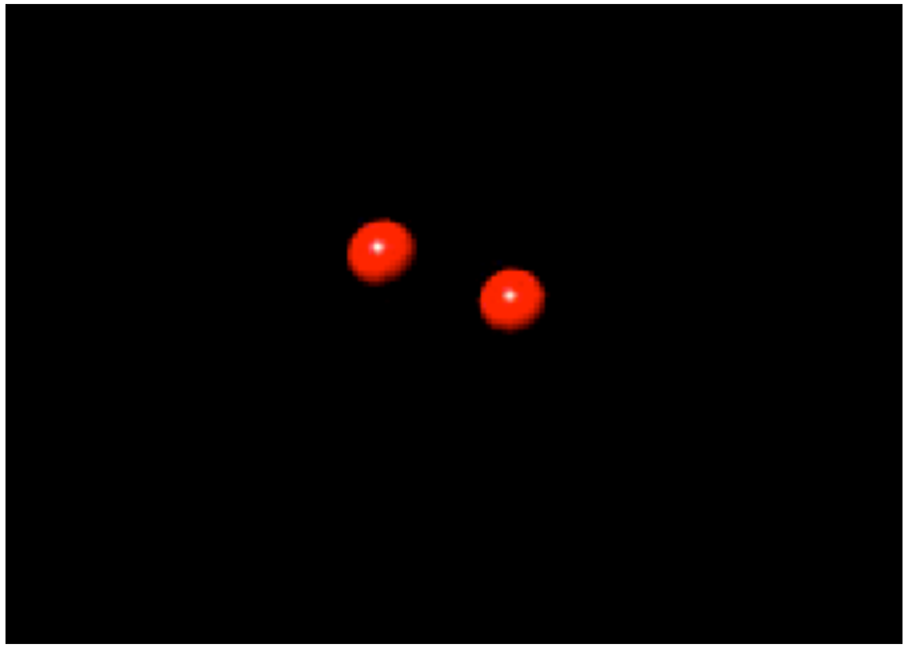
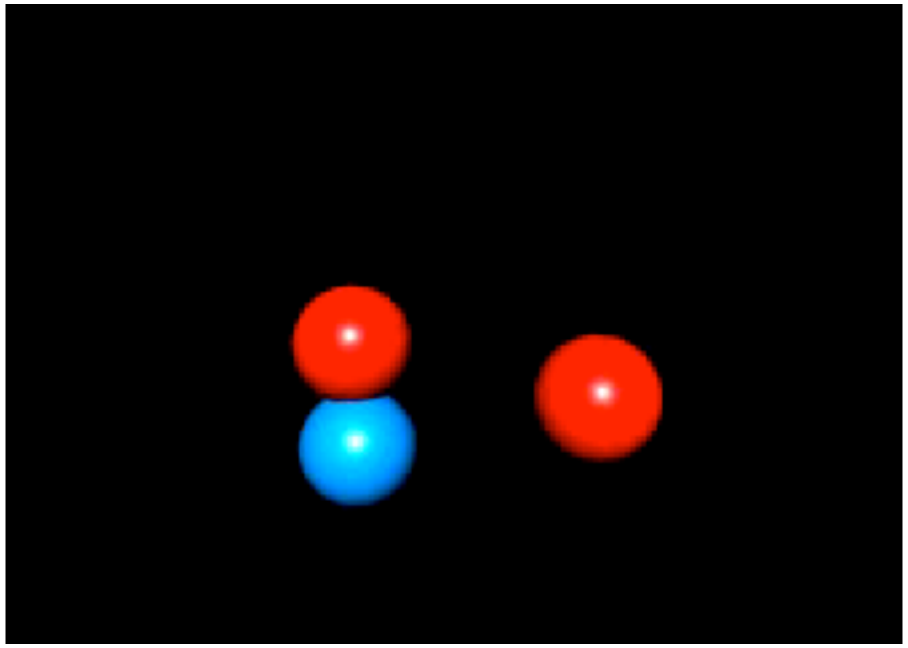
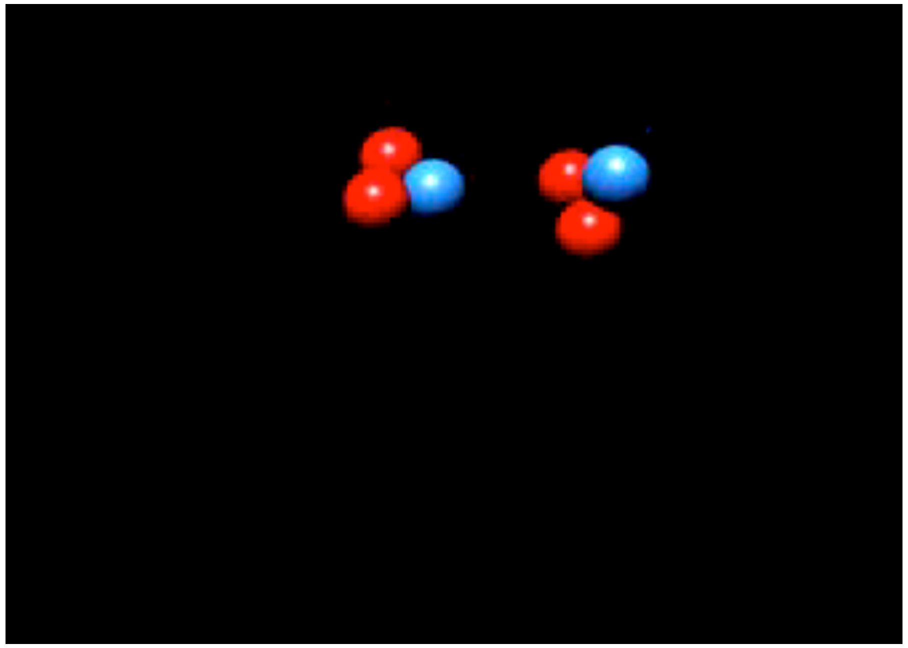
Lecture 9: Nuclear Reactions

Phillips Ch. 4







Animations of the proton-proton chain,
from http://atropos.as.arizona.edu/aiz/teaching/a250/pp.html and Prof. Greg Bothun

Physics 441/541 Spring 2022: Problem Set #3
due February 25 at 11:00 am in PDF format on Canvas

0. Identify the people you will work with on the group project for the end of the semester, where you will present an approximately 20-minute lecture on a topic. The groups should consist of three people (a few two-person groups may be necessary). Provide your top three choices for a group project topic. There is a list of potential topics on the course website, but you can also come up with your own ideas.

Potential topics for group presentations: Helio/asteroseismology. LIGO black holes and their progenitors. Population III stars. Metal-poor stars. Brown dwarfs. Stellar rotation/activity/age. Exoplanet host stars. Pulsars. Magnetars. Gamma-ray bursts. Stellar initial mass function. Stellar multiplicity. Stellar winds/mass-loss. Planetary nebulae. Numerical modeling (MESA). Stellar pulsation/variables. Standard candles (Cepheids, RR Lyrae, Mira). History of stellar classification.

if you want to operate this kitchen refrigerator using the fusion energy of the Sun what *volume* of the Sun would give you the needed power?

Nuclear reaction energies

We will see that the nuclear reaction powering the Sun is the fusion of 4 hydrogen atoms into 1 helium atom. What is the mass involved?

$$\begin{aligned} \text{4 hydrogen :} \qquad m_{4\mathrm{H}} &= 3755.13 \text{ MeV}/c^2 = 4.031300 \text{ u} \\ \text{helium :} \qquad m_{\mathrm{He}} &= 3728.40 \text{ MeV}/c^2 = 4.002603 \text{ u} \\ \Rightarrow \Delta m &= 26.73 \text{ MeV}/c^2 = 0.007\, m_{4\mathrm{H}} = 0.028697 \text{ u} \\ E &= \Delta m\, c^2 = 26.73 \text{ MeV} \end{aligned}$$

In other words, each hydrogen atom gives up 0.7% of its mass. This quantity is sometimes called the **efficiency** ϵ of a nuclear reaction

$$\epsilon = \frac{m_{\mathrm{start}} - m_{\mathrm{end}}}{m_{\mathrm{start}}} = \frac{\Delta m}{m_{\mathrm{start}}}$$

so that the total energy released starting with nuclear “fuel” of mass M is just

$$E = \Delta m\, c^2 = \epsilon M c^2$$

For hydrogen fusion into helium, the efficiency is $\epsilon = 0.007$.

Nuclear energy in the Sun

How much fusion energy is available? A first estimate is:

$$\begin{aligned} E_{\rm nuclear} &\sim 0.007 \times M_\odot\, c^2 = 1.3 \times 10^{52}\ {\rm erg} \\ t_{\rm nuclear} &\sim \frac{E_{\rm nuclear}}{L_\odot} \sim 10^{11}\ {\rm yr} \end{aligned}$$

Roughly 100 billion years. In fact, only about 10% of the Sun's mass is available for fusion (i.e., in a region where the temperature is high enough for fusion to occur). So the actual energy and lifetime are about a factor of 10 smaller (10 billion years). Nevertheless, that is still plenty to explain the Sun.

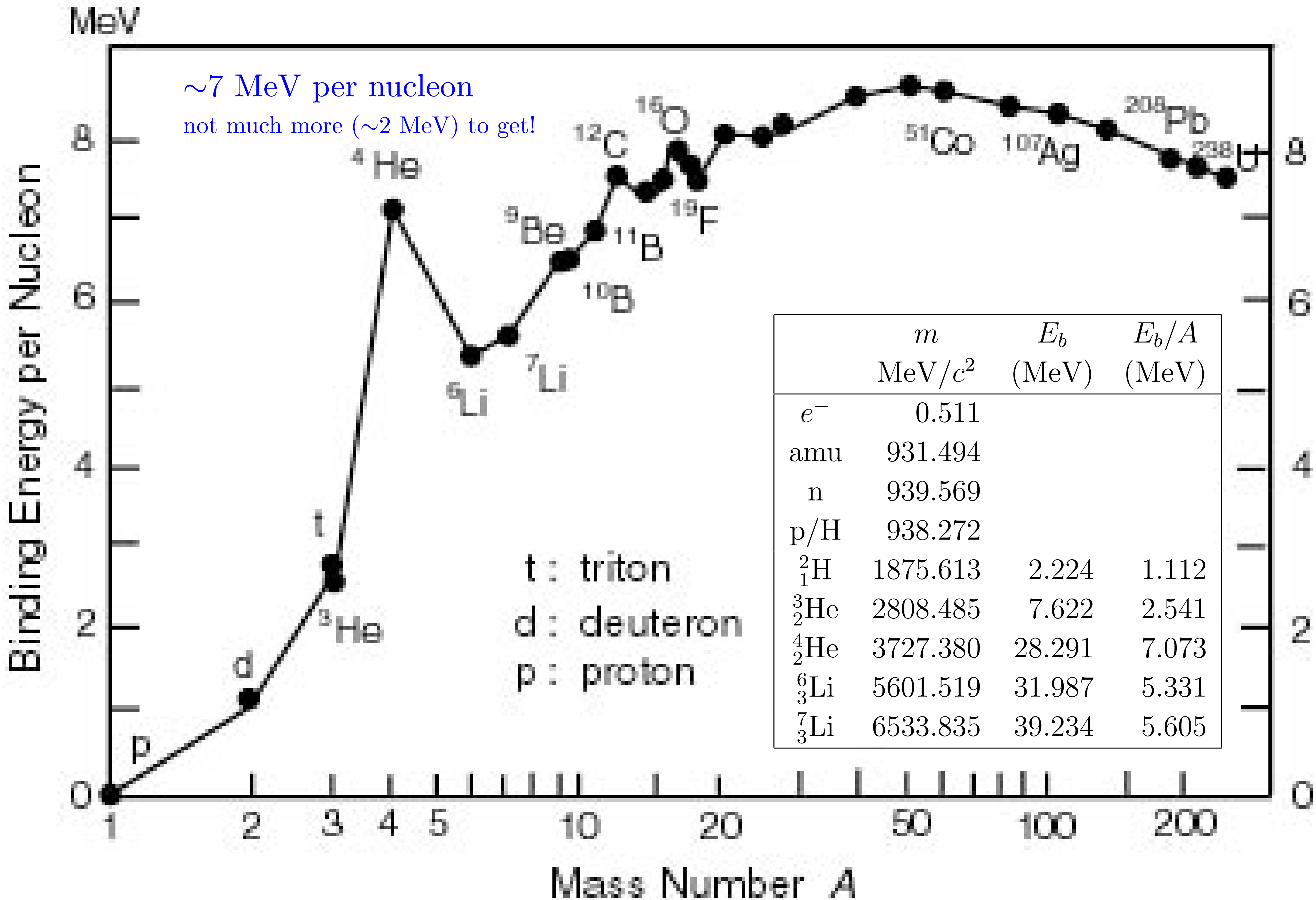
Nuclear binding energy

To understand how much energy can be released in fusion, let's tabulate some masses and energies. Define the **binding energy** to be the difference between the actual mass of a nucleus and the mass of the same number of isolated protons and neutrons:

$$E_b = \left[Z\, m_p + (A - Z)\, m_n - m_{\text{nuc}} \right] c^2$$

	m MeV$/c^2$	E_b (MeV)	E_b/A (MeV)
e^-	0.511		
n	939.569		
p/H	938.272		
${}^2_1$H	1875.613	2.224	1.112
${}^3_2$He	2808.485	7.622	2.541
${}^4_2$He	3727.380	28.291	7.073
${}^6_3$Li	5601.519	31.987	5.331
${}^7_3$Li	6533.835	39.234	5.605

Nuclear binding energy per nucleon



	m MeV/c^2	E_b (MeV)	E_b/A (MeV)
e^-	0.511		
amu	931.494		
n	939.569		
p/H	938.272		
$^2_1$H	1875.613	2.224	1.112
$^3_2$He	2808.485	7.622	2.541
$^4_2$He	3727.380	28.291	7.073
$^6_3$Li	5601.519	31.987	5.331
$^7_3$Li	6533.835	39.234	5.605

from http://www.kutl.kyushu-u.ac.jp/seminar/MicroWorld3_E/3Part2_E/3P23_E/nuclear_mass_E.htm

Nuclear cross section

$$\sigma(E) = \frac{\text{number of reactions / nucleus / time}}{\text{number of incident particles / area / time}}$$

units of area

mean free path $\quad \ell = \frac{1}{n\sigma}$

Gamow energy: $E_G = (\pi\alpha Z_A Z_B)^2 2 m_r c^2$ Probability of tunneling $\approx \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right]$

nuclear S factor

$$\sigma(E) = \frac{S(E)}{E} \underbrace{\exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right]}_{\text{probability of tunneling}}$$

effective "size" of nucleus (de Broglie wavelength)^2

nuclear S factor encapsulates all the details of the nuclear physics

$S(E)$ typical units of keV barn
1 barn = 10^{-24} cm^2

Nuclear reaction rates

see Phillips Ch. 4 for derivation

reaction rate between nucleus A and nucleus B

$$R_{AB} = n_A n_B \left[\frac{8}{\pi m_r}\right]^{1/2} \left[\frac{1}{kT}\right]^{3/2} \int_0^\infty E\sigma(E) \exp\left[-\frac{E}{kT}\right] \mathrm{d}E$$

n_A, n_B : number densities of A and B

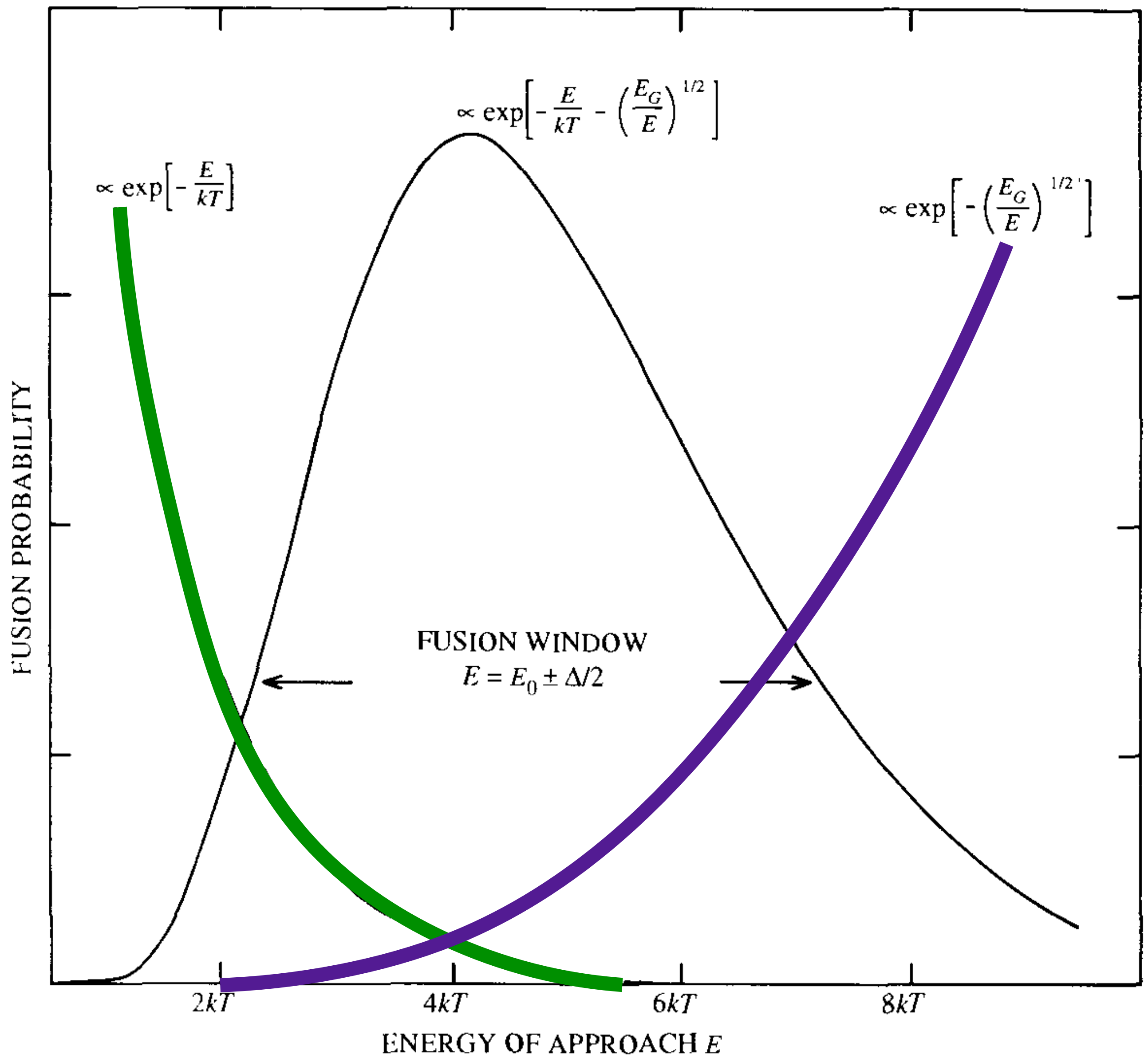
reduced mass $m_r = \dfrac{m_A m_B}{m_A + m_B}$

$$\sigma(E) = \frac{S(E)}{E} \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right]$$

$$R_{AB} = n_A n_B \left[\frac{8}{\pi m_r}\right]^{1/2} \left[\frac{1}{kT}\right]^{3/2} \int_0^\infty S(E) \underbrace{\exp\left[-\frac{E}{kT} - \left(\frac{E_G}{E}\right)^{1/2}\right]}_{\text{leads to “Gamow peak”}} \mathrm{d}E$$

Gamow peak

Phillips Figure 4.3



number of nuclei
decreases
at higher energy

tunneling probability
increases
at higher energy

George Gamow

Fig. 4.3 The energy window for the fusion of nuclei with a Gamow energy E_G and temperature T. To react at energy E, the nuclei need to borrow an energy E from the thermal environment, and the probability of a successful loan is proportional to the Boltzmann factor $\exp[-E/kT]$. To fuse, the nuclei must first penetrate the Coulomb barrier keeping them apart, and the probability of penetration is given by the factor $\exp[-(E_G/E)^{1/2}]$. The product of these two factors indicates that fusion mostly occurs in an energy window $E_0 \pm \Delta/2$. For the fusion of two protons at 2×10^7 K, $E_G = 290kT$, $E_0 = 4.2kT$, and $\Delta = 4.8kT$, as illustrated.

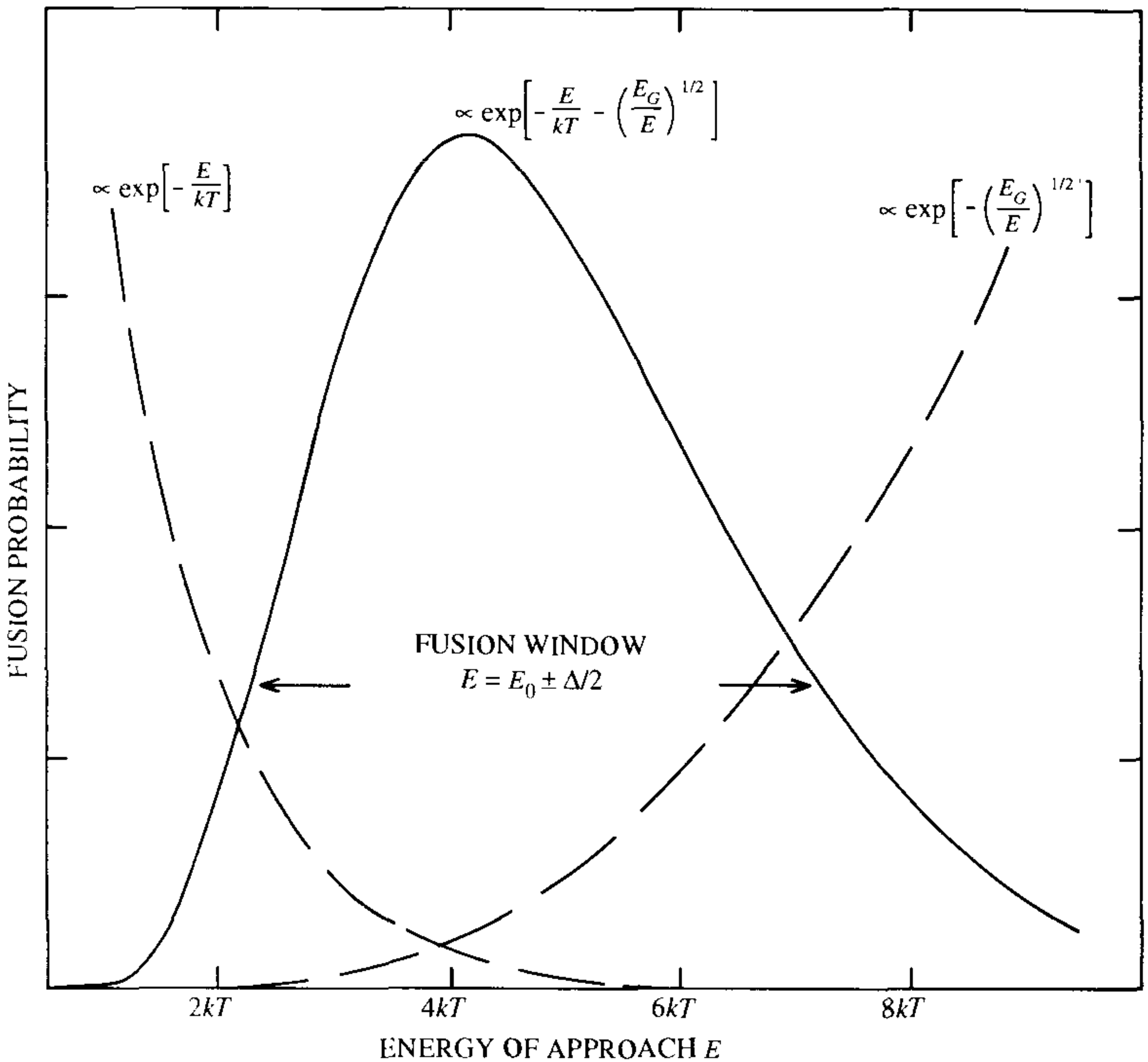
product of the two effects: Gamow peak

Gamow peak: energy where fusion occurs

$$\exp\left[-\frac{E}{kT}-\left(\frac{E_G}{E}\right)^{1/2}\right]$$

maximized at Gamow peak:

$$E_0=\left[\frac{E_G(kT)^2}{4}\right]^{1/3}$$



width of Gamow peak:

$$\Delta=\frac{4}{3^{1/2}2^{1/3}}E_G^{1/6}(kT)^{5/6}=1.83E_G^{1/6}(kT)^{5/6}$$

Strong temperature dependence

can approximate reaction rate as:

$$R_{AB} \propto n_A n_B S(E_0) \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]$$

temperature dependence (show this on problem set 3)

$$R_{AB} \propto T^{\alpha} \qquad\qquad \alpha = \left(\frac{E_G}{4kT}\right)^{1/3} - \frac{2}{3}$$

for proton-proton fusion, $\alpha \approx 4$
heavier nuclei fusion: higher Coulomb barrier, larger α

Hydrogen fusion to Helium

net reaction (not all in one step):

$$4\ {}^1_1\mathrm{H} \rightarrow {}^4_2\mathrm{He} + 2e^+ + 2\nu_e + 2\gamma$$

one way to do this: *p-p* chain, starts with

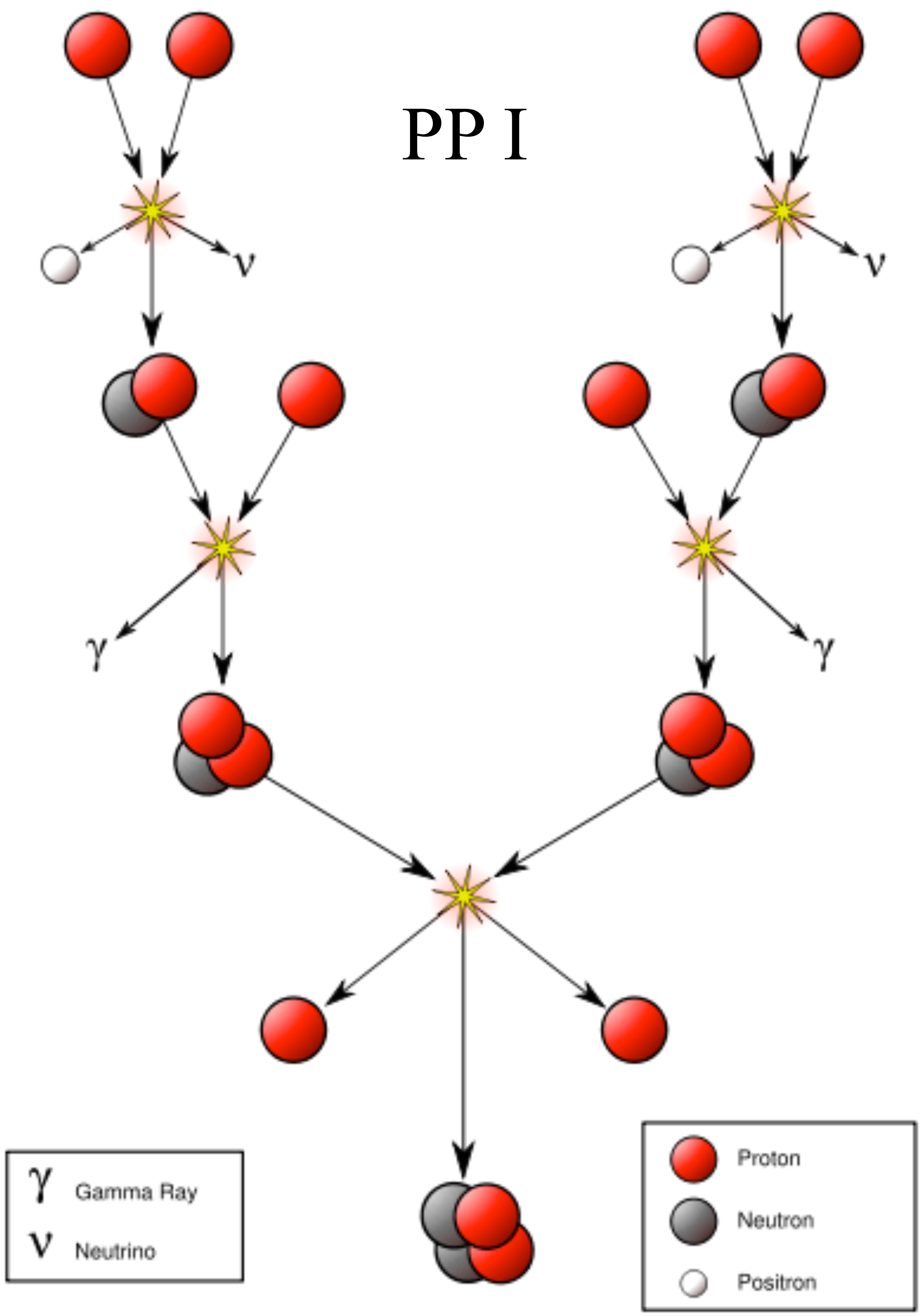
$$\begin{array}{lll} {}^1_1\mathrm{H} + {}^1_1\mathrm{H} & \rightarrow & {}^2_1\mathrm{H} + e^+ + \nu_e \quad [S(E) = 3.8 \times 10^{-22}\ \mathrm{keV\ barn}] \\ {}^2_1\mathrm{H} + {}^1_1\mathrm{H} & \rightarrow & {}^3_2\mathrm{He} + \gamma \quad [S(E) = 2.5 \times 10^{-4}\ \mathrm{keV\ barn}] \end{array}$$

$\sim 10^{18}\ \mathrm{s} \sim 10\ \mathrm{Gyr}$

$\sim 1\ \mathrm{s}$

first step involves *weak nuclear force*
it is slow ("rate-limiting step")
proton needs to convert into (heavier) neutron

p-p chain



REACTIONS OF THE PROTON-PROTON CHAIN

$p + p \to d + e^+ + \nu_e \qquad S_{pp}(0) = 3.8 \times 10^{-22}$ keV barns.

$p + d \to {}^3\mathrm{He} + \gamma \qquad S_{pd}(0) = 2.5 \times 10^{-4}$ keV barns,

$^3\mathrm{He} + {}^3\mathrm{He} \to {}^4\mathrm{He} + 2p$

$^3\mathrm{He} + {}^4\mathrm{He} \to {}^7\mathrm{Be} + \gamma$

$e^- + {}^7\mathrm{Be} \to {}^7\mathrm{Li} + \nu_e$

$p + {}^7\mathrm{Li} \to {}^4\mathrm{He} + {}^4\mathrm{He}$

$p + {}^7\mathrm{Be} \to {}^8\mathrm{B} + \gamma$

$^8\mathrm{B} \to {}^8\mathrm{Be}^* + e^+ + \nu_e$

$^8\mathrm{Be}^* \to {}^4\mathrm{He} + {}^4\mathrm{He}$

BRANCH I	BRANCH II	BRANCH III
$Q_{eff} = 26.2$ MeV	$Q_{eff} = 25.7$ MeV	$Q_{eff} = 19.1$ MeV
[85%]	[15%]	[0.02%]

Fig. 4.4 The three competing branches of the proton–proton chain with the net result $4p \to {}^4\mathrm{He} + Q_{eff}$. Here Q_{eff} is the effective energy released by the branch; it includes the energy from the annihilation of positrons, but it does not include any of the energy carried away by neutrinos. Note, a pre-existing ^{4}He nucleus acts as a catalyst in branches II and III, its destruction leading to two new ^{4}He nuclei. According to the Standard Solar Model, Bahcall (1989), the proton–proton chain in the sun is terminated by branch I 85% of the time, by branch II 15% of the time and by branch III 0.02% of the time.

from http://en.wikipedia.org/wiki/Proton-proton_chain_reaction and Phillips Figure 4.4

p-p chain

because of the low probability first step,
p-p chain proceeds slowly
$\sim 5 \times 10^{13}$ reactions per cubic meter per second in the Sun

averaging over different paths, 15 MeV released per *p-p* fusion

combining these, power per unit volume from fusion in the Sun:
120 W m^{-3}

if you want to operate this kitchen refrigerator using the fusion energy of the Sun what *volume* of the Sun would give you the needed power?

about the same size as the fridge!

note that volume could power the fridge for ~10 billion years

the Sun's fusion doesn't have a high power *density* but has a large power (luminosity) because the Sun (and its core) is so large

CNO cycle

REACTIONS OF THE CARBON–NITROGEN CYCLE

$p + {}^{12}C \rightarrow {}^{13}N + \gamma$ [$S(0) = 1.5$ keV barns]

$\quad \hookrightarrow {}^{13}C + e^+ + \nu_e$

$p + {}^{13}C \rightarrow {}^{14}N + \gamma$ [$S(0) = 5.5$ keV barns]

$p + {}^{14}N \rightarrow {}^{15}O + \gamma$ [$S(0) = 3.3$ keV barns]

$\quad \hookrightarrow {}^{15}N + e^+ + \nu_e$

$p + {}^{15}N \rightarrow {}^{12}C + {}^{4}He$ [$S(0) = 78$ keV barns]

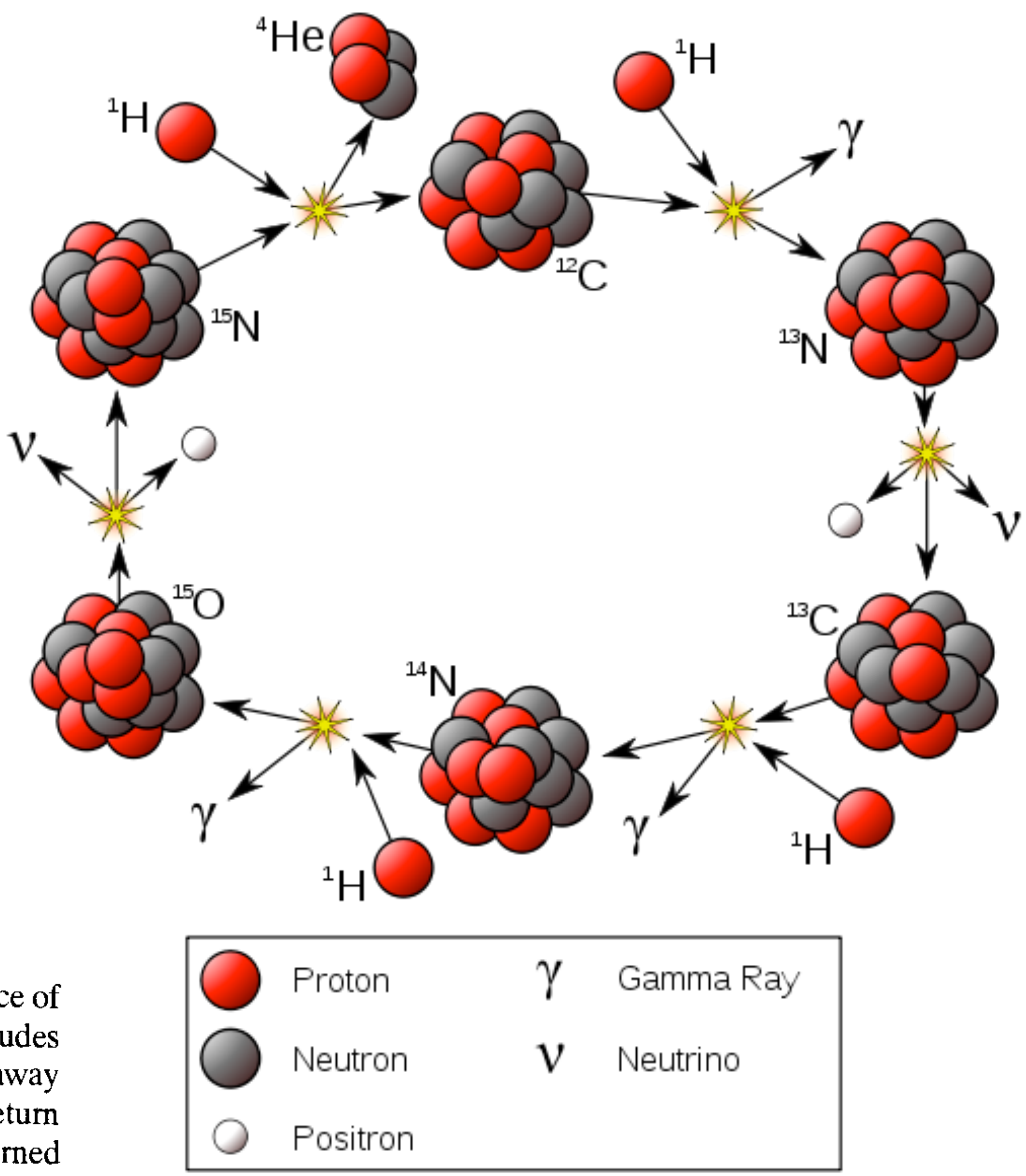
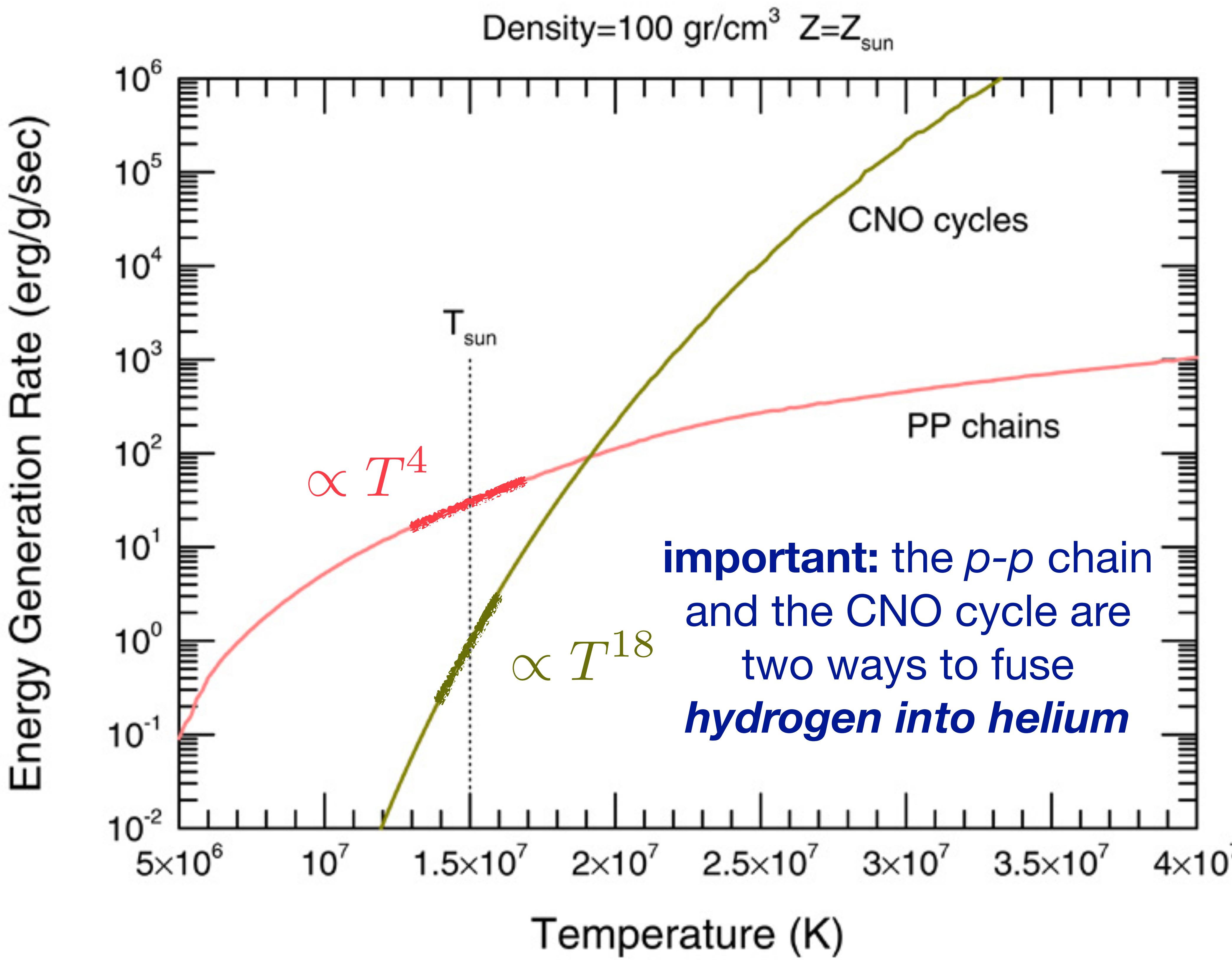


Fig. 4.5 Hydrogen burning by the carbon–nitrogen cycle. The net result of this sequence of reactions is $4p \rightarrow {}^{4}He + Q_{eff}$. The effective energy released Q_{eff} is 23.8 MeV; this includes the energy from the annihilation of positrons, but it does not include the energy carried away by neutrinos. Note that nuclei of carbon and nitrogen are temporarily transformed but return to take part in subsequent operations of the cycle. The rates for these reactions are governed by the relevant Coulomb barriers and the approximate S factors indicated.

net reaction is *still* 4 H → 1 He

Phillips Figure 4.5 and http://en.wikipedia.org/wiki/Cno_cycle

p-p chain or CNO cycle



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