Physics 444: Problem Set #1
due September 12, 2021

1. *This course will deal with equations that cannot be analytically integrated, but can be solved very straightforwardly using numeric methods. In later homeworks, I will ask you to use some of these methods to solve equations relevant for cosmology. This homework problem asks you to become familiar with one of two computer programs that can perform the necessary mathematical integration: either* MATHEMATICA *or* PYTHON *(other programs exist; I will offer technical support for these two, as they are the ones I am most familiar with).*

   MATHEMATICA *is a multi-purpose mathematics program designed and sold by Wolfram. Individual licenses are* **extremely** *expensive and* **under no circumstances** *does this course require you to purchase one. As Rutgers students, you can download a free license (along with the program itself) for both PC and Mac OSX from the Rutgers software portal* `https://software.rutgers.edu`*. However, the license will expire after a year, and so you may not have access to this program after you graduate.*

   PYTHON *is a free computer language which has a number of very powerful add-on packages which make it very useful for mathematical and scientific use. As a programming language, it has a somewhat higher barrier for entry than* MATHEMATICA*, however it is a tool you will always be able to access without paying for. I will provide support for programs in* PYTHON3 *(there is also* PYTHON2*, which is annoyingly not fully forwards-compatible). The language can be installed on both PC and Mac OSX, from* `www.python.org`*. In addition, you will likely want to install Numeric Python (*NUMPY*) from* `www.numpy.org`*, Scientific Python (*SCIPY*) from* `www.scipy.org`*, and matplotlib from* `matplotlib.org`*. The first two packages provide essential mathematical and scientific tools (including numeric integration). The last package allows you to plot functions. You may also find using* JUPYTER *notebooks useful to write* PYTHON *code, as this allows you to do dynamic coding (that is, you can execute code, and then rerun specific lines of code to fix bugs). If you want to install* JUPYTER*, you should install* PYTHON *using* CONDA *or* ANACONDA `docs.conda.io/projects/conda/en/latest/user-guide/install/`*.*

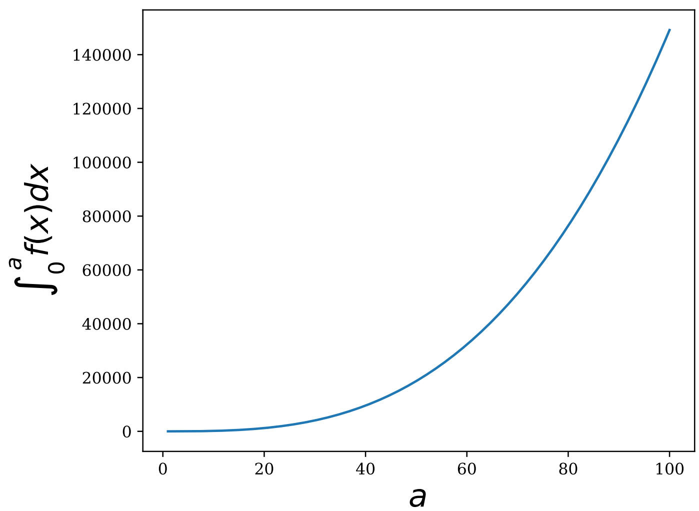
Download and install one of these two programs/languages. If needed, read the example files for each available on Canvas. Then:

(a) *Numerically integrate the function:*

$$f(x) = \sqrt{Ax^2 + Bx^4}$$

for $A = 0.8$ *and* $B = 0.2$ *from* $x = 0$ *to* a *in the range* $a = [0-100]$, *and plot the result.*

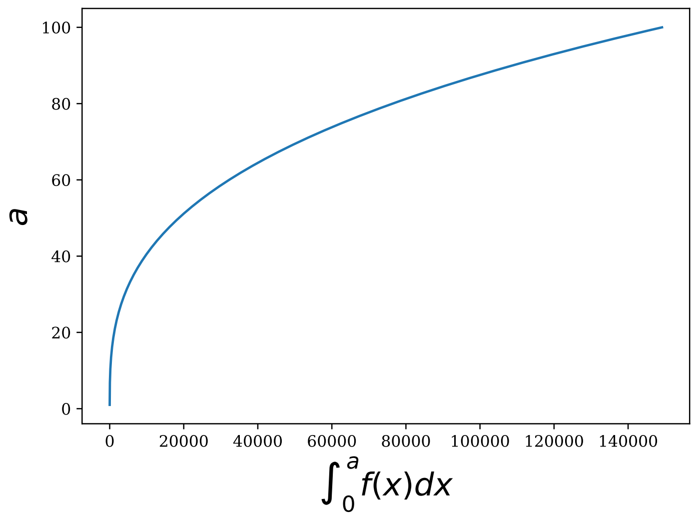
See code for a Python solution.



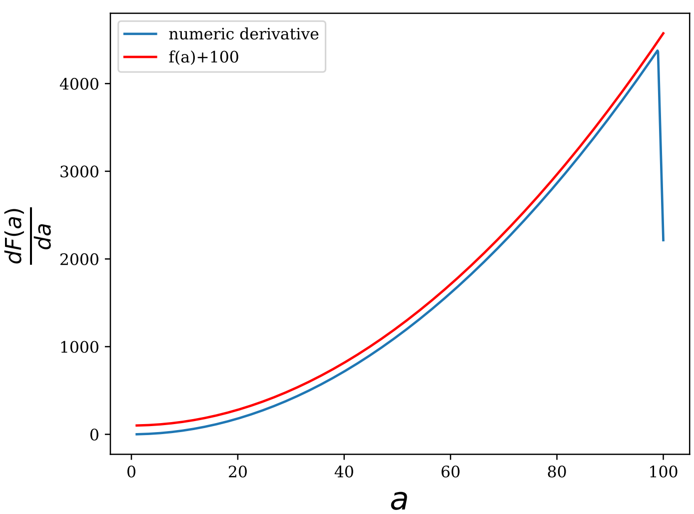
(b) *In the previous part, you have numerically calculated and plotted*

$$F(a) = \int_0^a f(x)dx$$

as a function of a. *Now, plot* a *as a function of* $F(a)$. *(Physically turning your previous plot sidewise does not qualify as a solution. The point of this problem is for you to figure out how to get the computer program to do this for you.)*



(c) *Plot the numerical derivative of $F(a)$ with respect to a evaluated on $a = [0, 100]$. For comparison, plot $f(a)$ on the same set of axes, offset so both lines are visible.*



2. *Let us work in the "spherical human" approximation, in which people can be assumed to be a simple ball of water. Human body temperature is 310 K.*

   (a) *Assuming you are a perfect blackbody, what is the rate at which you radiate energy, in watts?*

   The flux (energy per unit time per unit area) of blackbody radiation leaving a surface at temperature T is

$$\phi = \frac{c\epsilon_\gamma}{4} = \frac{\alpha c}{4} T^4.$$

   Notice that multiplying out the prefactor gives you

$$\phi = \sigma T^4,$$

   where $\sigma = 5.67 \times 10^{-8}$ W/m^2/K^4 is the Stefan-Boltzmann constant, so you could have just started here.

   For a human, this corresponds to

$$\begin{aligned} \phi &= \frac{1}{4}(7.566 \times 10^{-16}\ \mathrm{J/m^3/K^4})(3 \times 10^8\ \mathrm{m/s})(310\ \mathrm{K})^4 \\ &= 524\ \mathrm{W/m^2} \end{aligned}$$

   I mass about $m = 80$ kg, and the density of water is 1 g/cm^3 = 1000 kg/m^3. Therefore, if I was a sphere, I would have a radius of

$$\begin{aligned} 80\ \mathrm{kg} &= \frac{4\pi}{3}(1000\ \mathrm{kg/m^3})r^3 \\ r &= 0.27\ \mathrm{m}, \end{aligned}$$

   and a surface area of

$$A = 4\pi r^2 = 0.9\ \mathrm{m}^2.$$

   Therefore, I radiate away

$$\dot{E} = \phi \times A = 472\ \mathrm{W}.$$

   (b) *In a perfect vacuum, how long will it take your body temperature to drop to freezing (273.15 K)?*

To drop to freezing, I must radiate away

$$\begin{aligned}
c_w m(T_{\text{body}} - T_{\text{freeze}}) &= (4.181 \times 10^3 \text{ J/kg/K})(80 \text{ kg})(310 \text{ K} - 273 \text{ K}) \\
&= 1.2 \times 10^7 \text{ J}.
\end{aligned}$$

Here c_w is the specific heat of water. If I assumed that the energy loss was constant, and used the previous answer for the rate of energy loss, this would take

$$t = \frac{1.2 \times 10^7 \text{ J}}{472 \text{ W}} = 2.6 \times 10^4 \text{ s} = 7.2 \text{ hours}.$$

However, the energy loss is not constant because the temperature is changing as I cool. Therefore

$$\begin{aligned}
-c_w m dT &= \frac{\alpha c A}{4} T^4 dt \\
-\frac{dT}{T^4} &= \frac{\alpha c A}{4 c_w m} dt \\
t &= -\frac{4 c_w m}{\alpha c A} \int_{T_{\text{body}}}^{T_{\text{freeze}}} T^{-4} dT \\
&= \frac{4 c_w m}{3 \alpha c A} \left(T_{\text{freeze}}^{-3} - T_{\text{body}}^{-3} \right) \\
&= \frac{4(4.181 \times 10^3 \text{ J/kg/K})(80 \text{ kg})}{3(7.566 \times 10^{-16} \text{ J/m}^3/\text{K}^4)(3 \times 10^8 \text{ m/s})(0.9 \text{ m}^2)} \times \\
&\quad \left[(273 \text{ K})^{-3} - (310 \text{ K})^{-3} \right] \\
&= 3.4 \times 10^4 \text{ s} = 9.4 \text{ hours}.
\end{aligned}$$

3. *A hypothesis once used to explain the Hubble relation is the "tired light hypothesis." The tired light hypothesis states that the Universe is not expanding, but that photons simply lose energy as they move through space (by some unexplained means), with the energy loss per unit distance being given by the law*

$$\frac{dE}{dr} = -KE$$

where K is a constant.

(a) *Show that this hypothesis gives a distance-redshift relation that is linear in the limit $z \ll 1$. What must the value of K be in order to yield a Hubble constant of $H_0 = 70$ km s^{-1} Mpc^{-1}?*

Imagine a photon traveling towards to the Earth, emitted at distance R and energy $E_{\rm em}$, and reaching the Earth ($r = 0$) and energy $E_{\rm ob}$. Then we can solve the differential equation

$$\begin{aligned} \frac{dE}{E} &= -K dr \\ \int_{E_{\rm em}}^{E_{\rm ob}} \frac{dE}{E} &= -K \int_R^0 dr \\ \ln(E_{\rm ob}/E_{\rm em}) &= -KR. \end{aligned}$$

We are interested in redshift

$$z = \frac{\lambda_{\rm ob} - \lambda_{\rm em}}{\lambda_{\rm em}}$$

since $E = \frac{hc}{\lambda}$,

$$\begin{aligned} z &= \frac{\lambda_{\rm ob} - \lambda_{\rm em}}{\lambda_{\rm em}} \\ &= \frac{\lambda_{\rm ob}}{\lambda_{\rm em}} - 1 \\ &= \frac{E_{\rm em}}{E_{\rm ob}} - 1 \end{aligned}$$

so from our differential equation

$$\begin{aligned} -KR &= \ln(E_{\rm ob}/E_{\rm em}) \\ &= \ln[1/(1+z)] \end{aligned}$$

For small z, we can Taylor expand and find

$$\ln\left[\frac{1}{1+z}\right] \approx \ln(1) + \left(-\frac{1}{(1+z)^2}\right)\bigg|_{z=0} z = -z.$$

Therefore,

$$-KR = -z$$

or

$$z = KR.$$

This is Hubble's law, and we can identify K as H_0/c (see Eq. 2.5 in the text). Therefore,

$$K = \frac{H_0}{c} = \frac{70 \text{ km/s/Mpc}}{3 \times 10^5 \text{ km/s}} = 2.3 \times 10^{-4} \text{ Mpc}^{-1}.$$

(b) *Show that Hubble's Law predicts an identical relation for dE/dr. Is the energy of a photon conserved in an expanding Universe?*

Using

$$z = \frac{E_{\text{em}}}{E_{\text{ob}}} - 1$$

and Hubble's Law

$$z = \frac{H_0}{c} r,$$

we see that

$$\frac{E(r)}{E_{\text{ob}}} - 1 = \frac{H_0}{c} r$$

Therefore

$$\frac{dE(r)}{dr} = \frac{H_0}{c} E_{\text{ob}}$$

as in the previous problem (the difference in minus sign is due to the fact that in this problem, r is a measure away from the Earth, and in the previous, it was a measure of how far the photon was traveling – towards the Earth). Therefore, in an expanding Universe, photon energy is not conserved.