Physics 441/541 Spring 2022: Problem Set #1 Solutions

1. *The Planck spectrum has*

$$B_\lambda(T) = \frac{2\,hc^2/\lambda^5}{\exp(hc/\lambda kT) - 1}$$

(a) *Derive Wien's displacement law: show that the maximum of $B_\lambda(T)$ occurs at*

$$\lambda_{\rm max} T = 0.290 \text{ cm K}$$

To help you along, note that the equation $y = 5(1 - e^{-y})$ is solved by $y \approx 4.965$.

We need to find where $dB_\lambda/d\lambda = 0$. Let's define $y = hc/\lambda kT$ so $\lambda = hc/ykT$ and $d\lambda = -hc/y^2kT\,dy$. Then

$$B_\lambda = \frac{2hc^2y^5k^5T^5/h^5c^5}{e^y - 1} = \frac{2k^5T^5}{h^4c^3}\frac{y^5}{e^y - 1} = Q\frac{y^5}{e^y - 1}$$

where I collected all the constants in $Q \equiv 2k^5T^5/h^4c^3$. Then

$$\frac{dB_\lambda}{d\lambda} = -\frac{y^2kT}{hc}\frac{dB_\lambda}{dy} = \frac{y^2kTQ}{hc}\frac{d}{dy}\left[\frac{y^5}{1 - e^y}\right] = 0 \quad\Longrightarrow\quad \frac{d}{dy}\left[\frac{y^5}{1 - e^y}\right] = 0$$

$$\Longrightarrow \quad \frac{y^4}{(1 - e^y)^2}\left[ye^y + 5(1 - e^y)\right] = 0 \quad\Longrightarrow\quad y = 5(1 - e^{-y}) \quad\Longrightarrow\quad y \approx 4.9651$$

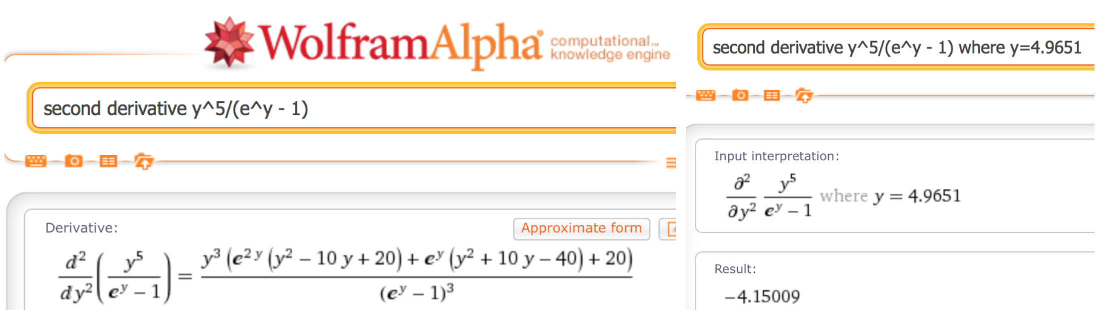
where in the last step we used the provided hint. Plugging in this value for y we get $\lambda_{\rm max} = hc/4.9651kT$, so then we get the desired result

$$\lambda_{\rm max}T = \frac{hc}{4.9651\,k} = \frac{(6.626\times10^{-27}\text{ cm}^2\text{ g s}^{-1})(3.0\times10^{10}\text{ cm s}^{-1})}{4.9651(1.38\times10^{-16}\text{ g cm}^2\text{ s}^{-2}\text{ K}^{-1})} = 0.290\text{ cm K}$$

To show this is a maximum (not a minimum or saddle point), we need to show that the second derivative $d^2B_\lambda/d\lambda^2$ is negative (concave down) at $\lambda_{\rm max}$.

$$\frac{d^2B_\lambda}{d\lambda^2} = \frac{y^4k^2T^2Q}{h^2c^2}\frac{d^2}{dy^2}\left[\frac{y^5}{e^y - 1}\right] = \ldots$$

Okay, clearly this is going to be some tedious algebra. Let's take the WolframAlpha shortcut [1] whereby we see the second derivative is negative as required.



[1] https://www.wolframalpha.com/input/?i=second+derivative+y%5E5%2F%28e%5Ey+-+1%29 and https://www.wolframalpha.com/input/?i=second+derivative+y%5E5%2F%28e%5Ey+-+1%29+where+y%3D4.9651

(b) *Show that the integral of the Planck function over all wavelengths is given by*

$$\int_0^\infty B_\lambda(T)\,d\lambda = \frac{2\pi^4 k^4}{15c^2h^3}T^4$$

Use the fact that $\int_0^\infty y^3/(e^y-1)\,dy = \pi^4/15$.

As before, take $y = hc/\lambda kT$, so $B_\lambda = Qy^5/(e^y-1)$ and $d\lambda = -hc/y^2kT\,dy$, thus

$$\int_{\lambda=0}^{\lambda=\infty} B_\lambda d\lambda = -Q\int_{y=\infty}^{y=0}\frac{y^5}{e^y-1}\frac{hc}{y^2kT}\,dy = \frac{Qhc}{kT}\int_0^\infty \frac{y^3\,dy}{e^y-1} = \frac{Qhc}{kT}\frac{\pi^4}{15} = \frac{2\pi^4k^4}{15c^2h^3}T^4$$

where I used the provided hint to do the integral.

(c) *Explain how and why the term that multiplies* T^4 *in your result above is related to the Stefan-Boltzmann constant.*

One form of the Stefan-Boltzmann law says that the bolometric flux at the surface of a blackbody emitter is given by $F = \sigma T^4$. In class we saw that for an isotropic blackbody source, the surface flux density was $F_\lambda = \pi B_\lambda$. So we have

$$F = \sigma T^4 = \pi\int_0^\infty B_\lambda d\lambda = \pi\left(\frac{2\pi^4k^4}{15c^2h^3}T^4\right) \implies \sigma = \pi\left(\frac{2\pi^4k^4}{15c^2h^3}\right) = \frac{2\pi^5k^4}{15c^2h^3}$$

Thus we see that the Stefan-Boltzmann constant is just π times the prefactor in front of T^4 from the Planck function integral. This is really π sterradians, as integrating the Planck function over wavelength gives a quantity (brightness or intensity) with CGS units of erg cm^{-2} s^{-1} sr^{-1} while the flux has CGS units of just erg cm^{-2} s^{-1}.

2. *(adapted from Lamers & Levesque, problem 2.3) The star* τ *Sco has an apparent visual magnitude* $m_V = +2.8$. *It is at a distance of 470 light years. Its spectral type is B0V, corresponding to an effective temperature* $T_{\text{eff}} = 30{,}000$ *K and a bolometric correction of* -3.2 *mag. Determine the following quantities for this star:*

(a) *its peak wavelength,* $\lambda_{\max}$

Assuming we can treat the star as a blackbody emitter, from the Wien displacement law we have

$$\lambda_{\max} = \frac{0.290\text{ cm K}}{T} = \frac{0.290\text{ cm K}}{30000\text{ K}} = 9.67\times10^{-6}\text{ cm} = 96.7\text{ nm} = 967\text{ Å}$$

(b) *its absolute visual magnitude,* M_V

The absolute magnitude of a star is the magnitude it would have if it were at a distance of 10 pc, so we have

$$M_V = m_V + 5\log\left(\frac{d}{10\text{ pc}}\right) = 2.8 - 5\log\left(\frac{470/3.26\text{ pc}}{10\text{ pc}}\right) = -3.0\text{ mag}$$

(c) *its distance modulus, μ*

The distance modulus is

$$\mu = m - M = 5\log\left(\frac{d}{10\ \text{pc}}\right) = 5\log\left(\frac{470/3.26\ \text{pc}}{10\ \text{pc}}\right) = 5.8\ \text{mag}$$

(d) *its apparent and absolute bolometric magnitudes, m_{bol} and M_{bol}*

We can just use the bolometric correction:

$$m_{\text{bol}} = m_V + BC = 2.8 + (-3.2) = -0.4\ \text{mag}$$

$$M_{\text{bol}} = M_V + BC = -3.0 + (-3.2) = -6.2\ \text{mag}$$

(e) i*ts bolometric luminosity in physical units (erg/s or W) and in solar units ($L_\odot$)*

We can use the fact that the Sun has a bolometric absolute magnitude of $M_{\text{bol},\odot} = 4.74$ to write

$$M_{\text{bol}} = -2.5\log\left(\frac{L}{L_\odot}\right) + 4.74$$

So then solving for L we have

$$L = 10^{-0.4(M_{\text{bol}}-4.74)}\ L_\odot = 10^{-0.4(-6.2-4.74)}\ L_\odot = 10^{4.38}\ L_\odot = 2.4\times10^4\ L_\odot$$

Using $L_\odot = 3.83\times10^{33}$ erg s^{-1} this corresponds to $L = 9.1\times10^{37}$ erg s^{-1} = 9.1×10^{30} W.

(f) *its radius in physical units (cm or m) and in solar units ($R_\odot$)*

From the Stefan-Boltzmann law $L = 4\pi R^2\sigma T_{\text{eff}}^4$ we can derive

$$\begin{aligned} R &= \left(\frac{L}{4\pi\sigma T_{\text{eff}}^4}\right)^{1/2} = \left[\frac{9.1\times10^{37}\ \text{erg s}^{-1}}{4\pi(5.67\times10^{-5}\ \text{erg s}^{-1}\ \text{cm}^{-2}\ \text{K}^{-4})(30000\ \text{K})^4}\right]^{1/2} \\ &= 4.0\times10^{11}\ \text{cm} = 4.0\times10^{9}\ \text{m} = 5.7\ R_\odot \end{aligned}$$

where in the last step I used $R_\odot = 6.96\times10^{10}$ cm.

(g) *the bolometric flux (erg/cm^2/s or W/m^2) at its surface*

The bolometric flux at the surface is

$$\begin{aligned} F &= \sigma T_{\text{eff}}^4 = (5.67\times10^{-5}\ \text{erg s}^{-1}\ \text{cm}^{-2}\ \text{K}^{-4})(30000\ \text{K})^4 \\ &= 4.6\times10^{13}\ \text{erg cm}^{-2}\ \text{s}^{-1} = 4.6\times10^{10}\ \text{W m}^{-2} \end{aligned}$$

(h) *its bolometric flux at the Earth (erg/cm^2/s or W/m^2); also compare this to the bolometric flux from the Sun at the Earth (the total solar irradiance).*

The received bolometric flux (or measured flux or apparent flux) is $f = \frac{L}{4\pi d^2}$ where L is the bolometric luminosity and d is the distance to the star. So we get

$$f = \frac{9.1\times10^{37}\ \text{erg s}^{-1}}{4\pi(470\times9.461\times10^{17}\ \text{cm})^2} = 3.66\times10^{-5}\ \text{erg cm}^{-2}\ \text{s}^{-1} = 3.66\times10^{-8}\ \text{W m}^{-2}$$

The bolometric flux of the Sun at Earth is $L_\odot/(4\pi[1\text{ AU}]^2) = 1.36\times10^6$ erg cm^{-2} s^{-1} = 1360 W m^{-2}, so the star τ Sco appears 2.7×10^{-11} times as bright as the Sun. Another way to say it is that the Sun appears nearly 40 billion (3.7×10^{10}) times as bright as τ Sco.

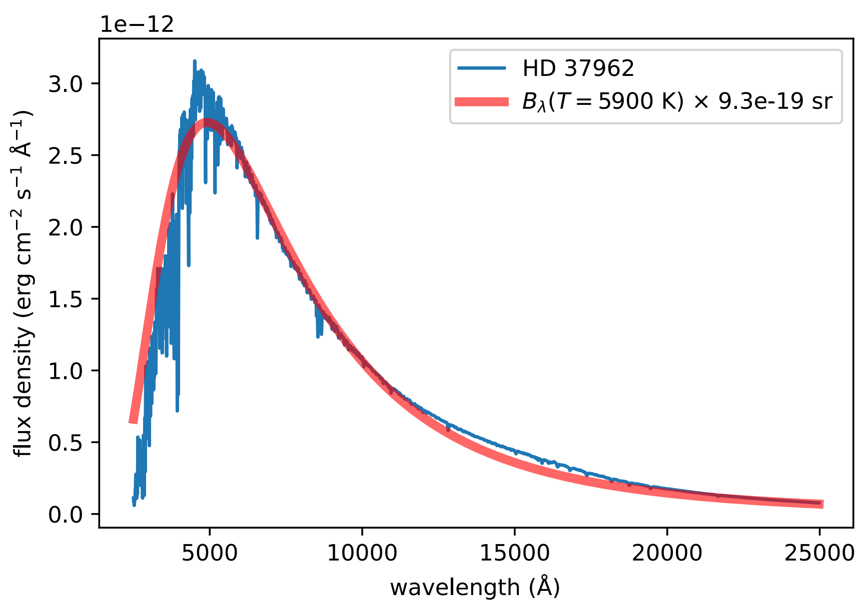
3. *On our Canvas site, under Files → Problem Set Resources, find a file called* `HD37962.txt`*. (There is also a CSV version of the same file.) This contains spectroscopic observations of the star HD 37962 taken with the Hubble Space Telescope; the file tabulates wavelength λ (in Angstroms) and observed flux density f_λ (in erg/cm^2/sec/Angstrom).*

   (a) *Using any plotting program or spreadsheet, plot the spectrum (f_λ versus λ). Be sure to label your axes, with correct units.*

   (b) *Using trial and error (or perhaps the Wien displacement law...), overplot a reasonable blackbody fit (Planck spectrum, B_λ) for this star. Label this curve with the temperature you used. You will need to multiply the Planck spectrum by some scale factor to get it close to the stellar spectrum. Use trial and error for this.* **Be careful with units!** *Make sure your B_λ is in erg/cm^2/sec/Angstrom/sterradian before multiplying by the scale factor (which will have units of sterradians) to match the data.*

   I have done this in an Jupyter notebook using python, which you can find at `http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps01q3.ipynb`. You can view the code in any browser. If you have python installed[2], you can download and run the notebooks interactively, edit the code, etc. I encourage you do this!

   Here is the figure from the notebook, where we see the scale factor required is $\sim 9.3 \times 10^{-19}$ sr.



[2] It's free and relatively easy to do; I use the `anaconda` distribution of Python 3.10: `https://docs.anaconda.com/anaconda/`.

(c) *The star HD 37962 has a parallax of 28.63 milliarcsec (measured by the Gaia satellite). What is its distance in parsecs? in cm?*

The distance in parsecs is just $d \approx 1/p$, where p is the parallax angle in arcseconds. One parsec is 3.086×10^{18} cm, so we have

$$d = 1/0.02863 = 34.93 \text{ pc} = 1.08 \times 10^{20} \text{ cm}$$

(d) *In class we saw that the radiant flux density at the surface of a spherical blackbody is $F_\lambda = \pi B_\lambda$. The luminosity per unit wavelength is the flux density times the surface area, $L_\lambda = 4\pi R^2 F_\lambda$. The measured flux density, f_λ, for an observer a distance d away is given by the inverse square law*

$$f_\lambda = \frac{L_\lambda}{4\pi d^2} = \frac{4\pi R^2 F_\lambda}{4\pi d^2} = F_\lambda \left(\frac{R}{d}\right)^2 = \pi B_\lambda \left(\frac{R}{d}\right)^2$$

This tells us the scale factor to go from the Planck function B_λ to the measured flux density f_λ is $\pi(R/d)^2$. Use the scale factor you estimated in part (b) and the distance from part (c) to derive the radius of HD 37962. Give your answer in cm and solar units ($R_\odot$).

We estimated a scale factor of 9.3×10^{-19} sr, so we have

$$\pi \left(\frac{R}{d}\right)^2 = 9.3 \times 10^{-19} \text{ sr} \quad \Longrightarrow$$

$$R = d\sqrt{\frac{9.3 \times 10^{-19} \text{ sr}}{\pi \text{ sr}}} = (1.08 \times 10^{20} \text{ cm})(5.4 \times 10^{-10}) = 5.9 \times 10^{10} \text{ cm} = 0.84\ R_\odot$$

(e) *From the data, estimate the apparent V magnitude (m_V), the apparent B magnitude (m_B), and the $B - V$ color ($m_B - m_V$) for HD 37962. There are two ways to do this:*

i. *Simpler: directly use the measured flux density near the central wavelengths of the bands (B: 4380 Å; V: 5450 Å).*

ii. *Better: measure an* average *flux density through the passband, e.g.,*

$$\langle f_\lambda \rangle = \frac{\int f_\lambda S_\lambda \, d\lambda}{\int S_\lambda \, d\lambda}$$

where S_λ is the passband response function (posted on Canvas as `B.txt` *and* `V.txt`*, or CSV versions). You will get extra credit if you do it this way (numerical integration).*

With either method, you will need the flux zeropoint *of each band, the flux density that corresponds to $m = 0$. The flux zeropoints are $f_0(B) = 6.32 \times 10^{-9}$ erg/cm^2/sec/Å and $f_0(V) = 3.63 \times 10^{-9}$ erg/cm^2/sec/Å.*

The `Jupyter` notebook linked above has the details of this calculation. The simpler method gives $B = 8.47$, $V = 7.80$, and so $B - V = 0.66$, while the more accurate method gives $B = 8.47$, $V = 7.83$, and $B - V = 0.64$.

(f) *Given your results in this problem, can you explain why HD 37962 is called a* solar analog*?*

We see that HD 37962 has a similar temperature to the Sun ($T_\odot \approx 5770$ K), a similar radius (about 15% smaller), and a similar $B - V$ color ($B_\odot - V_\odot \approx 0.65$).

4. *Let's get some practice working with solid angles.*

(a) *Integrate over the unit sphere to show that the total solid angle (of the whole sky, for example) is 4π sterradians.* Hint: *recall that in spherical coordinates $d\Omega = \sin\theta\, d\theta\, d\phi$ (see Lecture 1, slide 13).*

To cover the whole unit sphere or the sky (once) the ranges for the coordinates are $\theta : 0 \to \pi$ and $\phi : 0 \to 2\pi$. Thus the integral is

$$\int d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = 2\pi \left[-\cos\theta\right]_0^{\pi} = 2\pi\left[-(-1) - (-1)\right] = 4\pi$$

(b) *A* square degree *is a unit of solid angle one degree on each side. How many sterradians is this?* Hint: *just convert each side to radians and multiply.*

There are 180 degrees in π radians, so 1 degree is $\pi/180 \approx 0.017$ radians. Then 1 square degree is

$$1\,\mathrm{deg}^2 = \left(\frac{\pi}{180}\right)^2 \text{ sr} \approx 3.0 \times 10^{-4} \text{ sr}$$

(c) *How many square degrees cover the entire sky? The field of view of the upcoming Vera C. Rubin Observatory camera is 9.6 deg^2. Approximately how many pointings of this camera would be required to tile the entire sky? How about with the Near Infrared Camera (NIRCam) on the just-launched JWST, with a field of view of 9.7 arcmin^2 (square arcminutes)?*

The whole sky was 4π sterradians, so that is

$$4\pi \text{ sr} = \frac{4\pi}{(\pi/180)^2}\,\mathrm{deg}^2 = \frac{4(180)^2}{\pi}\,\mathrm{deg}^2 \approx 41,253\,\mathrm{deg}^2$$

For the Rubin Observatory camera to tile the sky would take $41253/9.6 \approx 4300$ pointings. Note that technically, because of the location of the Rubin Observatory in Chile, it cannot point to the extreme northern part of the celestial sphere, and thus can't cover all 4π sterradians. Also, the shape of the Rubin camera field of view is close to a circular, so it requires overlapped pointings to completely cover a contiguous area. But we'll ignore those minor technicalities! Our calculation gives the right order of magnitude estimate: a few thousand pointings of Rubin cover the sky.

The field of view for NIRCam on JWST is much smaller. Recall that there are 60 arcminutes in one degree, So 1 arcmin^2 = 1/3600 deg^2. For NIRCam then, covering the sky would take $41253/(9.7/3600) \approx 1.5 \times 10^7 = 15$ million pointings!

5. *(Based on Phillips, problem 1.1) Consider a sphere of mass M and radius R in hydrostatic equilibrium. Derive the following quantities:*

   - *the enclosed mass $m(r)$,*
   - *the gravitational potential energy (U or E_{pot}) in terms of M and R,*
   - *the average internal pressure $\langle P \rangle$ in terms of M and R, and*
   - *the pressure profile $P(r)$*

   for a constant density profile, $\rho(r) = \rho$ for $0 \leq r \leq R$ (and zero density for $r > R$).

   The enclosed mass for a spherically symmetric system is given by

$$m(r) = \int_{u=0}^{u=r} 4\pi u^2 \rho(u)\, du$$

   The gravitational potential energy is given by (Phillips eqn. 1.6):

$$E_{\mathrm{pot}} = U = \Omega = -\int_{m=0}^{m=M} \frac{G\, m(r)}{r}\, dm = -4\pi G \int_{r=0}^{r=R} m(r)\, \rho(r)\, r\, dr$$

   where $m(r)$ is the enclosed mass, and in the second step I used the fact that

$$\frac{dm}{dr} = 4\pi r^2 \rho \quad \Longrightarrow \quad dm = 4\pi r^2 \rho\, dr$$

   Then from the virial theorem, the average internal pressure needed for hydrostatic equilibrium is (Phillips eqn. 1.7)

$$\langle P \rangle = -\frac{1}{3}\frac{E_{\mathrm{pot}}}{V} = -\frac{E_{\mathrm{pot}}}{4\pi R^3}$$

   where I substituted in $V = 4/3\,\pi R^3$ for a sphere. Finally to calculate the pressure profile, we can use the equation of hydrostatic equilibrium (Phillips eqn. 1.5)

$$\frac{dP}{dr} = -\rho(r) g(r) = -\frac{G m(r) \rho(r)}{r^2}$$

   When we write the integral version of this, we need to be a little careful about the boundary conditions. For the enclosed mass, $m(r) = 0$ at $r = 0$, so it makes sense to start integrating from the center. But the pressure is maximum at the center and goes to zero at the surface of the star. So for the pressure profile, it makes more sense to integrate starting at the surface down to a given radius r. This yields

$$P(r) = -G \int_{u=R}^{u=r} \frac{m(u)\rho(u)}{u^2}\, du = G \int_{r}^{R} \frac{m(u)\rho(u)}{u^2}\, du$$

   where in the second step I flipped the sign and direction of integration. You don't have to do it this way; if you integrate from the center you just have to include a constant of integration, $P(0)$, that you can solve for given the boundary condition $P(R) = 0$.

Applying this to a constant density profile $\rho(r) = \rho$, we get for the enclosed mass

$$m(r) = \int_{u=0}^{u=r} 4\pi u^2 \rho(u)\, du = 4\pi\rho \int_0^r u^2\, du = 4\pi\rho \left[\frac{u^3}{3}\right]_0^r = \frac{4}{3}\pi r^3 \rho$$

which makes sense as just the constant density times the enclosed volume. Note that the total mass $M = m(r = R) = 4/3\, \pi R^3 \rho$ as expected and we can also write $m(r) = M\, (r/R)^3$.

The potential energy then is

$$\begin{aligned} E_{\rm pot} &= \Omega = -4\pi G \int_{r=0}^{r=R} m(r)\, \rho(r)\, r\, dr = -4\pi G \int_0^R \left(\frac{4\pi r^3 \rho}{3}\right) \rho\, r\, dr \\ &= -\frac{16\pi^2 G \rho^2}{3} \int_0^R r^4\, dr = -\frac{16\pi^2 G \rho^2}{3} \frac{R^5}{5} = -\frac{3}{5} \frac{GM^2}{R} \end{aligned}$$

where in the last step I substituted in $\rho = 3M/4\pi R^3$ so $\rho^2 = 9M^2/16\pi^2 R^6$.

The average internal pressure for hydrostatic equilibrium is

$$\langle P \rangle = -\frac{E_{\rm pot}}{4\pi R^3} = \frac{3}{20\pi} \frac{GM^2}{R^4}$$

The pressure profile is

$$P(r) = G \int_r^R \frac{m(u)\rho(u)}{u^2}\, du = G \int_r^R \left(\frac{4\pi u^3 \rho}{3}\right) \frac{\rho}{u^2}\, du = \frac{4\pi G \rho^2}{3} \int_r^R u\, du = \frac{2\pi G \rho^2}{3} \left(R^2 - r^2\right)$$

Substituting in for ρ^2 as above we can write

$$P(r) = \frac{3}{8\pi} \frac{GM^2}{R^6} \left[R^2 - r^2\right] = \frac{3}{8\pi} \frac{GM^2}{R^4} \left[1 - \left(\frac{r}{R}\right)^2\right]$$

6. (Required for 541; extra credit for 441) *Repeat the problem above, for a somewhat more realistic density profile that increases linearly towards the center, with*

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right).$$

Applying our results above to a *slightly* more realistic linear density profile $\rho(r) = \rho_0(1 - r/R)$ we get

$$m(r) = \int_0^r 4\pi u^2 \rho(u)\, du = 4\pi\rho_0 \int_0^r u^2 \left(1 - \frac{u}{R}\right) du = 4\pi\rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$

Note that $m(R) = M = 4\pi\rho_0(R^3/3 - R^4/4R) = 4\pi\rho_0 R^3/12 = \pi\rho_0 R^3/3$. So we can write ρ_0 in terms of M and R as $\rho_0 = 3M/(\pi R^3)$, and thus also

$$m(r) = 4\pi \frac{3M}{\pi R^3} \left(\frac{r^3}{3} - \frac{r^4}{4R}\right) = \frac{12M}{R^3} \left(\frac{r^3}{3} - \frac{r^4}{4R}\right) = 12M \left[\frac{1}{3}\left(\frac{r}{R}\right)^3 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right]$$

The potential energy is

$$\begin{aligned}
E_{\rm pot} &= \Omega = -4\pi G \int_0^R m(r)\,\rho(r)\,r\,dr \\
&= -4\pi G \int_0^R \left[4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^4}{4R}\right)\right]\left[\rho_0\left(1-\frac{r}{R}\right)\right] r\,dr \\
&= -16\pi^2 G\rho_0^2 \int_0^R \left[\frac{r^4}{3} - \frac{r^5}{4R} - \frac{r^5}{3R} + \frac{r^6}{4R^2}\right] dr \\
&= -16\pi^2 G\rho_0^2 \int_0^R \left[\frac{r^4}{3} - \frac{7r^5}{12R} + \frac{r^6}{4R^2}\right] dr = -16\pi^2 G\rho_0^2\left[\frac{R^5}{15} - \frac{7R^6}{72R} + \frac{R^7}{28R^2}\right] \\
&= -\frac{26\pi^2 G\rho_0^2}{315}R^5 = -\frac{26\pi^2 G R^5}{315}\left(\frac{9M^2}{\pi^2 R^6}\right) = -\frac{26}{35}\frac{GM^2}{R}
\end{aligned}$$

The average internal pressure for hydrostatic equilibrium is

$$\langle P\rangle = -\frac{E_{\rm pot}}{4\pi R^3} = \frac{13}{70\pi}\frac{GM^2}{R^4}$$

The pressure profile is

$$P(r) = G\int_r^R \frac{m(u)\rho(u)}{u^2}\,du = G\int_r^R \left[4\pi\rho_0\left(\frac{u^3}{3} - \frac{u^4}{4R}\right)\right]\left[\rho_0\left(1-\frac{u}{R}\right)\right]\frac{du}{u^2}$$

We can continue as before, but to show you another, often useful, way to do this, let's make our integral dimensionless, by setting $x = u/R$, so $u = Rx$ and $du = R\,dx$. Remembering to update our limits of integration, we now have

$$\begin{aligned}
P(r) &= 4\pi\rho_0^2 G \int_{x=r/R}^{x=1} \left(\frac{R^3x^3}{3} - \frac{R^4x^4}{4R}\right)\left(\frac{1-x}{R^2x^2}\right) R\,dx \\
&= 4\pi\rho_0^2 G R^2 \int_{r/R}^1 \left(\frac{x}{3} - \frac{x^2}{4}\right)(1-x)\,dx = 4\pi\rho_0^2 G R^2 \int_{r/R}^1 \left(\frac{x}{3} - \frac{7x^2}{12} + \frac{x^3}{4}\right) dx \\
&= 4\pi\rho_0^2 G R^2 \left[\frac{5}{144} - \frac{1}{6}\left(\frac{r}{R}\right)^2 + \frac{7}{36}\left(\frac{r}{R}\right)^3 - \frac{1}{16}\left(\frac{r}{R}\right)^4\right]
\end{aligned}$$

where I fed the dimensionless integral into Wolfram alpha. If we like, we can replace ρ_0 in favor of M

$$\begin{aligned}
P(r) &= 4\pi G R^2\left(\frac{9M^2}{\pi^2R^6}\right)\left[\frac{5}{144} - \frac{1}{6}\left(\frac{r}{R}\right)^2 + \frac{7}{36}\left(\frac{r}{R}\right)^3 - \frac{1}{16}\left(\frac{r}{R}\right)^4\right] \\
&= \frac{5}{4\pi}\frac{GM^2}{R}\left[1 - \frac{24}{5}\left(\frac{r}{R}\right)^2 + \frac{28}{5}\left(\frac{r}{R}\right)^3 - \frac{9}{5}\left(\frac{r}{R}\right)^4\right]
\end{aligned}$$

Note how tedious this became going just from a constant density profile to a linear one! You can see why for any realistic density profile the solution must be done numerically rather than analytically. In fact, to double check my work here, I made a quick numerical model of a linear density star (i.e., arrays of radius and density) and directly integrated it to determine the enclosed mass and pressure profiles; fortunately they matched the exact analytical solution here!