Physics 444: Final
December 23, 2019

Name:

You have 180 minutes for this exam. You may use a calculator and 1 pre-prepared sheet of equations. Each question is worth 10 points. Partial credit is given, so be sure to explain your reasoning when necessary.

1. Assuming that inflation occurred, place the following events in the history of the Universe in a timeline in the order in which they occurred.

   (a) Start of growth of baryonic structure

   (b) Electron-proton recombination

   (c) Start of growth of dark matter structure

   (d) Big Bang Nucleosynthesis

   (e) End of inflation

   (f) Matter-radiation equality

   (g) Formation of primordial fluctuations away from the average energy density.

   (h) Matter-Dark Energy equality

   (i) Scale Factor $a = 1$.

   (j) CMB decoupling

   (k) Latest time that the asymmetry between matter and antimatter could have formed.

   Let us work backwards in time. (i) is the present day. A few billion years ago, we had matter-dark energy equality, so (h) precedes (i).Therefore, we have (in order of increasing time):

   $$h \to i.$$

   Now we are up to the CMB era. The recombination of electrons with protons precedes (and instigates) the CMB decoupling, so (b) precedes (j). Matter-radiation equality precedes both, so (f) is earlier than (b) and (j). Dark matter structure growth starts immediately after matter domination begins, and baryonic structure starts forming immediately after decoupling. Therefore,

$$f \to c \to b \to j \to a \to h \to i.$$

Big Bang Nucleosynthesis is much earlier than matter radiation equality, and by the time BBN occurs, we must have already formed the asymmetry between protons and antiprotons, so (j) precedes (d). Before BBN was the inflationary epoch, so (e) must precede both (j) and (d). Finally, the primordial fluctuations must have been formed by fluctuations of the inflaton field so (g) is the earliest event. Therefore, the final answer is:

$$g \to e \to j \to d \to f \to c \to b \to j \to a \to h \to i.$$

2. There is a very unstable isotope of helium: ^{2}He, which has two protons and no neutrons, and is spin-0. It has a binding energy of $B_{^2\mathrm{He}} = 1.25$ MeV, but decays within a nanosecond. For reference, the deuterium binding energy is $B_D = 2.22$ MeV and $Q_n \equiv (m_n - m_p)c^2 = 1.29$ MeV, and you may assume $n_p = 0.8 n_{\mathrm{baryons}}$. (Note: there are equations on the equation sheet you can refer to.)

   (a) If we pretended that ^{2}He was stable, using the Saha equation, what would be the ratio of ^{2}He to *protons* as a function of temperature in the early Universe?

   From the number density equation, using $g_{^2\mathrm{He}} = 1$,

$$
\begin{aligned}
\frac{n_{^2\mathrm{He}}}{n_p n_p} &= \frac{1}{2^2}\left(\frac{m_{^2\mathrm{He}}}{m_p}\right)^{3/2}\left(\frac{m_p kT}{2\pi\hbar^2}\right)^{-3/2} e^{-(m_{^2\mathrm{He}}-2m_p)c^2/kT} \\
&= 2^{-1/2}\left(\frac{m_p kT}{2\pi\hbar^2}\right)^{-3/2} e^{B_{^2\mathrm{He}}/kT}
\end{aligned}
$$

   where I have used $(m_{^2\mathrm{He}} - 2m_p)c^2 = -B_{^2\mathrm{He}}$ and $\frac{m_{^2\mathrm{He}}}{m_p} \approx 2$. Then, using

$$
\begin{aligned}
\eta &= \frac{n_{\mathrm{baryon}}}{n_\gamma} \\
n_{\mathrm{baryon}} &= \eta n_\gamma \\
n_p &= \frac{4}{5}\eta n_\gamma = \frac{4\times 2.404}{5\times\pi^2}\eta\left(\frac{kT}{\hbar c}\right)^3,
\end{aligned}
$$

   we find

$$
\begin{aligned}
\frac{n_{^2\mathrm{He}}}{n_p} &= 2^{-1/2}\frac{4\times 2.404}{5\times\pi^2}\eta\left(\frac{m_p kT}{2\pi\hbar^2}\right)^{-3/2}\left(\frac{kT}{\hbar c}\right)^3 e^{B_{^2\mathrm{He}}/kT} \\
&= 2^{-1/2}(2\pi)^{3/2}\frac{4\times 2.404}{5\times\pi^2}\eta\left(\frac{kT}{m_p c^2}\right)^{3/2} e^{B_{^2\mathrm{He}}/kT} \\
\frac{n_{^2\mathrm{He}}}{n_p} &= 2.2\eta\left(\frac{kT}{m_p c^2}\right)^{3/2} e^{B_{^2\mathrm{He}}/kT}
\end{aligned}
$$

   Alternatively, you could have taken the equation for the deuterium-to-neutron ratio and replace $B_D \to B_{^2\mathrm{He}}$, $m_n \to m_p$ and note that the number of degrees of freedom here is 1 instead of 3, and thus the prefactor should be 1/3 as big.

(b) If we additionally assumed that neutrons were stable, using your answer (a) in what would be the ratio of ^{2}He to *neutrons* as a function of temperature in the early Universe?

Taking the result from (a), we note that

$$\frac{n_n}{n_p} = e^{-Q_n/kT}.$$

Thus,

$$\frac{n_{^2\mathrm{He}}}{n_n} = 2.2\eta \left(\frac{kT}{m_p c^2}\right)^{3/2} e^{(B_{^2\mathrm{He}}+Q_n)/kT}$$

(c) From your previous answer, if ^{2}He were stable, would there be more ^{2}He than deuterium in the Universe at the end of baryogenesis? (Again, there is an equation in the equation sheet you may want to consult here)

We can take the ratio of ^{2}He to deuterium, using the equation in the equation sheet, and note that

$$\frac{n_{^2\mathrm{He}}}{n_D} = \frac{2.2}{6.5} e^{(B_{^2\mathrm{He}}+Q_n-B_D)/kT}.$$

Numerically, the factor in the exponential is

$$B_{^2\mathrm{He}} + Q_n - B_D = (1.25 + 1.29 - 2.22)\ \mathrm{MeV} = 0.32\ \mathrm{MeV}.$$

That is, it is positive. Therefore, for T on the order of a MeV, the exponential factor is much much larger than $6.5/2.2 = 3$, and so dominates the ratio. As a result, there would be far more ^{2}He than deuterium in the Universe.

3. Let us assume that the scale factor of the Universe evolves as a flat, matter-only Universe with a present-day Hubble parameter of $H_0 = 70$ km/s/Mpc. We assume that the CMB decoupling still occurs at $z = 1100$, and that the physics of the decoupling is the same as in our own Universe, except for how the scale factor evolves.

   (a) Calculate the size of a causally-connected region of space at the time of CMB decoupling.

   The size of a causally connected region of space when the Universe has a scale factor a is the particle horizon size today, times the scale factor at decoupling:

$$d_{\rm hor} = c a_{\rm CMB} \int_0^{a_{\rm CMB}} \frac{1}{H_0\sqrt{\Omega_r a^-2 + \Omega_m a^{-1} + \Omega_\Lambda a^2 + (1-\Omega)}} \frac{da}{a}$$

   We are working with a flat, matter-only Universe ($\Omega_m = 1$), so

$$\begin{aligned} d_{\rm hor} &= \frac{c a_{\rm CMB}}{H_0} \int_0^{a_{\rm CMB}} \frac{da}{a^{-3/2}} \\ &= \frac{c a_{\rm CMB}}{H_0} \int_0^{a_{\rm CMB}} \frac{da}{a^{-1/2}} \\ &= \frac{2 c a_{\rm CMB}^{3/2}}{H_0} \\ &= \frac{2(3\times 10^5 \text{ km/s})(1/1101)^{3/2}}{70 \text{ km/s/Mpc}} = 0.23 \text{ Mpc.} \end{aligned}$$

   (b) How far away from us are the CMB photons originating?

   This is the comoving distance from $a_{\rm CMB}$ to $a = 1$:

$$\begin{aligned} d_p(t_0, t_{\rm CMB}) &= c \int_{a_{\rm CMB}}^1 \frac{1}{H_0\sqrt{\Omega_r a^-2 + \Omega_m a^{-1} + \Omega_\Lambda a^2 + (1-\Omega)}} \frac{da}{a} \\ &= \frac{c}{H_0} \int_{a_{\rm CMB}}^1 \frac{da}{a^{-1/2}} \\ &= \frac{2c(\sqrt{1} - \sqrt{a_{\rm CMB}})}{H_0} \\ &= \frac{2(3\times 10^5 \text{ km/s})(1 - \sqrt{1/1101})}{70 \text{ km/s/Mpc}} = 8310 \text{ Mpc.} \end{aligned}$$

(c) What is the approximate multipole moment ℓ corresponding to the first peak in the CMB spectrum?

The first multipole peak occurs at the angular size corresponding to the distance that a sound wave in the matter-radiation bath can cover. Since the sound speed in a radiation bath is $c/\sqrt{3}$, this distance is

$$d_{\rm speed} = \frac{d_{\rm hor}}{\sqrt{3}} = 0.14 \text{ Mpc}.$$

The angle that this subtends on the sky is

$$\begin{aligned} \theta &= \frac{d_{\rm sound}}{a_{\rm CMB} d_p(t_0, t_{\rm CMB})} \\ &= \frac{0.14 \text{ Mpc}}{1/1101 \times 8310 \text{ Mpc}} \\ &= 0.018 = 1.0^\circ. \end{aligned}$$

This corresponds to an ℓ of

$$\ell = \frac{\pi}{\theta} = \frac{\pi}{0.018} = 175.$$

4. Let us assume that, immediately after inflation ended, the Universe was reheated to a temperature of $kT = 10^{10}$ GeV.

   (a) How many e-foldings must inflation have continued for if the entire visible Universe once fit inside the classical radius of an electron ($r_e = 3 \times 10^{-15}$ m)?

   If inflation ended and the temperature was 10^{10} GeV, then the ratio of scale factors at the end of inflation to today is

$$\begin{aligned}
\frac{a_0}{a_f} &= \frac{T_f}{T_0} \\
&= \frac{10^{19}\ \text{eV}}{(2.7255\ \text{K})(8.62 \times 10^{-5}\ \text{eV/K})} \\
&= \frac{10^{19}\ \text{eV}}{2.34 \times 10^{-4}\ \text{eV}} = 4.26 \times 10^{22}.
\end{aligned}$$

   On top of which the ratio of scale factors from the beginning to the end of inflation is

$$\frac{a_f}{a_i} = e^N.$$

   The visible Universe has a comoving radius of $d_p(t_0, 0) = 14.4$ Gpc $= 4.38 \times 10^{26}$ m. For this to once have corresponded to a physical distance of less than r_e, it must be the case that

$$\begin{aligned}
r_e &= \frac{a_i}{a_0} d_p(t_0, 0) \\
&= \frac{a_i}{a_f} \frac{a_f}{a_0} d_p(t_0, 0) \\
&= e^{-N} \frac{a_f}{a_0} d_p(t_0, 0) \\
e^N &= a_f \frac{d_p(t_0, 0)}{r_e} \\
&= \frac{1}{4.26 \times 10^{22}} \frac{4.38 \times 10^{26}\ \text{m}}{3 \times 10^{-15}\ \text{m}} \\
&= 3.5 \times 10^{18} \\
N &= \ln 3.5 + 18 \ln 10 \\
&= 42.7.
\end{aligned}$$

(b) If the Universe was highly curved $|1-\Omega| = 1$ when the visible Universe was the size of an electron, what would the deviation from flatness be after the end of inflation?

From the Friedmann equation, if the Universe is dark-energy dominated (as it is when inflation is occurring), the deviation from flatness before and after inflation are related by

$$|1-\Omega_f| = |1-\Omega_i|e^{-2N}$$

Therefore, after inflation, the deviation from flatness is

$$\begin{aligned} |1-\Omega_f| &= |1-\Omega_i|e^{-2N} \\ &= 1 \times e^{-2\times 42.7} = 8.2 \times 10^{-38}. \end{aligned}$$

(c) Using your answer from part (b), what would the deviation from flatness be today (you may assume the benchmark Universe parameters for $\Omega_{r,0}$, $\Omega_{m,0}$, and $\Omega_{\Lambda,0}$)?

From the Friedmann equation, the curvature today is related to the curvature at some time t with a scale factor a by

$$|1-\Omega(t)| = \frac{|1-\Omega_0|a^2}{\Omega_{r,0} + a\Omega_{m,0} + a^4\Omega_{\Lambda,0}}.$$

We take $|1-\Omega(t)|$ as the result of part (b), and assume a scale factor of $a_f = (4.26 \times 10^{22})^{-1} = 2.3 \times 10^{-23}$. Then,

$$\begin{aligned} |1-\Omega_0| &= \frac{\Omega_{r,0} + a_f\Omega_{m,0} + a_f^4\Omega_{\Lambda,0}}{a_f^2}|1-\Omega(t)| \\ &= \frac{9.16 \times 10^{-5} + (2.3 \times 10^{-23})(0.315) + (2.3 \times 10^{-23})^4(0.685)}{(2.3 \times 10^{-23})^2}(8.2 \times 10^{-38}) \\ &= 14200. \end{aligned}$$

5. Suppose dark matter was composed of a "warm" sterile neutrino.

(a) If we discovered that the least-massive halo of dark matter has a mass of $10^5\ M_\odot$ (this mass includes both dark matter and baryons), what is the mass of the sterile neutrino?

We see that the halo must have been composed of material within a λ_0, given by

$$\begin{aligned}
M &= \frac{4\pi}{3} L_{\rm min}^3 \Omega_{m,0} \rho_{c,0} \\
L_{\rm min}^3 &= \frac{3M}{4\pi\Omega_{m,0}\rho_{c,0}} \\
&= \frac{3(10^5\ M_\odot)}{4\pi(0.315)(1.25 \times 10^{-4}\ M_\odot/\text{pc}^3)} \\
&= 6.06 \times 10^8\ \text{pc}^3 \\
L_{\rm min} &= 846\ \text{pc}
\end{aligned}$$

Therefore,

$$\begin{aligned}
8.46 \times 10^{-4}\ \text{Mpc} &= 60\ \text{Mpc} \left(\frac{m_\nu c^2}{2\ \text{eV}}\right)^{-2} \\
\left(\frac{m_\nu c^2}{2\ \text{eV}}\right)^2 &= 70900 \\
m_\nu c^2 &= 533\ \text{eV}
\end{aligned}$$

(b) What was the scale factor when this sterile neutrino became non-relativistic? (You may assume that this occurred when the Universe was radiation-dominated)

Given that the neutrino became non-relativistic when

$$3kT \sim m_\nu c^2$$

we can immediately see that

$$\begin{aligned}
a\nu &= \frac{T_0}{T_\nu} \\
&= \frac{T_0}{m_\nu c^2/3k} \\
&= \frac{3(8.62 \times 10^{-5}\ \text{eV/K})(2.7\ \text{K})}{533\ \text{eV}} \\
&= 1.3 \times 10^{-6}
\end{aligned}$$

Cheat Sheet

Physical constants and useful conversions:

$$\begin{aligned}
c &= 3 \times 10^{8} \text{ m/s} \\
G &= 6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \\
k &= 8.62 \times 10^{-5} \text{ eV/K} \\
h &= 6.63 \times 10^{-34} \text{ Js} = 4.14 \times 10^{-15} \text{ eVs} \\
1 \text{ Mpc} &= 3.09 \times 10^{22} \text{ m} \\
1 \text{ J} &= 6.24 \times 10^{18} \text{ eV} \\
1 \text{ km/s/Mpc} &= 3.24 \times 10^{-20} \text{ s}^{-1}
\end{aligned}$$

Benchmark Universe:

$$T_0 = 2.7255 \text{ K},\ H_0 = 67.3 \text{ km/s/Mpc},\ t_0 = 13.8 \text{ Gyr},$$
$$\Omega_{\Lambda,0} = 0.685,\ \Omega_{m,0} = 0.315,\ \Omega_{r,0} = 9.16 \times 10^{-5}, \Omega_b = 0.05$$
$$\epsilon_{c,0} = 4770 \text{ MeV/m}^3 = 8.50 \times 10^{-24} \text{ kg}\times c^2/\text{m}^3 = 1.25 \times 10^{-4}\ M_\odot/\text{pc}^3$$
$$d_p(t_0, t_{\text{CMB}}) = 13.9 \text{ Gpc},\ d_p(t_0, 0) = 14.4 \text{ Gpc}$$
$$\eta = 6.2 \times 10^{-10}$$

Equations:

$$z = \frac{\lambda_{\rm obs} - \lambda_{\rm emit}}{\lambda_{\rm emit}}, \qquad z = \frac{H}{c} r \text{ (Hubble's Law)}, \; 1 + z = \frac{1}{a}$$

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa^2 d\Omega^2 \right], \qquad S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0) & \kappa = +1 \\ r & \kappa = 0 \\ R_0 \sinh(r/R_0) & \kappa = -1 \end{cases}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{\kappa c^2}{R_0^2 a^2}, \qquad 1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}$$

$$\dot{\epsilon} + \frac{3\dot{a}}{a}(\epsilon + P) = 0, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^3}(\epsilon + 3P)$$

$$P = w\epsilon, \qquad \epsilon_w = \epsilon_{w,0} a^{-3(1+w)}$$

$$\epsilon_c = \frac{3c^2}{8\pi G} H^2 \qquad \Omega_w = \frac{\epsilon_w}{\epsilon_c}$$

$$d_p(t_0, t_e) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \qquad d_p(t_e) = c a(t_e) \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{w,0} a^{-(1+3w)} + (1 - \Omega_0)}}, \qquad a(t) = \begin{cases} \frac{t}{t_0} & \Omega_0 = 0, \kappa = -1 \\ (t/t_0)^{2/3(1+w)} & \Omega_{w,0} = 1 \kappa = 0 \end{cases}$$

$$d_p(t_0, t_e) \approx \frac{c}{H_0} z \left[1 - \left(\frac{1 + q_0}{2} \right) z \right], \qquad q_0 = \frac{1}{2} \sum_w \Omega_{w,0}(1 + 3w)$$

$$d_A = \frac{\ell}{\delta\theta} = \frac{S_\kappa(r)}{1 + z}, \qquad d_L = (1 + z) S_\kappa(r)$$

$$d_p(a_0, a_e) = \frac{c}{H_0} \int_{a_e}^{a_0} \frac{1}{\sqrt{\Omega_{r,0} a^{-2} + \Omega_{m,0} a^{-1} + \Omega_{\Lambda,0} a^2 + (1 - \Omega_0)}} \frac{da}{a}$$

$$M = \frac{\langle v^2 \rangle \langle r \rangle}{\alpha G}, (\alpha \sim 0.4) \qquad v(r)^2 = \frac{GM(<r)}{r},$$

$$X = \frac{n_e}{n_b} \qquad \Gamma = c\sigma_e X n_{b,0} a^{-3}$$

$$n = g\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} e^{-mc^2/kT}, \qquad n_\gamma = \frac{2.404}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$$

$$\frac{1-X}{X^2} = 3.84\eta\left(\frac{kT}{m_e c^2}\right)^{3/2} e^{Q/kT}, \qquad \theta \sim \frac{\pi}{\ell}$$

$$v_{\text{sound}} = \frac{c}{\sqrt{3}}, \qquad T(t) \sim 10^{10}\ \text{K}\left(\frac{t}{1\ \text{s}}\right)^{-1/2} \text{(Early U.)}$$

$$Y_{^4\text{He}} \equiv \frac{\rho(^4\text{He})}{\rho_b} = \frac{2n_n}{n_n + n_p}, \qquad \left.\frac{n_n}{n_p}\right|_{\text{freeze-out}} = \exp(-Q_n/kT)$$

$$\frac{n_D}{n_n} = 6.5\eta\left(\frac{kT}{m_n c^2}\right)^{3/2} e^{B_D/kT},$$

$$1 - \Omega(t) = (1-\Omega_0)\frac{H_0^2}{H(t)^2}a(t)^{-2}, \qquad \frac{H(t)^2}{H_0^2} = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}$$

$$\frac{a_f}{a_i} = e^N, \qquad d_{\text{hor}}(t_f) = 3ct_i e^N$$

$$t_{\text{dyn}} = \frac{1}{4\pi G\bar{\rho}}, \qquad \lambda_J = 5.13\sqrt{w}\frac{c}{H}$$

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0, \qquad \delta \propto D_1 t^{2/3} + D_2 t^{-1}$$

$$M \sim \frac{4\pi}{3}\lambda_0^3 \Omega_{m,0}\rho_{c,0}$$

$$ct_\nu = 7.1 \times 10^{-8}\ \text{Mpc}\left(\frac{m_\nu c^2}{1\ \text{keV}}\right)^{-2}, \qquad L_{\min} = 60\ \text{Mpc}(m_\nu c^2/2\ \text{eV})^{-2}$$