Physics 444: Problem Set #2
due September 19, 2019

1. *Imagine you are living in a 2-dimensional space. You look at a small object which is oriented so that its physical extent perpendicular to your line of sight is $d\ell$.*

   (a) *If the space you are living in is flat, what is the angular size $d\theta$ that the small object takes up in your field of view as a function of the distance $r \gg d\ell$ between you and the object?*

   In flat space, the metric for distances is:

   $$d\ell^2 = dr^2 + r^2 d\theta^2.$$

   For a distant object that is perpendicular to the line of sight, $dr = 0$ (both ends of the object are at the same distance r, assuming $r \gg d\ell$). Therefore, in this particular situation, the distance metric reduces to

   $$d\ell^2 = r^2 d\theta^2,$$

   or

   $$d\theta = d\ell/r.$$

   (b) *If instead you are living on a sphere of radius R, what is the angular size $d\theta$ that the small object takes up in your field of view, as a function of the distance measured on the surface of the sphere r? (You can imagine that the rays of light travel along the surface of the sphere.) Explain the behavior of $d\theta$ as $r \to \pi R$.*

   In this case, the metric is

   $$d\ell^2 = dr^2 + R^2 \sin^2(r/R) d\theta^2.$$

   Again setting $dr = 0$, the angular size is

   $$d\theta = \frac{d\ell}{R\sin(r/R)}.$$

   If you send $r \to \pi R$, then $\sin(r/R) \to 0$ and $d\theta \to \infty$. However, physically, this corresponds to the object being placed on the other size of the spherical surface from you (the "South Pole" if you are standing at the "North Pole"). In this case, every line of sight from you radially outwards would intersect the object, so it would take up the entire field of view.

2. *Imagine we lived in a* static *(that is, not expanding or collapsing) universe that was topologically a 3-sphere and had a radius of curvature of* $R = 5$ *Gpc.*

   (a) *We will eventually learn that the Cosmic Microwave Background has a special scale length that is particularly important. This length (in our Universe) is 250 kpc. Assuming the CMB is 4.2 Gpc away, what is the angular size of this length in our imagined curved, non-expanding universe? How would that compare to the angular size in a flat universe?*

   We can think of the CMB length as forming a right triangle with us, with the long leg having length $r_{\mathrm{CMB}} = 4.2 \times 10^3$ Mpc (the distance between the Earth and the CMB), and the short leg being the characteristic length $d\ell = 0.250$ Mpc. Since $d\ell \ll r_{\mathrm{CMB}}$, we can work approximate this as an infinitesimal distance. Using the metric of positively curved space,

$$d\ell^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left[d\theta^2 + \sin^2\theta d\phi^2\right],$$

   We can arrange our coordinate system so that the CMB length is at right-angles to us, so $dr = d\phi = 0$, and

$$\begin{aligned} d\ell^2 &= R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2 \\ d\theta &= \frac{d\ell}{R}\left[\sin\frac{r}{R}\right]^{-1} \\ d\theta &= \frac{d\ell}{R}\left[\sin\frac{r_{\mathrm{CMB}}}{R}\right]^{-1} \\ &= \frac{0.25\ \mathrm{Mpc}}{5 \times 10^3\ \mathrm{Mpc}}\left[\sin\frac{4.2 \times 10^3\ \mathrm{Mpc}}{5 \times 10^3\ \mathrm{Mpc}}\right]^{-1} = 6.7 \times 10^{-5}. \end{aligned}$$

   In the flat universe, the angular size is

$$\begin{aligned} d\ell^2 &= dr^2 + r^2\left[d\theta^2 + \sin^2\theta d\phi^2\right] \\ d\theta_{\mathrm{flat}} &= \frac{d\ell}{r} \\ &= \frac{0.25\ \mathrm{Mpc}}{4.2 \times 10^3\ \mathrm{Mpc}} \\ &= 6.0 \times 10^{-5} \end{aligned}$$

So in the curved universe, the spot size of the CMB is larger than in the flat universe.

(b) *Assume that you were really living in the curved universe, but thought you were living in the flat one. How far away would you think the CMB was?*

Taking the curved universe angle and plugging it into the flat universe distance metric, we would find a distance to the CMB of

$$\begin{aligned} r &= \frac{d\ell}{d\theta_{\text{curved}}} \\ &= \frac{0.250 \text{ Mpc}}{6.7 \times 10^{-5}} \\ &= 3.7 \text{ Gpc.} \end{aligned}$$

That is, because the spot size is larger, you think the CMB is closer than it "really" is.

(c) *Assuming you could see it, what would the apparent angular size of an object at a distance of πR be? Explain what your result means, physically.*

From our equation, the angular size of an object of length $d\ell$ at distance $r = \pi R$ would be

$$d\theta = \frac{d\ell}{R} [\sin \pi]^{-1} = \infty.$$

This is because an object at distance of $\pi \times R$ is sitting at the opposite "pole" of the universe, as far as is possible to go. Every direction one looks then has a line of sight ending at the star, and so the star covers the entire sky.

3. *Suppose you are a two-dimensional being, living on a sphere of radius R. Show that if you draw a circle of radius r, the circle's circumference will be*

$$C = 2\pi R \sin(r/R).$$

Idealize the Earth as a perfect circle of radius $R = 6371$ km. If you could measure distances with an error of ± 1 meter, how large a circle would you have to draw on the Earth's surface to convince yourself that the Earth is spherical rather than flat?

If the radius (along the curved surface of the Earth) is r, then the circle forms an arc (from the center of the Earth) with opening angle $\psi = r/R$. This arc has radius in Euclidean space of

$$r_e = R\sin\psi = R\sin r/R,$$

and so the circumference is

$$C = 2\pi R\sin(r/R).$$

If the Earth was flat, we would measure the circumference to be $C_e = 2\pi r$. We can measure r to within $\delta\ell = 1$ m, and we can measure the circumference to within $\delta\ell$ as well. Recalling that the Taylor expansion for $\sin x = x - \frac{x^3}{6}$,

$$\begin{aligned}
C_e - C &> \delta\ell \\
2\pi r - 2\pi R\sin r/R &> \delta\ell \\
2\pi R\left(\frac{r}{R} - \sin\frac{r}{R}\right) &> \delta\ell \\
2\pi R\left(\frac{r}{R} - \frac{r}{R} + \frac{r^3}{6R^3}\right) &> \delta\ell \\
\frac{\pi}{3}\frac{r^3}{R^2} &> \delta\ell \\
r &> 33,843 \text{ m.}
\end{aligned}$$

However, you can't measure the radius exactly either. The radius enters into this equation only at the 3rd order, so when we include the measurement error on r, we see that

$$\begin{aligned}
\frac{\pi}{3}\frac{(r\pm\delta\ell)^3}{R^2} &> \delta\ell \\
\frac{\pi}{3}\left(\frac{r^3}{R^2} + \frac{3(\pm\delta\ell)r^2}{R^2} + \frac{3(\pm\delta\ell)^2 r}{R^2} + \frac{(\pm\delta\ell)^3}{R^2}\right) &> \delta\ell
\end{aligned}$$

Dropping terms that are proportional to $(\delta\ell)^2$ (since they are very small compared to all other lengths), we have

$$\begin{aligned}
\frac{\pi}{3}\left(\frac{r^3}{R^2} \pm \frac{3\delta\ell r^2}{R^2}\right) &> \delta\ell \\
\frac{\pi}{3}\left(\frac{r^3}{R^2}\right) &> \left(1 \pm \pi\frac{r^2}{R^2}\right)\delta\ell \\
r &> 33,844 \text{ m.}
\end{aligned}$$

I picked the larger solution in the final step, because it is the more conservative answer.