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ASTR 401/501 Handout Number 12: Homologous Stars

Solving the equations of stellar structure in a detailed way generally requires computer modeling. But we can extract some back-of-the-envelope predictions by making the assumption that stars of different mass have *homologous* structure.

As an initial example, consider a reference star with mass $M_\circ$ and radius $R_\circ$. We now want to compare a second star with mass M and R to this reference. For an interior position given by radius r in the second star, this will be related to a position $r_\circ$ in our reference star by a simple proportional scaling:

$$r = \frac{R}{R_\circ} r_\circ \tag{1}$$

Similarly, the mass as a function of radius is given by

$$M(r) = \frac{M}{M_\circ} M_\circ(r) \quad . \tag{2}$$

Likewise,

$$dr = \frac{R}{R_\circ} dr_\circ \tag{3}$$

and

$$dM = \frac{M}{M_\circ} dM_\circ \quad . \tag{4}$$

We can write other similar scalings to relate other physical quantities (e.g. pressure) between the two stars.

Now consider the mass equation for our second star,

$$\frac{dM}{dr} = 4\pi r^2 \rho \quad . \tag{5}$$

Let's divide equation (5) by the mass equation for the reference star, which results in

$$\frac{dM}{dM_\circ} \frac{dr_\circ}{dr} = \frac{r^2}{r_\circ^2} \frac{\rho}{\rho_\circ} \quad . \tag{6}$$

If we substitute in for dr and dM from equations (3) and (4), we obtain

$$\frac{M}{M_\circ}\frac{R_\circ}{R} = \frac{r^2}{r_\circ^2}\frac{\rho}{\rho_\circ} \quad . \tag{7}$$

A further substitution from equation (1) implies

$$\frac{M}{M_\circ}\frac{R_\circ}{R} = \frac{R^2}{R_\circ^2}\frac{\rho}{\rho_\circ} \quad , \tag{8}$$

or

$$\frac{M}{M_\circ} = \frac{R^3}{R_\circ^3}\frac{\rho}{rho_\circ} \quad . \tag{9}$$

In these equations we can take ρ and $\rho_\circ$ as being average densities for the stars.

The quantities with subscripts can be taken as constants since they describe our reference star; we can therefore conclude that

$$M \propto R^3 \rho \quad , \tag{10}$$

which is no big surprise. But the more important point here is that we can generalize this approach to the other equations of stellar structure. The general method relies on noting

- the quantities with subscript $\circ$ are treated as constants, and
- we can replace differentials with the actual quantities.

Now let us proceed with this method:

1. Mass conservation

To repeat this derivation more succinctly, we adopt the following steps:

$$\frac{dM}{dr} = 4\pi r^2 \rho \quad , \tag{11}$$

from which it follows that

$$\frac{M}{R} \propto R^2 \rho \quad , \tag{12}$$

or

$$M \propto R^3 \rho \ . \tag{13}$$

2. Hydrostatic equilibrium

Recall

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \ , \tag{14}$$

which reduces to

$$\frac{P}{R} \propto \frac{M\rho}{R^3} \propto \frac{M^2}{R^5} \ , \tag{15}$$

or

$$P \propto \frac{M^2}{R^4} \tag{16}$$

so that

$$M^2 \propto R^4 P \ . \tag{17}$$

We can use the perfect gas law to eliminate P: $P \propto \rho T$, so that equation (17) becomes

$$M^2 \propto R^4 \rho T \ . \tag{18}$$

3. Energy equation

Recall

$$\frac{dL}{dr} = 4\pi r^2 \rho \epsilon \ , \tag{19}$$

which reduces to

$$\frac{L}{R} \propto R^2 \rho \epsilon \ , \tag{20}$$

or

$$L \propto M \epsilon \ . \tag{21}$$

We can express the dependence of energy generation as power laws, such that $\epsilon \propto \rho^\alpha T^\beta$, so that

$$L \propto M \rho^\alpha T^\beta \tag{22}$$

or

$$M^{-1} \propto \rho^{\alpha} T^{\beta} L^{-1} \quad . \tag{23}$$

4. Radiative Transfer

Recall that

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa\rho}{T^3} \frac{L}{4\pi r^2} \quad , \tag{24}$$

so that

$$\frac{T}{R} \propto \frac{\rho}{T^3} \frac{L}{R^2} \quad , \tag{25}$$

or

$$T^4 \propto \frac{ML}{R^4} \quad , \tag{26}$$

which means that

$$M \propto R^4 L^{-1} T^4 \quad . \tag{27}$$

5. Putting it all together

Mass is ultimately the fundamental parameter with the largest influence on stellar behavior. To combine the stellar structure results, let us then express our quantities R, ρ, T, and L as power laws of M. They become

$$r \propto M^{\alpha_R}, \rho \propto M^{\alpha_\rho}, T \propto M^{\alpha_T}, L \propto M^{\alpha_L} \quad . \tag{28}$$

Solving for these exponents thus provides us with fundamental scaling relations for the behavior of stars as a function of mass.

The method of solution involves equating exponents; for example, from equation (13) we have

$$M \propto R^3 \rho \propto M^{3\alpha_R} M^{\alpha_\rho} \quad , \tag{29}$$

which implies that

$$1 = 3\alpha_R + \alpha_\rho \quad . \tag{30}$$

Likewise, equation (18) implies

$$2 = 4\alpha_R + \alpha_\rho + \alpha_T \quad . \tag{31}$$

The results from equations (13), (18), (23), and (27) can thus be written as the matrix equation

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 1 \\ 0 & \alpha & -1 & \beta \\ 4 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} \alpha_R \\ \alpha_\rho \\ \alpha_L \\ \alpha_T \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} \quad . \tag{32}$$

Solution of equation (32) then gives us a basis for predicting stellar parameters as a function of mass. Note that the answers will depend on values adopted for α and β.

$\star$ Note that the temperature T should be treated as an average *interior* temperature. For comparison with observations, the *effective* temperature $T_{eff} \propto \left(\frac{L}{R^2}\right)^{1/4}$ should be used.

$T_{eff} \propto \left(\frac{L}{R^2}\right)^{1/4}$