Physics 444: Problem Set #7
due November 10, 2021

1. *Halos of dark matter tend have Navarro-Frenk-White (NFW) density distributions, with the density as a function of distance $\rho(r)$ given by*

$$\rho(r) = \frac{\rho_s}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2},$$

where ρ_s is some characteristic density and r_s is a scale radius.

(a) *What is the mass enclosed within a radius r for a NFW profile? (You may find Mathematica useful for evaluating the integrals)*

Integrating the mass distribution from $r = 0$ to a maximum r, we find:

$$\begin{aligned} M(<r) &= \int 4\pi r^2 dr \frac{\rho_s}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2} \\ &= 4\pi\rho_s r_s^3 \left[\log\left(1+\frac{r}{r_s}\right) - \frac{r}{r+r_s}\right] \end{aligned}$$

(b) *The Milky Way galaxy has a NFW-like dark matter distribution, with a r_s of perhaps 15 kpc (in reality, this value is actually not well known, but we will assume this value for this problem). The local density of dark matter near the Sun (located 8 kpc from the center) is approximately $0.01\,M_\odot/pc^3$. What is the dark matter mass enclosed within the radius of the Sun around the Galactic Center?*

We first need to determine the parameter ρ_s in the NFW profile, which we find by fitting to the local density:

$$\begin{aligned} 0.01\,M_\odot/\mathrm{pc}^3 &= \frac{\rho_s}{\left(\frac{8}{15}\right)\left(1+\frac{8}{15}\right)^2} \\ \rho_s &= 0.013\,M_\odot/\mathrm{pc}^3 \end{aligned}$$

Then we can evaluate the total mass enclosed:

$$\begin{aligned} M(<8\text{ kpc}) &= 4\pi(0.013\,M_\odot/\mathrm{pc}^3)(1.5\times10^4\text{ pc})^3\left[\log\left(1+\frac{8}{15}\right) - \frac{8}{8+15}\right] \\ &= 4.4\times10^{10}\,M_\odot. \end{aligned}$$

(c) *An NFW profile grows without end, so if you extend it to infinite radius, you find the mass of a galaxy to be infinite. We usually assume that the NFW profile is terminated when the dark matter density is approximately 200 times the critical density of the Universe ($\rho_c = 1.36 \times 10^{-7}\, M_\odot/pc^3$). Under this definition, what is the radial extent and total mass of the Milky Way's dark matter halo? (You may find using Mathematica or Python to solve a numeric equality to be useful)*

We first calculate the radial extend of the Milky Way, by setting the density equal to 200 times the critical:

$$\begin{aligned}
200 \times 1.36 \times 10^{-7}\, M_\odot/\mathrm{pc}^3 &= \frac{0.013\, M_\odot/\mathrm{pc}^3}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2} \\
2.1 \times 10^{-3} &= \frac{1}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2} \\
\frac{r}{r_s} &= 7.2 \\
r &= 108 \text{ kpc.}
\end{aligned}$$

The enclosed mass is

$$\begin{aligned}
M(< 108 \text{ kpc}) &= 4\pi(0.013\, M_\odot/\mathrm{pc}^3)(1.5 \times 10^4 \text{ pc})^3 \left[\log\left(1 + \frac{108}{15}\right) - \frac{108}{108+15}\right] \\
&= 6.8 \times 10^{11}\, M_\odot.
\end{aligned}$$

2. *The Milky Way Galaxy (and thus the Sun and the Earth) originated in a locally overdense region that would correspond to a hot spot in the CMB for some alien astronomer far away from us. Let's assume we are exactly in the center of this local overdense region.*

(a) *At the time of CMB decoupling, how far away (in physical distance) from the point that would eventually become the Earth was the edge of our local "hot patch?"*

We can interpret the size of the local hot patch in the CMB as the size of the horizon when the CMB decoupled. This is:

$$\begin{aligned}
d_{\rm hor}(t_{\rm CMB}) &= c a_{\rm CMB} \int_0^{t_{\rm CMB}} \frac{dt}{a(t)} \\
&= c a_{\rm CMB} \int_0^{a_{\rm CMB}} \frac{dt}{da}\frac{da}{a(t)} \\
&= \frac{c a_{\rm CMB}}{H_0} \int_0^{a_{\rm CMB}} \frac{1}{\sqrt{\Omega_{r,0}a^{-2} + \Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)}} \frac{da}{a(t)}
\end{aligned}$$

Plugging in the parameters for the Benchmark Model, and using $a_{\rm CMB} = 1/(z_{\rm CMB}+1) = 1/1101$, I find the local horizon distance (and thus the approximate size of our local hot patch) to be

$$d_{\rm hor}(t_{\rm CMB}) = 0.254 \text{ Mpc}.$$

(b) *Today, how far away is the edge of our local overdense CMB region?*

Distances scale as a, so today the horizon distance is

$$d_{\rm hor}(t_{\rm CMB})/a_{\rm CMB} = (1101)(0.254 \text{ Mpc}) = 279 \text{ Mpc}.$$

(c) *How far away would the alien astronomer have to be in order to see the CMB photons that were emitted by the atoms that would go on to form the Earth?*

This is the distance that the photons have traveled from the time of the CMB to today:

$$\begin{aligned}
d_{\rm p}(t_0, t_{\rm CMB}) &= \frac{c}{H_0} \int_{a_{\rm CMB}}^{1} \frac{1}{\sqrt{\Omega_{r,0}a^{-2} + \Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)}} \frac{da}{a(t)} \\
&= 13.9 \text{ Gpc}.
\end{aligned}$$

3. *Suppose the temperature T of a blackbody distribution is such that $kT \ll Q$, where $Q = 13.57\ eV$ is the ionization energy of hydrogen. (You may want to refer to Eq. (2.27) of the text for the blackbody spectrum for this problem. Also recall that* Mathematica *can very easily perform many analytic and numeric integrals.)*

   (a) *What fraction f of the blackbody photons have energy enough to ionize hydrogen?*

The blackbody spectrum of energy density for photons is

$$\epsilon(f)df = \frac{8\pi h}{c^3}\frac{f^3 df}{\exp(hf/kT) - 1}.$$

The energy of a photon is hf, so the energy density in terms of E (with $dE = hdf$)

$$\epsilon(E)dE = \frac{8\pi}{h^3c^3}\frac{E^3 dE}{\exp(E/kT) - 1}.$$

The number density is the energy density divided by the energy of each photon:

$$n(E)dE = \frac{8\pi}{h^3c^3}\frac{E^2 dE}{\exp(E/kT) - 1}.$$

If integrated over the entire range of energies, this yields

$$n = \int_0^\infty \frac{8\pi}{h^3c^3}\frac{E^2 dE}{\exp(E/kT) - 1} = \frac{16\pi\zeta(3)}{(hc)^3}(kT)^3$$

where $\zeta(3) = 1.202$.

The fraction of photons with energy greater than Q is

$$f = n^{-1}\int_Q^\infty \frac{8\pi}{h^3c^3}\frac{E^2 dE}{\exp(E/kT) - 1}$$

This integral has no analytic solution. However, when $E \gg kT$, the quantity $\exp(E/kT) - 1$ is very close to $\exp(E/kT)$, as the exponential is enormous. Therefore, (using MATHEMATICA to perform the integral)

$$\begin{aligned} f &\approx n^{-1}\int_Q^\infty \frac{8\pi}{h^3c^3}\frac{E^2 dE}{\exp(E/kT)} \\ &= n^{-1}\frac{8\pi}{(hc)^3}\left[2(kT)^3 + 2(kT)^2Q + (kT)Q^2\right]e^{-Q/kT} \\ &= \frac{1}{\zeta(3)}\left[1 + \frac{Q}{kT} + \frac{1}{2}\left(\frac{Q}{kT}\right)^2\right]e^{-Q/kT}. \end{aligned}$$

(b) *If $kT = kT_{\rm rec} = 0.323$ eV, what is the numerical value of f?*
Plugging in $kT = 0.323$ eV, this gives

$$f = 4.37 \times 10^{-16}.$$

That is, less than 1 photon per 10 quadrillion have enough energy to ionize a hydrogen atom. Alternatively, you could just do the numeric integral without approximation.

(c) *Given that $\eta = 6.2 \times 10^{-10}$, if $kT = kT_{\rm rec} = 0.323$ K, how many photons per baryon have enough energy to ionize hydrogen?*
Since

$$\eta = \frac{n_b}{n_\gamma} = 6.2 \times 10^{-10},$$

the number of photons per baryon that have enough energy to ionize a hydrogen atom is

$$\frac{n_\gamma(E > Q)}{n_b} = f\frac{n_\gamma}{n_b} = \frac{f}{\eta} = \frac{4.37 \times 10^{-16}}{6.2 \times 10^{-10}} = 7.05 \times 10^{-7}.$$