Lecture 3: Simple Stellar Models, Polytropes

AN INTRODUCTION TO THE STUDY OF STELLAR STRUCTURE

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TABLE 4

THE CONSTANTS OF THE LANE-EMDEN FUNCTIONS*

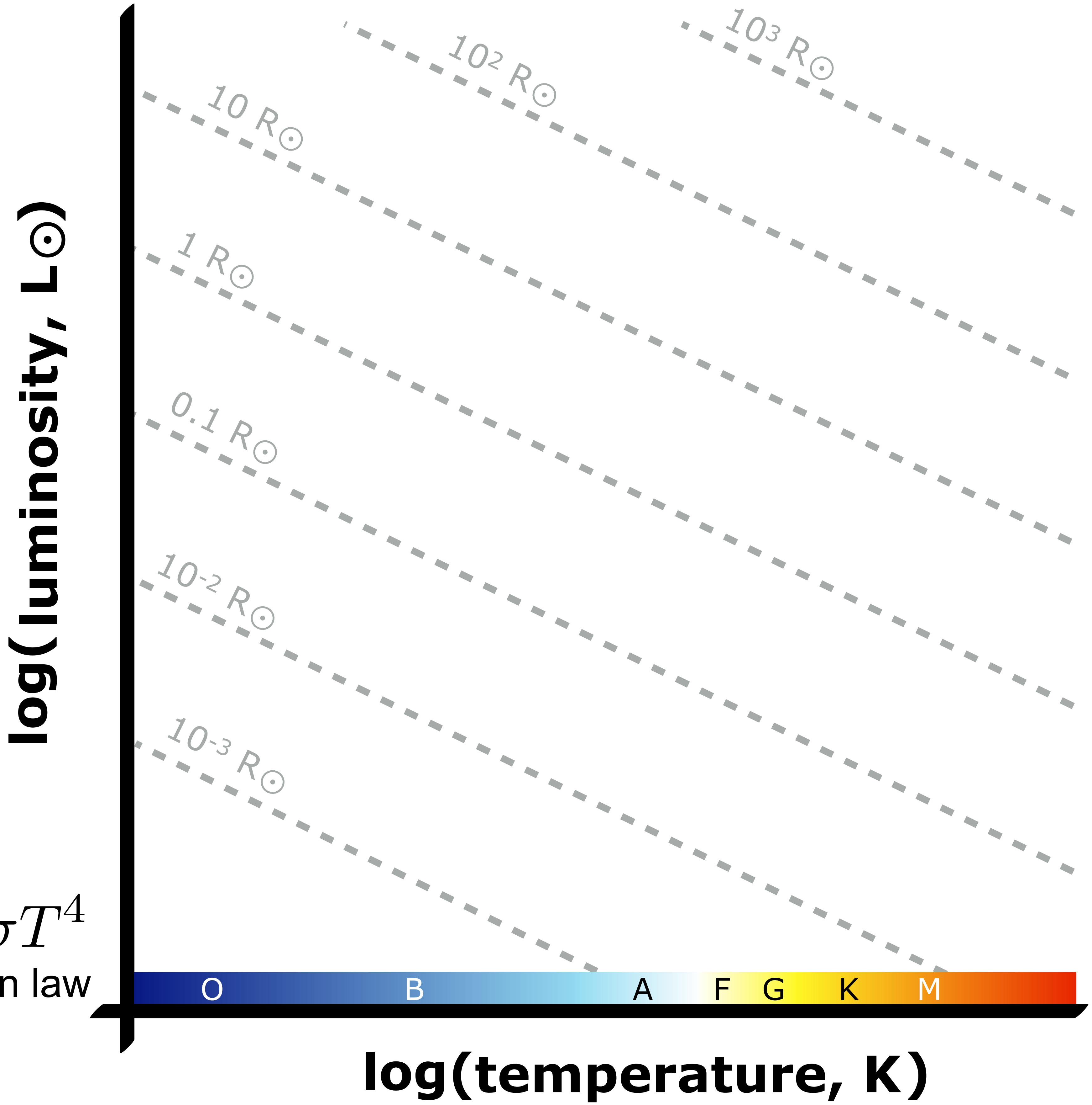
n	ξ_1	$-\xi_1^2\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$	${}_0\omega_n=-\xi_1^{\frac{n+1}{n-1}}\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	N_n		
0	2.4494	4.8988	1.0000	0.33333	....		
0.5	2.7528	3.7871	1.8361	0.02156	2.270		
1.0	3.14159	3.14159	3.28987	....	0.63662		
1.5	3.65375	2.71406	5.99071	132.3843	0.42422		
2.0	4.35287	2.41105	11.40254	10.4950	0.36475		
2.5	5.35528	2.18720	23.40646	3.82662	0.35150		
3.0	6.89685	2.01824	54.1825	2.01824	0.36394	11.05066	0.85432
3.25	8.01894	1.94980	88.153	1.54716	0.37898	20.365	0.96769
3.5	9.53581	1.89056	152.884	1.20426	0.40104	40.9098	1.12087
4.0	14.97155	1.79723	622.408	0.729202	0.47720	247.558	1.66606
4.5	31.83646	1.73780	6189.47	0.394356	0.65798	4922.125	3.33100
4.9	169.47	1.7355	934800	0.14239	1.340	3.693×10^6	16.550
5.0	∞	1.73205	∞	0	∞	∞	∞

* The values for $n = 0.5$ and 4.9 are computed from Emden's integrations of θ_n; for $n = 3.25$ an unpublished integration by Chandrasekhar has been used. $n = 5$ corresponds to the Schuster-Emden integral. For the other values of n the *British Association Tables*, Vol. II, has been used.

http://www.physics.rutgers.edu/ugrad/441

http://www.physics.rutgers.edu/grad/541

The Hertzsprung-Russell Diagram

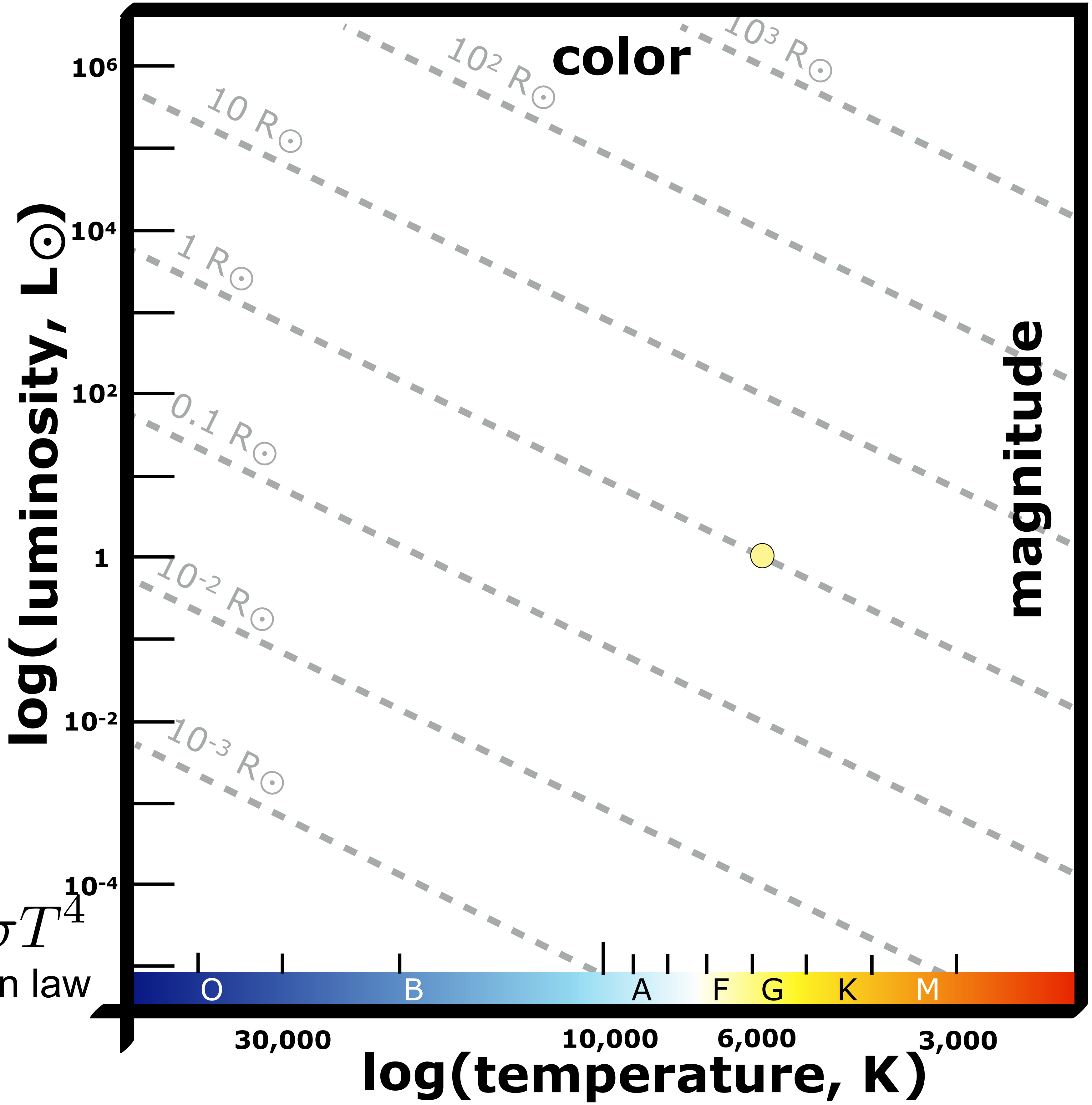


lines of constant radius

$$L = 4\pi R^2 \sigma T^4$$

Stefan-Boltzmann law

The Hertzsprung-Russell Diagram



$$L = 4\pi R^2 \sigma T^4$$

Stefan-Boltzmann law

The Color-Magnitude Diagram

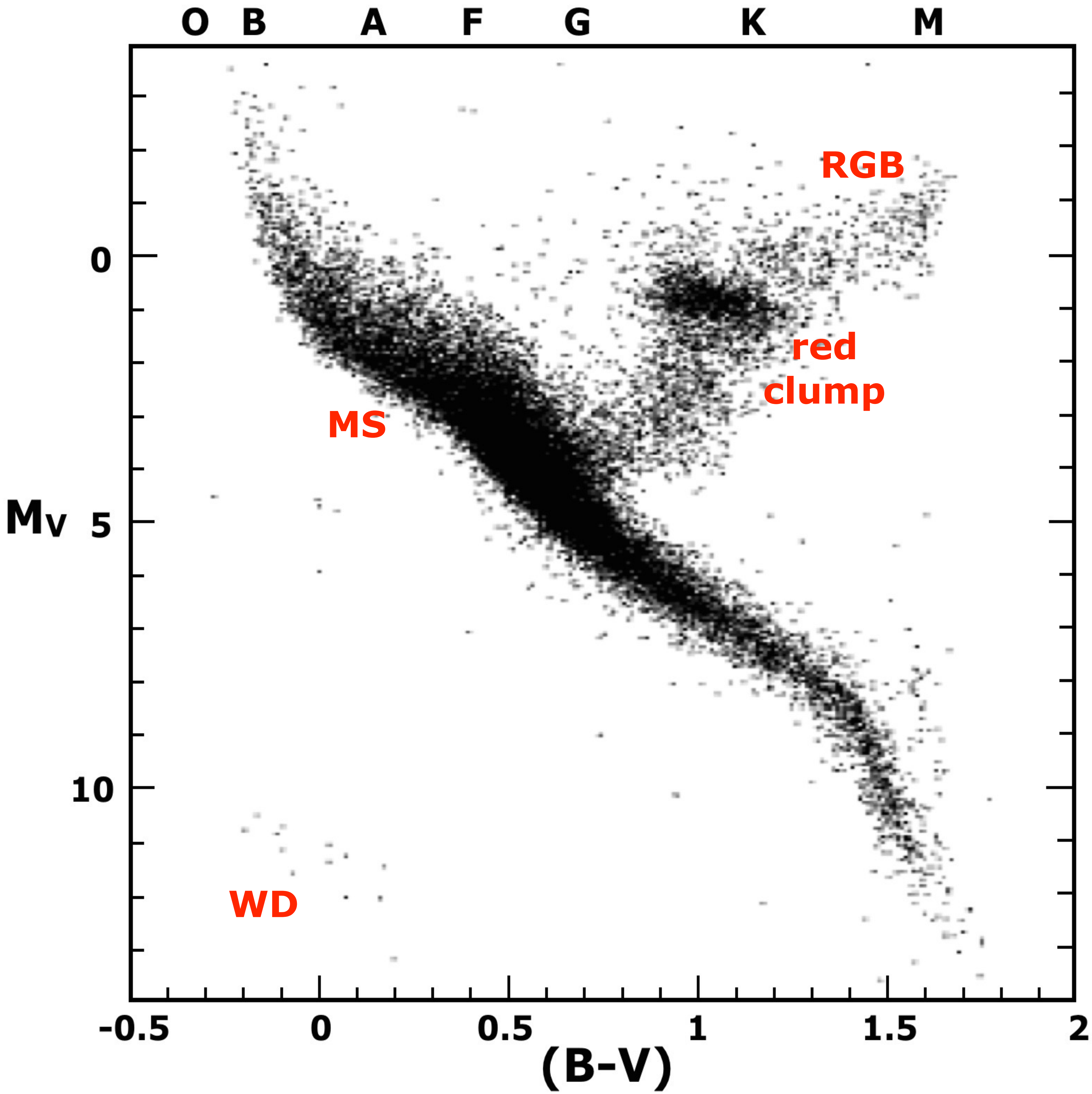
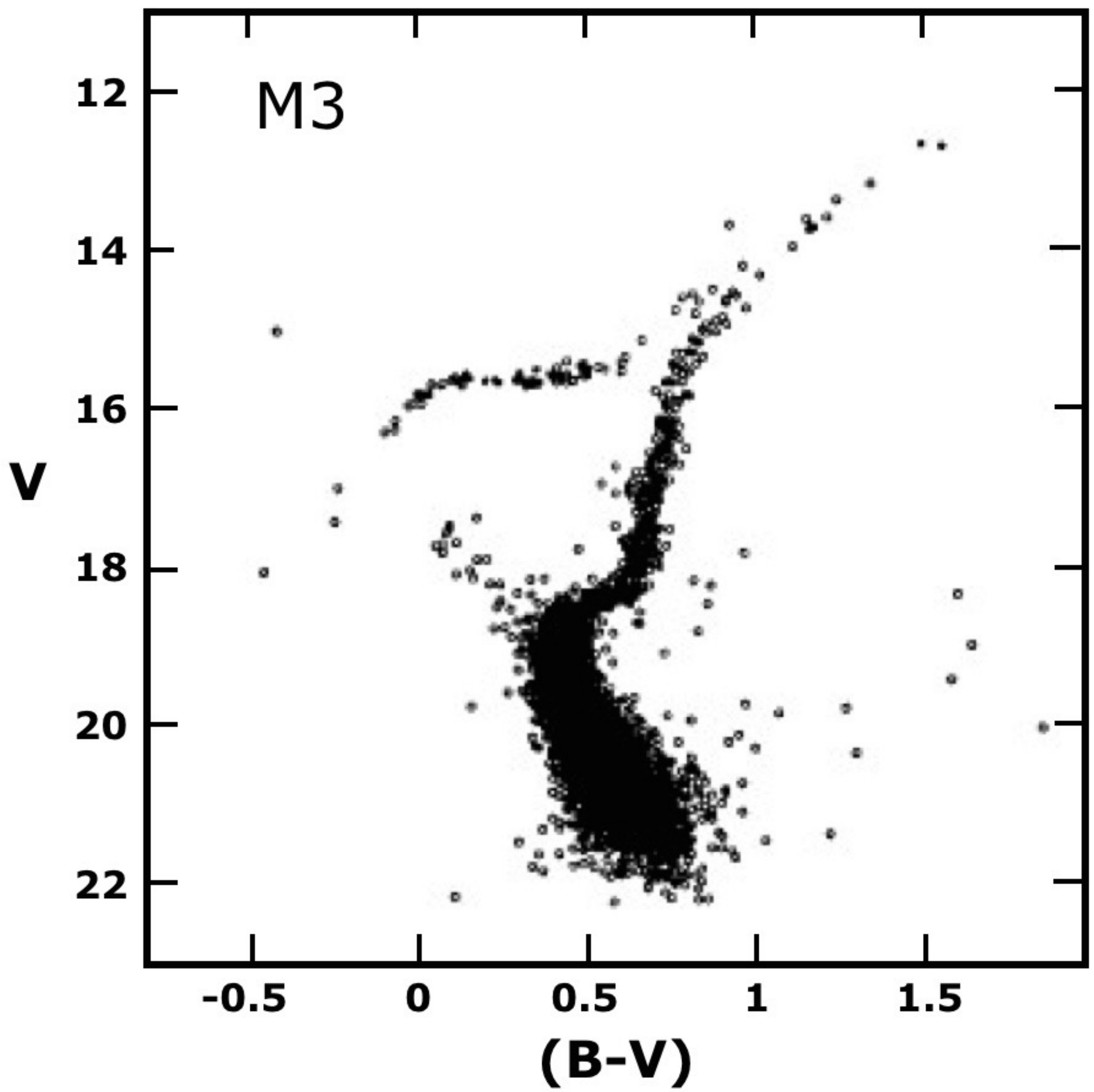
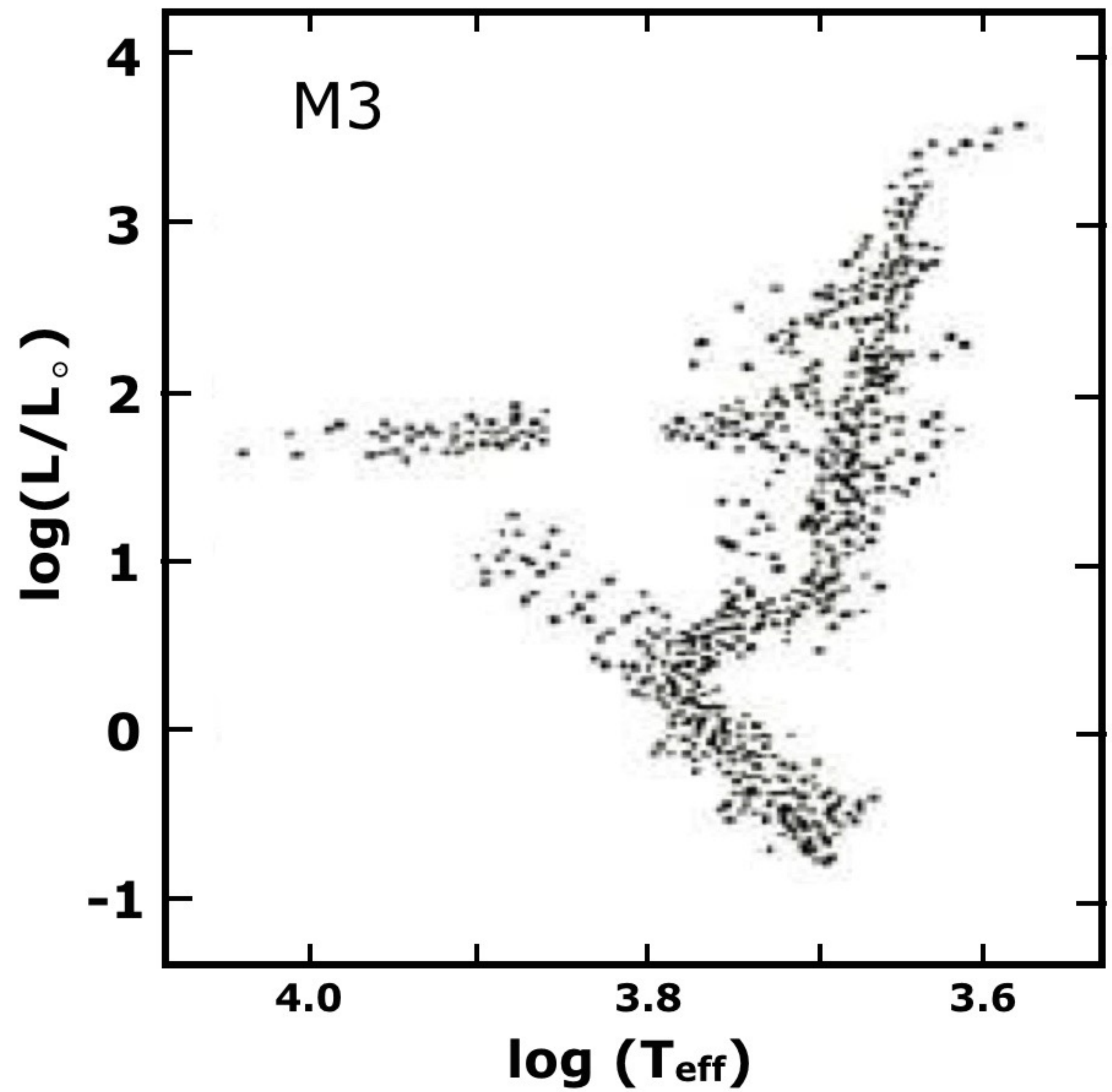


Figure courtesy of ESA.

The Color-Magnitude Diagram





1. on the blue/hot side stars cover a small (B-V), large T_{eff} range
2. on the red/cool side, stars cover a small T_{eff}, large (B-V) range
3. horizontal branch curves down strongly in the CMD

order of magnitude stellar physics

consider a star with mass M and radius R

applied to the Sun

average mass density $$\langle\rho\rangle = \bar{\rho} = \frac{M}{V} = \frac{M}{4/3\,\pi R^3} = \frac{3M}{4\pi R^3}$$ $\langle\rho\rangle_\odot = 1.4 \text{ g cm}^{-3}$

average number density $$n = \frac{\rho}{\langle m\rangle} = \frac{\rho}{\mu m_p}$$ $\langle n\rangle_\odot \approx 1.4 \times 10^{24} \text{ cm}^{-3}$

$\langle m\rangle = \rho/n$ is average particle mass = (mass of all particles)/(number of all particles)
$\mu = \langle m\rangle / m_p$ is mean molecular weight (avg particle mass in units of proton mass)
e.g., for ionized hydrogen (1 proton + 1 electron) $\mu \approx 0.5$

$\langle\mu\rangle_\odot \approx 0.62$
mostly ionized H + He

central temperature $$T_c \sim \frac{GM\mu m_p}{k_B R}$$ $T_{c,\odot} \sim 1.4 \times 10^7 \text{ K}$
very close to the actual value!

central pressure $$P_c \sim \frac{GM^2}{R^4}$$ $P_{c,\odot} \sim 1.1 \times 10^{16} \text{ dyne cm}^2$
about ~20 times lower than the actual value

hydrostatic equilibrium

$$P\Delta A - (P + \Delta P)\Delta A = g\Delta M = g\rho\Delta A\Delta r \quad \Rightarrow \quad -\Delta P = \rho g\Delta r \quad \Rightarrow \quad \frac{\Delta P}{\Delta r} = -\rho g$$

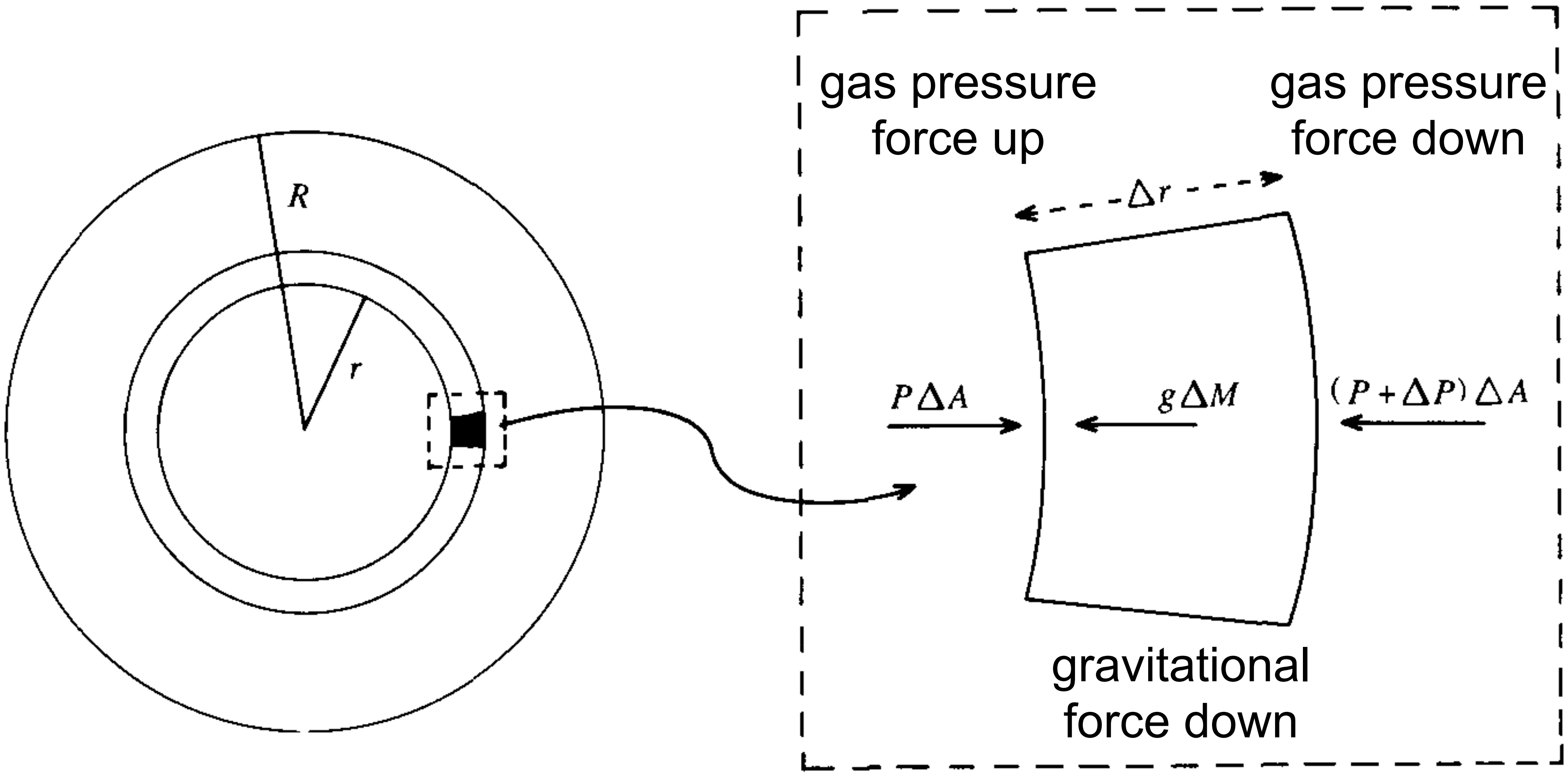
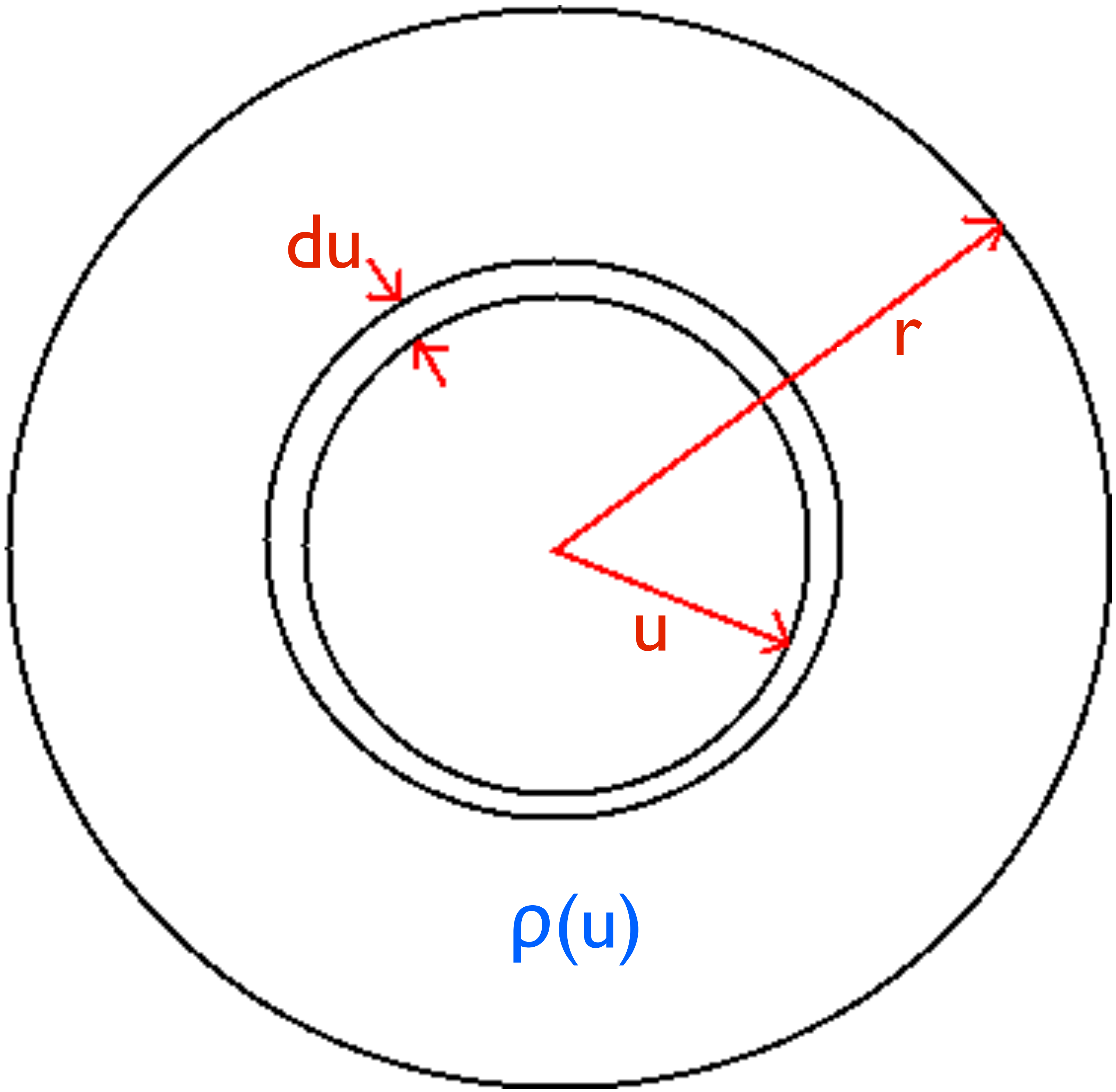


Fig. 1.1 A spherical system of mass M and radius R. The forces acting on a small element with volume $\Delta r\ \Delta A$ at distance r from the centre due to gravity and pressure are indicated. The gravitational attraction of the mass $m(r)$ within r produces an inward force which is equal to $g(r)\ \rho(r)\ \Delta r\ \Delta A = g(r)\ \Delta M$. If there is a non-zero pressure gradient at r, the difference in pressure on the inner and outer surfaces leads to an additional force which can oppose gravity.

Phillips Figure 1.1

integrating over spherical shells



enclosed mass:

$$m(r) = \int_{u=0}^{u=r} \rho(u)\, 4\pi u^2\, du$$
$$= M(r) = M(<r) = M_r$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$$
$$\frac{dm}{dr} = 4\pi r^2 \rho$$

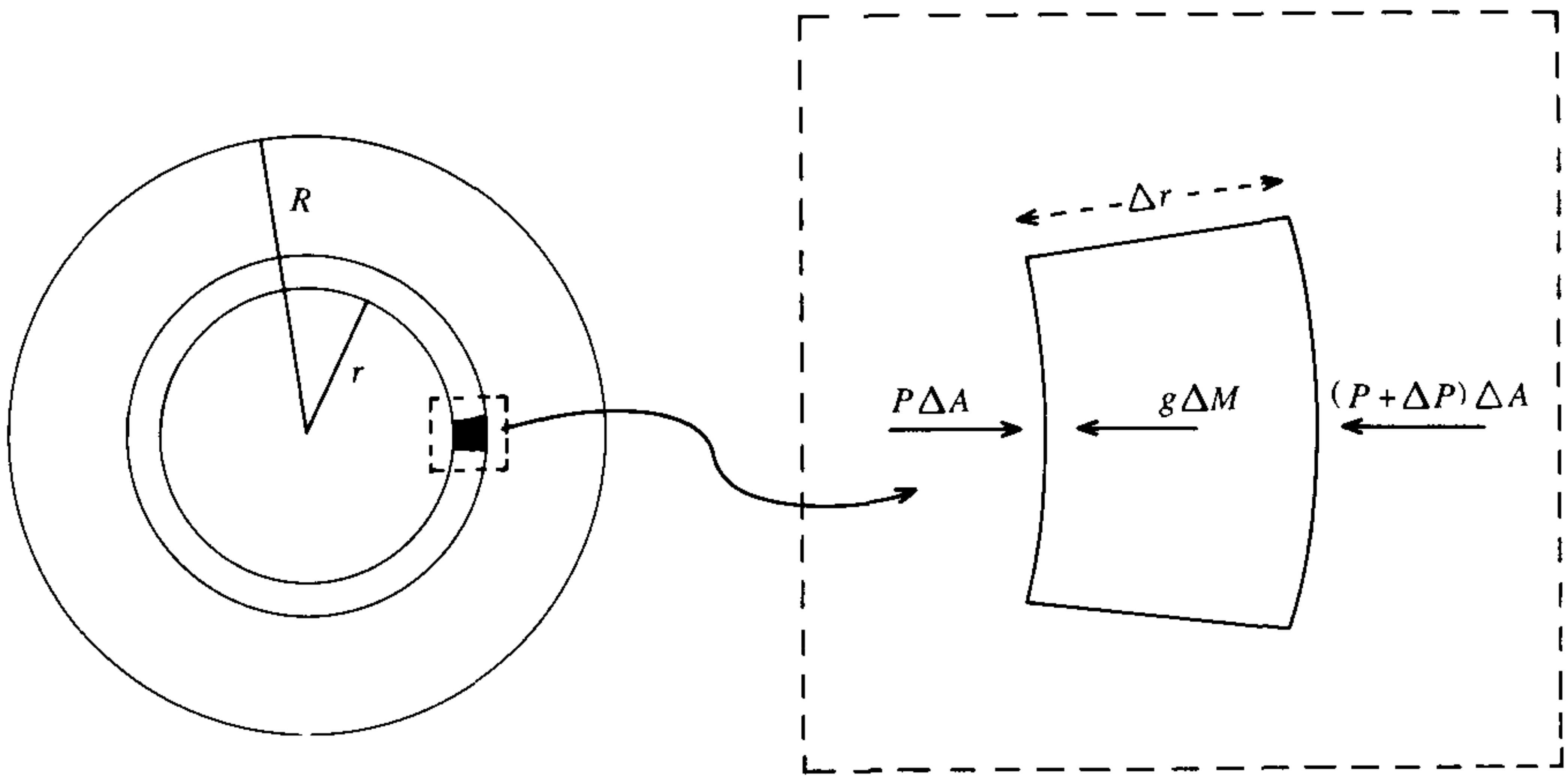
acceleration due to gravity

$$g(r) = \frac{G\, m(r)}{r^2}$$

from http://burro.cwru.edu/Academics/Astr221/StarPhys/gravity.html

hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\rho(r)g(r) = -\frac{Gm(r)\rho(r)}{r^2}$$



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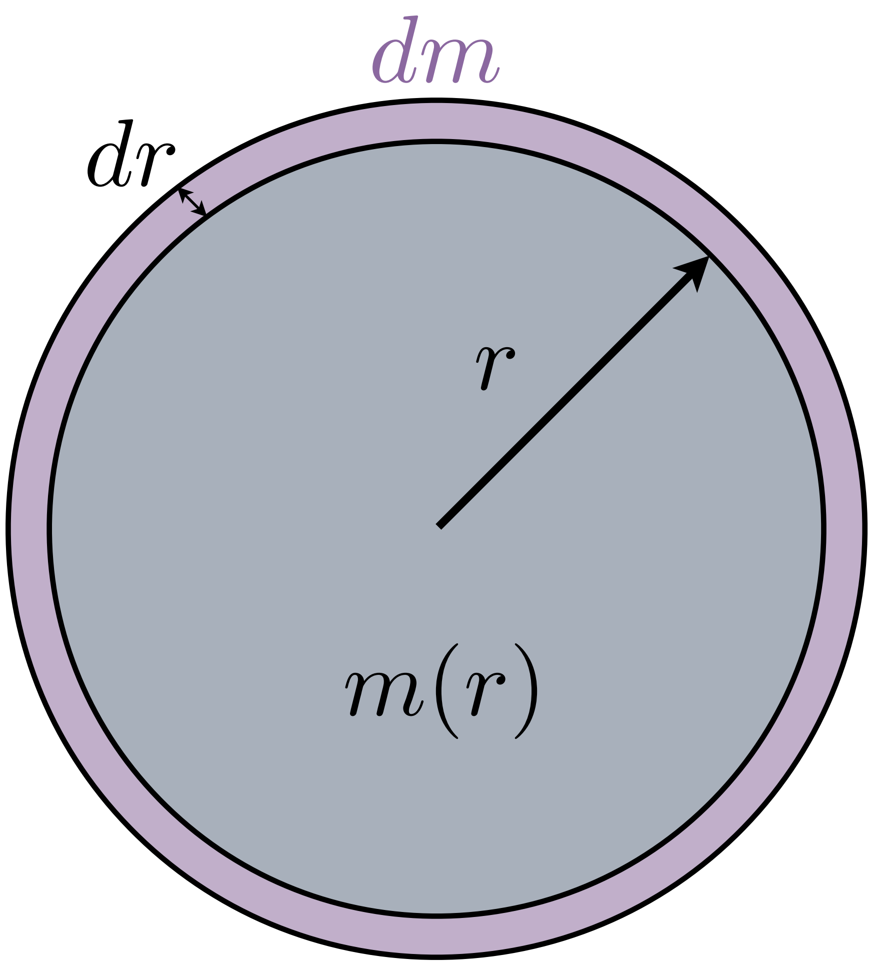
Phillips Figure 1.1

total gravitational potential energy of a spherical mass distribution



mass of infinitesimal spherical shell:

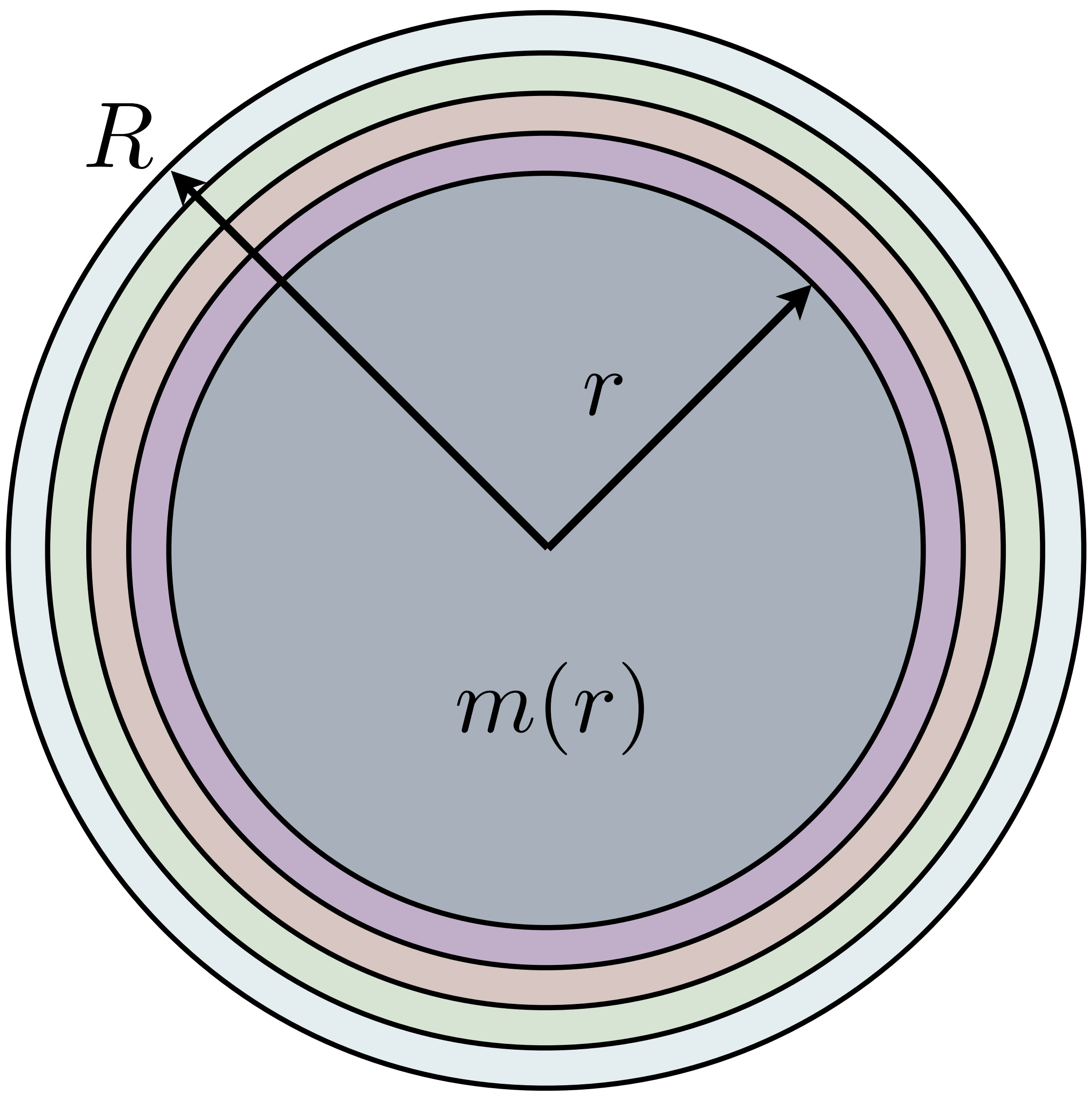
$$dm = 4\pi r^2 \, \rho(r) \, dr$$

energy needed to bring shell in from infinity:

$$dU(r) = -\frac{G \, m(r) \, dm}{r}$$

$$= -4\pi \, G \, m(r) \, \rho(r) \, r \, dr$$

total gravitational potential energy of a spherical mass distribution



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total gravitational potential energy:

$$U = -4\pi G \int_0^R m(r)\,\rho(r)\,r\,dr$$

notation choices for potential energy: $\Omega = U = E_{\mathrm{pot}} = \boldsymbol{E_{GR}}$

deriving the stellar virial theorem

start with hydrostatic equilibrium

$$\frac{dP(r)}{dr} = -\frac{G\, m(r)\, \rho(r)}{r^2}$$

multiply both sides by $4\pi r^3$ and integrate over dr from 0 to *R*

$$\int_0^R 4\pi r^3 \frac{dP(r)}{dr}\, dr = -\int_0^R \frac{G\, m(r)\, \rho(r)}{r^2} 4\pi r^3\, dr$$

note that the right hand side is now just the gravitational potential energy

$$U = -4\pi G \int_0^R m(r)\, \rho(r)\, r\, dr$$

so we have

$$\int_0^R 4\pi r^3 \frac{dP(r)}{dr}\, dr = U$$

integrate the left hand side by parts

$$= \underbrace{\cancel{\left[P(r)4\pi r^3\right]_0^R}}_{0} - 3\int_0^R P(r) 4\pi r^2\, dr$$

$$\int u\, dv = uv - \int v\, du$$

$$u = 4\pi r^3 \qquad v = P$$

$$du = 3(4\pi r^2)\, dr \qquad dv = dP = \frac{dP}{dr} dr$$

$$= -3\, \langle P \rangle\, V$$

volume-averaged pressure *volume*

Phillips 1.2

the stellar virial theorem

$$U = -3\langle P\rangle V \;\Rightarrow\; \langle P\rangle = -\frac{1}{3}\frac{U}{V}$$

gravitational potential energy

volume of star

average pressure

can relate this to the virial theorem used in other contexts
(relation between total kinetic energy and total potential energy)

$$-3\langle P\rangle V = E_{\rm pot}$$

$$PV = NkT \quad \text{ideal gas law}$$

$$E_{\rm kin} = 3/2NkT = 3/2PV$$

$$-3PV = -2E_{\rm kin}$$

$$\boxed{-E_{\rm pot} = 2E_{\rm kin}} \text{ or } \boxed{E_{\rm kin} = -\tfrac{1}{2}E_{\rm pot} > 0}.$$

$$\boxed{E_{\rm tot} = E_{\rm pot} + E_{\rm kin} = \tfrac{1}{2}E_{\rm pot} < 0}.$$

Phillips 1.2

equations of stellar structure (simplified)

enclosed mass (spherical) $$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$$

hydrostatic equilibrium $$\frac{dP(r)}{dr} = -\frac{G\, m(r)\, \rho(r)}{r^2}$$

there are more that we will explore later:
temperature $T(r)$, luminosity/energy generation, composition, etc. profiles

but if we ignore that for now and *assume* that **pressure depends only on density**
we can create a simplified stellar model: a *polytrope*

(note: this is not a good model for stars like the Sun, where the pressure depends on temperature, e.g., ideal gas law)

first we combine the two equations above into one "Poisson" equation
valid for any spherical star in hydrostatic equilibrium:

$$\left.\begin{aligned} &\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r) = -\frac{Gm}{r^2}\rho \quad \Longrightarrow \quad \frac{r^2}{\rho}\frac{dP}{dr} = -Gm \\ &\frac{dm}{dr} = 4\pi r^2 \rho \quad \Longrightarrow \quad \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} = -4\pi r^2 G\rho \end{aligned}\right\} \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

polytropes: summary

see also Phillips section 5.2
Lamers & Levesque chapter 11

Assume pressure depends only on density:

$$P = K\rho^{\gamma} \quad P = K\rho^{1+1/n} \qquad \gamma = 1 + 1/n \Leftrightarrow n = 1/(\gamma - 1)$$

Define dimensionless variables θ (1 at center, 0 at surface) and ξ (like radius, 0 at center), and α (not dimensionless: constant with units of length):

$$\rho = \rho_c \theta^n, \qquad P = P_c \theta^{n+1}, \qquad r = \alpha\xi, \qquad \alpha^2 = \frac{K(n+1)\rho_c^{\frac{1-n}{n}}}{4\pi G},$$

Then combined equations for hydrostatic equilibrium and enclosed mass can be rewritten:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \quad \Longrightarrow \quad \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

Lane-Emden equation

Use boundary conditions at center: $\theta(\xi = 0) = 1$ and $\left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0$

Then can solve equations (typically numerically) for a given value of polytropic index *n.*
Mathematical solutions of Lane-Emden equation just depend on *n* (and boundary conditions).

Physical solutions start at ξ = 0 (center) and extend to ξ_1 where $\theta(\xi_1) = 0$ (surface)
To get **physical units**, we **require two more parameters**. Choose **any two** of the following: total mass *M*, radius *R*, central density ρ_c, eqn of state const *K*, central pressure P_c, etc.
Once you specify two of those, all the others can be solved in terms of the rest, e.g.:

$$R = \alpha\xi_1 = \left[\frac{K}{G}\frac{n+1}{4\pi}\right]^{1/2} \rho_c^{\frac{1-n}{2n}} \underbrace{\xi_1} \qquad M = 4\pi\alpha^3\rho_c \underbrace{\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}} \qquad \Omega = E_{\text{pot}} = -\frac{3}{5-n}\frac{GM^2}{R}$$

look these up in tables, for a given value of *n*

polytropes (Chandrasekhar 1939)

TABLE 4

THE CONSTANTS OF THE LANE-EMDEN FUNCTIONS*

n	ξ_1	$-\xi_1^2\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$	${}_0\omega_n = -\xi_1^{\frac{n+1}{n-1}}\left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	N_n		
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* The values for $n = 0.5$ and 4.9 are computed from Emden's integrations of θ_n; for $n = 3.25$ an unpublished integration by Chandrasekhar has been used. $n = 5$ corresponds to the Schuster-Emden integral. For the other values of n the *British Association Tables*, Vol. II, has been used.

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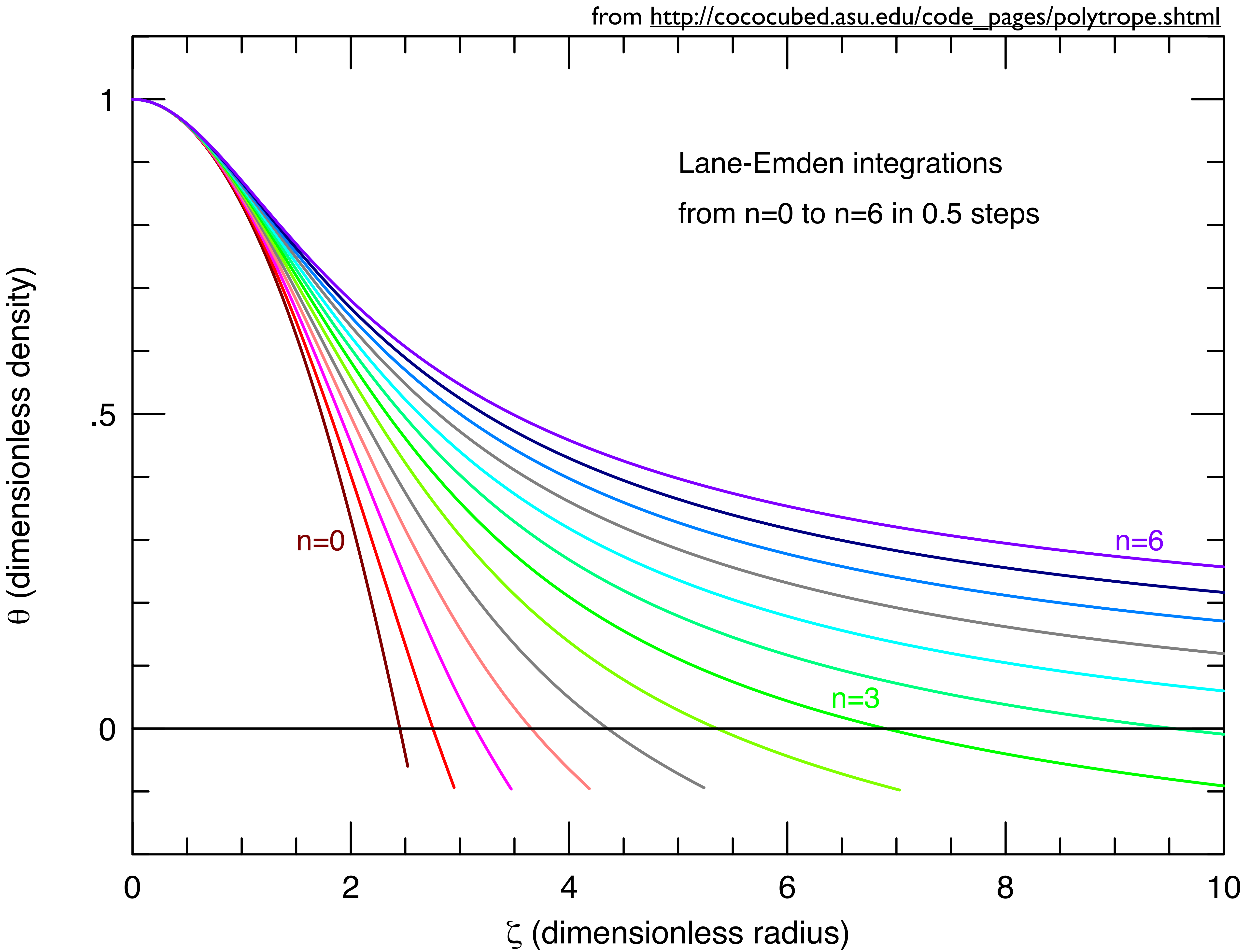

from http://cococubed.asu.edu/code_pages/polytrope.shtml
Lane-Emden integrations
from n=0 to n=6 in 0.5 steps
n=0
n=3
n=6
1
.5
0
0
2
4
6
8
10
θ (dimensionless density)
ζ (dimensionless radius)

The radius of the star is defined by $\rho = 0$, so $\theta(\xi_1) = 0$, and ξ_1 is the value of the ordinate where $\theta = 0$. This means that

$$R = \alpha\xi_1 \equiv \alpha R_{\mathrm{n}}. \tag{11.8}$$

The mass of the star is given by

typo in L&L eqn 11.9
fixed here

$$M = 4\pi \int_0^R \rho r^2 \, dr = 4\pi\alpha^3 \rho_c \int_0^{\xi 1} \xi^2 \theta^n \, d\xi. \tag{11.9}$$

Using Equation (11.7) results in

$$M = -4\pi\alpha^3 \rho_c \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi 1} \equiv 4\pi\alpha^3 \rho_c M_n, \tag{11.10a}$$

with

$$M_n = -\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi 1}. \tag{11.10b}$$

Table 11.1. Physical Parameters of Polytrope Models.

n	γ	$R_n = \xi_1$	$M_n = -\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$
1.00	2.00	3.14	3.14
1.50	5/3	3.65	2.71
3.00	4/3	6.90	2.02

from Lamers & Levesque, Ch. 11

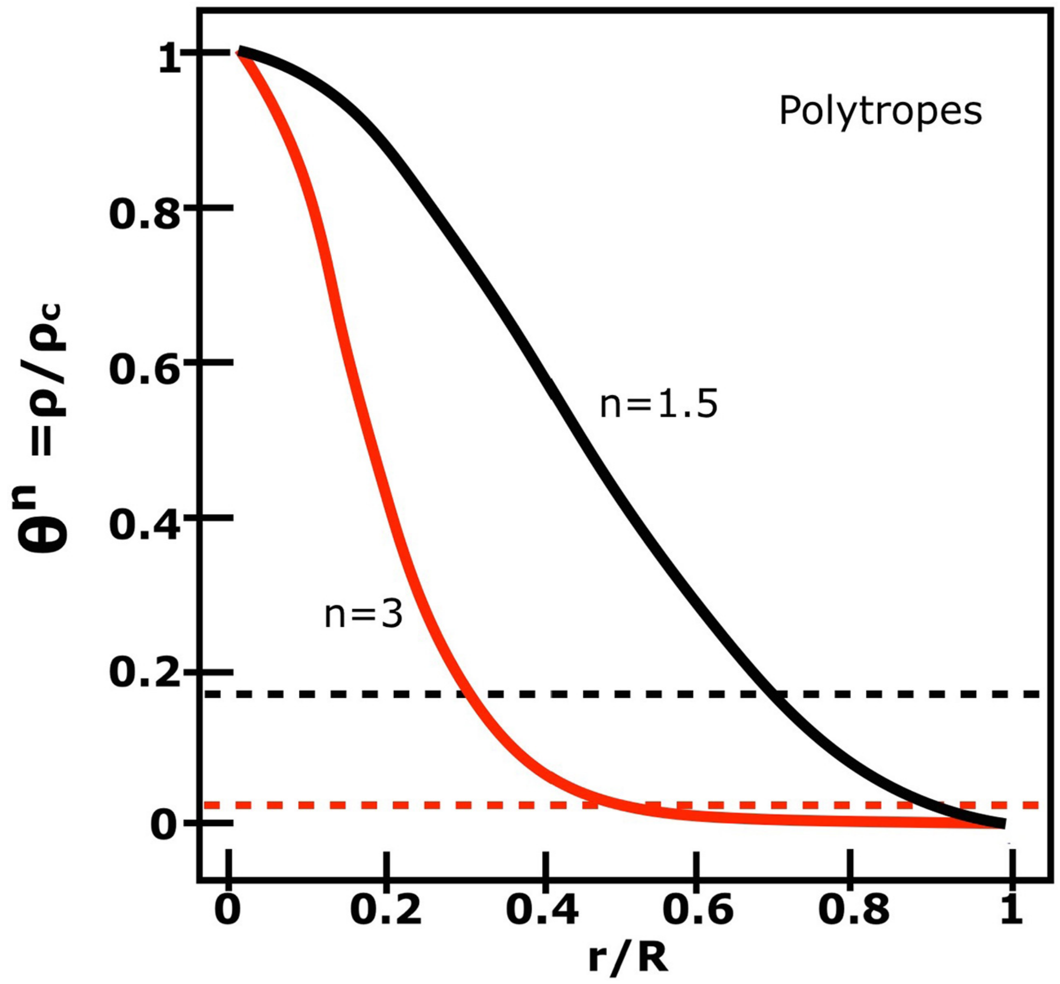

Figure 11.1. Density structure of polytropes for n = 3 (γ = 4/3) (red) and n = 1.5 (γ = 5/3) (black). The dashed lines show the mean density of the two models.

Table 11.2. Properties of Polytropic Stars

n	γ	Property	M–R Relation	Consequence
0	∞	Incompressible	$R \sim M^{1/3}$	Constant ρ
1	2	$P \sim \rho^2 \rightarrow T \sim \rho$	$R \sim M^{0}$	Constant R
1.5	5/3	Nonrelat. degeneracy	$R \sim M^{-1/3}$	Volume $\sim 1/M$
		Fully convective (ideal gas)		
3.0	4/3	Relativ. degeneracy	$R \sim M^{\infty}$	$M \sim R^{0}$
		Constant P_{gas}/P_{rad} (ideal gas)		
∞	1	Isothermal: $P \sim \rho$	$R \rightarrow \infty$	Infinite radius