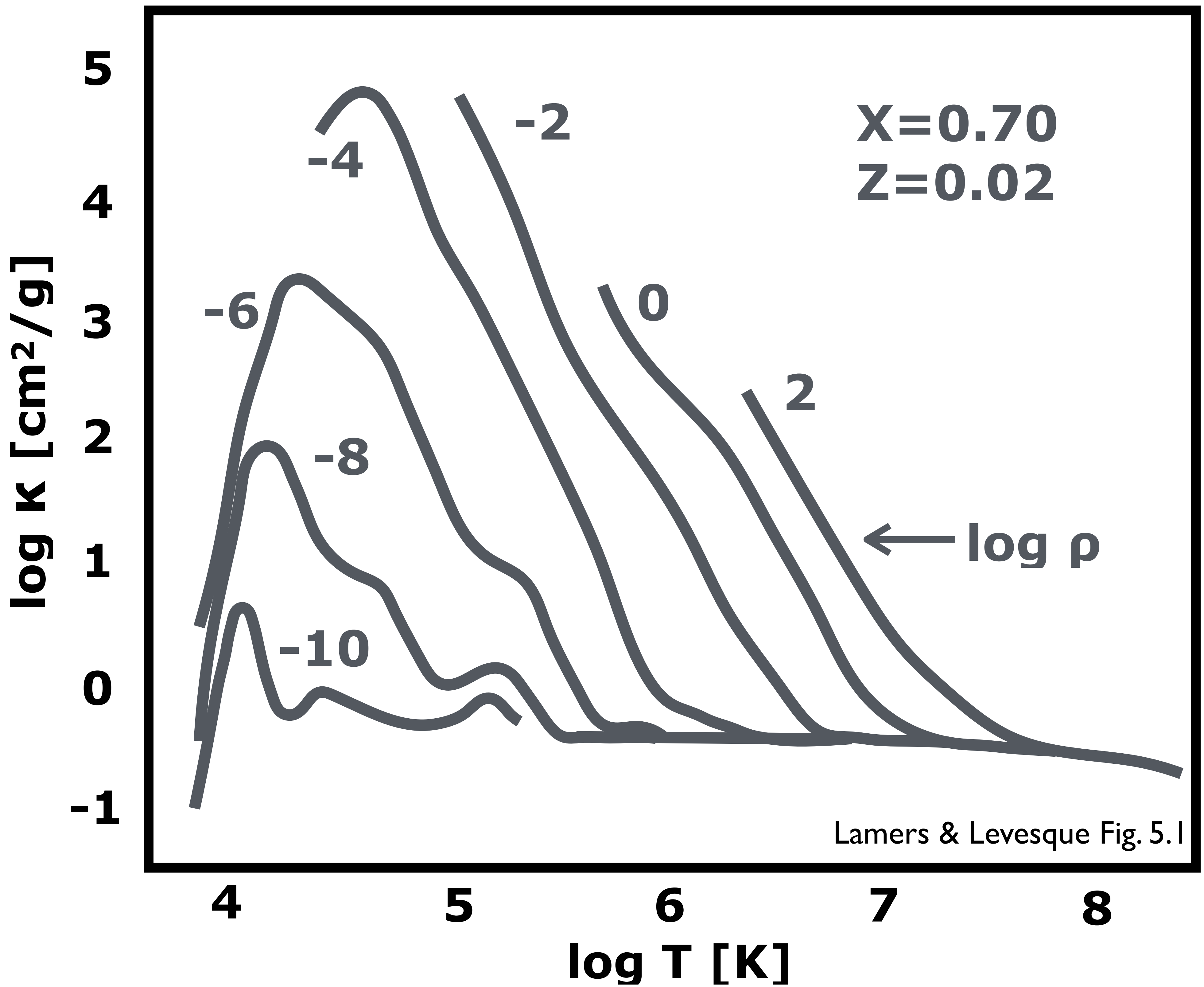
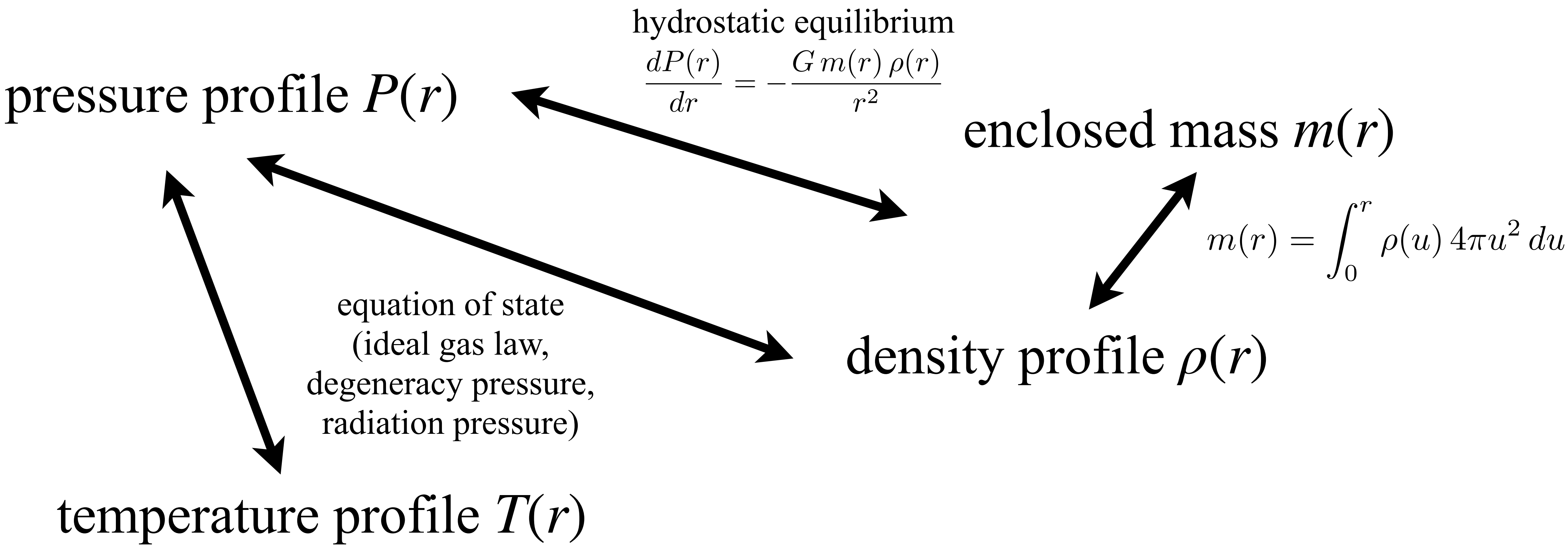
Lecture 7: Energy Transport — Radiation

Lamers & Levesque Ch. 6, 5
Phillips Ch. 3.1, 3.3



stellar structure



convection and hydrostatic equilibrium (recap)

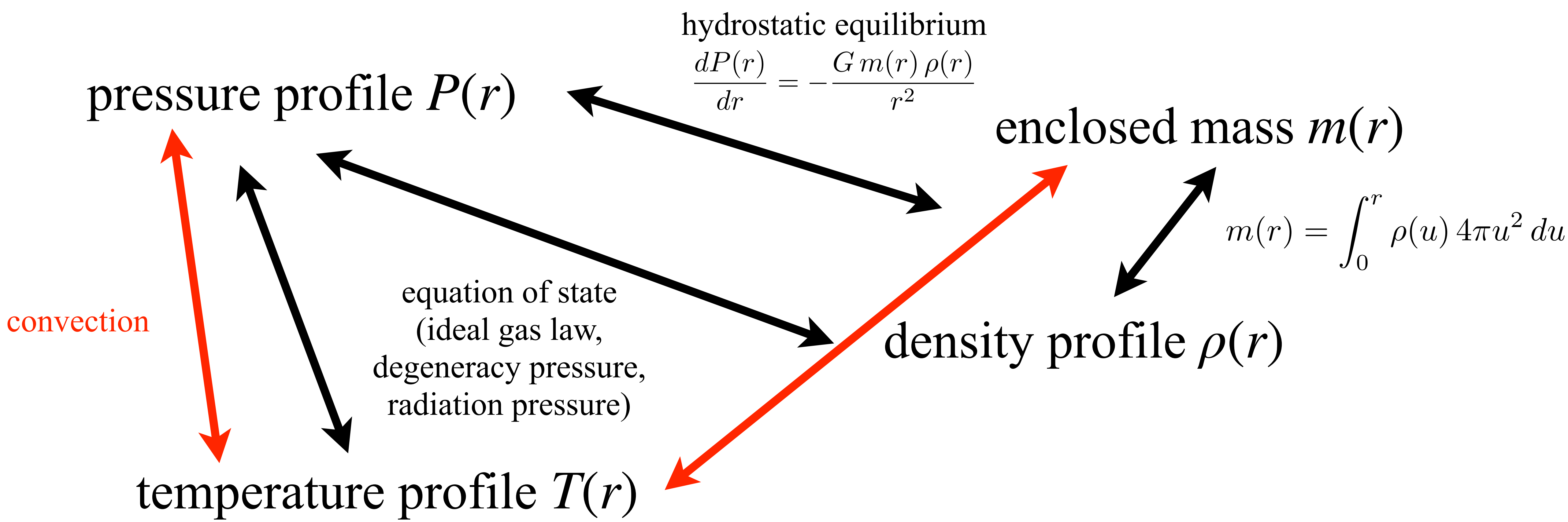
convection happens if

$$\left|\frac{dT}{dr}\right| > \underbrace{\frac{\gamma-1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right| \overbrace{= \frac{\gamma-1}{\gamma}\frac{\mu m_p}{k} g(r)}^{\text{for hydrostatic equilibrium}}}_{\equiv \left|\frac{dT}{dr}\right|_{\mathrm{ad}}}$$

alternate form

$$\boxed{\frac{d\ln T}{d\ln P} > \frac{\gamma-1}{\gamma}}$$

stellar structure



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alternate form

$$\boxed{\frac{d\ln T}{d\ln P} > \frac{\gamma-1}{\gamma}}$$

so convection will occur when
temperature gradient is strong
or *acceleration due to gravity is weak*

cores of high-mass stars

outer parts of low-mass stars

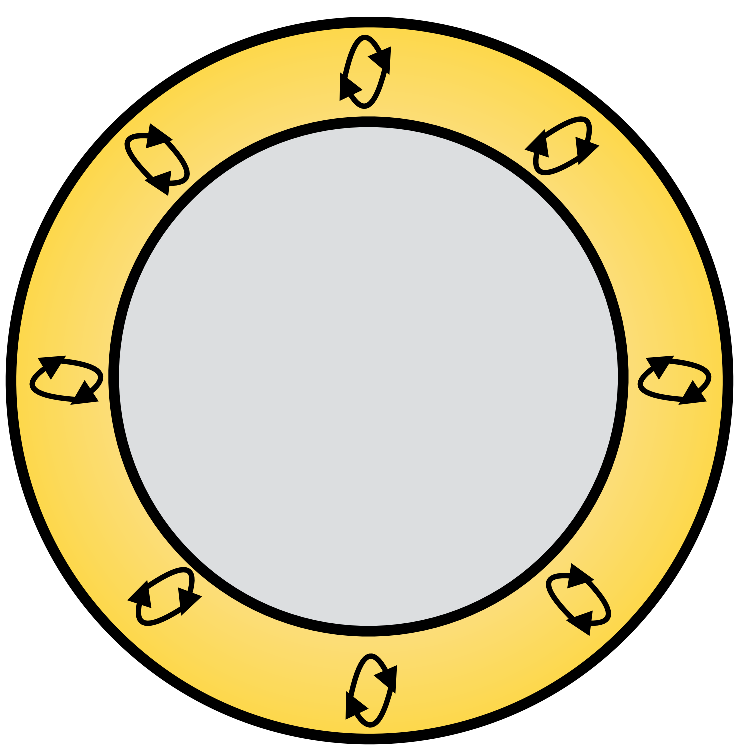
very low-mass stars ($M \lesssim 0.3\ M_\odot$)
are completely convective

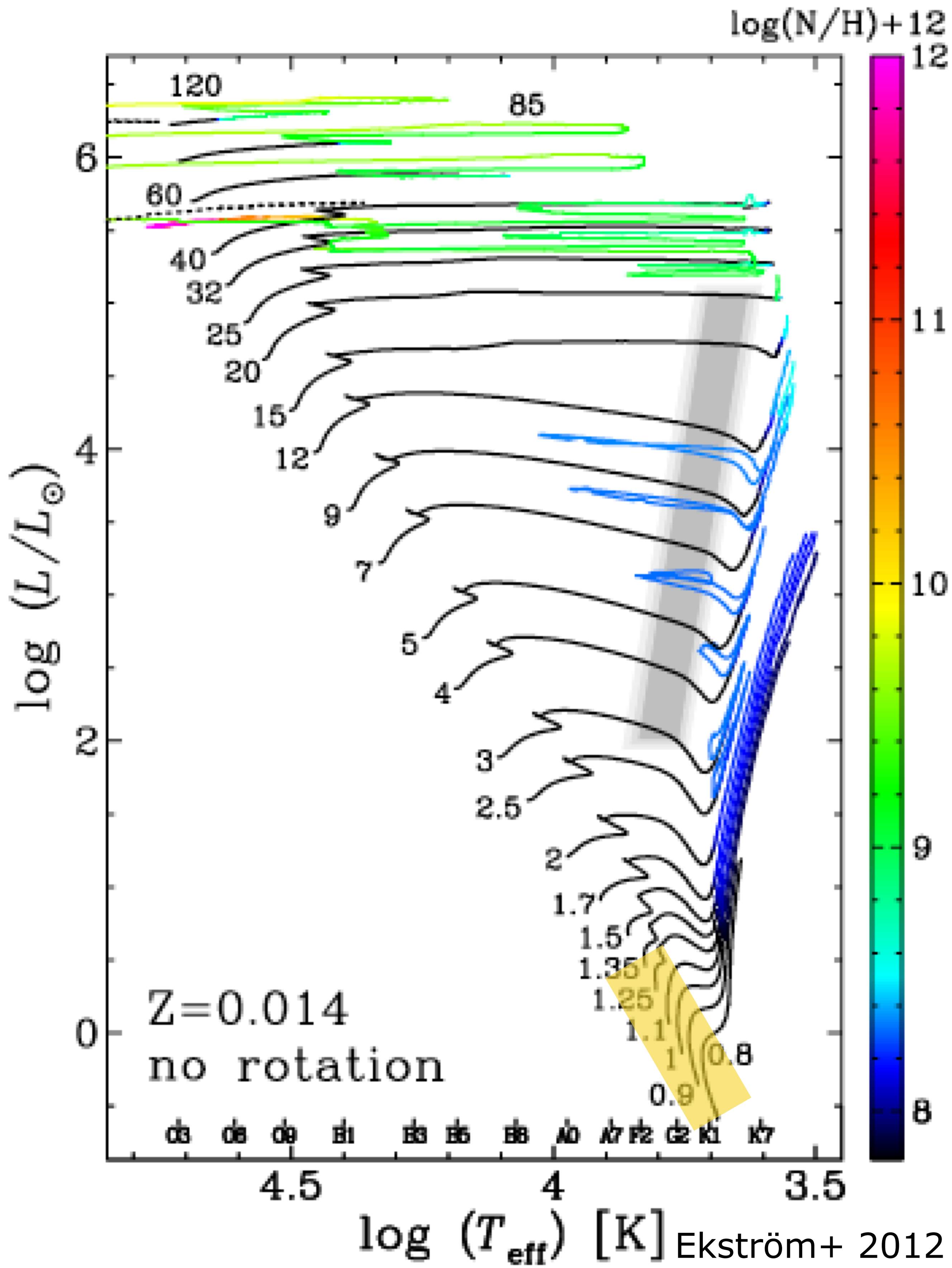
convection stabilizes the temperature gradient to $\left|\frac{dT}{dr}\right| = \left|\frac{dT}{dr}\right|_{\mathrm{ad}}$

Convection in Stars

<u>Where does convection actually happen?</u>

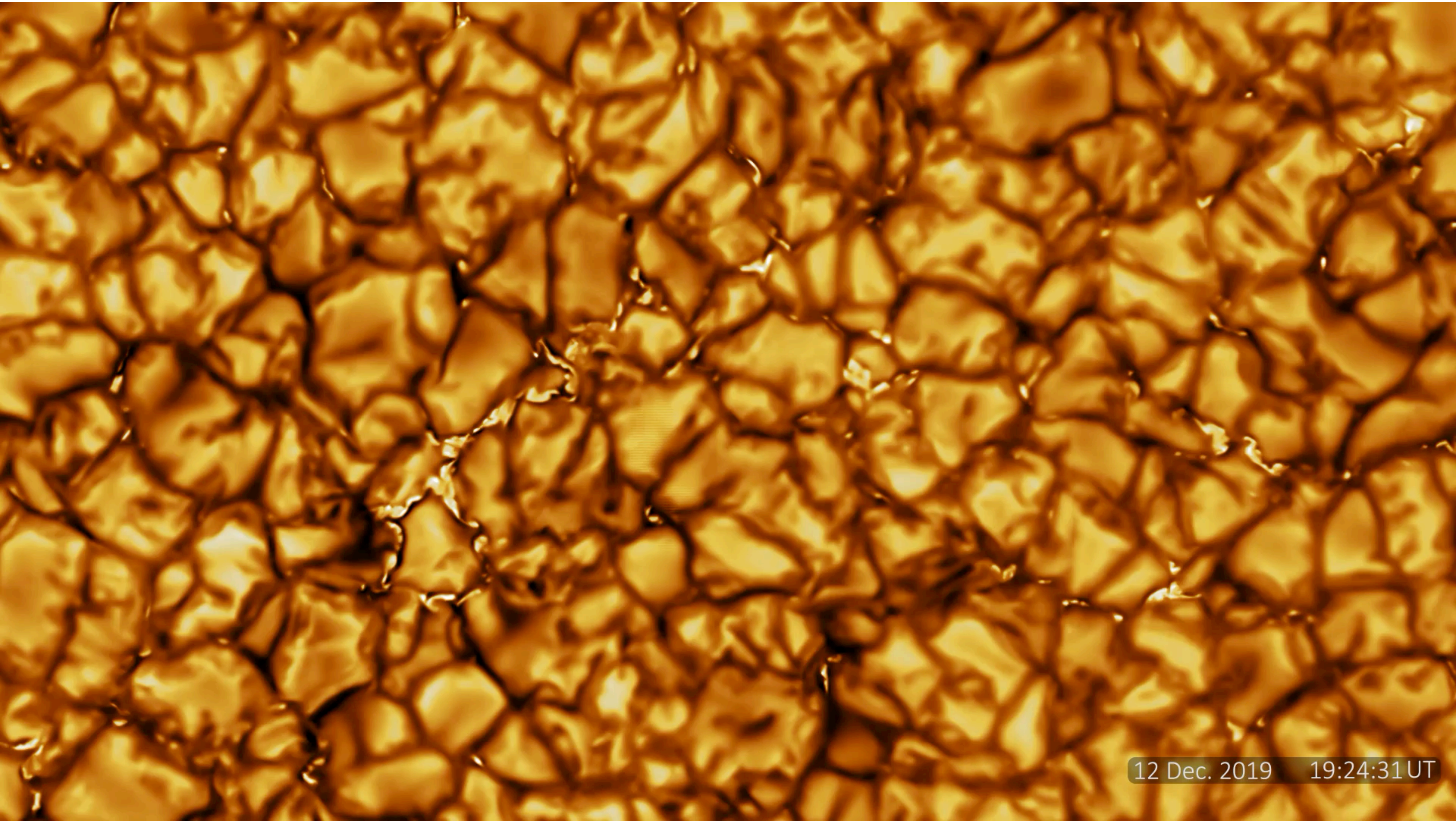
1. low gravitational acceleration
   —outer layers of cool stars





Ekström+ 2012

Daniel K. Inouye Solar Telescope

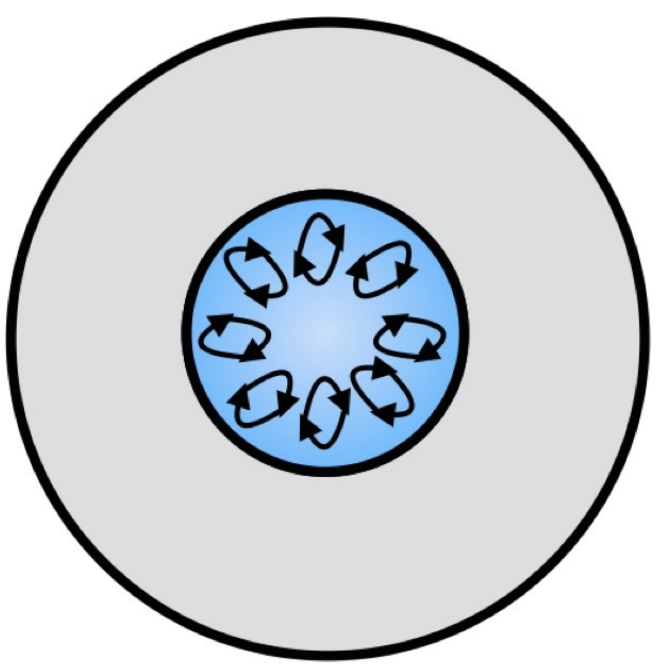


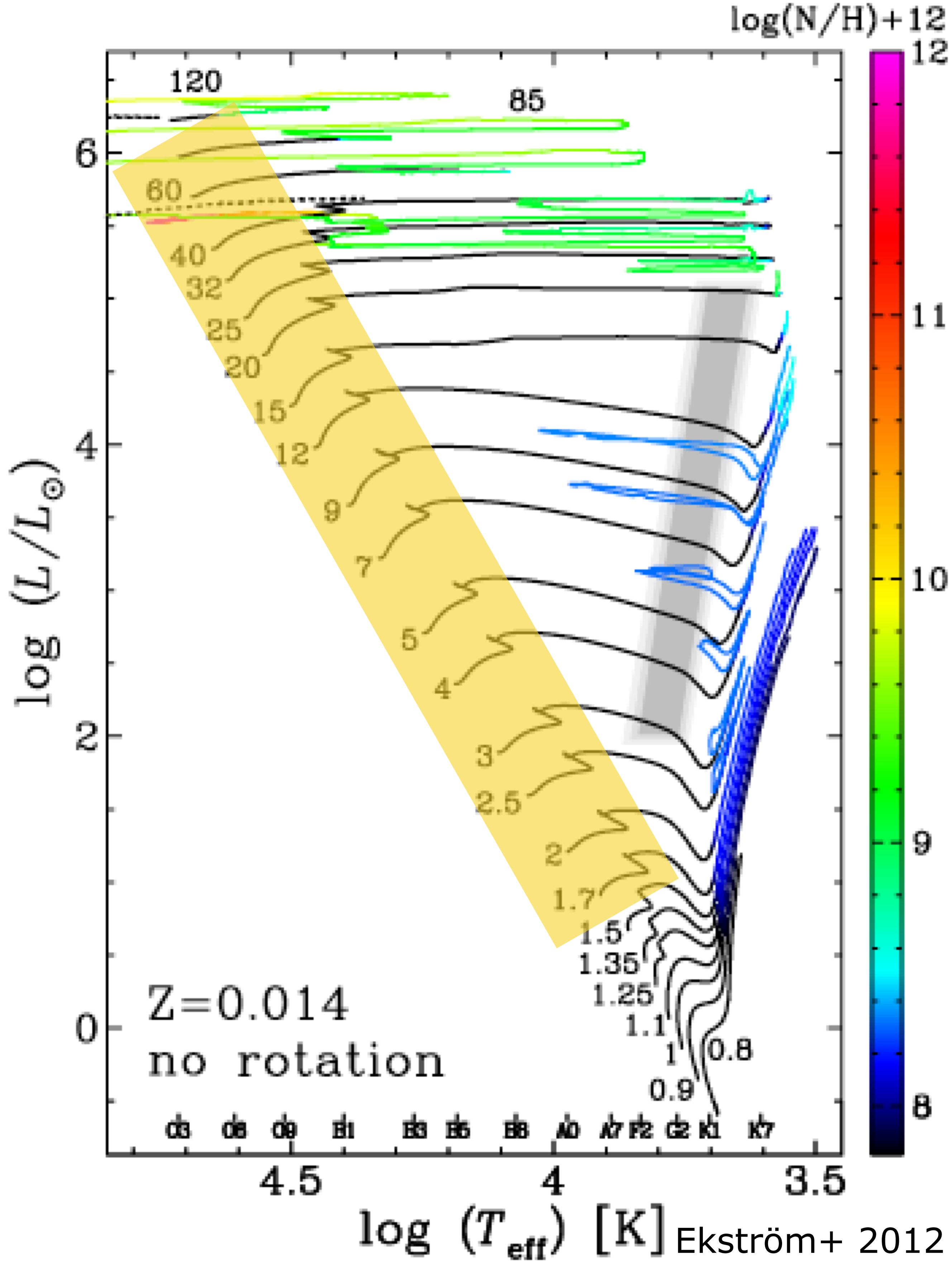
https://www.nso.edu/inouye-solar-telescope-first-light

Convection in Stars

<u>Where does convection actually happen?</u>

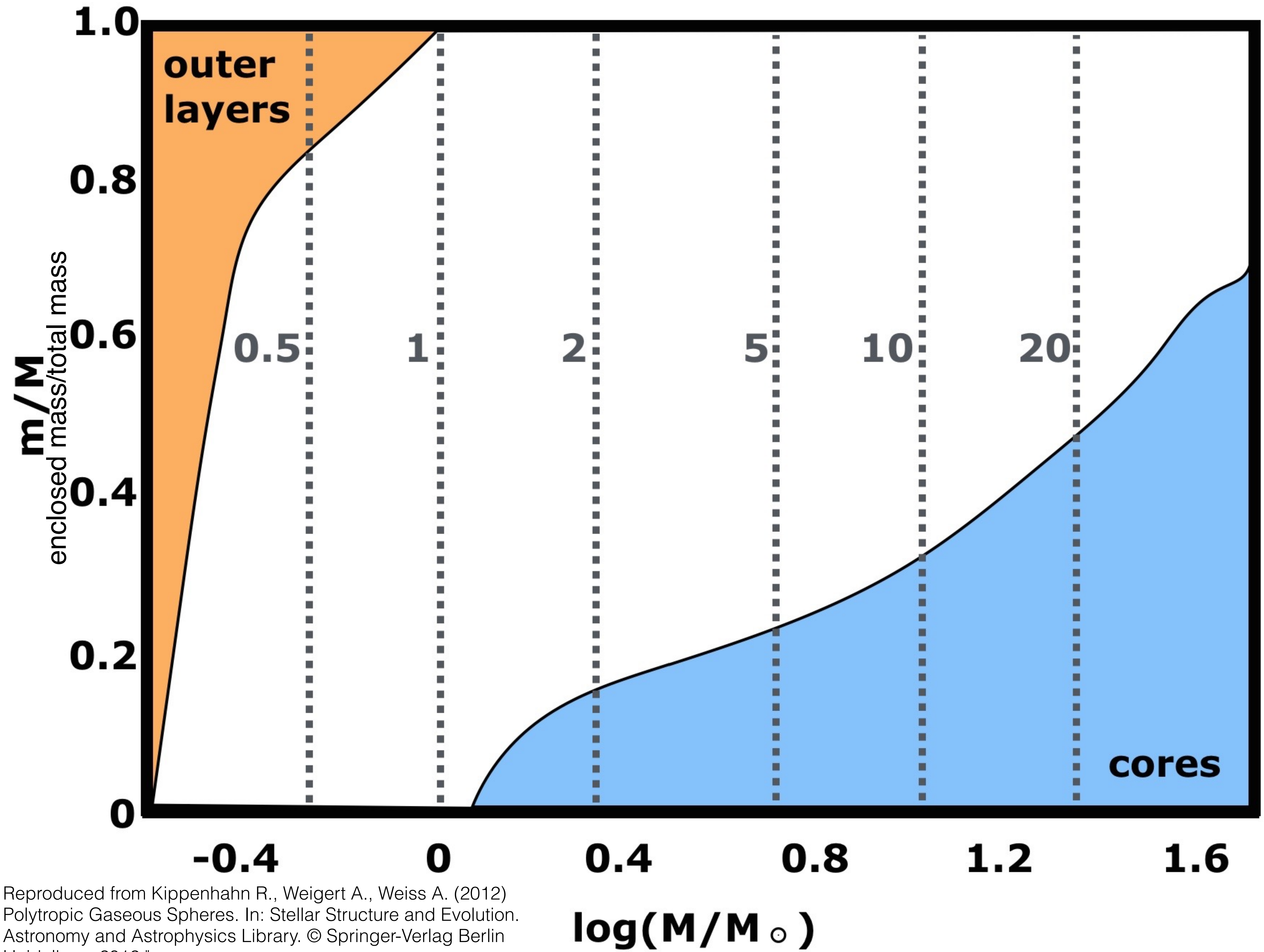
1. low gravitational accel.
   —outer layers of cool stars
2. large L: temp gradient
   —cores of massive stars





Ekström+ 2012

Convection in Stars



Reproduced from Kippenhahn R., Weigert A., Weiss A. (2012) Polytropic Gaseous Spheres. In: Stellar Structure and Evolution. Astronomy and Astrophysics Library. © Springer-Verlag Berlin Heidelberg 2012."

Convection in Stars

Convective overshooting

Convection is based on Archimedes force.

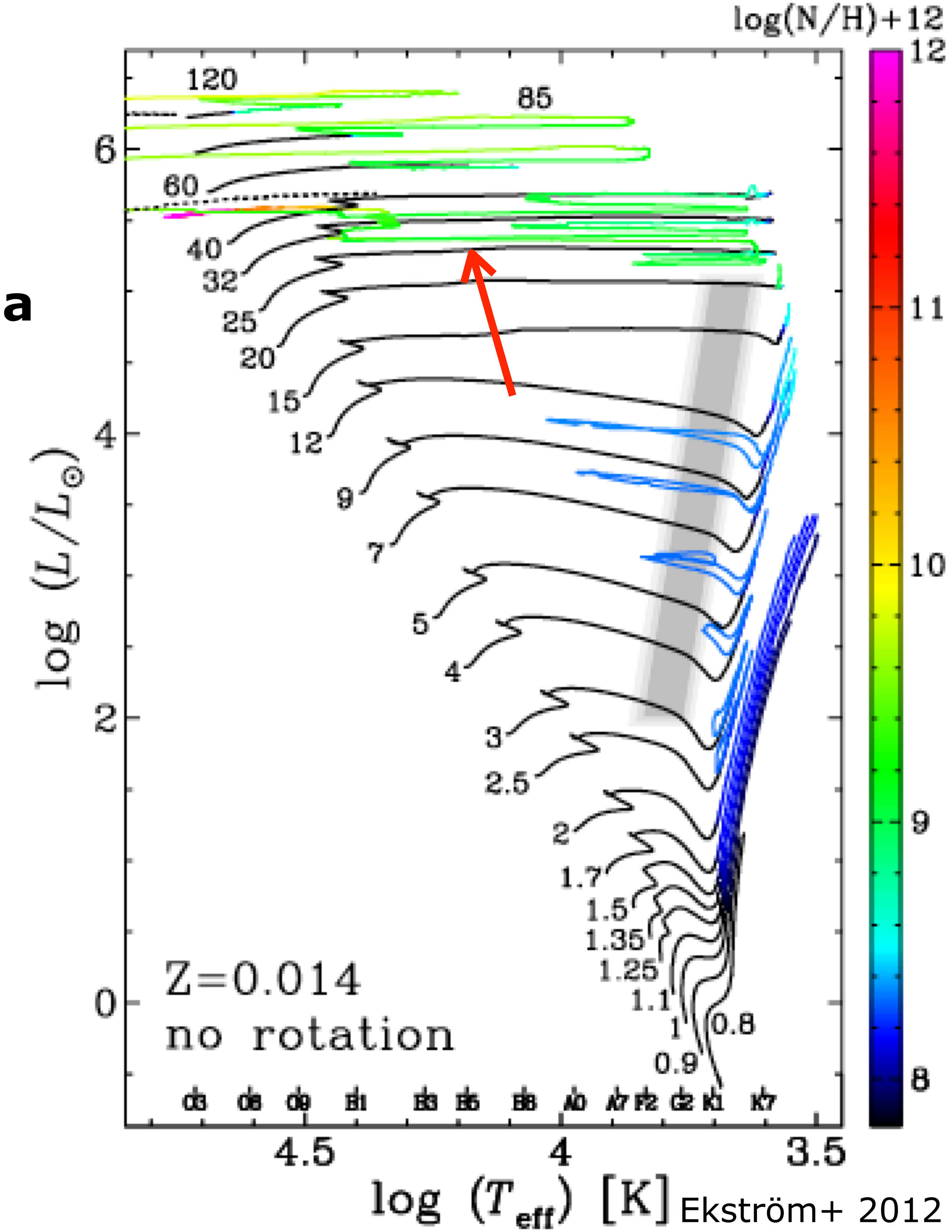
What would happen if you held a football underwater and let go?

Same thing happens in stars…

—nuclear products like He and N appear at massive stars surfaces very early…

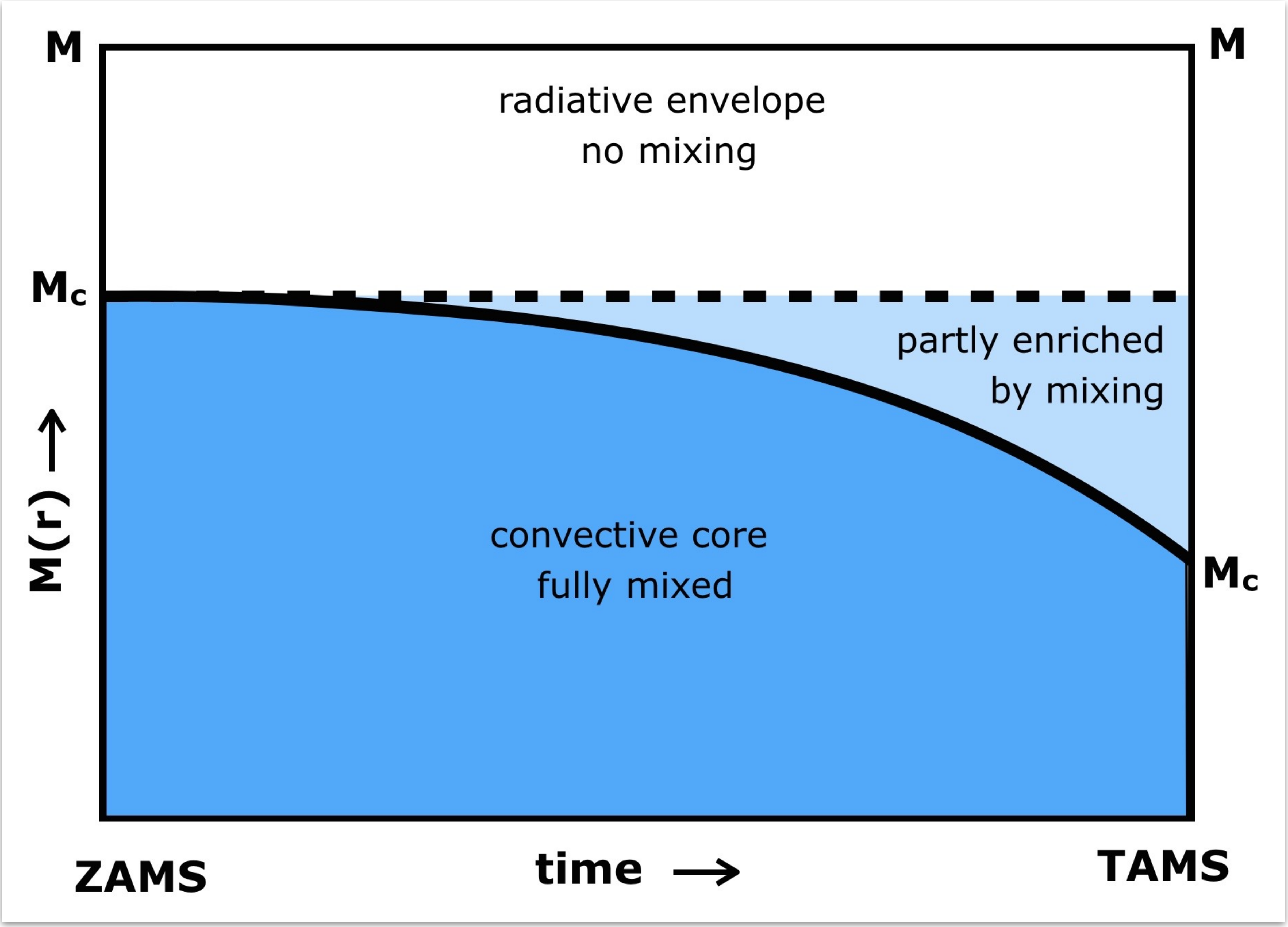
—convective mixing extends past the convective core

—difference is that convective overshooting doesn’t transport energy…



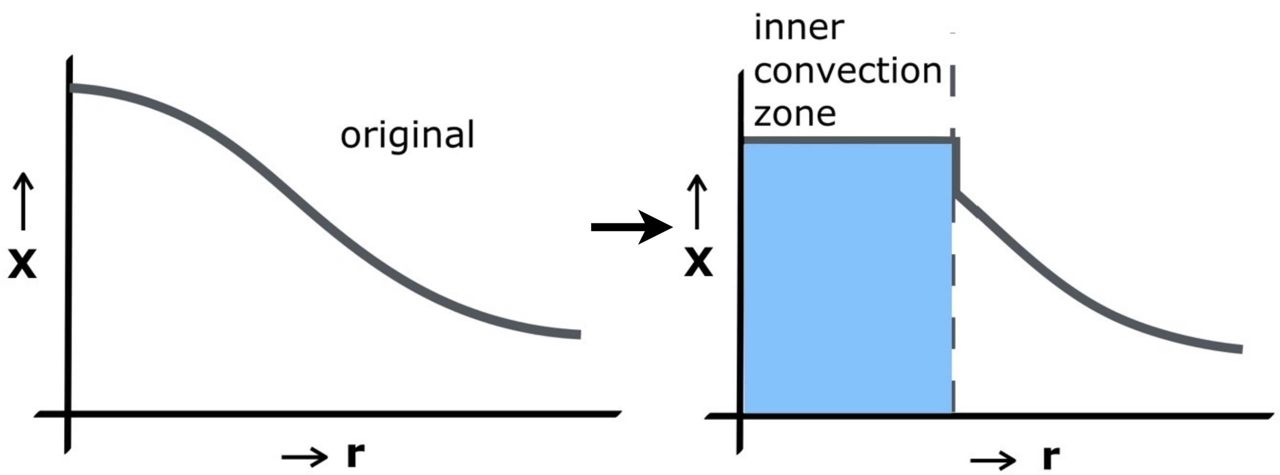
Ekström+ 2012

Convection in Stars

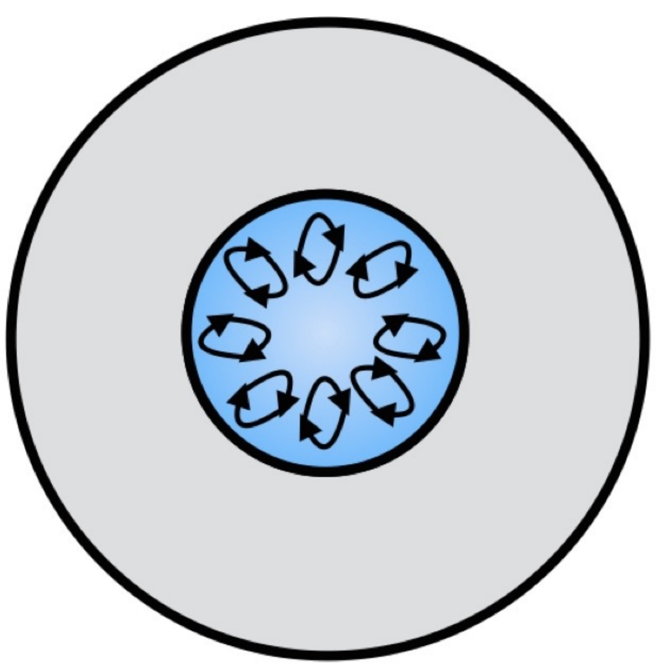


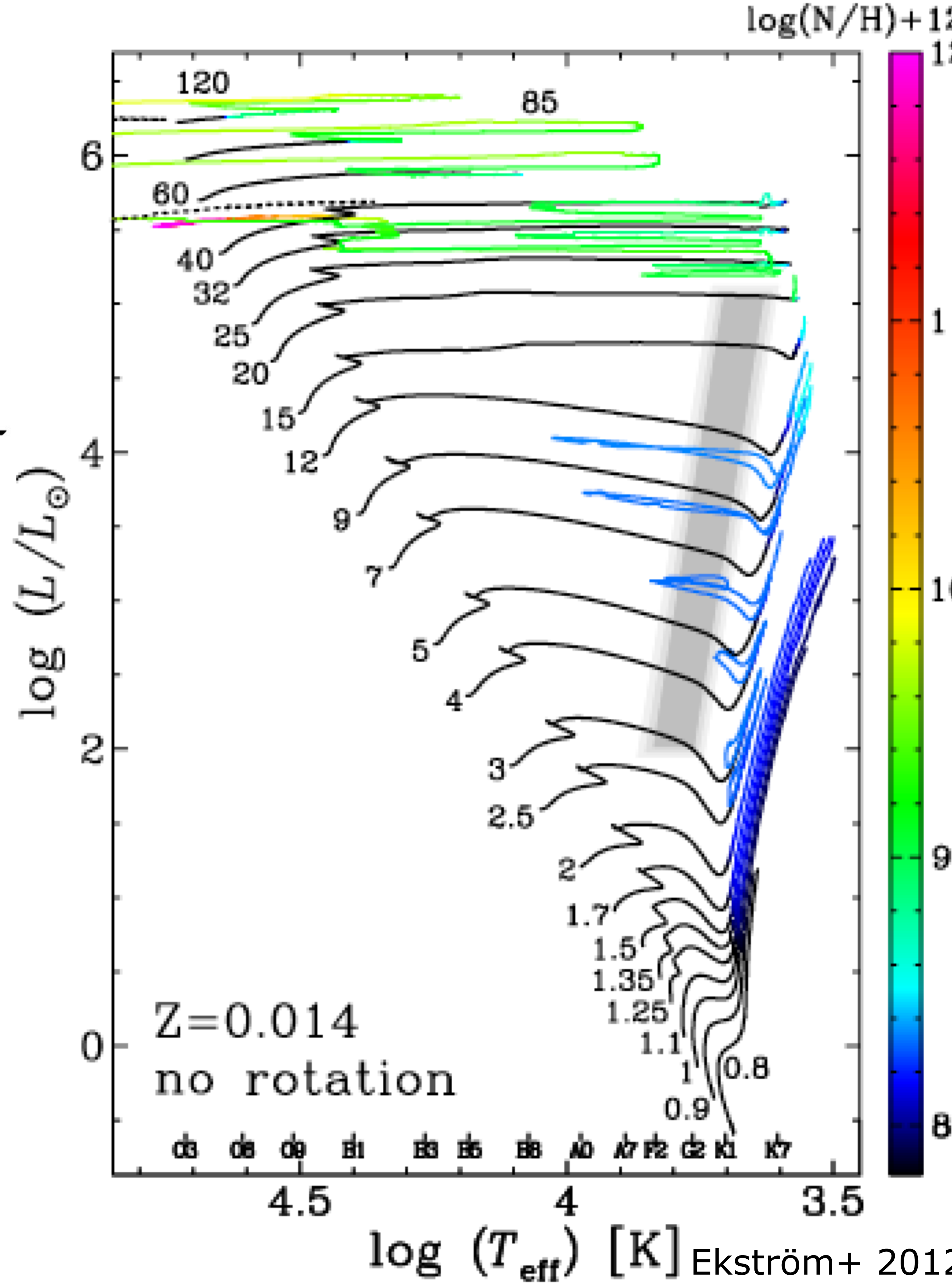
Convection in Stars

Convective mixing



Mixing in massive star cores extends their MS lifetimes…

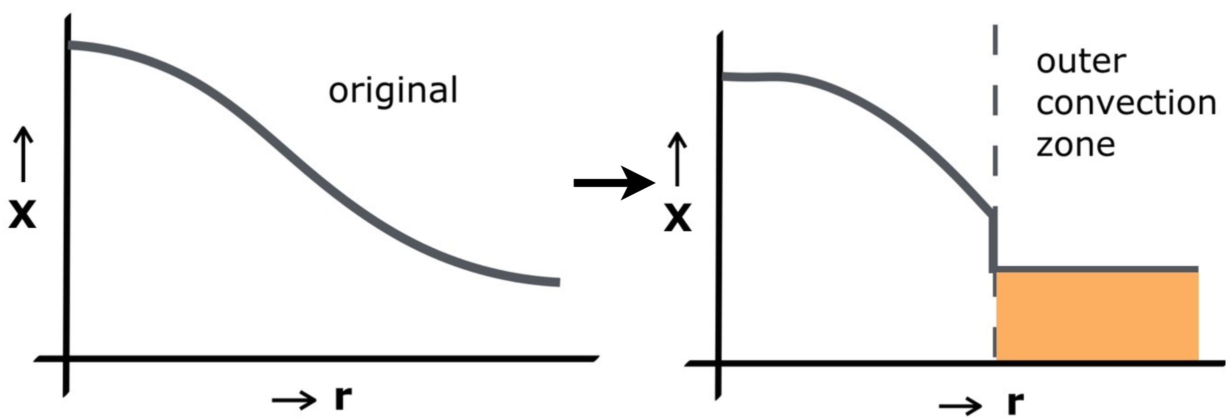




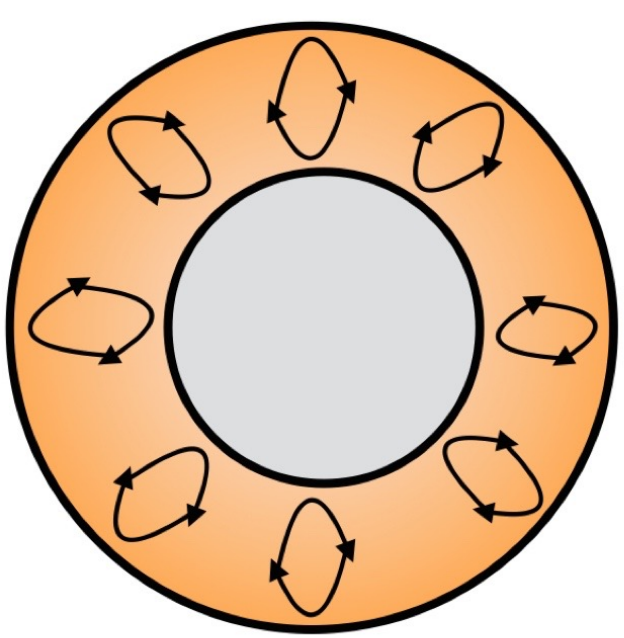
Ekström+ 2012

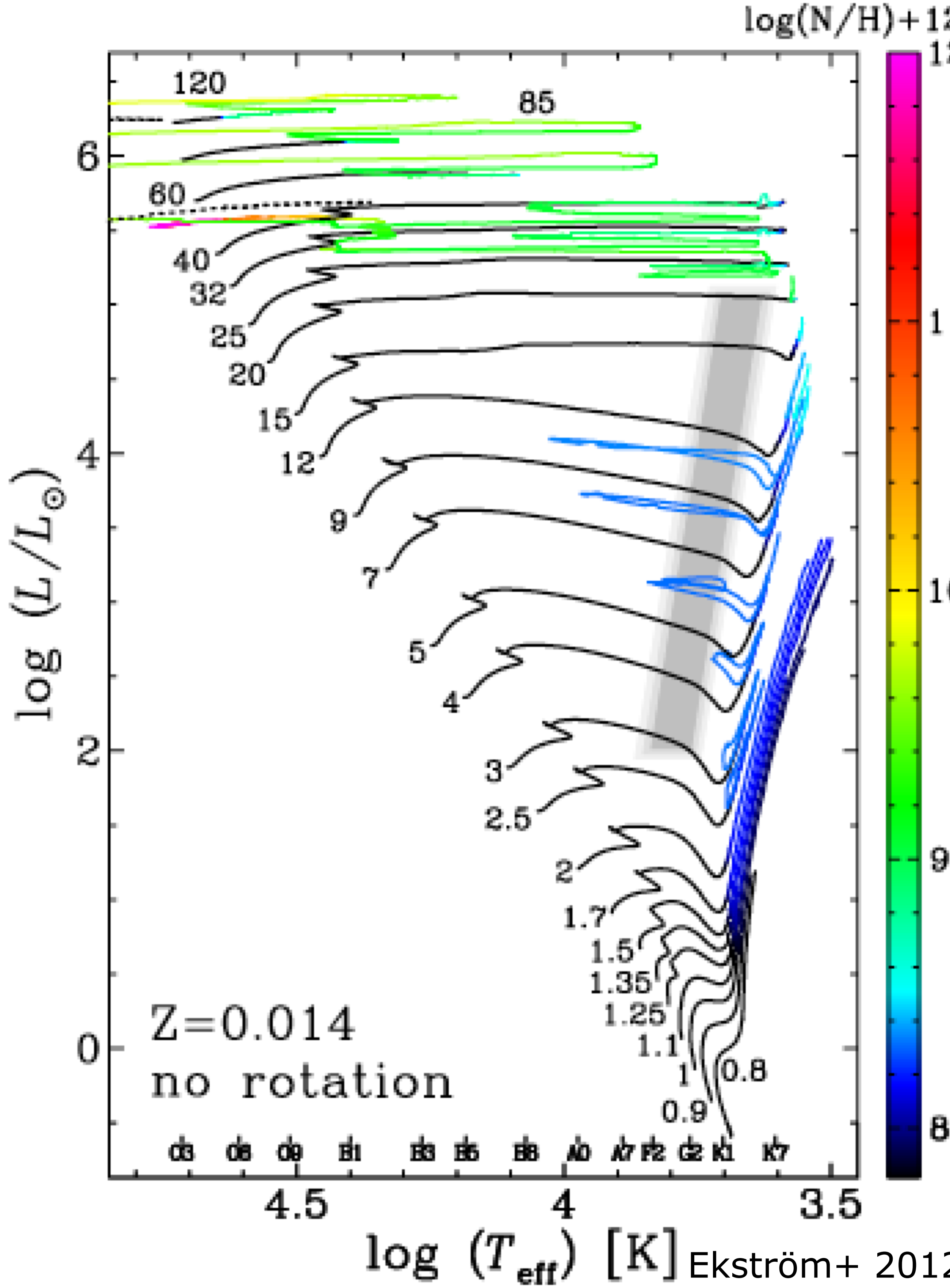
Convection in Stars

Convective mixing



Mixing in stars with large convective *outer* layers causes “dredge-up”…





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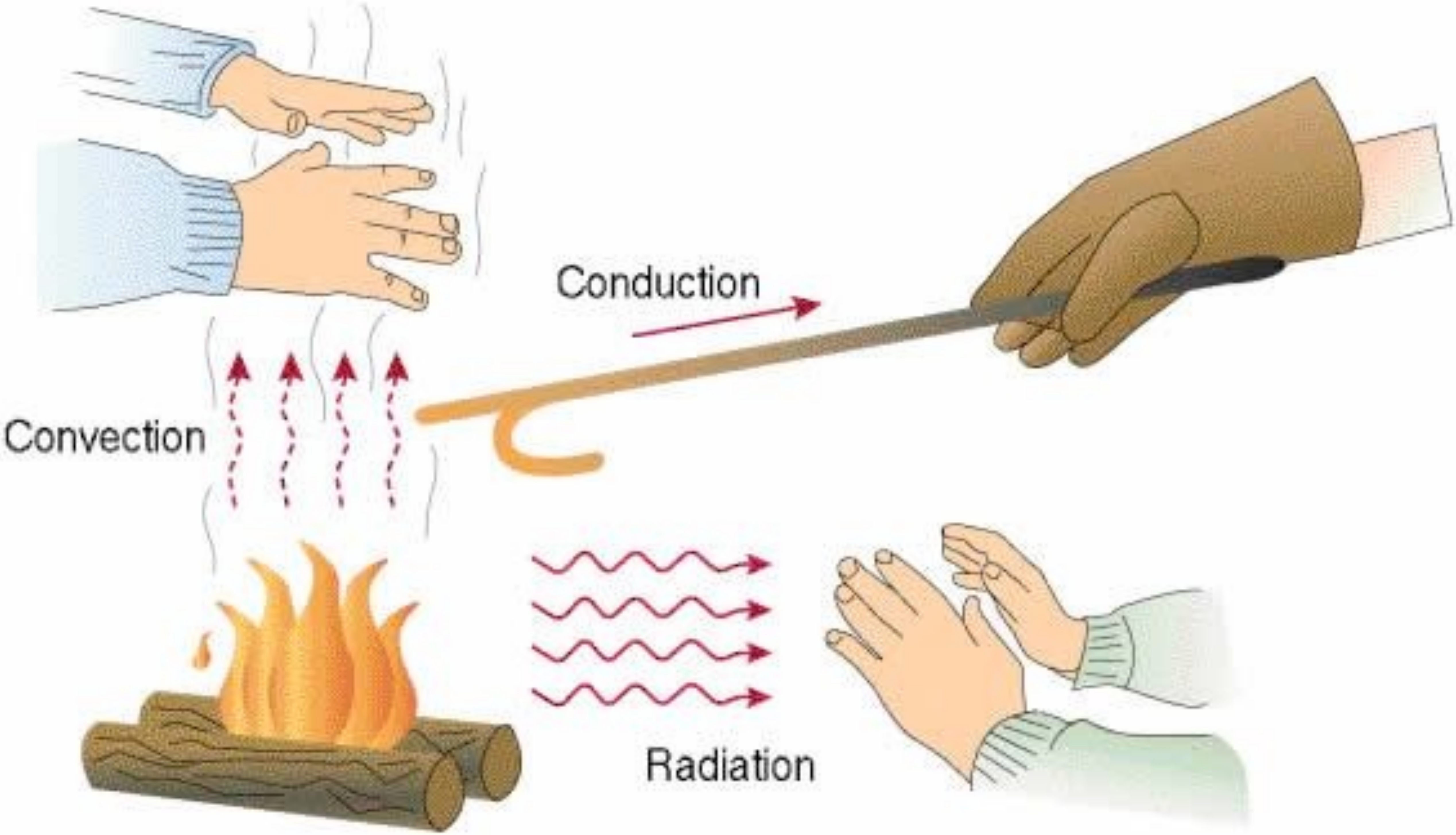

Conduction
Convection
Radiation

Mean free path

mean free path $$\ell = \frac{1}{n\sigma}$$

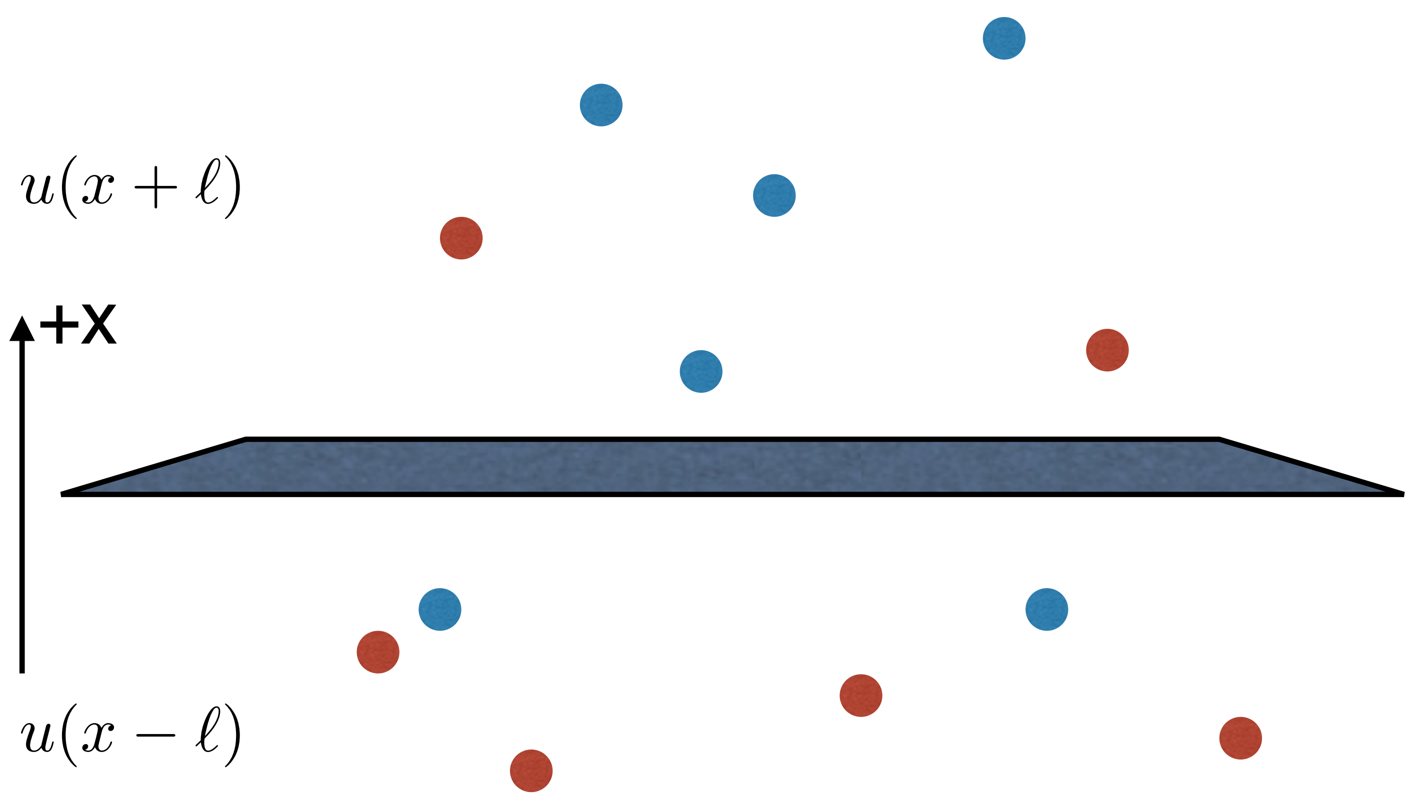
number density
of targets

cross-section
for interaction

in general, depends on frequency:

$$\ell_\nu = \frac{1}{n\sigma_\nu}$$

Conduction (diffusion)



typical speed: $\bar{v}$

typical length before interaction:

$\ell \equiv$ **mean free path**

energy density: $u(x)$

heat flux through surface: $j(x)$

time between interactions: $t \approx l/\bar{v}$

but only $\approx 1/6$ of particles are going in $+x$ direction

so rate of particles crossing "up" is $1/t \approx \bar{v}/6\ell$

rate of particles crossing "down" is same $1/t \approx \bar{v}/6\ell$

$$\begin{aligned} j(x) &\approx u(x-\ell)(\ell/t) - u(x+\ell)(\ell/t) \\ &\approx \frac{\bar{v}}{6}\left[u(x-\ell) - u(x+\ell)\right] \\ &\approx \frac{\bar{v}}{6}\left[\left(u(x) - \ell\frac{du}{dx}\right) - \left(u(x) + \ell\frac{du}{dx}\right)\right] \\ &\approx \frac{\bar{v}}{6}\left[-2\ell\frac{du}{dx}\right] = -\frac{\bar{v}\ell}{3}\frac{du}{dx} \end{aligned}$$

$$j(x) = -K\frac{dT}{dx} \quad \text{with} \quad K \approx \frac{1}{3}\bar{v}\ell C$$

thermal conductivity

$$\frac{du}{dx} = \frac{du}{dT}\frac{dT}{dx} = C\frac{dT}{dx}$$ C is heat capacity per unit volume

electron conduction

thermal conductivity

$$j(x) = -K\frac{dT}{dx} \quad \text{with} \quad K \approx \frac{1}{3}\bar{v}\ell C$$

for electrons at a temperature T, typical kinetic energy is $KE \approx \frac{3}{2}kT \approx \frac{1}{2}m_e v^2$

so average speed is $\bar{v} \approx \sqrt{3kT/m_e}$

electron energy density: $u_e = \frac{3}{2}n_e kT$ so heat capacity is $C = \frac{du_e}{dT} = \frac{3}{2}n_e k$

in a plasma electrons transfer energy through collisions with *ions*

mean free path $\ell = 1/(n_i\,\sigma)$, geometric interaction cross-section $\sigma \approx \pi r^2$

separation r set where typical kinetic energy ~ typical potential energy

$$\frac{Ze^2}{4\pi\epsilon_0 r} \approx kT \;\Rightarrow\; r \approx \frac{Ze^2}{4\pi\epsilon_0 kT}$$

Z is atomic number of the ions

$$\Rightarrow \; \ell \approx \frac{1}{n_i \pi r^2} \approx \frac{1}{\pi n_i}\left(\frac{4\pi\epsilon_0 kT}{Ze^2}\right)^2$$

electron conduction

thermal conductivity $$j(x) = -K\frac{dT}{dx} \quad \text{with} \quad K \approx \frac{1}{3}\bar{v}\ell C$$

$$\bar{v}_e \approx \left(\frac{3kT}{m_e}\right)^{1/2} \qquad \ell_e \approx \frac{1}{\pi n_i}\left(\frac{4\pi\epsilon_0 kT}{Ze^2}\right)^2 \qquad C_e = \frac{3}{2}n_e k$$

$$\Rightarrow \quad K_e \approx \frac{1}{3}\bar{v}_e\ell_e C_e \approx \frac{k}{2\pi}\frac{n_e}{n_i}\left(\frac{3kT}{m_e}\right)^{1/2}\left(\frac{4\pi\epsilon_0 kT}{Ze^2}\right)^2$$

conduction by ions is much smaller ($n_e \leftrightarrow n_i \quad m_e \rightarrow m_i$) and can be neglected

in the end, electron conduction turns out to be ineffective at transporting energy in normal stars.

electron conduction is more relevant in white dwarfs (but need to calculate with degenerate equation of state)

radiation (radiative diffusion = photon conduction)

thermal conductivity

$$j(x) = -K\frac{dT}{dx} \quad \text{with} \quad K \approx \frac{1}{3}\bar{v}\ell C$$

radiation constant
(see Lecture 1, slide 17)

$$a = \frac{8\pi^5 k^4}{15h^3c^3}$$

$$\bar{v} = c \qquad u = aT^4 \quad C = \frac{du}{dT} = 4aT^3$$

The heat flux density due to radiation is

$$j(x) = -K_r\frac{dT}{dx} \qquad \text{with} \qquad K_r \approx \frac{4}{3}claT^3$$

mean free path from free electron scattering (*Thomson scattering*)

$$\bar{l} = \frac{1}{n_e\sigma_T} \quad \text{where} \quad \sigma_T = \frac{8\pi}{3}\left[\frac{e^2}{4\pi\epsilon_o m_e c^2}\right]^2$$

$$P_r = \frac{1}{3}aT^4 \qquad P_e = n_e kT \qquad K_r \approx \frac{4}{3}claT^3 = \sqrt{3}Z\left(\frac{m_ec^2}{kT}\right)^{5/2} K_e\frac{P_r}{P_e}$$

$$\boxed{\frac{K_r}{K_e} = \sqrt{3}Z\left(\frac{m_ec^2}{kT}\right)^{5/2}\frac{P_r}{P_e}.}$$

ratio of conduction by photons
to conduction by electrons

for the Sun's interior, radiation dominates: $K_r/K_e \approx 2\times10^5$

radiation (radiative diffusion = photon conduction)

thermal conductivity

$$j(x) = -K\frac{dT}{dx} \quad \text{with} \quad K \approx \frac{1}{3}\bar{v}\ell C$$

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in general we write the photon mean free path in terms of the *opacity* κ

$$\boxed{j(x) = -\frac{4ac}{3}\frac{T^3}{\rho\kappa}\frac{dT}{dx} \qquad \text{where} \qquad l = \frac{1}{\rho\kappa}}$$

Opacity

mean free path $$\ell = \frac{1}{n\sigma} = \frac{1}{\rho\kappa}$$

number density of targets

cross-section for interaction

total mass density

opacity

in general, depends on frequency:

$$\ell_\nu = \frac{1}{n\sigma_\nu} = \frac{1}{\rho\kappa_\nu}$$

average the mean free path (or opacity) over frequency: **Rosseland** mean

Sources of Opacity

1) Electron scattering

In stellar interiors (and the hottest atmospheres) gas is fully ionized, so free e^-s are a dominant source of opacity (Thomson scattering)

$$\kappa_{\mathrm{es}} = \frac{n_e \sigma_T}{\rho} = \frac{\sigma_T}{\mu_e m_p}$$

$\sigma_T = 6.65 \cdot 10^{-25}$ cm^2/e$^-$

n_e = # of e^-/cm^3

$\mu_e m_H$ = mean particle mass/e$^-$

$$\boxed{\kappa_{es} = \frac{\sigma_T}{m_H} \cdot \frac{1}{\mu_e} = \frac{0.40}{\mu_e} \simeq 0.40\,\frac{1+X}{2} \approx 0.2(1+X)\ \mathrm{cm}^2/\mathrm{g}}$$

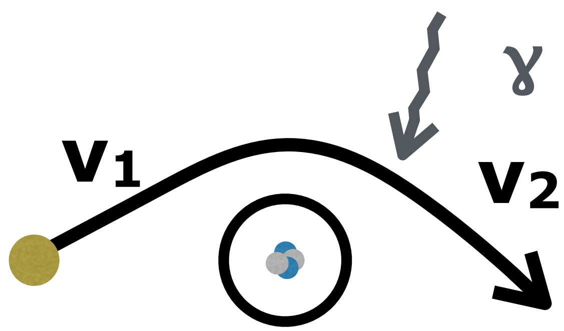
Thomson opacity is independent of λ

X is hydrogen fraction,
X=0.7 for the Sun

Sources of Opacity

2) Free-free absorption

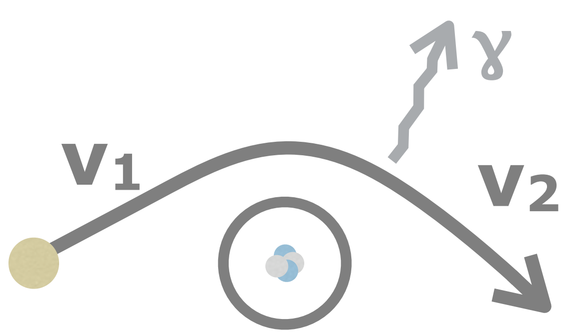
A free e^- near an ion increases its velocity and absorbs a photon…



$v_2 > v_1$

($hc/\lambda = 1/2mv_2^2 - 1/2mv_1^2$)

(opposite of brehmsstrahlung:)



$v_2 < v_1$

Sources of Opacity

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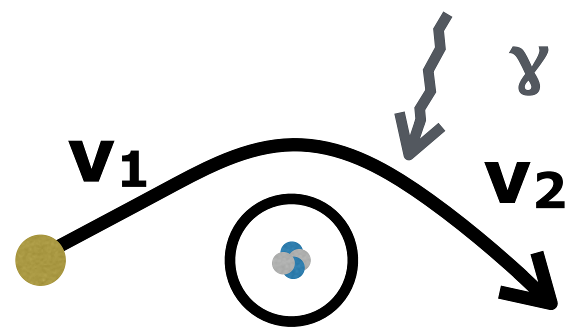


$v_2 > v_1$

$(hc/\lambda = 1/2mv_2^2 - 1/2mv_1^2)$

Z = charge of ions
n_i = ion density (cm^{-3}) = $\rho/\mu_i m_H$
n_e = e^- density (cm^{-3}) = $\rho/\mu_e m_H$

$$\kappa_{\mathrm{ff}} \text{ per cm}^3 \sim Z^2 . n_i n_e . T^{-7/2}$$

Kramer's opacity law

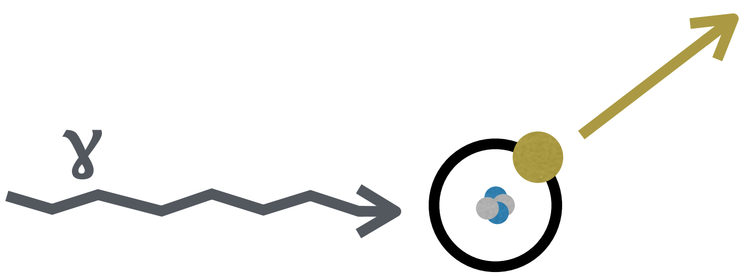
So $$\boxed{\kappa_{\mathrm{ff}} = 7.5 \cdot 10^{22} \left(\frac{1+X}{2}\right) \langle\frac{Z^2}{A}\rangle \rho T^{-7/2} \text{ cm}^2/\text{g}}$$

ρ in $\mathrm{g/cm^3}$, T in K

Sources of Opacity

3) Bound-free absorption

A photon is energetic enough to kick an electron completely out of an atom; photoionization



$\lambda \leq hc/X_n$
(X_n = excitation of nth orbital)

Also dependent on Kramer's opacity law; if we sum all possible bound-free transitions we get:

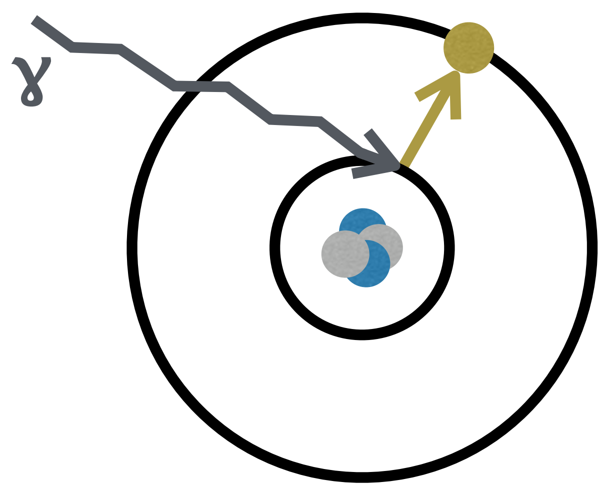
$$\boxed{\kappa_{bf} = 4.3 \times 10^{25}\,(1+X)\;Z\cdot\rho\cdot T^{-7/2}\;\left(\mathrm{cm^2/g}\right)}$$

ρ in $\mathrm{g/cm^3}$, T in K

Sources of Opacity

4) Bound-bound absorption

A photon is absorbed by an e^- transition to a higher energy level.



$$hc/\lambda = X_{upper} - X_{lower}$$

K_{bb} = VERY DIFFICULT.

Calculating K_{bb} would need to include all possible transitions of all atoms/ions/isotopes/molecules plus ρ dependence.

OPAL: http://opalopacity.llnl.gov/opal.html

Sources of Opacity

1) Electron scattering

$$\kappa_{es} = \frac{\sigma_T}{m_H} \cdot \frac{1}{\mu_e} = \frac{0.40}{\mu_e} \simeq 0.40\,\frac{1+X}{2} \approx 0.2(1+\mathrm{X})\ \mathrm{cm}^2/\mathrm{g}$$

2) Free-free absorption

$$\kappa_{\mathrm{ff}} = 7.5 \cdot 10^{22} \left(\frac{1+X}{2}\right) \left\langle \frac{Z^2}{A} \right\rangle \rho T^{-7/2}\ \ \mathrm{cm}^2/\mathrm{g}$$

ρ in g/cm^3, T in K

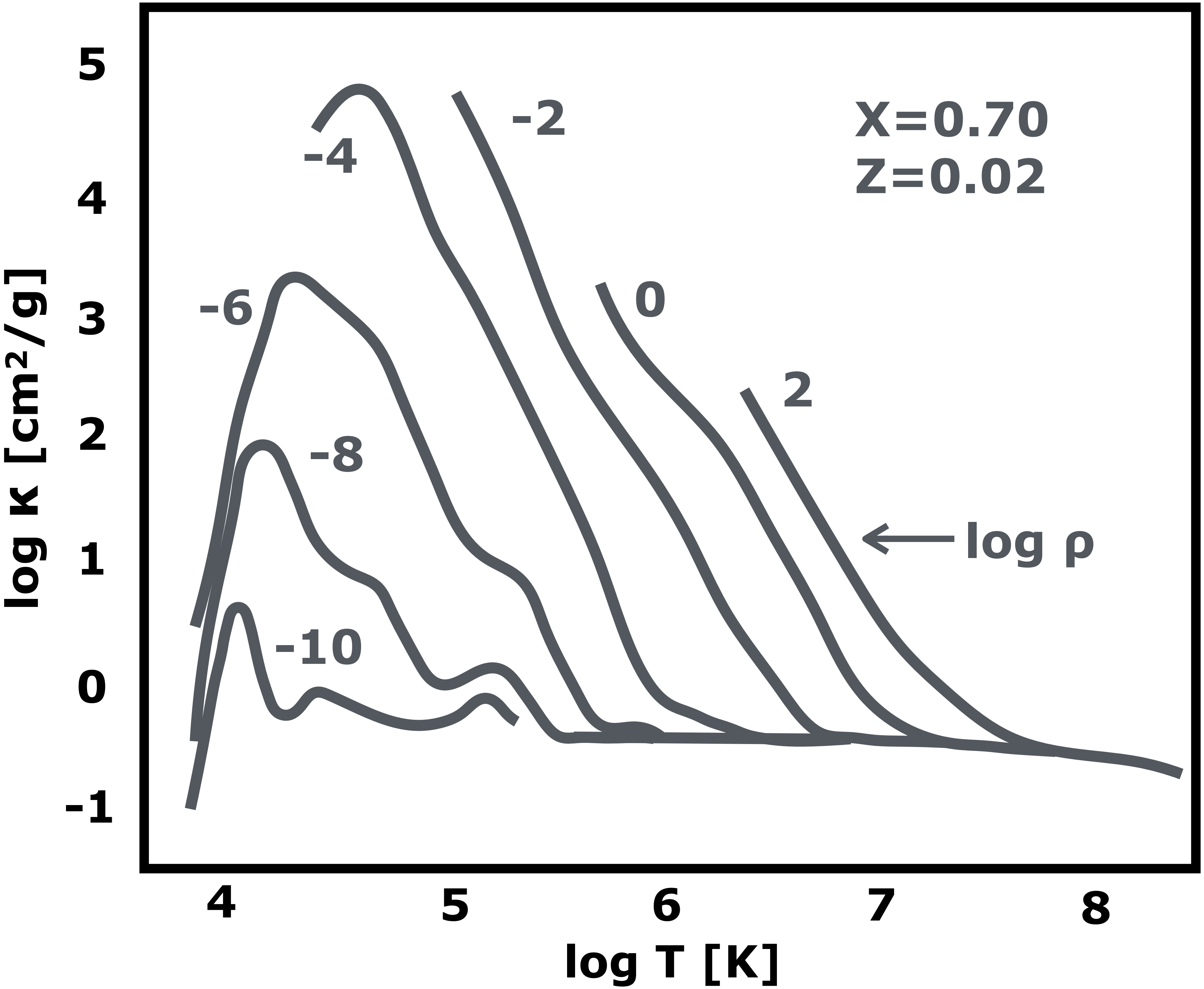
3) Bound-free absorption

$$\kappa_{\mathrm{bf}} = 4.3 \times 10^{25}\,(1+\mathrm{X})\ \ \mathrm{Z} \cdot \rho \cdot \mathrm{T}^{-7/2}\ \ \left(\mathrm{cm}^2/\mathrm{g}\right)$$

4) Bound-bound absorption

$\kappa_{\mathrm{bb}} =$ unpleasant

Sources of Opacity



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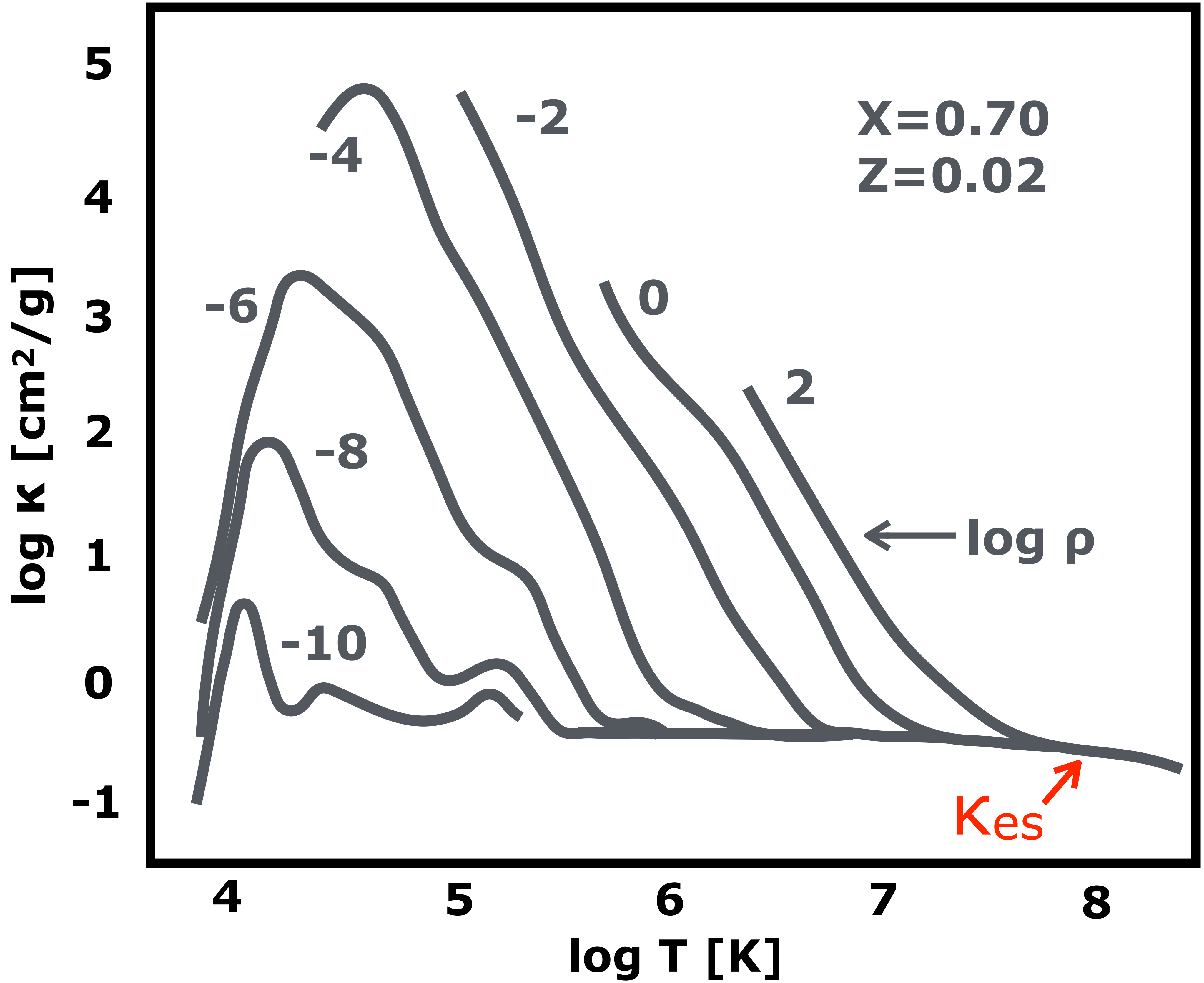

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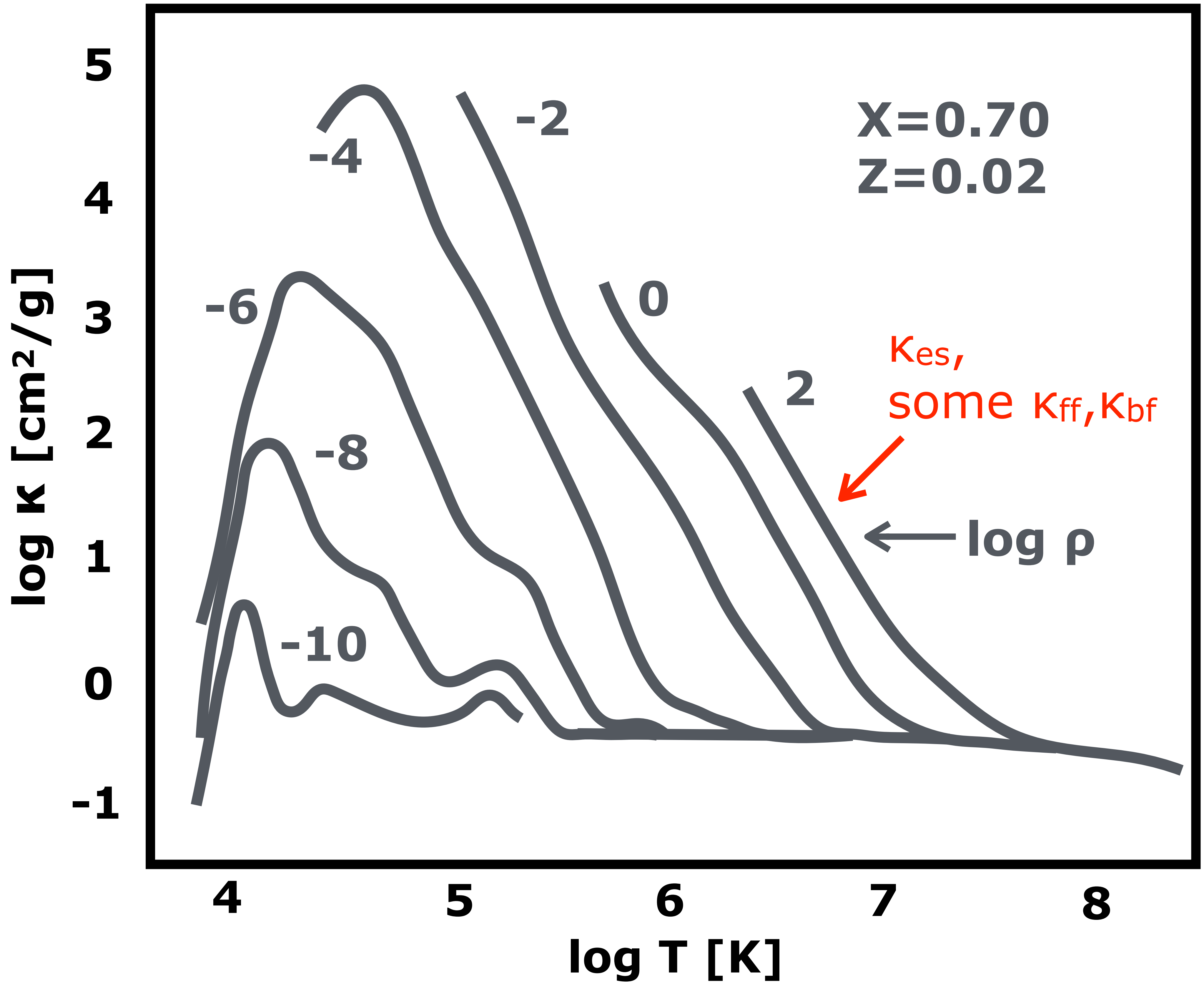

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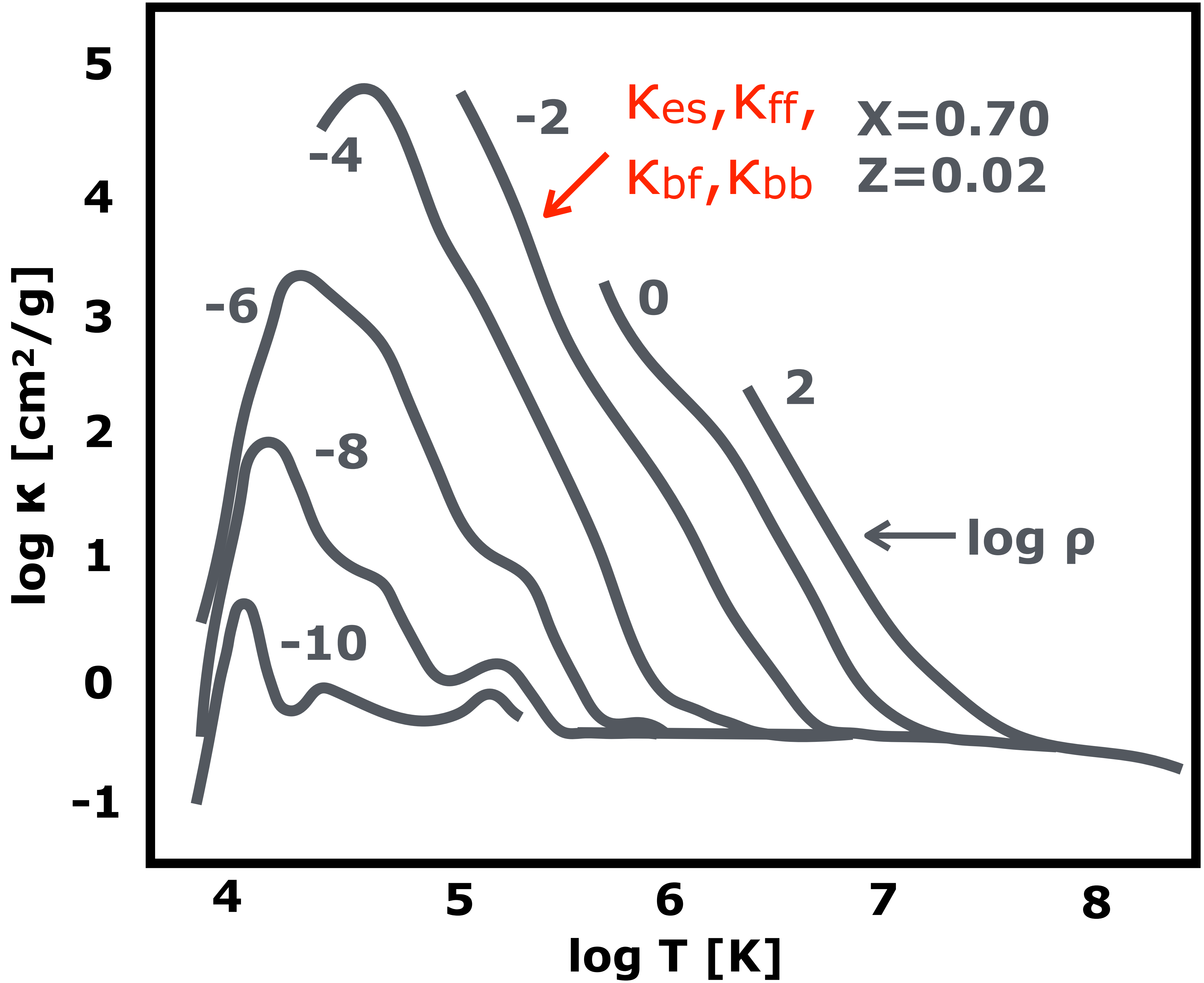

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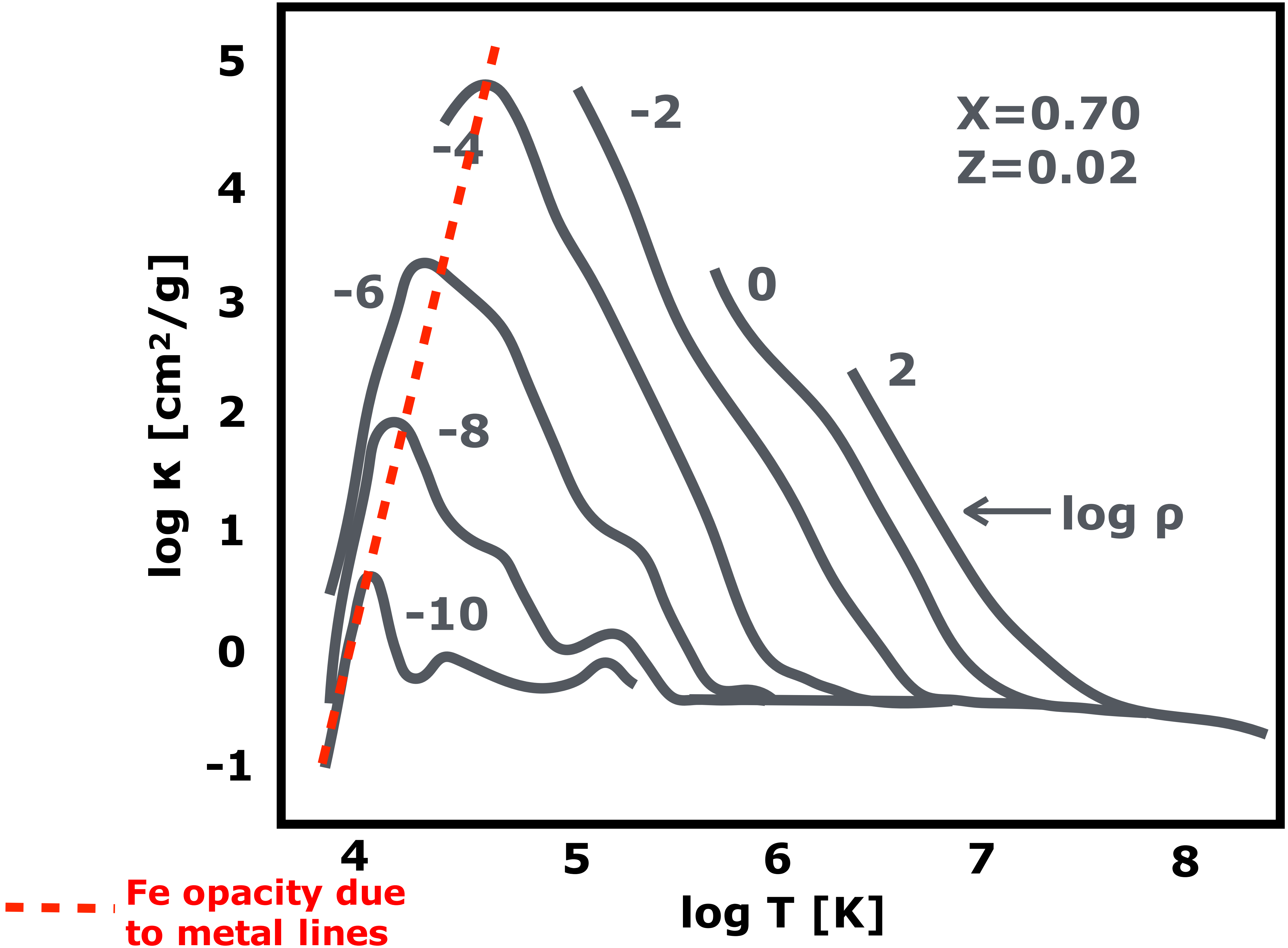


radiative temperature gradient

Temperature in a Star

First, we need to talk about luminosity. We will call $L(r)$ the amount of energy that flows through a spherical surface of radius r for a star. For stars powered by nuclear fusion, $L(r)$ increases outward until we reach the radius where nuclear fusion is no longer occurring. We will call $\epsilon(r)$ the nuclear power generated per unit volume at radius r. Then the power generated in a shell between r and $r+dr$ is

$$\epsilon(r)4\pi r^2 dr.$$

Because this power generated adds to the overall energy flow, we have

$$dL = 4\pi r^2 \epsilon(r) dr$$
$$\frac{dL}{dr} = 4\pi r^2 \epsilon(r).$$

After the radius where nuclear energy stops,

$$\epsilon(r) = 0,$$

and

$$\frac{dL}{dr} = 0.$$

After this radius, the $L(r)$ is a constant, and is equivalent to the surface luminosity.

If radiative diffusion is the dominant energy transport, then

$$L(r) = 4\pi r^2 j(r).$$

Therefore,

$$\frac{L(r)}{4\pi r^2} = -\frac{4ac}{3}\frac{T^3(r)}{\rho(r)\kappa(r)}\frac{dT}{dr},$$

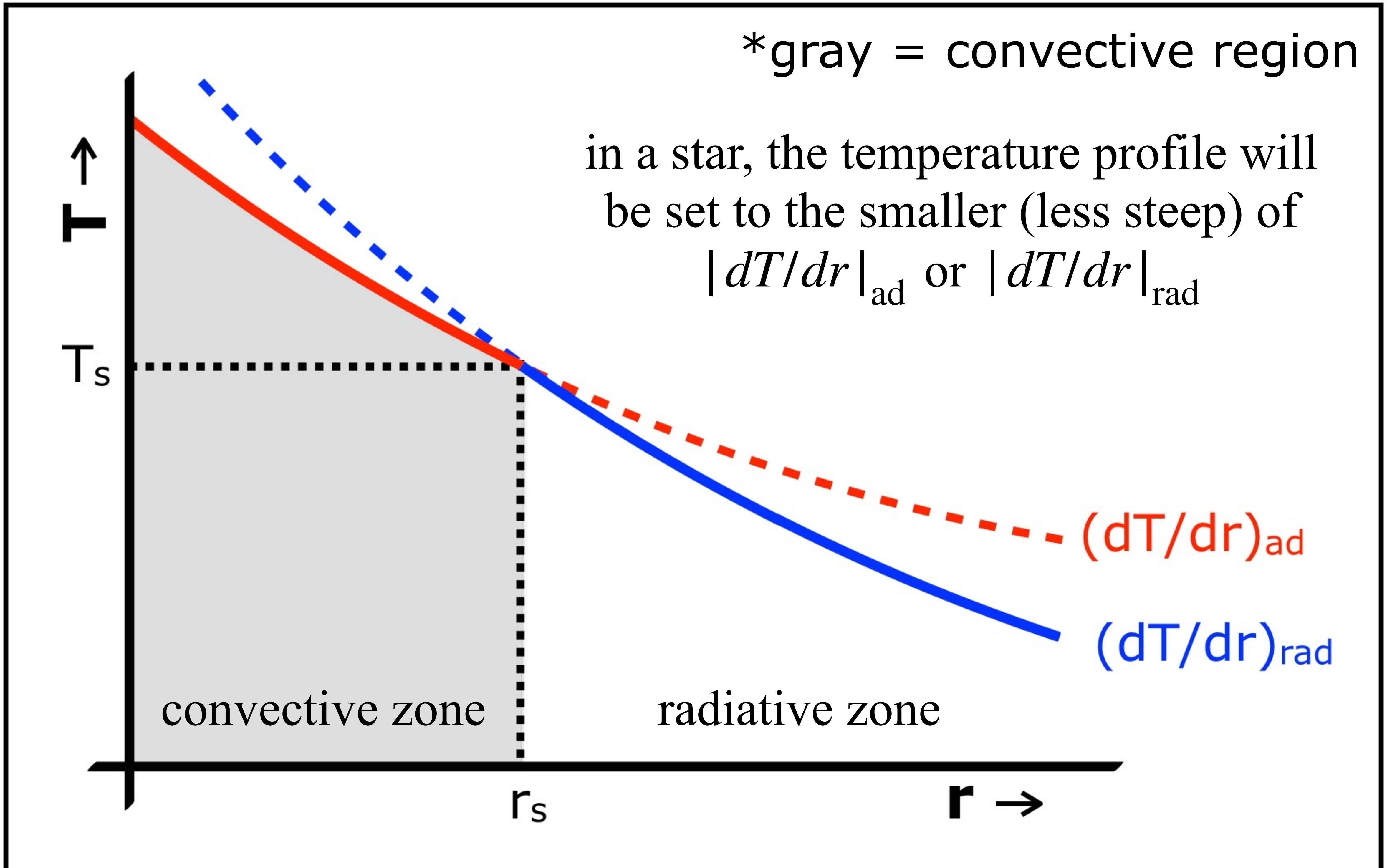
and

$$\boxed{\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa(r)\rho(r)}{T^3(r)}\frac{L(r)}{4\pi r^2}} \qquad \left|\frac{dT}{dr}\right|_{\mathrm{rad}}$$

radiative temperature gradient

Radiation vs. Convection

<u>Schwarzschild criterion</u> $$|dT/dr|_{ad} < |dT/dr|_{rad}$$



stellar structure