Physics 444: Problem Set #3
due September 29, 2021

1. *The principle of wave-particle duality tells us that a particle with momentum p has an associated de Broglie wavelength $\lambda = h/p$; this wavelength increases as $\lambda \propto a$ as the Universe expands.*

   (a) *Suppose a particle of mass m was moving relative to the comoving coordinates of the Universe with speed $v = \beta_0 c$ at time t_0. Calculate v as a function of $a(t)$.*

   In terms of the speed, $p = \gamma\beta mc$. However, p is also equal to h/λ, and $\lambda = \lambda a$. Therefore, at some future time t,

$$\begin{aligned}
\frac{\beta(t)}{\sqrt{1-\beta(t)^2}}mc &= \frac{h}{\lambda_0 a(t)} \\
\frac{\beta(t)}{\sqrt{1-\beta(t)^2}}mc &= \frac{\beta_0}{\sqrt{1-\beta_0^2}}mca(t)^{-1} \\
\beta(t) &= \frac{1}{\sqrt{1-a(t)^2(1-\beta_0^{-2})}}
\end{aligned}$$

   (b) *Calculate the acceleration of the particle (relative to the comoving coordinates) as a function of $a(t)$ and the Hubble parameter $H(t) = \dot{a}/a$.*

   The acceleration is the time derivative of the velocity, so

$$\begin{aligned}
\frac{d\beta(t)}{dt} &= \frac{d}{dt}\frac{1}{\sqrt{1-a(t)^2(1-\beta_0^{-2})}} \\
&= \frac{a(t)(1-\beta_0^{-2})}{[a(t)^2(\beta_0^{-2}-1)+1]^{3/2}}\frac{da(t)}{dt} \\
&= \frac{a(t)^2(1-\beta_0^{-2})}{[a(t)^2(\beta_0^{-2}-1)+1]^{3/2}}\frac{\dot{a}(t)}{a(t)} \\
&= \frac{a(t)^2(1-\beta_0^{-2})}{[a(t)^2(\beta_0^{-2}-1)+1]^{3/2}}H(t)
\end{aligned}$$

2. *Suppose the energy density of the cosmological constant is equal to the present critical density $\epsilon_\Lambda = \epsilon_{c,0} = 4870\ MeV/m^3$. What is the total energy of the cosmological constant within a sphere 1 AU in radius?*

What is the rest energy of the Sun? Comparing these two numbers, do you expect the cosmological constant to have a significant effect on the motion of planets within the Solar System?

How does the energy in the cosmological constant within a sphere of 1 AU centered on the Sun compare to the energy contained in the sunlight within that radius?

1 AU is 1.496×10^{11} m, so the total energy of the cosmological constant within the radius of Earth's orbit is

$$\begin{aligned} E_\Lambda &= \epsilon_\Lambda \times \left(\frac{4\pi}{3} r^3\right) \\ &= (4870 \text{ MeV/m}^3)\left(\frac{4\pi}{3}(1.496 \times 10^{11} \text{ m})^3\right) \\ &= 6.83 \times 10^{37} \text{ MeV.} \end{aligned}$$

The mass of the Sun is 2×10^{30} kg, and, via $E = mc^2$,

$$\begin{aligned} E_\odot &= (2 \times 10^{30} \text{ kg})(3 \times 10^8 \text{ m/s})^2 \\ &= 1.8 \times 10^{47} \text{ J} \times \frac{6.24 \times 10^{12} \text{ MeV}}{1 \text{ J}} \\ &= 1.12 \times 10^{60} \text{ MeV.} \end{aligned}$$

Since the cosmological constant within in the Earth's orbital radius is a factor of 6.1×10^{-23} times smaller than the rest energy of the Sun, we expect the cosmological constant to have absolutely no effect on the orbit of the Earth.

To calculate the energy in the sunlight, we need the luminosity of the Sun:

$$L_\odot = 3.85 \times 10^{26}\ W.$$

That is, every second, the Sun emits 3.85×10^{26} J. This energy suffuses the 1 AU sphere around the Sun. It takes light

$$t = \frac{1 \text{ AU}}{c} = \frac{1.496 \times 10^{11} \text{ m}}{3 \times 10^8 \text{ m/s}} = 499 \text{ s}$$

to travel 1 AU. Therefore, the solar energy in the sphere is

$$\begin{aligned} E_{\text{sunlight}} &= L_\odot \times t \\ &= (3.85 \times 10^{26}\ W)(499 \text{ s}) \\ &= 1.9 \times 10^{29} \text{ J} = 1.2 \times 10^{42} \text{ MeV.} \end{aligned}$$

So even sunlight is more important than the cosmological constant by a factor of 1.76×10^4.

3. *The visible Universe consists of about 10^{53} kg of baryonic matter, of which only about 25% is in galaxies. Let us assume that all of the mass in galaxies is in stars, all of which are just like our sun, with a luminosity $L_\odot = 3.8 \times 10^{26}$ W (this assumption massively overestimates the number of stars and their average luminosity). Let us further assume that all of these stars have been burning continuously since the beginning of the Universe, 13.8 billion years ago (this assumption again overestimates the length of time stars have been burning). Finally, let us assume that none of the photons emitted by these stars are absorbed by other matter (again, clearly a ridiculous assumption). What is the energy in photons in the Universe today as a result of stellar fusion? Compare this to the energy in the stars themselves.*

   The total mass of stars is

$$0.25 \times 10^{53} \text{ kg} = 2.5 \times 10^{52} \text{ kg}.$$

   The Sun has a mass of 2×10^{30} kg, so there are

$$N = \frac{2.5 \times 10^{52} \text{ kg}}{2 \times 10^{30} \text{ kg}} = 1.25 \times 10^{22}$$

   stars in the visible Universe (under this set of assumptions). Each has been burning for 13.8 billion years, with an energy output rate of $L_\odot$, so the total energy emitted in starlight is

$$\begin{aligned} E_\gamma &= N \Delta t L_\odot \\ &= (1.25 \times 10^{22})(13.8 \times 10^9 \text{ yr})(3.15 \times 10^6 \text{ s/yr})(3.8 \times 10^{26} \text{ J/s}) \\ &= 2.1 \times 10^{65} \text{ J}. \end{aligned}$$

   Converting the rest energy of stars to an energy equivalent via $E = mc^2$,

$$E_* = (2.5 \times 10^{52} \text{ kg})(3 \times 10^8 \text{ m/s})^2 = 2.3 \times 10^{69} \text{ J}.$$

   Therefore, the ratio of energy in starlight to the energy in stars is

$$\frac{E_\gamma}{E_*} = \frac{2.1 \times 10^{65} \text{ J}}{2.3 \times 10^{69} \text{ J}} \sim 10^{-4}.$$