Physics 444: Problem Set #6
due October 20, 2021

1. *Let us consider the benchmark Universe:* $\Omega_0 = 1$*,* $\Omega_{r,0} = 9.03 \times 10^{-5}$*,* $\Omega_{m,0} = 0.306$*, and* $\Omega_{\Lambda,0} = 0.692$*, with* $H_0 = 67.8$ *km/s/Mpc=(14.43 Gyr)*$^{-1}$*.*

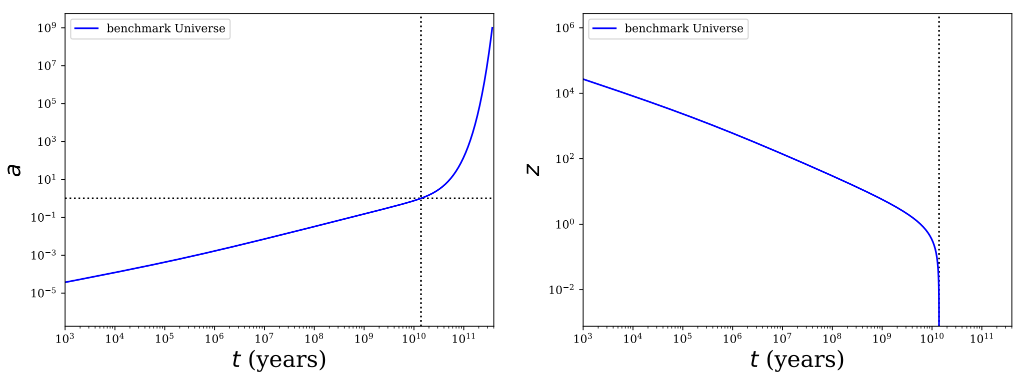
   (a) *Using the analytic formula for a Universe with only matter and a cosmological constant, calculate the age of the benchmark Universe today, ignoring radiation.*

   Using
   $$t_0 = \frac{2H_0^{-1}}{3\sqrt{1-\Omega_{m,0}}} \ln\left[a_{m\Lambda}^{-3/2} + \sqrt{1 + a_{m\Lambda}^{-3}}\right]$$
   with
   $$a_{m\Lambda} = (\Omega_{m,0}/\Omega_{\Lambda,0})^{1/3} = (0.306/0.692)^{1/3} = 0.762,$$
   and $H_0^{-1} = (67.8 \text{ km/s/Mpc})^{-1} = 14.43$ Gyr, we see that
   $$\begin{aligned} t_0 &= \frac{2(14.43 \text{ Gyr})}{3\sqrt{0.694}} \ln\left[0.762^{-3/2} + \sqrt{1 + 0.762^{-3}}\right] \\ &= 13.82 \text{ Gyr} \end{aligned}$$

   (b) *Now, using a numerical differential equation integrator (as contained in* Mathematica *or* Python*), solve the exact differential equation for the evolution of the scale factor* a *as a function of time for our Universe. Make two log-log plots: one of* a *as a function of* t *and the 2nd of* z *as a function of* t *for* $t = 10^3$ *to* 4×10^{11} *years.*

   There are a number of ways to do this. I include a python script which does it via one particular method. Here, I use the relation between the scale factor a and the time t when the Universe has that scale factor
   $$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{r,0}a^{-2} + \Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)}}.$$
   In my python script, I can use this to determine the age of the Universe t_0 when the Universe has $a = 1$ (see lines 41 and 43),

but also to create a list of scale factors and then, by performing the integral with that list as the upper limit, I can construct a list of times which correspond to the age of the Universe for each of those scale factors (see lines 50-60). Essentially, instead of a as a function of t, I have t as a function of a. I can flip the axes though, and plot this a t vs. a.



(c) *From your numeric solution, what is the age of the Universe, including radiation?*

From my code (see python script) I calculate

$$t_0 = 13.84 \text{ Gyr},$$

which agrees to this level of accuracy with the answer in part (a)

(d) *If you set $\Omega_{r,0} = 0$ in your numeric solution, how much does the age of the Universe change, relative to your answer in part c)?*

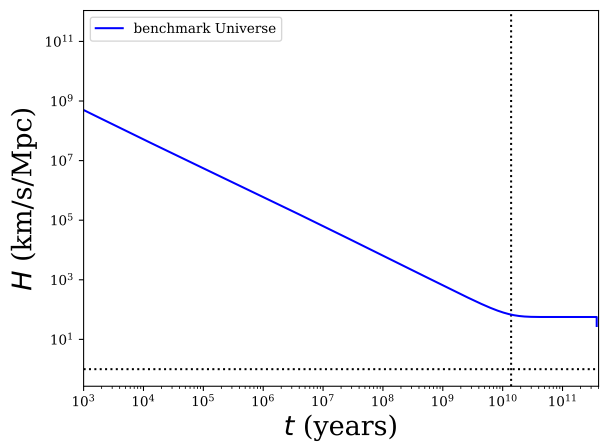
Taking the difference between the numeric solution with radiation and without radiation I find:

$$\Delta t = t_0 - t_0(\Omega_{r,0} = 0) = -5.9 \times 10^6.$$

Thus, the Universe is 5.9 million years older if we neglect radiation.

(e) *Calculate and plot the Hubble parameter H as a function of time t in physical units (km/s/Mpc) for $t = 10^3$ to 4×10^{11} years. Explain the behavior of your result at large t.*

My technique for getting t as a function of a saved me time in getting the previous answers, but does leave me in a difficult position if I want to start taking a derivative of a with respect to t. My solution starts at line 115 in my code. I define an interpolating function, which interpolates my list of a values as a function of t. This gives me the function I want: $a(t)$ (I could have done this much earlier, but didn't find it necessary). Then I can take the derivative of a with respect to t, and divide by a, and plot the Hubble parameter.



At large t, we see the Hubble parameter asymptotes to a constant (ignore the numeric instability in the last bin, this is the result of taking a numeric derivative with a step size of zero). This is expected, since at large time the Universe is dominated by the cosmological constant. In such a Universe, the Hubble parameter does not change.

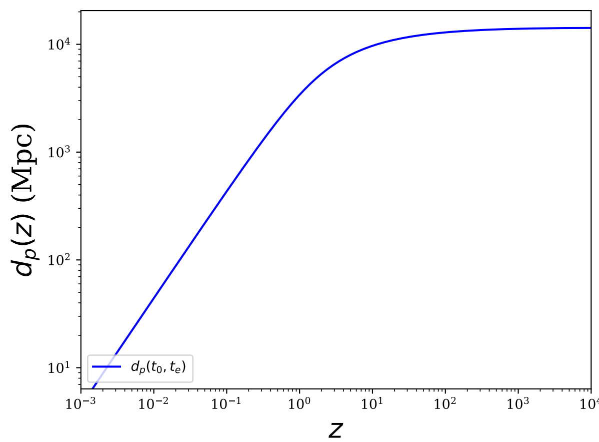
(f) *Using your numeric solution, calculate and plot the comoving distance to a light source at redshift z for values of z from 10^{-3} to 10^4.*

The quantity asked for is the comoving distance, which is equivalent to the proper distance at $t = t_0$ when $a = 1$:

$$d_p(t_0, t_e) = c \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

I then need to translate t_e into an equivalent z. Again, there are a number of ways to implement a solution to this problem.

To do the actual integral, I chose to work with an interpolating function of $1/a(t)$ (see lines 153-155). Then for each time in my list of times, I could perform the integral, treating that time as the emission time, and get a distance. I have a list of distances, each corresponding to an emission time. Each time in my list has an associated scale factor in my list of scale factors, so I already know the z for each time ($z = 1/a - 1$). Therefore, I can plot the comoving distance versus z fairly easily.



2. *As we will see in the next few weeks, the redshift $z = 1100$ is a very important moment in cosmological history*

   (a) *What are the density parameters Ω of radiation, matter, and cosmological constant at $z = 1100$ in the benchmark Universe?*

   The density parameters evolve as:

$$
\begin{aligned}
\Omega_r &= \frac{\Omega_{r,0}a^{-4}}{\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}} \\
\Omega_m &= \frac{\Omega_{m,0}a^{-3}}{\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}} \\
\Omega_\Lambda &= \frac{\Omega_{\Lambda,0}}{\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}}
\end{aligned}
$$

Using the benchmark parameters and $z = a^{-1} - 1$ or $a = (z+1)^{-1}$,

$$\begin{aligned}
\Omega_r &= \frac{(9.03 \times 10^{-5})(1101)^4}{(9.03 \times 10^{-5})(1101)^4 + (0.306)(1101)^3 + 0.692} = 0.245 \\
\Omega_m &= \frac{(0.306)(1101)^3}{(9.03 \times 10^{-5})(1101)^4 + (0.306)(1101)^3 + 0.692} = 0.755 \\
\Omega_\Lambda &= \frac{0.692}{(9.03 \times 10^{-5})(1101)^4 + (0.306)(1101)^3 + 0.692} = 1.28 \times 10^{-9}
\end{aligned}$$

(b) *What is the temperature of the Universe at $z = 1100$?*

Temperature goes as the inverse of the scale factor, so

$$T(z) = T_0(z+1).$$

Thus,

$$T(1100) = (2.7556\ \text{K})(1101) = 3034\ \text{K} = 0.26\ \text{eV}.$$

(c) *Using your numeric calculations from the previous problem, approximately what is the age of the Universe when light with a redshift of $z = 1100$ is emitted?*

Using my numeric code I find the age of the Universe at this time to be

$$t(z = 1100) = 3.68 \times 10^5\ \text{years}$$

(d) *Using your numeric calculations from the previous problem, approximately what is the comoving distance to a light source at redshift $z = 1100$?*

The comoving distance to an object which emitted light now seen with $z = 1100$ is

$$d_p(z = 1100, t_0) = 14.0\ \text{Gpc}$$

(e) *Using your numeric calculations from the previous problem, approximately what was the physical distance to a light source at emission time to a light source at redshift $z = 1100$?*

At the time of emission, said objects would have been

$$d_p(z = 1100) = 12.7\ \text{Mpc}$$

away from us.