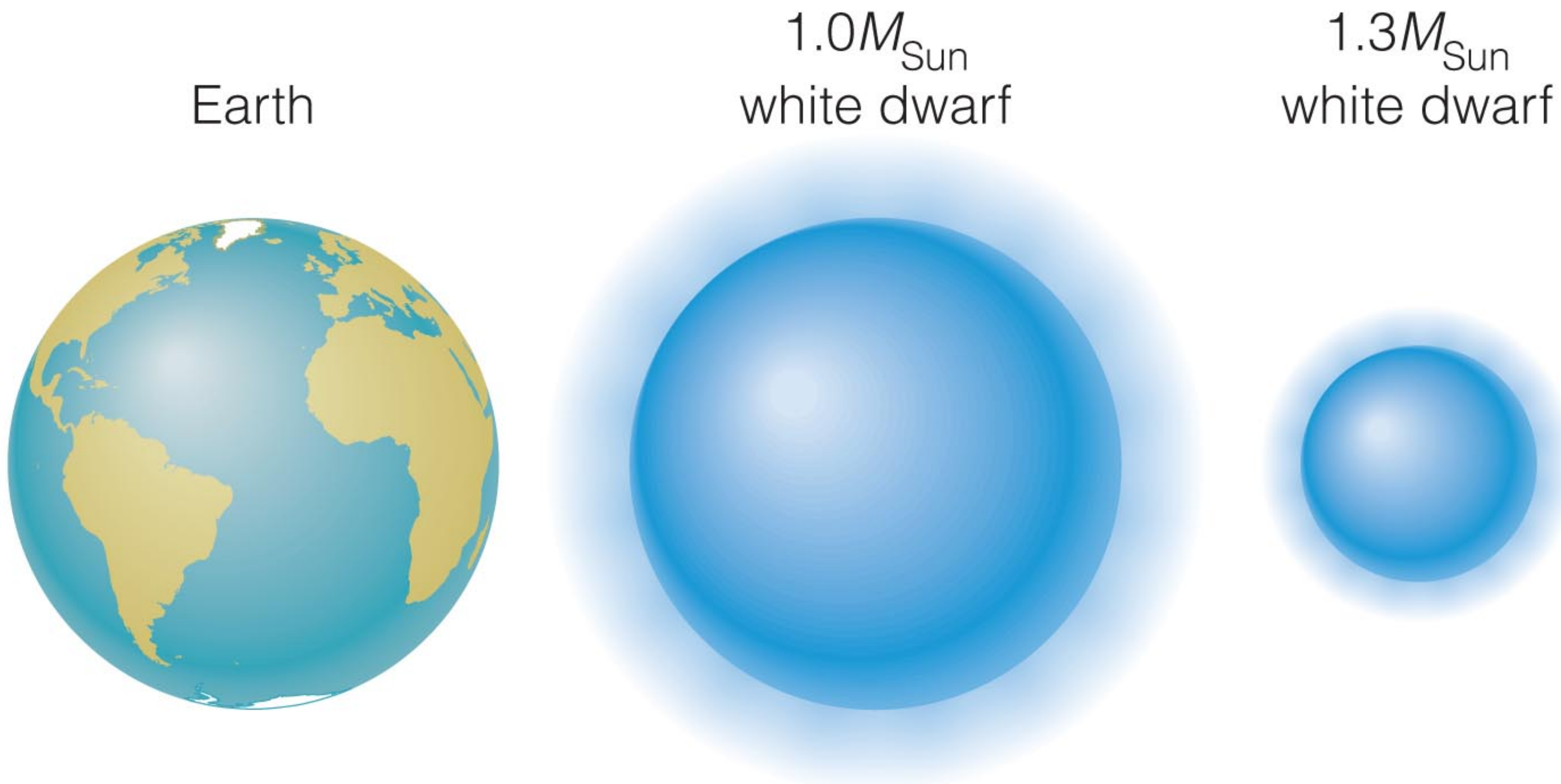
$$R = 0.012 \; M^{-1/3} \, (\mu_e/2)^{-5/3}$$

For fully non-relativistic degenerate WDs the values are:

H - WD	$\mu_e = 1$	$R = 0.046 \times (M_{WD}/0.5M_{\odot})^{-1/3} \; R_{\odot}$
He or CO-WD	$\mu_e = 2$	$R = 0.014 \times (M_{WD}/0.5M_{\odot})^{-1/3} \; R_{\odot}$

This shows that R will decrease as M increases…

Size of a White Dwarf



- White dwarfs with same mass as Sun are about same size as Earth.
- Higher-mass white dwarfs are smaller.

White dwarf central density, mass

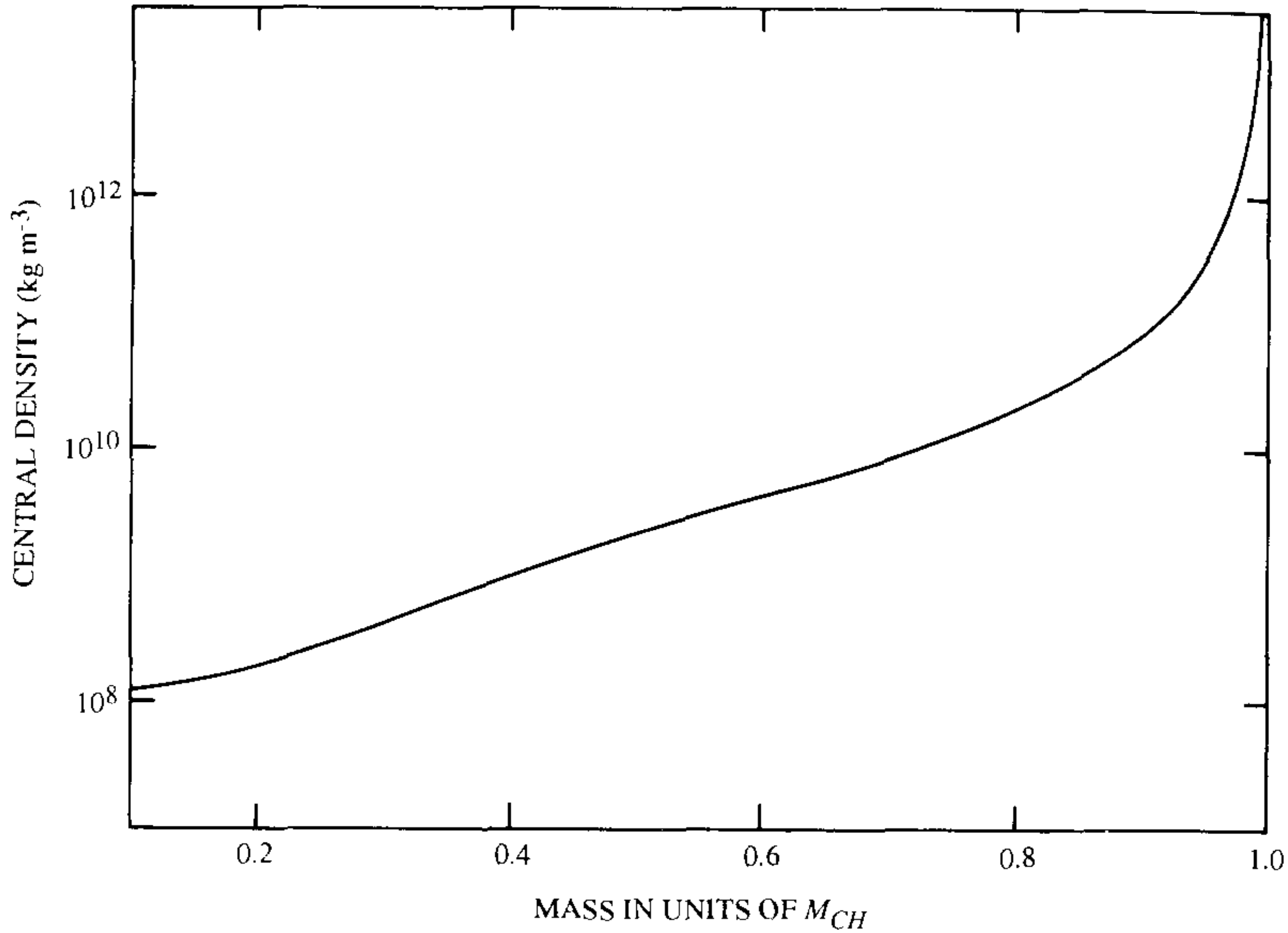


Fig. 6.1 The density at the centre of a white dwarf of mass M supported by the pressure of an ideal gas of degenerate electrons. Note the density tends to infinity as the mass approaches the Chandrasekhar mass M_{CH}.

Phillips Figure 6.1

The Chandrasekhar mass

let's look at the ultra-relativistic case more closely:

$$n_e = \frac{\rho}{\mu_e m_p} \quad P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3} = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3}\left[\frac{1}{\mu_e m_p}\right]^{4/3}\rho^{4/3}$$

L&L Table 11.1 (few slides ago)

$$\text{polytrope } K = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3}\left[\frac{1}{\mu_e m_p}\right]^{4/3} \qquad n = 3 \quad \xi_1 = 6.90 \quad -\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1} = 2.02$$

from polytrope relations: use K to get α, plug in α to get M (do this on PS #4):

$$\alpha^2 \propto \rho_c^{\frac{1-n}{n}} = \rho_c^{-2/3} \Rightarrow \alpha \propto \rho_c^{-1/3} \Rightarrow \alpha^3 \propto \rho_c^{-1} \Rightarrow M \propto \alpha^3 \rho_c \propto \rho_c^0 = \text{ constant}$$

ρ_c cancels out of mass, so there's only one value of the mass that works, and we call this **the Chandrasekhar mass**

$$M_{\mathrm{Ch}} = 5.8\ M_\odot \left(\frac{1}{\mu_e}\right)^2 = 5.8\ M_\odot Y_e^2 \qquad Y_e \equiv \frac{1}{\mu_e} \approx \frac{1+X}{2}$$

for He or C/O white dwarfs, X = 0, so μ_e = 2 or Y_e = 0.5: $M_{\mathrm{Ch}} \approx 1.4\ M_\odot$

Degenerate Stars

White Dwarfs

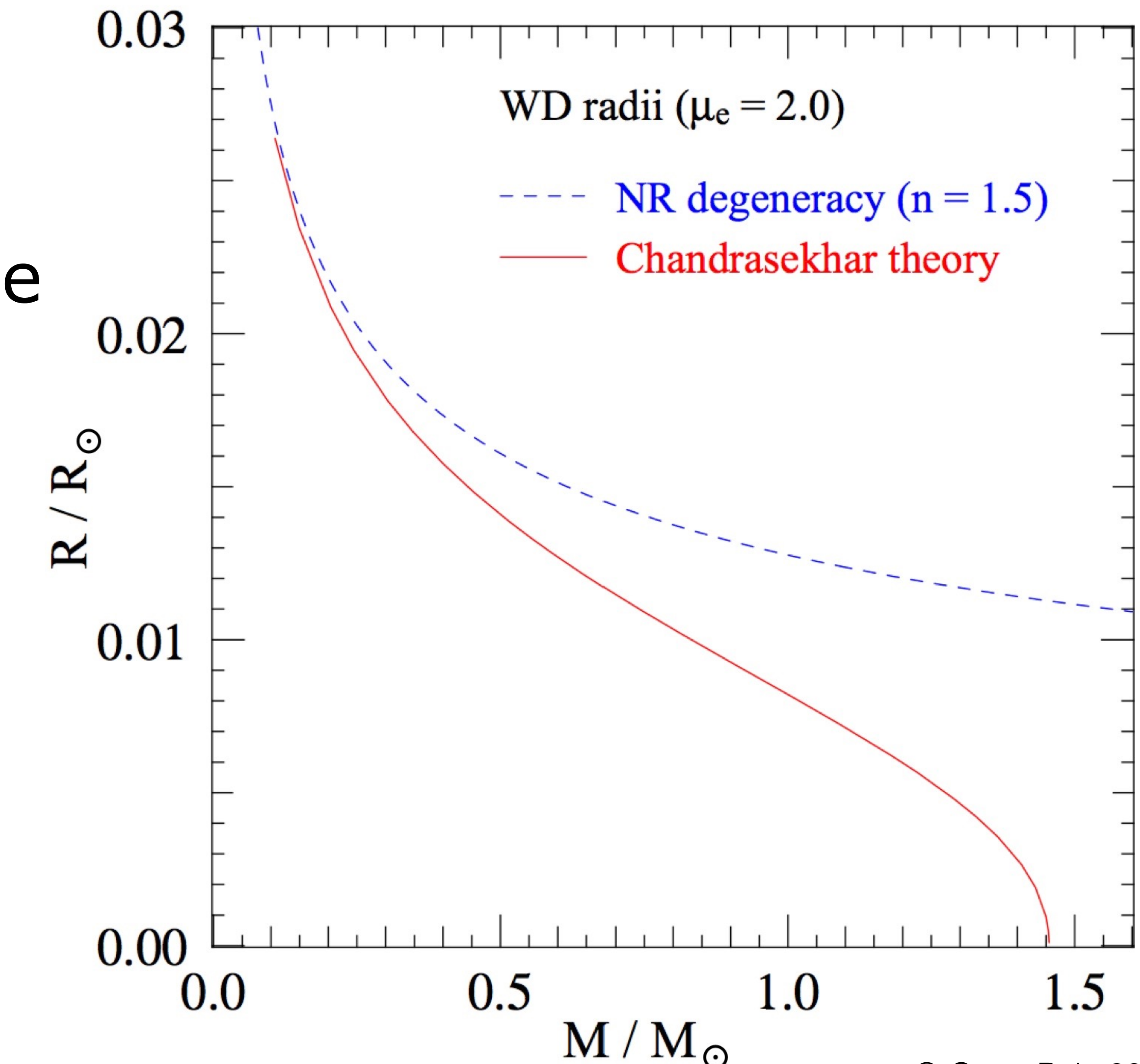
This shows that R will decrease as M increases…

Mean density increases as $\rho \sim M^2$. Eventually ρ in the center of a WD will be high enough to become relativistic degenerate, where $P \sim K\rho^{4/3}$, n=3.

We've demonstrated that such polytropes can only exist for one specific mass, the **Chandrasekhar mass**:

$$\boxed{M_{ch} = 1.46(2/\mu_e)^2 M_\odot}$$

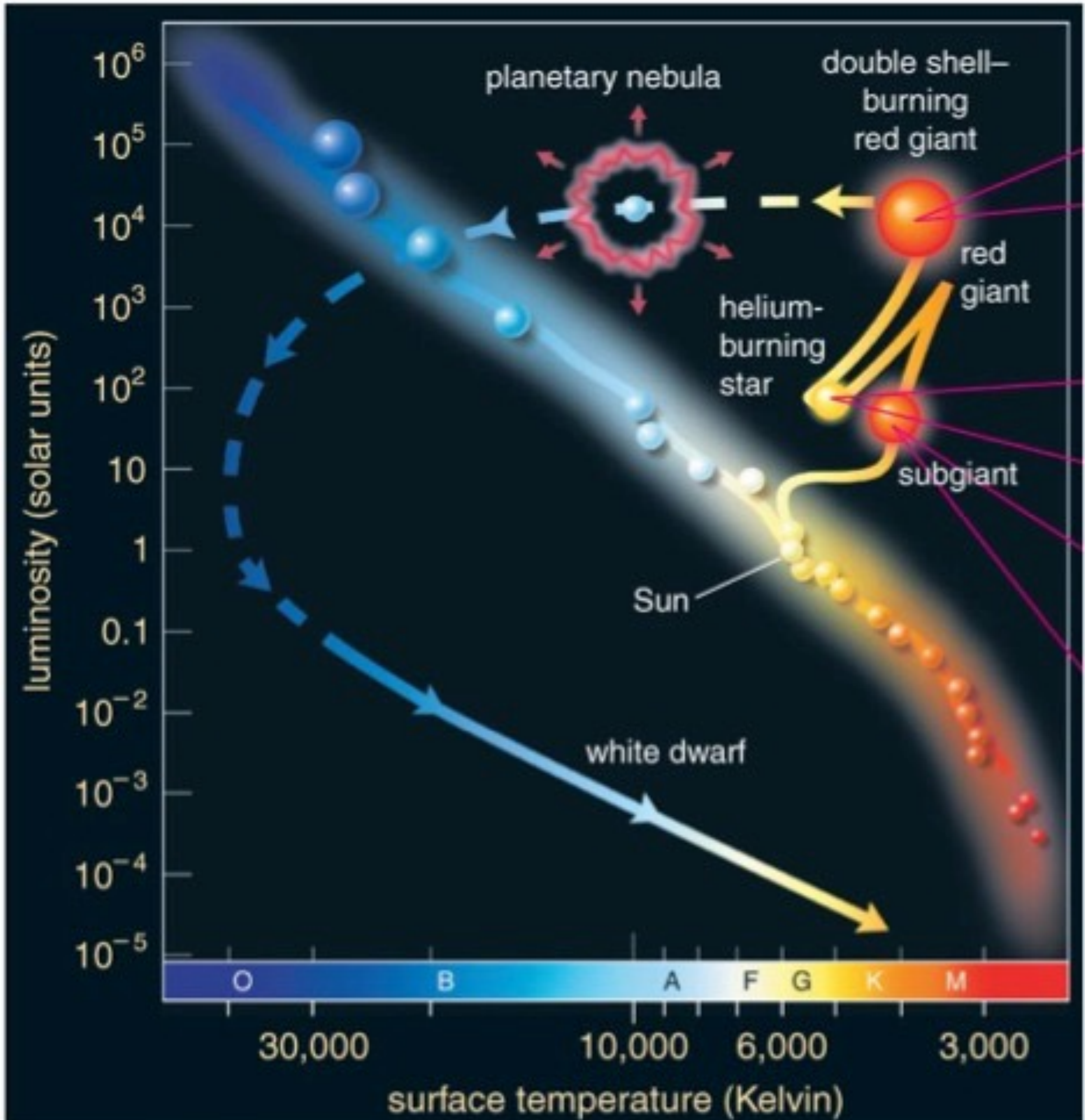
μ_e=1 for H-WDs
μ_e=2 for He- and CO-WDs



WDs with $M > M_{ch}$ cannot exist.

the Chandrasekhar limit

https://www.google.com/doodles/s-chandrasekhars-107th-birthday



White dwarfs cool off and grow dimmer with time.

White Dwarf Cooling

Phillips section 3.4
Lamers & Levesque section 20.4

WD luminosity comes from ion cooling; thermal energy is transported outward via conduction and radiated by a thin non-degenerate atmosphere.

Initial T is $\sim 10^8$ K and the WD cools fast with $L \sim 10^{-1} L_\odot$. As ions cool L decreases; cooling slows and L decreases over time.

L-T dependence can be derived if we assume the WD consists of a fully degenerate core and a thin photosphere of ideal gas with a Kramer's opacity law ($\kappa = \kappa_0 \rho T^{-7/2} \sim P T^{-9/2}$).

Eddington equation for RT:

$$\frac{dT}{dr} = -\frac{3}{4} \cdot \frac{1}{ac} \cdot \frac{\kappa\rho}{T^3} \cdot \frac{L_r}{4\pi r^2}$$

White Dwarf Cooling

Phillips section 3.4
Lamers & Levesque section 20.4

Phillips equations 3.43 and 3.45 are what we need here. Let's start with the differential equation for the internal temperature evolution $T_I(t)$ (eqn. 3.45)

$$\frac{dT_I}{dt} = -\alpha \left[\frac{T_I}{T_{\rm ref}}\right]^{7/2} \text{ where } T_{\rm ref} = 7 \times 10^7 \text{ K and } \alpha \approx \frac{2}{3k}\left[\frac{12 m_p}{M_\odot}\right] L_\odot$$

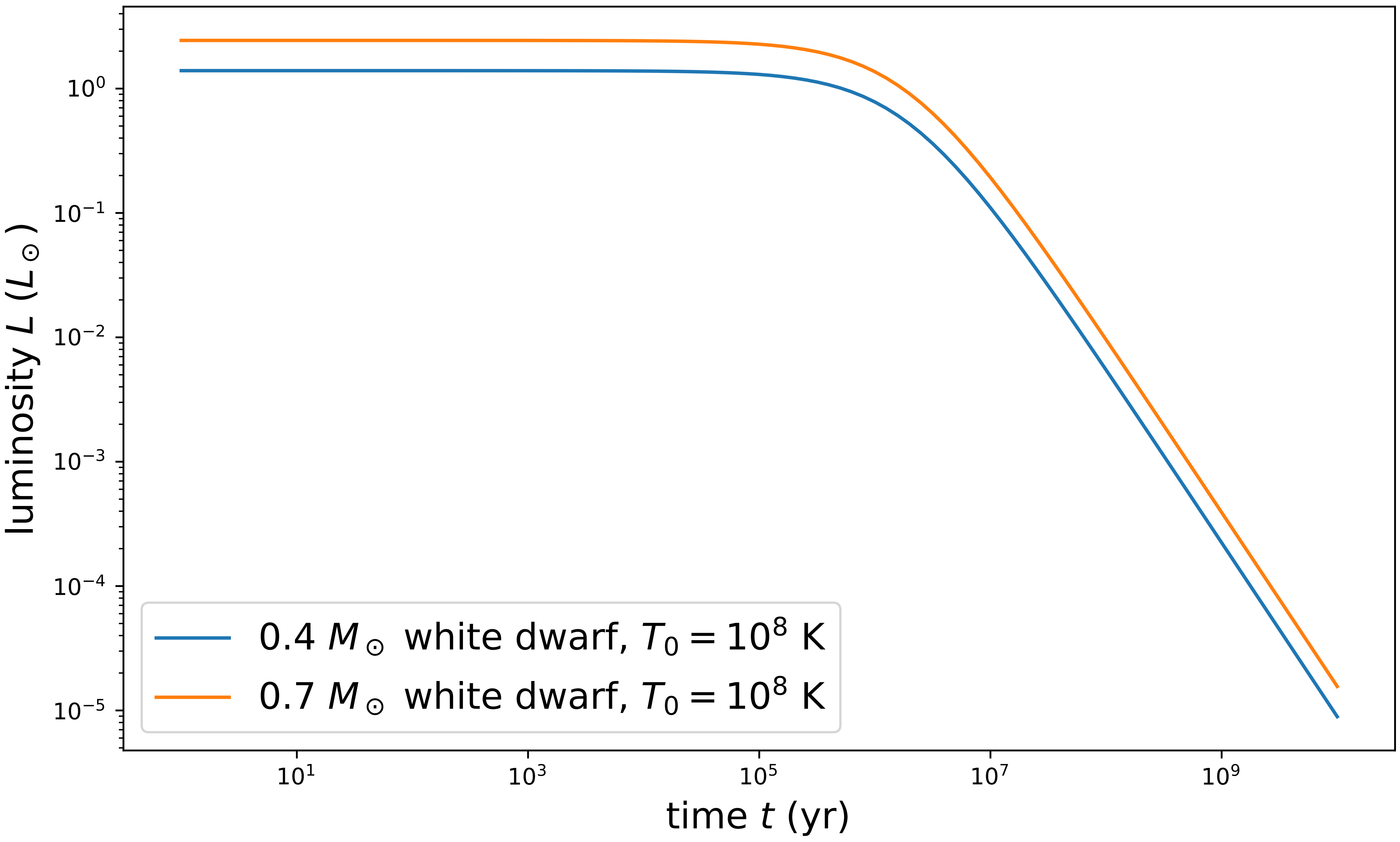
$$T_I(t) = T_{\rm ref}\left[\frac{5}{2}\frac{\alpha t}{T_{\rm ref}} + \left(\frac{T_0}{T_{\rm ref}}\right)^{-5/2}\right]^{-2/5}$$

We can then plug this into Phillips eqn. 3.43 to get the luminosity evolution

$$L(t) \approx \left[\frac{T_I}{T_{\rm ref}}\right]^{7/2}\left[\frac{M}{M_\odot}\right] L_\odot = \left[\frac{5}{2}\frac{\alpha t}{T_{\rm ref}} + \left(\frac{T_0}{T_{\rm ref}}\right)^{-5/2}\right]^{-7/5}\left[\frac{M}{M_\odot}\right] L_\odot$$

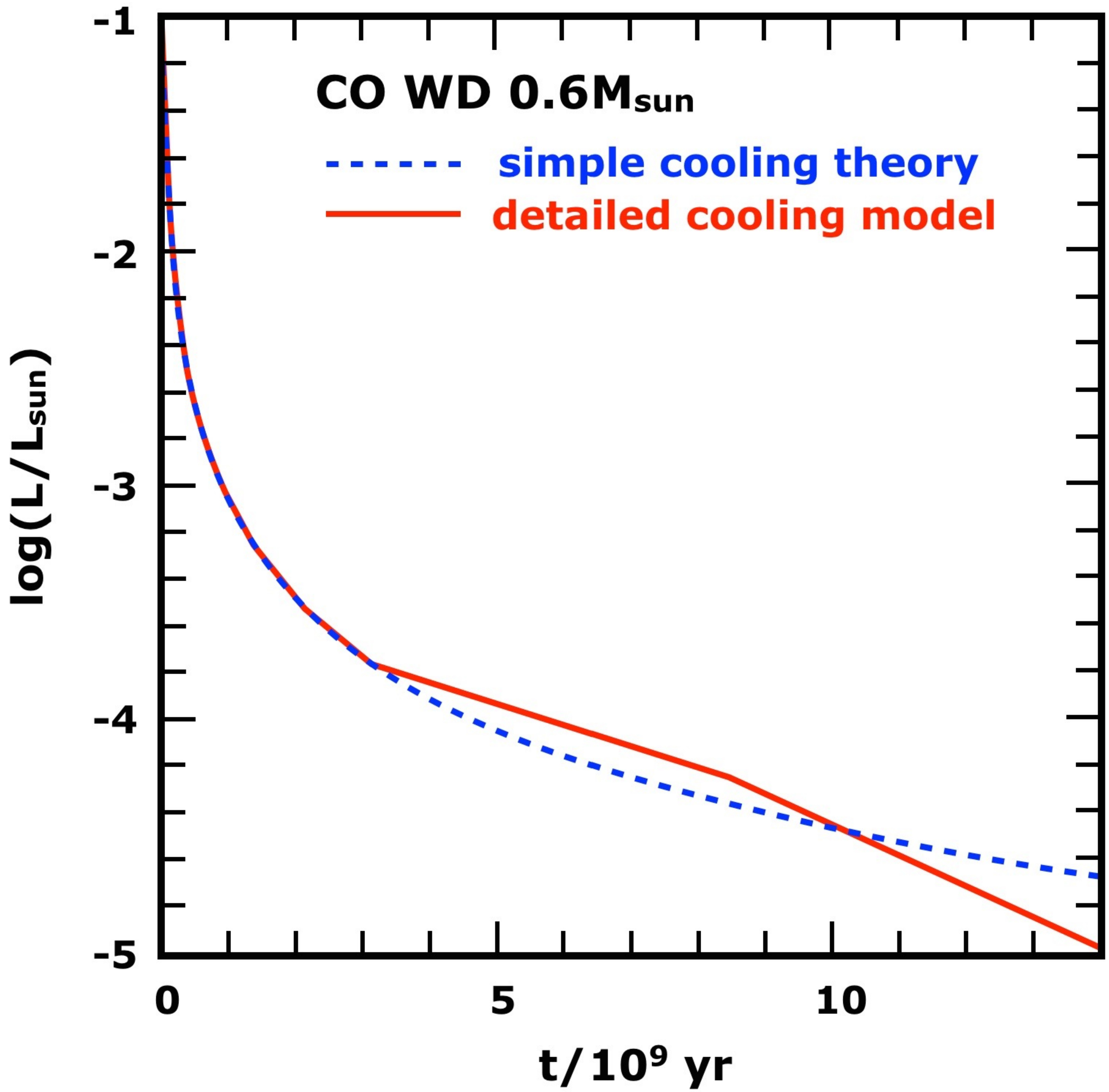
White Dwarf Cooling

Phillips section 3.4
Lamers & Levesque section 20.4

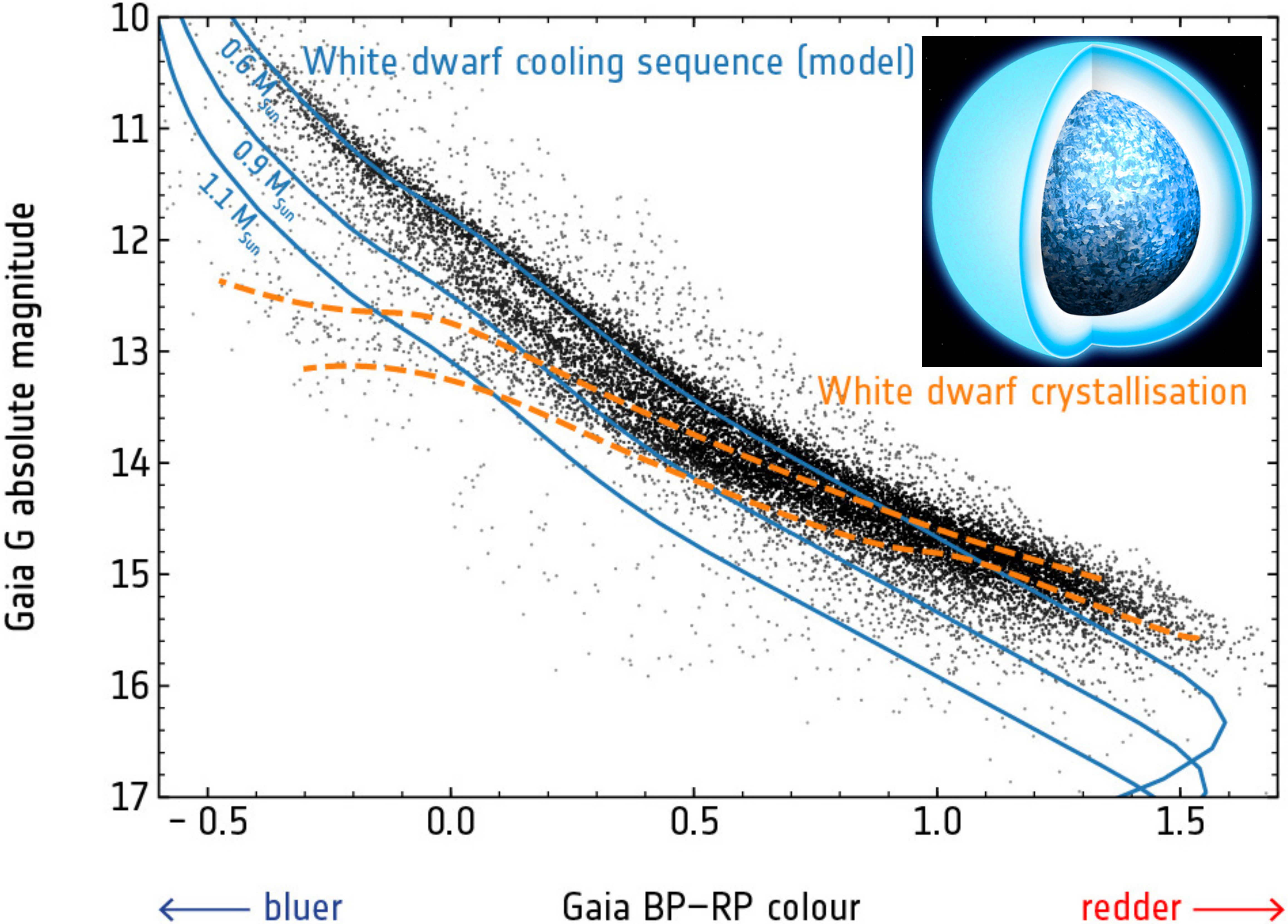


White Dwarf Cooling

Phillips section 3.4
Lamers & Levesque section 20.4



White Dwarf Crystallization



from https://sci.esa.int/web/gaia/-/61044-gaia-reveals-how-sun-like-stars-turn-solid-after-their-demise