Physics 441/541: Stars and Star Formation
Spring 2022 Midterm Solutions

some possibly useful formulas

(for which I am on purpose not providing more details)

$$\frac{dP}{dr} = -\rho(r)\,g(r) = -\frac{Gm(r)\rho(r)}{r^2} \qquad \langle P\rangle = -\frac{1}{3}\frac{E_{\rm pot}}{V} \qquad T_c \sim \frac{GM\mu m_p}{kR} \qquad P_c \sim \frac{GM^2}{R^4}$$

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT)-1} \qquad F = \sigma T^4 \qquad L = 4\pi R^2\sigma T_{\rm eff}^4 \qquad f = L/4\pi d^2 \qquad d = 1/p$$

$$m = -2.5\log(f/f_0) \qquad \mu = m - M = 5\log(d/10\ {\rm pc}) \qquad M_{\rm bol} = -2.5\log(L/L_\odot) + 4.74$$

$$P = K\rho^\gamma \qquad \gamma = 1 + 1/n \qquad R \propto M^{(1-n)/(3-n)} \qquad E_{\rm pot} = -\frac{3}{5-n}\frac{GM^2}{R}$$

$$P = \frac{1}{3}aT^4 \qquad P = nkT = \frac{\rho kT}{\mu m_p} \qquad P = \frac{h^2}{5m_e}\left[\frac{3}{8\pi}\right]^{2/3} n_e^{5/3} \qquad P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n_e^{4/3}$$

$$E_n = -13.6\ {\rm eV}\left(\frac{Z^2}{n^2}\right) \qquad g_n = 2n^2 \qquad \frac{n_m}{n_n} = \frac{g_m}{g_n}\exp\left(-\frac{E_m - E_n}{kT}\right)$$

$$\frac{n_{II}}{n_I} = \frac{2Z_{II}}{n_e Z_I}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_I}{kT}\right) \qquad \left|\frac{dT}{dr}\right| > \frac{\gamma-1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right| = \frac{\gamma-1}{\gamma}\frac{\mu m_p}{k}g$$

$$\ell = \frac{1}{n\sigma} = \frac{1}{\rho\kappa} \qquad \frac{dT}{dr} = -\frac{3}{4ac}\frac{\rho\kappa}{T^3}\frac{L}{4\pi r^2} \qquad E = \Delta mc^2 = \epsilon Mc^2 \qquad E_G = (\pi\alpha Z_A Z_B)^2\, 2m_r c^2$$

$$\text{probability} \approx \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right] \qquad E_0 = \left[\frac{E_G(kT)^2}{4}\right]^{1/3} \qquad \alpha = \left(\frac{E_G}{4kT}\right)^{1/3} - \frac{2}{3}$$

$$R_{AB} = n_A n_B\left[\frac{8}{\pi m_r}\right]^{1/2}\left[\frac{1}{kT}\right]^{3/2}\int_0^\infty S(E)\exp\left[-\frac{E}{kT} - \left(\frac{E_G}{E}\right)^{1/2}\right]dE \qquad R_{AB} \propto T^\alpha$$

$$\frac{dm}{dr} = 4\pi r^2\rho \qquad \frac{dL}{dr} = 4\pi r^2\rho\epsilon \qquad \kappa_{\rm es} = \frac{n_e\sigma_T}{\rho} \approx 0.2(1+X)\ {\rm cm^2\ g^{-1}} \qquad L_{\rm Edd} = \frac{4\pi GMm_p c}{\sigma_T}$$

$$t_{\rm dyn} = \frac{1}{\sqrt{G\rho}} \qquad t_{\rm KH} = \frac{GM^2}{RL} \qquad t_{\rm nuc} = \frac{E_{\rm nuc}}{L} \approx \frac{f_M\epsilon Mc^2}{L} \qquad t_{\rm dyn} \ll t_{\rm conv} \ll t_{\rm KH} \ll t_{\rm nuc}$$

Brief calculation/derivation [25 pts total]

1. *[2 pts] The star Altair has an apparent visual magnitude $m_V = +0.8$ and is at a distance of 5 pc from Earth. What would its apparent visual magnitude be if viewed from 50 pc away?*

   There are a few ways to approach this. For instance you can use the relationship $m - M = 5\log(d/10\ \text{pc})$ to determine the absolute magnitude for the current distance, and then figure out the apparent magnitude at the new distance. Another approach is to see that if the star is $10\times$ farther away, the apparent brightness (flux) will be $1/10^2 = 1/100$ as much. The change in magnitude would then be $\Delta m = -2.5\log(1/100) = -2.5(-2) = +5$. Thus the new apparent magnitude would be $m_V = +0.8 + 5 = +5.8$.

It's a useful rule of thumb to remember that a factor of 100 in flux corresponds to 5 magnitudes.

2. *[4 pts] Write an expression for the temperature at which radiation pressure equals gas pressure for an ideal gas with density ρ, mean molecular weight μ, and constants.*

$$P_{\rm rad} = \frac{1}{3}aT^4 \quad P_{\rm gas} = nkT = \frac{\rho kT}{\mu m_p} \qquad P_{\rm rad} = P_{\rm gas} \Rightarrow T = \left(\frac{3\rho k}{a\mu m_p}\right)^{1/3}$$

3. *[4 pts] Here are the radii and effective temperatures for three fictitious stars:*

star	R	$T_{\rm eff}$
Mirabel	1 $R_\odot$	6000 K
Luisa	160 $R_\odot$	3000 K
Isabela	5 $R_\odot$	12000 K

(a) *Arrange these stars in order of their color (peak wavelength of emitted light) from blue to red.*

The Wien displacement law says higher temperatures peak at shorter (bluer) wavelengths, so from blue to red we have Isabela, Mirabel, Luisa.

(b) *Arrange these stars in order of their luminosity from faintest to brightest.*

The Stefan-Boltzmann law says $L = 4\pi R^2\sigma T_{\rm eff}^4$, so we just need to compare the product of $R^2 T_{\rm eff}^4$ for the three stars. Isabela has both a larger radius and higher effective temperature than Mirabel, so Isabela must be more luminous than Mirabel. Luisa has $160/5 = 32$ the radius of Isabela, but Isabela has $12000/3000 = 4$ times higher temperature. Note that $32^2 > 4^4 = 16^2$ so Luisa must have a higher luminosity than Isabela. Thus the ordering from faintest to brightest is Mirabel, Isabela, Luisa.

4. *[6 pts] Show that the criterion for convection can be written as*

$$\frac{d\ln T}{d\ln P} > s$$

where s is a number that depends on the adiabatic index. Assuming a typical value of the adiabatic index in normal stars (a monoatomic, nonrelativistic ideal gas), what is the numerical value of s? Hint: *recall that $d\ln x = dx/x$.*

$$\left|\frac{dT}{dr}\right| > \frac{\gamma-1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right| \quad\Rightarrow\quad \frac{1}{T}\left|\frac{dT}{dr}\right| > \frac{\gamma-1}{\gamma}\frac{1}{P}\left|\frac{dP}{dr}\right|$$

$$\Rightarrow\quad \left|\frac{d\ln T}{dr}\right| > \frac{\gamma-1}{\gamma}\left|\frac{d\ln P}{dr}\right| \quad\Rightarrow\quad \frac{d\ln T}{d\ln P} > \frac{\gamma-1}{\gamma} \quad\Rightarrow\quad s = \frac{\gamma-1}{\gamma}$$

Both T and P decrease with r so no need for the absolute value when looking at $(d\ln T/d\ln P)$. For a monoatomic, nonrelativistic ideal gas, $\gamma = 5/3$, so $s = (5/3 - 1)/(5/3) = (2/3)/(5/3) = 2/5$.

5. *[4 pts] What particle or nucleus fills in the blank in these stellar nuclear reactions? For reference, here are some elements from the periodic table: 1. hydrogen (H), 2. helium (He), 3. lithium (Li), 4. beryllium (Be), 5. boron (B), 6. carbon (C), 7. nitrogen (N), 8. oxygen (O), 9. fluorine (F), 10. neon (Ne), 11. sodium (Na), 12. magnesium (Mg).*

   (a) $p \;+\; p \;\rightarrow\; {}^2_1\mathrm{H} \;+\; \underline{\hspace{3em}} + \;\nu_e$

   We need another positive charge on the right hand side, and it needs to have a lepton number of -1 to cancel out the lepton number from the neutrino, so the missing particle is a **positron**, e^+. This reaction is the first step of the p-p chain.

   (b) $e^- \;+\; {}^7_4\mathrm{Be} \;\rightarrow\; \underline{\hspace{3em}} + \;\nu_e$

   The total electric charge on the left hand side is $-1+4 = 3$, so we need a nucleus with atomic number 3 on the right hand side. There are 7 nucleons on the left, so the missing nucleus is ${}^7_3\mathrm{Li}$. This reaction is part of branch II of the p-p chain.

   (c) $p \;+\; {}^{15}_7\mathrm{N} \;\rightarrow\; {}^{12}_6\mathrm{C} \;+\; \underline{\hspace{3em}}$

   We have 16 nucleons ad 8 positive charges on the left hand side, so with the carbon the missing particle must be an α particle, ${}^4_2\mathrm{He}$. This reaction is the "last step" of the CNO cycle.

   (d) ${}^{12}_6\mathrm{C} \;+\; {}^{12}_6\mathrm{C} \;\rightarrow\; {}^{23}_{12}\mathrm{Mg} \;+\; \underline{\hspace{3em}}$

   We have 24 nucleons and 12 positive charges on the left hand side. All the charges are accounted for on the right hand side, but one nucleon is missing. The only neutral possibility is the **neutron**, n. This reaction is one part of carbon fusion.

6. *[5 pts] Write down the net nuclear reaction for hydrogen fusion into helium (for example, from the PP I branch). You don't need to write down all the steps, just the net reaction, so there should be only protons on the left hand side and a helium nucleus (same as an alpha particle) plus other particles on the right hand side.*

$$4\,p \;\rightarrow\; {}^4_2\mathrm{He} + 2\,e^+ + 2\,\nu_e + \text{ photons}$$

Short answer (one or two sentences are enough for each) [13 pts total]

7. *[2 pts] The typical kinetic energy of protons in the Sun's core is much lower than the* Coulomb barrier *for fusion, so how do protons manage to fuse?*

   Protons fuse because of **quantum tunneling**.

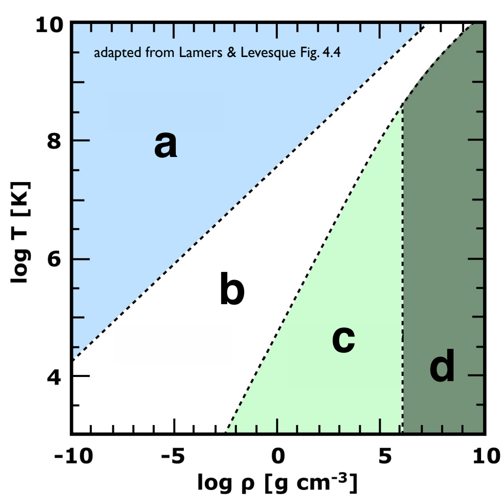
8. *[3 pts] A few percent of the energy released in hydrogen fusion goes into neutrinos. Why are the neutrinos different than the other products of the fusion that contribute to the hydrostatic equilibrium of the star?*

   The neutrinos have a very small interaction cross section compared to the other particles, so they largely escape from the star. The kinetic energy of the neutrinos is thus not used to heat up the gas and provide thermal pressure support against gravity (hydrostatic equilibrium).

9. *[4 pts] In what way are the p-p chain and the CNO cycle the same thing? Which one requires a higher temperature? Why?*

   Both the *p*-*p* chain and the CNO cycle are ways to fuse **hydrogen** into **helium**. The CNO cycle just uses other nuclei as a "catalyst" for this process. The CNO cycle requires a higher temperature because the fusion of a proton with one of those nuclei has to overcome a higher Coloumb barrier (higher positive charge for those nuclei compared to another proton).

10. *[4 pts] Describe in a few words (or with an equation) the dominant source of the pressure for each of the regions a, b, c, and d in the figure below.*



   (a) radiation pressure ($P = aT^4$).

   (b) ideal gas (thermal) pressure $P = nkT = \rho kT/\mu m_p$.

   (c) non-relativistic degeneracy pressure $P \propto n_e^{5/3}$.

   (d) relativistic degeneracy pressure $P \propto n_e^{4/3}$.

Slightly longer answer (a few sentences are enough for each) [12 pts total]

11. *[6 pts] Hydrostatic equilibrium implies that higher mass stars must have a higher central pressure than lower mass stars. Explain how this ultimately means that higher mass stars have a shorter main-sequence lifetime than lower mass stars.*

    Main-sequence stars are fusing hydrogen to helium in their cores, and the pressure support is provided by either ideal gas pressure or for massive stars, radiation pressure. For either of these, producing a higher pressure requires a higher temperature: thus, higher-mass main-sequence stars have higher central temperatures. The fusion rate of hydrogen to helium (either by the p-p chain or the CNO cycle) is a strongly increasing function of temperature, so higher temperature means a much higher fusion rate. This is more than enough to counteract the fact that higher-mass stars have more fuel to fuse. The high fusion rate (evidenced by a high luminosity) means that high-mass main-sequence stars exhaust their supply of core hydrogen more quickly than low-mass main-sequence stars, and thus have shorter main-sequence lifetimes.

12. *[6 pts] Which one of the three types of energy transport we discussed in class is common in stars and depends on the* opacity, *κ, of the material? What elementary particle is largely responsible for the opacity at high temperatures? What one equation would you use to help determine how many of those particles are free, and thus, for example, the relative importance of free-free opacity versus bound-bound opacity?*

    Of the three types of energy transport (convection, conduction, and radiation), **radiation** is the one that depends on the opacity (i.e. how opaque or transparent the material is to light). At high temperatures, the opacity is dominated by electron scattering, meaning that (free) **electrons** are largely responsible for the opacity. The **Saha equation** tells us about the ionization state of the matter, and thus how many free electrons we would expect.