**Physics 444: Problem Set #8**
**due November 17, 2021**

1. *The ionization fraction $X(t)$ (the ratio of unbound protons in the Universe to bound hydrogen atoms) obeys the Saha equation. In class, we derived that the Saha equation gives the result*

$$\frac{1-X}{X^2} = 3.84\eta \left(\frac{kT}{m_e c^2}\right)^{3/2} e^{Q/kT},$$

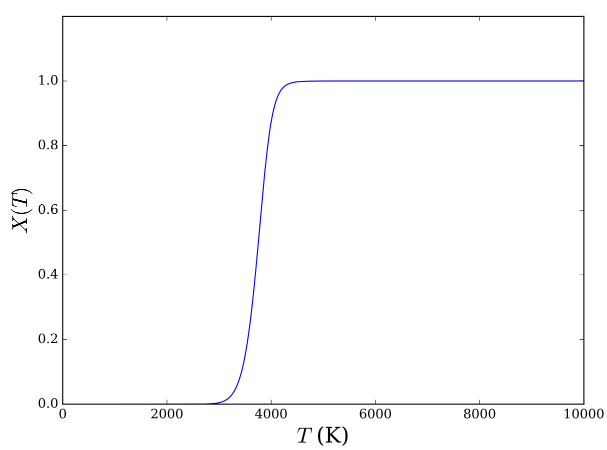
*where $T$ is the temperature of the Universe (and is itself a function of time), $\eta = 6.2 \times 10^{-10}$ is the baryon-to-photon ratio, $m_e = 511\ keV/c^2$ is the mass of an electron, and $Q = 13.57\ eV$ is the binding energy of hydrogen.*

   (a) *Using* Mathematica *or* Python*, plot $X$ as a function of temperature $T$ from $T = 0$ to $T = 10^4$ K. (Notice that the Saha equation is just a quadratic equation in $X$, so you can solve $X$ as a function of $T$ by hand.)*

   If we call the right-hand side of the Saha equation $R(T)$ (which is just a number that depends on $T$), then the positive solution for $X(T)$ is

$$X(T) = \frac{\sqrt{4R(T)+1}-1}{2R(T)}$$

   From this, it is relatively straightforward to plot $X(T)$:

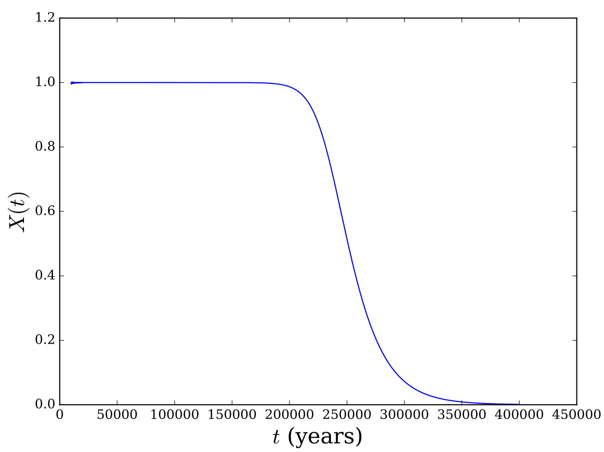


(b) *In Problem Set 6, you numerically calculated $a$ as a function of time $t$. Using this, and the fact that $T \propto a^{-1}$, plot $X$ as a function of $t$ from $10^4$ to $3 \times 10^5$ years. (You may use the code provided in the solution set to Problem Set 6, if you choose).*

Reusing my old code from problem set 6, I have an interpolating function that returns $a$ as a function of time. Therefore, I can easily write a function that returns the temperature as function of time:

$$T(t) = \frac{T_0}{a(t)},$$

where $T_0 = 2.7255$ K is the temperature today. Feeding this into my solution for $X(T)$, I now have a function $X(t)$.

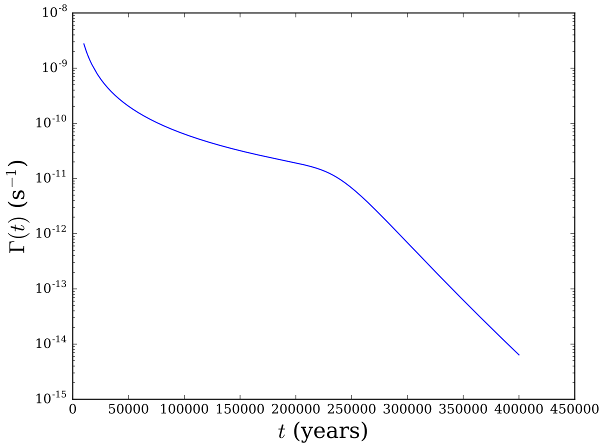



(c) *The rate for photons to scatter off of the ionized electrons in the Universe is*

$$\Gamma(t) = X(t) c \sigma_e n_{b,0} a(t)^{-3} = (5.0 \times 10^{-21}\ s^{-1}) X(t) a(t)^{-3}.$$

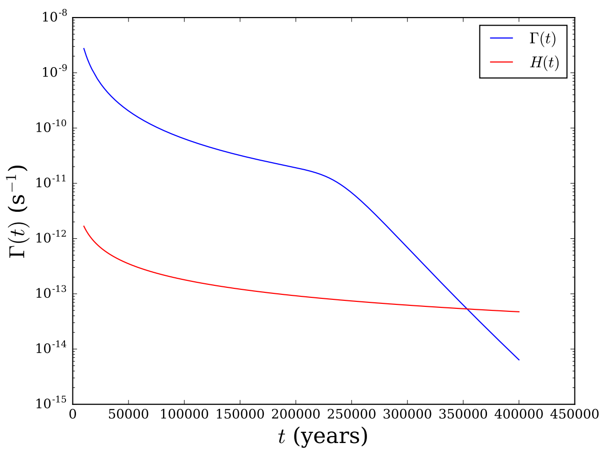
*Using your previous results, plot $\Gamma(t)$ as a function of time.*

I have $X(t)$ and $a(t)$ from the previous parts of this problem, so I can straightforwardly multiply them together and get




(d) *Photon decoupling occurs approximately when the rate of scattering (the $\Gamma(t)$ you calculated in part (c)) becomes smaller than the rate of expansion of the Universe, $H(t)$. Plot $\Gamma(t)$ and $H(t)$ on the same set of axis, and estimate the age of the Universe $t$ when decoupling occurs.*

Adding the function $H(t)$ (reused from problem set 6) gives




The two lines cross (and thus decoupling occurs) around $t \sim 354,000$ years.

(e) *We discussed in class that the actual $z$ of decoupling is actually slightly smaller than the result of solving the Saha equation, due to the rate of electron-photon scattering departing for equilibrium. Given this, do you think your numeric solution for the age of the*

*Universe at decoupling is earlier or later than the actual answer?*

A smaller $z$ corresponds to a later time, so the real answer for the age of the Universe at decoupling should be later than the result in part (d). The actual $z$ of decoupling $z = 1100$ corresponds to 366,000 years.

2. *Suppose the neutron decay time was $\tau_n = 88$ seconds, rather than the actual value (880 s), with all other physical parameters unchanged. Estimate $Y_{\rm max}$, the maximum possible mass fraction in $^4$He, assuming that all available neutrons were bound into helium.*

   Keeping all other parameters constant means that the freeze-out ratio of neutrons to protons is unchanged

$$\left.\frac{n_n}{n_p}\right|_{\rm freeze-out} = e^{-1.29 \text{ MeV}/0.8 \text{ MeV}} = 0.2.$$

   Furthermore, the time of deuterium synthesis will remain the same

$$1 = 6.5\eta \left(\frac{kT_{\rm nuc}}{m_n c^2}\right)^{3/2} e^{B_D/kT_{\rm nuc}}$$

   which occurs when $T_{\rm nuc} = 7.7 \times 10^8$ K or $t = 170$ s. However, now at this time, the number of neutrons that have decayed will be higher, so the fraction of neutrons remaining to be bound up into deuterium and thence into helium-4 is

$$\left.\frac{n_n}{n_p}\right|_{t=170 \text{ s}} = \frac{e^{-170/88}}{5 + (1 - e^{-170/88})} = 0.025$$

   so the max helium yield is

$$Y_{\rm max} = \frac{2 \times 0.025}{1 + 0.025} = 0.048,$$

   down from 0.27.

   Note that the book uses $t = 200$ s for the time of deuterium synthesis. If you use that answer, you will find

$$\left.\frac{n_n}{n_p}\right|_{t=200 \text{ s}} = \frac{e^{-170/88}}{5 + (1 - e^{-200/88})} = 0.017$$

so the max helium yield is

$$Y_{\max} = \frac{2 \times 0.025}{1 + 0.025} = 0.034.$$

Either answer is ok for this problem.

3. *Suppose that the difference in rest energy of the neutron and proton was* 0.129 *MeV, rather than the actual value (*1.29 *MeV), with all other physical parameters unchanged. Estimate* $Y_{\max}$*, the maximum possible mass fraction in* $^4$*He, assuming that all available neutrons were bound into helium.*

   In changing the difference in rest energy of the proton and neutron, we change the freeze-out abundance ratio

$$\left.\frac{n_n}{n_p}\right|_{\text{freeze-out}} = e^{-0.129 \text{ MeV}/0.8 \text{ MeV}} = 0.85.$$

   If we proceed as before, we would say that

$$\begin{aligned} n_p &= n_{\text{baryons}} - n_n \\ 1.85 n_p &= n_{\text{baryons}} \\ n_p &= 0.54 n_{\text{baryons}} = 0.54 \eta g_\gamma \end{aligned}$$

   Thus, the Saha equation results in

$$\begin{aligned} \frac{n_D}{n_n} &= 6(0.54\eta)\left[0.243\left(\frac{kT}{\hbar c}\right)^3\right]\left(\frac{m_n kT}{\pi \hbar^2}\right)^{-3/2} e^{B_D/kT} \\ &= 4.4\eta\left(\frac{kT}{m_n c^2}\right)^{3/2} e^{B_D/kT} \end{aligned}$$

   Setting the $n_D/n_n$ ratio to 1, I can solve this numerically, and find essentially the same answer as I did for the answer in the real Universe, $kT = 0.0659$ MeV, so $T = 7.65 \times 10^8$ K and $t = 170$ s.

   Therefore, after the deuterium synthesis stops, at $t = 170$ s, the ratio of neutrons to protons would be

$$\left.\frac{n_n}{n_p}\right|_{t=170 \text{ s}} = \frac{e^{-170/880}}{(0.85)^{-1} + (1 - e^{-170/880})} = 0.61,$$

$$Y_{\rm max} = \frac{2 \times 0.61}{1 + 0.61} = 0.76.$$

However, since the mass splitting is less than the mass of an electron, the neutron cannot decay, and so the freeze-out abundance does not change, and so

$$Y_{\rm max} = \frac{2 \times 0.85}{1 + 0.85} = 0.92.$$