Physics 444: Problem Set #4
due October 6, 2021

1. Consider Einstein's static universe, in which the attractive force of the matter density ρ is exactly cancelled by the repulsive force of the cosmological constant $\Lambda = 4\pi G\rho$. Suppose that some of the matter is converted into radiation (by stars, for instance). Will the universe start to expand or contract? Explain your answer.

2. The current limit on the sum of the masses of the three neutrino species is $\sum m_\nu \leq 0.23$ eV/c^2 (this limit comes from measurements of the Universe's properties in early times, as we will see). The two mass-squared differences between the three species of neutrinos are known to be

$$\Delta m_{12}^2 = 7.53 \times 10^{-5} \text{ eV}^2/c^4,\ \Delta m_{23}^2 = 2.44 \times 10^{-3} \text{ eV}^2/c^4.$$

Here Δm_{ij}^2 means $\left|m_i^2 - m_j^2\right|$. (These values come from neutrino oscillation measurements on Earth.)

   (a) Assuming $m_3 > m_2 > m_1$ (this is ordering is actually not known), what are the minimum neutrino masses compatible with the measured mass squared differences?

   (b) When the temperature of the CMB is $T_{\rm CMB} \sim 5.8 \times 10^6$ K $=$ 500 eV, the neutrinos are all relativistic (meaning you can act as if their masses are all zero), and have a temperature of $(4/11)^{1/3} T_{\rm CMB}$. The energy density of a relativistic fermion (such as a neutrino) is

$$\epsilon(T) = \frac{\pi^2}{15} \frac{k^4}{\hbar^3 c^3} \frac{7}{8} T^4,$$

and the number density is

$$n(T) = \frac{2.404}{\pi^2} \frac{k^3}{\hbar^3 c^3} \times \frac{3}{4} T^3.$$

Note the factors of 7/8 and 3/4 relative to the text's equations (2.26-2.27) and (2.28-2.29). What is the energy density and number density of neutrinos (summing over all three species) when the Universe's temperature is 500 eV? Compare this to the energy density and number density of photons at this time.

(c) What is the redshift z_{500} and scale factor a_{500} when $T_{\rm CMB} = 5.8 \times 10^6$ K $= 500$ eV?

(d) When a neutrino is relativistic, its energy is approximately $E_\nu = pc$. Including the neutrino mass, the energy of a neutrino is

$$E_\nu = \sqrt{m_\nu^2 c^4 + p^2 c^2},$$

where the momentum changes with the scale factor $p \propto a^{-1}$. Including the mass, the energy density of a neutrino species evolves with a as

$$\epsilon_1(a) = \sqrt{m_\nu^2 c^4 + p_{500}^2 c^2 (a_{500}/a)^2} \times n_{500}(a_{500}/a)^3$$

where p_{500} and n_{500} are the average momentum and number density when the CMB temperature is 500 eV. Plot (on a log-log plot) the evolution of the energy density of each neutrino species versus scale factor a from a_{500} until today, assuming their masses are the minimum possible values you found in part (a).

(e) The maximum masses of the three neutrino species consistent with the limit $\sum m_\nu \leq 0.23$ eV$/c^2$ and the measured mass-squared differences are

$$m_1 = 0.0712 \text{ eV}/c^2, \; m_2 = 0.0717 \text{ eV}/c^2, \; m_3 = 0.0871 \text{ eV}/c^2.$$

Plot (on a log-log plot) the evolution of the energy density of each neutrino species versus a from a_{500} until today, assuming their masses are the maximum possible values.

(f) What are the density parameters Ω_ν today if the neutrino masses are the minimum and maximum possible?

3. For the Benchmark Universe,

(a) what is the total mass of all the matter within our horizon? (You will need the value from Eq. (5.115) in the text.)

(b) what is the total energy of all the photons within our horizon?

(c) how many baryons are within the horizon (Here you may assume all baryons have the mass of a neutron: 940 MeV)?