Physics 444: Problem Set #5
due October 10, 2019

1. *Consider an expanding, positively curved universe containing only a cosmological constant* $\Omega_0 = \Omega_{\Lambda,0} > 1$.

   (a) *Show that such a universe underwent a "Big Bounce" (i.e., the universe was contracting, but stops, then starts expanding) at a scale factor*

$$a_{\rm bounce} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2},$$

Remember to prove that this is a minimum

Using the form of the Friedman equation in terms of density parameters, we can solve for the curvature R_0 in terms of Ω_0 (setting $\kappa = +1$):

$$\begin{aligned} 1 - \Omega_0 &= -\frac{c^2}{R_0^2 a_0^2 H_0^2} \\ \frac{c^2}{R_0^2} &= H_0^2(\Omega_0 - 1) \end{aligned}$$

A Big Bounce will occur when $\dot{a} = 0$ and $\ddot{a} > 0$. From the Friedman equation in terms of $\dot{a}$,

$$\begin{aligned} 0 = \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3c^2}\epsilon_\Lambda - \frac{c^2}{R_0^2 a_{\rm bounce}^2} \\ &= \frac{8\pi G}{3c^2}\epsilon_\Lambda - (\Omega_0 - 1)H_0^2 a_{\rm bounce}^{-2} \\ (\Omega_0 - 1)a_{\rm bounce}^{-2} &= \frac{8\pi G}{3c^2 H_0^2}\epsilon_\Lambda. \end{aligned}$$

However, the definition of the critical density is

$$\epsilon_c = \frac{3c^2}{8\pi G}H^2.$$

Therefore,

$$\begin{aligned} (\Omega_0 - 1)a_{\rm bounce}^{-2} &= \frac{\epsilon_\Lambda}{\epsilon_{c,0}} \equiv \Omega_0 \\ a_{\rm bounce} &= \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2} \end{aligned}$$

We have to show that this is a minimum, not a maximum, so, using the acceleration equation and the equation of state of dark energy,

$$\begin{aligned}\frac{\ddot{a}_{\text{bounce}}}{a_{\text{bounce}}} &= -\frac{4\pi G}{3c^2}(\epsilon_0 - 3\epsilon_0) \\ &= +\frac{8\pi G}{3c^2}\epsilon_0 > 0\end{aligned}$$

So this is a minimum.

(b) *and that the scale factor as a function of time is*

$$a(t) = a_{\text{bounce}} \cosh[\sqrt{\Omega_0} H_0(t - t_{\text{bounce}})],$$

where t_{bounce} is the time at which the Big Bounce occurred.

Following the technique of "assume the answer and then prove it works," we can plug this formula for the scale factor into the Friedman equation. Starting with the Friedman equation, and using the results from the previous part that

$$(\Omega_0 - 1)a_{\text{bounce}}^{-2} = \frac{8\pi G}{3c^2 H_0^2}\epsilon_\Lambda, \;\; \frac{c^2}{R_0^2} = H_0^2(\Omega_0 - 1)$$

we can write the Friedman equation as

$$\begin{aligned}\dot{a}^2 &= \frac{8\pi G}{3c^2}\epsilon_\Lambda a^2 - \frac{c^2}{R_0^2} \\ \dot{a}^2 &= H_0^2(\Omega_0 - 1)\frac{a^2}{a_{\text{bounce}}^2} - H_0^2(\Omega_0 - 1)\end{aligned}$$

The left-hand side is

$$\begin{aligned}\dot{a} &= \frac{d}{dt}a_{\text{bounce}} \cosh[\sqrt{\Omega_0} H_0(t - t_{\text{bounce}})] \\ &= a_{\text{bounce}}\sqrt{\Omega_0} H_0 \sinh[\sqrt{\Omega_0} H_0(t - t_{\text{bounce}})] \\ \dot{a}^2 &= a_{\text{bounce}}^2 \Omega_0 H_0^2 \sinh^2[\sqrt{\Omega_0} H_0(t - t_{\text{bounce}})] \\ &= \left(\frac{\Omega_0 - 1}{\Omega_0}\right)\Omega_0 H_0^2 \sinh^2[\sqrt{\Omega_0} H_0(t - t_{\text{bounce}})] \\ &= H_0^2(\Omega_0 - 1)\sinh^2[\sqrt{\Omega_0} H_0(t - t_{\text{bounce}})].\end{aligned}$$

The right-hand side is

$$\begin{aligned}
H_0^2(\Omega_0 - 1)\frac{a^2}{a_{\text{bounce}}^2} - H_0^2(\Omega_0 - 1) &= \\
H_0^2(\Omega_0 - 1)\frac{a_{\text{bounce}}^2 \cosh^2[\sqrt{\Omega_0}H_0(t - t_{\text{bounce}})]}{a_{\text{bounce}}^2} - H_0^2(\Omega_0 - 1) &= \\
H_0^2(\Omega_0 - 1)\left[\cosh^2[\sqrt{\Omega_0}H_0(t - t_{\text{bounce}})] - 1\right] &= \\
H_0^2(\Omega_0 - 1)\sinh^2[\sqrt{\Omega_0}H_0(t - t_{\text{bounce}})]. &
\end{aligned}$$

Thus, the left- and right-hand sides are equal at all times, and so this is a valid solution for the scale factor.

(c) *What is the time $t_0 - t_{\text{bounce}}$ that has elapsed since the Big Bounce, expressed as a function of H_0 and Ω_0.*

The scale factor today is 1. Therefore, from our solution, and defining $T \equiv t - t_{\text{bounce}}$

$$\begin{aligned}
1 &= a_{\text{bounce}} \cosh[\sqrt{\Omega_0}H_0 T] \\
\cosh[\sqrt{\Omega_0}H_0 T] &= a_{\text{bounce}}^{-1} = \left(\frac{\Omega_0}{\Omega_0 - 1}\right)^{1/2} \\
T &= \Omega_0^{-1/2} H_0^{-1} \cosh^{-1}\left[\left(\frac{\Omega_0}{\Omega_0 - 1}\right)^{1/2}\right]
\end{aligned}$$

2. *Prove that, in a Universe filled with a single type of energy with equation of state w, if $\Omega(t_0) = 1$ then $\Omega = 1$ for all times.*

In complete generality, this can be proved by using the definition of

$$\epsilon_c(t) = \frac{3c^2}{8\pi G}H(t)^2$$

and the relation that

$$\epsilon_w(t) = \epsilon_{w,0}a(t)^{-3(1+w)}.$$

Therefore,

$$\Omega(t) = \frac{8\pi G\epsilon_{w,0}}{3c^2}\frac{a^{-3(1+w)}}{H(t)^2}.$$

But from the Friedmann equation, for a flat Universe,

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon_{w,0}a^{-3(1+w)},$$

so

$$\Omega(t) = \frac{8\pi G\epsilon_{w,0}}{3c^2}a^{-3(1+w)}\left(\frac{8\pi G}{3c^2}\epsilon_{w,0}a^{-3(1+w)}\right)^{-1} = 1.$$

For a Universe with w between -1 and $+1$ (but not including -1), we can use the explicit formulae for $a(t)$ to prove this. The density parameter is defined as

$$\Omega(t) = \frac{\epsilon_w(t)}{\epsilon_c(t)}.$$

If $\Omega(t_0) = 1$, the Universe is flat.

In a flat Universe, the critical density evolves as

$$\begin{aligned}
\epsilon_c(t) &= \frac{3c^2}{8\pi G}H(t)^2 \\
&= \frac{3c^2}{8\pi G}\frac{4}{9(1+w)^2}t^{-2} \\
&= \frac{c^2}{6\pi(1+w)^2 G}t^{-2}
\end{aligned}$$

While

$$\begin{aligned}
\epsilon_w(t) &= \epsilon_0 a(t)^{-3(1+w)} \\
&= \epsilon_0\left[\left(\frac{t}{t_0}\right)^{2/3(1+w)}\right]^{-3(1+w)} \\
&= \epsilon_0\left(\frac{t}{t_0}\right)^{-2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Omega(t) &= \frac{\epsilon_w(t)}{\epsilon_c(t)} \\
&= \frac{\epsilon_0 t_0^2 t^{-2}}{\frac{c^2}{6\pi(1+w)^2G}t^{-2}} \\
&= \frac{6\pi G(1+w)^2}{c^2}\epsilon_0 t_0^2.
\end{aligned}$$

That is, $\Omega(t)$ is independent of time. By assumption, we know that $\Omega(t_0) = 1$, so

$$\Omega(t) = \frac{6\pi G(1+w)^2}{c^2}\epsilon_0 t_0^2 = \Omega(t_0) \equiv 1.$$

3. *If the Universe has a form of energy with $w < -1$, something very drastic will occur. Let us calculate this.*

   (a) *Show that the general solution to the Friedmann equation for $w < -1$ is of the form*

   $$a(t) = A(t_R - t)^x.$$

   What are x, and A in terms of ϵ_0, w, and fundamental constants?

   The Friedmann equation is

   $$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon_0 a^{-(1+3w)}.$$

   Using the general solution, the left hand side of the Friedmann equation is

   $$\begin{aligned} \dot{a} &= -Ax(t_R - t)^{x-1} \\ \dot{a}^2 &= A^2x^2(t_R - t)^{2x-2}. \end{aligned}$$

   The right hand side is

   $$\frac{8\pi G\epsilon_0}{3c^2}a^{-(1+3w)} = \frac{8\pi G\epsilon_0}{3c^2}A^{-(1+3w)}(t_R - t)^{-x(1+3w)}$$

   Setting the two sides equal

   $$A^2x^2(t_R - t)^{2x-2} = \frac{8\pi G\epsilon_0}{3c^2}A^{-(1+3w)}(t_R - t)^{-x(1+3w)}.$$

   If this equation is to be true, the time dependence must be the same on both sides of the equation. Thus

   $$\begin{aligned} 2x - 2 &= -x(1+3w) \\ x &= \frac{2}{3(1+w)} \end{aligned}$$

Thus, looking at the non-time dependent parts of the Friedmann equation,

$$\begin{aligned}
\frac{4A^2}{9(1+w)^2} &= \frac{8\pi G\epsilon_0}{3c^2}A^{-(1+3w)} \\
A^{3(1+w)} &= \frac{6\pi G\epsilon_0}{c^2}(1+w)^2 \\
A &= \left[\frac{6\pi G\epsilon_0}{c^2}(1+w)^2\right]^{1/3(1+w)}
\end{aligned}$$

So

$$a(t) = \left[\frac{6\pi G\epsilon_0}{c^2}(1+w)^2\right]^{1/3(1+w)} (t_R - t)^{\frac{2}{3(1+w)}} = \left[\frac{6\pi G\epsilon_0}{c^2}(1+w)^2(t_R-t)^2\right]^{1/3(1+w)}$$

(b) *What happens to the scale factor as $t \to t_R$ when $w < -1$? Why do you think this scenario is known as "the Big Rip"?*

From our solution, we see that

$$a(t) \propto (t_R - t)^{\frac{2}{3(1+w)}}.$$

When $w < -1$,

$$\frac{2}{3(1+w)} < 0.$$

Defining $\alpha = \left|\frac{2}{3(1+w)}\right|$,

$$a(t) \propto \frac{1}{(t_R - t)^{\alpha}}.$$

Therefore, as $t \to t_R$, $a(t) \to \infty$. Therefore, the scale factor grows to infinity in a finite amount of time. As the scale factor approaches infinity, every object will be dissociated, and rip apart from Hubble expansion: thus, the name "Big Rip."

(c) *Does this Universe have a beginning? (that is, does $a(t) = 0$ at a finite time?)*

From our solution,

$$a(t) \propto \frac{1}{(t_R - t)^{\alpha}},$$

we see that the scale factor goes to zero when $(t_R - t) \to \infty$, or when $t \to -\infty$. Thus, there is no finite time when this Universe was at zero size.

(d) *Recalling that $a(t_0) = 1$, what is t_R in terms of t_0, ϵ_0, w, and fundamental constants?*

Setting $a(t_0) = 1$, we see that

$$\begin{aligned}
1 &= \left[\frac{6\pi G\epsilon_0}{c^2}(1+w)^2\right]^{1/3(1+w)} (t_R - t_0)^{\frac{2}{3(1+w)}} \\
&= \left[\frac{6\pi G\epsilon_0}{c^2}(1+w)^2(t_R - t_0)^2\right]^{1/3(1+w)} \\
1 &= \frac{6\pi G\epsilon_0}{c^2}(1+w)^2(t_R - t_0)^2 \\
t_R - t_0 &= \left(\frac{6\pi G\epsilon_0}{c^2}(1+w)^2\right)^{-1/2} \\
t_R &= \left(\frac{6\pi G\epsilon_0}{c^2}(1+w)^2\right)^{-1/2} + t_0.
\end{aligned}$$

(e) *Assuming that our own Universe (with $t_0 = 13.8$ Gyr and $H_0 = \sqrt{8\pi G\epsilon_0/3c^2} = 67.3$ km/s/Mpc) is flat and composed only of energy with $w < -1$. For $w = -1.1$ and $w = -2$, calculate t_R and plot the evolution of the scale factor from $t = 0$ to t_R.*

Using

$$H_0 = \sqrt{8\pi G\epsilon_0/3c^2} = 67.3\ \text{km/s/Mpc} = (14.5\ \text{Gyr})^{-1},$$

we see that

$$\begin{aligned}
t_R &= \left(\frac{9}{4}H_0^2(1+w)^2\right)^{-1/2} + t_0 \\
&= \frac{2}{3|1+w|}H_0^{-1} + t_0 \\
t_R(w=-1.1) &= \frac{2}{3|-0.1|}(14.5\ \text{Gyr}) + 13.8\ \text{Gyr} = 110\ \text{Gyr} \\
t_R(w=-2) &= \frac{2}{3|-1|}(14.5\ \text{Gyr}) + 13.8\ \text{Gyr} = 23.5\ \text{Gyr}
\end{aligned}$$

In terms of H_0, the evolution of the scale factor is:

$$\begin{aligned} a(t) &= \left[\frac{6\pi G\epsilon_0}{c^2}(1+w)^2\right]^{1/3(1+w)} (t_R - t)^{\frac{2}{3(1+w)}} \\ &= \left[\frac{9H_0^2}{4}(1+w)^2\right]^{1/3(1+w)} (t_R - t)^{\frac{2}{3(1+w)}} \end{aligned}$$

Plotting the two scale factors, I see the following (where the solid red is the $w = -1.1$ curve and the dotted blue is the $w = -2$ curve).