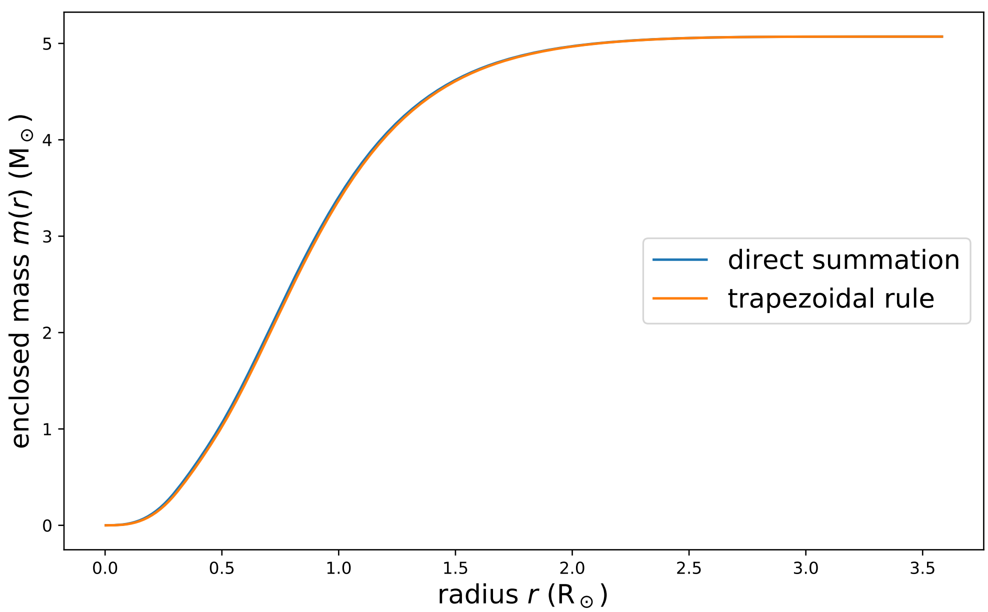
Physics 441/541 Spring 2022: Problem Set #3 Solutions

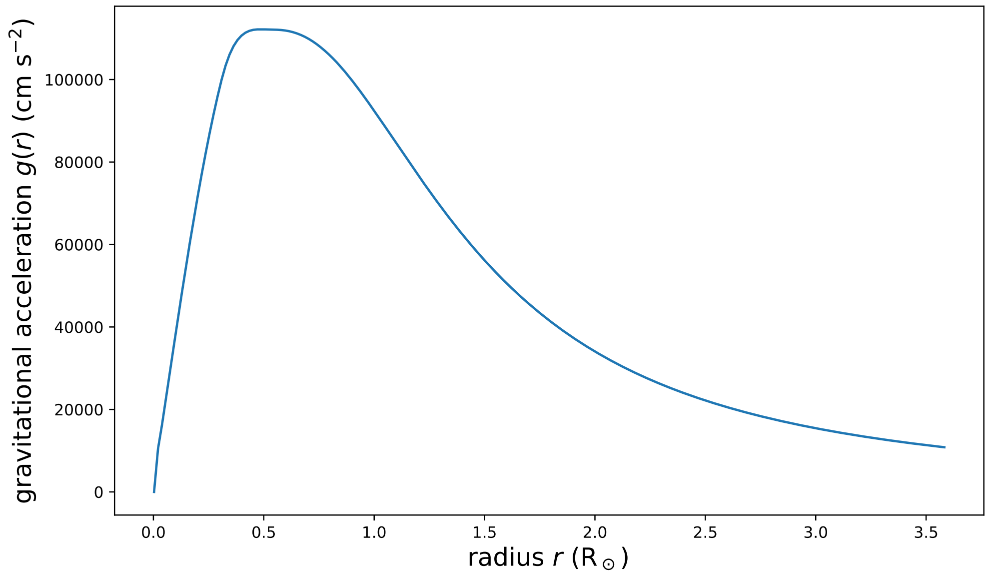
1. *On our Canvas site, under Files–Problem Set Resources, you will find a file called* `stellarmodel-ps03.txt` *that gives the density* ρ *(in* g cm^{-3}*) and temperature* T *(in K) of a star as function of radius* r *(in cm). There is also a comma-separated-value file* `stellarmodel-ps03.csv` *with the same data, if that is more convenient for you.*

   (a) *Using the density profile* $\rho(r)$ *calculate and plot the enclosed mass* $m(r)$ *for the star from the center to the surface. Use solar units* ($R_\odot$, $M_\odot$) *for the* x *and* y *axes, respectively.*

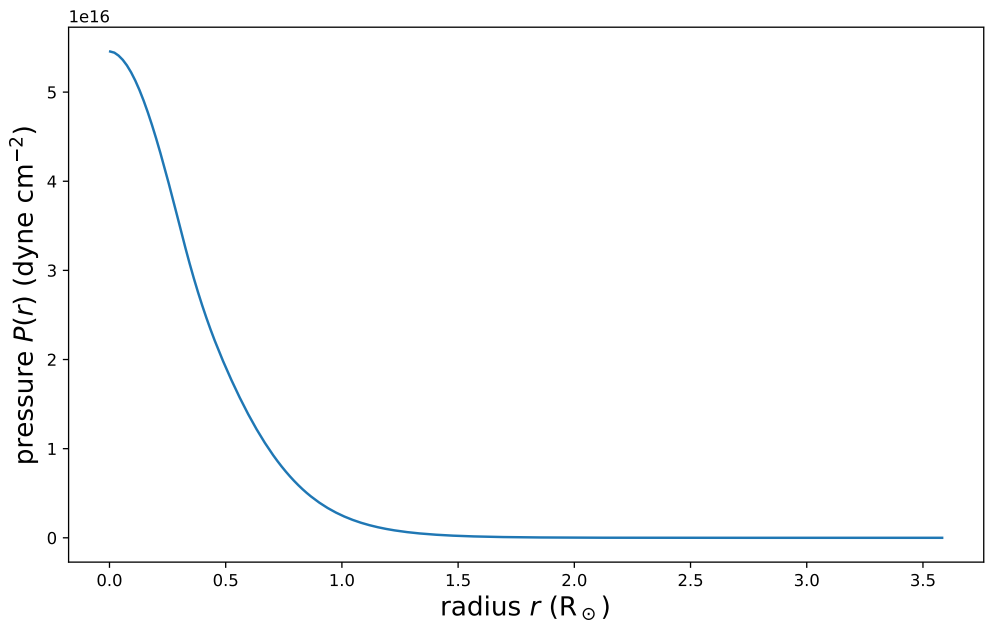
   I've made a python notebook at http://nbviewer.jupyter.org/url/www.physics.rutgers.edu/ugrad/441/notebooks/ps03q1.ipynb with the relevant code. The resulting plots are included here.



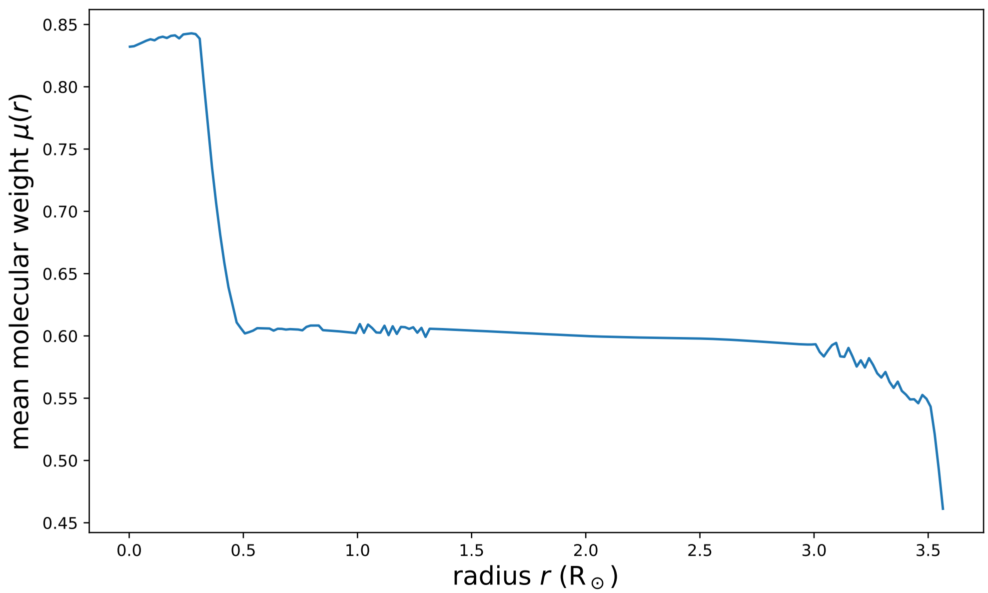
   (b) *Calculate and plot the local gravitational acceleration* $g(r)$*; use CGS units for this quantity (keep the* x*-axis in* $R_\odot$*).*



(c) *Assume hydrostatic equilibrium to derive and plot the pressure profile $P(r)$ in CGS units (keep the x-axis in $R_\odot$ again and remember to use the correct boundary condition; see Problem Set 1).*



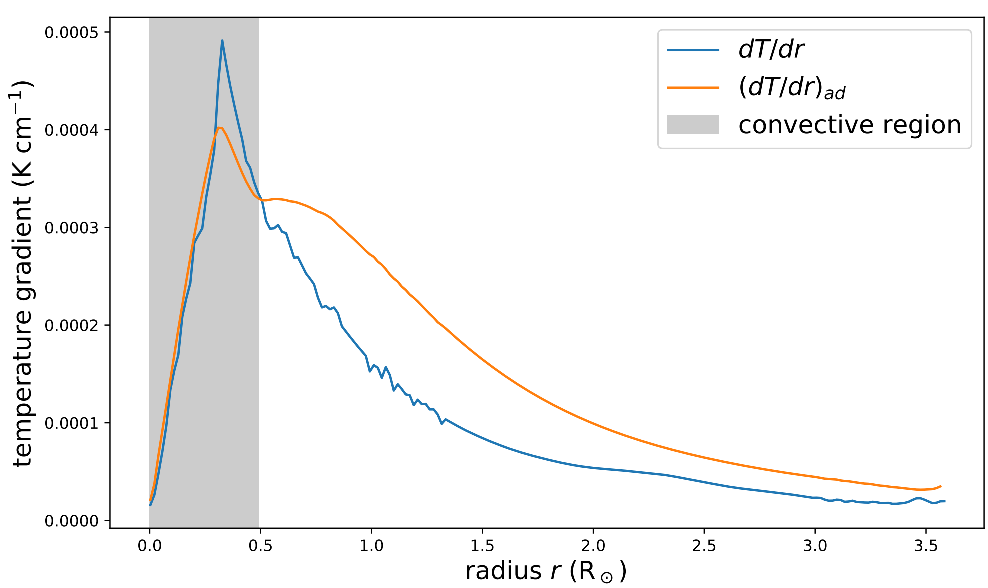
(d) *Assuming an ideal gas throughout the star, plot the mean molecular weight $\mu(r)$ versus radius (keep the x-axis in $R_\odot$).*



(e) *Show, on a single plot versus radius on the x-axis (again in units of $R_\odot$), the quantities:*

$$\left|\frac{dT}{dr}\right| \quad \text{and} \quad \left|\frac{dT}{dr}\right|_{\mathrm{ad}} \equiv \frac{\gamma-1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|$$

in CGS units. Use an appropriate value for γ. On your plot, mark the region of the star that is convective.



(f) *Based on all of your results, what kind of star is this? Be as specific as you can.*

We can see that the total mass of the star is approximately 5 $M_\odot$. The mean molecular weight is $\mu \approx 0.6$ in most of the interior; that corresponds to relatively typical stellar composition (e.g., 70% hydrogen by mass, helium for most of the rest, and some small amount of heavy elements, all ionized). The core has $\mu \approx 0.8$, so it must be enriched in something heavier than hydrogen, and this presumably is a sign that hydrogen fusion to helium is happening in the core. That's the definition of a star on the *main sequence*, like the Sun is today. We also see that the core is convective, which is expected on the main sequence for these masses (due to the high luminosity of the CNO cycle; see Lecture 6 slides 17 and 18).

Altogether, we can infer this is a $\sim 5\ M_\odot$ star on the main sequence. That's exactly what I requested from the `MESA` code at http://mesa-web.asu.edu.

2. (a) *(adapted from Phillips 3.3) Recall that the radiation pressure for a blackbody is given by $P_{\rm rad} = aT^4/3$. Write an expression for the radiation pressure gradient $dP_{\rm rad}/dr$ in terms of the temperature gradient dT/dr.*

   Given the expression for the radiation pressure, we have

$$\frac{dP_{\rm rad}}{dr} = \frac{d}{dr}\left(\frac{aT^4}{3}\right) = \frac{4aT^3}{3}\frac{dT}{dr}$$

   (b) *Near the surface of a star with luminosity L and radius R, the radiative temperature gradient is given by*

$$\left[\frac{dT}{dr}\right]_{\rm rad} = -\frac{3\rho\kappa}{4acT^3}\frac{L}{4\pi R^2}$$

   Show that this implies a radiation pressure gradient

$$\left[\frac{dP}{dr}\right]_{\rm rad} = -\frac{\rho\kappa}{c}\frac{L}{4\pi R^2}$$

   Plugging $[dT/dr]_{\rm rad}$ into our answer from part (a) we get the desired result:

$$\left[\frac{dP}{dr}\right]_{\rm rad} = -\frac{4aT^3}{3}\frac{3\rho\kappa}{4acT^3}\frac{L}{4\pi R^2} = -\frac{\rho\kappa}{c}\frac{L}{4\pi R^2}$$

   (c) *Show that the maximum luminosity that the star can have and still be in hydrostatic equilibrium for a surface gravity $g = GM/R^2$ is $L_{\rm max} = 4\pi GMc/\kappa$.*

   For the star to be in hydrostatic equilibrium, we must have $dP/dr = -\rho g$. From our result in part (b), we see that a higher luminosity results in a higher pressure gradient. So we see there is a maximum luminosity for hydrostatic equilbrium, in which radiation would provide all of the pressure support (leaving nothing from other sources of pressure, e.g., gas pressure). This would happen when

$$\left[\frac{dP}{dr}\right]_{\rm rad} = -\rho g = -\frac{\rho\kappa}{c}\frac{L_{\rm max}}{4\pi R^2} \quad \Longrightarrow \quad L_{\rm max} = \frac{4\pi g R^2 c}{\kappa} = \frac{4\pi GMc}{\kappa}$$

   where in the last step I substituted in $g = GM/R^2$, so $gR^2 = GM$.

(d) *Assuming an ionized hydrogen atmosphere with the opacity dominated by electron scattering, show that this maximum luminosity corresponds to the* Eddington luminosity *(or the* Eddington limit*)*

$$L_{\rm Edd} = \frac{4\pi G M m_p c}{\sigma_T} = 1.26 \times 10^{38}\ {\rm erg\ s^{-1}} \times \left(\frac{M}{M_\odot}\right) = 3.27 \times 10^4\ L_\odot \times \left(\frac{M}{M_\odot}\right)$$

Above this limit the radiation pressure is strong enough to push away any ionized hydrogen; this is very important in regulating accretion *onto astrophysical objects.*

For opacity dominated by electron scattering we can write the mean free path as $l = (\rho\kappa_{\rm es})^{-1} = (n_e\sigma_T)^{-1}$ where σ_T is the Thomson cross section. That means $\kappa_{\rm es} = n_e\sigma_T/\rho = \sigma_T/(\mu_e m_p)$ where μ_e is the mean molecular weight per free electron (see Lecture 7 slide 22). For ionized hydrogen, there is one proton for every electron so, $\mu_e \approx 1$.

Substituting $\kappa_{\rm es} = \sigma_T/(\mu_e m_p)$ into our result from part (c), and taking $\mu_e = 1$, we get the desired result

$$\begin{aligned}
L_{\rm Edd} &= \frac{4\pi G M m_p c}{\sigma_T} = \left(\frac{4\pi G M_\odot m_p c}{\sigma_T}\right)\frac{M}{M_\odot} \\
&= \frac{4\pi(6.67\times10^{-8}\ {\rm cm^3\ g^{-1}\ s^{-2}})(1.99\times10^{33}\ {\rm g})(1.67\times10^{-24}\ {\rm g})(3.0\times10^{10}\ {\rm cm\ s^{-1}})}{6.65\times10^{-25}\ {\rm cm^2}}\frac{M}{M_\odot} \\
&= 1.26\times10^{38}\ {\rm g\ cm^2\ s^{-3}} \times \left(\frac{M}{M_\odot}\right) = 1.26\times10^{38}\ {\rm erg\ s^{-1}} \times \left(\frac{M}{M_\odot}\right) \\
&= 3.27\times10^4\ L_\odot \times \left(\frac{M}{M_\odot}\right)
\end{aligned}$$

What this is saying is for a star with mass 1 $M_\odot$, its maximum stable luminosity is $3.27\times10^4\ L_\odot$; above that the radiation pressure would start to blow away ionized hydrogen off the surface.

3. *The central temperature of the Sun is* $T_c \approx 1.5\times10^7$ K.

(a) *Use the radius of the Sun to (very roughly) approximate the average temperature gradient dT/dr in the Sun. Then, using the Sun's luminosity, estimate the average opacity in the Sun's interior. How does this average opacity compare to the electron scattering opacity $\kappa_{\rm es}$?*

A rough approximation to the temperature gradient over the whole Sun would be

$$\frac{dT}{dr} \sim \frac{\Delta T}{\Delta r} = \frac{T_c - T_{\rm surface}}{R_\odot} \approx \frac{T_c}{R_\odot} = \frac{1.5\times10^7\ {\rm K}}{6.96\times10^{10}\ {\rm cm}} = 2\times10^{-4}\ {\rm K\ cm^{-1}}$$

. The radiative temperature gradient is given by

$$\left[\frac{dT}{dr}\right]_{\rm rad} = -\frac{3\rho\kappa}{4acT^3}\frac{L}{4\pi R^2}$$

Solving for the average opacity we have

$$
\begin{aligned}
\kappa &= \frac{4acT^3}{3\rho}\frac{4\pi R^2}{L}\left|\frac{dT}{dr}\right|_{\rm rad} \\
&\approx \frac{4(7.566\times10^{-15}\ {\rm erg\ cm^{-3}\ K^{-4}})(3.0\times10^{10}\ {\rm cm\ s^{-1}})(7.5\times10^6\ {\rm K})^3}{3(1.4\ {\rm g\ cm^{-3}})} \\
&\quad\times \frac{4\pi(6.96\times10^{10}\ {\rm cm})^2}{3.83\times10^{33}\ {\rm erg\ s^{-1}}}\times 2\times10^{-4}\ {\rm K\ cm^{-1}} = 290\ {\rm cm^2\ g^{-1}}
\end{aligned}
$$

where I used a rough average value of the temperature $T = T_c/2 = 7.5\times10^6$ K and an average density for the Sun $\rho = 3M_\odot/4\pi R_\odot^3 = 1.4$ g cm^{-3}. You may have made different choices here, but should still estimate an average opacity at this order of magnitude.

The electron scattering opacity is $\kappa_{es} \approx 0.2(1+X)$ cm^2 g^{-1} $= 0.34$ cm^2 g^{-1}, where we take the hydrogen fraction $X \approx 0.7$ for the Sun. We see that the true average opacity in the Sun is three orders of magnitude larger than the electron scattering opacity. This is consistent with the opacity plots shown in class (e.g., Lecture 7, slide 28).

(b) *Based on the average opacity, what is the mean free path ℓ for a photon in the Sun's interior? In radiative diffusion, photons undergo a* random walk *where after N interactions they travel an average distance $d = \ell\sqrt{N}$. How many interactions does a typical photon have getting from the center of the Sun to the surface? How long does this take? How does the time compare to the light travel time in the absence of interactions?*

The mean free path is

$$\ell = \frac{1}{\rho\kappa} \approx \frac{1}{(1.4\ {\rm g\ cm^{-3}})(290\ {\rm cm^2\ g^{-1}})} = 2.5\times10^{-3}\ {\rm cm}$$

That is a short distance! Photons undergo many interactions on their way from the center of the Sun to the surface:

$$d = \ell\sqrt{N} \quad\Longrightarrow\quad N = \left(\frac{d}{\ell}\right)^2 \approx \left(\frac{6.96\times10^{10}\ {\rm cm}}{2.5\times10^{-3}\ {\rm cm}}\right)^2 = 7.8\times10^{26}$$

The photons travel at the speed of light (obvs) between the interactions, so the total time this takes is

$$t = N\left(\frac{\ell}{c}\right) \approx 7.8\times10^{26}\left(\frac{2.5\times10^{-3}\ {\rm cm}}{3.0\times10^{10}\ {\rm cm\ s^{-1}}}\right) = 6.5\times10^{13}\ {\rm s} = 2.1\times10^6\ {\rm yr}$$

That's 2 million years for photons to slowly diffuse out of the Sun! If the photons flew straight out from the core (like the neutrinos do), it would only take $t = R_\odot/c = 2.3$ seconds! The diffusion takes 20 trillion (2×10^{13}) times longer.

4. *Let's look at some nuclear reactions.*

(a) *For each of these there is a mystery particle X or missing value y. Use conservation laws to determine X and y in each case (and explain the reasoning for your answers).*

In class we saw three conservation laws for typical nuclear reactions in stars: (1) electric charge, (2) lepton number, and (3) nucleon number. Also, along with photons, we only need to be concerned with the first "generation" of matter particles: protons, neutrons, electrons/positrons, and electron neutrinos/antineutrinos, and not other flavors (muons, tau neutrinos, charm quarks, etc.). *This problem was originally from Prof. Ryan Chornock.*

i. ${}^{8}_{5}\mathrm{B} \rightarrow {}^{y}_{4}\mathrm{Be} + \nu_e + X$

The electric charge on the left hand side is +5 so we know that X must have a charge of +1. The lepton number on the left hand side is 0, but the right hand side has a neutrino ν_e (lepton number of +1); so X must have a lepton number of −1 (an antilepton, either a positron or antineutrino). Taking these together, X must be a **positron**. Conserving nucleon number gives $\mathbf{y = 8}$.

ii. $\gamma \ + \ {}^{28}_{14}\mathrm{Si} \ \rightarrow \ {}^{24}_{y}\mathrm{Mg} + X$

There are no leptons on the left hand side, so X cannot be a lepton. By definition, magnesium has atomic number 12, so we know that $\mathbf{y = 12}$. Conserving nucleon number and charge, we know that X must have 4 nucleons with a charge of +2. That means X is **an alpha particle, $\mathbf{{}^{4}_{2}He}$**.

iii. ${}^{16}_{8}\mathrm{O} + {}^{16}_{8}\mathrm{O} \ \rightarrow \ {}^{31}_{y}\mathrm{P} + X$

Again we don't have any leptons involved. Phosphorous is element number 15, so $\mathbf{y = 15}$. Conserving nucleon number and charge, we know that X must have 1 nucleon and charge +1, so X is **a proton, p** or $\mathbf{{}^{1}_{1}H}$.

iv. ${}^{21}_{10}\mathrm{Ne} + {}^{4}_{2}\mathrm{He} \ \rightarrow \ {}^{24}_{12}\mathrm{Mg} + X$

Still no leptons here. X has no electric charge, but $21 + 4 - 24 = 1$ nucleon, so it must be **a neutron, n**.

v. ${}^{27}_{14}\mathrm{Si} \ \rightarrow \ {}^{y}_{13}\mathrm{Al} + e^{+} + X$

The lepton number on the left is 0, but the positron on the right has lepton number of −1, so X must have a lepton number of +1 (electron or neutrino). The electric charges already balance, so X cannot be charged, thus X is **a neutrino**, $\boldsymbol{\nu_e}$. Conserving nucleon number requires $\mathbf{y = 27}$.

(b) *The energy released (or absorbed) in a nuclear reaction is the Q value (defined as positive if energy is released). Calculate the Q values (in MeV) for each of the following reactions; indicate whether energy is released or absorbed. You should look up the nuclear masses or binding energies online; cite your source(s).*

I will use nuclear masses from https://wwwndc.jaea.go.jp/NuC/index.html; these are given in amu. Recall that 1 amu = 1 u = 931.5 MeV. Note that it is important to keep several digits of precision in the masses because we will be subtracting numbers that are close to each other.

i. ${}^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \;\rightarrow\; {}^{2}_{1}\mathrm{H} + e^{+} + \nu_e$
$m({}^{1}_{1}\mathrm{H}) = m_p = 1.007825$ u, $m({}^{2}_{1}\mathrm{H}) = 2.014102$ u, $m(e^{+}) = 0.000549$ u. So $Q = (2(1.007825) - 2.014102 - 0.000549)$ u $= 0.000999$ u $= +0.931$ MeV. This is positive, so energy is released. Note that the positron will quickly annihilate with a free electron, so sometimes that energy release is included in the Q value, which would give 1.442 MeV. Either is fine.

ii. ${}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} \;\rightarrow\; {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}$
$m({}^{15}_{7}\mathrm{N}) = 15.000109$ u, $m({}^{1}_{1}\mathrm{H}) = 1.007825$ u, $m({}^{12}_{6}\mathrm{C}) = 12$ u, $m({}^{4}_{2}\mathrm{He}) = 4.002603$. So $Q = 15.000109 + 1.007825 - 12 - 4.002603$ u $= 0.005331$ u $= +4.966$ MeV. Energy is released.

iii. ${}^{19}_{9}\mathrm{F} + {}^{1}_{1}\mathrm{H} \;\rightarrow\; {}^{16}_{8}\mathrm{O} + {}^{4}_{2}\mathrm{He}$
$m({}^{19}_{9}\mathrm{F}) = 18.998403$ u, $m({}^{1}_{1}\mathrm{H}) = 1.007825$ u, $m({}^{16}_{8}\mathrm{O}) = 15.994915$ u, $m({}^{4}_{2}\mathrm{He}) = 4.002603$ u. So $Q = 18.998403 + 1.007825 - 15.994915 - 4.002603$ u $= 0.008710$ u $= +8.113$ MeV. Energy is released.

iv. ${}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \;\rightarrow\; {}^{8}_{4}\mathrm{Be}$
$m({}^{4}_{2}\mathrm{He}) = 4.002603$ u, $m({}^{8}_{4}\mathrm{Be}) = 8.005305$ u. So $Q = 2(4.002603) - 8.005305$ u $= -0.000099$ u $= -0.092$ MeV. The Q value is negative, so energy is absorbed. Note this means that it is energetically favorable for ${}^{8}_{4}\mathrm{Be}$ to decay into two alpha particles (${}^{4}_{2}\mathrm{He}$).

v. ${}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \;\rightarrow\; {}^{16}_{8}\mathrm{O} + {}^{4}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He}$
$m({}^{12}_{6}\mathrm{C}) = 12$ u, $m({}^{16}_{8}\mathrm{O}) = 15.994915$ u, $m({}^{4}_{2}\mathrm{He}) = 4.002603$ u. So $Q = 2(12) - 15.994915 - 2(4.002603)$ u $= -0.000121$ u $= -0.113$ MeV. Energy is absorbed.

5. *The goal of this problem is to understand how the fusion reaction rate depends on temperature. We can write the reaction rate as (see Phillips eqn. 4.22)*

$$r_{AB} = n_A n_B \left[\frac{8}{\pi m_r}\right]^{1/2} \left[\frac{1}{kT}\right]^{3/2} \int_0^\infty S(E)\, e^{-f(E)}\, dE \quad \text{where} \quad f(E) = \frac{E}{kT} + \left(\frac{E_G}{E}\right)^{1/2}$$

The function $e^{-f(E)}$ is sharply peaked near the Gamow peak E_0. Assuming $S(E)$ is reasonably constant near the Gamow peak, we pull it out of the integral and write

$$r_{AB} = n_A n_B \left[\frac{8}{\pi m_r}\right]^{1/2} \left[\frac{1}{kT}\right]^{3/2} S(E_0) \int_0^\infty e^{-f(E)}\, dE$$

Our task is to estimate the remaining integral. Here are the steps to do that. Warning: *The algebra can get a bit messy for this problem; don't get discouraged!*

(a) *Recall that the Taylor series approximation of a function $g(x)$ around the point $x = a$ is given by*

$$g(x) = g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2 + \ldots \;=\; \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(x-a)^n$$

Expand the function $f(E)$ as a Taylor series around the Gamow peak, $E = E_0$. This means you can approximate f with the form

$$f(E) \approx b_0 + b_1(E - E_0) + b_2(E - E_0)^2 + \ldots$$

What you will need to do is determine the coefficients b_0, b_1, and b_2. Hint. *In your derivation, you should find that $b_1 = 0$ because $f(E)$ is minimized at E_0.*

Here we want to expand the function $f(E)$ around $E = E_0$ and we will work to second order only (i.e., approximating the function around the Gamow peak as a parabola). Looking at the Taylor series formula, we can immediately identify the coefficients:

$$b_0 = f(E_0) \qquad b_1 = f'(E_0) \qquad b_2 = \frac{f''(E_0)}{2}$$

Also, we saw in class an expression for the Gamow peak energy (Phillips eqn. 4.23)

$$E_0 = \left[\frac{1}{4}E_G(kT)^2\right]^{1/3} = 4^{-1/3}E_G^{1/3}(kT)^{2/3}$$

Thus the coefficients are

$$\begin{aligned}
b_0 &= f(E_0) \;=\; \left(\frac{E_G}{E_0}\right)^{1/2} + \frac{E_0}{kT} \\
&= \frac{E_G^{1/2}}{4^{-1/6}E_G^{1/6}(kT)^{1/3}} + \frac{4^{-1/3}E_G^{1/3}(kT)^{2/3}}{kT} \\
&= 4^{1/6}E_G^{1/3}(kT)^{-1/3} + 4^{-1/3}E_G^{1/3}(kT)^{-1/3} \\
&= (2^{1/3} + 2^{-2/3})\left(\frac{E_G}{kT}\right)^{1/3} \;=\; \left(\frac{2^{1/3}2^{2/3}+1}{2^{2/3}}\right)\left(\frac{E_G}{kT}\right)^{1/3} \;=\; \frac{3}{2^{2/3}}\left(\frac{E_G}{kT}\right)^{1/3} \\
&= 3\left(\frac{E_G}{4kT}\right)^{1/3}
\end{aligned}$$

$$\begin{aligned}
b_1 &= f'(E_0) \;=\; \left.\frac{df}{dE}\right|_{E=E_0} \;=\; \left[E_G^{1/2}\left(-\frac{1}{2}E^{-3/2}\right) + \frac{1}{kT}\right]_{E=E_0} \\
&= -\frac{E_G^{1/2}}{2}E_0^{-3/2} + \frac{1}{kT} \\
&= -2^{-1}E_G^{1/2}[4^{1/2}E_G^{-1/2}(kT)^{-1}] + (kT)^{-1} \;=\; -(kT)^{-1} + (kT)^{-1} \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
b_2 &= \frac{f''(E_0)}{2} = \frac{1}{2}\frac{d^2 f}{dE^2}\bigg|_{E=E_0} = \frac{1}{2}\left[E_G^{1/2}\left(\frac{3}{4}E^{-5/2}\right)\right]_{E=E_0} = \frac{3}{8}E_G^{1/2}E_0^{-5/2} \\
&= \frac{3}{8}E_G^{1/2}[4^{5/6}E_G^{-5/6}(kT)^{-5/3}] = \frac{3}{2^{4/3}}E_G^{-1/3}(kT)^{-5/3} \\
&= \frac{3}{2}[2E_G(kT)^5]^{-1/3}
\end{aligned}
$$

Note that b_1 has to be zero, because that's how we derived E_0 in the first place; E_0 the energy at which $\exp[-f(E)]$ peaks so $f(E)$ is minimized at that energy, and df/dE must be zero there.

(b) *Using the approximate form for f, and with $y = E - E_0$, the integral becomes*

$$
\int_0^\infty e^{-f(E)}\ dE \approx e^{-b_0}\int e^{-b_2\, y^2}\ dy
$$

(I am not being very careful about the limits of integration, because as long as they are far from the peak they don't matter very much.) The integrand is now a Gaussian, and the integral of a Gaussian is simple:

$$
\int e^{-b_2\, y^2}\ dy = \left(\frac{\pi}{b_2}\right)^{1/2}
$$

Use this with your values of b_0 *and* b_2 *to write an approximation for* r_{AB}.

So the integral becomes

$$
\int_0^\infty e^{-f(E)}\ dE \ \approx\ e^{-b_0}\int e^{-b_2\, y^2}\ dy \ =\ e^{-b_0}\left(\frac{\pi}{b_2}\right)^{1/2}
$$

And the reaction rate becomes

$$
\begin{aligned}
r_{AB} &\approx n_A\, n_B\left[\frac{8}{\pi m_r}\right]^{1/2}\left[\frac{1}{kT}\right]^{3/2} S(E_0)\int_0^\infty e^{-f(E)}\ dE \\
&\approx n_A\, n_B\left[\frac{8}{\pi m_r}\right]^{1/2}\left[\frac{1}{kT}\right]^{3/2} S(E_0)\ e^{-b_0}\left(\frac{\pi}{b_2}\right)^{1/2} \\
&\approx n_A\, n_B\left[\frac{8}{\pi m_r}\right]^{1/2}\left[\frac{1}{kT}\right]^{3/2} S(E_0)\ e^{-3(E_G/4kT)^{1/3}}\left(\frac{2\pi}{3}[2E_G(kT)^5]^{1/3}\right)^{1/2} \\
&\approx n_A\, n_B\left[\frac{8}{\pi m_r}\right]^{1/2}\left[\frac{1}{kT}\right]^{3/2}\frac{(2\pi)^{1/2}}{3^{1/2}}\ S(E_0)\ (2E_G)^{1/6}\ (kT)^{5/6}\ e^{-3(E_G/4kT)^{1/3}} \\
&\approx \frac{4}{3^{1/2}}\frac{n_A n_B}{m_r^{1/2}}\ (2E_G)^{1/6}\ S(E_0)\ (kT)^{-3/2}\ (kT)^{5/6}\ e^{-3(E_G/4kT)^{1/3}} \\
&\approx \left[\frac{4 n_A n_B}{(3m_r)^{1/2}}\ (2E_G)^{1/6}\ S(E_0)\right]\ (kT)^{-2/3}\ e^{-3(E_G/4kT)^{1/3}}
\end{aligned}
$$

(c) *Now let's examine the temperature dependence of r_{AB}. As discussed in class, if we approximate the temperature dependence as a power law, $r_{AB} \propto T^\alpha$, we can* *find the power law index with any of the following:*

$$\alpha = \frac{T}{r_{AB}} \frac{dr_{AB}}{dT} = T \frac{d \ln r_{AB}}{dT} = \frac{d \ln r_{AB}}{d \ln T}$$

Using your approximation for r_{AB}, show that

$$\alpha = \left(\frac{E_G}{4kT}\right)^{1/3} - \frac{2}{3}$$

Since we only care about the temperature dependence, let's collect all the terms that don't depend on temperature (in the brackets above) into a constant I will call C. Note that technically $S(E_0)$ changes with temperature (because E_0 does), but we were already assuming that $S(E)$ was nearly constant. So we can rewrite

$$r_{AB} = C\ (kT)^{-2/3}\ e^{-3(E_G/4kT)^{1/3}}$$

I'm going to use the second form of the equation for $\alpha = Td(\ln r_{AB}/dT)$. Taking the natural log of r_{AB} we get

$$\ln r_{AB} = \ln C - \frac{2}{3} \ln k - \frac{2}{3} \ln T - 3 \left(\frac{E_G}{4kT}\right)^{1/3}$$

The derivatives of the first two terms vanish, so then we can get the desired result

$$\alpha = T\frac{d \ln r_{AB}}{dT} = T \left[-\frac{2}{3}\frac{1}{T} - 3 \left(\frac{E_G}{4k}\right)^{1/3} \left(-\frac{1}{3}\right) T^{-4/3} \right] = \left(\frac{E_G}{4kT}\right)^{1/3} - \frac{2}{3}$$

(d) *Using this expression, estimate α for*

i. *the p-p reaction in the Sun (take $T = 1.5 \times 10^7$ K)*

The Gamow energy is defined (Lecture 8 slide 16; Phillips 4.10)

$$E_G = (\pi \alpha Z_A Z_B)^2 (2 m_r c^2)$$

where $\alpha \approx 1/137$ is the fine structure constant, Z_A and Z_B are the atomic numbers of the two nuclei, and $m_r = m_A m_B/(m_A + m_B)$ is the reduced mass. For two protons, $Z_A = Z_B = 1$ and $m_r = m_p/2 = 469$ MeV$/c^2$ so

$$E_G = \left[\frac{\pi(1)(1)}{137}\right]^2 [2(469 \text{ MeV})] = 0.493 \text{ MeV} = 493 \text{ keV}$$

For $T = 1.5 \times 10^7$ K we can write

$$kT = (8.62 \times 10^{-5} \text{ eV K}^{-1})(1.5 \times 10^7 \text{ K}) = 1.29 \text{ keV}$$

and thus

$$\alpha_{pp} = \left[\frac{493 \text{ keV}}{4(1.29 \text{ keV})}\right]^{1/3} - \frac{2}{3} = 3.9$$

ii. *the* ${}^4_2\mathrm{He} + {}^8_4\mathrm{Be}$ *reaction that is part of the triple alpha process to make carbon (take* $T = 10^8$ K*).*

Here we have $Z_A = 2$ and $Z_B = 4$. The nuclear masses are 3727.379 MeV/c^2 for the helium nucleus (alpha particle) and 7454.850 MeV/c^2 for the ${}^8_4\mathrm{Be}$ (see problem 4b), so the reduced mass is

$$m_r = \frac{(3727.379)(7454.850)}{3727.379 + 7454.850} = 2484.93 \text{ MeV}/c^2$$

though note that because we are not calculating an energy difference here, we need not be so precise. Just doing

$$m_r \approx \frac{(4m_p)(8m_p)}{4m_p + 8m_p} = \frac{8}{3}m_p = 2502 \text{ MeV}/c^2$$

would have been fine. The Gamow energy is

$$E_G = \left[\frac{\pi(2)(4)}{137}\right]^2 [2(2484.93 \text{ MeV})] = 167 \text{ MeV}$$

For $T = 1 \times 10^8$ K we can write

$$kT = (8.62 \times 10^{-5} \text{ eV K}^{-1})(1 \times 10^8 \text{ K}) = 8.62 \text{ keV}$$

and thus

$$\alpha_{3\alpha} = \left[\frac{167000 \text{ keV}}{4(8.62 \text{ keV})}\right]^{1/3} - \frac{2}{3} = 16.3$$

We see that this reaction is much more temperature sensitive than the p-p reaction, with the reaction rate $\propto T^{16}$ compared to $\propto T^4$ for p-p.

6. *(Required for 541; extra credit for 441)*

(a) *(adapted from Phillips problem 4.1) Find the classical distance of closest approach for two protons with an energy of approach equal to 2 keV. Estimate the probability that the protons penetrate the Coulomb barrier tending to keep them apart. Compare this probability with the corresponding probability for two* ^{4}*He nuclei with the same energy of approach. By what factor would you need to increase the approach energy (or temperature) for the* ^{4}*He nuclei to have the same probability as the p-p case?*

The closest approach distance r_C comes when the kinetic energy is entirely converted to potential energy. Phillips (eqn. 4.1) writes this energy of approach as

$$E = \frac{Z_A Z_B e^2}{4\pi\epsilon_0 r_C} \quad \Longrightarrow \quad r_C = \frac{Z_A Z_B e^2}{4\pi\epsilon_0 E}$$

Plugging in $Z_A = Z_B = 1$ for two protons with approach energy $E = 2$ keV,

$$\begin{aligned} r_C &= \frac{(1)(1)(1.602 \times 10^{-19} \text{ C})^2}{4\pi(8.854 \times 10^{-12} \text{ C}^2 \text{ s}^2 \text{ kg}^{-1} \text{ m}^{-3})(2000 \text{ eV} \times 1.6 \times 10^{-19} \text{ J eV}^{-1})} \\ &= 7.2 \times 10^{-13} \text{ m} = 720 \text{ fm} \end{aligned}$$

This is well away from the ~1 fm needed for nuclear fusion, so quantum tunneling is required.

The probability of tunneling is given by Phillips eqn. 4.12,

$$P \approx \exp\left[-\left(\frac{E_G}{E}\right)^{1/2}\right] \quad \text{where} \quad E_G = (\pi \alpha Z_A Z_B)^2 \, (2 m_r c^2)$$

For two protons, $Z_A = Z_B = 1$, and the reduced mass is $m_r = m_p^2/2m_p = m_p/2 = (938 \text{ MeV}/c^2)/2 = 469 \text{ MeV}/c^2$, so the Gamow energy E_G is

$$E_G = \left[\frac{\pi(1)(1)}{137}\right]^2 [2(469 \text{ MeV})] = 0.493 \text{ MeV} = 493 \text{ keV}$$

That makes the probability of tunneling for two protons with approach energy $E = 2$ keV

$$P \approx \left[-\left(\frac{E_G}{E}\right)^{1/2}\right] = \exp\left[-\left(\frac{493 \text{ keV}}{2 \text{ keV}}\right)^{1/2}\right] = \exp(-15.7) = 1.5 \times 10^{-7}$$

For two helium nuclei, $Z_A = Z_B = 2$, and the reduced mass is $m_r = m(\text{He})/2 = (3727.4 \text{ MeV}/c^2)/2 = 1863.7 \text{ MeV}/c^2$. That gives a Gamow energy

$$E_G = \left[\frac{\pi(2)(2)}{137}\right]^2 [2(1863.7 \text{ MeV})] = 31.3 \text{ MeV}$$

and tunneling probability

$$P \approx \exp\left[-\left(\frac{31300 \text{ keV}}{2 \text{ keV}}\right)^{1/2}\right] = \exp(-125.1) = 4.7 \times 10^{-55}$$

That is a very small number, 48 orders of magnitude less likely than proton fusion at this energy.

The Gamow energy for the helium nuclei is 64 times higher than the Gamow energy for the two protons. This includes a factor of 16 from $(Z_A Z_B)^2$ and another factor of 4 from the increased mass. That means we would also need to increase the approach energy (or temperature) by a factor of 64 to get the same probability of tunneling.

(b) *(adapted from Phillips problem 4.3) Assume that the solar luminosity of* 3.8×10^{26} W *is due to hydrogen fusion by the p-p chain illustrated in Figure 4.4. How many neutrinos per second are emitted by the Sun? From this, calculate the neutrino flux (number of neutrinos per cm*^{2} *per second) at the Earth. What are the maximum neutrino energies (in MeV) for the PP I, PP II, and PP III branches?*

The net reaction from hydrogen fusion is

$$4\,{}^1_1\text{H} \;\rightarrow\; {}^4_2\text{He} + 2\,e^+ + 2\,\nu_e + \text{photons}$$

The mass difference between the 4 protons and the helium nucleus corresponds to an energy release of 26.7 MeV per reaction. Then, from the Sun's luminosity, $L_\odot = 3.83 \times 10^{33}$ erg s^{-1}, we can calculate the number of reactions per second, each of which produces two neutrinos.

One (minor) complication: the solar luminosity is defined as its *photon* luminosity; neutrinos are not included. So we should not include the neutrino energy released in the reaction; Phillips Fig. 4.4 gives the effective energy release Q_{eff} for each branch of the p-p chain (more on this below) and this averages to about $\langle Q_{\text{eff}} \rangle =$ 26 MeV per reaction. That's only a few percent difference.

The neutrino emission rate is

$$N = 2\frac{L_\odot}{\langle Q_{\text{eff}} \rangle} = \frac{2(3.83 \times 10^{33}\ \text{erg s}^{-1})}{(26 \times 10^{6}\ \text{eV})(1.6 \times 10^{-12}\ \text{erg eV}^{-1})} = 1.84 \times 10^{38}\ \text{neutrinos s}^{-1}$$

The neutrino flux at Earth is then

$$f = \frac{N}{4\pi d^2} = \frac{1.84 \times 10^{38}\ \text{neutrinos s}^{-1}}{4\pi(1.496 \times 10^{13}\ \text{cm})^2} = 6.5 \times 10^{10}\ \text{neutrinos cm}^{-2}\ \text{s}^{-1}$$

That's 65 billion neutrinos going through every square centimeter at Earth every second!

To calculate the neutrino energies, we need to look at the reactions in more detail. For PP I, both neutrinos come from the initial p-p fusion

$$p + p \ \rightarrow\ {}^{2}_{1}\text{H} + e^{+} + \nu_e$$

We can calculate the maximum neutrino energy by assuming zero kinetic energy for the products. The rest mass difference between the input and outputs is

$$2(938.272\ \text{MeV}/c^2) - (1875.613\ \text{MeV}/c^2 + 0.511\ \text{MeV}/c^2) = 0.420\ \text{MeV}/c^2$$

Thus the maximum energy for the neutrino in PP I is 0.420 MeV.

Note that for this calculation we have to use the mass of the deuteron (the deuterium nucleus), not a deuterium atom (which includes the orbiting electron). You can look these up online, or see the table in Lecture 8 slides 23 or 24.

What about the kinetic energy of the incoming protons? The higher their kinetic energy, the higher the neutrino energy could be. However, recall that the fusing protons have an energy near the Gamow peak E_0; in the Sun this is just several keV, so that would change the answer only in the third decimal place.

For PP II, one of the neutrinos comes from the same reaction above, but the other comes from

$$e^{-} + {}^{7}_{4}\text{Be} \ \rightarrow\ {}^{7}_{3}\text{Li} + \nu_e$$

The atomic masses are 7.016928707 u for ^{7}Be and 7.016003427 u for ^{7}Li, as tabulated at http://wwwndc.jaea.go.jp/NuC/, where the atomic mass unit is $1u =$ 931.494 MeV$/c^2$. Note that these are *atomic* masses, which include the masses of the orbiting electrons, whereas for the nuclear reactions only the nuclei are involved. To get the nuclear masses, we need to subtract off the right number of electron masses. So the nuclear mass of ${}^7_4$Be is $7.016928707(931.494) - 4(0.511) = 6534.183$ MeV$/c^2$, and the nuclear mass of ${}^7_3$Li is $7.016003427(931.494) - 3(0.511) = 6533.832$ MeV$/c^2$. So the maximum neutrino energy from PP II is

$$0.511 \text{ MeV} + 6534.183 \text{ MeV} - 6533.832 \text{ MeV} = 0.862 \text{ MeV}$$

For PP III, the reaction that produces the second neutrino is the decay of boron-8[1]

$${}^8_5\text{B} \rightarrow {}^8_4\text{Be}^* + e^+ + \nu_e$$

The atomic mass of ${}^8_5$B is 8.024607326 u, corresponding to a nuclear mass of $8.024607326(931.494) - 5(0.511) = 7472.319$ MeV$/c^2$. The atomic mass of ${}^8_4$Be is 8.005305102 u, corresponding to a nuclear mass of $8.005305102(931.494) - 4(0.511) = 7454.850$ MeV$/c^2$. So the maximum neutrino energy would be

$$7472.319 \text{ MeV} - (7454.850 \text{ MeV} + 0.511 \text{ MeV}) = 16.958 \text{ MeV}$$

except that the reaction has Be*, meaning that the beryllium nucleus starts in an excited (higher energy) state, leaving a bit less energy for the neutrino. It turns out the maximum for PP III is closer to 15 MeV, but note that this is still much higher than for PP I or PP II, and given that these high energy neutrinos are the easiest to detect with experiments here on Earth, this is an important channel to understand.

[1] My copy of Phillips has some typos for PP III in Fig. 4.4: the arrow is in the wrong place on the last two reactions. A correct version of this figure is in Lecture 9 slide 14.