Physics 441/541 Spring 2022: Problem Set #4 Solutions

1. *On Problem Set #2, we looked at stellar models from Ekström et al. (2012, A&A, 537, 146), and we will use these again here. You can access the paper via* `https://ui.adsabs.harvard.edu/abs/2012A%26A...537A.146E/abstract`*. A direct link to the PDF is* `https://www.aanda.org/articles/aa/pdf/2012/01/aa17751-11.pdf`).

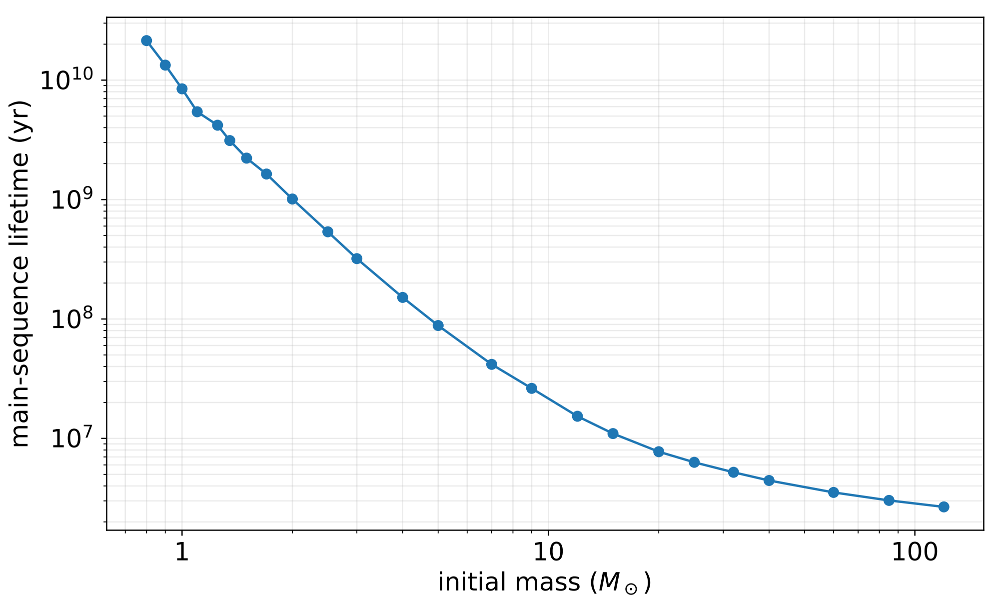
   (a) *I encourage you to skim through the whole paper! For now we need to grab one important piece of information: what is the initial hydrogen mass fraction X for the stellar models? (See the beginning of section 2 in the article.)*

   The initial hydrogen mass fraction is $X = 0.720$.

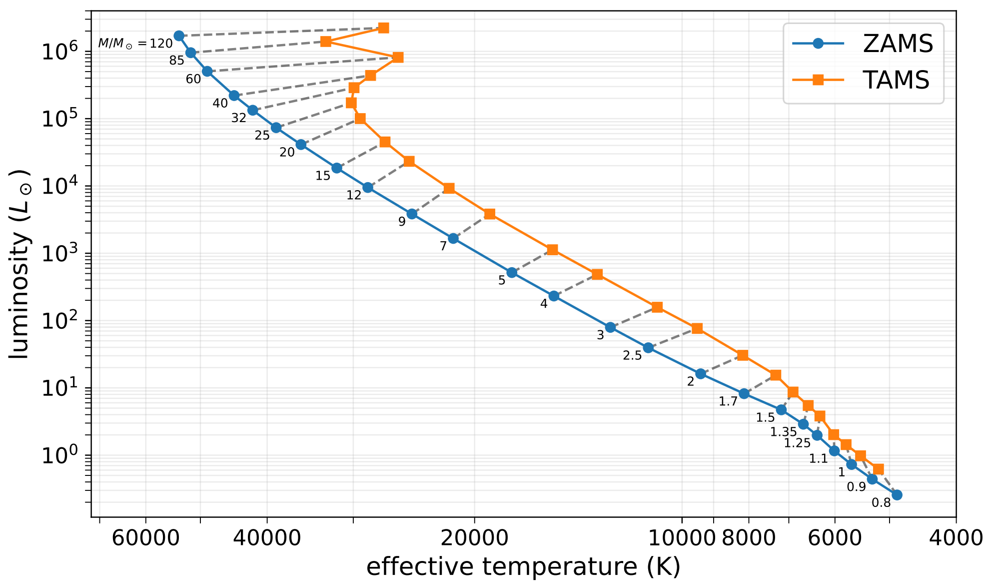
   From this paper, I have downloaded data for the (non-rotating) stellar models at two points in each star's evolution: the ***zero-age main sequence*** *(ZAMS) when hydrogen fusion starts in the core and the* ***terminal-age main sequence*** *(TAMS, also called the "turn-off") when the star runs out of hydrogen to fuse in the center. You can grab the data either at this Google sheet or on Canvas under Files → Problem Set Resources as zams-tams.xlsx or zams-tams.csv. The data give the initial mass for each star (in solar masses) and the time (in years after formation), the $\log_{10}$ of the luminosity in solar units: $\log_{10}(L/L_\odot)$, and the $\log_{10}$ of the effective temperature in Kelvin, for both the ZAMS (start of the main-sequence) and TAMS (end of the main sequence).*

   (b) *Using this data, make a plot of the stars' main-sequence lifetime (between ZAMS and TAMS) as a function of their initial mass. Use logarithmic x and y axes.*

   I've made a python notebook at `http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps04q1.ipynb` for this problem.



(c) *Make an H-R diagram of this data with log Te on the x-axis (increasing to the left) and log L on the y-axis. Plot the ZAMS data with one set of symbols/color and the TAMS data with another set of symbols/color, all on the same plot.*

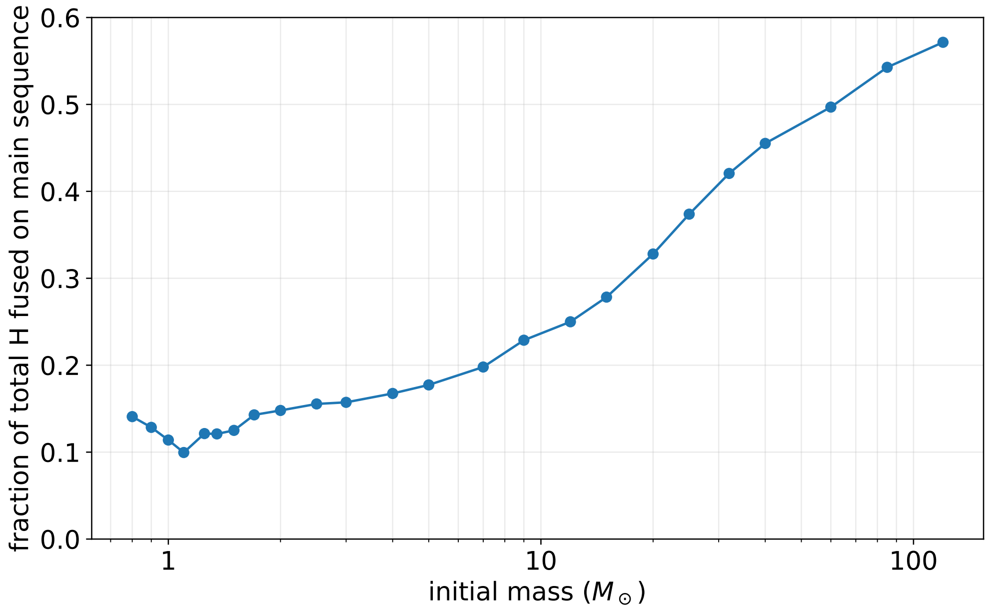


(d) *Let's look at what fraction of the star's hydrogen is fused on the main sequence. First calculate the average luminosity of each star during this time, as*

$$(\log L)_{\text{avg}} = \left[\frac{(\log L)_{\text{ZAMS}} \;+\; (\log L)_{\text{TAMS}}}{2}\right] \qquad L_{\text{avg}} = 10^{(\log L)_{\text{avg}}}$$

Then calculate the total energy released during the main sequence for each star, by multiplying this average luminosity by the main-sequence lifetime from part (b). Next, use this to estimate the mass of hydrogen fused on the main sequence by each star. Finally divide this mass by the total starting hydrogen mass of the star, making use of the initial mass and hydrogen mass fraction from part (a). Make a plot of the fraction of each star's hydrogen that is fused on the main sequence versus stellar mass (use a logarithmic x-axis). Comment on your results.

See the notebook for the details of the calculation. In a nutshell, we multiply the average luminosity (energy/time) by the main sequence duration from part (b) to get the total energy released on the main sequence. We convert this into a mass of hydrogen fused using the efficiency $\epsilon \approx 0.007$, and then compare to the total hydrogen mass of the star ($M_{\text{ini}} \times X$).



We see that solar mass stars fuse about 10% of their hydrogen on the main sequence, and this fraction increases with mass. Those stars have higher central temperatures, so there is more mass of the star above the $\sim 10^7$ K temperature required for hydrogen fusion. In addition, the more massive stars have convective cores, causing more hydrogen to be brought into the fusion region.

2. *The data below come from two locations in the interior of a single main-sequence stellar model, tabulating the radius, enclosed mass, enclosed luminosity, temperature, density, and opacity at the specified locations:*

r	$m(r)$	$L(r)$	$T(r)$	$\rho(r)$	$\kappa(r)$
$0.242\ R_\odot$	$0.199\ M_\odot$	$340\ L_\odot$	2.52×10^7 K	1.88×10^4 kg m^{-3}	0.044 m^2 kg^{-1}
$0.670\ R_\odot$	$2.487\ M_\odot$	$528\ L_\odot$	1.47×10^7 K	6.91×10^3 kg m^{-3}	0.059 m^2 kg^{-1}

(a) *Is energy transport at each of these two locations in the star radiative or convective? You may assume that radiation pressure is negligible, so that the gas behaves like a monoatomic ideal gas. Take the mean molecular weight to be $\mu = 0.7$.*

Recall that the Schwarzschild criterion says convection occurs when the temperature gradient exceeds (is more negative) than the "adiabatic" temperature gradient

$$\left|\frac{dT}{dr}\right| > \left|\frac{dT}{dr}\right|_{\rm ad} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|$$

We are not given the pressure in the table, but we are told to assume an ideal gas, so we can write $P = \rho kT/\mu m_p$ and use hydrostatic equilibrium $dP/dr = -\rho g = -Gm/r^2$. With this, the adiabatic temperature gradient is

$$\left|\frac{dT}{dr}\right|_{\rm ad} = \frac{\gamma - 1}{\gamma} T \left(\frac{\mu m_p}{\rho k T}\right) \rho g = \frac{\gamma - 1}{\gamma}\left(\frac{\mu m_p}{k}\right) g = \frac{\gamma - 1}{\gamma}\left(\frac{\mu m_p}{k}\right)\left(\frac{Gm}{r^2}\right)$$

The radiative temperature gradient is

$$\left|\frac{dT}{dr}\right|_{\rm rad} = \frac{3}{4ac}\frac{\rho\kappa}{T^3}\frac{L}{4\pi r^2}$$

Taking $\mu = 0.7$ as given and $\gamma = 5/3$ for a monoatomic non-relativistic gas (so $(\gamma-1)/\gamma = 2/5 = 0.4$), we can plug in constants and values for the inner location to find

$$\left|\frac{dT}{dr}\right|_{\rm rad} = \frac{3(1.88\times10^4\ {\rm kg\ m^{-3}})(0.044\ {\rm m^2\ kg^{-1}})(340\times3.83\times10^{26}\ {\rm J\,sec^{-1}})}{4(7.5657\times10^{-16}\ {\rm J\ m^{-3}\ K^{-4}})(3.0\times10^8\ {\rm m\,sec^{-1}})(2.52\times10^7\ {\rm K})^3(4\pi)(0.242\times6.96\times10^8\ {\rm m})^2}$$

$$\Rightarrow \left|\frac{dT}{dr}\right|_{\rm rad} = 0.062\ {\rm K\ m^{-1}}$$

$$\begin{aligned}\left|\frac{dT}{dr}\right|_{\rm ad} &= 0.4\frac{(0.7)(1.67\times10^{-27}\ {\rm kg})(6.67\times10^{-11}\ {\rm m^3\ kg^{-1}\,sec^{-2}})(0.199\times1.99\times10^{30}\ {\rm kg})}{(1.38\times10^{-23}\ {\rm kg\ m^2\,sec^{-2}\ K^{-1}})(0.242\times6.96\times10^8\ {\rm m})^2}\\ &= 0.032\ {\rm K\ m^{-1}}\end{aligned}$$

So we see that the radiative temperature gradient would be larger (steeper) than the adiabatic temperature gradient, and so the star would be convective here.

Repeating this calculation with the values further out in the star ($r = 0.670\ R_\odot$), we find

$$\left|\frac{dT}{dr}\right|_{\rm rad} = 0.031\ {\rm K\ m^{-1}} \qquad \left|\frac{dT}{dr}\right|_{\rm ad} = 0.051\ {\rm K\ m^{-1}}$$

and here the energy transport is by radiation (radiative diffusion), not convection.

(b) *Based on your results, what kind of star is this?*

This main-sequence star has a convective core, so it must be somewhat massive (with CNO cycle fusion; see lecture 7 slide 9). Indeed the enclosed mass in the outer position is $m(r) \approx 2.5\ M_\odot$, so of course the star must have at least that mass. In fact, it turns out these points come from a $M = 5\ M_\odot$ model.

3. *Let's examine the structure of the Sun after the main sequence when it moves onto the red giant branch (RGB) of the H-R diagram. As a red giant, the Sun will have a dense, degenerate helium core surrounded by a low density envelope.*

(a) *The gravitational potential energy of a sphere with constant density is given by $\Omega = U = E_{\rm pot} = -\frac{3}{5}GM^2/R$. Calculate this value for the Sun today, assuming a constant density. Also calculate it for when the Sun expands as a red giant to $R \approx 100\ R_\odot$ (again assuming a constant density, and ignoring any mass loss).*

Plugging in values we get

$$\begin{aligned}U_\odot &= -\frac{3}{5}\frac{(6.67\times10^{-8}\ {\rm cm^3\ g^{-1}\,sec^{-2}})(1.99\times10^{33}\ {\rm g})^2}{6.96\times10^{10}\ {\rm cm}} = -2.3\times10^{48}\ {\rm erg}\\ U_{\rm RG} &= \frac{U_\odot}{100} = -2.3\times10^{46}\ {\rm erg}\end{aligned}$$

(b) *The virial theorem says that in equilibrium, the total energy of the star is given by* $E_{\rm tot} = U/2$. *How much does the total energy change between the Sun today and the red-giant Sun based on your calculation above? Does the Sun's total energy really change by this much or have we made a major error in our assumptions? Explain.*

The virial theorem says $\langle U \rangle = -2\langle K \rangle$, so $E_{\rm tot} \approx U + K = U - U/2 = U/2$, assuming we are in equilbrium and so the instantaneous energies are approximately equal to their average values. The total energy is negative for bound systems with

$$E_\odot = U_\odot/2 = -1.1 \times 10^{48} \text{ erg} \qquad E_{\rm RG} = -1.1 \times 10^{46} \text{ erg}$$

If this were right, the Sun would somehow have to *increase* its total energy by a large factor. Our faulty conclusion is due to the assumption of a constant density. Indeed as the Sun becomes a red giant its core *shrinks* while the envelope expands; this means the density contrast between the core and envelope increases even further compared to what it is today.

4. *The radius of a non-relativistic, fully degenerate star is given by*

$$R = 2.7 \times 10^4 \; \mu_e^{-5/3} \left(\frac{M}{M_\odot}\right)^{-1/3} \text{ km} \tag{1}$$

where μ_e *is the mean molecular weight per free electron,* $\mu_e = \rho/(n_e m_p)$.

(a) *Show that for fully ionized material, we can approximate* $\mu_e = 2/(1+X)$, *where* X *is the hydrogen mass fraction.*

Assuming things are *fully* ionized, then each hydrogen atom contributes 1 electron and 1 proton (with mass $\approx m_p$), each helium atom 2 electrons and one helium nucleus (with mass $\approx 4\,m_p$), etc. For a "metal" with atomic mass $A\,m_p$, we get approximately $A/2$ electrons. (It's fine if you only considered hydrogen and helium for this problem.)

The electrons contribute negligible mass, and there is one electron for every proton, so we can write $n_e \approx n_{\rm H} + 2n_{\rm He} + (A/2)n_{\rm metals}$. The mass density fraction is X for hydrogen, Y for helium, and Z for other metals. Also we know that $X + Y + Z = 1$ so that $Y + Z = (1 - X)$. Thus we can write

$$\frac{n_e m_p}{\rho} \approx X + 2(Y/4) + (A/2)(Z/A) = X + Y/2 + Z/2 = X + (1-X)/2 = \frac{1+X}{2}$$

And the reciprocal gives us $\mu_e = \rho/(n_e m_p) \approx 2/(1+X)$.

(b) *Assume that the red giant Sun has a pure helium core with a mass* $M = 0.25\ M_\odot$. *What is the radius of the core? What is its average density? Compare your result to the average density of the Sun today* ($\bar{\rho}_\odot = 1.4$ g cm^{-3}) *and the central density of the Sun today* ($\rho_c \approx 150$ g cm^{-3}).

Taking just the core to be fully degenerate and setting $\mu_e \approx 2/(1+X) = 2$ for fully ionized pure helium, we get

$$\begin{aligned} R &= 2.7 \times 10^4\ (2)^{-5/3}\ (0.25)^{-1/3}\ \text{km} = 1.4 \times 10^4\ \text{km} \\ \bar{\rho}_{\text{core}} &= \frac{3M}{4\pi R^3} = \frac{3 \times 0.25 \times 1.99 \times 10^{33}\ \text{g}}{4\pi(1.4 \times 10^9\ \text{cm})^3} = 4.3 \times 10^4\ \text{g cm}^{-3} \end{aligned}$$

This density is more than four orders of magnitude higher than today's Sun's average density, and about 290 times the Sun's current central density.

(c) *What is the average density of the envelope (everything outside the helium core) of the red giant Sun? Compare this to the typical density of air at Earth's surface.*

The envelope mass is the total mass minus the core mass: $1 - 0.25 = 0.75\ M_\odot$. Similarly, the volume of the envelope is the volume of the star minus the volume of the core:

$$\begin{aligned} V_{\text{env}} &= \frac{4\pi}{3}\left(R^3 - R_{\text{core}}^3\right) \\ &= \frac{4\pi}{3}\left[\left(100 \times 6.96 \times 10^{10}\ \text{cm}\right)^3 - \left(1.4 \times 10^9\ \text{cm}\right)^3\right] = 1.41 \times 10^{39}\ \text{cm}^3 \end{aligned}$$

Note that the volume of the core is negligible, $V_{\text{env}} \approx V_\star$. The average density is

$$\bar{\rho}_{\text{env}} = \frac{0.75 \times 1.99 \times 10^{33}\ \text{g}}{1.41 \times 10^{39}\ \text{cm}^3} = 1.1 \times 10^{-6}\ \text{g cm}^{-3}$$

This is 10 orders of magnitude less than the average core density! So you can see that assuming a constant density for the whole star is a terrible assumption. The density of air at the Earth's surface[1] is approximately 1.2×10^{-3} g cm^{-3}, meaning the red giant envelope is a thousand times less dense.

5. *The equation of state for an ultra-relativistic white dwarf is*

$$P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3}$$

(a) *Replace the electron number density to rewrite the pressure in terms of the mass density ρ and mean molecular weight per free electron μ_e. The result is a polytrope, with $P = K\rho^\gamma$. What are K, γ, and the polytropic index n?*

Recall from problem 4a above, the definition $\mu_e = \rho/(n_e m_p)$, so $n_e = \rho/(\mu_e m_p)$, and we can write

$$P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3}\left(\frac{1}{\mu_e m_p}\right)^{4/3}\rho^{4/3}$$

In polytropic form this gives

$$K = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3}\left(\frac{1}{\mu_e m_p}\right)^{4/3} \qquad \gamma = 4/3 \qquad n = \frac{1}{\gamma - 1} = 3$$

[1] https://en.wikipedia.org/wiki/Density_of_air

(b) *Recalling our previous results about polytropes, show that there is a unique value of the mass of such a star, with*

$$M \approx 2.02 \left(\frac{\sqrt{3\pi}}{2}\right)\left(\frac{1}{\mu_e m_p}\right)^2 \left(\frac{\hbar c}{G}\right)^{3/2}$$

Hint: *look at Lecture 3, slide 15. Derive α and plug in to the expression for M.*

Following the hint, we have these relations for polytropes:

$$M = 4\pi\alpha^3\rho_c \left[-\xi^2 \frac{d\theta}{d\xi}\right]_{\xi=\xi_1} \qquad \text{and} \quad \alpha^2 = \frac{K(n+1)\rho_c^{\frac{1-n}{n}}}{4\pi G}$$

So we get

$$\begin{aligned}
\alpha^2 &= \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3}\left(\frac{1}{\mu_e m_p}\right)^{4/3}\frac{(4)\rho_c^{-2/3}}{4\pi G} = \frac{hc}{4\pi G}\left[\frac{3}{8\pi}\right]^{1/3}\left(\frac{1}{\mu_e m_p}\right)^{4/3}\rho_c^{-2/3} \\
\Rightarrow \alpha^3 &= \left[\frac{hc}{4\pi G}\right]^{3/2}\left[\frac{3}{8\pi}\right]^{1/2}\left(\frac{1}{\mu_e m_p}\right)^2 \rho_c^{-1} \\
\Rightarrow M &= 4\pi\left[\frac{hc}{4\pi G}\right]^{3/2}\left[\frac{3}{8\pi}\right]^{1/2}\left(\frac{1}{\mu_e m_p}\right)^2 \rho_c^{-1}\rho_c(2.02) \\
&= (2.02)4\pi\left[\frac{\hbar c}{2G}\right]^{3/2}\left[\frac{3}{8\pi}\right]^{1/2}\left(\frac{1}{\mu_e m_p}\right)^2 = 2.02\left(\frac{\sqrt{3\pi}}{2}\right)\left(\frac{1}{\mu_e m_p}\right)^2\left(\frac{\hbar c}{G}\right)^{3/2}
\end{aligned}$$

where we see the central density ρ_c cancels, and have used the Lame-Emden equation constant $M_3 = 2.02$ (see L & L Table 11.1 or Lecture 3, slide 16 or 18).

(c) *Plug in constants to derive the Chandrasekhar limit for a carbon-oxygen white dwarf, $M_{\mathrm{Ch}} \approx 1.44\ M_\odot$.*

For a C/O white dwarf, and assuming full ionization, $\mu_e \approx 2/(1+X) \approx 2$. So

$$\begin{aligned}
M &= \left(\frac{2.02\sqrt{3\pi}}{2[2(1.67\times10^{-24}\ \mathrm{g})]^2}\right)\left(\frac{(1.05\times10^{-27}\ \mathrm{cm^2\ g\,sec^{-1}})(3.0\times10^{10}\ \mathrm{cm\,sec^{-1}})}{6.67\times10^{-8}\ \mathrm{cm^3\ g^{-1}\ sec^{-2}}}\right)^{3/2} \\
&= 2.85\times10^{33}\ \mathrm{g} = 1.43\ M_\odot
\end{aligned}$$

6. (Required for 541; extra credit for 441) *Use the procedure from problem 5, except this time for a non-relativistic, fully-degenerate equation of state*

$$P = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} n_e^{5/3}$$

to determine the mass-radius relation for non-relativistic white dwarfs. As in that problem, derive your result with symbols first. Then at the end plug in the constants to get a numerical value that you can compare to equation (1) of problem 4.

We can rewrite the pressure in terms of the mass density

$$P = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3}\left(\frac{1}{\mu_e m_p}\right)^{5/3}\rho^{5/3} \;\Rightarrow\; K = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3}\left(\frac{1}{\mu_e m_p}\right)^{5/3}$$

with $\gamma = 5/3$ and thus $n = 1/(\gamma - 1) = 3/2$. The polytropic equations can be written

$$R = \alpha\xi_1 \quad M = 4\pi\alpha^3\rho_c\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1} \qquad \alpha^2 = \frac{K(n+1)\rho_c^{\frac{1-n}{n}}}{4\pi G}$$

We need to combine these, getting rid of α and ρ_c to get the mass-radius relationship. For $n = 3/2$ we have

$$\begin{aligned}
\alpha^2 &= \frac{K(5/2)\rho_c^{-1/3}}{4\pi G} \quad\Rightarrow\quad \rho_c = \left(\frac{5K}{8\pi G\alpha^2}\right)^3 \\
\Rightarrow M &= 4\pi\alpha^3\left(\frac{5K}{8\pi G\alpha^2}\right)^3\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1} = \frac{4\pi\xi_1^3}{R^3}\left(\frac{5K}{8\pi G}\right)^3\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1} \\
\Rightarrow R &= \left(\frac{4\pi}{M}\right)^{1/3}\left(\frac{5K\xi_1}{8\pi G}\right)\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}^{1/3}
\end{aligned}$$

Plugging in K from above, and $\xi_1 = 3.65$ and $\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1} = 2.71$ for $n = 3/2$ (see Lecture 3, slide 18 and L&L Table 11.1), we get

$$\begin{aligned}
R &= \left(\frac{4\pi}{M}\right)^{1/3}(3.65)(2.71)^{1/3}\left(\frac{5}{8\pi G}\right)\frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3}\left(\frac{1}{\mu_e m_p}\right)^{5/3} \\
&= 5.09\left(\frac{9}{16\pi}\right)^{1/3}\frac{h^2}{8\pi G m_e}\left(\frac{1}{\mu_e m_p}\right)^{5/3} M^{-1/3}
\end{aligned}$$

Plugging in numbers, we get

$$\begin{aligned}
R &= 5.09\left(\frac{9}{16\pi}\right)^{1/3}\frac{(6.626\times10^{-27}\ \mathrm{cm^2\ g\,sec^{-1}})^2}{8\pi(6.67\times10^{-8}\ \mathrm{cm^3\ g^{-1}\,sec^{-2}})(9.11\times10^{-28}\ \mathrm{g})} \\
&\quad\times\left(\frac{1}{\mu_e(1.67\times10^{-24}\ \mathrm{g})}\right)^{5/3}\left(1.99\times10^{33}\ \mathrm{g}\right)^{-1/3}\left(\frac{M}{M_\odot}\right)^{-1/3} \\
&= 2.8\times10^4\ \mu_e^{-5/3}\left(\frac{M}{M_\odot}\right)^{-1/3}\ \mathrm{km}
\end{aligned}$$

which is close to Equation 1 that we were using.