

1 Linear Algebra Basics

1.1 Boolean Function Spaces

Let \mathbb{F}_2 be the field of two elements $\{0, 1\}$ where addition is $\pmod 2$ and multiplication is AND. \mathbb{F}_2^n is the n dimensional vector space over \mathbb{F}_2 consisting of n tuples of \mathbb{F}_2 . Let \mathcal{F} be the set of all functions with domain \mathbb{F}_2^n and codomain \mathbb{F}_2 . It is easy to verify that \mathcal{F} is a 2^n dimensional vector space over \mathbb{F}_2 with the natural scalar multiplication and vector addition ($\pmod 2$). Assume $n \geq 2$.

a.) For any subset S of $\{1, \dots, n\}$, let $\text{PARITY}_S : \mathbb{F}_2^n \rightarrow \mathbb{F}_2 \in \mathcal{F}$ be defined as:

$$\text{PARITY}_S(x) = \bigoplus_{i \in S} x_i.$$

For $S = \emptyset$, define $\text{PARITY}_S(x) = 1$ (constant 1 function). Show that the following set of 2^n parity functions are linearly dependent:

$$\{\text{PARITY}_S : S \subseteq \{1, \dots, n\}\}$$

b.) For any subset S of $\{1, \dots, n\}$, let $\text{AND}_S : \mathbb{F}_2^n \rightarrow \mathbb{F}_2 \in \mathcal{F}$ be defined as:

$$\text{AND}_S(x) = \bigwedge_{i \in S} x_i.$$

For $S = \emptyset$, define $\text{AND}_S(x) = 1$ (constant 1 function). Show that the following set of 2^n functions forms a basis for \mathcal{F} :

$$\{\text{AND}_S : S \subseteq \{1, \dots, n\}\}$$

1.2 Infinite Dimensional Vector Spaces

a.) Consider the set of all functions with domain \mathbb{R} and codomain \mathbb{R} as a vector space over \mathbb{R} . Define a set of basis functions. Are they countable or uncountable? If so why?

b.) Consider \mathbb{R} as a vector space over the field \mathbb{Q} (rational numbers). Is there a basis set for the above vector space that is countable. (Remember countable and uncountable

sets from your discrete math course). Explain why?

1.3 Rank over different Fields

Let \mathbb{K}, \mathbb{F} be fields such that $\mathbb{F} \subset \mathbb{K}$ and the addition, multiplication operations in \mathbb{F} is the same as that in \mathbb{K} . For example \mathbb{K} can be \mathbb{R} and \mathbb{F} can be \mathbb{Q} (or \mathbb{C}, \mathbb{R} respectively). $\mathbb{F}^{n \times m}$ is the set of $n \times m$ matrices with entries in \mathbb{F} . For any matrix $M \in \mathbb{F}^{n \times m}$, we can define rank with respect to \mathbb{F} as well as \mathbb{K} . The rank with respect to \mathbb{K} denoted by $\text{rank}_{\mathbb{K}}(M) = \dim(\text{span}_{\mathbb{K}}(\text{columns}(M)))$ where $\text{span}_{\mathbb{K}}(S)$ denotes the vector space spanned by S by taking linear combinations with scalars from \mathbb{K} . Similarly we define $\text{rank}_{\mathbb{F}}(M)$.

- a.) Show that for $M \in \mathbb{F}^{n \times m}$, $\text{rank}_{\mathbb{F}}(M) = \text{rank}_{\mathbb{K}}(M)$.
- b.) Given a binary matrix $M \in \{0,1\}^{m \times n}$, show that $\text{rank}_{\mathbb{R}}(M) \geq \text{rank}_{\mathbb{F}_2}(M)$. Note that addition and multiplication over \mathbb{F}_2 is different from \mathbb{R} . $\text{rank}_{\mathbb{R}}, \text{rank}_{\mathbb{F}_2}$ are defined as earlier with the respective definition of addition and multiplication in \mathbb{R}, \mathbb{F}_2 (ie normal arithmetic and mod 2 arithmetic).

1.4 Help Alice & Bob Communicate

Alice and Bob needs to compute a known function $f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$. They know the function beforehand and can agree upon a plan. Later Bob will go to Mars. Then both of them will be given some $x, y \in \{0,1\}^n$ (not known beforehand) respectively and Alice will be allowed to sent a message to Bob. Alice will have access to only x , Bob will have access to only y and they do not know the other persons input. Every bit of message Alice communicates is expensive. After getting Alice's message Bob should be able to find out $f(x, y)$.

Let $M \in \{0,1\}^{2^n \times 2^n}$ (binary matrix) be defined as $M_{i,j} = f(\text{bin}(i), \text{bin}(j))$, where $\text{bin}(i)$ is the n bit binary representation of i ($0 \leq i, j < 2^n$). Can you design a protocol for them such that Alice only needs to sent $\text{rank}_{\mathbb{F}_2}(M)$ bits of communication?

2.1 Random Walks

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Consider an **undirected graph** $G = (V, E)$ without any isolated vertices, where V is a set of n nodes and

$$E \subseteq \{\{a, b\} : a \neq b \text{ and } a, b \in V\}$$

is a set of edges. The random walk matrix of G is a matrix M defined by

$$M_{a,b} = \begin{cases} 1/d_b & \text{if } \{a, b\} \in E \\ 0 & \text{otherwise} \end{cases} \quad \text{where } a, b \in V \text{ and } d_b = |\{\{a, b\} \in E : a \in V\}|$$

d_b is called the degree of the vertex b .

- a.) Show that if λ is a real eigenvalue ($\in \mathbb{R}$) of M then $-1 \leq \lambda \leq 1$.

Hint 1 Need to use the facts that a.) eigenvalues of $M = \text{eigenvalues of } M^T$ b.) columns of M sum upto 1. Consider an eigenvector v of λ of M^T . Let i be the coordinate of v , which has the highest absolute value. This coordinate is going to be crucial for the proof to work.

- b.) Show that the column vector v defined by $v_a = d_a / (\sum_{b \in V} d_b)$, $\forall a \in V$ is an eigenvector of M with eigenvalue 1. That is $Mv = v$, for any graph G .
- c.) Show that the maximal number of linearly independent eigenvectors with eigenvalue 1 is equal to the number of connected components in G .
- d.) Show that -1 is an eigenvalue of M if and only if G is a **bipartite graph**.

Hint 2 Try to show that $LHS \Rightarrow RHS$ and $RHS \Rightarrow LHS$ separately for the last two questions.

2.2 Polynomials

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Let \mathcal{P}_n be the set of polynomials (on one variable) of degree less than n . As you know $p \in \mathcal{P}_n$, can be written as a linear combination of the standard monomial basis as follows $p = \sum_{d=0}^{n-1} p_d x^d$, where p_d 's are coordinates with respect to this basis.

- a.) For any polynomial $q \in \mathcal{P}_n$ (having coordinates q_0, \dots, q_{n-1} in standard monomial basis), define the function $T_q : \mathcal{P}_n \rightarrow \mathcal{P}_{2n-1}$, which maps $p \mapsto q \times p$ (ie. polynomial

multiplication). Is T_q a linear transformation? If so what is the matrix of the transformation in the standard monomial basis ie $\{1, x, x^2, x^3, \dots, x^{n-1}\}$? Give the formula for each entry of the matrix for general n , in the standard monomial basis.

- b.) Let $n = 4$. Consider the change of basis, which maps the d th standard basis to the column vector $[1, \omega^d, \omega^{2 \cdot d}, \omega^{3 \cdot d}]$, where $\omega = e^{i \cdot \frac{2\pi}{4}}$ (a complex number; $i = \sqrt{-1}$). What is the matrix of T_q with respect to this basis? What is the change of basis matrix for changing coordinates from this new basis back to the standard monomial basis?

2.3 Invariance of Eigenvalues

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- a.) Let $M \in \mathbb{R}^{n \times n}$. We can define eigenvalues from the left and the right as follows. λ is left eigenvalue of M iff there exists a nonzero row vector v , such that $vM = \lambda v$. Similarly λ is a right eigenvalue of M iff there exists a nonzero column vector v , such that $Mv = \lambda v$.
- Show that the set of left eigenvalues and right eigenvalues of any matrix are equal.
 - Are the left and right eigenvectors (similarly defined) the same (by taking transpose)?
- b.) Let M, M' be matrices corresponding to the same linear operator $T : V \rightarrow V$ (V is a n dimensional vector space over some field) with respect to different basis. Also assume that T is a rank n operator and M has n eigenvalues $\lambda_1, \dots, \lambda_n$.
- Show that set of eigenvalues of M is equal to the set of eigenvalues of M' .
 - Show that $\det(M) = \det(M') = \prod_{i=1}^n \lambda_i$.
 - Define trace of a matrix, as the sum of diagonal entries. ie $\text{trace}(M) = \sum_{j=1}^n M_{jj}$. Show that $\text{trace}(M) = \text{trace}(M') = \sum_{i=1}^n \lambda_i$.

3.1 An Orthonormal Basis for Boolean Functions

Consider the set of functions with domain $\{+1, -1\}^n$ and range \mathbb{R} . Observe that it is a vector space over \mathbb{R} of dimension 2^n . Consider the inner product and norm defined by

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \{+1, -1\}^n} f(x)g(x) \quad \text{and} \quad \|f\| = \sqrt{\langle f, f \rangle}.$$

a.) Define the following set of functions,

$$\{\chi_S\}_{S \subseteq \{1, \dots, n\}} \quad \text{where} \quad \chi_S(x) = \prod_{i \in S} x_i.$$

For $S = \emptyset$, χ_S is the constant 1 function. Show that these functions form an orthonormal basis under the inner product defined.

b.) Let f be any function in this space with range $\{+1, -1\}$ such that

$$f = \sum_{S \subseteq \{1, \dots, n\}} \hat{f}_S \chi_S \quad \text{where} \quad \forall S \subseteq \{1, \dots, n\}, \hat{f}_S \in \mathbb{R}$$

That is $(\hat{f}_S)_{S \subseteq \{1, \dots, n\}}$ are the coordinates with respect to the χ_S basis. Show that

$$\sum_{S \subseteq \{1, \dots, n\}} (\hat{f}_S)^2 = 1.$$

3.2

Question 5, Review Problems 14.7, page 274 in [\[CDTW\]](#).

There is a typo in the question. $V = \text{Span}\{\sin(t), \sin(2t), \sin(3t), \sin(4t)\}$.

3.3

Question 14, Review Problems 14.7, page 276 in [\[CDTW\]](#).

4 Deep Quiz 2

4.1 Fixed Points

Let M be a matrix given by

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

Given any vector $v(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$, we can create an infinite sequence of vectors $v(1), v(2), \dots$ by using the rule

$$v(t+1) = Mv(t) \quad \text{for all natural numbers } t$$

- a.) Find all vectors $v(0)$ such that

$$v(0) = v(1) = v(2) = v(3) = \dots$$

- b.) Find all vectors $v(0)$, such that $v(0), v(1), v(2), v(3), \dots$, belongs to a 1 dimensional subspace.

4.2 Commuting Matrices

Let A, B be commuting matrices of dimension $n \times n$ (ie $AB = BA$) and suppose A is diagonalizable with n distinct eigenvalues.

- a.) Show that if v is an eigenvector of A with eigenvalue λ , then Bv is also an eigenvector of A with eigenvalue λ .
- b.) Show that if v is an eigenvector of A , the v is also an eigenvector of B . Should the eigenvalues be the same?
- c.) Explain why the above implies that there is a single change of basis such that A, B are both diagonal in the same basis.

4.3 Decomposition

- a.) Let Q be an $n \times n$ orthonormal matrix (columns form an orthonormal basis). For any vectors $v, u \in \mathbb{R}^n$, show that

$$u \cdot v = (Qu) \cdot (Qv) \quad (\text{dot product})$$

b.) Let $\{u_1, \dots, u_n\}$ be column vectors $\in \mathbb{R}^n$ (ie $n \times 1$ dimensional) that are orthonormal. Suppose $M \in \mathbb{R}^{n \times n}$ ($n \times n$ dimensional matrix) defined by:

$$M = \sum_{i=1}^n \alpha_i u_i u_i^T \quad \text{where } \alpha_i \text{ are scalars.}$$

Note that $u_i u_i^T$ are $n \times n$ dimensional. What are the eigenvectors and eigenvalues of M ? (need to explain why)

