Physics 441/541 Spring 2022: Problem Set #2 Solutions

1. (a) *Show that the mass-radius relationship for a polytrope with index n is given by*

$$R \propto M^{(1-n)/(3-n)}$$

All the information we need is on the polytropes summary slide from class: Lecture 3, slide 15.

For a given value of n, we see that $R \propto \rho_c^{(1-n)/2n}$. The formula for M looks like it only has a linear dependence on ρ_c but if you look carefully, you will see that it also depends on α^3, and $\alpha^2 \propto \rho_c^{(1-n)/n}$. So we have

$$R \propto \rho_c^{\frac{1-n}{2n}} \quad M \propto \rho_c^{\frac{3}{2}\frac{1-n}{n}} \rho_c \propto \rho_c^{\frac{3}{2}\frac{1-n}{n}+1} \propto \rho_c^{\frac{3-3n}{2n}+1} \propto \rho_c^{\frac{3-n}{2n}}$$

Thus, if we raise M to the $(1-n)/(3-n)$ power we get the desired result

$$M^{\frac{1-n}{3-n}} \propto \rho_c^{\frac{3-n}{2n}\frac{1-n}{3-n}} \propto \rho_c^{\frac{1-n}{2n}} \propto R$$

(b) *Show that the ratio of the central density to the mean density of a polytrope of index n is given by*

$$\frac{\rho_c}{\bar{\rho}} = \rho_c \left(\frac{4\pi R^3}{3M}\right) = \frac{-\xi_1}{3\left(\frac{d\theta}{d\xi}\right)_{\xi_1}}$$

We can just plug in for R^3 and M:

$$R^3 = \alpha^3 \xi_1^3 \quad M = 4\pi\alpha^3\rho_c \left[-\xi^2 \frac{d\theta}{d\xi}\right]_{\xi=\xi_1} \Longrightarrow \frac{1}{\bar{\rho}} = \frac{4\pi R^3}{3M} = \frac{4\pi\alpha^3\xi_1^3}{3\left(-4\pi\alpha^3\rho_c\xi_1^2\left[\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}\right)}$$

So we arrive at

$$\frac{\rho_c}{\bar{\rho}} = \frac{-\xi_1}{3\left[\frac{d\theta}{d\xi}\right]_{\xi_1}}$$

(c) *Solutions to the Lane-Emden equation are given in the table below:*

n	ξ_1	$-\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}$
1.5	3.65	2.71
3	6.90	2.02

Using your result from part (b) and the constants above, calculate the ratio of the central density to the mean density, $\rho_c/\bar{\rho}$, for the $n = 1.5$ and $n = 3$ polytropes.

Given the way the constants are tabulated, it is easier to multiply top and bottom by ξ_1^2 and write $\rho_c/\bar{\rho} = \xi_1^3/3(-\xi_1^2[d\theta/d\xi]_{\xi_1})$. So we get

$$n = 1.5: \ \frac{\rho_c}{\bar{\rho}} = \frac{\xi_1^3}{3\left(-\xi_1^2\left[\frac{d\theta}{d\xi}\right]_{\xi_1}\right)} = \frac{3.65^3}{3(2.71)} = 5.98 \qquad n = 3: \ \frac{\rho_c}{\bar{\rho}} = \frac{6.90^3}{3(2.02)} = 54.2$$

In fact this quantity is tabulated as D_n in Lamers & Levesque Table 11.1 or Table 4 in Chandrasekhar's book *An Introduction to the Study of Stellar Structure* which I reproduced on the title slide for Lecture 3. Note that the larger value of n corresponds to a density profile more concentrated towards the center, so the ratio of central density to average density increases with increasing n (see, for example, Lamers & Levesque Figure 11.1).

(d) *The central pressure for a star can be written as*

$$P_c \quad = \quad \underline{\qquad\qquad} \left(\frac{GM^2}{R^4}\right) \quad = \quad \underline{\qquad\qquad\qquad} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4$$

where I've left out the numerical factors that go in front. Calculate these numerical values (i.e., fill in the blanks) for polytropes with $n = 1.5$ and $n = 3$. Don't forget to include units if necessary! Hint: *recall that $P = K\rho^\gamma$, so $P_c = K\rho_c^\gamma$, where $\gamma = (n+1)/n$; you can derive K in terms of the mass M and radius R.*

There are a number of ways to do this; following the hint, we have that $P_c = K\rho_c^{(n+1)/n}$, so we could rewrite K and ρ_c in terms of M and R. But as a shortcut, we know for pressure we ultimately want to get to something that goes as GM^2/R^4, so let's see what that combination gives us

$$\begin{aligned}
\frac{GM^2}{R^4} &= \frac{(16\pi^2 G\alpha^6\rho_c^2)[+\xi_1^4(d\theta/d\xi)^2_{\xi_1}]}{\alpha^4\xi_1^4} = 16\pi^2 G\alpha^2\rho_c^2[d\theta/d\xi]^2_{\xi_1} \\
&= 16\pi^2 G\left(\frac{K(n+1)\rho_c^{\frac{1-n}{n}}}{4\pi G}\right)\rho_c^2[d\theta/d\xi]^2_{\xi_1} = 4\pi(n+1)[d\theta/d\xi]^2_{\xi_1} K\rho_c^{\frac{1-n}{n}+2} \\
&= 4\pi(n+1)\left[\frac{d\theta}{d\xi}\right]^2_{\xi_1} K\rho_c^{\frac{1+n}{n}} = 4\pi(n+1)\left[\frac{d\theta}{d\xi}\right]^2_{\xi_1} P_c
\end{aligned}$$

where in middle step I substituted in for α^2, and in the last step we see the central pressure has appeared. Thus, moving the prefactors to the opposite side we get

$$P_c = \frac{1}{4\pi(n+1)\left[\frac{d\theta}{d\xi}\right]^2_{\xi_1}} \frac{GM^2}{R^4}$$

We need to rewrite this in terms of our tabulated constants, so note that

$$\left[\frac{d\theta}{d\xi}\right]^2_{\xi_1} = \frac{\left(-\xi_1^2\left[\frac{d\theta}{d\xi}\right]_{\xi_1}\right)^2}{\xi_1^4}$$

So we have

$$n = 1.5 \quad : \quad P_c = \frac{1}{4\pi(2.5)\frac{2.71^2}{3.654^4}} \frac{GM^2}{R^4} = 0.77\, \frac{GM^2}{R^4}$$
$$n = 3 \quad : \quad P_c = \frac{6.90^4}{4\pi(4)(2.02^2)} \frac{GM^2}{R^4} = 11.05\, \frac{GM^2}{R^4}$$

To get these answers in solar units, plug in $G = 6.67 \times 10^{-8}$ cm^3 g^{-1} s^{-2}, $M = 1.99 \times 10^{33}$ g, and $R = 6.96 \times 10^{10}$ cm, which gives:

$$n = 1.5 \quad : \quad P_c = 0.77\, \frac{GM^2}{R^4} = 8.7 \times 10^{15} \text{ dyne cm}^{-2} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4$$
$$n = 3 \quad : \quad P_c = 11.05\, \frac{GM^2}{R^4} = 1.2 \times 10^{17} \text{ dyne cm}^{-2} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4$$

If you use SI units, with pressures in N m^{-2} (also known as Pascals: Pa), the prefactors are 8.7×10^{14} Pa and 1.2×10^{16} Pa, respectively. We see that the more centrally concentrated density profile gives a higher central pressure as well.

2. *(Adapted from Lamers & Levesque problems 4.2 and 4.4) Ekström et al. (2012, A&A, 537, 146) have published a set of stellar models over a range from 0.8 to 120 $M_\odot$. You can access the paper at* https://ui.adsabs.harvard.edu/abs/2012A%26A...537A.146E/abstract. *You can get the stellar model data associated with this paper at* https://vizier.cds.unistra.fr/viz-bin/VizieR-3?-source=J/A%2bA/537/A146/tables.

   We are interested in the stars at the beginning of their lives, the zero-age main sequence *or ZAMS, and we will look at non-rotating stars for this problem. To get just this data, put in a constraint of "n" for the "Rot" parameter, and set "Line" to be 1 for ZAMS data. Check the "Show" boxes for the initial mass (Mini) and further down, the $\log_{10}$ of the central density (log(rhoc)), the $\log_{10}$ of the central temperature (logTc), and the central mass fractions of hydrogen (Xc) and helium (Yc). Click Submit and if all goes well you should see a table with 24 rows each corresponding to a different mass star.*

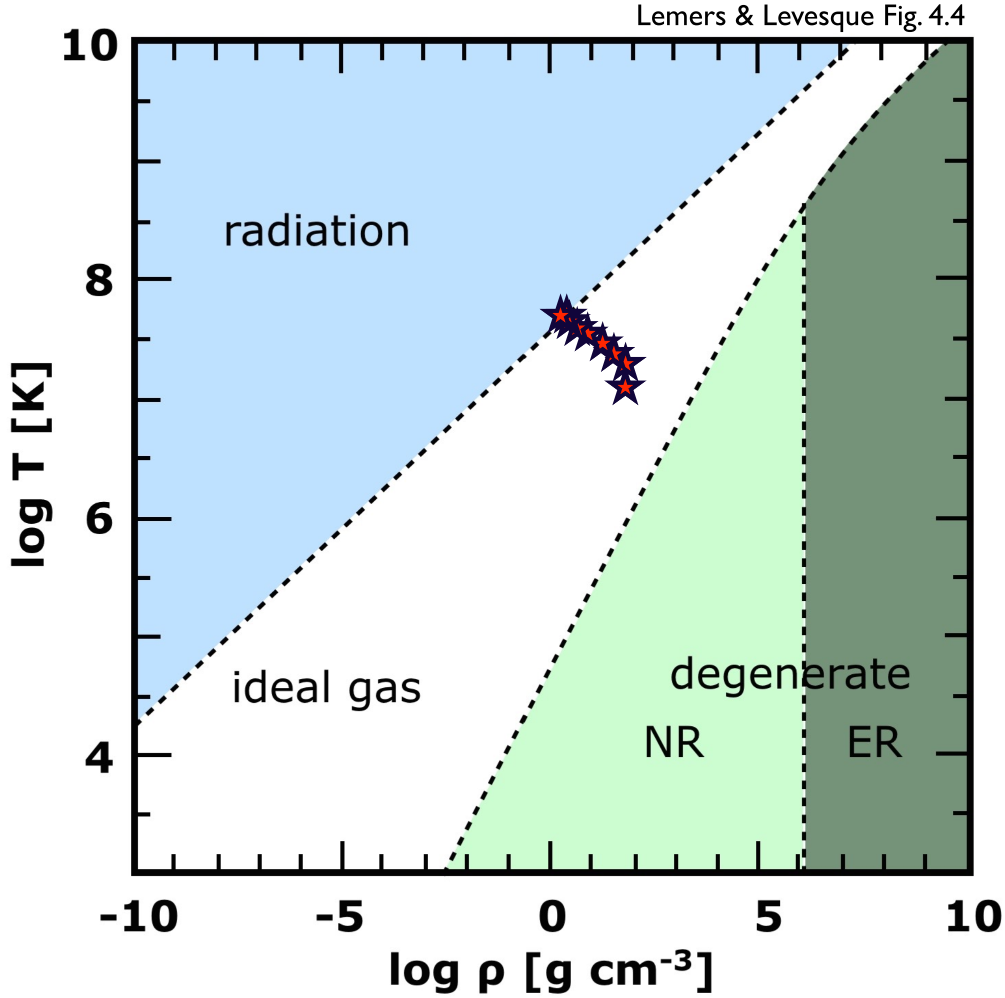
   (a) *On a copy of Lamers & Levesque Figure 4.4 (there's one on Canvas under Problem Set Resources called* LLFig4-4.pdf*), mark points for the central density and temperature of M = 1, 2, 4, 7, 12, 20, 40, 60, and 120 $M_\odot$ stars. In what regions of the diagram do the centers of these stars reside?*

   Below I show a table of the data after following the steps. Then I give an annotated version of the plot where I tried estimate the correct position of each star. The general trend is to lower central density and higher central temperature with mass.

   The cores of the stars are largely in the ideal gas regime, with radiation pressure being important for the most massive stars.

Full	Mini Msun	log(rhoc) [g/cm3]	logTc [K]	Xc	Yc
1	0.80	1.898	7.061	0.7167	0.2691
2	0.90	1.903	7.098	0.7169	0.2688
3	1.00	1.910	7.130	0.7162	0.2694
4	1.10	1.923	7.161	0.7168	0.2687
5	1.25	1.933	7.199	0.7169	0.2687
6	1.35	1.937	7.226	0.7170	0.2686
7	1.50	1.934	7.261	0.7170	0.2686
8	1.70	1.913	7.298	0.7169	0.2687
9	2.00	1.864	7.336	0.7169	0.2688
10	2.50	1.768	7.374	0.7170	0.2686
11	3.00	1.673	7.400	0.7170	0.2687
12	4.00	1.515	7.433	0.7169	0.2688
13	5.00	1.395	7.457	0.7169	0.2688
14	7.00	1.215	7.489	0.7168	0.2689
15	9.00	1.090	7.511	0.7169	0.2688
16	12.00	0.964	7.538	0.7169	0.2687
17	15.00	0.872	7.557	0.7169	0.2688
18	20.00	0.760	7.578	0.7169	0.2688
19	25.00	0.680	7.592	0.7169	0.2688
20	32.00	0.598	7.606	0.7170	0.2687
21	40.00	0.529	7.617	0.7169	0.2688
22	60.00	0.419	7.636	0.7169	0.2689
23	85.00	0.335	7.651	0.7169	0.2688
24	120.00	0.257	7.664	0.7170	0.2688

Lemers & Levesque Fig. 4.4



(b) *Using ρ_c, T_c, X_c, and Y_c, write a formula for the ratio of radiation pressure to gas pressure, $P_{\rm rad}/P_{\rm gas}$, in the center of a star. Taking all 24 rows of the online data, make a plot of $P_{\rm rad}/P_{\rm gas}$ versus M using any numerical/plotting package you like.*

The radiation pressure at the center is $P_{\rm rad} = aT_c^4/3$. The (ideal) gas pressure is $P_{\rm gas} = nkT = \rho_c kT_c/\mu_c m_p$.

Thus we need to know the mean molecular weight at the center; this depends on the composition. The central hydrogen and helium fraction are given by X_c and Y_c. At the high temperatures in the centers of stars, we can safely assume all of the material is fully ionized (see later problems). For ionized hydrogen, we have 1 proton and 1 election, giving an average particle mass $\langle m \rangle = (m_p+m_e)/2 \approx m_p/2$ and a mean molecular weight of $\mu = \langle m \rangle / m_p \approx 0.5$. For ionized helium, we have 1 helium nucleus ($m_{\rm He} \approx 4m_p$) and two free electrons so an average particle mass $\langle m \rangle = (4m_p + m_e + m_e)/3 \approx 4/3 m_p$ and thus $\mu = 4/3$.

If we have a mixture of ionized hydrogen and helium, the mean molecular weight will be in between. We have to be a little bit careful here, because X and Y are the **mass fractions** of hydrogen and helium. Because helium has approximately 4 times the mass of hydrogen, the relative *number* of helium ions is $Y/4$.

So if we have X hydrogen ions and $Y/4$ helium ions, their total mass is approximately $Xm_p + (Y/4)(4m_p) = (X+Y)m_p$, where I am ignoring the mass of the electrons. The number of particles would be X protons + X electrons + $(Y/4)$ helium nuclei + $2(Y/4)$ electrons from the helium. That's a total number of particles of $2X + 3Y/4$. Thus the mean molecular weight for this composition would be $\mu \approx (X+Y)/(2X+3Y/4)$.

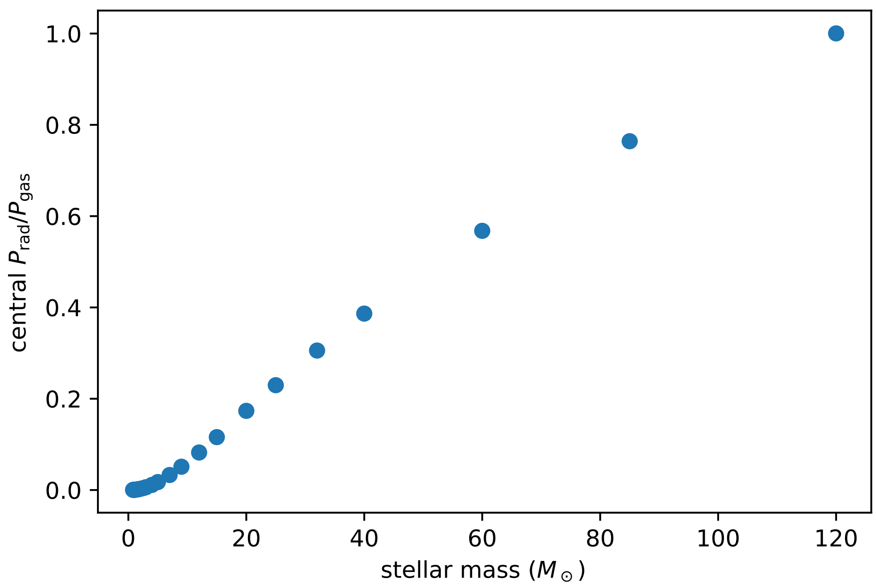
As an example, for the first row, with $X_c = 0.7161$ and $Y_c = 0.2691$, this gives $\mu \approx 0.60$.

All of this discussion about the mean molecular weight and the precise definitions of the mass fractions are a bit subtle, so don't worry if you didn't do it exactly this way. It doesn't make a huge difference in the end, so any reasonable approach to estimating the mean molecular weight for a mixed composition is fine.

Putting it all together we can write a direct formula for the pressure ratio in terms of what we have in our table and fundamental constants:

$$\frac{P_{\rm rad}}{P_{\rm gas}} = \frac{aT_c^4/3}{\rho_c kT_c/\mu m_p} = \frac{aT_c^4}{3}\frac{\mu m_p}{\rho_c kT_c} = \frac{aT_c^4}{3}\frac{(X_c+Y_c)m_p}{(2X_c+3Y_c/4)\rho_c kT_c} = \frac{aT_c^3 m_p(X_c+Y_c)}{3\rho_c k(2X_c+3Y_c/4)}$$

In the python notebook at `http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps02q2.ipynb` I calculate out the pressures and their ratio, and check that this single-step answer matches. Here are the results:



(c) *Do your results in part (b) correspond to the diagram in part (a)?*

Yes, most of the stars are dominated by gas pressure except for perhaps the most massive star, which is encroaching into the blue "radiation" territory on the plot.

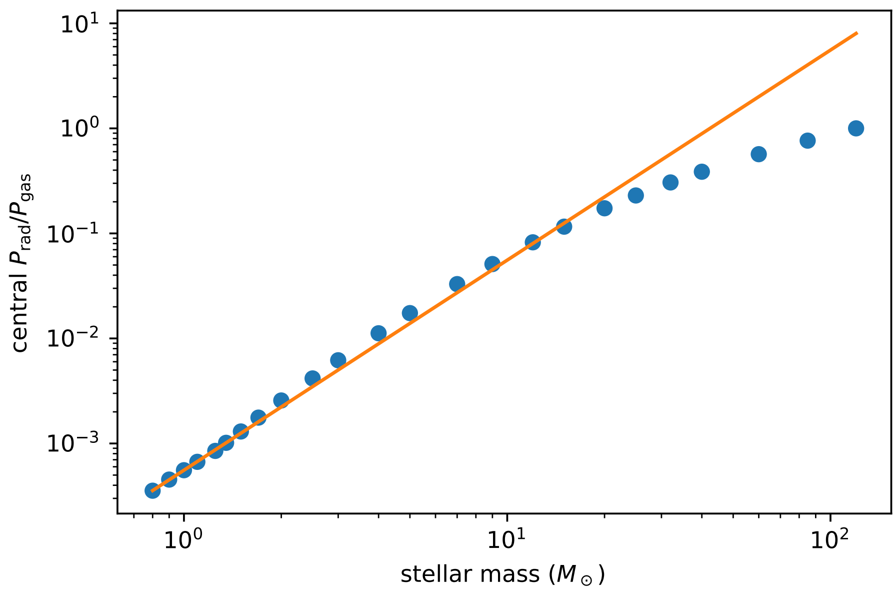
(d) *In class we derived $P_{\rm rad}/P_{\rm gas} \propto M^2$. Describe how to obtain this scaling relation. Is this prediction validated by your plot? Explain.*

We made dimensional analysis estimates of the density and central temperature for stars in terms of their mass and radius:

$$\rho \propto \frac{M}{R^3} \text{ and } T_c \sim \frac{GM\mu m_p}{kR} \propto \frac{M}{R} \quad \Longrightarrow \quad P_{\rm rad} \propto T^4 \propto \frac{M^4}{R^4} \text{ and } P_{\rm gas} \propto \rho T \propto \frac{M^2}{R^4}$$

Thus, we get $P_{\rm rad}/P_{\rm gas} \propto M^2$.

To check this expectation against the data, we plot the pressure ratio vs mass on a log-log plot and see if it is consistent with a straight line of slope 2. The result is below, where we see a reasonable match for the lower mass stars, but the high mass stars have a somewhat lower radiation pressure (so lower central temperature) than our simple estimate would suggest.



3. (a) *Show that the pressure integral $P = 1/3 \int_0^\infty p\, v(p)\, n(p)\, dp$ for a completely degenerate electron gas with Fermi momentum $p_F = (3h^3 n_e/8\pi)^{1/3}$ can be written*

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{(p^2c^2 + m_e^2 c^4)^{1/2}}\, dp$$

This is a relativistically correct formula, so use the fact that the relativistic energy and momentum for an electron with Lorentz factor γ are $E = \gamma m_e c^2 = \sqrt{p^2c^2 + m_e^2c^4}$ and $p = \gamma m_e v$. Hint: *start by writing v in terms of p and E.*

Following the hint, we see that $p/E = \gamma m_e v/\gamma m_e c^2 = v/c^2$. So $v = pc^2/E$.

For complete degeneracy the electron phase space density is given by $2/h^3$ (because electrons are spin 1/2 and have two spin states) for all states up to the Fermi momentum and there is no occupation above that. So that means $n(p)dp = (2/h^3)(4\pi p^2)\, dp = (8\pi/h^3)\, p^2\, dp$ for $0 \le p \le p_F$ and $n(p)\, dp = 0$ for $p > p_F$.

Thus the pressure integral becomes

$$\begin{aligned} P &= \frac{1}{3}\int_0^\infty p\, v(p)\, n(p)\, dp = \frac{1}{3}\int_0^{p_F} p \left(\frac{pc^2}{E}\right)\left(\frac{8\pi p^2}{h^3}\right) dp \\ &= \frac{8\pi}{3h^3}\int_0^{p_F} \frac{p^4c^2}{(p^2c^2 + m_e^2c^4)^{1/2}}\, dp \end{aligned}$$

as desired, where in the last step I substitute in the relativistic energy formula.

(b) *In the non-relativistic limit, $p \approx mv \ll mc$, so $p^2c^2 \ll m_e^2c^4$ and you can simplify the denominator. Show this gives the non-relativistic degenerate equation of state*

$$P = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} n^{5/3}$$

We can simplify the denominator keeping only the $m_e^2c^4$ term, which gives

$$\begin{aligned}
P &= \frac{8\pi}{3h^3}\int_0^{p_F}\frac{p^4c^2}{m_ec^2}\,dp = \frac{8\pi}{3h^3m_e}\int_0^{p_F}p^4\,dp \\
&= \frac{8\pi}{3h^3m_e}\frac{p_F^5}{5} = \frac{1}{5h^3m_e}\left(\frac{8\pi}{3}\right)\left(\frac{3}{8\pi}h^3\,n_e\right)^{5/3} \\
&= \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} n_e^{5/3}
\end{aligned}$$

(c) *Show that for the ultra-relativistic limit, where $p^2c^2 \gg m_e^2c^4$ we get the ultra-relativistic degenerate equation of state*

$$P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}$$

Similarly as above, this time we simplify the denominator keeping only the p^2c^2 term, so we have

$$\begin{aligned}
P &= \frac{8\pi}{3h^3}\int_0^{p_F}\frac{p^4c^2}{(p^2c^2)^{1/2}}\,dp = \frac{8\pi c}{3h^3}\int_0^{p_F}p^3\,dp = \frac{8\pi c}{3h^3}\frac{p_F^4}{4} = \frac{c}{4h^3}\left(\frac{8\pi}{3}\right)\left(\frac{3}{8\pi}h^3\,n_e\right)^{4/3} \\
&= \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3}
\end{aligned}$$

(d) *What kind of degenerate equation of state (non-relativistic or ultra-relativistic) leads to a mass-radius relation of the form $R \propto M^{-1/3}$? Explain qualitatively why for those white dwarfs, a higher mass leads to a smaller radius.* Hint: *look back to polytropes and problem 1.*

Both of these are polytropic equations of state, where the pressure only depends on the density as a power-law. Going back to problem 1a, we saw that for a polytrope with index n the mass-radius relation could be written $R \propto M^{(1-n)/(3-n)}$.
To get $R \propto M^{-1/3}$ we need a polytropic index of $n = 3/2$, which corresponds to $\gamma = 1 + 1/n = 5/3$. This corresponds to the *non-relativistic* degenerate equation of state ($P \propto \rho^{5/3} \propto n_e^{5/3}$).
Qualitatively, the degeneracy pressure gets stronger as the density increases. So for degeneracy pressure to support a larger mass against gravity, the density needs to be higher, and the white dwarf needs to have a physically smaller radius.

4. *Let us explore the Saha equation (and one of its quirks) and apply it to the Sun. In this problem we will assume a pure hydrogen composition ($X = 1, Y = 0$); this is not a good idea (we ignore helium at our peril, especially in the core of the Sun), but it simplifies things in a useful way. In class we wrote the Saha equation for hydrogen ionization as*

$$\frac{n_{II}}{n_I} = \frac{2Z_{II}}{n_eZ_I}\left(\frac{2\pi m_ekT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

(a) *There are three densities here:* n_I*, the number density of neutral hydrogen atoms;* n_{II}*, the number density of hydrogen ions (protons); and* n_e*, the number density of free electrons. These are all linked; rewrite each of these in terms of* n_H*, the total number density of hydrogen (neutral + ionized) and the ionization fraction* $y = n_{II}/n_H$*. Note that Phillips calls the ionization fraction* $x(H)$ *rather than* y*.*

From the definition, we have $n_{II} = yn_H$. Similarly, every free electron comes from an ionized hydrogen atom in this pure hydrogen composition, so $n_e = n_{II} = yn_H$. Finally, we know that $n_H = n_I + n_{II}$, so $n_I/n_H = 1 - n_{II}/n_H = 1 - y$, and thus $n_I = (1 - y)n_H$.

(b) *Show that*

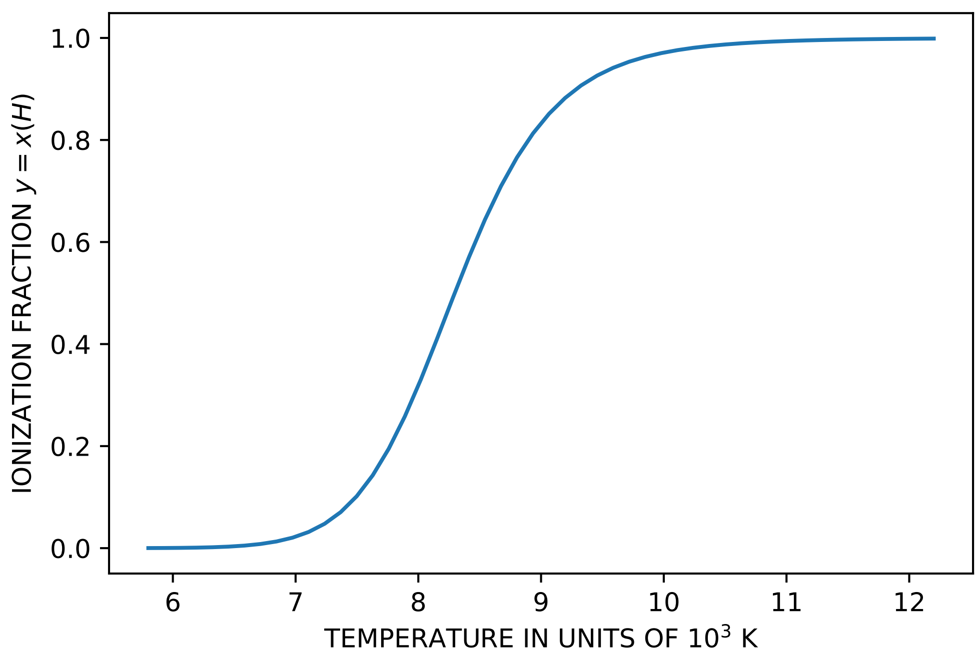
$$y = \frac{n_{II}/n_I}{1 + (n_{II}/n_I)}$$

Use this and the Saha equation above to reproduce the top panel of Phillips Figure 2.4, for a free electron density of $n_e = 10^{19}\ \mathrm{m}^{-3} = 10^{13}\ \mathrm{cm}^{-3}$ *(a typical value in the solar atmosphere). You may use any plotting package you like.*

$$y = \frac{n_{II}}{n_H} = \frac{n_{II}}{n_I + n_{II}} = \frac{n_{II}/n_I}{1 + (n_{II}/n_I)}$$

where in the last step I divided through top and bottom by n_I.

My python notebook is at `http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps02q4b.ipynb` and produces this plot:



(c) *Use the Saha equation and your results from part (a) to derive the following expression for the ionization fraction*

$$\frac{y^2}{1-y} = \frac{2m_p Z_{II}}{\rho Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

From part (a), we see that we can rewrite the left hand side of the Saha equation

$$\frac{n_{II}}{n_I} = \frac{yn_H}{(1-y)n_H} = \frac{y}{1-y}$$

Similarly we can replace $n_e = yn_H$ on the right hand side of the Saha equation, so we get

$$\frac{y}{1-y} = \frac{2}{yn_H}\frac{Z_{II}}{Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

Bringing the y in the denominator on the right over to the numerator on the left, and recognizing that $n_H = \rho/m_p$, we get the desired result:

$$\frac{y^2}{1-y} = \frac{2m_p}{\rho}\frac{Z_{II}}{Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

(d) *Using this result, calculate the ionization fraction expected at the center of the Sun, where $T_c = 1.6\times10^7$ K and $\rho_c = 150$ g cm^{-3} $= 1.5\times10^5$ kg m^{-3}. Take $Z_I \approx 2$ and $Z_{II} = 1$. Is this what you would have predicted given your plot in part (b)? Explain.*

Note that we have a quadratic equation for y; call the right hand side w, so

$$\frac{y^2}{1-y} = w \;\Rightarrow\; y^2 = w - wy \;\Rightarrow\; y^2 + wy - w = 0 \;\Rightarrow\; y = \frac{-w \pm \sqrt{w^2+4w}}{2}$$

The ionization fraction must be positive, so we can choose the positive root.

Plugging in the Sun numbers we get (I used eV units in the exponential, and note that $kT \gg 13.6$ eV, so the exponential is very close to $e^{-0} = 1$),

$$\begin{aligned}
\frac{y^2}{1-y} &= \frac{2(1.67\times10^{-24}\text{ g})(1)}{(150\text{ g cm}^{-3})(2)}\exp\left[-\frac{13.6\text{ eV}}{(8.62\times10^{-5}\text{ eV K}^{-1})(1.6\times10^7\text{ K})}\right]\times \\
&\quad \left[\frac{2\pi(9.11\times10^{-28}\text{ g})(1.38\times10^{-16}\text{ g cm}^2\text{ s}^{-2}\text{ K}^{-1})(1.6\times10^7\text{ K})}{(6.626\times10^{-27}\text{ cm}^2\text{ g s}^{-1})^2}\right]^{3/2} \\
&= 1.7 \\
\Rightarrow y &= \frac{-1.7+\sqrt{1.7^2+4(1.7)}}{2} = 0.71
\end{aligned}$$

The Saha equation says that hydrogen in the center of the Sun should only be $\sim$70% ionized! This is definitely at odds with the expectation from Phillips Fig. 2.4 where we would have expected nearly 100% ionization.

In the formula what is happening is that the exponential has saturated at nearly 1, and then ionization fraction increases as $T^{3/2}$ but decreases as the electron density increases. In the center of the Sun $n_e \sim 10^{25}$ cm^{-3}, which is 12 orders of magnitude higher than what we had in part (b), representing the Sun's atmosphere.

So does that mean that the Sun's interior is not fully ionized?! It turns out that for the high densities inside the Sun or other stars, the hydrogen and helium nuclei are packed closely together enough that we can't treat them in isolation, as the Saha equation does. Neighboring positive charges help liberate the electrons, and the Sun's interior is fully ionized. The atomic bound states are are strongly affected by the electrostatic potentials of nearby ions. This is called *pressure ionization*. The Saha equation works well for stellar atmospheres, but pressure ionization is responsible for keeping stellar interiors ionized. *This problem and explanation is from Prof. Ryan Chornock.*

5. *Atomic absorption in stellar atmospheres involves both discrete lines (from electronic transitions between bound states) and* continuum absorption*.*

   (a) *One form of continuum absorption is "bound-free" transitions, in which an electron moves from a bound state to an unbound (ionized) state. These transitions require only a minimum photon energy, because the free electron can have any (positive) kinetic energy. Even though most hydrogen in the Sun's atmosphere is neutral, with the electron in the ground state ($n = 1$), the dominant source of bound-free absorption of visible light in the Sun's atmosphere comes from excited hydrogen with the electron in the $n = 3$ state. Explain why (and be quantitative).*

   The energy levels of hydrogen are given by $E_n = -13.6/n^2$ eV, yielding $E_1 = -13.6$ eV, $E_2 = -3.40$ eV, and $E_3 = -1.51$ eV. For a bound-free transition of an electron from the ground ($n = 1$) state, we need a photon with $E > 13.6$ eV, corresponding to a wavelength $\lambda = hc/E < 912$ Å (the so-called Lyman limit), far into the ultraviolet. Thus, even though most of the hydrogen is in the ground state, visible light photons cannot photoionize such atoms.

   Similarly, only photons with $E > 3.4$ eV, corresponding to $\lambda < 3647$ Å (the Balmer limit), can ionize excited hydrogen atoms with an electron in the $n = 2$ level. This too is ultraviolet light, not visible light.

   For hydrogen with the electron in $n = 3$, photons with $E > 1.51$ eV or $\lambda < 8205$ Å (the Paschen limit), including all visible light photons, can lead to photoionization.

   (b) *Another source of continuum absorption in the Sun's atmosphere is from the negative ion H^-, in which an extra electron is added to neutral hydrogen. This electron is only weakly bound, with a binding energy of 0.754 eV.*

      i. *Show that all visible light photons have sufficient energy to liberate an electron from H^-.*

As above, to "ionize" H^- requires photons with $E > 0.7$ eV, corresponding[1]. to $\lambda = hc/E < 17712$ Å. This in the near-infrared, so all visible light photons can liberate the "extra" electron[2] in H^-.

ii. *What is the expected abundance of H^- ions relative to neutral hydrogen in the Sun's atmosphere ($T = 5770$ K, free electron pressure $P_e = 15$ dyne cm^{-2})?* Hint: *Treat neutral hydrogen as the "ionized" version of H^-.*

Following the hint, we can use the Saha equation, with $n_e = P_e/kT$. The "ionized" species is H I (with partition function $Z_I = 2$), while the "unionized" species is H^- (with partition function $Z_- = 1$, because there is only one way to arrange the two electrons). So we have

$$\begin{aligned}
\frac{n_I}{n_-} &= \frac{2 Z_I kT}{Z_- P_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_-/kT} \\
&= \frac{2(2)(1.38\times10^{-16}\text{ g cm}^2\text{ s}^{-2}\text{ K}^{-1})(5770\text{ K})}{1(15\text{ g cm}^{-1}\text{ s}^{-2})} \\
&\quad \times \left[\frac{2\pi(9.11\times10^{-28}\text{ g})(1.38\times10^{-16}\text{ g cm}^2\text{ s}^{-2}\text{ K}^{-1})(5770\text{ K})}{(6.626\times10^{-27}\text{ cm}^2\text{ g s}^{-1})^2}\right]^{3/2} \\
&\quad \times \exp\left[-\frac{0.7\text{ eV}}{(8.62\times10^{-5}\text{ eV K}^{-1})(5770\text{ K})}\right] = 5.5\times10^7
\end{aligned}$$

We see that neutral hydrogen dominates over H^-; flipping this result over we have $n_-/n_I = 1.8\times10^{-8}$.

(c) *What dominates the continuum absorption of visible light in the Sun's atmosphere, bound-free transitions of neutral hydrogen or "ionization" of H^-? Explain quantitatively.*

We saw that all visible light photons can remove the electron from neutral hydrogen in the $n = 3$ (or higher) state, and similarly all visible light photons can "ionize" H^-, so we just need to see which population is larger. We saw that $n^-/n_I \approx 1.8\times10^{-8}$. To see how many electrons are in the $n = 3$ state of neutral hydrogen, we use the Boltzmann factor:

$$\frac{n_3}{n_1} = \frac{g_3}{g_1}\exp\left(-\frac{E_3 - E_1}{kT}\right) = \frac{18}{2}\exp\left[-\frac{(13.6 - 1.5)\text{ eV}}{(8.62\times10^{-5}\text{ eV K}^{-1})(5770\text{ K})}\right] = 2.4\times10^{-10}$$

So both excited hydrogen with $n = 3$ and the negative hydrogen ion H^- are rare in the Sun compared to neutral hydrogen in the ground state, but there about 100 times as many H^- ions compared to $n = 3$ excited neutral H atoms, and thus H^- dominates the continuum opacity.

[1]The H^- binding energy is more precisely 0.754 eV, so this would mean $\lambda < 16443$ Å.

[2]It is interesting to consider where the electrons to make the negative ions come from. As we see, the hydrogen is almost entirely neutral, as is helium in the Sun's atmosphere. The free electrons largely come from the trace amount of alkali metals like Na and K (as well as elements like Ca), where the ionization potential to liberate the outermost electron is low, a ∼few eV.

6. ***(Required for 541; extra credit for 441)***

(a) *(Phillips 2.3) Using the pressure integral from problem 3a, show that the general expression for the pressure in a completely degenerate electron gas can be written*

$$P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}\, I(x)$$

where $x = p_F/(m_e c)$ *and*

$$I(x) = \frac{3}{2x^4}\left\{x\left(1+x^2\right)^{1/2}\left(\frac{2x^2}{3}-1\right) + \ln\left[x + \left(1+x^2\right)^{1/2}\right]\right\}.$$

Let's make the integral dimensionless and define $y \equiv p/m_e c$, so $p = m_e c y$, $dy = dp/m_e c$ and thus $dp = m_e c\, dy$. Substituting in we get

$$P = \frac{8\pi}{3h^3}\int_0^{p_F/m_e c} \frac{(m_e c y)^4 c^2}{\left[(m_e c y)^2 c^2 + m_e^2 c^4\right]^{1/2}}\,(m_e c\, dy) = \frac{8\pi}{3h^3}\frac{m_e^5 c^7}{m_e c^2}\int_0^x \frac{y^4}{\left[y^2+1\right]^{1/2}}\,dy$$

where I followed the hint and set $x = p_F/m_e c$. Wolfram Alpha tells me that

$$\int \frac{y^4\, dy}{\sqrt{y^2+1}} = \frac{1}{8}\left[y\sqrt{y^2+1}(2y^2-3) + 3\sinh^{-1} y\right] + const$$

The expression evaluates to zero for $y = 0$ and $\sinh^{-1} y = \ln(y + \sqrt{y^2+1})$, so the definite integral evaluates to

$$P = \frac{8\pi}{3h^3}\frac{3m_e^4 c^5}{8}\left[x\sqrt{x^2+1}\left(\frac{2x^2}{3}-1\right) + \ln\left(x + \sqrt{x^2+1}\right)\right]$$

To get the prefactor in the right form, we can write $m_e^4 = (p_F/xc)^4$ so we have

$$\begin{aligned}
P &= \frac{\pi c^5}{h^3}\frac{(3h^3 n_e/8\pi)^{4/3}}{x^4 c^4}\left[x\sqrt{x^2+1}\left(\frac{2x^2}{3}-1\right) + \ln\left(x + \sqrt{x^2+1}\right)\right] \\
&= \frac{3^{4/3}}{16}\frac{hc}{\pi^{1/3} x^4} n_e^{4/3}\left[x\sqrt{x^2+1}\left(\frac{2x^2}{3}-1\right) + \ln\left(x + \sqrt{x^2+1}\right)\right] \\
&= \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3}\frac{3}{2x^4}\left[x\sqrt{x^2+1}\left(\frac{2x^2}{3}-1\right) + \ln\left(x + \sqrt{x^2+1}\right)\right] \\
&= \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3} I(x)
\end{aligned}$$

(b) *Confirm that this expression leads to the correct non-relativistic (problem 3b) and ultra-relativistic (problem 3c) equations of state.*

In the ultra-relativistic limit, we take $x \gg 1$. In that case $\sqrt{x^2+1} \approx x$ so

$$\begin{aligned}
\lim_{x\to\infty} I(x) &\approx \frac{3}{2x^4}\left[x^2\left(\frac{2x^2}{3}-1\right)+\ln(2x)\right] = \frac{3}{2x^4}\left[\frac{2x^4}{3}-x^2+\ln(2x)\right] \\
&= 1-\frac{3}{2x^2}+\frac{3\ln(2x)}{2x^4} = 1
\end{aligned}$$

Taking $I(x) = 1$, we recover the ultra-relativistic limit for the pressure.

In the non-relativistic limit, we take $x \ll 1$. Wolfram Alpha tells me that the series expansion of $I(x) \approx 4x/5$ near $x = 0$. This is clearly what we need, but let's dig in a bit to see where that comes from. For small x,

$$\sqrt{x^2+1} \approx 1+x^2/2-x^4/8+\ldots \quad \text{and} \quad \ln(x+\sqrt{x^2+1}) \approx x-x^3/6+3x^5/40-\ldots$$

(again these are from Wolfram alpha). So

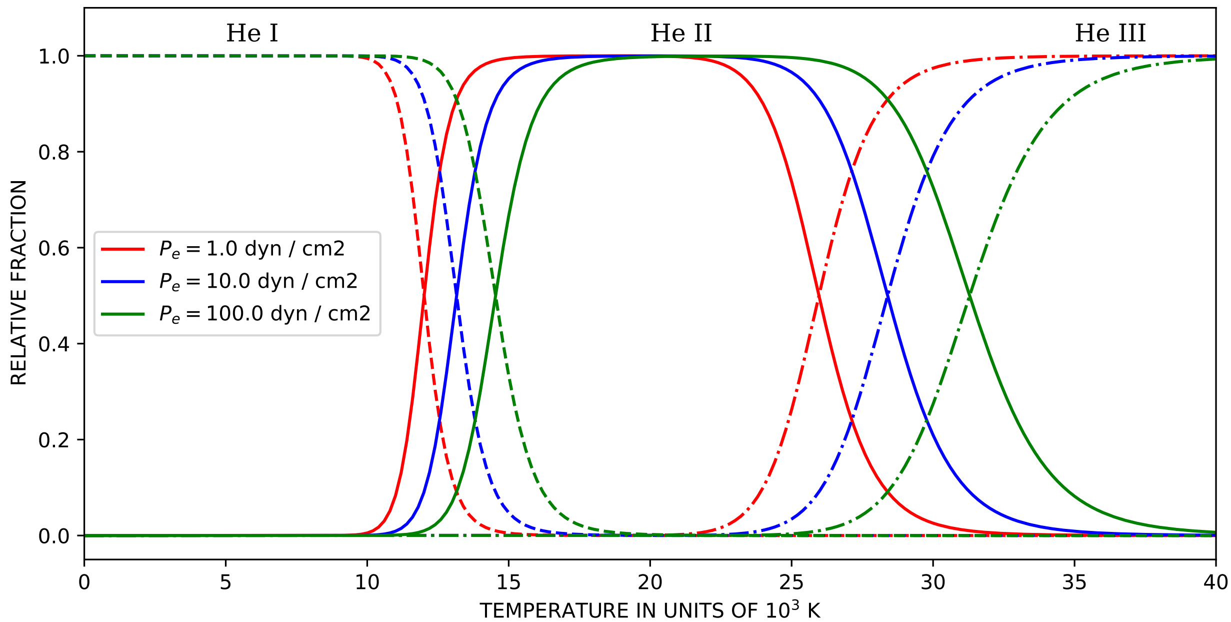
$$\begin{aligned}
\lim_{x\to 0} I(x) &\approx \frac{3}{2x^4}\left[x\left(1+\frac{x^2}{2}-\frac{x^4}{8}\right)\left(\frac{2x^2}{3}-1\right)+\left(x-\frac{x^3}{6}+\frac{3x^5}{40}\right)\right] \\
&= \frac{3}{2x^4}\left[\left(-x+\frac{2x^3}{3}-\frac{x^3}{2}+\frac{x^5}{3}+\frac{x^5}{8}-\frac{x^7}{12}\right)+\left(x-\frac{x^3}{6}+\frac{3x^5}{40}\right)\right] \\
&= \frac{3}{2x^4}\left[\frac{8x^5}{15}+\ldots\right] \approx \frac{4x}{5}
\end{aligned}$$

Thus we get the desired non-relativistic pressure

$$\begin{aligned}
P &= \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3}\left(\frac{4p_F}{5m_ec}\right) = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3}\left(\frac{4(3h^3n_e/8\pi)^{1/3}}{5m_ec}\right) \\
&= \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} n_e^{5/3}
\end{aligned}$$

(c) *The ionization energies for helium are 24.6 eV (to get singly ionized He II), and an additional 54.4 eV (to get doubly ionized He III). Make a single plot showing the relative fractions of He I, He II, and He III as a function of temperature from 0 to 40000 K. Show results for three values of the free electron pressure: $P_e = 1$, 10, and 100* dyne cm^{-2}.

This is a straightforward application of the Saha equation. Here is a python notebook with the calculation `http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps02q6c.ipynb` that produces the following plot.



He I
He II
He III
RELATIVE FRACTION
TEMPERATURE IN UNITS OF 10^3 K
$P_e = 1.0$ dyn / cm2
$P_e = 10.0$ dyn / cm2
$P_e = 100.0$ dyn / cm2