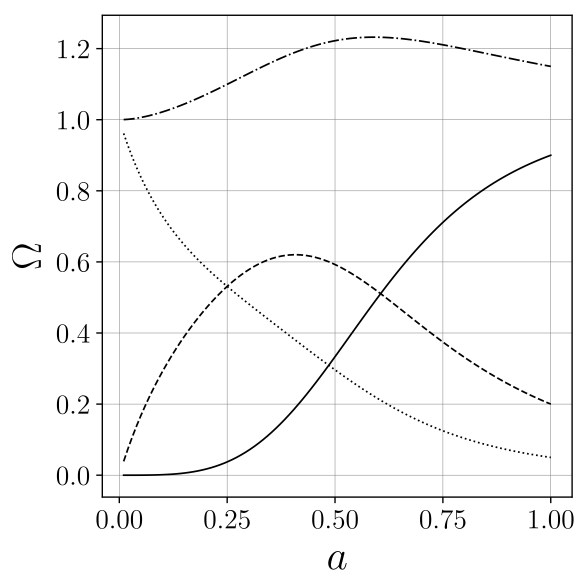
Physics 444: Midterm
October 29, 2021

Name:

You have 80 minutes for this exam. You may use a calculator and 1 pre-prepared sheet of equations. Each question is worth 10 points. Partial credit is given, so be sure to explain your reasoning. Unless explicitly stated, do not assume the parameters of the universe assumed in a problem are those of the Benchmark Universe.

1. *Consider a universe (not our own) which at the present day consists of radiation, matter, and dark energy.* **Do Not** *assume that the values of* $\Omega_{r,0}$, $\Omega_{m,0}$, *and* $\Omega_{\Lambda,0}$ *are the same as those in our own Benchmark Universe.*

   (a) *Here I show a plot of the evolution of* Ω_r, Ω_m, Ω_Λ, *and the total* Ω *as a function of scale factor for this universe. On the plot, label which curve corresponds to the density parameters for radiation, matter, dark energy, and the sum.*



The dashed-dotted curve is strictly larger than all others, and is equal at any a value to the sum of the other three curves, so this is the total Ω. The solid curve is increasing its share of Ω over time, so it is Ω_Λ. Radiation dilutes slower than matter, so radiation is more important at earlier times. Thus the dotted line is Ω_r and the dashed line is Ω_m.

(b) *Is this universe spatially open, closed, or flat?*

The total line is > 1, so the Universe is closed.

(c) *Calculate Ω_Λ when $a = 2$.*

Ω_Λ evolves as

$$\Omega_\Lambda(a) = \frac{\Omega_{\Lambda,0}}{\Omega_{\Lambda,0} + \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + (1 - \Omega_0)a^{-2}}.$$

Reading from the plot, $\Omega_{\Lambda,0} = 0.9$, $\Omega_{m,0} = 0.2$, and $\Omega_{r,0} = 0.05$. Therefore, $\Omega_0 = 0.9 + 0.2 + 0.05 = 1.15$ and

$$\Omega_\Lambda(2) = \frac{0.9}{0.9 + 0.05 \times (2)^{-4} + 0.2 \times (2)^{-3} + (1 - 1.15)(2)^{-2}} = 1.01053$$

2. *At the present day, the temperature of the CMB photons in* **the Benchmark Universe** *is $T_0 = 2.7255$ K.*

   (a) *What is the energy density in CMB photons today?*

   Using

$$\epsilon_\gamma = \frac{\pi^2}{15}\frac{(kT)^4}{\hbar^3 c^3},$$

   and plugging in the numeric values from the cheat sheet, we find

$$\begin{aligned} \epsilon_\gamma &= \frac{\pi^2}{15}\frac{(2.7255\ \mathrm{K})^4(8.62\times10^{-5}\ \mathrm{eV/K})^4}{(1.98\times10^{-7}\ \mathrm{eVm})^3} \\ &= 2.88\times10^5\ \mathrm{eV/m^3} = 0.288\ \mathrm{MeV/m^3} \end{aligned}$$

   (b) *What was the energy density in photons when the Universe had a scale factor of $a = 10^{-10}$?*

   Energy density in radiation scales like a^{-4}, so

$$\begin{aligned} \epsilon_\gamma(10^{-10}) &= \epsilon_{\gamma,0}\times(10^{-10})^{-4} \\ &= 10^{40}\times0.288\ \mathrm{MeV/m^3} = 2.88\times10^{39}\ \mathrm{MeV/m^3} \end{aligned}$$

   (c) *If we assumed for the moment that photons were the only form of energy relevant when* **the Benchmark Universe** *had a scale factor of $a = 10^{-10}$, what would be the Hubble parameter H at this time?*

   From the Friedmann equation, assuming a spatially flat Benchmark Universe,

$$\begin{aligned} H^2 &= \frac{8\pi G}{3c^2}\epsilon \\ &= \frac{8\pi G}{3c^2}(2.88\times10^{39}\ \mathrm{MeV/m^3}) \\ &= \frac{8\pi(6.67\times10^{-11}\ \mathrm{m^3/kg/s^2}}{3(3\times10^8\ \mathrm{m/s})^2}\left[2.88\times10^{45}\ \mathrm{eV/m^3}\times(6.24\times10^{18}\ \mathrm{eV/J})^{-1}\right] \\ &= 2.87\ \mathrm{s^{-2}} \\ H &= 1.69\ \mathrm{s^{-1}} = 5.22\times10^{19}\ \mathrm{km/s/Mpc}. \end{aligned}$$

3. *In the* **Benchmark Universe***, a galaxy with diameter of 10 kpc has an angular size on our sky of* 5.57×10^{-6} *radians, and has a redshift of* $z = 1.7$

   (a) *How far away (in physical units) is this galaxy today?*

   The angular distance is

$$d_A = \frac{\ell}{\delta\theta} = \frac{10 \text{ kpc}}{5.57 \times 10^{-6}} = (1800 \text{ Mpc})$$

   This is related to $S_\kappa(r) = r$ (for $\kappa = 0$) by

$$r = (1+z)d_A = (1+1.7)(1800 \text{ Mpc}) = 4850 \text{ Mpc}$$

   (b) *A supernova is observed in this galaxy. The intrinsic luminosity of a supernova is* $L = 4 \times 10^{36}$ *W. What is the measured flux of this supernova on Earth?*

   The luminosity distance can be written in two ways:

$$d_L = \sqrt{\frac{L}{4\pi f}} = (1+z)S_\kappa(r).$$

   Using our value for $S_\kappa(r) = r$ from the previous problem,

$$\begin{aligned}
\frac{L}{4\pi f} &= (1+z)^2 r^2 \\
f &= \frac{L}{4\pi(1+z)^2 r^2} \\
& \quad r = 4850 \text{ Mpc} = 1.50 \times 10^{26} \text{ m} \\
f &= \frac{4 \times 10^{36} \text{ W}}{4\pi(1+1.7)^2(1.50 \times 10^{26} \text{ m})^2} = 1.94 \times 10^{-18} \text{ W/m}^2.
\end{aligned}$$

   (c) *How far away (in physical units) was the galaxy from us when the supernova occurred?*

   The scale factor at emission time was $a = 1/(1+z) = 1/2.7 = 0.37$. Therefore

$$d_p(t_e) = a(t_e)r = 1800 \text{ Mpc}.$$

4. *The top quark is the heaviest particle known in the Standard Model. It has a mass of* 175 GeV/c^2.

   (a) *Estimate the minimum temperature (in K) at which the top quark would be relativistic.*

   The temperature should be somewhere close to the mass of the particle for it to be relativistic. Assuming $kT = mc^2$, I find

$$\begin{aligned} T &= (1.75 \times 10^{11} \text{ eV})(8.62 \times 10^{-5} \text{ eV/K})^{-1} \\ &= 2.03 \times 10^{15} \text{ K}. \end{aligned}$$

   (b) *Using your estimate from part a), and assuming the top quarks were the same temperature as the photons which go on to form the CMB, what is the redshift of the Universe at the latest time the top quarks were relativistic.*

   The temperature $T \propto a^{-1}$, so using $T_0 = 2.7255$ K and $a = 1/(1+z)$,

$$\begin{aligned} a &= \frac{2.7255}{2.03 \times 10^{15} \text{ K}} = 1.34 \times 10^{-15} \\ z \approx (1+z) &= 7.45 \times 10^{14}. \end{aligned}$$

5. *Suppose you had a spatially flat universe filled with energy density that has $w = +1/2$. The present-day Hubble parameter is H_0.* **Do Not** *assume that the values of $\Omega_{r,0}$, $\Omega_{m,0}$, $\Omega_{\Lambda,0}$, and H_0 are the same as those in our own Benchmark Universe.*

   (a) *What is the age of this universe, in terms of H_0?*

   From the cheat-sheet:

$$\begin{aligned} t_0 &= \frac{2}{3(1+\frac{1}{2})} H_0^{-1} \\ &= \frac{4}{9} H_0^{-1}. \end{aligned}$$

   (b) *What is the distance to the particle horizon in this universe, in terms of H_0?*

   From the definition of the proper distance:

$$d_p(t_0, t_e) = c \int_{t_e}^{t_0} \frac{dt}{a(t)},$$

   setting $t_e = 0$ and $a(t) = (t/t_0)^{4/9}$,

$$\begin{aligned} d_h &= c \int_0^{t_0} (t/t_0)^{-4/9} dt \\ &= \frac{9c}{5} t_0 \left. (t/t_0)^{5/9} \right|_0^{t_0} \\ &= \frac{9c}{5} t_0 = \frac{4c}{5} H_0^{-1} \end{aligned}$$

Cheat Sheet

Physical constants and useful conversions:

$$
\begin{aligned}
c &= 3 \times 10^{8} \text{ m/s} \\
G &= 6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \\
k &= 8.62 \times 10^{-5} \text{ eV/K} \\
h &= 6.63 \times 10^{-34} \text{ Js} = 4.14 \times 10^{-15} \text{ eVs} \\
\hbar c &= 1.98 \times 10^{-7} \text{ eVm} \\
1 \text{ Mpc} &= 3.09 \times 10^{22} \text{ m} \\
1 \text{ J} &= 6.24 \times 10^{18} \text{ eV} \\
1 \text{ km/s/Mpc} &= 3.24 \times 10^{-20} \text{ s}^{-1} \\
1 \text{ year} &= 3.15 \times 10^{7} \text{ s} \\
1 \text{ parsec} &= 3.26 \text{ ly} \\
1\, M_\odot &= 2 \times 10^{30} \text{ kg} \\
1\, L_\odot &= 3.83 \times 10^{26} \text{ W} = 3.83 \times 10^{33} \text{ erg/s}
\end{aligned}
$$

Benchmark Universe:

$$
T_0 = 2.7255 \text{ K},\ H_0 = 67.8 \text{ km/s/Mpc},\ t_0 = 13.8 \text{ Gyr}, \epsilon_{c,0} = 4870 \text{ MeV/m}^3, \\
\Omega_{\Lambda,0} = 0.692,\ \Omega_{m,0} = 0.306,\ \Omega_{r,0} = 9.03 \times 10^{-5}
$$

Equations:

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}}, \qquad z = \frac{H}{c} r \text{ (Hubble's Law)}, \ 1 + z = \frac{1}{a}$$

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa^2 d\Omega^2 \right], \qquad S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0) & \kappa = +1 \\ r & \kappa = 0 \\ R_0 \sinh(r/R_0) & \kappa = -1 \end{cases}$$

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \epsilon - \frac{\kappa c^2}{R_0^2 a^2}, \qquad 1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}$$

$$\dot{\epsilon} + \frac{3\dot{a}}{a}(\epsilon + P) = 0, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^3}(\epsilon + 3P)$$

$$P = w\epsilon, \qquad \epsilon_w = \epsilon_{w,0} a^{-3(1+w)}$$

$$\epsilon_\gamma(T) = \frac{\pi^2}{15} \frac{(kT)^4}{\hbar^3 c^3} \qquad n_\gamma(T) = \frac{2.4041}{\pi^2} \frac{(kT)^3}{\hbar^3 c^3}$$

$$\epsilon_c = \frac{3c^2}{8\pi G} H^2 \qquad \Omega_w = \frac{\epsilon_w}{\epsilon_c}$$

$$d_p(t_0, t_e) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \qquad d_p(t_e) = c a(t_e) \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{w,0} a^{-(1+3w)} + (1 - \Omega_0)}}, \qquad a(t) = \begin{cases} \frac{t}{t_0} & \Omega_0 = 0, \kappa = -1 \\ (t/t_0)^{2/3(1+w)} & \Omega_{w,0} = 1, \kappa = 0 \\ e^{H_0(t-t_0)} & \Omega_\Lambda = 1, \kappa = 0 \end{cases}$$

$$t_0 = \frac{2}{3(1+w)} H_0^{-1} \qquad (\Omega_{w,0} = 1, \kappa = 0, w > -1)$$

$$d_A = \frac{\ell}{\delta\theta} = \frac{S_\kappa(r)}{1+z} \qquad d_L = \sqrt{\frac{L}{4\pi f}} = (1+z) S_\kappa(r)$$

$$d_L(z) \approx \frac{cz}{H_0} \left[1 + \left(\frac{1 - q_0}{2} \right) z \right] \qquad q_0 = \frac{1}{2} \sum_w \Omega_{w,0}(1 + 3w)$$