Physics 444: Problem Set #4
due October 6, 2021

1. *Consider Einstein's static universe, in which the attractive force of the matter density ρ is exactly cancelled by the repulsive force of the cosmological constant $\Lambda = 4\pi G\rho$. Suppose that some of the matter is converted into radiation (by stars, for instance). Will the universe start to expand or contract? Explain your answer.*

   Using the acceleration equation,

   $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{\Lambda}{3},$$

   we see that if the Universe is composed only of matter ($\epsilon = \rho c^2$ and $P = 0$) with $\Lambda = 4\pi G\rho$, then

   $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{\Lambda}{3} = -\frac{4\pi G}{3c^2}\left(\rho c^2\right) + \frac{4\pi G\rho}{3} = 0.$$

   Thus, the acceleration of the scale factor is zero, and the universe is indeed static. Now, we transform some amount of matter density into relativistic photons: $\rho = \rho - \delta\rho$, $\epsilon_\gamma = \delta\rho c^2$. These photons have $P = w\epsilon_\gamma = \delta\rho c^2/3$. Therefore,

   $$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3c^2}\left[(\rho - \delta\rho)c^2 + \epsilon_\gamma + 3 \times \epsilon_\gamma/3\right] + \frac{4\pi G\rho}{3} \\ &= -\frac{4\pi G}{3c^2}\left[(\rho - \delta\rho)c^2 + \delta\rho c^2 + \delta\rho c^2\right] + \frac{4\pi G\rho}{3} \\ &= -\frac{4\pi G}{3c^3}\delta\rho c^2.\end{aligned}$$

   That is, now that some of the energy resides in radiation, the $\ddot{a}$ of the universe is negative. This means the scale factor is decreasing: the universe is starting to contract. The occurs because, even though the cosmological constant is still perfectly canceling the energy in the matter plus radiation, it is not canceling the acceleration caused by the photon pressure.

2. *The current limit on the sum of the masses of the three neutrino species is $\sum m_\nu \leq 0.23$ eV/c^2 (this limit comes from measurements of the*

Universe's properties in early times, as we will see). The two mass-squared differences between the three species of neutrinos are known to be

$$\Delta m_{12}^2 = 7.53 \times 10^{-5}\ eV^2/c^4,\ \Delta m_{23}^2 = 2.44 \times 10^{-3}\ eV^2/c^4.$$

Here Δm_{ij}^2 means $\left|m_i^2 - m_j^2\right|$. (These values come from neutrino oscillation measurements on Earth.)

(a) *Assuming $m_3 > m_2 > m_1$ (this is ordering is actually not known), what are the minimum neutrino masses compatible with the measured mass squared differences?*

The minimum possible mass sum occurs if one of the neutrino masses is zero, so $m_1 = 0$. Then

$$m_2^2 = \Delta m_{12}^2 = 7.53 \times 10^{-5}\ \mathrm{eV}/c^4,$$

and

$$m_2 = \sqrt{7.53 \times 10^{-5}\ \mathrm{eV}^2/c^4} = 8.78 \times 10^{-3}\ \mathrm{eV}/c^2.$$

Then,

$$\begin{aligned} m_3^2 - m_2^2 &= 2.44 \times 10^{-3}\ \mathrm{eV}^2/c^4 \\ m_3^2 &= 2.44 \times 10^{-3}\ \mathrm{eV}^2/c^4 + 7.53 \times 10^{-5}\ \mathrm{eV}^2/c^4 \\ m_3 &= 0.0502\ \mathrm{eV}/c^2. \end{aligned}$$

(b) *When the temperature of the CMB is $T_{\mathrm{CMB}} \sim 5.8 \times 10^6\ K = 500\ eV$, the neutrinos are all relativistic (meaning you can act as if their masses are all zero), and have a temperature of $(4/11)^{1/3}T_{\mathrm{CMB}}$. The energy density of a relativistic fermion (such as a neutrino) is*

$$\epsilon(T) = \frac{\pi^2}{15}\frac{k^4}{\hbar^3 c^3}\frac{7}{8}T^4,$$

and the number density is

$$n(T) = \frac{2.404}{\pi^2}\frac{k^3}{\hbar^3 c^3} \times \frac{3}{4}T^3.$$

Note the factors of 7/8 and 3/4 relative to the text's equations (2.26-2.27) and (2.28-2.29). What is the energy density and number density of neutrinos (summing over all three species) when the

Universe's temperature is 500 eV? Compare this to the energy density and number density of photons at this time.

Comparing to Eq. (2.27) of the text, we see that the energy density of a single species of relativistic neutrino is

$$\epsilon_1 = \frac{7}{8}\alpha T^4,$$

so, using $T_\nu = (4/11)^{1/3} T_{\rm CMB}$,

$$\begin{aligned}\epsilon_1 &= \left(7.56\times10^{-16}\ \mathrm{Jm^{-3}K^{-4}}\right)\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\left(5.8\times10^{6}\ \mathrm{K}\right)^4 \\ &= 1.94\times10^{11}\ \mathrm{J/m^3} = 1.21\times10^{24}\ \mathrm{MeV/m^3}.\end{aligned}$$

The number density of this species of neutrino is

$$n_1 = \frac{3}{4}\beta T^3,$$

so

$$\begin{aligned}n_1 &= \frac{3}{4}\left(2.03\times10^{7}\ \mathrm{m^{-3}K^{-3}}\right)\frac{3}{4}\left(\frac{4}{11}\right)\left(5.8\times10^{6}\ \mathrm{K}\right)^3 \\ &= 1.08\times10^{27}\ \mathrm{m^{-3}}.\end{aligned}$$

However, these are just the energy and number densities of a single species of neutrino. There are three species of neutrino, so

$$\begin{aligned}\epsilon_\nu &= 3\epsilon_1 = 5.83\times10^{11}\ \mathrm{J/m^3} = 3.63\times10^{24}\ \mathrm{MeV/m^3} \\ n_\nu &= 3.24\times10^{27}\ \mathrm{m^{-3}}.\end{aligned}$$

The energy density and number density of the CMB photons at $T = 500$ eV are

$$\begin{aligned}\epsilon_{\rm CMB} &= \alpha T_{\rm CMB}^4 \\ &= \left(7.56\times10^{-16}\ \mathrm{Jm^{-3}K^{-4}}\right)\left(5.8\times10^{6}\ \mathrm{K}\right)^4 \\ &= 8.56\times10^{11}\ \mathrm{J/m^3} = 1.47\epsilon_\nu \\ n_{\rm CMB} &= \beta T_{\rm CMB}^3 \\ &= \frac{3}{4}\left(2.03\times10^{7}\ \mathrm{m^{-3}K^{-3}}\right)\left(5.8\times10^{6}\ \mathrm{K}\right)^3 \\ &= 3.96\times10^{27}\ \mathrm{m^{-3}} = 1.22 n_\nu.\end{aligned}$$

(c) *What is the redshift* z_{500} *and scale factor* a_{500} *when* $T_{\rm CMB} = 5.8 \times 10^6\ K = 500\ eV$*?*

Today, $T_{\rm CMB,0} = 2.755$ K and $a_0 = 1$. The ratio of scale factors is just the inverse ratio of temperatures:

$$\begin{aligned}\frac{a_0}{a_{500}} &= \frac{5.8 \times 10^6 \text{ K}}{2.755 \text{ K}} \\ &= 2.1 \times 10^6 \\ a_{500} &= 4.75 \times 10^{-7}.\end{aligned}$$

The redshift z_{500} is related to the scale factor by $1 + z_{500} = a_{500}^{-1}$, so

$$z_{500} = a_{500}^{-1} - 1 = 2.1 \times 10^6.$$

(d) *When a neutrino is relativistic, its energy is approximately* $E_\nu = pc$*. Including the neutrino mass, the energy of a neutrino is*

$$E_\nu = \sqrt{m_\nu^2 c^4 + p^2 c^2},$$

where the momentum changes with the scale factor $p \propto a^{-1}$*. Including the mass, the energy density of a neutrino species evolves with* a *as*

$$\epsilon_1(a) = \sqrt{m_\nu^2 c^4 + p_{500}^2 c^2 (a_{500}/a)^2} \times n_{500}(a_{500}/a)^3$$

where p_{500} *and* n_{500} *are the average momentum and number density when the CMB temperature is* $500\ eV$*. Plot (on a log-log plot) the evolution of the energy density of each neutrino species from* a_{500} *to today, assuming their masses are the minimum possible values you found in part (a).*

We have calculated n_{500} in part (b), it is

$$n_{500} = n_1 = 1.08 \times 10^{27} \text{ m}^{-3}.$$

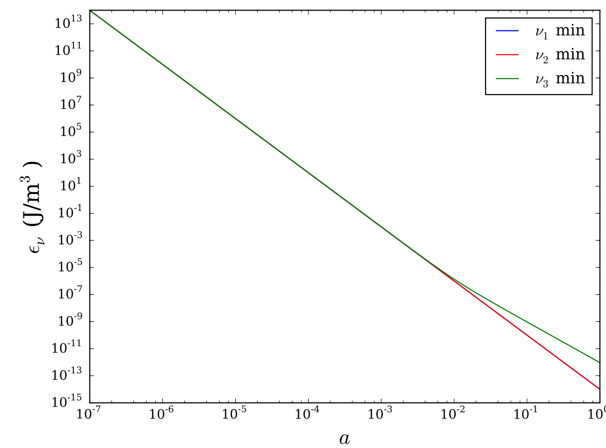
We need p_{500}. We can use the fact that the neutrinos are relativistic at 500 eV and our result for the energy density from (b) to say that

$$\begin{aligned}\epsilon_1 &= (p_{500}c)n_{500} \\ p_{500}c &= \frac{\epsilon_1}{n_1} \\ &= \frac{1.94 \times 10^{11} \text{ J/m}^3}{1.08 \times 10^{27} \text{ m}^{-3}} = 1.80 \times 10^{-16} \text{ J} = 1120 \text{ eV}\end{aligned}$$

Converting to joules, the minimum masses of the three neutrino species are:

$$m_1c^2 = 0,\ m_2c^2 = 1.20 \times 10^{-23}\ J,\ m_3c^2 = 8.04 \times 10^{-21}\ J.$$

Therefore, the evolution of the energy density of the three species is:



(e) *The maximum masses of the three neutrino species consistent with the limit $\sum m_\nu \leq 0.23\ eV/c^2$ and the measured mass-squared differences are*

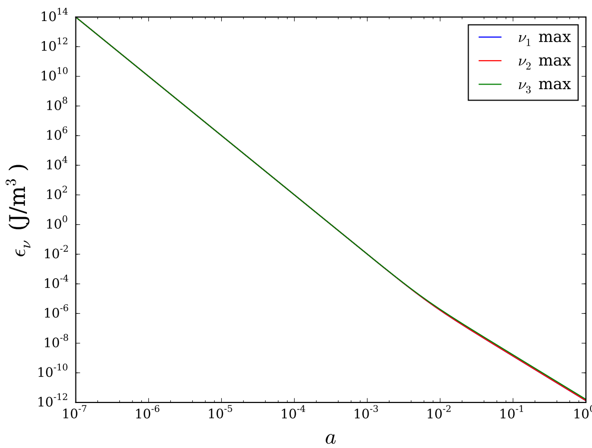
$$m_1 = 0.0712\ eV/c^2,\ m_2 = 0.0717\ eV/c^2,\ m_3 = 0.0871\ eV/c^2.$$

Plot (on a log-log plot) the evolution of the energy density of each neutrino species from a_{500} to today, assuming their masses are the maximum possible values.

In joules, these masses are

$$m_1c^2 = 1.14 \times 10^{20}\ \text{J},\ m_3c^2 = 1.15 \times 10^{20}\ \text{J},\ m_3c^2 = 1.40 \times 10^{20}\ \text{J},$$

Using the same values as part (d), we find an evolution of the energy density of:



(f) *What is the density parameter Ω_ν today if the neutrino masses are the minimum and maximum possible?*

Assuming the minimum possible neutrino masses and the equations plotted in part (d), I find

$$\begin{aligned}
\epsilon_{\nu,0}^{\text{min}} &= \epsilon_{1,0}^{\text{min}} + \epsilon_{2,0}^{\text{min}} + \epsilon_{3,0}^{\text{min}} \\
&= 9.90 \times 10^{-15}\ \text{J/m}^3 + 9.99 \times 10^{-15}\ \text{J/m}^3 + 9.31 \times 10^{-13}\ \text{J/m}^3 \\
&= 9.50 \times 10^{-13}\ \text{J/m}^3 = 5.93\ \text{MeV/m}^3.
\end{aligned}$$

The critical density today is $\epsilon_{c,0} = 4870\ \text{MeV/m}^3$, so

$$\Omega_{\nu,0}^{\text{min}} = 5.93/4870 = 1.22 \times 10^{-3}.$$

The maximum possible neutrino masses give (from part (e)),

$$\begin{aligned}
\epsilon_{\nu,0}^{\text{max}} &= \epsilon_{1,0}^{\text{max}} + \epsilon_{2,0}^{\text{max}} + \epsilon_{3,0}^{\text{max}} \\
&= 1.32 \times 10^{-12}\ \text{J/m}^3 + 1.33 \times 10^{-12}\ \text{J/m}^3 + 1.62 \times 10^{-12}\ \text{J/m}^3 \\
&= 4.27 \times 10^{-12}\ \text{J/m}^3 = 26.7\ \text{MeV/m}^3.
\end{aligned}$$

so

$$\Omega_{\nu,0}^{\text{max}} = 26.7/4770 = 5.48 \times 10^{-3}.$$

3. *For the Benchmark Universe,*

(a) *what is the total mass of all the matter within our horizon? (You will need the value from Eq. (5.115) in the text.)*

The critical energy density of our Universe is

$$\epsilon_{c,0} = 4870 \text{ MeV/m}^3,$$

which converts to a mass density of

$$\begin{aligned}
\rho_{c,0} &= \frac{\epsilon_{c,0}}{c^2} = \frac{(4.870 \times 10^9 \text{ eV/m}^3)(1.6 \times 10^{-19} \text{ J/eV})}{(3 \times 10^8 \text{ m/s})^2} \\
&= 8.66 \times 10^{-27} \text{ kg/m}^3.
\end{aligned}$$

The horizon for our Universe is at a distance of 14,000 Mpc, or

$$d_{\text{hor}}(t_0) = 4.32 \times 10^{26} \; m.$$

Since $\Omega_{m,0} = 0.31$, the total mass inside the horizon is

$$\begin{aligned}
M &= \Omega_{m,0}\rho_{c,0}\frac{4\pi}{3}d_{\text{hor}}^3 \\
&= \frac{4\pi}{3}(0.31)(8.66 \times 10^{-27} \text{ kg/m}^3)(4.32 \times 10^{26} \text{ m})^3 \\
&= 9.07 \times 10^{53} \text{ kg}.
\end{aligned}$$

(b) *what is the total energy of all the photons within our horizon?*
The density parameter for CMB photons is $\Omega_{\text{CMB},0} = 5.35 \times 10^{-5}$, so

$$\begin{aligned}
E_\gamma &= \Omega_{CMB,0}\epsilon_{c,0}\frac{4\pi}{3}d_{\text{hor}}^3 \\
&= \frac{4\pi}{3}(5.25 \times 10^{-5})(4870 \text{ MeV/m}^3)(4.32 \times 10^{26} \text{ m})^3 \\
&= 8.8 \times 10^{79} \text{ MeV}.
\end{aligned}$$

(c) *how many baryons are within the horizon? (Here you may assume all baryons have the mass of a neutron: 940 MeV)*
The density parameter for baryons is only 0.048 (not the 0.31 for all matter used in part a). Each baryon weighs about $m_n \sim$ 940 MeV, so

$$\begin{aligned}
N_b &= \frac{E_b}{m_n} \\
&= \frac{\frac{4\pi}{3}(0.048)(4870 \text{ MeV/m}^3)(4.32 \times 10^{26} \text{ m})^3}{940 \text{ MeV}} \\
&= 8.4 \times 10^{79}.
\end{aligned}$$

So there are nearly 10^{80} protons and neutrons in the visible Universe.