Physics 441/541 Spring 2022: Problem Set #6
due Friday April 29 at 11:00 am in PDF format on Canvas

The Group Project assignment has been uploaded to Canvas. Presentations will start Friday April 22; see the posted schedule. **A draft of your presentation is due a week before your presentation date.**

Lecture Date	Group	Members: Topic
Apr 22 (Fri)	1	Haonan Cheng, Yoon Choi, Matthew Wang: Stellar Initial Mass Function
Apr 22 (Fri)	2	Frank Genty, Anthony Pizzarelli, Khovesh Ramdin: Brown Dwarfs
Apr 22 (Fri)	3	Barbara Benda, Avery Kiihne, Harshill Patel: First Stars and Reionization
Apr 26 (Tue)	4	George Kharchilava, Geet Purohit, Anish Seth: Exoplanet Host Stars
Apr 26 (Tue)	5	Ava Marie Friedrich, Seung Hee Sung: Helioseismology
Apr 26 (Tue)	6	Rujuta Mokal, Michael Wozniak, Orion Yeung: Standard Candles
Apr 29 (Fri)	7	Aidan Boyce, Kailash Raman: MESA code
Apr 29 (Fri)	8	Arya Lakshmanan, Ina Park, Brandon Shane: Magnetars
Apr 29 (Fri)	9	Bradley Butler, Christine Carvajal, Connor Lane: LIGO Black Holes

You are encouraged to work in groups on these problems, but you must write up the solutions individually. In your writeup, list your collaborators and cite any external sources you used. You **may not** consult previous solution sets for this class or other similar classes.

1. Consider a binary star system with masses M_1 and M_2 (and $M_1 > M_2$) in a circular orbit with separation a. We can write the total orbital kinetic energy as $K = M_1 v_1^2/2 + M_2 v_2^2/2 = \mu v^2/2$, where $\mu = M_1 M_2/(M_1 + M_2)$ is the reduced mass, and $v = v_1 + v_2$ is the relative orbital speed. We can also write the potential energy as $U = -GM_1M_2/a = -GM\mu/a$ where $M = M_1 + M_2$ is the total mass. The virial theorem applies here ($E_{\rm tot} = -K = U/2$); use it to solve for v in terms of the masses and separation.

   Now imagine that M_1 instantaneously (and spherically symmetrically) loses ΔM of its mass (e.g., in a supernova explosion), so that the relative velocity v and separation a are the same before and immediately after the explosion. For what values of ΔM does the system become unbound? *Hint:* recall that a bound system has $E_{\rm tot} < 0$.

2. (Lamers & Levesque problem 12.2)

   (a) Estimate the radii at the beginning and end of the Hayashi contraction phase and at the beginning and end of the pre-main-sequence contraction for stars of 0.1, 0.3, 1.0, 3, 10, 30, and 100 $M_\odot$. *Hint:* review L&L sections 12.6, 12.7, and 12.8.

   (b) Estimate the duration of the Hayashi contraction phase and of the pre-main-sequence contraction for stars of 0.1, 0.3, 1.0, 3, 10, 30, and 100 $M_\odot$. *Hint:* see L&L page 12-10, including eqn. 12.17.

3. **This problem is longer than the others so it will count for double the points.** Type Ia supernovae are the thermonuclear explosions of accreting white dwarfs that approach the Chandrasekhar limit. In this problem, consider an exploding white dwarf of mass 1.4 $M_\odot$ and radius 10^4 km.

   (a) The explosive fusion occurs in several steps, but the ultimate result is that carbon and oxygen are fused into nickel: $2\ {}^{12}_{6}\mathrm{C} + 2\ {}^{16}_{8}\mathrm{O} \rightarrow {}^{56}_{28}\mathrm{Ni}$. The atomic masses are 12 amu (for carbon-12), 15.994915 amu (for oxygen-16) and 55.942132 amu (for nickel-56). Assuming that the entire white dwarf is half carbon and half oxygen that fuses to nickel, how much total energy is released in the explosion?

   (b) What is the gravitational binding energy of the white dwarf? Assume a polytrope model with a relativistic equation of state (and look back through our class notes for a relatively simple formula for the gravitational potential energy of a polytrope with index n). Compare the explosive fusion energy with the binding energy.

   (c) The optical light we see from a type Ia supernova is produced mainly by the radioactive decay of ^{56}Ni, first to ^{56}Co, and then to ^{56}Fe. Let's examine the first step, from nickel to cobalt. This reaction releases energy, because the atomic mass of cobalt-56 is 55.939839 amu. Determine the total amount of energy, E_{decay}, released from the radioactive decay of all the nickel to cobalt.

   (d) The radioactive energy is not released all at once, because the half-life of this decay is $t_{1/2} = 6.075$ days. We can write the number of radioactive nickel atoms remaining at any time as $N(t) = N_0\, e^{-t/\tau}$, where N_0 is the number of radioactive nickel atoms at $t = 0$. The half-life is defined as $N(t_{1/2}) = N_0/2$. Use the half-life to determine τ.

   (e) The rate of radioactive energy release is the bolometric luminosity, with $L(t) = L_0\, e^{-t/\tau}$. Use your results from parts (b) and (c), and the fact that $E_{\mathrm{decay}} = \int_0^\infty L(t)\, dt$ to calculate the initial bolometric luminosity L_0 in units of $L_\odot$. What is the initial bolometric absolute magnitude?

   (f) According to this simplistic model, the type Ia supernova should be brightest at the time of explosion, but in actuality the supernova light curve rises for a few weeks because of its *opacity*. Review our class discussion of radiative diffusion to provide an explanation of the delayed peak.

   (g) Compare the energy released in a Type Ia supernova to the energy released in a core-collapse supernova (with a $\sim 1.4\ M_\odot$ iron core collapsing down to a neutron star with $R = 12$ km). Given your result, briefly explain how a typical Type Ia supernova can have a higher luminosity than a typical core-collapse supernova.

4. (Required for 541; extra credit for 441) On our Canvas site, under Files → Problem Set Resources, there is a handout called homology.pdf from Prof. Joe Shields at Ohio University. "Homology" is an approach to stellar structure where we approximate the relevant variables as power laws of density, temperature, etc., and grossly simplify all the derivatives, allowing us to derive interesting (though very approximate) scaling relations between stellar masses, luminosities, radii, etc. Read through this document before proceeding with this question.

(a) To apply this methodology, we need to parameterize the nuclear energy generation rate (per unit mass) as $\epsilon \propto \rho^{\alpha} T^{\beta}$. List α and β for

   i. hydrogen fusion via the *p*-*p* chain (see Lamers & Levesque, eqn. 8.13),

   ii. hydrogen fusion via the CNO cycle (see L&L, eqn. 8.14), and

   iii. helium fusion via the triple-α process (see L&L, eqn. 8.18a).

(b) Now, set up a matrix equation following equation 32 in the document, for each of the three cases above, and solve for α_R, α_ρ, α_L, and α_T for each case.

(c) This matrix approach seems a bit like magic, so let's double check things are making sense. We expect the interior temperature T to scale as M/R (recall our order of magnitude estimate for the central temperature was $T_c \approx GM\mu m_p/kR$). What relation does that imply between α_R and α_T? Is that relation satisfied by your results?

(d) We cannot directly observe the interior temperature, T. Rather, we observe the surface effective temperature T_{eff}. Use the scaling implied by the Stefan-Boltzmann law and your homology results to derive the scaling between effective temperature and mass for these three cases.

(e) Compare your results for the *mass-luminosity* relationship (given by α_L) with real observations of stars on the main sequence (e.g., L&L Figure 2.3). How well do your homology results work? Comment specifically on very low-mass stars, medium mass stars ($\sim 1\ M_\odot$), and higher-mass stars.