Physics 444: Problem Set #10
due December 9, 2021

1. *One suggested type of "warm" dark matter is the sterile neutrino: a hypothetical new particle that is similar to the known neutrinos, but lacks an interaction via the weak nuclear force (thus the name "sterile"). Such particles could have masses in the range of a few keV, and would while relativistic would erase the primordial structure by "free streaming" out of primordial overdensities.*

   (a) *Suppose that sterile neutrinos compose 100% of the dark matter, have a mass m_ν, and are initially in thermal equilibrium with the photon bath. Estimate the comoving length scale λ at which the primordial power spectrum would be cut-off, as a function of m_ν. (You can assume this occurs when the Universe is radiation-dominated.)*

   The free streaming stops when the sterile neutrinos are no longer relativistic, when

$$\begin{aligned} 3kT_\nu &= m_\nu c^2 \\ T_\nu &= \frac{m_\nu c^2}{3k} = (3.89 \times 10^6 \text{ K}) \left(\frac{m_\nu c^2}{1 \text{ keV}}\right) \end{aligned}$$

   In a radiation-dominated Universe, the time t and temperature T are related by

$$T = (10^{10} \text{ K})(t/1 \text{ s})^{-1/2}.$$

   Therefore, the time at which the sterile neutrinos stop free-streaming is

$$\begin{aligned} t_\nu &= (1 \text{ s}) \left(\frac{T_\nu}{10^{10} \text{ K}}\right)^{-2} \\ &= (1\text{s}) \left(\frac{3.89 \times 10^6 \text{ K}}{10^{10} \text{ K}}\right)^{-2} \left(\frac{m_\nu c^2}{1 \text{ keV}}\right)^{-2} \\ &= 7.34 \times 10^7 \text{ s} \left(\frac{m_\nu c^2}{1 \text{ keV}}\right)^{-2} \end{aligned}$$

   which corresponds to a horizon size of

$$ct_\nu = 2.20 \times 10^{15} \text{ m} \left(\frac{m_\nu c^2}{1 \text{ keV}}\right)^{-2} = 7.13 \times 10^{-8} \text{ Mpc} \left(\frac{m_\nu c^2}{1 \text{ keV}}\right)^{-2}$$

This is a physical distance at time t_ν, so to get a comoving distance (distance today), we must divide by the scale factor at that time

$$\begin{aligned}
\lambda &= \frac{ct_\nu}{a_\nu} \\
&= \frac{T_\nu}{T_0} ct_\nu \\
&= \left(\frac{3.89 \times 10^6 \text{ K}}{2.7255 \text{ K}}\right)\left(\frac{m_\nu c^2}{1 \text{ keV}}\right)(7.13 \times 10^{-8} \text{ Mpc})\left(\frac{m_\nu c^2}{1 \text{ keV}}\right)^{-2} \\
&= (0.106 \text{ Mpc})\left(\frac{m_\nu c^2}{1 \text{ keV}}\right)^{-1}
\end{aligned}$$

(b) *What is the dark matter mass (in units of solar masses) of the smallest dark matter haloes that would survive in this sterile neutrino model, as a function of neutrino mass m_ν?*

The minimum halo mass would be the mass of dark matter in a sphere of radius λ, assuming a uniform distribution of dark matter with density $\Omega_{\mathrm{DM},0}\epsilon_{0,c}$. Since the density parameter of matter (dark matter plus baryons) is 0.315, and the density parameter of baryons is 0.0499, the density parameter of dark matter is

$$\Omega_{\mathrm{DM}} = 0.315 - 0.0499 = 0.265.$$

The critical density today is 4770 MeV/m^3 or 2.50×10^{41} kg/Mpc^3, or $1.25 \times 10^{11}\ M_\odot/\mathrm{Mpc}^3$. Therefore,

$$\begin{aligned}
M_{\min} &= \frac{4\pi}{3}\lambda^3 \Omega_{\mathrm{DM}}\epsilon_{c,0} \\
&= \frac{4\pi}{3}(0.265)\left(1.25 \times 10^{11}\ M_\odot/\mathrm{Mpc}^3\right)\left[0.106 \text{ Mpc}\left(\frac{m_\nu c^2}{1 \text{ keV}}\right)^{-1}\right]^3 \\
&= 1.65 \times 10^8\ M_\odot \left(\frac{m_\nu c^2}{1 \text{ keV}}\right)^{-3}.
\end{aligned}$$

(c) *We are reasonably certain that dwarf galaxies with masses of* $10^9\ M_\odot$ *were formed in by the primordial overdensities. Estimate the minimum mass of sterile neutrinos that are allowed by this.*

We require $M_{\min} < 10^9\ M_\odot$. Therefore,

$$\begin{aligned}
1.65 \times 10^8\ M_\odot \left(\frac{m_\nu c^2}{1\ \text{keV}}\right)^{-3} &< 10^9\ M_\odot \\
1.65 \times 10^2 &< \left(\frac{m_\nu c^2}{1\ \text{keV}}\right)^{3} \\
0.548\ \text{keV} &< m_\nu.
\end{aligned}$$

Therefore, a model of sterile neutrinos which has them relativistic in the Early Universe must have neutrinos heavier than about 5.5 keV.

2. *Within the Coma cluster, galaxies have a root mean squared velocity of $\langle v^2 \rangle^{1/2} \approx 1520$ km/s relative to the center of mass of the cluster. The half-mass radius of the Coma cluster is $r_h \approx 1.5$ Mpc.*

   (a) *Estimate the minimum time $t_{\min}$ that it would take the for the Coma Cluster to form by gravitational collapse.*
   The dynamical time for the collapse of an object with average density $\bar{\rho}$ is

   $$t_{\text{dyn}} = \frac{1}{(G\bar{\rho})^{1/2}}.$$

   So, we need the average density of the Coma cluster.
   Using the virial theorem, the mass of the Coma cluster is

   $$M \sim \frac{\langle v^2 \rangle r_h}{\alpha G},$$

   where $\alpha \sim 0.4$. Therefore,

   $$\begin{aligned}
   t_{\text{dyn}} &= \left(G \frac{4\pi M}{3 r_h^3}\right)^{-1/2} \\
   &= \left(G \frac{3}{4\pi r_h^3} \frac{\langle v^2 \rangle r_h}{\alpha G}\right)^{-1/2} \\
   &= \left(\frac{3}{4\pi r_h^2} \frac{\langle v^2 \rangle}{\alpha}\right)^{-1/2} \\
   &= \left(\frac{3}{4\pi(0.4)} \frac{(1520\ \text{km/s})^2}{(1.5\ \text{Mpc} \times 3.09 \times 10^{19}\ \text{km/Mpc})^2}\right)^{-1/2} \\
   &= 3.95 \times 10^{16}\ \text{s} = 1.2 \times 10^9\ \text{years}.
   \end{aligned}$$

(b) *The Milky Way galaxy has a* $\langle v^2 \rangle^{1/2} \sim 220$ *km/s and a virial radius of* ~ 100 *kpc. Do Milky Way-type galaxies form before or after Coma cluster-like objects?*

The dynamical time for the collapse of the Milky Way galaxy would be (assuming $\alpha \sim 0.4$)

$$\begin{aligned} t_{\rm dyn} &= \left(\frac{3}{4\pi r_h^2} \frac{\langle v^2 \rangle}{\alpha} \right)^{-1/2} \\ &= \left(\frac{3}{4\pi (0.4)} \frac{(220 \text{ km/s})^2}{(0.1 \text{ Mpc} \times 3.09 \times 10^{19} \text{ km/Mpc})^2} \right)^{-1/2} \\ &= 1.82 \times 10^{16} \text{ s} = 5.7 \times 10^8 \text{ years}, \end{aligned}$$

so galaxies form before the clusters.

3. *Consider an empty, negatively curved, expanding universe. If a dynamically insignificant amount of matter* ($\Omega_m \ll 1$*, small enough to not affect the expansion of the Universe) is present in this universe, how do density fluctuations in the matter evolve with time? That is, what is the functional form of* $\delta(t)$*?*

   You will want to use the differential equation

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0$$

   and use the $H(t)$ *of a negatively curved empty universe.*

   In the empty, negatively curved Universe,

$$a(t) = t/t_0,$$

   so

$$H(t) = \frac{\dot{a}}{a} = t^{-1}.$$

   The growth of density perturbations is then

$$\begin{aligned} \ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta &= 0 \\ \ddot{\delta} + 2t^{-1}\dot{\delta} - \frac{3}{2} t^{-2} \Omega_m \delta &= 0 \end{aligned}$$

Guessing a solution of the form $\delta = Dt^n$, we see that this is

$$\begin{aligned} n(n-1)Dt^{n-2} + 2nDt^{n-2} - \frac{3}{2}\Omega_m Dt^{n-2} &= 0 \\ n(n-1) + 2n - \frac{3}{2}\Omega_m &= 0 \\ n &= -\frac{1}{2}\left(1 \pm \sqrt{6\Omega_m + 1}\right) \end{aligned}$$

In the limit of $\Omega_m \to 0$, these two solutions are $\propto t^{-1}$ and $\propto t^{3\Omega_m/2}$. Therefore,

$$\delta = D_1 t^{-1} + D_2 t^{3\Omega_m/2}.$$

The decaying solution will disappear, leaving

$$\delta = D_2 t^{3\Omega_m/2}.$$