Physics 441/541 Spring 2022: Problem Set #1
due January 28 at 11:00 am in PDF format on Canvas

You are encouraged to work in groups on these problems, but you must write up the solutions individually. In your writeup, list your collaborators and cite any external sources you used. You **may not** consult previous solutions for this, class or other similar classes. *This problem set will take some time! I strongly urge you to start early.*

1. The Planck spectrum has

$$B_\lambda(T) = \frac{2\,hc^2/\lambda^5}{\exp(hc/\lambda kT) - 1}$$

   (a) Derive Wien's displacement law: show that the maximum of $B_\lambda(T)$ occurs at

$$\lambda_{\rm max} T = 0.290 \text{ cm K}$$

   To help you along, note that the equation $y = 5(1 - e^{-y})$ is solved by $y \approx 4.965$.

   (b) Show that the integral of the Planck function over all wavelengths is given by

$$\int_0^\infty B_\lambda(T)\, d\lambda = \frac{2\pi^4 k^4}{15 c^2 h^3} T^4$$

   Use the fact that $\int_0^\infty y^3/(e^y - 1)\, dy = \pi^4/15$.

   (c) Explain how and why the term that multiplies T^4 in your result above is related to the Stefan-Boltzmann constant.

2. (adapted from Lamers & Levesque, problem 2.3) The star τ Sco has an apparent visual magnitude $m_V = +2.8$. It is at a distance of 470 light years. Its spectral type is B0V, corresponding to an effective temperature $T_{\rm eff} = 30{,}000$ K and a bolometric correction of -3.2 mag. Determine the following quantities for this star:

   (a) its peak wavelength, $\lambda_{\rm max}$
   (b) its absolute visual magnitude, M_V
   (c) its distance modulus, μ
   (d) its apparent and absolute bolometric magnitudes, $m_{\rm bol}$ and $M_{\rm bol}$
   (e) its bolometric luminosity in physical units (erg/s or W) and in solar units ($L_\odot$)
   (f) its radius in physical units (cm or m) and in solar units ($R_\odot$)
   (g) the bolometric flux (erg/cm^2/s or W/m^2) at its surface
   (h) its bolometric flux at the Earth (erg/cm^2/s or W/m^2); also compare this to the bolometric flux from the Sun at the Earth (the total solar irradiance).

3. On our Canvas site, under Files $\rightarrow$ Problem Set Resources, find a file called `HD37962.txt`. (There is also a CSV version of the same file.) This contains spectroscopic observations of the star HD 37962 taken with the Hubble Space Telescope; the file tabulates wavelength λ (in Angstroms) and observed flux density f_λ (in erg/cm^2/sec/Angstrom).

   (a) Using any plotting program or spreadsheet, plot the spectrum (f_λ versus λ). Be sure to label your axes, with correct units.

   (b) Using trial and error (or perhaps the Wien displacement law...), overplot a reasonable blackbody fit (Planck spectrum, B_λ) for this star. Label this curve with the temperature you used. You will need to multiply the Planck spectrum by some scale factor to get it close to the stellar spectrum. Use trial and error for this. **Be careful with units!** Make sure your B_λ is in erg/cm^2/sec/Angstrom/sterradian before multiplying by the scale factor (which will have units of sterradians) to match the data.

   (c) The star HD 37962 has a parallax of 28.63 milliarcsec (measured by the Gaia satellite). What is its distance in parsecs? in cm?

   (d) In class we saw that the radiant flux density at the surface of a spherical blackbody is $F_\lambda = \pi B_\lambda$. The luminosity per unit wavelength is the flux density times the surface area, $L_\lambda = 4\pi R^2 F_\lambda$. The measured flux density, f_λ, for an observer a distance d away is given by the inverse square law

   $$f_\lambda = \frac{L_\lambda}{4\pi d^2} = \frac{4\pi R^2 F_\lambda}{4\pi d^2} = F_\lambda \left(\frac{R}{d}\right)^2 = \pi B_\lambda \left(\frac{R}{d}\right)^2$$

   This tells us the scale factor to go from the Planck function B_λ to the measured flux density f_λ is $\pi(R/d)^2$. Use the scale factor you estimated in part (b) and the distance from part (c) to derive the radius of HD 37962. Give your answer in cm and solar units ($R_\odot$).

   (e) From the data, estimate the apparent V magnitude (m_V), the apparent B magnitude (m_B), and the $B-V$ color ($m_B - m_V$) for HD 37962. There are two ways to do this:

      i. Simpler: directly use the measured flux density near the central wavelengths of the bands (B: 4380 Å; V: 5450 Å).

      ii. Better: measure an *average* flux density through the passband, e.g.,

      $$\langle f_\lambda \rangle = \frac{\int f_\lambda \, S_\lambda \, d\lambda}{\int S_\lambda \, d\lambda}$$

      where S_λ is the passband response function (posted on Canvas as `B.txt` and `V.txt`, or CSV versions). You will get extra credit if you do it this way (numerical integration).

With either method, you will need the *flux zeropoint* of each band, the flux density that corresponds to $m = 0$. The flux zeropoints are $f_0(B) = 6.32 \times 10^{-9}$ erg/cm^2/sec/Å and $f_0(V) = 3.63 \times 10^{-9}$ erg/cm^2/sec/Å.

(f) Given your results in this problem, can you explain why HD 37962 is called a *solar analog*?

4. Let's get some practice working with solid angles.

   (a) Integrate over the unit sphere to show that the total solid angle (of the whole sky, for example) is 4π sterradians. *Hint:* recall that in spherical coordinates $d\Omega = \sin\theta \, d\theta \, d\phi$ (see Lecture 1, slide 13).

   (b) A *square degree* is a unit of solid angle one degree on each side. How many sterradians is this? *Hint:* just convert each side to radians and multiply.

   (c) How many square degrees cover the entire sky? The field of view of the upcoming Vera C. Rubin Observatory camera is 9.6 deg^2. Approximately how many pointings of this camera would be required to tile the entire sky? How about with the Near Infrared Camera (NIRCam) on the just-launched JWST, with a field of view of 9.7 arcmin^2 (square arcminutes)?

5. (Based on Phillips, problem 1.1) Consider a sphere of mass M and radius R in hydrostatic equilibrium. Derive the following quantities:

   - the enclosed mass $m(r)$,
   - the gravitational potential energy (U or $E_{\rm pot}$) in terms of M and R,
   - the average internal pressure $\langle P \rangle$ in terms of M and R, and
   - the pressure profile $P(r)$

   for a constant density profile, $\rho(r) = \rho$ for $0 \leq r \leq R$ (and zero density for $r > R$).

6. **(Required for 541; extra credit for 441)** Repeat the problem above, for a somewhat more realistic density profile that increases linearly towards the center, with

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right).$$