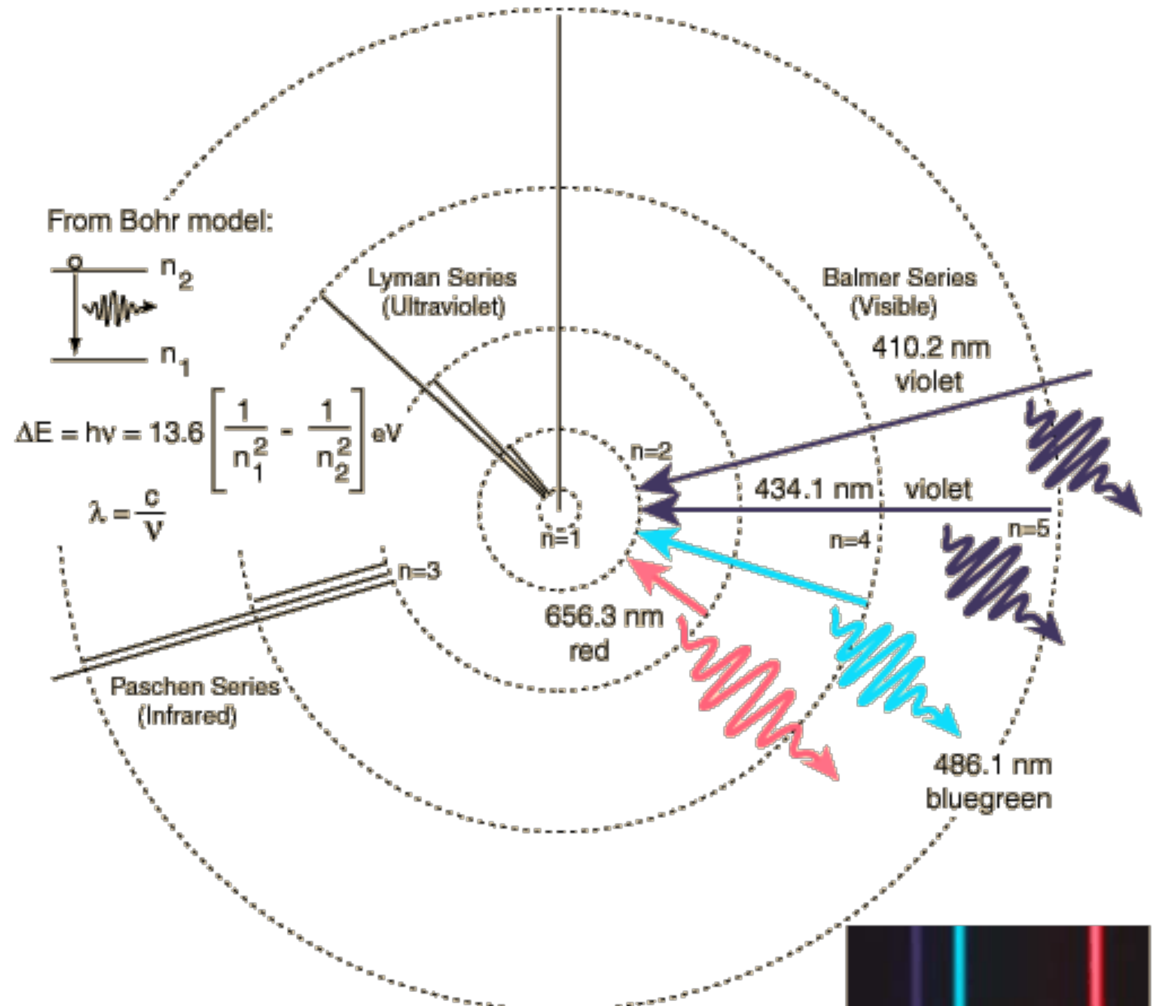
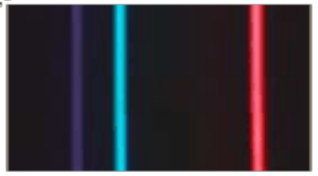
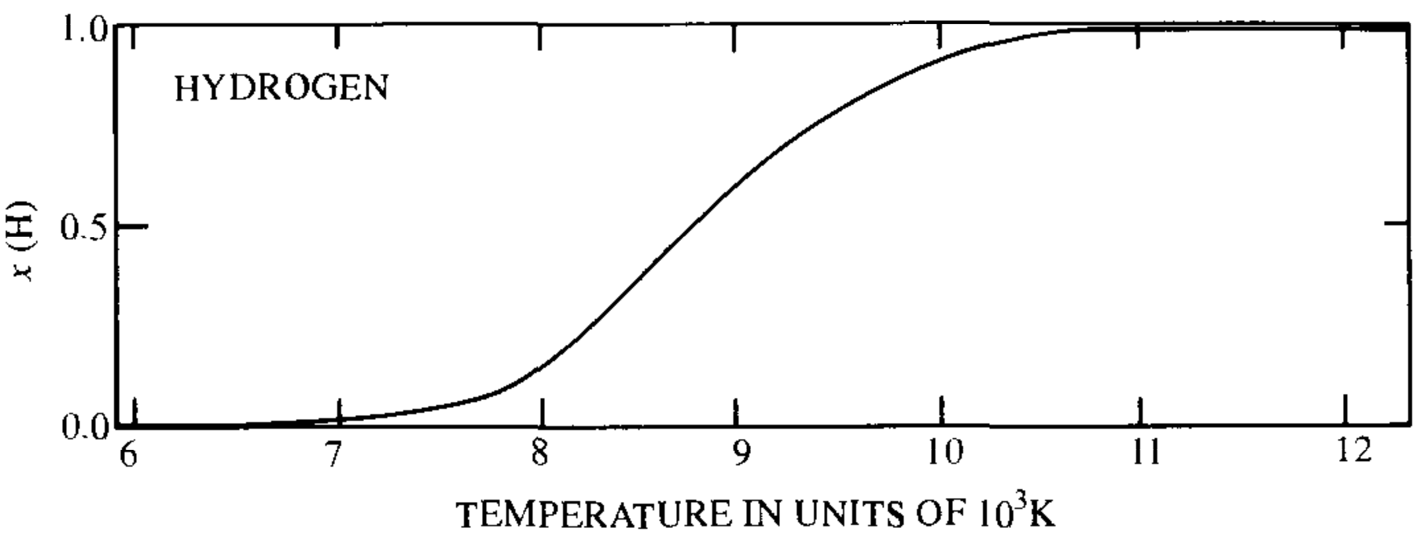
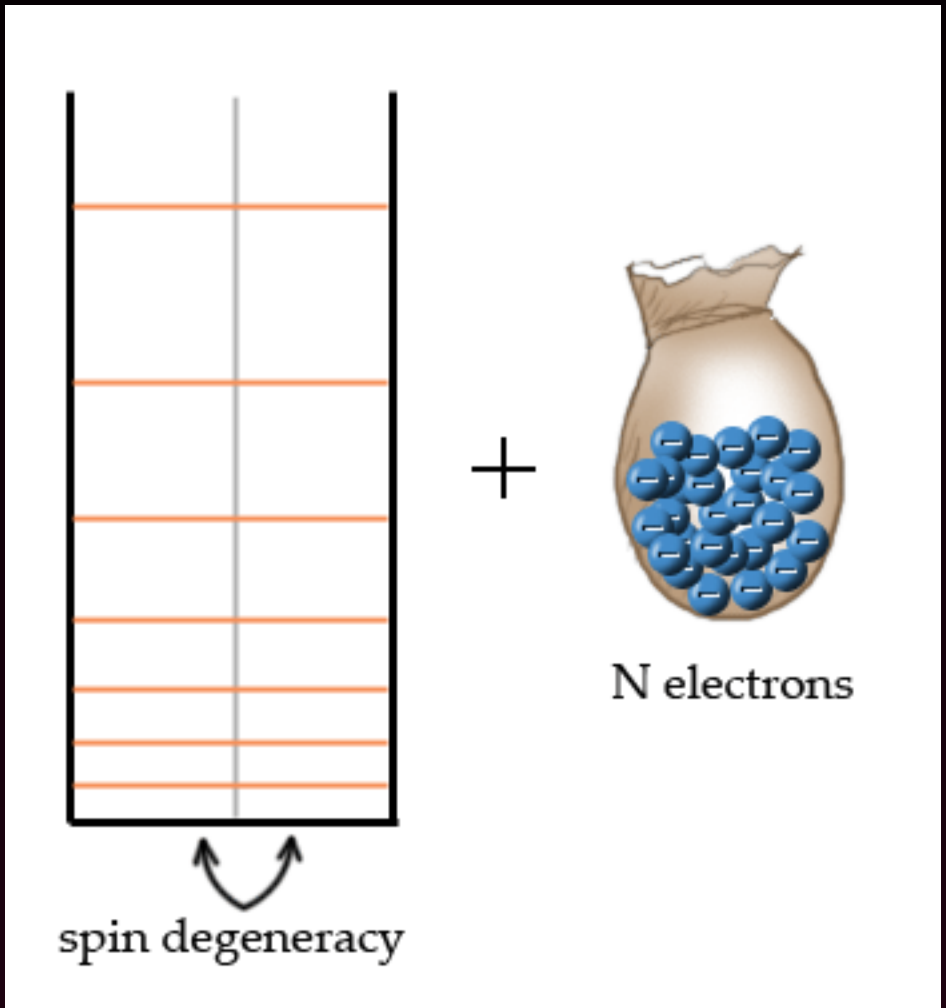
Lecture 5: Degeneracy & Ionization



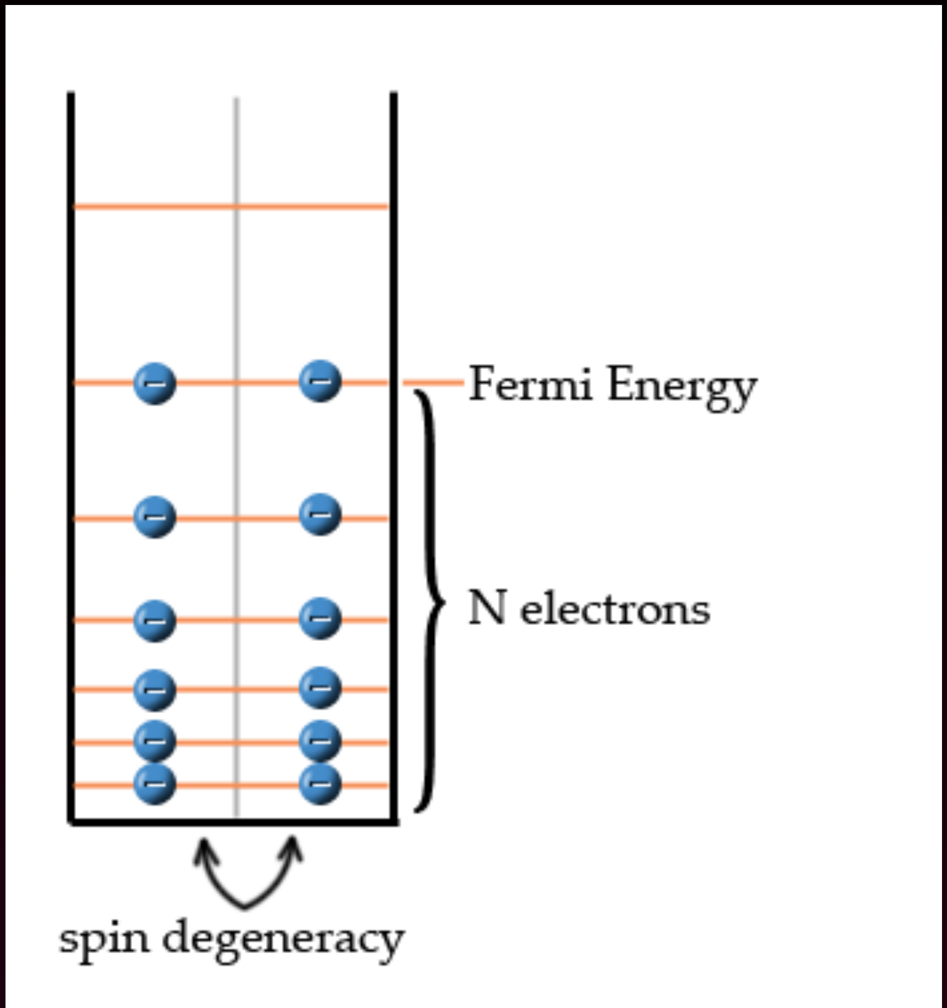




Fermi momentum and energy

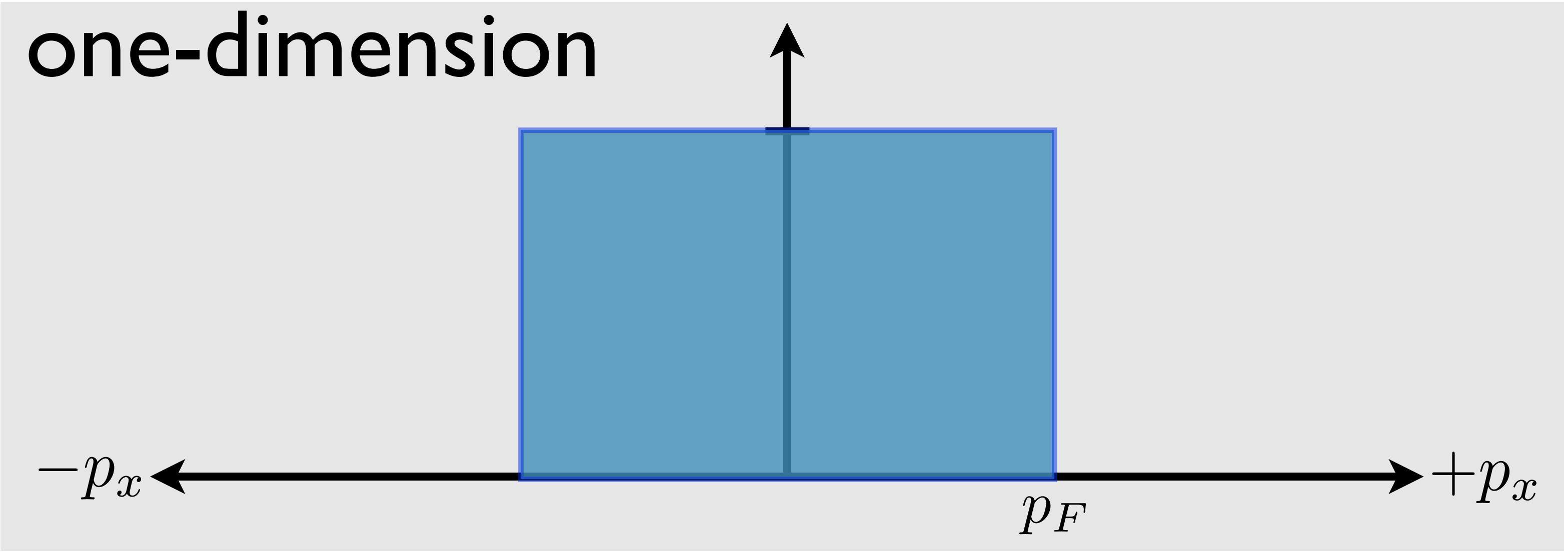


Pauli exclusion principle

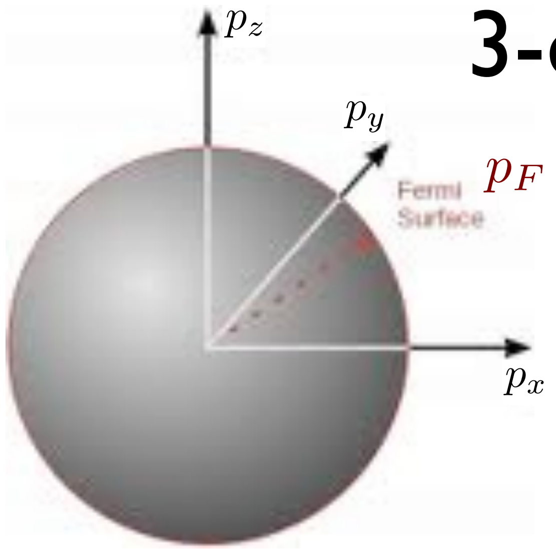


from http://www.ugrad.physics.mcgill.ca/wiki/index.php/PHYS-558

one-dimension



3-d



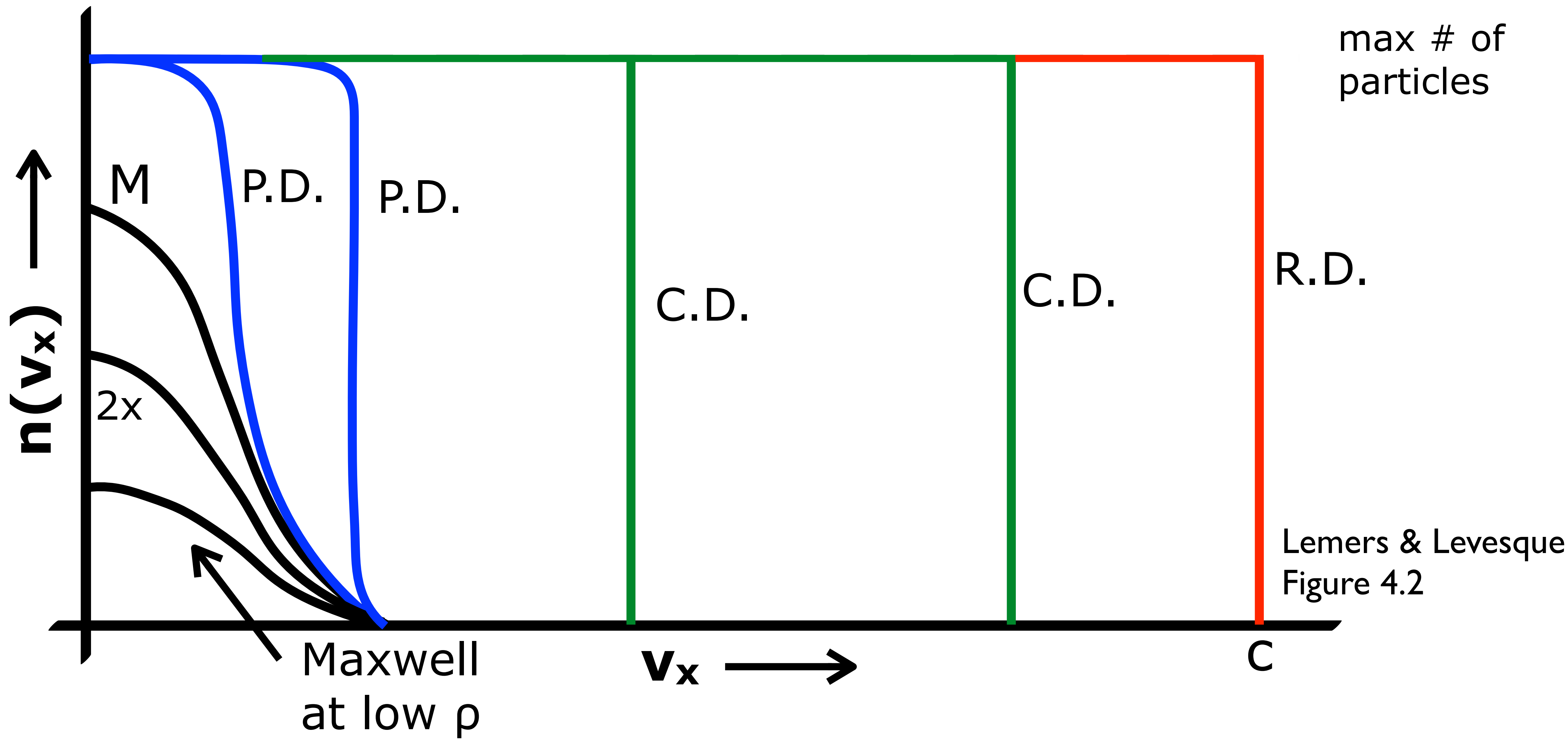
$$p_F = (3\pi^2\hbar^3 n)^{1/3}$$

$$E_F = p_F^2/2m$$

Equation of State

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…



M: Maxwellian
P.D.: partial degeneracy

C.D.: complete degeneracy
R.D.: relativistic degeneracy

Equation of State

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…

C.D. - COMPLETE DEGENERACY

$P_e \sim n_e^{5/3}$ (independent of T!)

R.D. - RELATIVISTIC DEGENERACY

$P_e \sim n_e^{4/3}$ (independent of T!)

P.D. - PARTIAL DEGENERACY

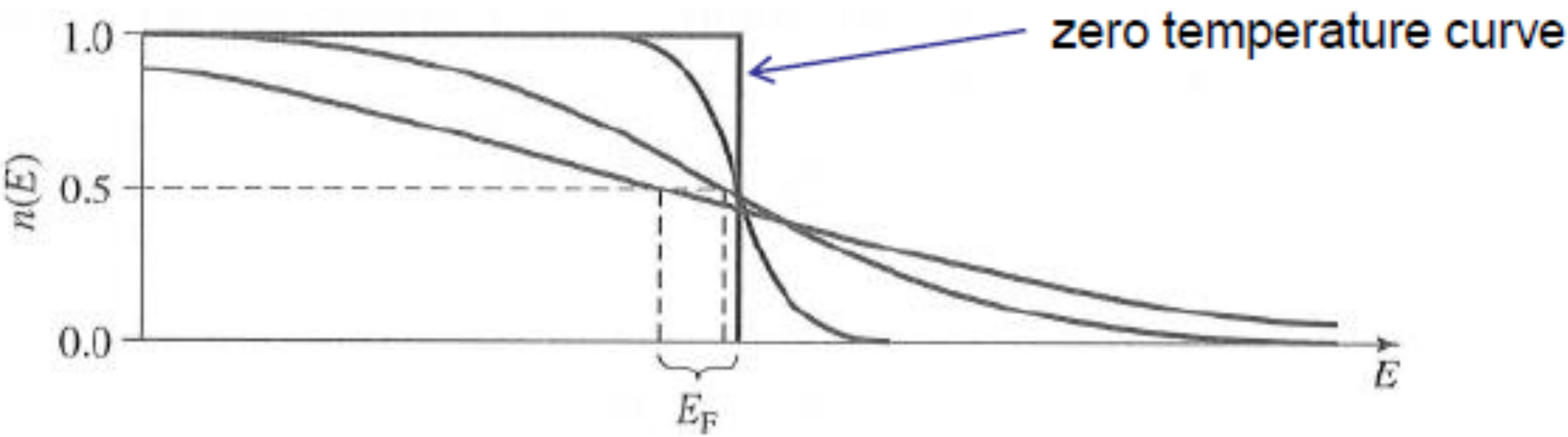
$$P_e \sim \frac{8\pi}{3h^3}(2m_e kT)^{3/2}\cdot kT\cdot F_{3/2}(\psi)$$

n(p) is not a rectangular profile; has a Maxwell tail…

Note: all of this is for e^- degeneracy…

partial degeneracy (non-zero temperature)

$$n(E) = \frac{e^{(E_F - E)/k_B T}}{1 + e^{(E_F - E)/k_B T}} = \frac{1}{e^{(E - E_F)/k_B T} + 1}$$

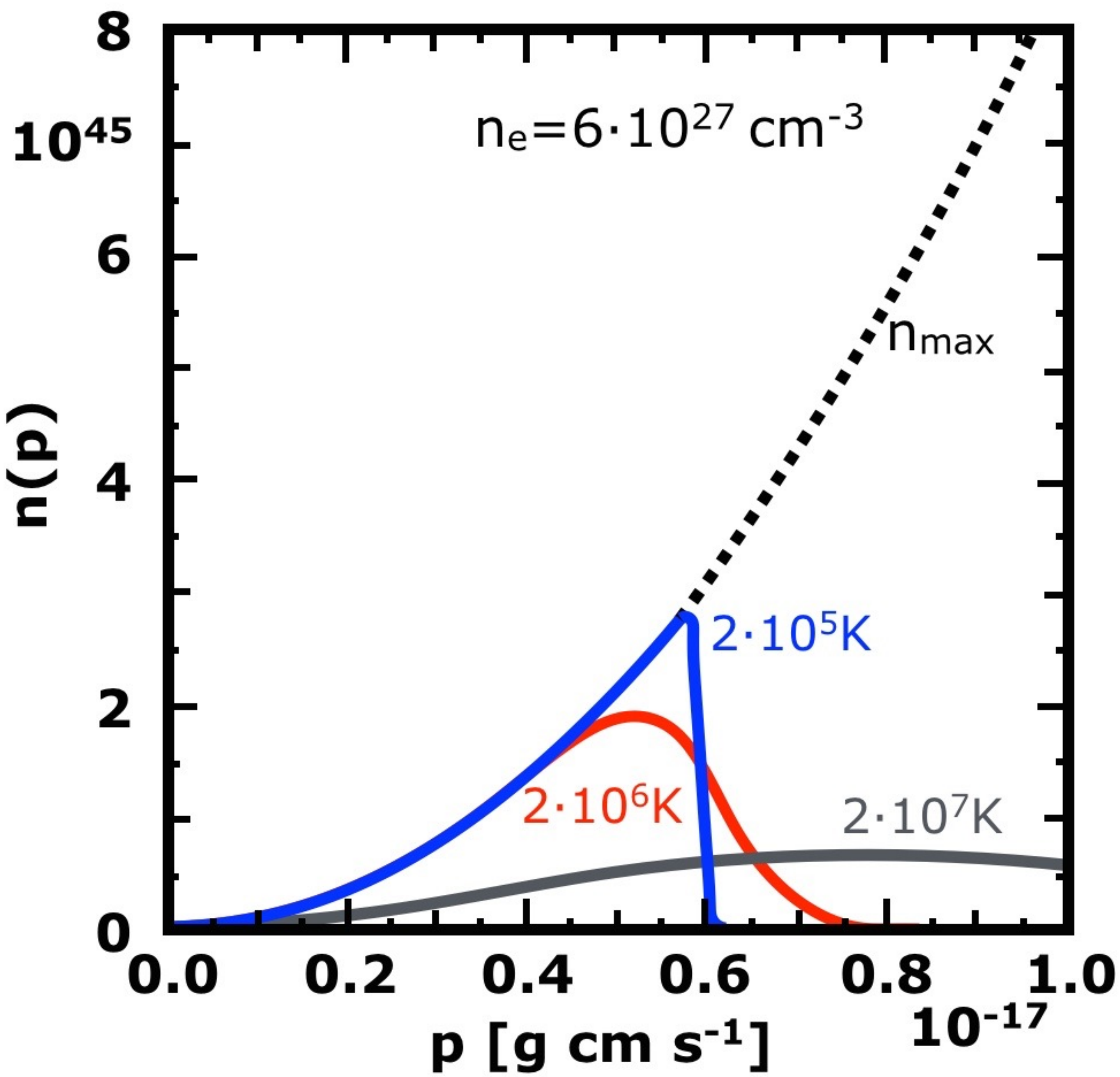


from https://www.doitpoms.ac.uk/tlplib/semiconductors/chemical_potential.php

Equation of State

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…

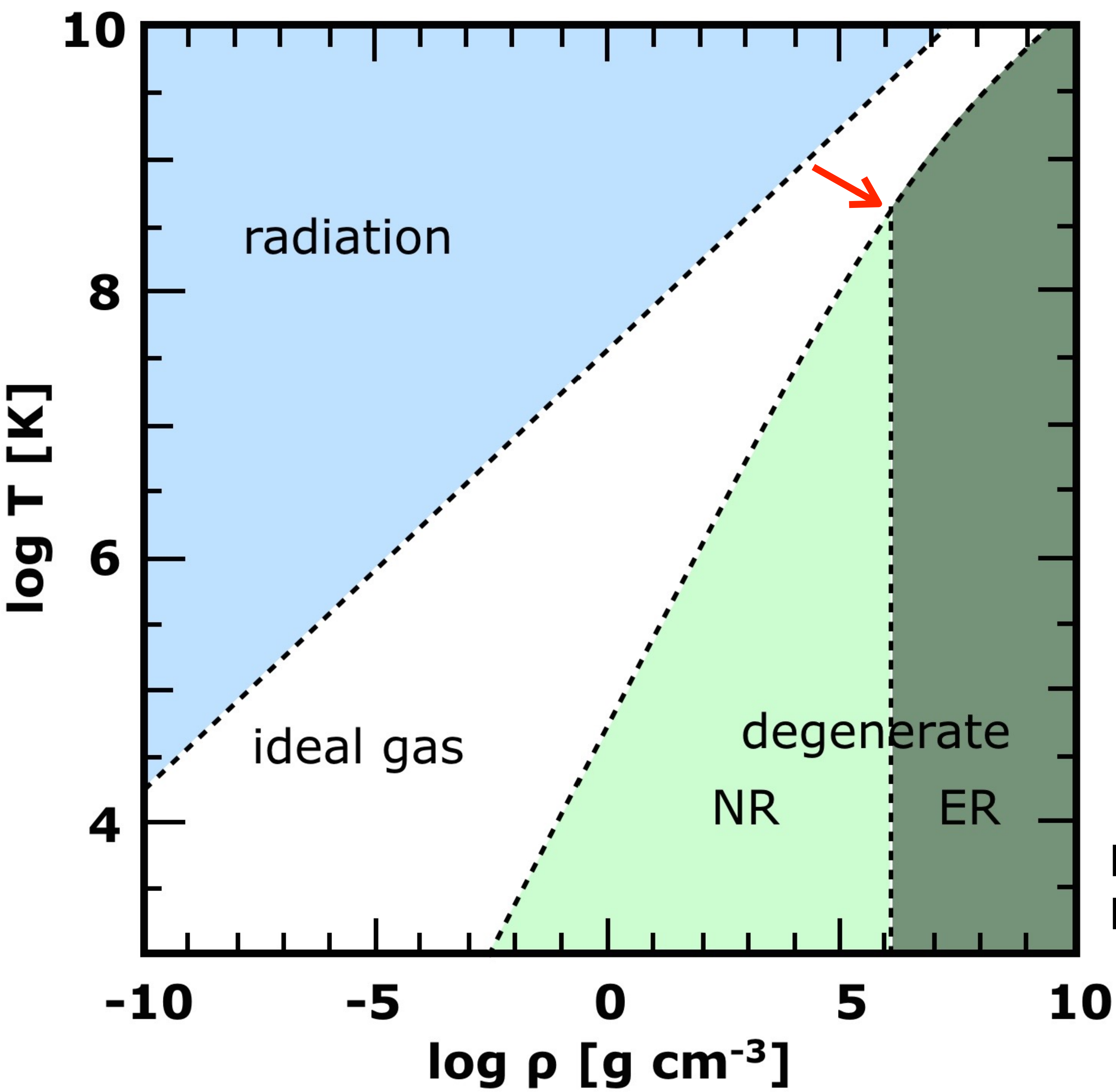


Lemers & Levesque Fig. 4.3

Equation of State

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…



Lamers & Levesque
Figure 4.4

- equation of state for completely degenerate e^-
- transition from 5/3 to 4/3 slope is smooth, but takes place around $\rho_{tr} \approx 10^6/\mu_e$ g cm^{-3}

Adapted from Onno Pols, 2011.

mean molecular weight

mean molecular weight
average particle mass in units of proton mass

$$\mu = \frac{\langle m \rangle}{m_p} = \frac{\rho}{n m_p}$$

mean molecular weight *per free electron*
mass per free electron in units of proton mass

$$\mu_e = \frac{\rho}{n_e m_p}$$

mean molecular weight *per ion*
mass per ion in units of proton mass

$$\mu_i = \frac{\rho}{n_i m_p}$$

if the material is ionized (no neutral atoms): $n = n_i + n_e \Rightarrow \frac{1}{\mu} = \frac{1}{\mu_i} + \frac{1}{\mu_e}$

Equation of State

4.6 The Equation of State (EoS) for Electron Gas

We have seen that the pressure of CD electron gas has two limiting cases. In the nonrelativistic case, $P_e = K_1 \times {n_e}^{5/3}$, and in the relativistic case, $P_e = K_2 \times {n_e}^{4/3}$. We can derive the electron density ${n_e}^{\mathrm{crit}}$ at which the gas goes from CD to RD by requiring

$$\begin{aligned} P_e(CD) = P_e(\mathrm{RD}) &\rightarrow K_1(\rho/\mu_e)^{5/3} = K_2(\rho/\mu_e)^{4/3} \rightarrow \\ &\rightarrow (\rho/\mu_e)_{\mathrm{crit}} = (K_2/K_1)^3 = 1.91 \times 10^6 \ \mathrm{g\ cm^{-3}}. \end{aligned} \tag{4.24}$$

Similarly, the boundary between the ideal gas law and CD follows from

$$n_e\ kT = K_1(\rho/\mu_e)^{5/3} \rightarrow T = \frac{K_1 m_{\mathrm{H}}}{k}(\rho/\mu_e)^{2/3} = 1.21 \times 10^5\ (\rho/\mu_e)^{2/3}\ \mathrm{K}. \tag{4.25}$$

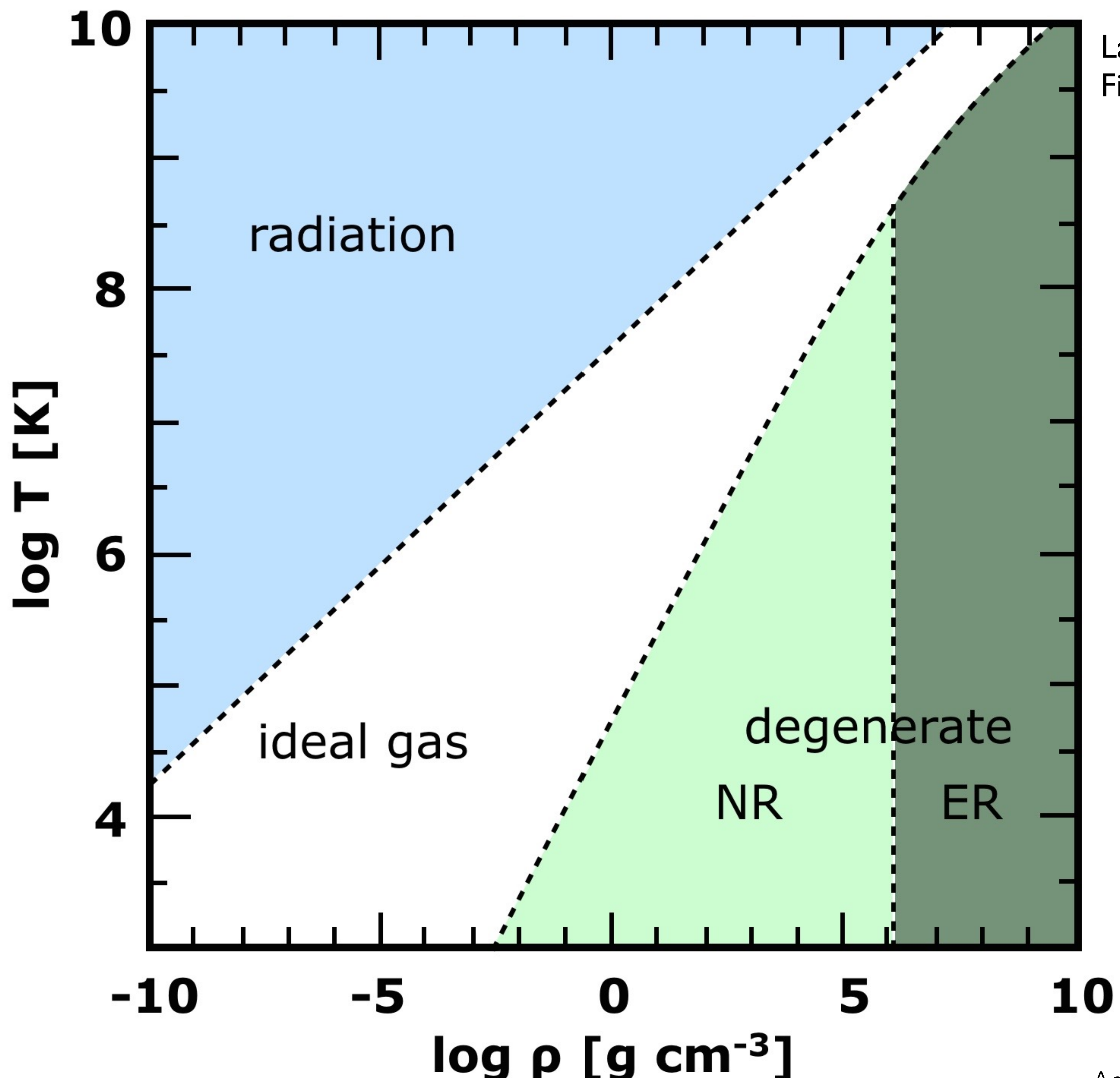
The boundary between the ideal gas law and RD follows from

$$n_e\ kT = K_2(\rho/\mu_e)^{4/3} \rightarrow T = \frac{K_2 m_{\mathrm{H}}}{k}(\rho/\mu_e)^{1/3} = 1.50 \times 10^7\ (\rho/\mu_e)^{1/3}\ \mathrm{K}. \tag{4.26}$$

The boundary between radiation pressure and the ideal gas law follows from

$$n_e\ kT = \frac{a}{3}T^4 \rightarrow T = \left(\frac{3k}{a m_{\mathrm{H}}}\right)^{1/3}(\rho/\mu_e)^{1/3} = 3.20 \times 10^7\ (\rho/\mu_e)^{1/3}\ \mathrm{g\ cm^{-3}}. \tag{4.27}$$

Equation of State

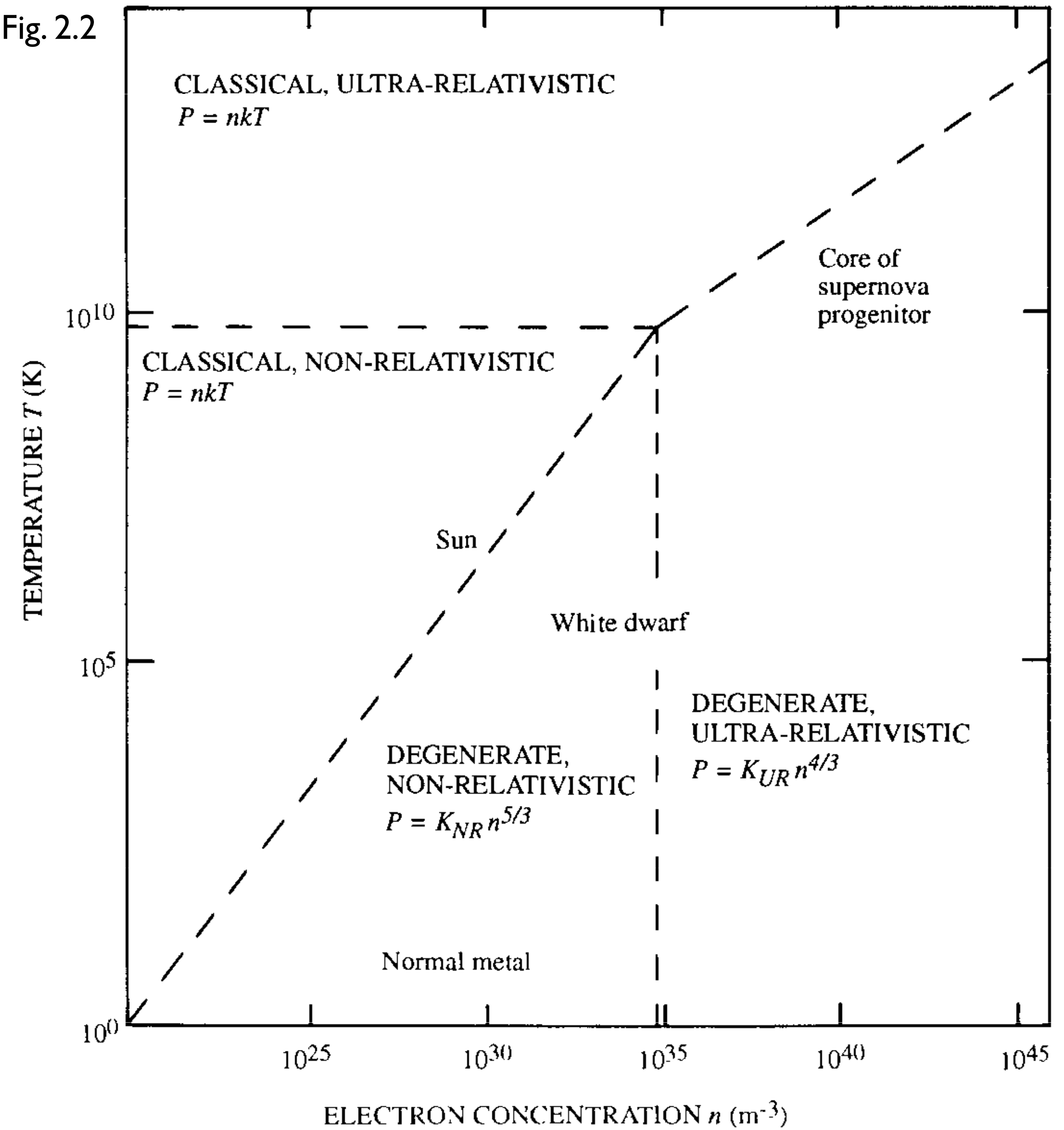


Lamers & Levesque
Figure 4.4

Adapted from Onno Pols, 2011.

Equation of State

Phillips Fig. 2.2



Neutron Degeneracy

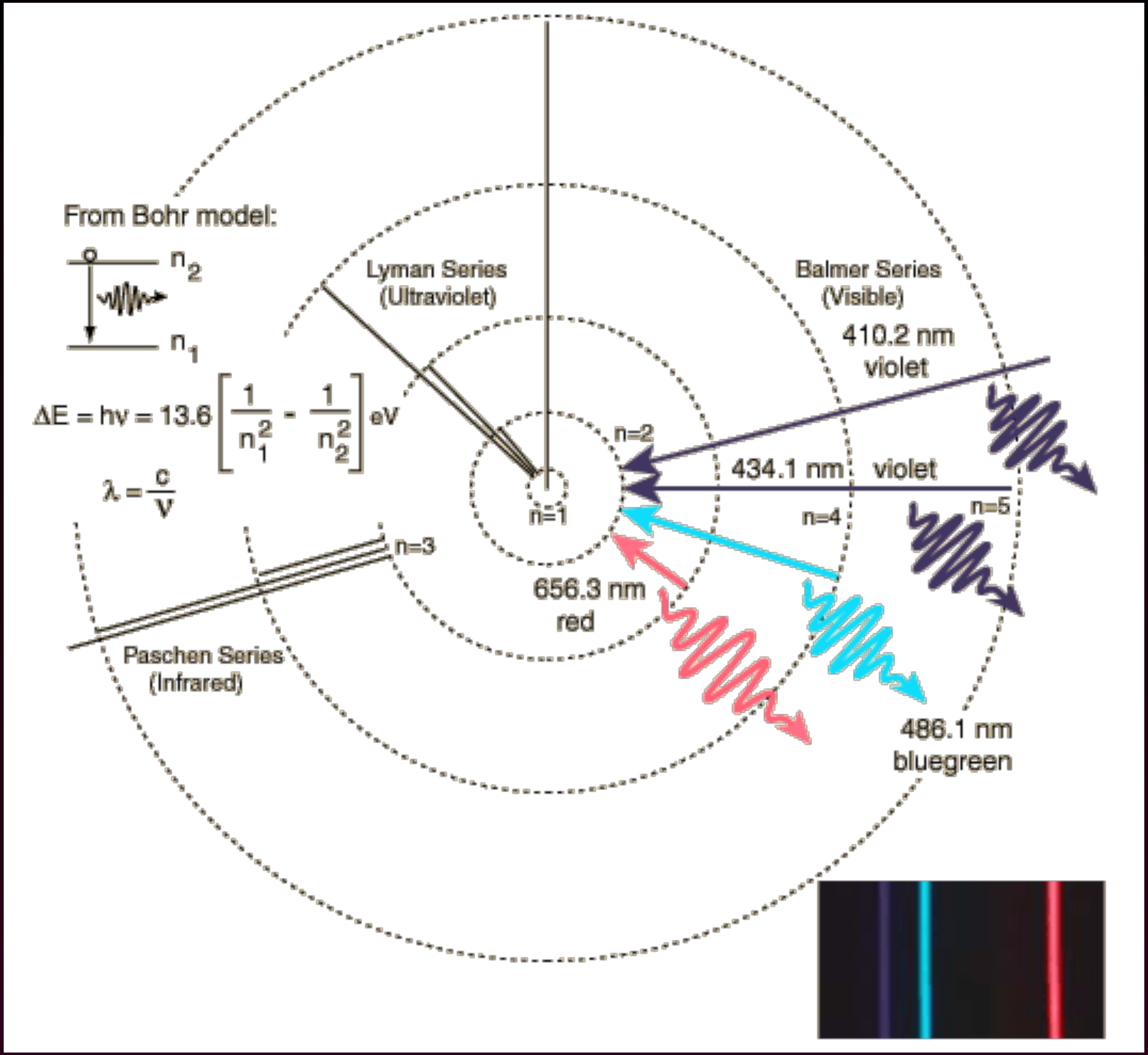
At $\rho > 3\times10^{7}$ g cm^{-3} the Fermi energy of the degenerate electrons overcomes the energy difference between n and p (1.29 MeV). $p+e \rightarrow n+\nu$ and n density increases dramatically.

For neutrons (similar to electrons, but much higher ρ):

$$P_{\rm n}({\rm CD}) = K_{1,{\rm n}}(\rho/\mu_{\rm n})^{5/3} \quad \text{for } 3\times10^{7} < \rho < 6\times10^{15} \text{ g/cm}^3$$
$$P_{\rm n}({\rm RD}) = K_{2,{\rm n}}(\rho/\mu_{\rm n})^{4/3} \quad \text{for } \rho > 6\times10^{15} \text{ g/cm}^3$$

The similarity of these equations to those for electron degeneracy impose similar rules on white dwarfs and neutron stars (both have mass-radius relations and maximum stable masses…)

recall the Bohr model



from http://outreach.atnf.csiro.au/education/senior/astrophysics/spectroscopyhow.html

Bohr model; energy levels

$$F = \frac{kq_1q_2}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2} \quad \text{SI, charge in Coulombs}$$

$$F = \frac{q_1q_2}{r^2} \quad \text{Gaussian CGS, charge in esu = statC}$$

one electron (charge $-e$)
orbiting nucleus with atomic number Z (charge $+Ze$)
Bohr model: quantized orbits, electron can only occupy fixed *energy levels*

$$r_n = \frac{4\pi\epsilon_0\hbar^2}{e^2 m_e}\frac{n^2}{Z} \qquad E_n = -\frac{1}{2}\frac{Ze^2}{4\pi\epsilon_0 r_n} = -\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{m_e}{2\hbar^2}\frac{Z^2}{n^2} \approx -13.6\ \text{eV}\left(\frac{Z^2}{n^2}\right)$$

for example, hydrogen ($Z = 1$) $\quad E_n = \dfrac{-13.6\ \text{eV}}{n^2}$

statistical weight $\quad g_n = 2n^2$

number of "spots" for the electron

energy level occupation: Boltzmann factor

$$\mathrm{Probability}(n) \propto g_n \, e^{-E_n/kT} \qquad \frac{n_m}{n_n} = \frac{g_m}{g_n} e^{-(E_m - E_n)/kT}$$

energy level occupation depends on temperature
compare *kT* with energy difference between levels

for example, hydrogen ($Z = 1$) $\quad E_n = \dfrac{-13.6 \text{ eV}}{n^2}$

statistical weight $\quad g_n = 2n^2$

number of “spots” for the electron

ionization: Saha equation

ionization stages (labeled with Roman numerals)

I: neutral   II: singly ionized   III: doubly ionized   …

one electron removed   two electrons removed

mean molecular weight

mean molecular weight
average particle mass in units of proton mass

$$\mu = \frac{\langle m \rangle}{m_p} = \frac{\rho}{n m_p}$$

mean molecular weight *per free electron*
mass per free electron in units of proton mass

$$\mu_e = \frac{\rho}{n_e m_p}$$

mean molecular weight *per ion*
mass per ion in units of proton mass

$$\mu_i = \frac{\rho}{n_i m_p}$$

if the material is ionized (no neutral atoms): $n = n_i + n_e \Rightarrow \frac{1}{\mu} = \frac{1}{\mu_i} + \frac{1}{\mu_e}$

fully ionized hydrogen: 1 proton for every free electron, $\mu_e = 1$
fully ionized helium: 2 protons, 2 neutrons, 2 free electrons, $\mu_e \approx 2$

ionization: Saha equation

ionization stages (labeled with Roman numerals)

I: neutral   II: singly ionized   III: doubly ionized   …

one electron removed   two electrons removed

electron probability distribution now depends on Boltzmann factors for bound states (discrete energy levels) and *continuum* of free electron states free electron energies described by *Maxwell-Boltzmann distribution*

number of neutral vs. ionized depends on temperature, *ionization potential* (energy needed to liberate an electron), and *free electron density* (more free electrons, more *recombination*)

Saha equation:

$$\frac{n_{II}}{n_I} = \frac{2Z_{II}}{n_e Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

partition function:

describes bound states occupation

$$Z = \sum_{n=1}^{\infty} g_n \, e^{-(E_n - E_1)/kT}$$

Saha equation example: Sun

Saha equation: $$\frac{n_{II}}{n_I} = \frac{2Z_{II}}{n_e Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

if electrons can be treated as an ideal gas, then $P_e = n_e kT$

example, hydrogen near the Sun's surface:

$$Z_I \approx 2 \quad Z_{II} = 1$$

From the ground state, the ionization energy is $\chi_I = 13.6$ eV. For the Sun, $T_\odot = 5770$ K and P_e is about 15 g cm^{-1} s^{-2} near the surface. So the Saha equation gives

$$\frac{n_{II}}{n_I} = \frac{2Z_{II}kT}{P_e Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT} = 7.4 \times 10^{-5}$$

almost all of the hydrogen is neutral in the Sun's atmosphere

in the interior of stars, higher temperature, gas (plasma) nearly fully ionized

ionization fraction

$$y = x(\text{element}) = \frac{n_{\text{ionized}}}{n_{\text{neutral}}}$$

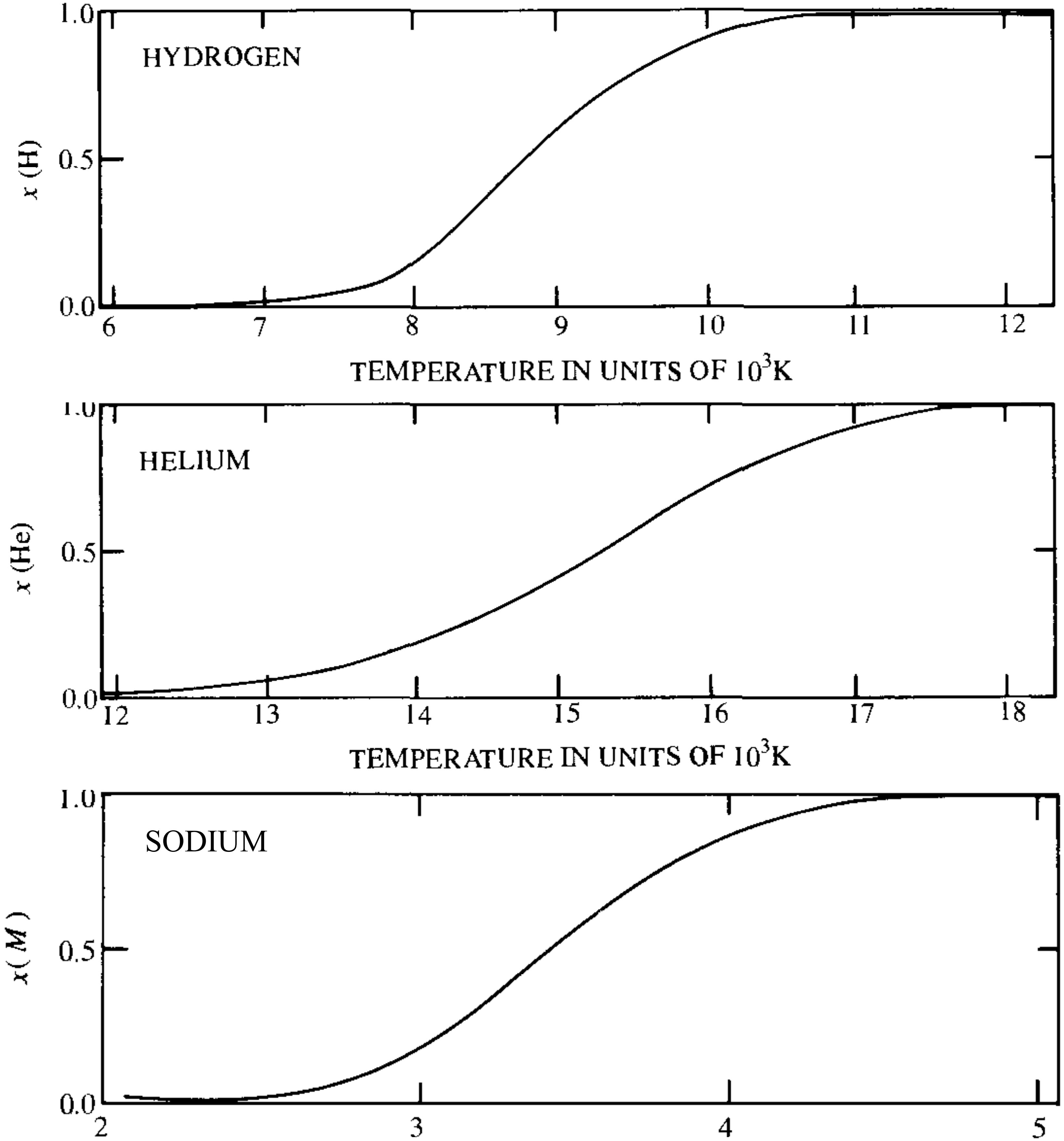
for example, hydrogen: $\quad y = x(\text{H}) = \dfrac{n_{II}}{n_I + n_{II}}$

$\approx 7 \times 10^{-5}$ at the Sun's surface

helium at the Sun's surface is also nearly completely neutral,
but there are some elements with low ionization potential ($\chi_I \simeq kT$)
that are ionized (e.g., calcium)

Ionization: Saha equation

$$n_e = 10^{19}\ \mathrm{m}^{-3}$$
$$= 10^{13}\ \mathrm{cm}^{-3}$$



Phillips Fig 2.4

Phillips Fig 2.5

generalized Saha equation

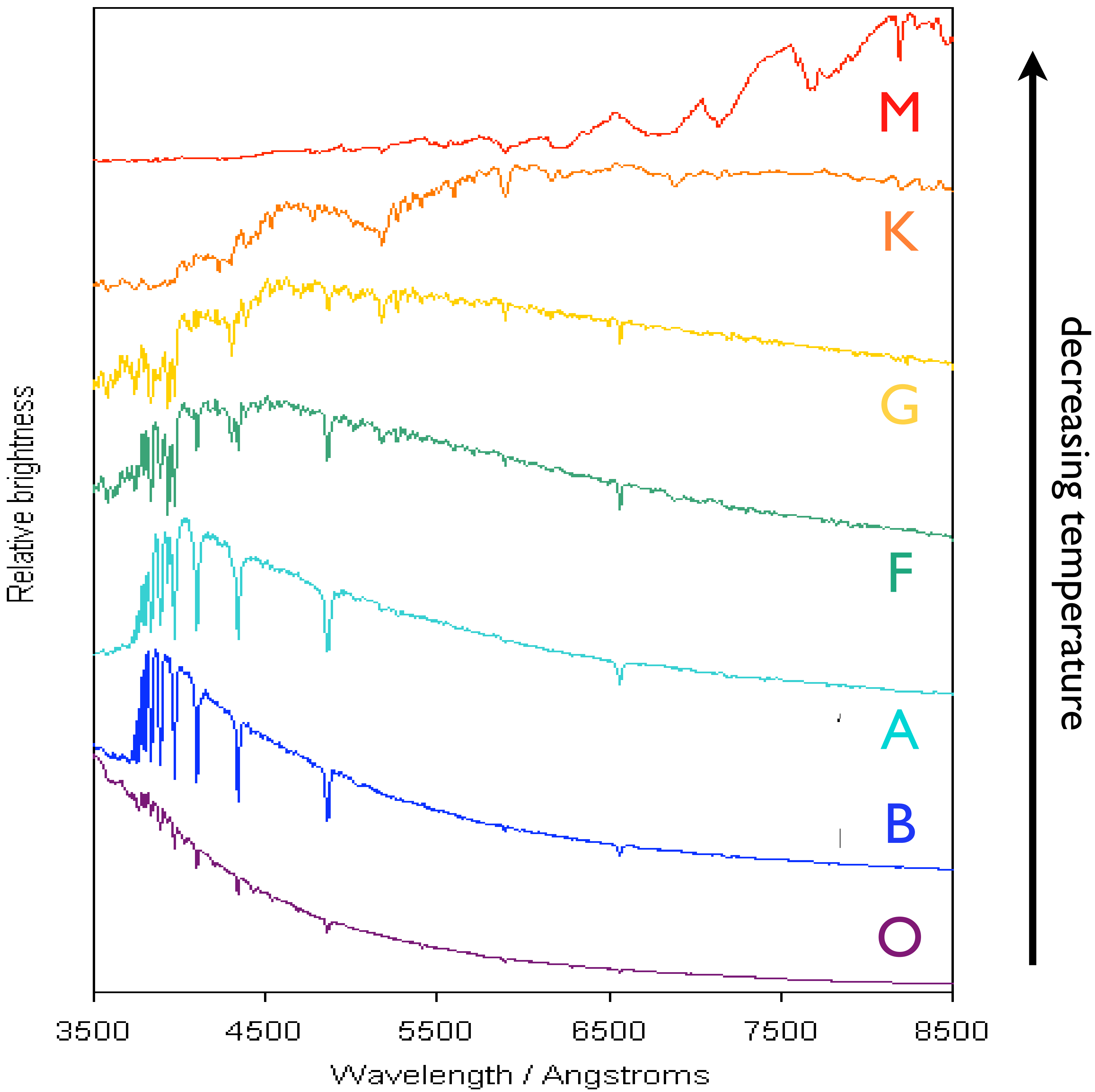
Saha equation:

$$\frac{n_{II}}{n_I} = \frac{2Z_{II}}{n_e Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_I/kT}$$

can also apply to higher ionization states, e.g., $n_{II} \rightarrow n_{III}$
just replace $n_I \rightarrow n_{II}$, $Z_{II} \rightarrow Z_{III}$, $Z_I \rightarrow Z_{II}$, and $\chi_I \rightarrow \chi_{II}$

higher ionization states have higher ionization potentials
(harder to remove each subsequent electron)
so require high temperatures

Pioneers of Stellar Classification



from http://www.jb.man.ac.uk/distance/life/sample/stars/spectra.gif

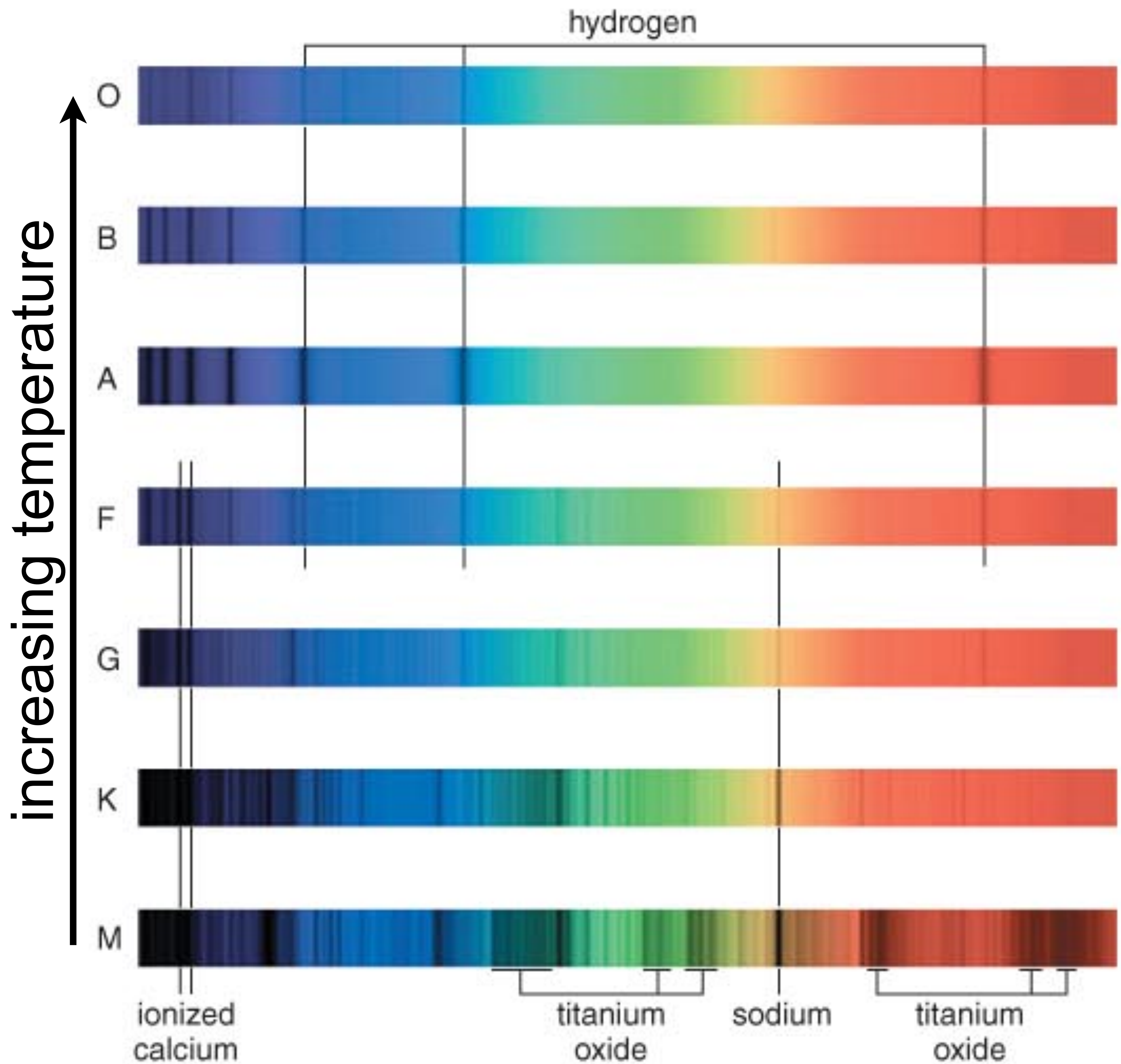
Annie Jump Cannon
from http://www.twu.edu/dsc/

Stellar spectral types

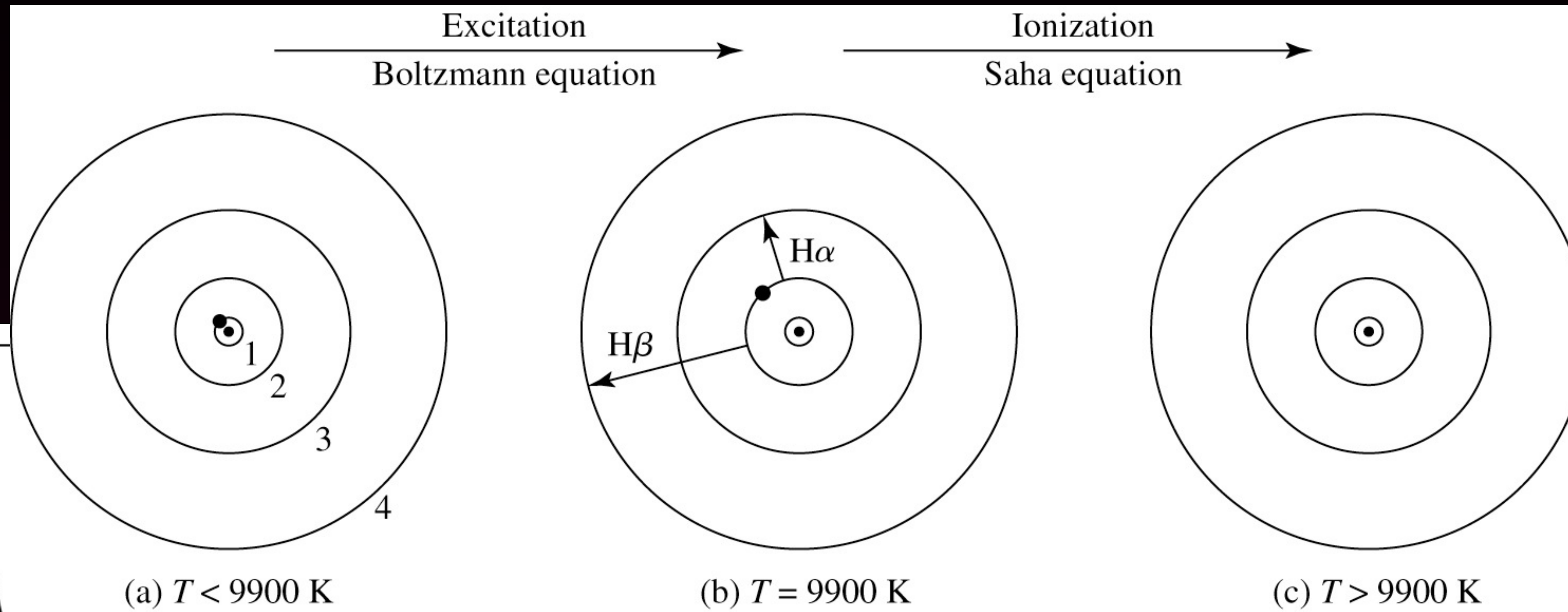
Lines in a star's spectrum correspond to a ***spectral type*** that reveals its temperature.

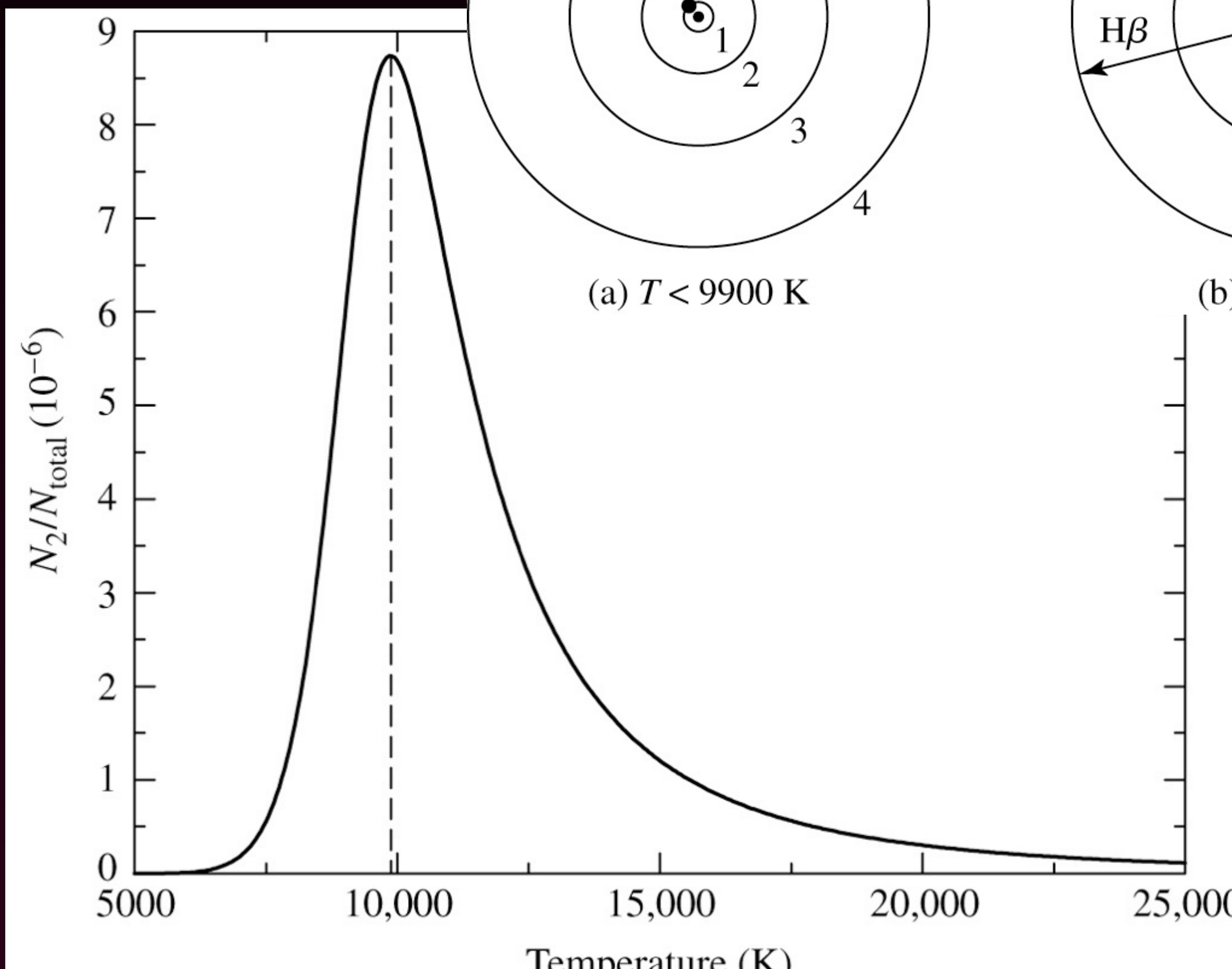
O B A F G K M

(from hottest to coldest)



Balmer line strength (n=2 occupation fraction)

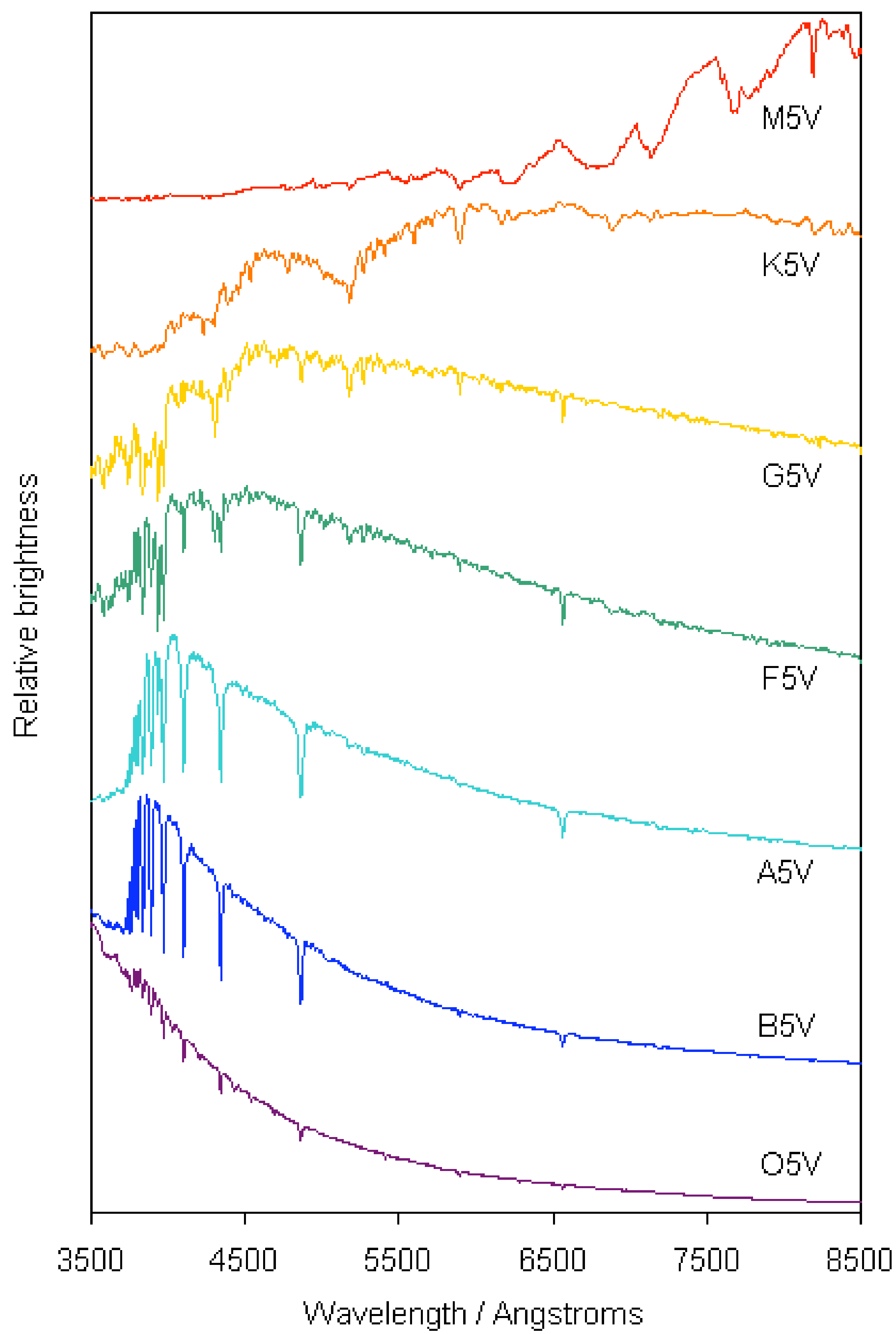




Carroll & Ostlie Figures 8.9 and 8.10

understanding stars from spectra

from http://www.jb.man.ac.uk/distance/life/sample/stars/spectra.gif and Carroll & Ostlie Figure 8.11



Type	T (K)	Spectral Lines
O	>25000	ionized He
B	11000–25000	neutral He, some H
A	7500–11000	strong H; some ionized metals (Mg II, Ca II)
F	6000–7500	weaker H; ionized metals (Ca II, Fe II, Fe I)
G	5000–6000	ionized and neutral metals (Fe II, Fe I)
K	3500–5000	strong metals (Ca I, Sr II)
M	<3500	molecules (TiO)

What we can learn about stellar atmospheres from spectra?

1. Absorption lines $\Rightarrow$ certain atoms and molecules are present.
2. Line strengths $\Leftarrow$ temperature and composition.