Physics 441/541 Spring 2022: Problem Set #2
due February 11 at 11:00 am in PDF format on Canvas

You are encouraged to work in groups on these problems, but you must write up the solutions individually. In your writeup, list your collaborators and cite any external sources you used. You **may not** consult previous solution sets for this class or other similar classes.

1. (a) Show that the mass-radius relationship for a polytrope with index n is given by

$$R \propto M^{(1-n)/(3-n)}$$

   (b) Show that the ratio of the central density to the mean density of a polytrope of index n is given by

$$\frac{\rho_c}{\bar{\rho}} = \rho_c \left(\frac{4\pi R^3}{3M}\right) = \frac{-\xi_1}{3\left(\frac{d\theta}{d\xi}\right)_{\xi_1}}$$

   (c) Solutions to the Lane-Emden equation are given in the table below:

n	ξ_1	$-\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}$
1.5	3.65	2.71
3	6.90	2.02

   Using your result from part (b) and the constants above, calculate the ratio of the central density to the mean density, $\rho_c/\bar{\rho}$, for the $n = 1.5$ and $n = 3$ polytropes.

   (d) The central pressure for a star can be written as

$$P_c \quad = \quad \underline{\hspace{3cm}} \left(\frac{GM^2}{R^4}\right) \quad = \quad \underline{\hspace{4cm}} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R_\odot}{R}\right)^4$$

   where I've left out the numerical factors that go in front. Calculate these numerical values (i.e., fill in the blanks) for polytropes with $n = 1.5$ and $n = 3$. Don't forget to include units if necessary! *Hint:* recall that $P = K\rho^\gamma$, so $P_c = K\rho_c^\gamma$, where $\gamma = (n+1)/n$; you can derive K in terms of the mass M and radius R.

2. (Adapted from Lamers & Levesque problems 4.2 and 4.4) Ekström et al. (2012, A&A, 537, 146) have published a set of stellar models over a range from 0.8 to 120 $M_\odot$. You can access the paper at https://ui.adsabs.harvard.edu/abs/2012A%26A...537A.146E/abstract. You can get the stellar model data associated with this paper at https://vizier.cds.unistra.fr/viz-bin/VizieR-3?-source=J/A%2bA/537/A146/tables.

   We are interested in the stars at the beginning of their lives, the *zero-age main sequence* or ZAMS, and we will look at non-rotating stars for this problem. To get just this data, put in a constraint of "n" for the "Rot" parameter, and set "Line" to be 1 for ZAMS data. Check the "Show" boxes for the initial mass (Mini) and further down, the $\log_{10}$ of the central density (log(rhoc)), the $\log_{10}$ of the central temperature (logTc), and the central mass fractions of hydrogen (Xc) and helium (Yc). Click Submit and if all goes well you should see a table with 24 rows each corresponding to a different mass star.

(a) On a copy of Lamers & Levesque Figure 4.4 (there's one on Canvas under Problem Set Resources called `LLFig4-4.pdf`), mark points for the central density and temperature of $M = 1, 2, 4, 7, 12, 20, 40, 60$, and $120\ M_\odot$ stars. In what regions of the diagram do the centers of these stars reside?

(b) Using ρ_c, T_c, X_c, and Y_c, write a formula for the ratio of radiation pressure to gas pressure, $P_{\rm rad}/P_{\rm gas}$, in the center of a star. Taking all 24 rows of the online data, make a plot of $P_{\rm rad}/P_{\rm gas}$ versus M using any numerical/plotting package you like. For easier copying/pasting, you may want to change the table format from HTML to tab-separated values or ascii using the dropdown menu on the left.

(c) Do your results in part (b) correspond to the diagram in part (a)?

(d) In class we derived $P_{\rm rad}/P_{\rm gas} \propto M^2$. Describe how to obtain this scaling relation. Is this prediction validated by your plot? Explain.

3. (a) Show that the pressure integral $P = 1/3 \int_0^\infty p\, v(p)\, n(p)\, dp$ for a completely degenerate electron gas with Fermi momentum $p_F = (3h^3 n_e/8\pi)^{1/3}$ can be written

$$P = \frac{8\pi}{3h^3} \int_0^{p_F} \frac{p^4 c^2}{(p^2c^2 + m_e^2 c^4)^{1/2}}\, dp$$

This is a relativistically correct formula, so use the fact that the relativistic energy and momentum for an electron with Lorentz factor γ are $E = \gamma m_e c^2 = \sqrt{p^2c^2 + m_e^2c^4}$ and $p = \gamma m_e v$. *Hint:* start by writing v in terms of p and E.

(b) In the non-relativistic limit, $p \approx mv \ll mc$, so $p^2c^2 \ll m_e^2c^4$ and you can simplify the denominator. Show this gives the non-relativistic degenerate equation of state

$$P = \frac{h^2}{5m_e} \left[\frac{3}{8\pi}\right]^{2/3} n^{5/3}$$

(c) Show that for the ultra-relativistic limit, where $p^2c^2 \gg m_e^2c^4$ we get the ultra-relativistic degenerate equation of state

$$P = \frac{hc}{4} \left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}$$

(d) What kind of degenerate equation of state (non-relativistic or ultra-relativistic) leads to a mass-radius relation of the form $R \propto M^{-1/3}$? Explain qualitatively why for those white dwarfs, a higher mass leads to a smaller radius. *Hint:* look back to polytropes and problem 1.

4. Let us explore the Saha equation (and one of its quirks) and apply it to the Sun. In this problem we will assume a pure hydrogen composition ($X = 1, Y = 0$); this is not a good idea (we ignore helium at our peril, especially in the core of the Sun), but it simplifies things in a useful way. In class we wrote the Saha equation for hydrogen ionization as

$$\frac{n_{II}}{n_I} = \frac{2Z_{II}}{n_e Z_I} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_I/kT}$$

   (a) There are three densities here: n_I, the number density of neutral hydrogen atoms; n_{II}, the number density of hydrogen ions (protons); and n_e, the number density of free electrons. These are all linked; rewrite each of these in terms of n_H, the total number density of hydrogen (neutral + ionized) and the ionization fraction $y = n_{II}/n_H$. Note that Phillips calls the ionization fraction $x(H)$ rather than y.

   (b) Show that

$$y = \frac{n_{II}/n_I}{1 + (n_{II}/n_I)}$$

   Use this and the Saha equation above to reproduce the top panel of Phillips Figure 2.4, for a free electron density of $n_e = 10^{19}$ m^{-3} = 10^{13} cm^{-3} (a typical value in the solar atmosphere). You may use any plotting package you like.

   (c) Use the Saha equation and your results from part (a) to derive the following expression for the ionization fraction

$$\frac{y^2}{1 - y} = \frac{2m_p Z_{II}}{\rho Z_I} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_I/kT}$$

   (d) Using this result, calculate the ionization fraction expected at the center of the Sun, where $T_c = 1.6 \times 10^7$ K and $\rho_c = 150$ g cm^{-3} = 1.5×10^5 kg m^{-3}. Take $Z_I \approx 2$ and $Z_{II} = 1$. Is this what you would have predicted given your plot in part (b)? Explain.

5. Atomic absorption in stellar atmospheres involves both discrete lines (from electronic transitions between bound states) and *continuum absorption*.

   (a) One form of continuum absorption is "bound-free" transitions, in which an electron moves from a bound state to an unbound (ionized) state. These transitions require only a minimum photon energy, because the free electron can have any (positive) kinetic energy. Even though most hydrogen in the Sun's atmosphere is neutral, with the electron in the ground state ($n = 1$), the dominant source of bound-free absorption of visible light in the Sun's atmosphere comes from excited hydrogen with the electron in the $n = 3$ state. Explain why (and be quantitative).

   (b) Another source of continuum absorption in the Sun's atmosphere is from the negative ion H$^-$, in which an extra electron is added to neutral hydrogen. This electron is only weakly bound, with a binding energy of 0.754 eV.

i. Show that all visible light photons have sufficient energy to liberate an electron from H^-.

ii. What is the expected abundance of H^- ions relative to neutral hydrogen in the Sun's atmosphere ($T = 5770$ K, free electron pressure $P_e = 15$ dyne cm^{-2})? *Hint:* Treat neutral hydrogen as the "ionized" version of H^-.

(c) What dominates the continuum absorption of visible light in the Sun's atmosphere, bound-free transitions of neutral hydrogen or "ionization" of H^-? Explain quantitatively.

6. (Required for 541; extra credit for 441)

(a) (Phillips 2.3) Using the pressure integral from problem 3a, show that the general expression for the pressure in a completely degenerate electron gas can be written

$$P = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3} n^{4/3}\, I(x)$$

where $x = p_F/(m_e c)$ and

$$I(x) = \frac{3}{2x^4}\left\{x\left(1+x^2\right)^{1/2}\left(\frac{2x^2}{3}-1\right) + \ln\left[x + \left(1+x^2\right)^{1/2}\right]\right\}.$$

(b) Confirm that this expression leads to the correct non-relativistic (problem 3b) and ultra-relativistic (problem 3c) equations of state.

(c) The ionization energies for helium are 24.6 eV (to get singly ionized He II), and an additional 54.4 eV (to get doubly ionized He III). Make a single plot showing the relative fractions of He I, He II, and He III as a function of temperature from 0 to 40000 K. Show results for three values of the free electron pressure: $P_e = 1$, 10, and 100 dyne cm^{-2}.