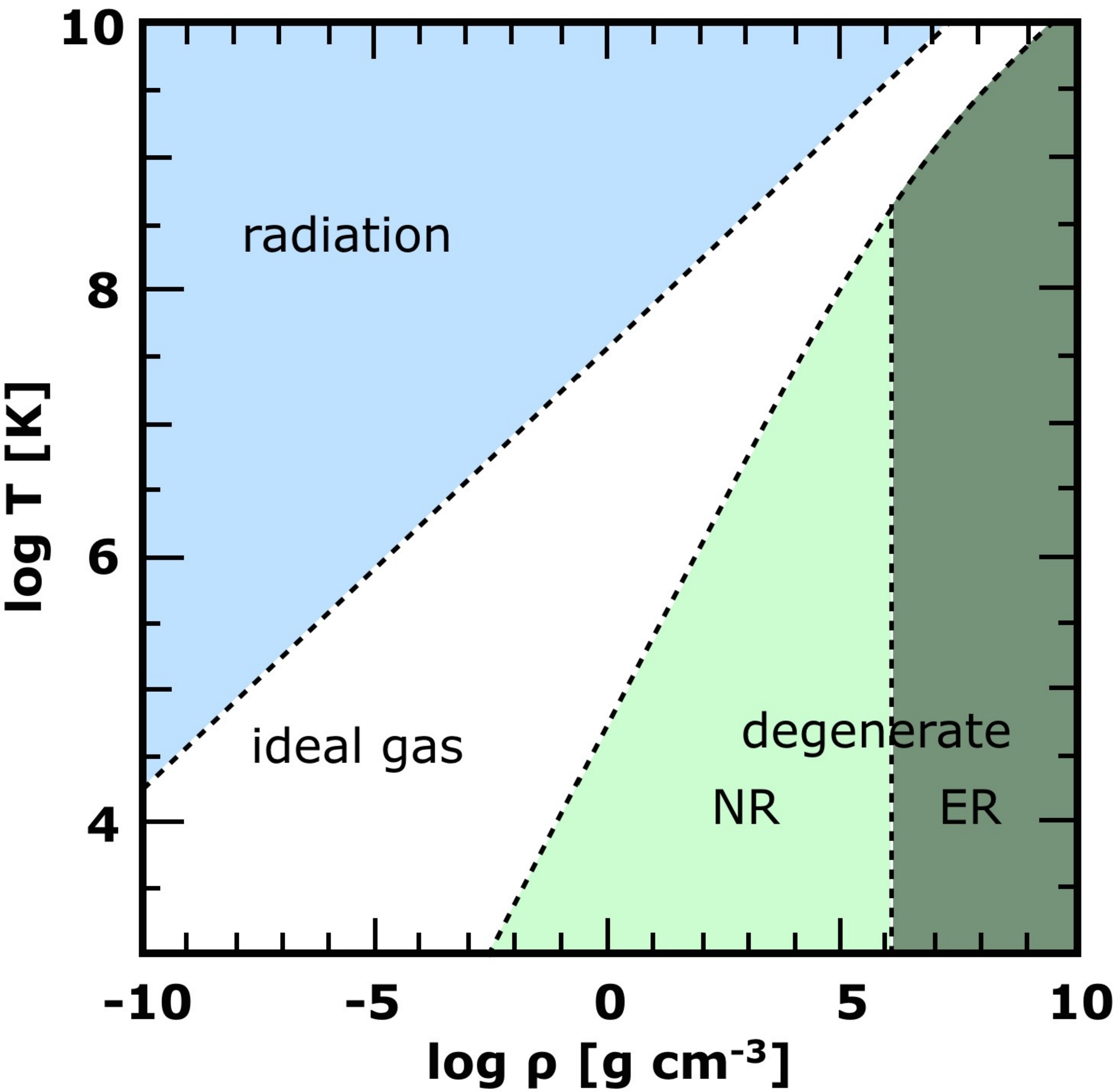
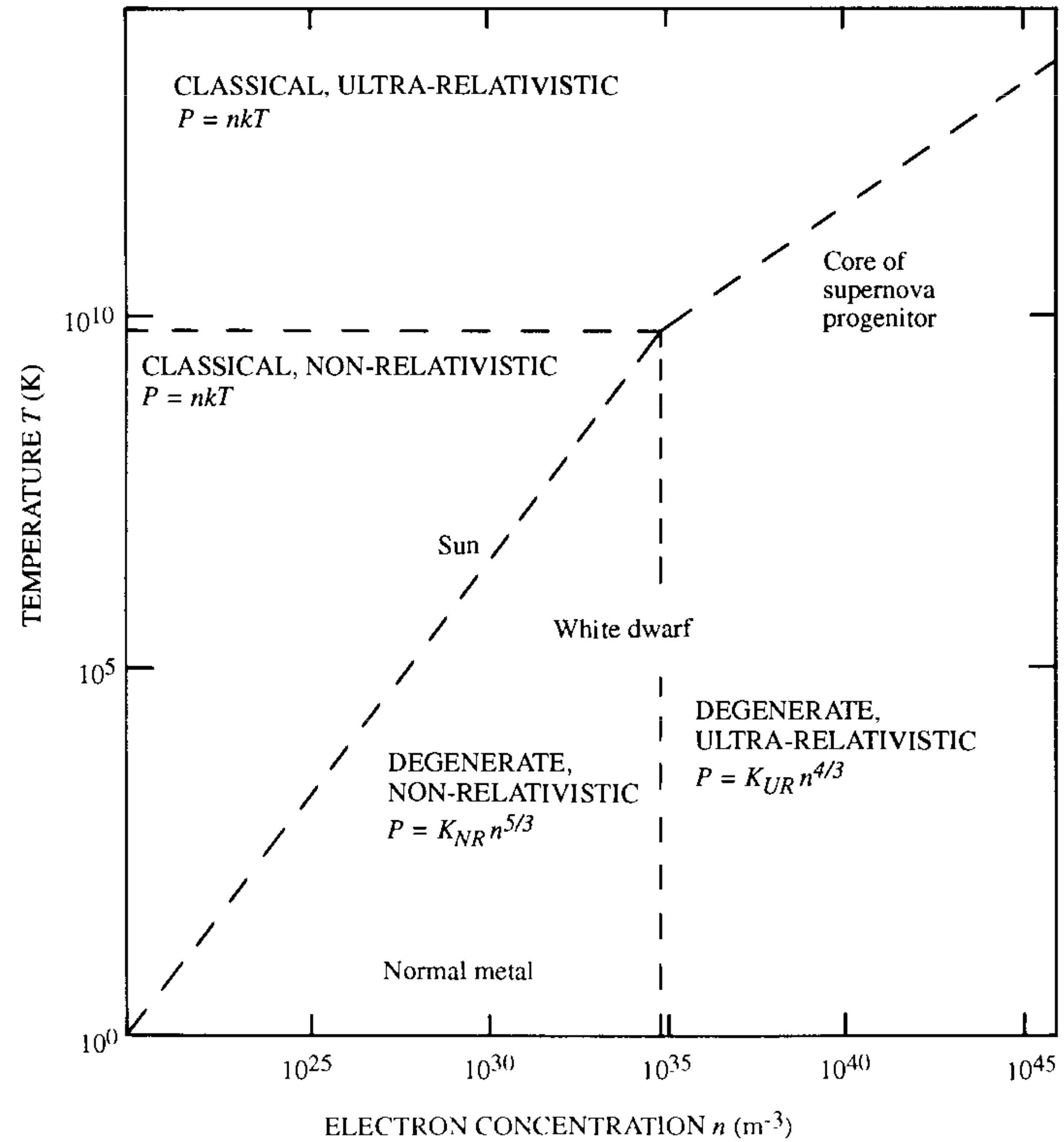
Lecture 4: Equations of State

Lamers & Levesque Ch. 4
Phillips Ch. 2





http://www.physics.rutgers.edu/ugrad/441

http://www.physics.rutgers.edu/grad/541

equation of state

equation of state relates the *pressure* to other quantities:

$$P(\rho, T, \text{composition})$$

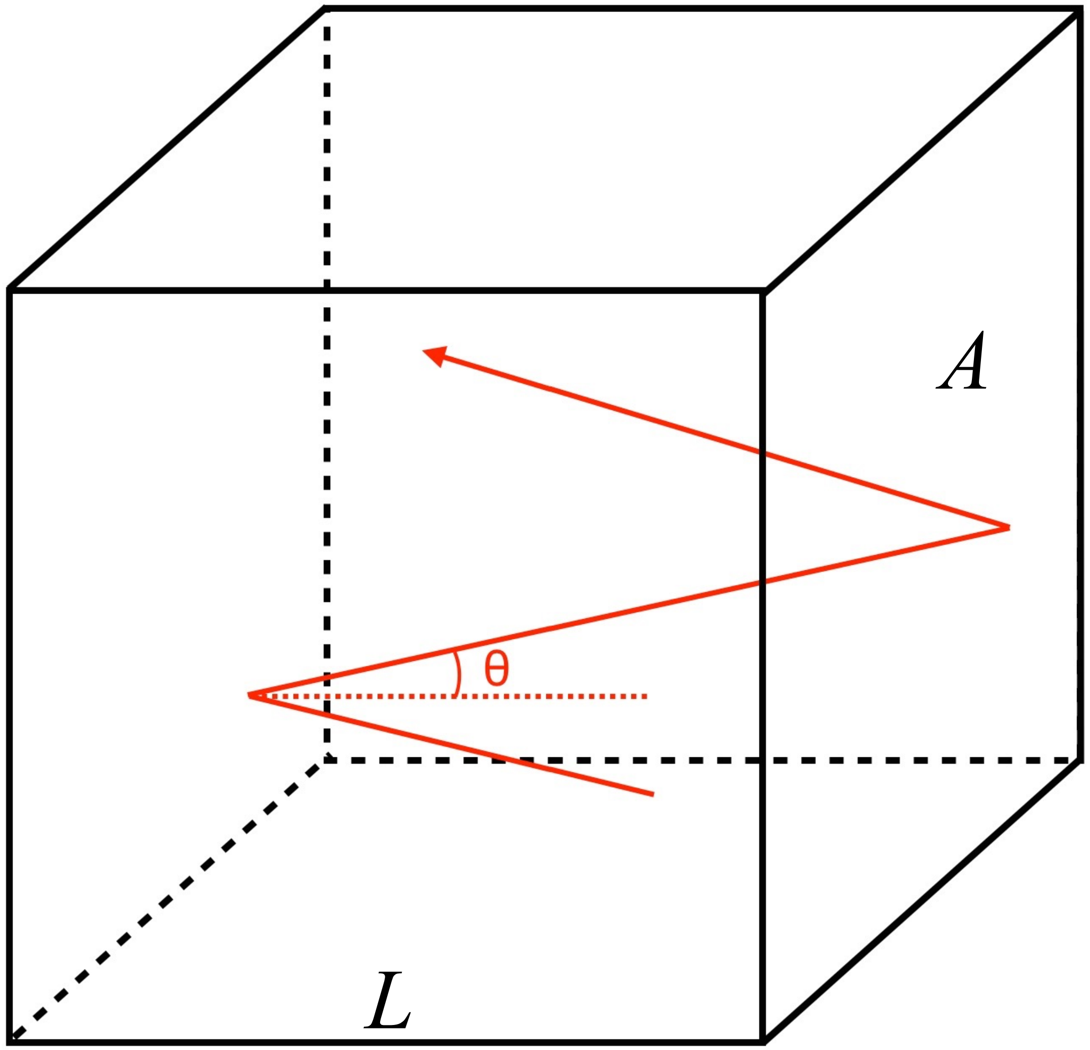
combine this with *hydrostatic equilibrium* to build stellar models

$$\frac{dP(r)}{dr} = -\frac{G\, m(r)\, \rho(r)}{r^2}$$

for example, last time we saw that an equation of state where pressure depends only on density $P \propto \rho^{\gamma}$ leads to a *polytrope* model;
in general the pressure depends on other quantities also (e.g., temperature)

pressure integral

Lamers & Levesque Fig. 4.1



pressure is *momentum flux*:

$$P = \frac{F}{A} \quad F = \frac{\Delta p}{\Delta t} \quad \Longrightarrow \quad P = \frac{\Delta p}{A \Delta t}$$

$$\Delta t = \frac{2L}{v \cos\theta} \qquad \Delta p = 2p\cos\theta$$

$$P = \frac{\Delta p}{A\Delta t} = \frac{2pv\cos^2\theta}{2AL} = \frac{pv}{V}\cos^2\theta = \frac{pv}{3V}$$

$$\text{assume isotropic} : \langle \cos^2\theta \rangle = \frac{\int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta}{\int_0^{\pi/2} \sin\theta \, d\theta} = 1/3$$

that's (average) pressure for **one** particle with velocity *v* and momentum *p*.
for *N* particles with velocity *v* and momentum *p* in the box, we have:

$$P = \frac{N}{V}\frac{pv}{3} = \frac{npv}{3}$$

n is **number density**

finally, for a *distribution* of momenta,
we get the general *pressure integral*:

$$\boxed{P = \frac{1}{3}\int_0^\infty p\, v(p)\, n(p)\, dp}$$

radiation pressure

general *pressure integral*: $$\boxed{P = \frac{1}{3}\int_0^\infty p\, v(p)\, n(p)\, dp}$$

for photons: $E = h\nu = hc/\lambda \qquad p = E/c = h\nu/c = h/\lambda$

$$v = c$$

in thermal equilibrium we can get the distribution function of photons $n(\nu)$ from the Planck function. Integrating, we get the *radiation pressure:*

$$P_{\rm rad} = \frac{1}{3} a T^4$$

$$a \equiv \frac{4\sigma}{c} = \frac{8\pi^5 k^4}{15 c^3 h^3} = 7.56\times10^{-15}\ {\rm erg\ cm^{-3}\ K^{-4}} = 7.56\times10^{-15}\ {\rm dyne\ cm^{-2}\ K^{-4}}$$

note: radiation pressure depends only on temperature

ideal gas pressure (non-relativistic)

general *pressure integral*: $\boxed{P = \frac{1}{3}\int_0^\infty p\, v(p)\, n(p)\, dp}$

$$v = p/m \qquad n(p)\,dp = n\frac{4\pi p^2 dp}{(2\pi mkT)^{3/2}} e^{-\frac{p^2}{2mkT}} \quad \text{Maxwell-Boltzmann dist}$$

$$P = \frac{1}{3}\int_0^\infty \frac{p^2}{m}\frac{4\pi p^2 n}{(2\pi mkT)^{3/2}} e^{-\frac{p^2}{2mkT}}\,dp = \frac{4n}{3m\pi^{1/2}}\int_0^\infty \frac{p^4}{A^{3/2}} e^{-\frac{p^2}{A}}\,dp$$

$$= \frac{4n}{3m\pi^{1/2}}\frac{3\pi^{1/2}A}{8} = \frac{nA}{2m} = \frac{n(2mkT)}{2m} = nkT$$

ideal gas law

$$\boxed{P = nkT = \frac{k}{\mu m_p}\rho T}$$

recall $n = \frac{\rho}{\langle m\rangle} = \frac{\rho}{\mu m_p}$

$n \equiv N/V$ number density

$PV = NkT$

if there are multiple species, just sum the pressures

ideal gas pressure (relativistic)

general *pressure integral*: $$\boxed{P = \frac{1}{3}\int_0^{\infty} p\, v(p)\, n(p)\, dp}$$

here we need the relativistic form of the Maxwell-Boltzmann distribution, and we can set $v \approx c$.
In the end we find the same formula as the non-relativistic case:

ideal gas law

$$\boxed{P = nkT = \frac{k}{\mu m_p}\rho T}$$

recall $n = \dfrac{\rho}{\langle m \rangle} = \dfrac{\rho}{\mu m_p}$

see Phillips, p. 53 for an alternate derivation

radiation vs. gas pressure

recall our order-of-magnitude estimate of a star's central temperature:

$$T_c \approx \frac{GM\mu m_p}{kR} \qquad \text{Sun: } T_c \approx 1.4 \times 10^7 \text{ K}$$

Sun's central density $\rho_c \approx 150$ g cm^{-3} and mean molecular weight $\mu \approx 0.62$ so we can calculate the ideal gas pressure and radiation pressure:

$$P_{\rm gas} = nkT = \frac{k}{\mu m_p}\rho T \qquad P_{\rm rad} = \frac{1}{3}aT^4$$

$$P_{\rm gas} \approx \frac{1.38 \times 10^{-16} \text{ g cm}^2 \text{ s}^{-2} \text{ K}^{-1}}{0.62 \times 1.67 \times 10^{-24} \text{ g}} \left(150 \text{ g cm}^{-3}\right)\left(1.4 \times 10^7 \text{ K}\right) = 2.8 \times 10^{17} \text{ dyne cm}^{-2}$$

$$P_{\rm rad} = \frac{aT^4}{3} \approx \frac{(7.56 \times 10^{-15} \text{ dyne cm}^{-2} \text{ K}^{-4})(1.4 \times 10^7 \text{ K})^4}{3} = 9.7 \times 10^{13} \text{ dyne cm}^{-2}$$

so for the Sun, gas pressure dominates; ~300 times the radiation pressure
is that true for all stars?

radiation vs. gas pressure

use order of magnitude estimates to get scaling relations

$$P_{\rm gas} = nkT = \frac{k}{\mu m_p}\rho T \qquad P_{\rm rad} = \frac{1}{3}aT^4$$

$$T_c \approx \frac{GM\mu m_p}{kR} \propto \frac{M}{R} \quad \Longrightarrow \quad P_{\rm rad} \propto \frac{M^4}{R^4}$$

$$P_{\rm gas} = \frac{k}{\mu m_p}\rho T \propto \frac{M}{R^3}\frac{M}{R} = \frac{M^2}{R^4}$$

recall our order-of-magnitude estimate of central pressure was $P \sim GM^2/R^4$ that assumes $P_{\rm rad} \ll P_{\rm gas}$

let's look at the ratio of radiation to gas pressure:

$$\frac{P_{\rm rad}}{P_{\rm gas}} \propto \frac{M^4/R^4}{M^2/R^4} = M^2$$

for the Sun, gas pressure was ~300 times the radiation pressure
thus for stars with $M \gtrsim 15 - 20\ {\rm M}_\odot$ radiation pressure is important!
total pressure is $P = P_{\rm gas} + P_{\rm rad}$

degeneracy pressure

for ideal gas we've ignored any complications from quantum mechanics

another way to say this is that we've assume that the separation between particles is large compared to their *de Broglie wavelength:* $\lambda = h/p$
typical separation of particles is $n^{-1/3}$
so $n^{-1/3} \gg h/p$ implies $n \ll (p/h)^3$ for ideal gas description to be valid

as density increases, lowest mass particles become degenerate first: *electrons*

in the non-relativistic regime:

$$p = mv \sim m\sqrt{kT/m} = \sqrt{mkT} \qquad (p/h)^3 \sim \frac{(mkT)^{3/2}}{h^3}$$

define *quantum concentration:* $$n_{Q,\mathrm{NR}} = \left[\frac{2\pi mkT}{h^2}\right]^{3/2}$$

see Phillips eqn. 2.22 for full derivation with constants

ideal gas equation of state is valid when $n \ll n_Q$

degeneracy pressure

for ideal gas we've ignored any complications from quantum mechanics

another way to say this is that we've assume that the separation between particles is large compared to their *de Broglie wavelength:* $\lambda = h/p$
typical separation of particles is $n^{-1/3}$
so $n^{-1/3} \gg h/p$ implies $n \ll (p/h)^3$ for ideal gas description to be valid

as density increases, lowest mass particles become degenerate first: *electrons*

in the *ultra-relativistic* ($v \approx c$) regime:

$$p = E/c \sim kT/c \quad (p/h)^3 \sim \left(\frac{kT}{hc}\right)^3$$

define ultra-relativistic quantum concentration:

$$n_{Q,\mathrm{UR}} = 8\pi \left[\frac{kT}{hc}\right]^3$$

see Phillips eqn. 2.24 for full derivation with constants

ideal gas equation of state is valid when $n \ll n_Q$

Degeneracy Pressure

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…

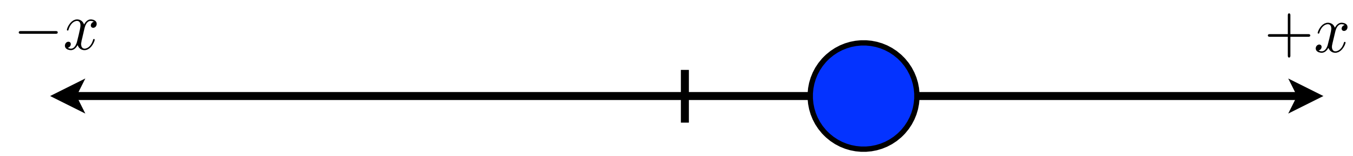
Heisenberg uncertainty principle: $\Delta x \cdot \Delta p > h$ (in 1D)

$\Delta vol \cdot \Delta^3 p > h^3$ (in 3D)

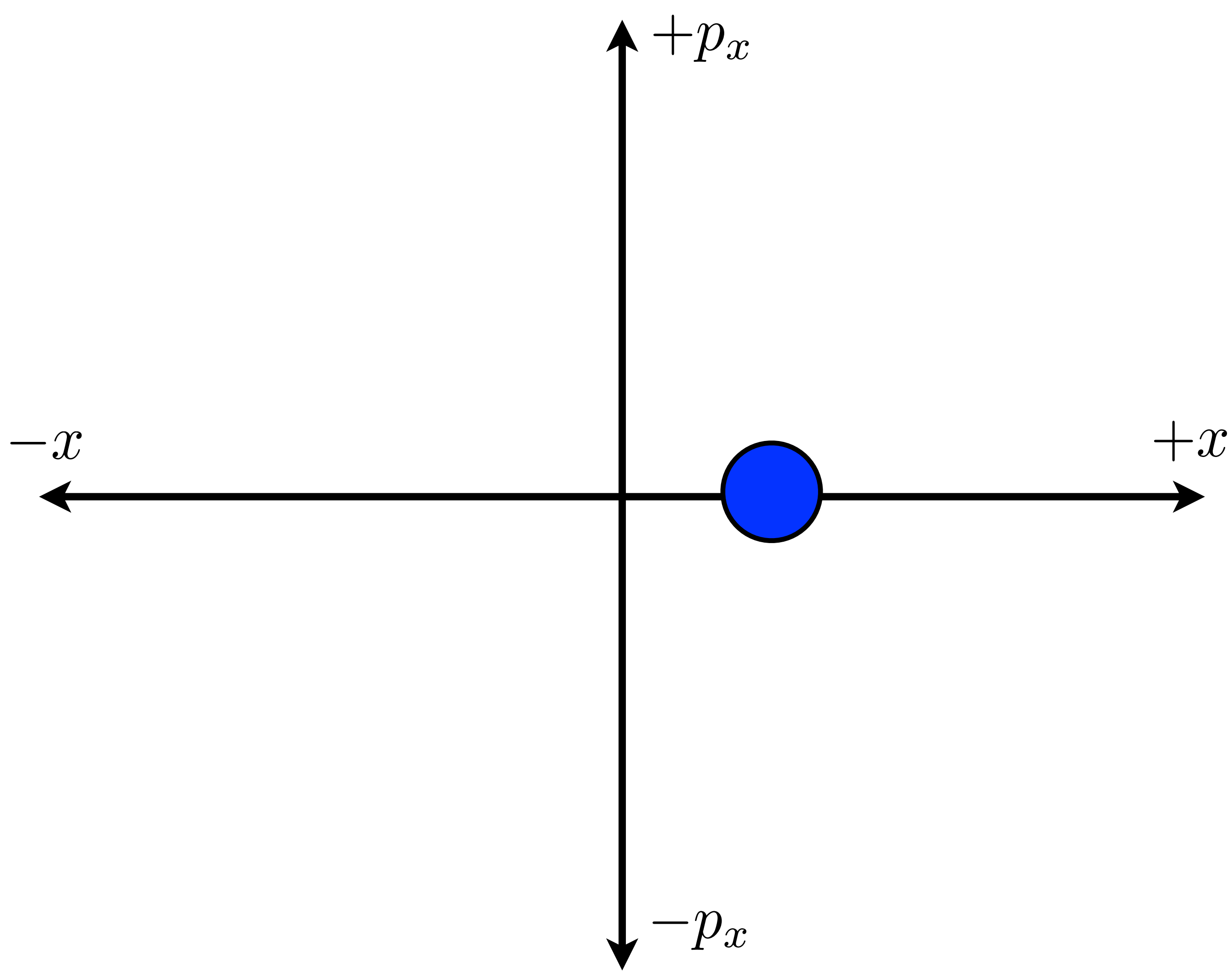
Pauli exclusion principle: no two identical fermions (same quantum-state) can exist at same time & place (i.e. in same phase-space volume h^3)

one-dimensional motion: 2d *phase space*

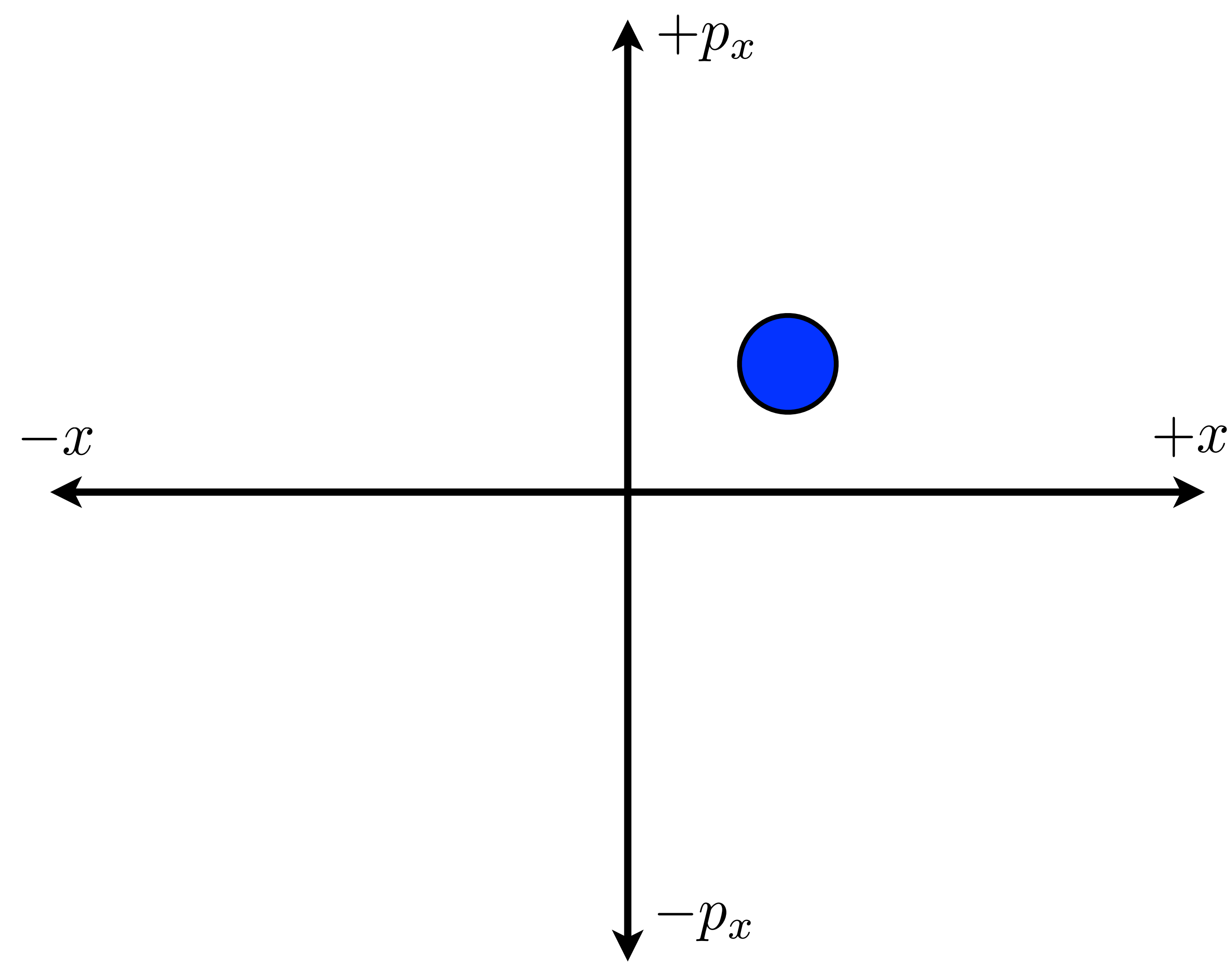
real space:



phase space:

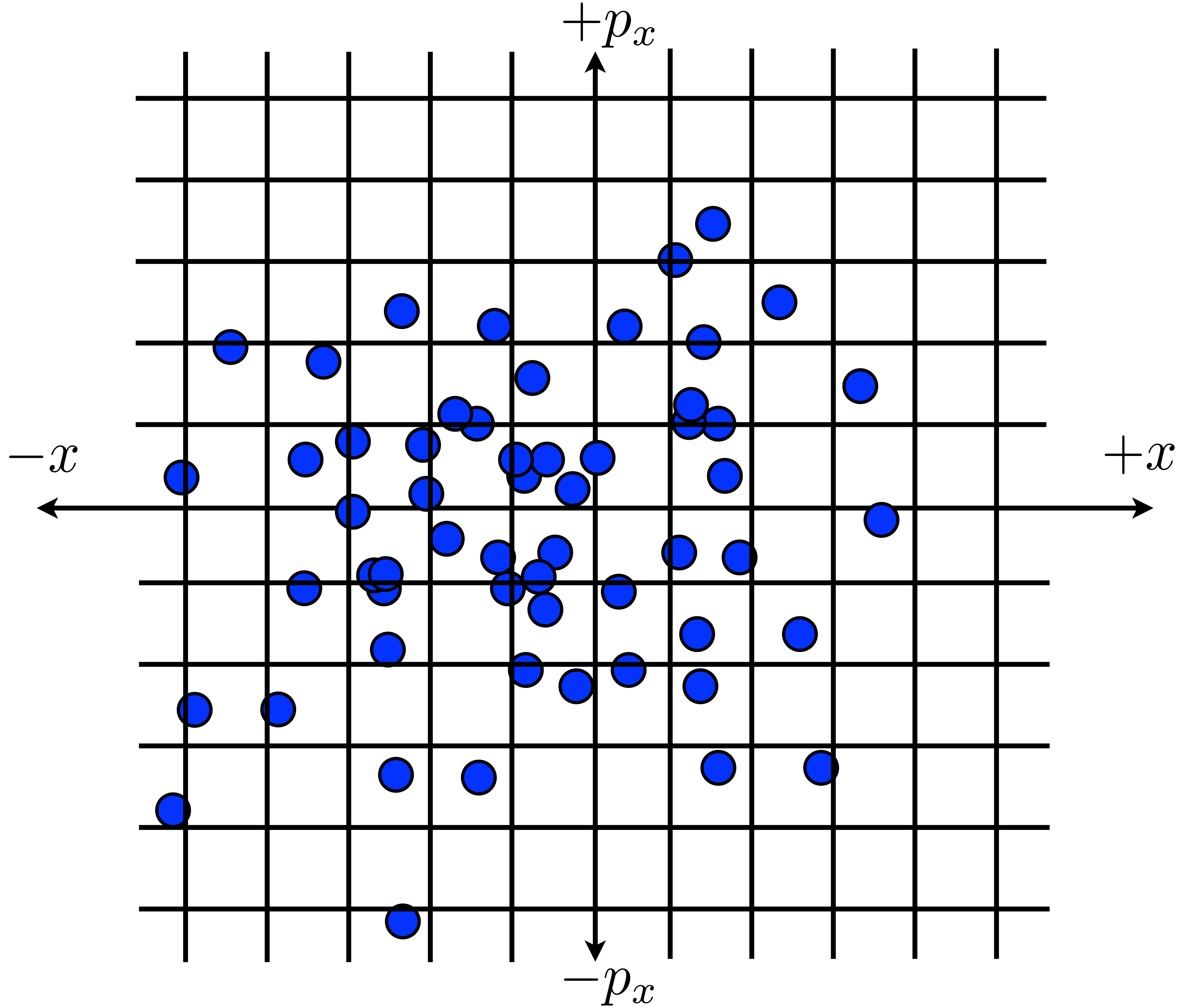


ball at rest



ball moving to the right

phase space distribution function (aka phase space density)

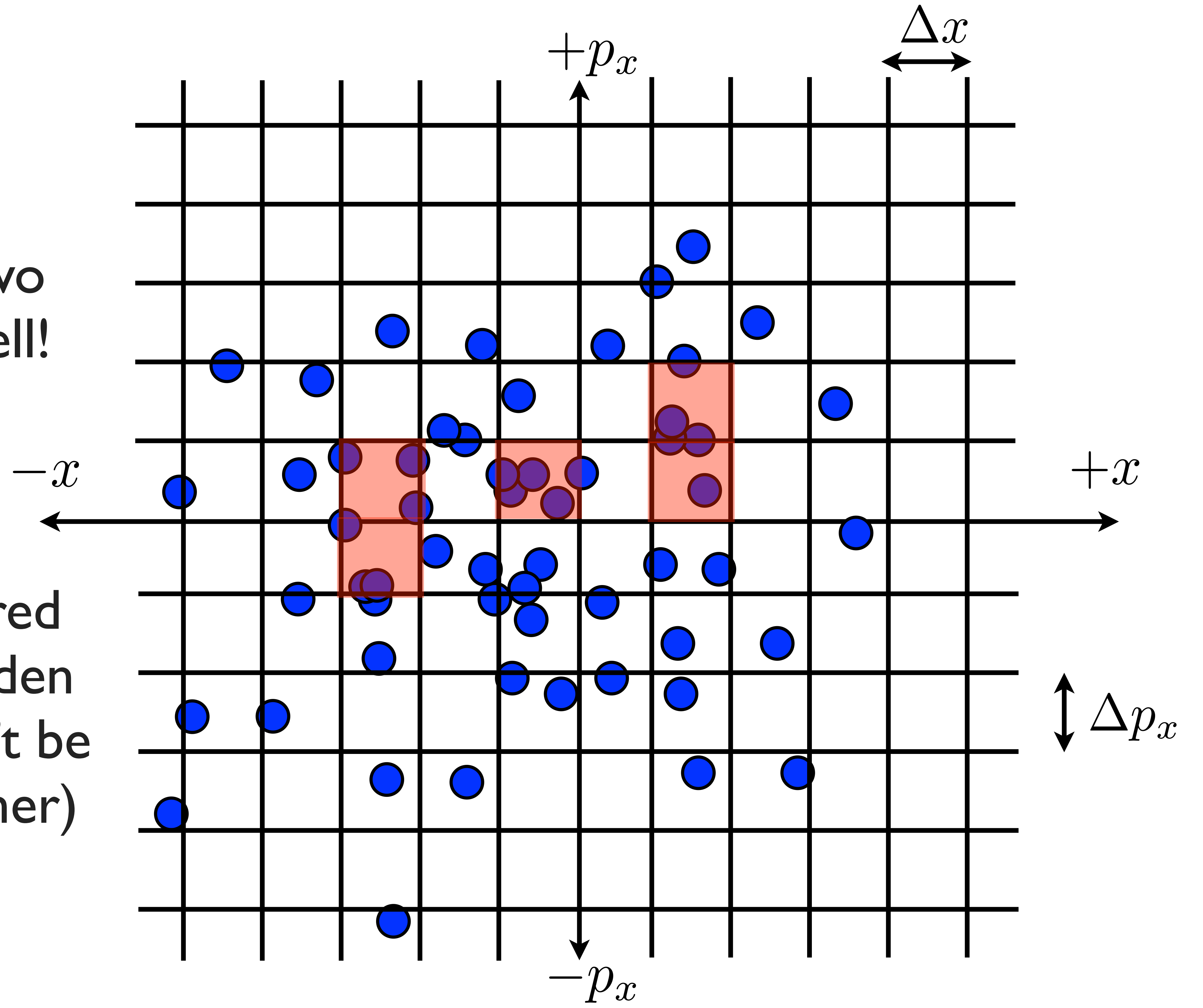


quantum mechanics
Pauli exclusion principle for fermions

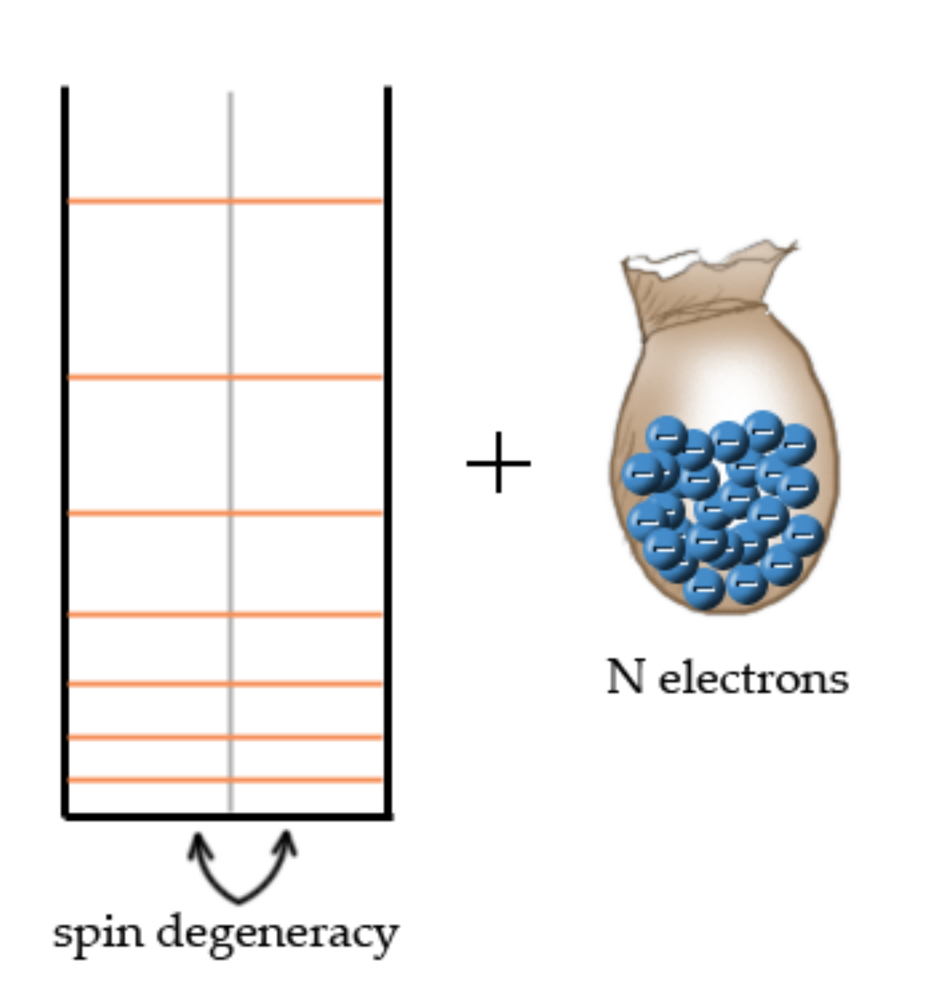
$$\Delta x \Delta p_x = h$$

maximum of two fermions per cell!

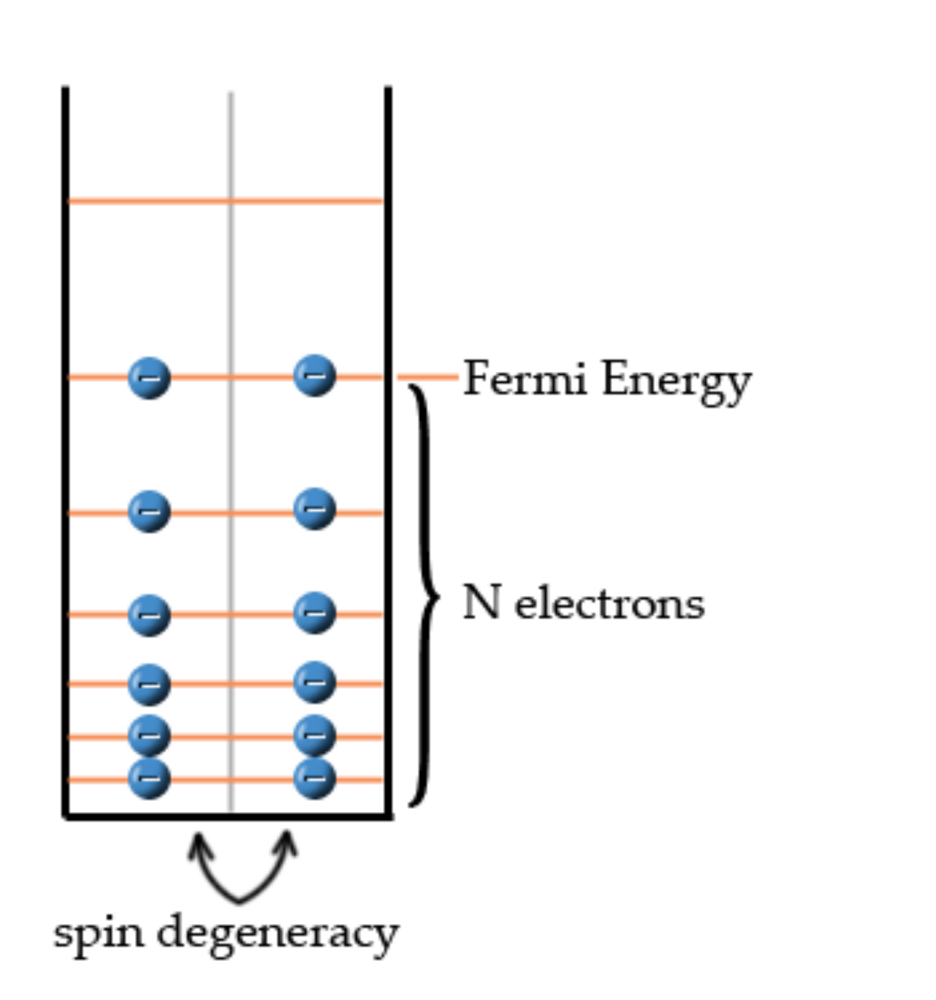
cells shaded in red would be forbidden (particles couldn't be that close together)



Fermi momentum and energy

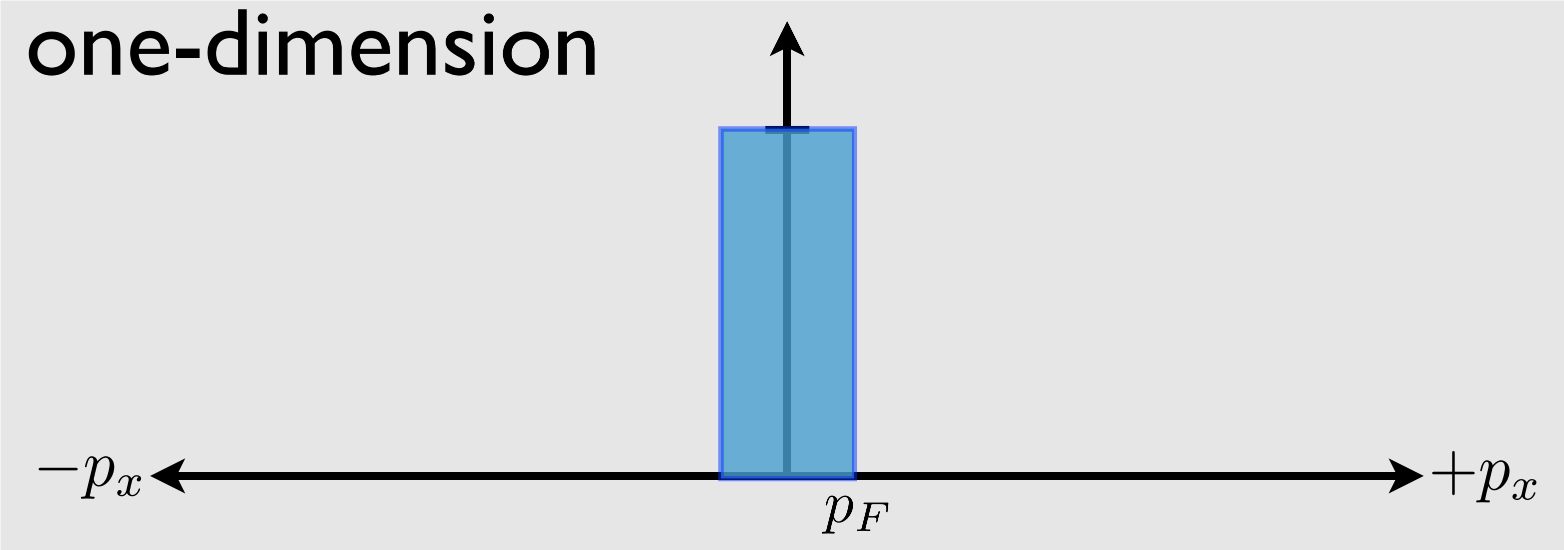


Pauli exclusion principle



from http://www.ugrad.physics.mcgill.ca/wiki/index.php/PHYS-558

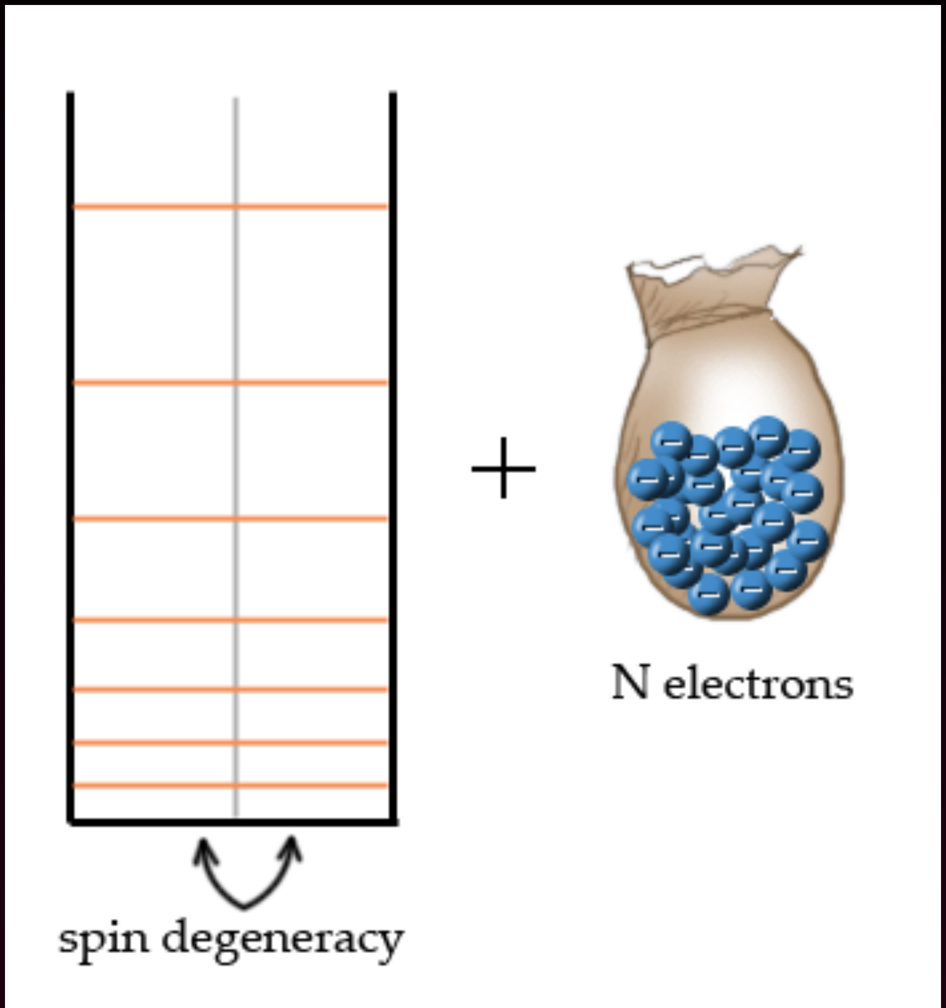
one-dimension



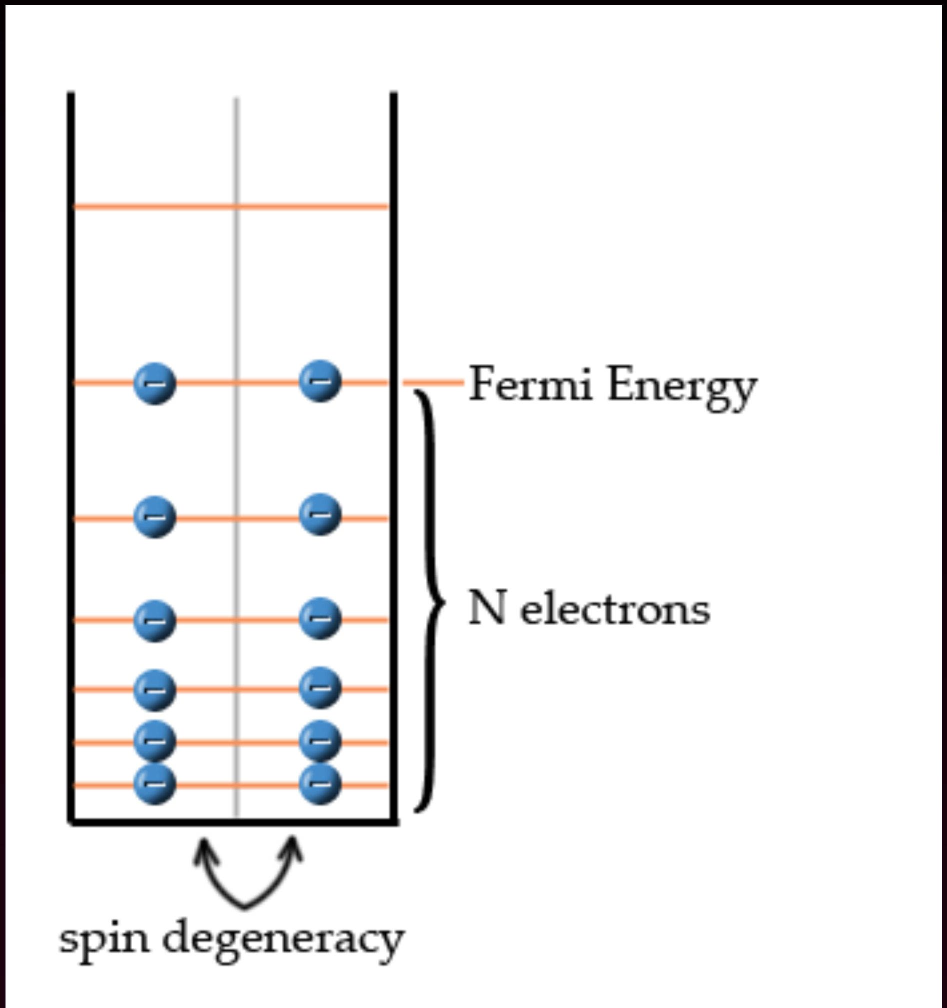
$$p_F = (3\pi^2\hbar^3 n)^{1/3}$$

$$E_F = p_F^2/2m$$

Fermi momentum and energy



Pauli exclusion principle



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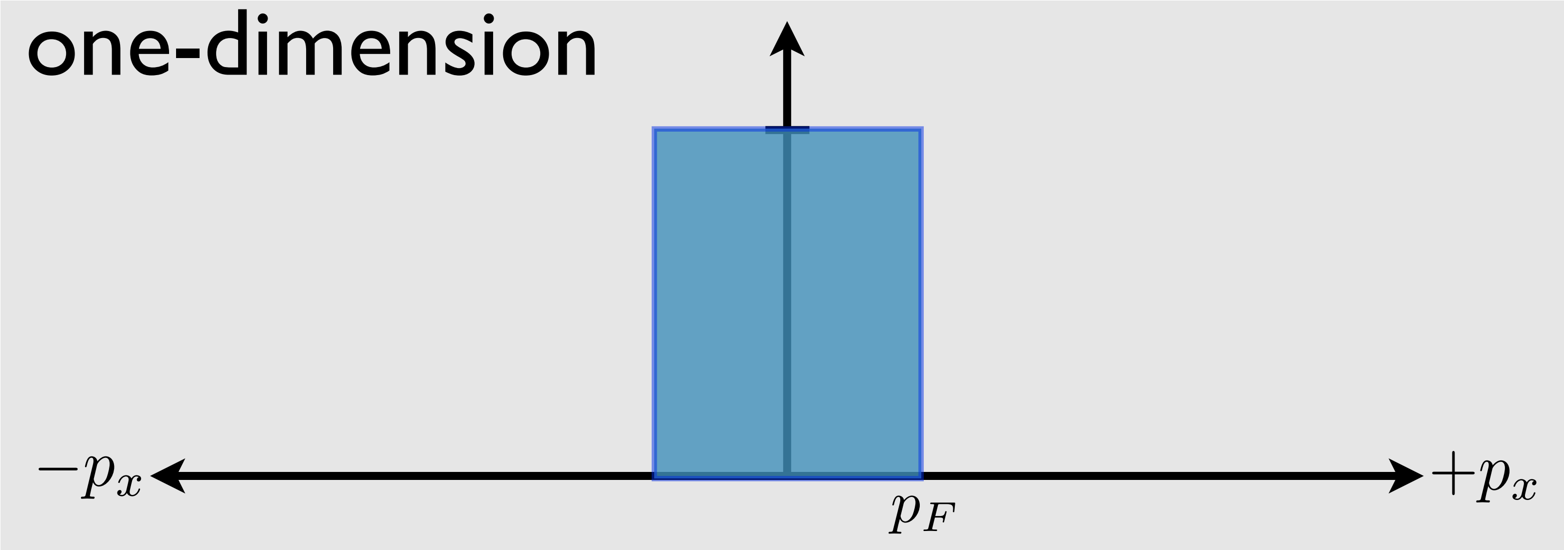

$$p_F = (3\pi^2\hbar^3 n)^{1/3}$$

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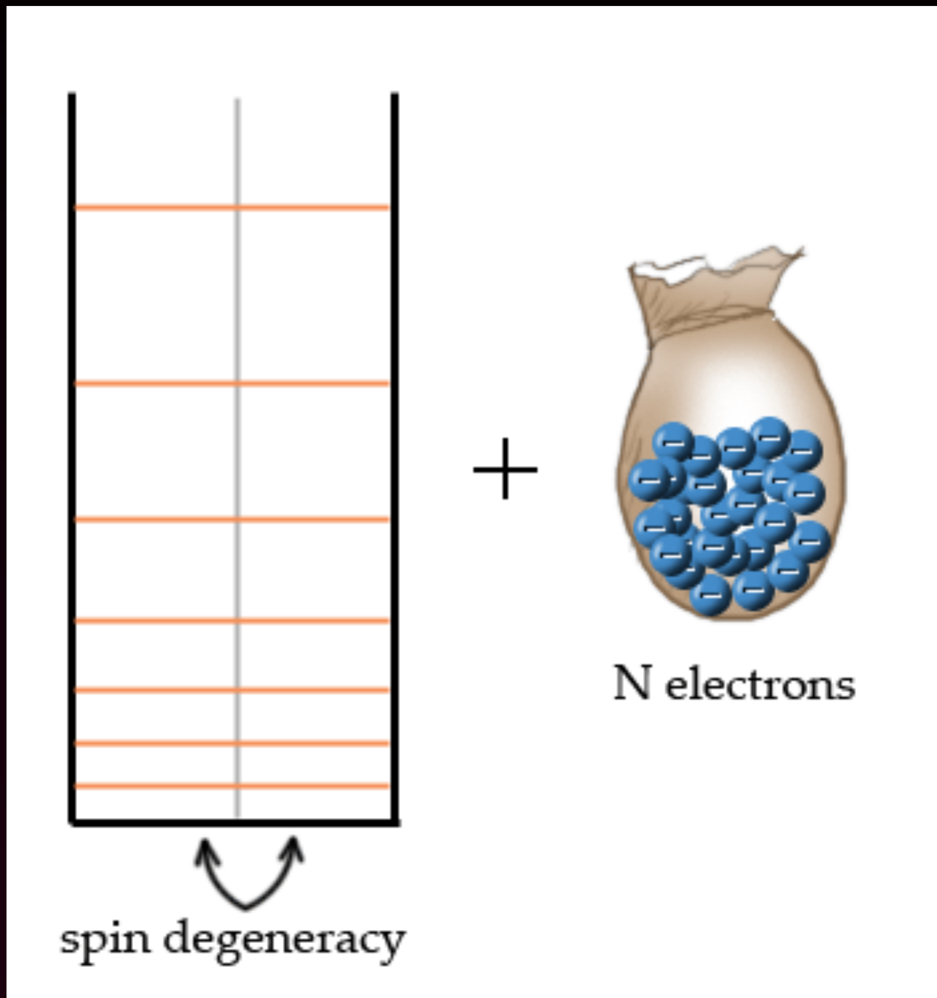
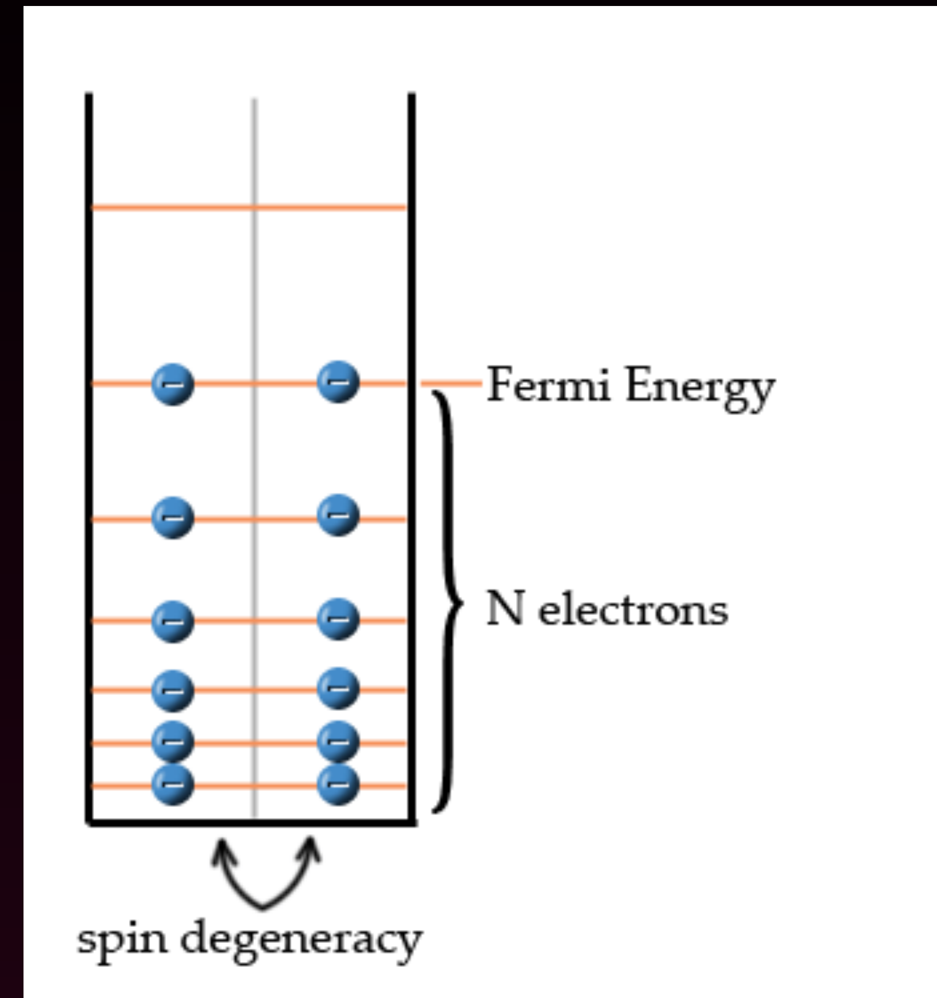
Fermi momentum and energy



Pauli exclusion principle



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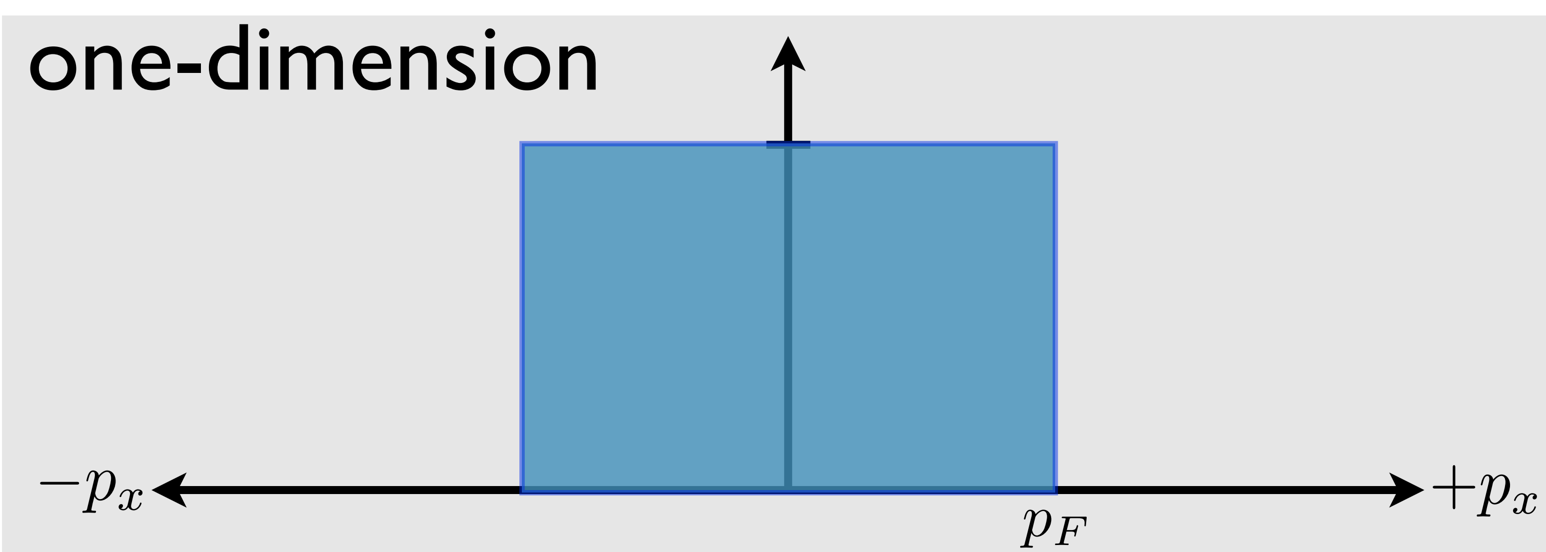
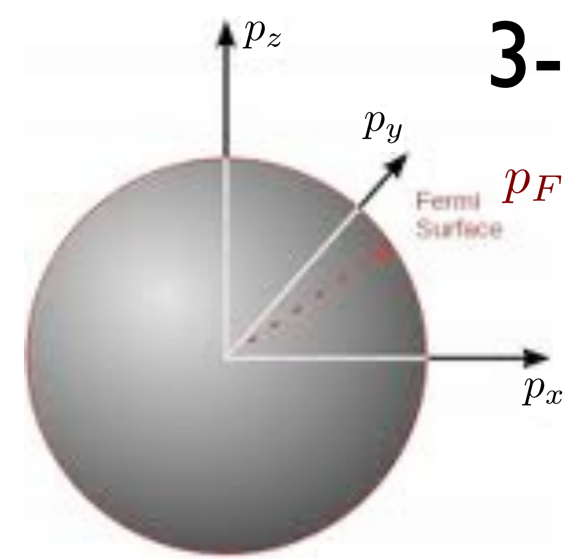
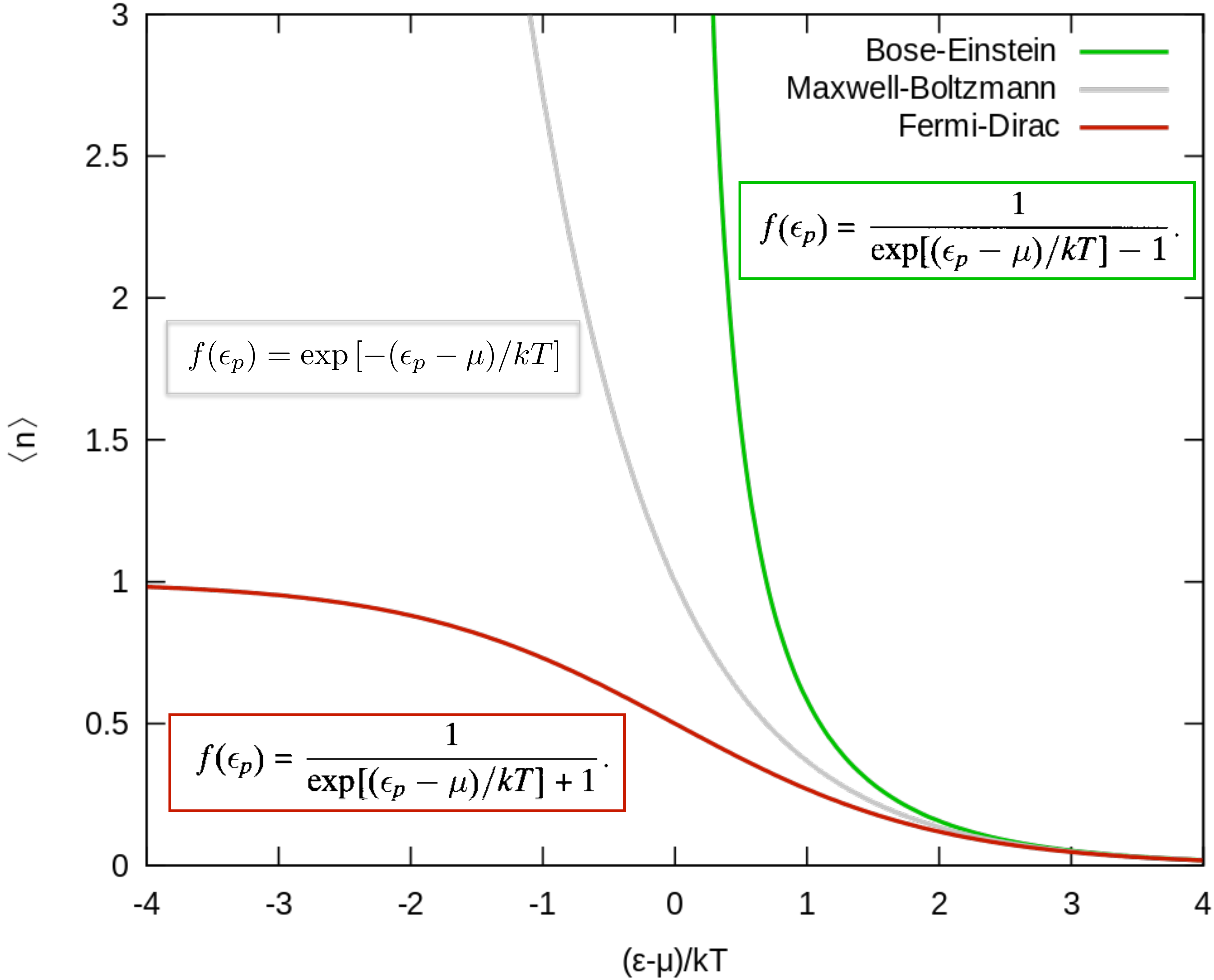


$$p_F = (3\pi^2\hbar^3 n)^{1/3}$$

3-d



$$E_F = p_F^2/2m$$

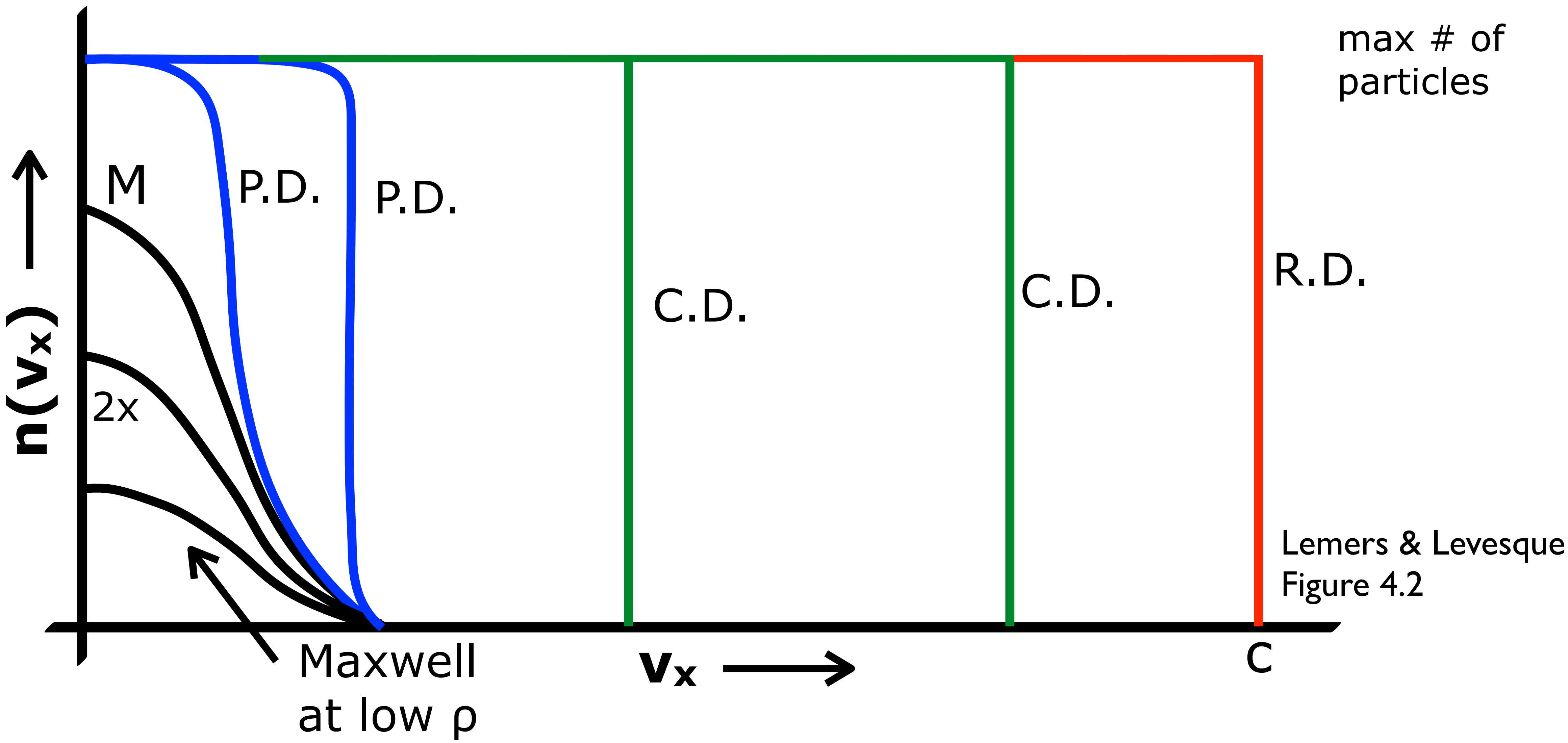


Bose-Einstein
Maxwell-Boltzmann
Fermi-Dirac
$f(\epsilon_p) = \dfrac{1}{\exp[(\epsilon_p - \mu)/kT] - 1}.$
$f(\epsilon_p) = \exp\left[-(\epsilon_p - \mu)/kT\right]$
$f(\epsilon_p) = \dfrac{1}{\exp[(\epsilon_p - \mu)/kT] + 1}.$
⟨n⟩
(ε-μ)/kT
3
2.5
2
1.5
1
0.5
0
-4
-3
-2
-1
0
1
2
3
4

Equation of State

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…



M: Maxwellian
P.D.: partial degeneracy

C.D.: complete degeneracy
R.D.: relativistic degeneracy

complete degeneracy (non-relativistic)

4.5.1 Nonrelativistic Complete Degeneracy (CD)

In the case of complete degeneracy, the $n(p)$ distribution is rectangular for $p < p_F$. So the electron distribution is described by

$$n_e(p)\, d^3p = \frac{2}{h^3}(4\pi p^2\, dp) \qquad \text{if } p < p_F \tag{4.16a}$$

$$n_e(p)\, d^3p = 0 \qquad \text{if } p \geqslant p_F. \tag{4.16b}$$

Using these equations, we can derive the Fermi momentum for a total electron density n_e.

$$n_e = \int_0^{p_F} n_e(p)\; d^3p = \int_0^{p_F} \frac{2}{h^3}\; 4\pi p^2\; dp = \frac{8\pi}{3h^3}\, p_F^3. \tag{4.17}$$

So

$$p_F = \left(\frac{3h^3\, n_e}{8\pi}\right)^{1/3}. \tag{4.18}$$

Now we can find P_e, with $v = p/m_e$

$$P_e = \frac{1}{3}\int p\ \mathrm{v}\ n_e(p)\, dp = \frac{1}{3}\int_0^{p_F} \frac{p^2}{m_e}\frac{2}{h^3}\; 4\pi p^2\; dp = \frac{8\pi}{15h^3} p_F^5. \tag{4.19}$$

$$P = K_{NR} n^{5/3}, \quad \text{where} \quad K_{NR} = \frac{h^2}{5m}\left[\frac{3}{8\pi}\right]^{2/3}.$$

note that pressure is independent of temperature!
only depends on density: *polytrope* with $\gamma = 5/3 \leftrightarrow n = 3/2$

complete degeneracy (ultra-relativistic)

$$P = K_{UR} n^{4/3}, \quad \text{where} \quad K_{UR} = \frac{hc}{4}\left[\frac{3}{8\pi}\right]^{1/3}.$$

pressure is still independent of temperature!
still only depends on density: polytrope with $\gamma = 4/3 \leftrightarrow n = 3$

Equation of State

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…

C.D. - COMPLETE DEGENERACY

$$P_e \sim n_e^{5/3} \text{ (independent of T!)}$$

R.D. - RELATIVISTIC DEGENERACY

$$P_e \sim n_e^{4/3} \text{ (independent of T!)}$$

P.D. - PARTIAL DEGENERACY

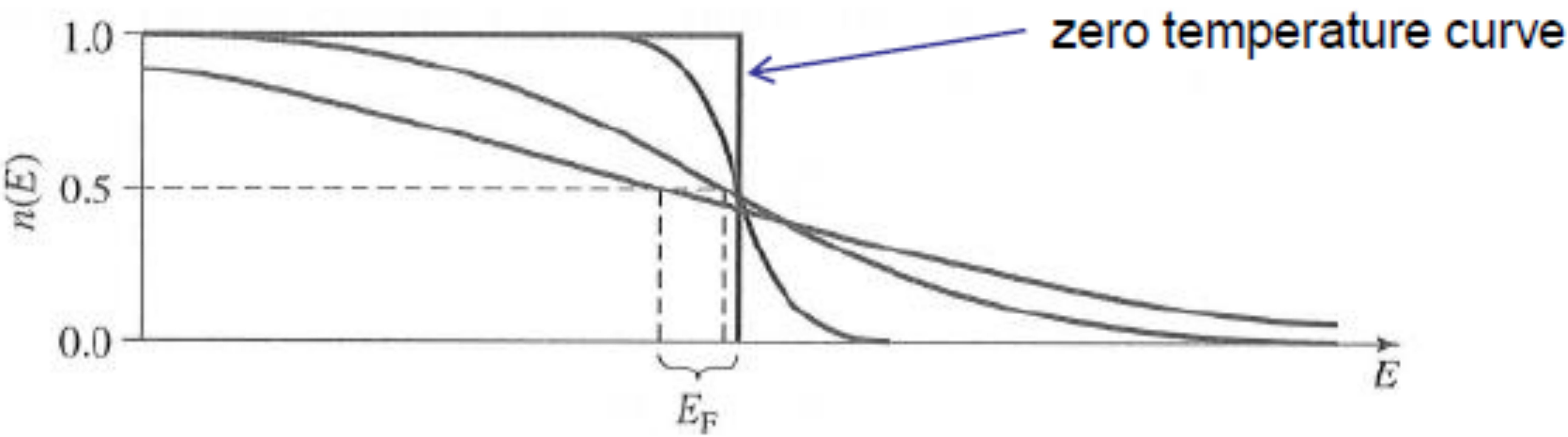
$$P_e \sim \frac{8\pi}{3h^3}(2m_e kT)^{3/2} \cdot kT \cdot F_{3/2}(\psi)$$

n(p) is not a rectangular profile; has a Maxwell tail…

Note: all of this is for e^- degeneracy…

partial degeneracy (non-zero temperature)

$$n(E) = \frac{e^{(E_F - E)/k_B T}}{1 + e^{(E_F - E)/k_B T}} = \frac{1}{e^{(E - E_F)/k_B T} + 1}$$

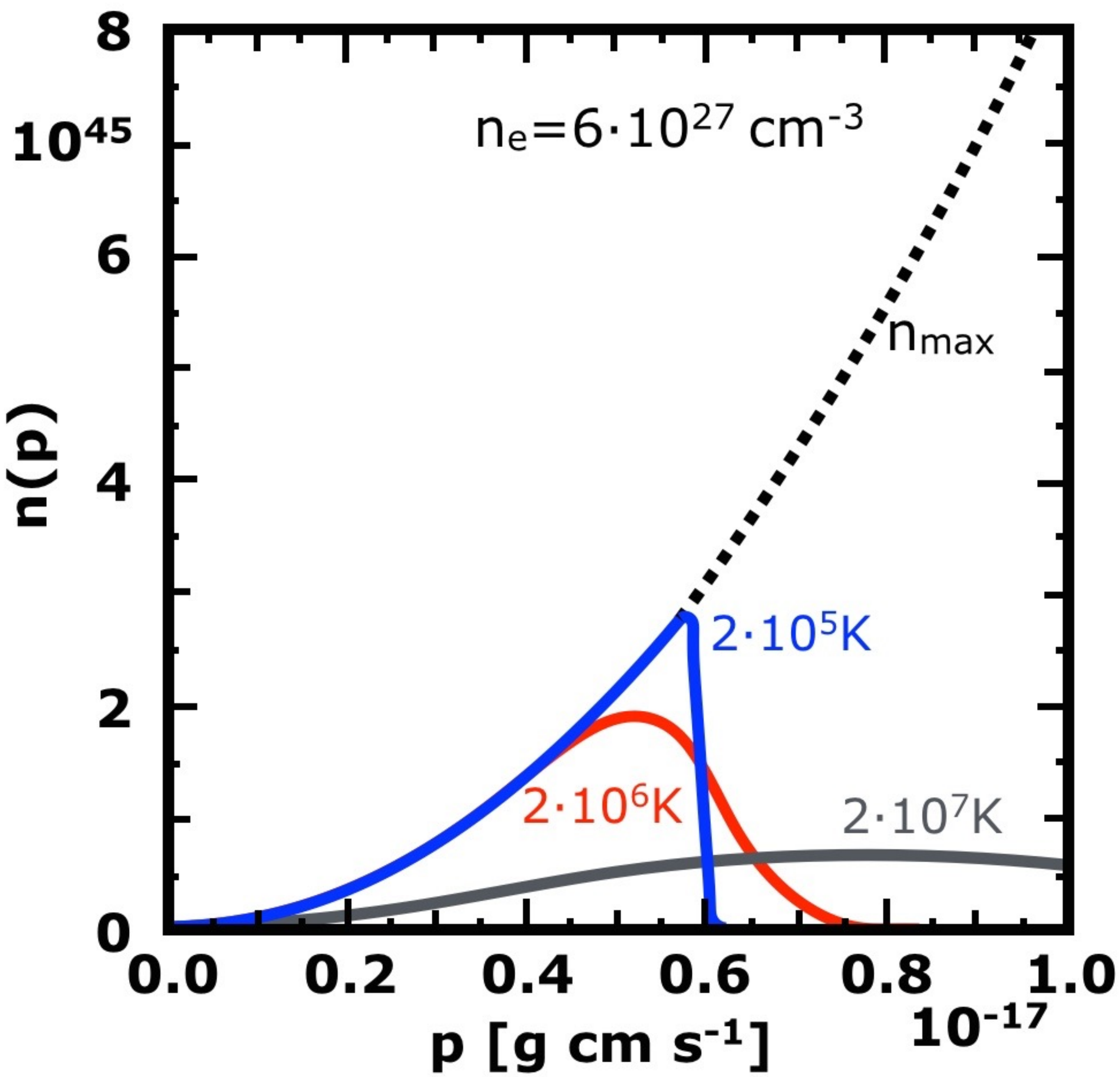


from https://www.doitpoms.ac.uk/tlplib/semiconductors/chemical_potential.php

Equation of State

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state...

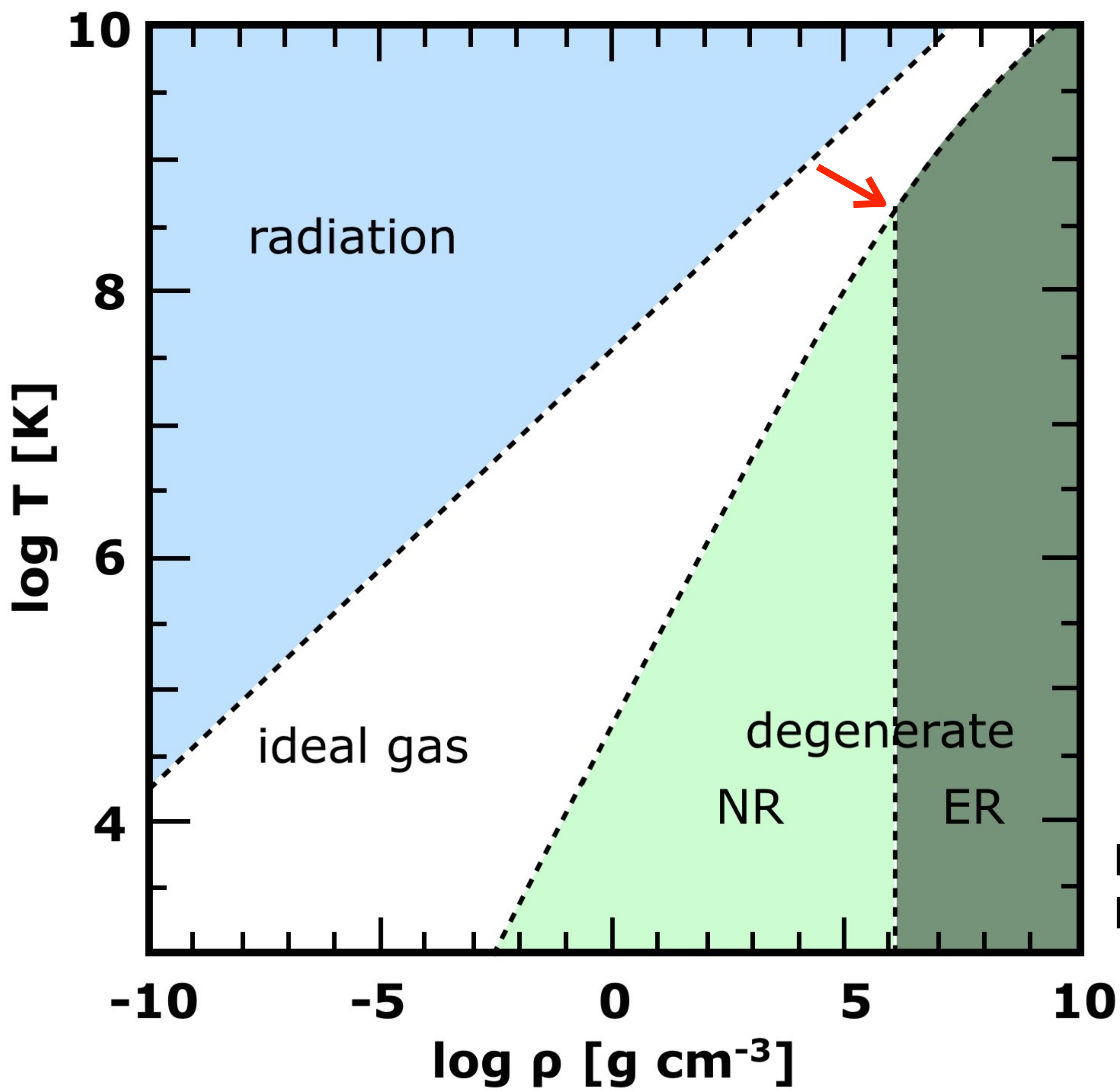


Lemers & Levesque Fig. 4.3

Equation of State

Degeneracy

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…



Lamers & Levesque
Figure 4.4

- equation of state for completely degenerate e^-
- transition from 5/3 to 4/3 slope is smooth, but takes place around $\rho_{tr} \approx 10^6/\mu_e$ g cm^{-3}

Adapted from Onno Pols, 2011.

Equation of State

4.6 The Equation of State (EoS) for Electron Gas

We have seen that the pressure of CD electron gas has two limiting cases. In the nonrelativistic case, $P_e = K_1 \times {n_e}^{5/3}$, and in the relativistic case, $P_e = K_2 \times {n_e}^{4/3}$. We can derive the electron density ${n_e}^{\mathrm{crit}}$ at which the gas goes from CD to RD by requiring

$$\begin{aligned} P_e(CD) = P_e(\mathrm{RD}) \rightarrow K_1(\rho/\mu_e)^{5/3} = K_2(\rho/\mu_e)^{4/3} \rightarrow \\ \rightarrow (\rho/\mu_e)_{\mathrm{crit}} = (K_2/K_1)^3 = 1.91 \times 10^6\ \mathrm{g\ cm^{-3}}. \end{aligned} \tag{4.24}$$

Similarly, the boundary between the ideal gas law and CD follows from

$$n_e\ kT = K_1(\rho/\mu_e)^{5/3} \rightarrow T = \frac{K_1 m_{\mathrm{H}}}{k}(\rho/\mu_e)^{2/3} = 1.21 \times 10^5\ (\rho/\mu_e)^{2/3}\ \mathrm{K}. \tag{4.25}$$

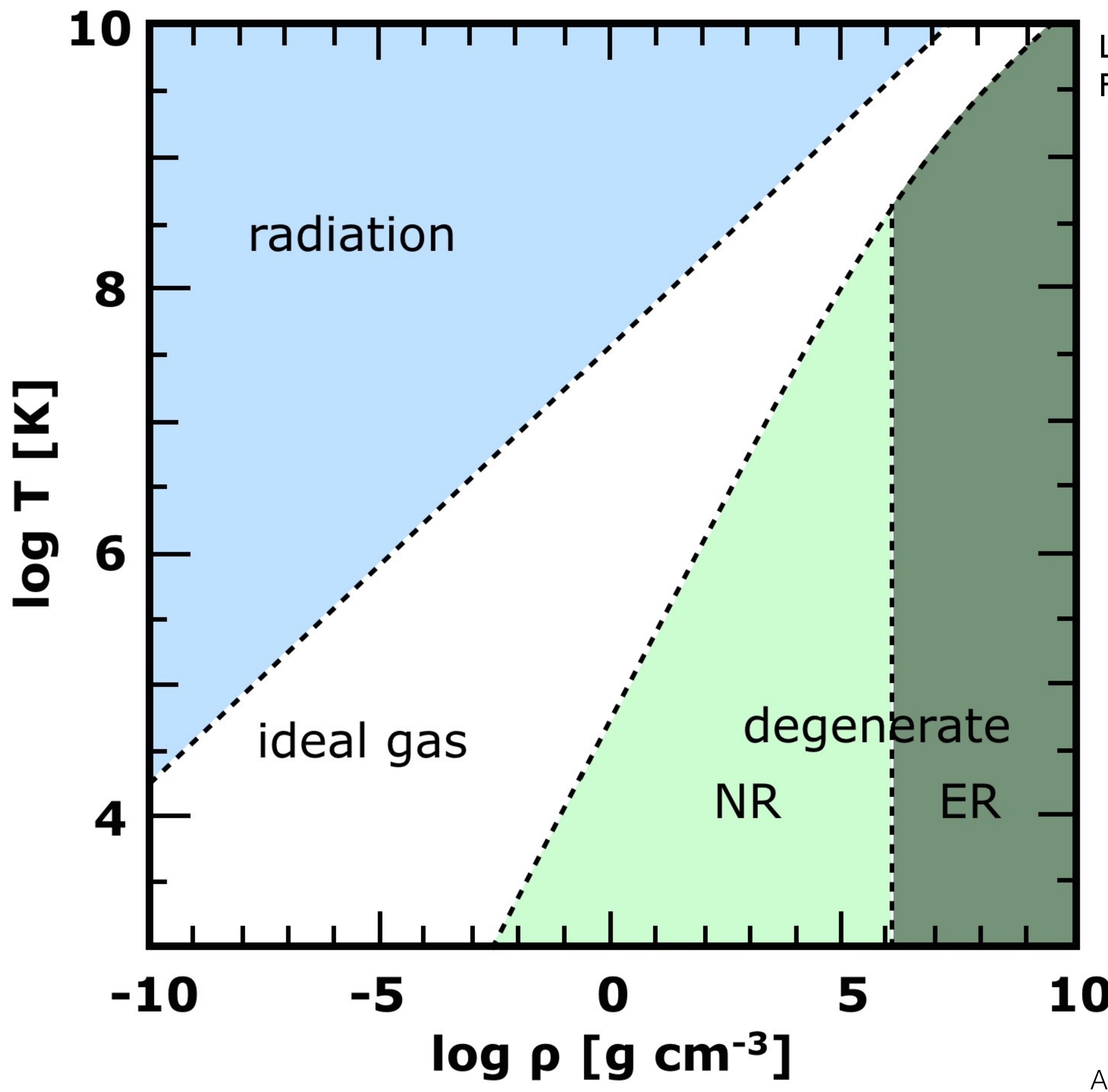
The boundary between the ideal gas law and RD follows from

$$n_e\ kT = K_2(\rho/\mu_e)^{4/3} \rightarrow T = \frac{K_2 m_{\mathrm{H}}}{k}(\rho/\mu_e)^{1/3} = 1.50 \times 10^7\ (\rho/\mu_e)^{1/3}\ \mathrm{K}. \tag{4.26}$$

The boundary between radiation pressure and the ideal gas law follows from

$$n_e\ kT = \frac{a}{3}T^4 \rightarrow T = \left(\frac{3k}{a m_{\mathrm{H}}}\right)^{1/3}(\rho/\mu_e)^{1/3} = 3.20 \times 10^7\ (\rho/\mu_e)^{1/3}\ \mathrm{g\ cm^{-3}}. \tag{4.27}$$

Equation of State

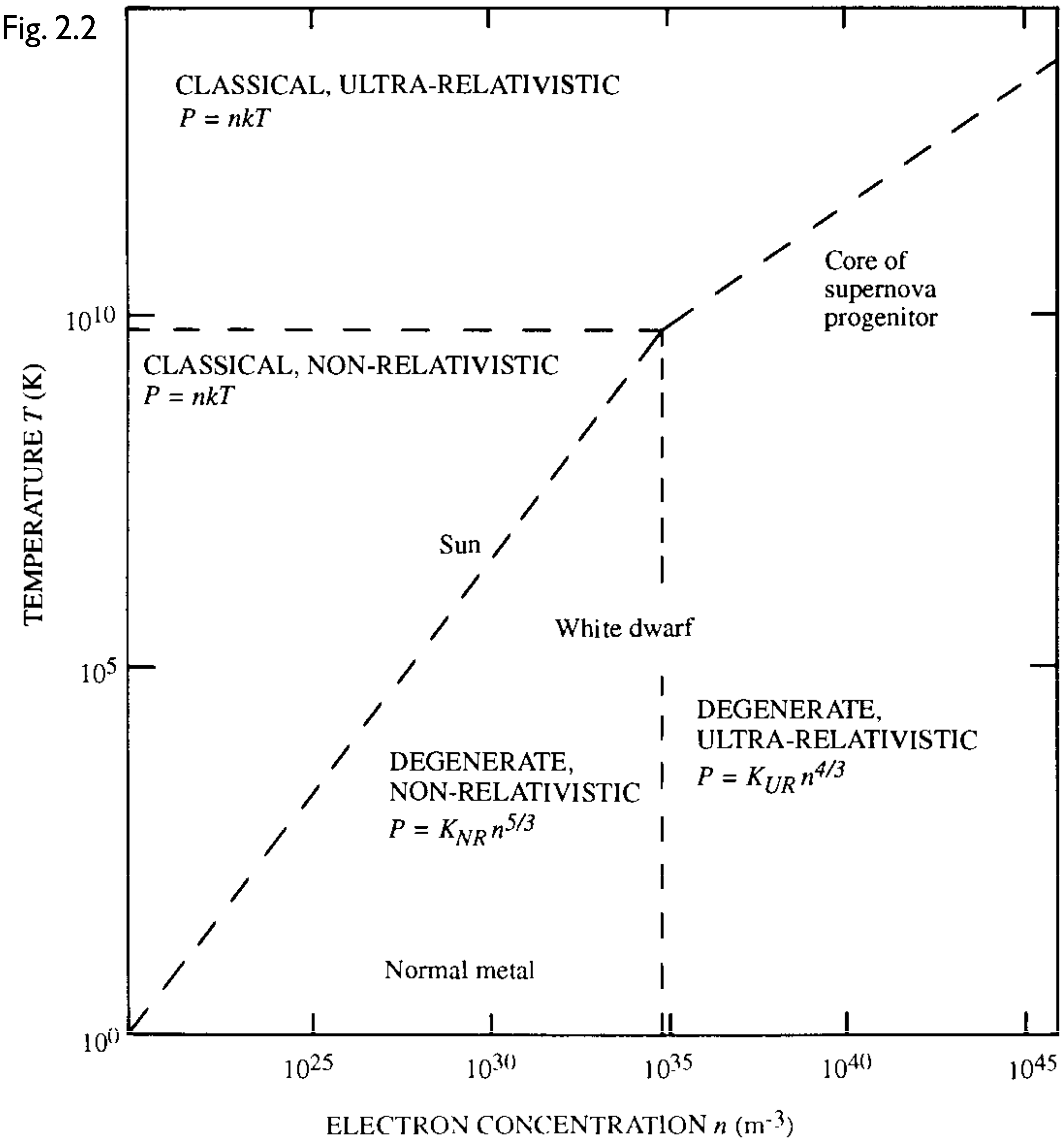


Lamers & Levesque
Figure 4.4

Adapted from Onno Pols, 2011.

Equation of State

Phillips Fig. 2.2



Neutron Degeneracy

<u>Degeneracy</u>

At high ρ (or low T!) quantum mechanical effects become important and changes the equation of state…

At $\rho > 3\times10^7$ g cm^{-3} the Fermi energy of the degenerate electrons overcomes the energy difference between n and p (1.29 MeV). $p+e \rightarrow n+\nu$ and n density increases dramatically.

Similar to electrons:

$$P_{\rm n}({\rm CD}) = K_{1,{\rm n}}(\rho/\mu_{\rm n})^{5/3} \quad \text{for } 3\times10^7 < \rho < 6\times10^{15} \ \ {\rm g/cm^3}$$

$$P_{\rm n}({\rm RD}) = K_{2,{\rm n}}(\rho/\mu_{\rm n})^{4/3} \quad \text{for } \rho > 6\times10^{15} \ \ {\rm g/cm^3}$$

The similarity of these equations to those for electron degeneracy impose similar rules on white dwarfs and neutron stars (both have mass-radius relations and maximum stable masses…)