Physics 444: Problem Set #9
due December 1, 2021

1. *In the lecture, I said that, around the time of BBN, the Universe was radiation-dominated, with*

$$T(t) \approx 10^{10}\ K \left(\frac{t}{1\ s}\right)^{-1/2}.$$

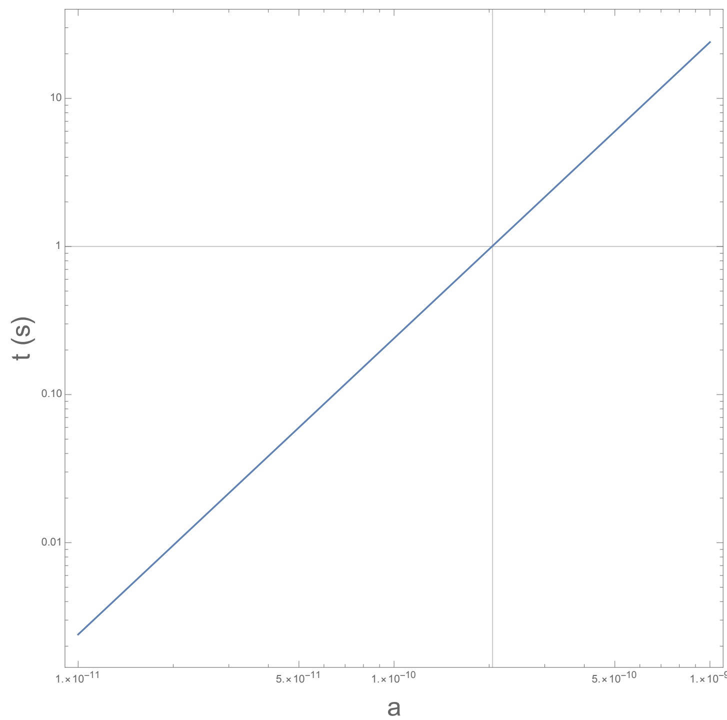
Using our numeric solution in the Benchmark Universe, calculate the actual temperature when the Universe was 1 second old. What was the scale factor a at 1 second? What are the density parameters Ω_r, Ω_m, and Ω_Λ at this time? What was the energy density of radiation at this time ϵ_r?

Numerically, we know that the relation between time and scale factor is

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{r,0}a^{-2} + \Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)}}.$$

Given the number in class, I know that the relevant scale factors must be around $a \sim 2.7255$ K$/10^{10}$ K $\sim 10^{-10}$. Therefore, I can numerically solve for time t with upper limits of integration around this value.

Using $H_0 = 67.3$ km/s/Mpc$= 2.18 \times 10^{-18}$ s, $\Omega_{r,0} = 9.18 \times 10^{-5}$, $\Omega_{m,0} = 0.315$, and $\Omega_{\Lambda,0} = 0.685$, I plot the time versus the scale factor for scale factors from 10^{-11} to 10^{-9}, I find the following plot:



By inspection, I find that the age of the Universe is 1 second when $a = 2.05 \times 10^{-10}$. Explicit calculation bears this out.

Thus, the temperature of the Universe at 1 second is

$$T(1 \text{ s}) = \frac{T_0}{a(1 \text{ s})} = (2.7255 \text{ K})(2.05 \times 10^{-10})^{-1} = 1.33 \times 10^{10} \text{ K}.$$

We can calculate the density parameters using $a = 2.05 \times 10^{-10}$ and that fact that the Universe is flat:

$$\begin{aligned}
\Omega_{r,\text{BBN}} = \frac{\Omega_{r,\text{BBN}}}{\Omega_{r,\text{BBN}} + \Omega_{m,\text{BBN}} + \Omega_{\Lambda,\text{BBN}}} &= \frac{\Omega_{r,0} a^{-4}}{\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}} \\
&= 1 - 7.06 \times 10^{-7} \\
\Omega_{m,\text{BBN}} = \frac{\Omega_{m,\text{BBN}}}{\Omega_{r,\text{BBN}} + \Omega_{m,\text{BBN}} + \Omega_{\Lambda,\text{BBN}}} &= \frac{\Omega_{m,0} a^{-3}}{\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}} \\
&= 7.06 \times 10^{-7} \\
\Omega_{\Lambda,\text{BBN}} = \frac{\Omega_{\Lambda,\text{BBN}}}{\Omega_{r,\text{BBN}} + \Omega_{m,\text{BBN}} + \Omega_{\Lambda,\text{BBN}}} &= \frac{\Omega_{\Lambda,0}}{\Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}} \\
&= 1.32 \times 10^{-35}
\end{aligned}$$

We can calculate the energy density of radiation by using

$$\epsilon_r(t) = \epsilon_{r,0} a(t)^{-4}$$

where $\epsilon_{r,0} = \epsilon_{c,0}\Omega_{r,0} = (4770 \text{ MeV/m}^3)(9.18 \times 10^{-5}) = 0.438 \text{ MeV/m}^3$. Thus,

$$\epsilon(1 \text{ s}) = (0.438 \text{ MeV/m}^3)(2.05 \times 10^{-10})^{-4} = 2.48 \times 10^{38} \text{ MeV/m}^3.$$

2. *Suppose that the number of inflationary e-foldings was the value typically assumed: $N = 62$. Assume further that inflation started at a time $t_i = 10^{-36}$ s when the Hubble parameter was $H_i = t_i^{-1}$. and that prior to this time, the Universe was radiation-dominated.*

   (a) *Calculate the physical particle horizon size at t_i (just as inflation starts) $d_{\rm hor}(t_i)$, and after inflation ends $d_{\rm hor}(t_f)$.*
   The particle horizon at some time t is

$$d_{\rm hor}(t) = ca(t) \int_0^t \frac{dt}{a(t)}.$$

   If the Universe was radiation dominated, $a(t) = a(t_i)(t/t_i)^{1/2}$. Thus,

$$\begin{aligned} d_{\rm hor}(t_i) &= ca(t_i) \int_0^{t_i} t_i^{1/2} t^{-1/2} \frac{dt}{a(t_i)} \\ &= 2ct_i \\ &= 2(3 \times 10^8 \text{ m/s})(10^{-36} \text{ s}) = 6 \times 10^{-28} \text{ m}. \end{aligned}$$

   During inflation, the scale factor's dependence on time goes as

$$a(t) = a(t_i) e^{H_i(t-t_i)}.$$

   The number of e-foldings is

$$e^N = a(t_f)/a(t_i) = e^{H_i(t_f - t_i)} = e^{62}.$$

Therefore, after inflation,

$$\begin{aligned}
d_{\text{hor}}(t_f) &= ca(t_f)\int_0^{t_f}\frac{dt}{a(t)} \\
&= ca(t_f)\left(\int_0^{t_i}\frac{dt}{a(t)}+\int_{t_i}^{t_f}\frac{dt}{a(t)}\right) \\
&= c\frac{a(t_f)}{a(t_i)}\left(2t_i+\int_{t_i}^{t_f}e^{-H_i(t-t_i)}dt\right) \\
&= c\frac{a(t_f)}{a(t_i)}\left(2t_i+H_i^{-1}[1+e^{-H_i(t_f-t_i)}]\right) \\
&= ce^N t_i\left(3+e^{-N}\right) \\
&= 3(3\times10^8\ \text{m/s})(10^{-36}\ \text{s})e^{62} = 0.76\ \text{m}.
\end{aligned}$$

(b) *Today, the distance to the CMB is $d_p(t_0, t_{\text{CMB}}) = 13.9$ Gpc. In order for all of the volume that we now see to have been in causal contact before inflation, the horizon size at the end of inflation must translate to a physical size today at least as large as $d_p(t_0, t_{\text{CMB}})$. What must have been the maximum scale factor after inflation? What was the minimum temperature (in K) must the Universe have been immediately after inflation? What energy does this correspond to (in GeV)?*

The distance $d_{\text{hor}}(t_f)$ today will have a physical size increased by the ratio of scale factors:

$$\frac{a(t_0)}{a(t_f)}d_{\text{hor}}(t_f) = a(t_f)^{-1}d_{\text{hor}}(t_f).$$

We are demanding that

$$a(t_f)^{-1}d_{\text{hor}}(t_f) > d_p(t_0, t_{\text{CMB}}).$$

Therefore

$$\begin{aligned}
a(t_f) &< \frac{d_{\text{hor}}(t_f)}{d_p(t_0, t_{\text{CMB}})} \\
&= \frac{0.76\ \text{m}}{(13.9\times10^9\ \text{pc})(3.09\times10^{16}\ \text{m/pc})} \\
a(t_f) &< 1.8\times10^{-27}.
\end{aligned}$$

We can then determine the minimum temperature after inflation:

$$\begin{aligned} T(t_f) &= \frac{T_0}{a(t_f)} \\ &> (2.7255 \text{ K})(1.8 \times 10^{-27})^{-1} = 1.5 \times 10^{27} \text{ K} \end{aligned}$$

Using $k = 8.62 \times 10^{-5}$ eV/K, this gives

$$E(t_f) = kT(t_f) > (8.62\times10^{-5} \text{ eV/K})(1.5\times10^{27} \text{ K}) = 1.3\times10^{14} \text{ GeV}.$$

(c) *Suppose that, instead of 62 e-foldings, the Universe inflated for 60 e-foldings, starting at the same time as previously assumed in part (a) and still ending with horizon size after inflation translating to a physical size today at least as large as* $d_p(t_0, t_{\rm CMB})$*.*

What now is the minimum temperature (in K) and energy (in GeV) of the Universe after inflation? Is this minimum temperature larger or smaller than the temperature you calculated assuming the minimum number of e-foldings?

We can now see that the size of the horizon after inflation is

$$\begin{aligned} d_{\rm hor}(t_f) &= ce^N t_i \left(3 + e^{-N}\right) \\ &= 3(3 \times 10^8 \text{ m/s})(10^{-36} \text{ s})e^{60} = 0.10 \text{ m}. \end{aligned}$$

Tracing out the same logic as in (c), the maximum scale factor after inflation as

$$\begin{aligned} a(t_f) &< \frac{d_{\rm hor}(t_f)}{d_p(t_0, t_{\rm CMB})} \\ &= \frac{0.10 \text{ m}}{(13.9 \times 10^9 \text{ pc})(3.09 \times 10^{16} \text{ m/pc})} \\ a(t_f) &< 2.4 \times 10^{-28}. \end{aligned}$$

and

$$\begin{aligned} T(t_f) &= \frac{T_0}{a(t_f)} \\ &> (2.7255 \text{ K})(2.4 \times 10^{-28})^{-1} = 1.1 \times 10^{28} \text{ K} = 9.5 \times 10^{14} \text{ GeV} \end{aligned}$$

We can see that the Universe can have a higher temperature after inflation if there were a fewer number of e-foldings.

(d) *Let us continue assuming that there were 60 e-foldings, rather than 62. However, rather than the temperature after inflation being the minimum calculated in part (c), let us assume that, after inflation, the Universe has the minimum temperature which you calculated in part (b). In that case, what is the physical distance today corresponding to the horizon distance after inflation $d_{\rm hor}(t_f)$? In this case, would we be seeing regions in the CMB which were never in causal contact?*

The horizon distance after inflation is

$$\begin{aligned} d_{\rm hor}(t_f) &= ce^N t_i \left(3 + e^{-N}\right) \\ &= 3(3 \times 10^8 \text{ m/s})(10^{-36} \text{ s})e^{60} = 0.10 \text{ m}. \end{aligned}$$

If the temperature of the Universe after inflation is 1.5×10^{27} K, this corresponds to an $a(t_f) = 1.8 \times 10^{-27}$. Therefore, today, the horizon distance at the end of inflation corresponds to the physical distance

$$\frac{a(t_0)}{a(t_f)} d_{\rm hor}(t_f) = (1.8 \times 10^{-27})^{-1}(0.1 \text{ m}) = 5.6 \times 10^{25} \text{ m} = 1.8 \text{ Gpc}.$$

That is, today, regions within 1.8 Gpc of each other would all have been in causal contact before inflation. Since the distance to the CMB is ~ 14 Gpc, not all of the CMB would have been in causal contact before inflation.

3. *The dark energy active in the Universe today shares many similarities to inflation. Assuming that we are dark energy-dominated today, how long would we have to wait for 62 e-foldings to occur? How far away would the present-day CMB surface-of-last-scattering be after this many e-foldings?*

If we are dark-energy dominated today, then the scale factor increases as

$$a(t) = e^{H_0(t-t_0)}$$

where H_0 is the Hubble parameter today, 67.3 km/s/Mpc or $(14.5 \text{ Gyr})^{-1}$. For 62 e-foldings,

$$t - t_0 = 62H_0^{-1} = 62(14.5 \text{ Gyr}) = 899 \text{ Gyr}.$$

After this many e-foldings, the scale factor will be

$$a(t_f) = e^{62} a_0 = e^{62} = 10^{27}.$$

Presently, the CMB surface-of-last scattering is at a distance of 13.9 Gpc. This is both the physical distance and the comoving coordinate (since those two numbers agree by definition at t_0). After 62 e-foldings, the physical distance to the present-day surface of last scattering will increase by the scale factor:

$$d_{\mathrm{CMB}} = 10^{27}(13.9 \text{ Gpc}) = 1.39 \times 10^{28} \text{ Gpc}.$$