Physics 441/541 Spring 2022: Problem Set #3
due February 25 at 11:00 am in PDF format on Canvas

You are encouraged to work in groups on these problems, but you must write up the solutions individually. In your writeup, list your collaborators and cite any external sources you used. You **may not** consult previous solution sets for this class or other similar classes.

0. Identify the people you will work with on the group project for the end of the semester, where you will present an approximately 20-minute lecture on a topic. The groups should consist of three people (a few two-person groups may be necessary). Provide your top three choices for a group project topic. There is a list of potential topics on the course website, but you can also come up with your own ideas.

1. On our Canvas site, under Files–Problem Set Resources, you will find a file called `stellarmodel-ps03.txt` that gives the density ρ (in g cm^{-3}) and temperature T (in K) of a star as function of radius r (in cm). There is also a comma-separated-value file `stellarmodel-ps03.csv` with the same data, if that is more convenient for you.

   (a) Using the density profile $\rho(r)$ calculate and plot the enclosed mass $m(r)$ for the star from the center to the surface. Use solar units ($R_\odot$, $M_\odot$) for the x and y axes, respectively.

   (b) Calculate and plot the local gravitational acceleration $g(r)$; use CGS units for this quantity (keep the x-axis in $R_\odot$).

   (c) Assume hydrostatic equilibrium to derive and plot the pressure profile $P(r)$ in CGS units (keep the x-axis in $R_\odot$ again and remember to use the correct boundary condition; see Problem Set 1).

   (d) Assuming an ideal gas throughout the star, plot the mean molecular weight $\mu(r)$ versus radius (keep the x-axis in $R_\odot$)

   (e) Show, on a single plot versus radius on the x-axis (again in units of $R_\odot$), the quantities:
   $$\left|\frac{dT}{dr}\right| \quad \text{and} \quad \left|\frac{dT}{dr}\right|_{\text{ad}} \equiv \frac{\gamma-1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|$$
   in CGS units. Use an appropriate value for γ. On your plot, mark the region of the star that is convective.

   (f) Based on all of your results, what kind of star is this? Be as specific as you can.

2. (a) (adapted from Phillips 3.3) Recall that the radiation pressure for a blackbody is given by $P_{\rm rad} = aT^4/3$. Write an expression for the radiation pressure gradient $dP_{\rm rad}/dr$ in terms of the temperature gradient dT/dr.

   (b) Near the surface of a star with luminosity L and radius R, the radiative temperature gradient is given by

$$\left[\frac{dT}{dr}\right]_{\rm rad} = -\frac{3\rho\kappa}{4acT^3}\frac{L}{4\pi R^2}$$

   Show that this implies a radiation pressure gradient

$$\left[\frac{dP}{dr}\right]_{\rm rad} = -\frac{\rho\kappa}{c}\frac{L}{4\pi R^2}$$

   (c) Show that the maximum luminosity that the star can have and still be in hydrostatic equilibrium for a surface gravity $g = GM/R^2$ is $L_{\rm max} = 4\pi GMc/\kappa$.

   (d) Assuming an ionized hydrogen atmosphere with the opacity dominated by electron scattering, show that this maximum luminosity corresponds to the *Eddington luminosity* (or the *Eddington limit*)

$$L_{\rm Edd} = \frac{4\pi GMm_pc}{\sigma_T} = 1.26\times10^{38}\ {\rm erg\ s^{-1}}\times\left(\frac{M}{M_\odot}\right) = 3.27\times10^4\ L_\odot\times\left(\frac{M}{M_\odot}\right)$$

   Above this limit the radiation pressure is strong enough to push away any ionized hydrogen; this is very important in regulating *accretion* onto astrophysical objects.

3. The central temperature of the Sun is $T_c \approx 1.5\times10^7$ K.

   (a) Use the radius of the Sun to (very roughly) approximate the average temperature gradient dT/dr in the Sun. Then, using the Sun's luminosity, estimate the average opacity in the Sun's interior. How does this average opacity compare to the electron scattering opacity $\kappa_{\rm es}$?

   (b) Based on the average opacity, what is the mean free path ℓ for a photon in the Sun's interior? In radiative diffusion, photons undergo a *random walk* where after N interactions they travel an average distance $d = \ell\sqrt{N}$. How many interactions does a typical photon have getting from the center of the Sun to the surface? How long does this take? How does the time compare to the light travel time in the absence of interactions?

4. Let's look at some nuclear reactions.

   (a) For each of these there is a mystery particle X or missing value y. Use conservation laws to determine X and y in each case (and explain the reasoning for your answers).

      i. ${}^{8}_{5}\mathrm{B} \rightarrow {}^{y}_{4}\mathrm{Be} + \nu_e + X$
      ii. $\gamma + {}^{28}_{14}\mathrm{Si} \rightarrow {}^{24}_{y}\mathrm{Mg} + X$
      iii. ${}^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} \rightarrow {}^{31}_{y}\mathrm{P} + X$
      iv. ${}^{21}_{10}\mathrm{Ne} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{24}_{12}\mathrm{Mg} + X$
      v. ${}^{27}_{14}\mathrm{Si} \rightarrow {}^{y}_{13}\mathrm{Al} + e^{+} + X$

   (b) The energy released (or absorbed) in a nuclear reaction is the Q value (defined as positive if energy is released). Calculate the Q values (in MeV) for each of the following reactions; indicate whether energy is released or absorbed. You should look up the nuclear masses or binding energies online; cite your source(s).

      i. ${}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{2}_{1}\mathrm{H} + e^{+} + \nu_e$
      ii. ${}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}$
      iii. ${}^{19}_{9}\mathrm{F} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{16}_{8}\mathrm{O} + {}^{4}_{2}\mathrm{He}$
      iv. ${}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{8}_{4}\mathrm{Be}$
      v. ${}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \rightarrow {}^{16}_{8}\mathrm{O} + {}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He}$

5. The goal of this problem is to understand how the fusion reaction rate depends on temperature. We can write the reaction rate as (see Phillips eqn. 4.22)

$$r_{AB} = n_A n_B \left[\frac{8}{\pi m_r}\right]^{1/2} \left[\frac{1}{kT}\right]^{3/2} \int_0^\infty S(E)\, e^{-f(E)}\, dE \quad \text{where} \quad f(E) = \frac{E}{kT} + \left(\frac{E_G}{E}\right)^{1/2}$$

The function $e^{-f(E)}$ is sharply peaked near the Gamow peak E_0. Assuming $S(E)$ is reasonably constant near the Gamow peak, we pull it out of the integral and write

$$r_{AB} = n_A n_B \left[\frac{8}{\pi m_r}\right]^{1/2} \left[\frac{1}{kT}\right]^{3/2} S(E_0) \int_0^\infty e^{-f(E)}\, dE$$

Our task is to estimate the remaining integral. Here are the steps to do that. *Warning:* The algebra can get a bit messy for this problem; don't get discouraged!

   (a) Recall that the Taylor series approximation of a function $g(x)$ around the point $x = a$ is given by

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \ldots \;=\; \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(x-a)^n$$

   Expand the function $f(E)$ as a Taylor series around the Gamow peak, $E = E_0$. This means you can approximate f with the form

$$f(E) \approx b_0 + b_1(E - E_0) + b_2(E - E_0)^2 + \ldots$$

What you will need to do is determine the coefficients b_0, b_1, and b_2. *Hint.* In your derivation, you should find that $b_1 = 0$ because $f(E)$ is minimized at E_0.

(b) Using the approximate form for f, and with $y = E - E_0$, the integral becomes

$$\int_0^\infty e^{-f(E)}\, dE \approx e^{-b_0} \int e^{-b_2\, y^2}\, dy$$

(I am not being very careful about the limits of integration, because as long as they are far from the peak they don't matter very much.) The integrand is now a Gaussian, and the integral of a Gaussian is simple:

$$\int e^{-b_2\, y^2}\, dy = \left(\frac{\pi}{b_2}\right)^{1/2}$$

Use this with your values of b_0 and b_2 to write an approximation for r_{AB}.

(c) Now let's examine the temperature dependence of r_{AB}. As discussed in class, if we approximate the temperature dependence as a power law, $r_{AB} \propto T^\alpha$, we can find the power law index with any of the following:

$$\alpha = \frac{T}{r_{AB}}\frac{dr_{AB}}{dT} = T\,\frac{d\ln r_{AB}}{dT} = \frac{d\ln r_{AB}}{d\ln T}$$

Using your approximation for r_{AB}, show that

$$\alpha = \left(\frac{E_G}{4kT}\right)^{1/3} - \frac{2}{3}$$

(d) Using this expression, estimate α for

i. the p-p reaction in the Sun (take $T = 1.5 \times 10^7$ K), and for

ii. the ${}^4_2\mathrm{He} + {}^8_4\mathrm{Be}$ reaction that is part of the triple alpha process to make carbon (take $T = 10^8$ K).

6. (Required for 541; extra credit for 441)

(a) (adapted from Phillips problem 4.1) Find the classical distance of closest approach for two protons with an energy of approach equal to 2 keV. Estimate the probability that the protons penetrate the Coulomb barrier tending to keep them apart. Compare this probability with the corresponding probability for two ^{4}He nuclei with the same energy of approach. By what factor would you need to increase the approach energy (or temperature) for the ^{4}He nuclei to have the same probability as the p-p case?

(b) (adapted from Phillips problem 4.3) Assume that the solar luminosity of 3.8×10^{26} W is due to hydrogen fusion by the p-p chain illustrated in Figure 4.4. How many neutrinos per second are emitted by the Sun? From this, calculate the neutrino flux (number of neutrinos per cm^2 per second) at the Earth. What are the maximum neutrino energies (in MeV) for the PP I, PP II, and PP III branches?