Physics 444: Final
December 23, 2019

Name:

You have 180 minutes for this exam. You may use a calculator and 1 pre-prepared sheet of equations. Each question is worth 10 points. Partial credit is given, so be sure to explain your reasoning when necessary.

1. Assuming that inflation occurred, place the following events in the history of the Universe in a timeline in the order in which they occurred.

   (a) Start of growth of baryonic structure

   (b) Electron-proton recombination

   (c) Start of growth of dark matter structure

   (d) Big Bang Nucleosynthesis

   (e) End of inflation

   (f) Matter-radiation equality

   (g) Formation of primordial fluctuations away from the average energy density.

   (h) Matter-Dark Energy equality

   (i) Scale Factor $a = 1$.

   (j) CMB decoupling

   (k) Latest time that the asymmetry between matter and antimatter could have formed.

2. There is a very unstable isotope of helium: ^{2}He, which has two protons and no neutrons, and is spin-0. It has a binding energy of $B_{^2\text{He}} = 1.25$ MeV, but decays within a nanosecond. For reference, the deuterium binding energy is $B_D = 2.22$ MeV and $Q_n \equiv (m_n - m_p)c^2 = 1.29$ MeV, and you may assume $n_p = 0.8 n_{\text{baryons}}$. (Note: there are equations on the equation sheet you can refer to.)

   (a) If we pretended that ^{2}He was stable, using the Saha equation, what would be the ratio of ^{2}He to *protons* as a function of temperature in the early Universe?

   (b) If we additionally assumed that neutrons were stable, using your answer (a) in what would be the ratio of ^{2}He to *neutrons* as a function of temperature in the early Universe?

   (c) From your previous answer, if ^{2}He were stable, would there be more or less ^{2}He than deuterium in the Universe at the end of baryogenesis? (Again, there is an equation in the equation sheet you may want to consult here)

3. Let us assume that the scale factor of the Universe evolves as a flat, matter-only Universe with a present-day Hubble parameter of $H_0 = 70$ km/s/Mpc. We assume that the CMB decoupling still occurs at $z = 1100$, and that the physics of the decoupling is the same as in our own Universe, except for how the scale factor evolves.

   (a) Calculate the size of a causally-connected region of space at the time of CMB decoupling.

   (b) How far away from us are the CMB photons originating?

   (c) What is the approximate multipole moment ℓ corresponding to the first peak in the CMB spectrum?

4. Let us assume that, immediately after inflation ended, the Universe was reheated to a temperature of $kT = 10^{10}$ GeV.

   (a) How many e-foldings must inflation have continued for if the entire visible Universe once fit inside the classical radius of an electron ($r_e = 3 \times 10^{-15}$ m)?

   (b) If the Universe was highly curved $|1 - \Omega| = 1$ when the visible Universe was the size of an electron, what would the deviation from flatness be after the end of inflation?

   (c) Using your answer from part (b), what would the deviation from flatness be today (you may assume the benchmark Universe parameters for $\Omega_{r,0}$, $\Omega_{m,0}$, and $\Omega_{\Lambda,0}$)?

5. Suppose dark matter was composed of a “warm” sterile neutrino.

   (a) If we discovered that the least-massive halo of dark matter has a mass of $10^5\, M_\odot$ (this mass includes both dark matter and baryons), what is the mass of the sterile neutrino?

   (b) What was the scale factor when this sterile neutrino became non-relativistic? (You may assume that this occurred when the Universe was radiation-dominated)

Cheat Sheet

Physical constants and useful conversions:

$$\begin{aligned}
c &= 3 \times 10^8 \text{ m/s} \\
G &= 6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \\
k &= 8.62 \times 10^{-5} \text{ eV/K} \\
h &= 6.63 \times 10^{-34} \text{ Js} = 4.14 \times 10^{-15} \text{ eVs} \\
1 \text{ Mpc} &= 3.09 \times 10^{22} \text{ m} \\
1 \text{ J} &= 6.24 \times 10^{18} \text{ eV} \\
1 \text{ km/s/Mpc} &= 3.24 \times 10^{-20} \text{ s}^{-1}
\end{aligned}$$

Benchmark Universe:

$$T_0 = 2.7255 \text{ K},\ H_0 = 67.3 \text{ km/s/Mpc},\ t_0 = 13.8 \text{ Gyr},$$
$$\Omega_{\Lambda,0} = 0.685,\ \Omega_{m,0} = 0.315,\ \Omega_{r,0} = 9.16 \times 10^{-5}, \Omega_b = 0.05$$
$$\epsilon_{c,0} = 4770 \text{ MeV/m}^3 = 8.50 \times 10^{-24} \text{ kg}\times c^2/\text{m}^3 = 1.25 \times 10^{-4}\ M_\odot/\text{pc}^3$$
$$d_p(t_0, t_{\text{CMB}}) = 13.9 \text{ Gpc},\ d_p(t_0, 0) = 14.4 \text{ Gpc}$$
$$\eta = 6.2 \times 10^{-10}$$

Equations:

$$z = \frac{\lambda_{\rm obs} - \lambda_{\rm emit}}{\lambda_{\rm emit}}, \qquad z = \frac{H}{c} r \text{ (Hubble's Law)}, \;\; 1 + z = \frac{1}{a}$$

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa^2 d\Omega^2\right], \qquad S_\kappa(r) = \begin{cases} R_0 \sin(r/R_0) & \kappa = +1 \\ r & \kappa = 0 \\ R_0 \sinh(r/R_0) & \kappa = -1 \end{cases}$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon - \frac{\kappa c^2}{R_0^2 a^2}, \qquad 1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}$$

$$\dot\epsilon + \frac{3\dot a}{a}(\epsilon + P) = 0, \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3c^3}(\epsilon + 3P)$$

$$P = w\epsilon, \qquad \epsilon_w = \epsilon_{w,0} a^{-3(1+w)}$$

$$\epsilon_c = \frac{3c^2}{8\pi G} H^2 \qquad \Omega_w = \frac{\epsilon_w}{\epsilon_c}$$

$$d_p(t_0, t_e) = c \int_{t_e}^{t_0} \frac{dt}{a(t)} \qquad d_p(t_e) = c a(t_e) \int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$$H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{w,0} a^{-(1+3w)} + (1 - \Omega_0)}}, \qquad a(t) = \begin{cases} \frac{t}{t_0} & \Omega_0 = 0, \kappa = -1 \\ (t/t_0)^{2/3(1+w)} & \Omega_{w,0} = 1 \kappa = 0 \end{cases}$$

$$d_p(t_0, t_e) \approx \frac{c}{H_0} z \left[1 - \left(\frac{1 + q_0}{2}\right) z\right], \qquad q_0 = \frac{1}{2} \sum_w \Omega_{w,0}(1 + 3w)$$

$$d_A = \frac{\ell}{\delta\theta} = \frac{S_\kappa(r)}{1 + z}, \qquad d_L = (1 + z) S_\kappa(r)$$

$$d_p(a_0, a_e) = \frac{c}{H_0} \int_{a_e}^{a_0} \frac{1}{\sqrt{\Omega_{r,0} a^{-2} + \Omega_{m,0} a^{-1} + \Omega_{\Lambda,0} a^2 + (1 - \Omega_0)}} \frac{da}{a}$$

$$M = \frac{\langle v^2 \rangle \langle r \rangle}{\alpha G}, (\alpha \sim 0.4) \qquad v(r)^2 = \frac{GM(< r)}{r},$$

$$X = \frac{n_e}{n_b} \qquad \Gamma = c\sigma_e X n_{b,0} a^{-3}$$

$$n = g \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2} e^{-mc^2/kT}, \qquad n_\gamma = \frac{2.404}{\pi^2} \left(\frac{kT}{\hbar c} \right)^3$$

$$\frac{1-X}{X^2} = 3.84\eta \left(\frac{kT}{m_e c^2} \right)^{3/2} e^{Q/kT}, \qquad \theta \sim \frac{\pi}{\ell}$$

$$v_{\text{sound}} = \frac{c}{\sqrt{3}}, \qquad T(t) \sim 10^{10}\ \text{K} \left(\frac{t}{1\ \text{s}} \right)^{-1/2} \text{(Early U.)}$$

$$Y_{^4\text{He}} \equiv \frac{\rho(^4\text{He})}{\rho_b} = \frac{2n_n}{n_n + n_p}, \qquad \left. \frac{n_n}{n_p} \right|_{\text{freeze-out}} = \exp(-Q_n/kT)$$

$$\frac{n_D}{n_n} = 6.5\eta \left(\frac{kT}{m_n c^2} \right)^{3/2} e^{B_D/kT},$$

$$1 - \Omega(t) = (1 - \Omega_0) \frac{H_0^2}{H(t)^2} a(t)^{-2}, \qquad \frac{H(t)^2}{H_0^2} = \Omega_{r,0} a^{-4} + \Omega_{m,0} a^{-3} + \Omega_{\Lambda,0}$$

$$\frac{a_f}{a_i} = e^N, \qquad d_{\text{hor}}(t_f) = 3ct_i e^N$$

$$t_{\text{dyn}} = \frac{1}{4\pi G\bar{\rho}}, \qquad \lambda_J = 5.13\sqrt{w} \frac{c}{H}$$

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0, \qquad \delta \propto D_1 t^{2/3} + D_2 t^{-1}$$

$$M \sim \frac{4\pi}{3} \lambda_0^3 \Omega_{m,0} \rho_{c,0}$$

$$ct_\nu = 7.1 \times 10^{-8}\ \text{Mpc} \left(\frac{m_\nu c^2}{1\ \text{keV}} \right)^{-2}, \qquad L_{\min} = 60\ \text{Mpc}(m_\nu c^2/2\ \text{eV})^{-2}$$