Physics 444: Midterm
October 31, 2019

Name:

You have 80 minutes for this exam. You may use a calculator and 1 pre-prepared sheet of equations. Each question is worth 10 points. Partial credit is given, so be sure to explain your reasoning. Unless explicitly stated, do not assume the parameters of the Universe assumed in a problem are those of the Benchmark Universe.

1. Suppose you lived in a spatial flat universe filled only with photons in a Maxwell-Boltzmann distribution with temperature T that changes with time and the scale factor. What is the Hubble parameter in terms of fundamental constants and the temperature T?

   The Friedmann equation for a flat universe is

   $$H(T)^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon.$$

   Substituting the energy density of a Maxwell-Boltzmann distribution of massless particles, we find

   $$\epsilon = \frac{\pi^2}{15}\frac{(kT)^4}{\hbar^3 c^3},$$

   therefore,

   $$\begin{aligned} H(T)^2 &= \frac{8\pi G}{3c^2}\epsilon \\ &= \frac{8\pi G}{3c^2}\frac{\pi^2}{15}\frac{(kT)^4}{\hbar^3 c^3} \\ &= \frac{8\pi^3 G}{45\hbar^3 c^5}(kT)^4 \\ H(T) &= \left(\frac{8\pi^3 G}{45\hbar^3 c^5}\right)^{1/2}(kT)^2. \end{aligned}$$

2. In general, we assume that energy densities must have $|w| \leq 1$, but let us consider for a moment a spatially-flat universe filled with energy density that has $w = +2$.

   (a) If you had a box full of this material, with dimensions of the box that changed with the scale factor a, how much energy would be in the box when $a = 1/3$ as compared to when $a = 1$?

   The energy density scales as

$$\epsilon = \epsilon_0 a^{-3(1+w)} = \epsilon_0 a^{-9}.$$

   while the volume scales as

$$V = V_0 a^3.$$

   When $a = 0$, the enclosed energy is

$$E(a = 1) = \epsilon_0 V_0.$$

   When $a = 1/3$, the energy is

$$\begin{aligned} E(a = 1/3) &= \epsilon_0 a^{-9} V_0 a^3 \\ &= \epsilon_0 V_0 a^{-6} \\ &= 729 \epsilon_0 V_0. \end{aligned}$$

   Thus, the box has 729 times as much energy when the size of each side was 1/3rd the current value.

   (b) From the Friedmann Equation, determine the time-dependence of a for this universe. (You must prove this time-dependence, not just assert an answer from your equation sheet)

From the Friedmann equation,

$$
\begin{aligned}
\frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3c^2}\epsilon \\
&= \frac{8\pi G}{3c^2}\epsilon_0 a^{-9} \\
\dot{a} &= \sqrt{\frac{8\pi G}{3c^2}\epsilon_0}a^{-7/2} \\
\int_0^a a^{7/2}da &= \sqrt{\frac{8\pi G}{3c^2}\epsilon_0}\int_0^t dt \\
\frac{2a^{9/2}}{9} &= \sqrt{\frac{8\pi G}{3c^2}\epsilon_0}t \\
a &= \left(\frac{9}{2}\sqrt{\frac{8\pi G}{c^2}\epsilon_0}t\right)^{2/9}
\end{aligned}
$$

(This is the same power as you'd have gotten from the equation sheet: you could have plugged in the $t^{2/9}$ dependence into the Friedmann Equation to demonstrate that it works.)

3. Consider a spatially-flat universe filled with matter, with a present-day Hubble parameter of $H_0 = 70$ km/s/Mpc. In this universe, there is a galaxy located at a comoving coordinate of $r = 1$ Gpc from you.

   (a) A supernova went off in this galaxy, which is seen by you today (time t_0.) At what time t_e did the supernova occur?

   Since this is a matter-dominated Universe, $a(t) = (t/t_0)^{2/3}$ and $t_0 = 2/3H_0^{-1}$. The proper distance between the galaxy and us as a function of t_e can be written as

$$
\begin{aligned}
d_p(t_0, t_e) &= c\int_{t_e}^{t_0} \frac{dt}{a(t)} \\
&= ct_0^{2/3}\int_{t_e}^{t_0} t^{-2/3}dt \\
&= 3ct_0^{2/3}(t_0^{1/3} - t_e^{1/3}) \\
&= 3ct_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right].
\end{aligned}
$$

   This must be equal to the proper distance today, $r = 1$ Gpc. Note that

$$
\begin{aligned}
t_0 &= 2/3H_0^{-1} \\
&= \frac{2}{3}(70 \times 3.24 \times 10^{-20}\ \mathrm{s}^{-1})^{-1} \\
&= 2.94 \times 10^{17}\ \mathrm{s} = 9.33 \times 10^9\ \text{years}.
\end{aligned}
$$

   Therefore,

$$
\begin{aligned}
1\ \mathrm{Gpc} = 3.26 \times 10^9\ \mathrm{ly} &= (3 \times 9.33 \times 10^9\ \mathrm{ly})\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right] \\
1 - \left(\frac{t_e}{t_0}\right)^{1/3} &= 0.116 \\
t_e/t_0 &= (1 - 0.116)^3 = 0.690 \\
t_e &= 0.690 \times 9.33 \times 10^9\ \text{years} \\
&= 6.43 \times 10^9\ \text{years}
\end{aligned}
$$

(b) What is the luminosity distance of this supernova?

The luminosity distance is

$$d_L = (1+z)r = (1+z)(1 \text{ Gpc}),$$

so now we just need z.

We see that

$$\begin{aligned} \frac{1}{1+z} &= a \\ &= (0.690)^{2/3} \\ &= 0.781 \\ 1+z &= 1.28. \end{aligned}$$

Therefore

$$d_L = (1+z)r = 1.28 \text{ Gpc})$$

4. In the **Benchmark Universe**, what was the Hubble parameter at matter-radiation equality?

   Matter-radiation equality occurred when

$$
\begin{aligned}
\Omega_r &= \Omega_m \\
\Omega_{r,0} a^{-4} &= \Omega_{m,0} a^{-3} \\
a_{rm} &= \frac{\Omega_{r,0}}{\Omega_{m,0}} \\
&= \frac{9.03 \times 10^{-5}}{0.306} = 2.95 \times 10^{-4}.
\end{aligned}
$$

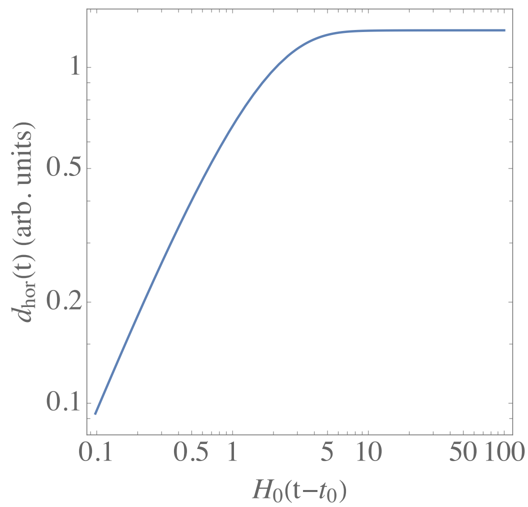
   The Hubble parameter evolves, via the Friedmann Equation, as

$$
\begin{aligned}
H(t)^2 &= \frac{8\pi G}{3c^2} \sum \epsilon_w \\
&= \frac{8\pi G}{3c^2} \sum \epsilon_{w,0} a^{-3(1+w)} \\
&= H_0^2 \sum_w \Omega_{w,0} a^{-3(1+w)} \\
&= H_0^2 \left(\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0} \right).
\end{aligned}
$$

   Therefore,

$$
\begin{aligned}
H_{rm}^2 &= H_0^2 \left[(9.03 \times 10^{-5})(2.95 \times 10^{-4})^{-4} + (0.306)(2.95 \times 10^{-4})^{-3} + 0.692 \right) \\
&= (2.38 \times 10^{10}) H_0^2 \\
H_{rm} &= (1.54 \times 10^5) H_0 = 1.05 \times 10^7 \text{ km/s/Mpc}.
\end{aligned}
$$

5. Below is a plot of the distance today ($t = t_0$) to the furthest object that can be seen at a future time t. Does this universe contain dark energy (i.e., $\Omega_\Lambda \neq 0$)? Explain your reasoning. (Do **not** assume this universe is the Benchmark Universe)



For a universe with dark energy, we know that, in the far future, all other forms of energy density will dilute away. At that point, the dark energy will dominate, and

$$a(t) \propto e^{H_0(t-t_0)}.$$

As a result, the present-day distance to the particle horizon will asymptote to

$$\begin{aligned} d_{\rm hor}(t) &= c\int_{t_0}^{t} \frac{dt}{a(t)} \\ &= c\int_{t_0}^{t} dt e^{-H_0(t-t_0)} \\ &= cH_0^{-1}\left(1 - e^{-H_0(T-t_0)}\right) \end{aligned}$$

As T increases, this will approach a constant value, which matches what we see in the plot. Thus, this universe must contain some amount of dark energy.

Cheat Sheet

Physical constants and useful conversions:

$$
\begin{aligned}
c &= 3\times 10^{8}\ \text{m/s} \\
G &= 6.67\times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2} \\
k &= 8.62\times 10^{-5}\ \text{eV/K} \\
h &= 6.63\times 10^{-34}\ \text{Js} = 4.14\times 10^{-15}\ \text{eVs} \\
1\ \text{Mpc} &= 3.09\times 10^{22}\ \text{m} \\
1\ \text{J} &= 6.24\times 10^{18}\ \text{eV} \\
1\ \text{km/s/Mpc} &= 3.24\times 10^{-20}\ \text{s}^{-1} \\
1\ \text{year} &= 3.15\times 10^{7}\ \text{s} \\
1\ \text{parsec} &= 3.26\ \text{ly} \\
1\, M_\odot &= 2\times 10^{30}\ \text{kg} \\
1\, L_\odot &= 3.83\times 10^{26}\ \text{W} = 3.83\times 10^{33}\ \text{erg/s}
\end{aligned}
$$

Benchmark Universe:

$$T_0 = 2.7255\ \text{K},\ H_0 = 67.8\ \text{km/s/Mpc},\ t_0 = 13.8\ \text{Gyr}, \epsilon_{c,0} = 4870\ \text{MeV/m}^3,$$
$$\Omega_{\Lambda,0} = 0.692,\ \Omega_{m,0} = 0.306,\ \Omega_{r,0} = 9.03\times 10^{-5}$$

Equations:

$$z = \frac{\lambda_{\rm obs} - \lambda_{\rm emit}}{\lambda_{\rm emit}}, \qquad z = \frac{H}{c} r \text{ (Hubble's Law)}, \ 1 + z = \frac{1}{a}$$

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa^2 d\Omega^2\right], \qquad S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0) & \kappa = +1 \\ r & \kappa = 0 \\ R_0 \sinh(r/R_0) & \kappa = -1 \end{cases}$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{\kappa c^2}{R_0^2 a^2}, \qquad 1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}$$

$$\dot\epsilon + \frac{3\dot a}{a}(\epsilon + P) = 0, \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3c^3}(\epsilon + 3P)$$

$$P = w\epsilon, \qquad \epsilon_w = \epsilon_{w,0} a^{-3(1+w)}$$

$$\epsilon_\gamma(T) = \frac{\pi^2}{15}\frac{(kT)^4}{\hbar^3 c^3} \qquad n_\gamma(T) = \frac{2.4041}{\pi^2}\frac{(kT)^3}{\hbar^3 c^3}$$

$$\epsilon_c = \frac{3c^2}{8\pi G} H^2 \qquad \Omega_w = \frac{\epsilon_w}{\epsilon_c}$$

$$d_p(t_0, t_e) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \qquad d_p(t_e) = c a(t_e) \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{w,0} a^{-(1+3w)} + (1 - \Omega_0)}}, \qquad a(t) = \begin{cases} \frac{t}{t_0} & \Omega_0 = 0, \kappa = -1 \\ (t/t_0)^{2/3(1+w)} & \Omega_{w,0} = 1, \kappa = 0 \\ e^{H_0(t - t_0)} & \Omega_\Lambda = 1, \kappa = 0 \end{cases}$$

$$t_0 = \frac{2}{3(1+w)} H_0^{-1} \qquad (\Omega_{w,0} = 1, \kappa = 0, w > -1)$$

$$d_A = \frac{\ell}{\delta\theta} = \frac{S_\kappa(r)}{1+z} \qquad d_L = (1+z) S_\kappa(r)$$

$$d_L(z) \approx \frac{cz}{H_0}\left[1 + \left(\frac{1 - q_0}{2}\right) z\right] \qquad q_0 = \frac{1}{2}\sum_w \Omega_{w,0}(1 + 3w)$$