Electronic Commerce 096211

(a quick) Introduction to Game Theory

Omer Ben Porat (The slides are adopted from Prof. Reshef Meir)

Outline

- Background – game theory, non-cooperative games
  - Strategies and utilities, dominated strategies, Nash equilibrium, mixed Nash equilibrium
- Find PNE, ~~MNE~~
- Optimality
- ~~Tree/repeated games~~
- Evolutionary games

Game Theory

- The study of mathematical models of strategic interactions among rational agents
  - applications in social science, logic, systems science and computer science
- In the context of electronic commerce :
  - Auctions, sponsored search
  - Information markets
  - Decision making
  - Crypto
- Topics: <u>Non-cooperative</u> vs cooperative games, <u>mechanism design</u>, one-shot vs sequential, perfect information vs <u>imperfect information</u>, discrete vs continuous games, …

Phases of Non-cooperative Game (matrix game)

- Agents select actions simultaneously
- An outcome is realized
- Each agent gets a utility based on the outcome

- Examples of 2X2 games:
  - Prisoner’s dilemma
  - Battle of the sexes
  - The teacher-student game

The teacher / student game

Student: Teacher:		
Prepare slides	10, 10	-5, 5
Copy-paste	-10, 0	0, 5

(adapted from Vince Conitzer)

Definitions

- A game is a tuple $(N, (A_i)_i, (u_i)_i)$ where:
  - N is the set of player (indices)
  - A_i is the set of agent i's actions
  - u_i is the utility function of agent i
- A profile: $a = (a_1, \dots, a_n), \forall i \in N, a_i \in A_i$
- Denote $a_{-i} = (a_1, a_2, \dots a_{i-1}, a_{i+1}, \dots a_n)$
  - (all actions except the action of i)
  - $u(a), u(a_{-i}, a_i'), \dots$
- Complete information!

The teacher / student game

Student (2): Teacher (1):		
Prepare slides	10, 10	-5, 5
Copy-paste	-10, 0	0, 5

$$N = \{1, 2\}$$
$$A_1 = \{Prepare, Copy\}$$
$$A_2 = \{Listen, AngryB\}$$

$$u_1(P, AB) = -5$$

(adapted from Vince Conitzer)

The teacher / student game

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Copy-paste	-10, 0	0, 5

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$$u_1(P, AB) = -5$$
$$u_2(P, AB) = 5$$

(adapted from Vince Conitzer)

The teacher / student game

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$$N = \{1, 2\}$$
$$A_1 = \{Prepare, Copy\}$$
$$A_2 = \{Listen, AngryB\}$$

$$u_1(P, AB) = -5$$
$$u_2(P, AB) = 5$$
$$u_2(C, AB) = 5$$

(adapted from Vince Conitzer)

Definitions

- $a_i{}'$ is a better response of i in profile a if
$$u_i(a_{-i}, a_i{}') > u_i(a_{-i}, a_i)$$
- $a_i{}'$is a best response of i in profile a_{-i} if
$$a_i{}' = argmax_{a_i''{}' \in A_i} u_i(a_{-i}, a_i'')$$
- We can draw better responses and best responses as arrows from a to $(a_{-i}, a_i{}')$

דוגמא

u1, u2	L	M	R
T	3,4	8,2	2,6
B	3,5	7,7	5,8

Better response graph:

u1, u2	L	M	R
T	3,4	8,2	2,6
B	3,5	7,7	5,8

Best response graph:

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T	3,4	8,2	2,6
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שימו לב שתועלת שחקן השורות תמיד משמאל!

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שימו לב שתועלת שחקן השורות תמיד משמאל!

Player 2 *always* prefers R over M!

Definitions

- Dominated strategies:
  - Action $a_i{}'$ very weakly dominates a_i if for all a_{-i}

$$u_i(a_i', a_{-i}) \geq u_i(a_i, a_{-i})$$

  - a_i strictly dominated: if all equalities are strict
    - a'_i is a better-response of i in any profile (a_{-i}, a_i)
  - a_i weakly dominated: if some inequalities are strict

- Players would never player weakly dominated strategies

Definitions

- Profile a is a Pure Nash equilibrium (PNE) if

$$\forall i \in N \text{ , } a_i \text{ is a best-response to } a_{-i}$$

- Equivalently: at profile a , not agent has a better-response

דוגמא

u1, u2	L	M	R
T	3,4	8,2	2,6
B	3,5	7,7	5,8

Better response graph:

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Best response graph:

u1, u2	L	M	R
T	3,4	8,2	2,6
B	3,5	7,7	5,8

The only pure Nash equilibrium is (B,R). L is strictly dominated by R. There are no other mixed equilibria

Definitions

- A mixed strategy $s_i \in \Delta(A_i)$ is a lottery over pure actions (specifies the probability of each action)

- $s = (s_1, \dots, s_n)$ is a mixed profile

$$u_i(s) = E_{a\sim s}[u_i(a)] = \sum_{a\in A_1\times\cdots\times A_n} \prod_{i\in N} \Pr_{s_i}[a_i]\ u_i(a)$$

- Note that all previous definitions extend to mixed strategies

Parity game

ODD: / EVEN:	0	1
0	1,0	0,1
1	0,1	1,0

No pure equilibrium!

Claim: (0.5,0.5) is an MNE

Proof:

Row player:
0->0.5*1+0.5*0=0.5
1->0.5*0+0.5*1=0.5

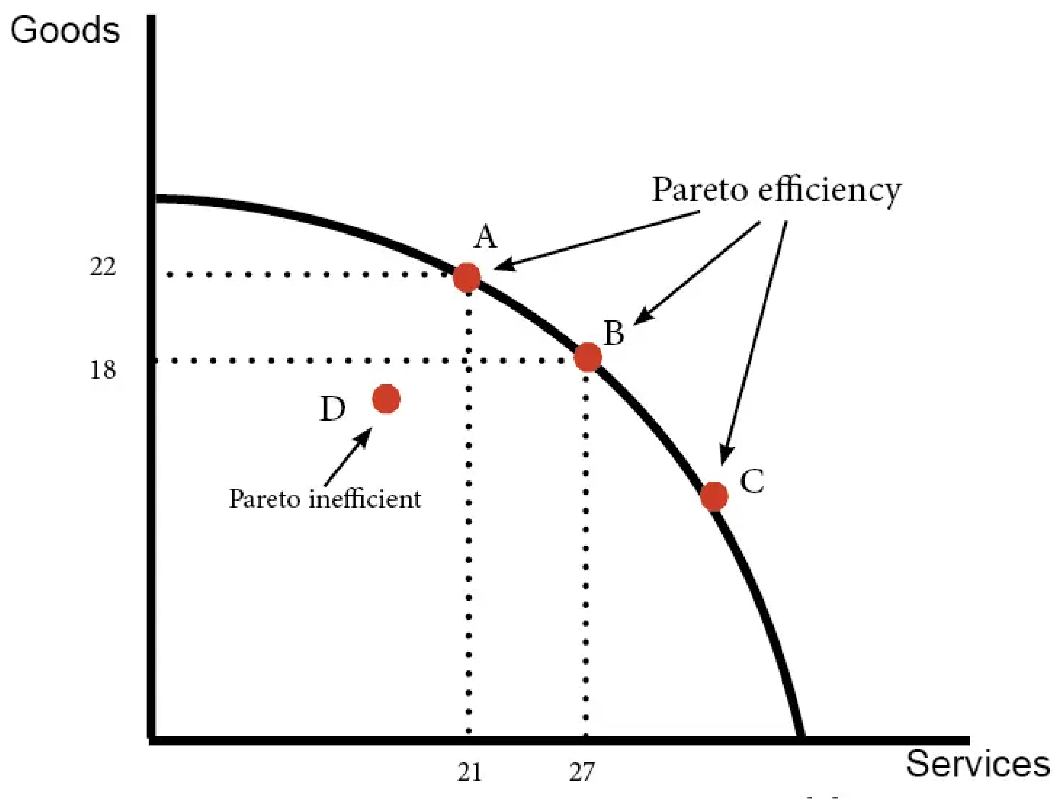
The teacher / student game

Student: / Teacher:		
Prepare slides	10, 10	-5, 5
Copy-paste	-10, 0	0, 5

Two pure Nash equilibria, and one MNE

Optimality

- Pareto Optimality
  - A profile is Pareto-optimal if there is no other profile that all players weakly prefer, and some players strictly prefer
- Social Optimality
  - A profile is socially-optimal if there is no other profile where the sum of utilities is higher



דוגמאות למשחקים מוכרים

שימו לב שהמספרים המדוייקים לא חשובים.
רק הגרף שמתקבל. הכותרות הן קישורים לוויקיפדיה.

Prisoner's dilemma	C	D
C	5, 5	0,7
D	7, 0	2, 2

Battle of the sexes	Poetry	Football
Poetry	2,1	-1, -1
Football	0, 0	1 , 2

Chicken	C	D
C	0, 0	-3,3
D	3,-3	-10, -10

Stag hunt	Hunt	Stay
Hunt	8,8	0,3
Stay	3,0	3,3

Matching pennies	Odd	Even
Odd	1, 0	0, 1
Even	0, 1	1, 0

Student \ Prof. game	Prepare talk	Copy-paste slides
Listen to talk	10, 10	0, -10
Play Angry Birds	5, -5	5, 0

Non cooperative games

- Correlated equilibrium
  - A mediator flips coins and suggests actions to players
  - All players are happy with following their suggested action
  - Example: Battle of the sexes
- Could this be achieved via Nash equilibrium?

Battle of the sexes	Poetry	Football
Poetry	2,1	-1, -1
Football	0, 0	1, 2

Extensive form games

- Repeated games
  - Iterated prisoner's dilemma
  - Axelrod's tournament
  - Emergence of cooperation?

Prisoner's dilemma	C	D
C	5, 5	0,7
D	7, 0	2, 2

- Tree-form games
  - Play in turns

Evolutionary games

- The lion and the zebra

Lion

Zebra

u1, u2	Chase	Pass
Taunt	-10, 10	0 , 0
Run	-5 , -5	-5, 0

Evolutionary games

- The lion and the zebra

Lion

Zebra

u1, u2	Chase	Pass
Taunt	-10, 10	0 , 0
Run	-5 , -5	-5, 0

Unique NE: (0.5,0.5)
What does it mean?

מה נדרש מכם לדעת בתורת המשחקים?

- להיות מסוגלים לכתוב בצורה פורמלית משחק שמתאר סיטואציה פשוטה
  - כלומר מי השחקנים, מה הפעולות האפשריות, מה התועלת לכל שחקן בכל צירוף פעולות
- לצייר את הגרף המכוון שמתאר את ה better response ואת ה best response
- לדעת למצוא אסטרטגיות שולטות ונשלטות בכל משחק
  - להבין את ההבדל בין "שולט חזק", "שולט חלש" ו"שולט חלש מאוד"
  - כולל אסטרטגיות שנשלטות ע"י אסטרטגיה מעורבת
  - לדעת להסיר אסטרטגיות נשלטות באופן איטרטיבי (עד שלא ניתן להסיר יותר)
- למצוא את כל שיוויי המשקל הטהורים בכל משחק
- למצוא פרופילים פארטו-אופטימליים

SOLVING A 2X2 GAME

Parity game

ODD: / EVEN:	0	1
0	1,0	0,1
1	0,1	1,0

No pure equilibrium!

The teacher / student game

Student: Teacher:		
Prepare slides	10, 10	-5, 5
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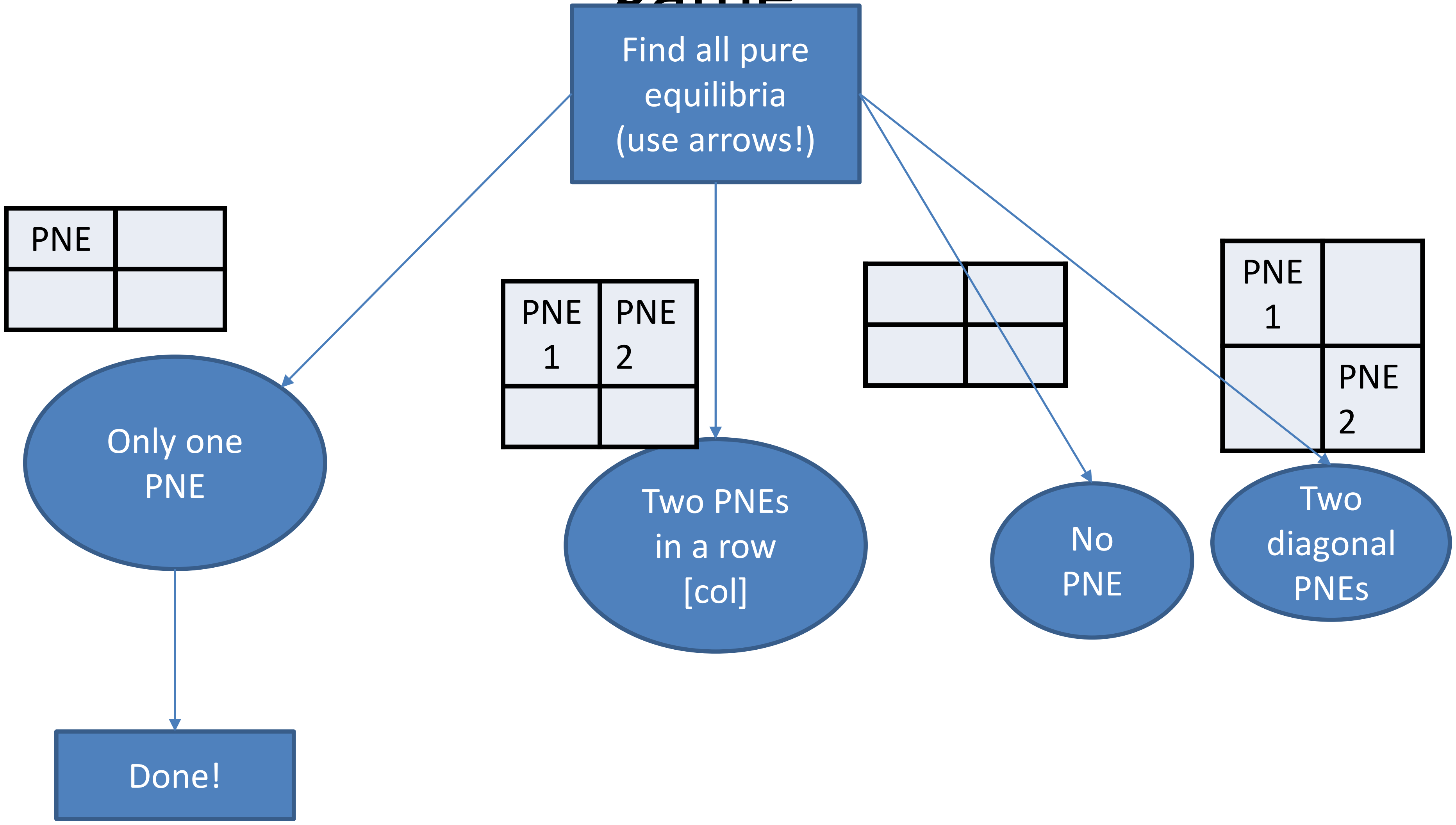
Two pure Nash equilibria, and one MNE

Finding all Nash equilibria in a 2X2 game*



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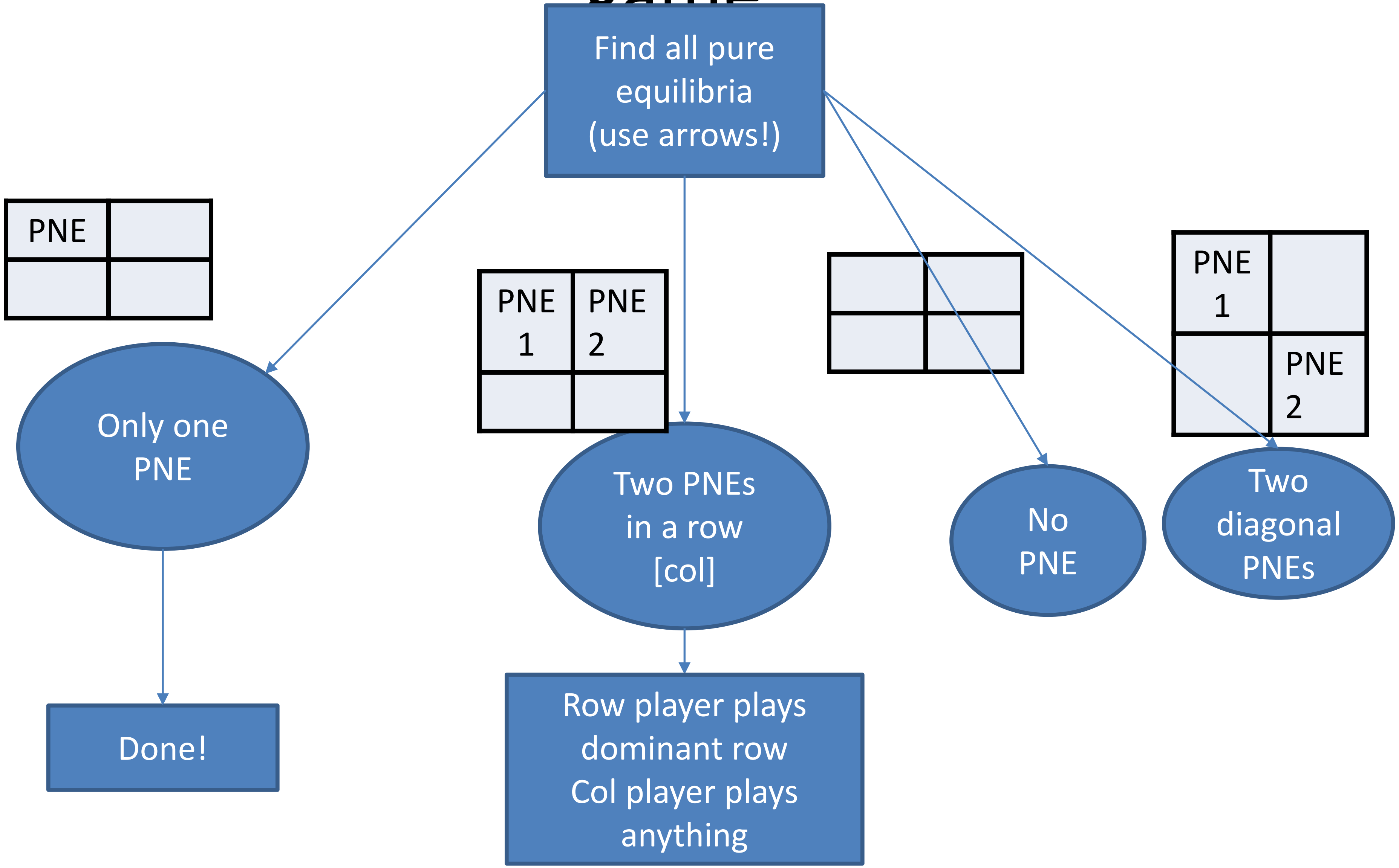

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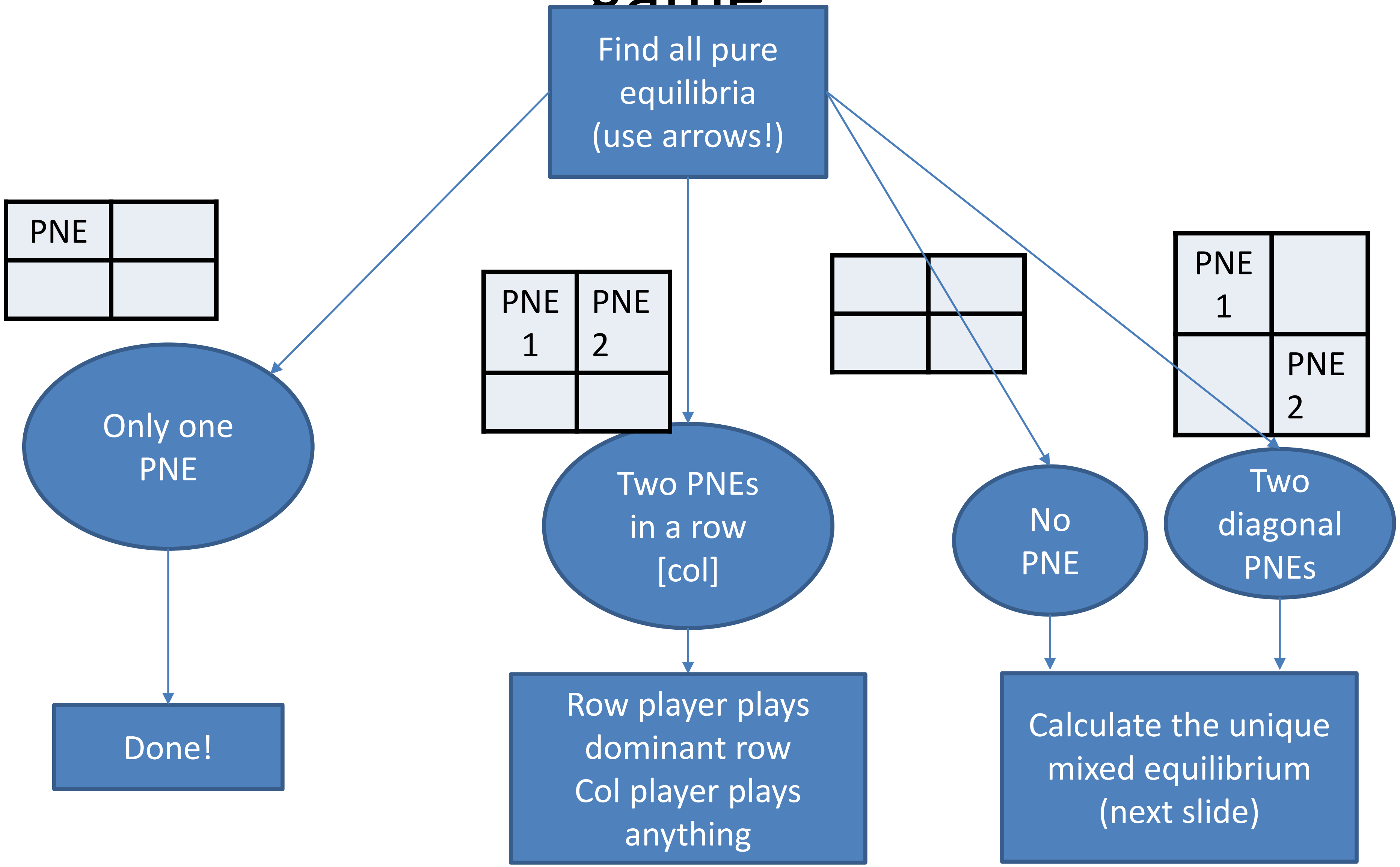

Calculating Mixed NE in 2X2 games

- Only works if MNE exists! (see previous slide)
- Key idea: in equilibrium, each player is **indifferent** between her actions
  - Row player: need to play so that col player is indifferent

p	8 , 10	5 , 4
1-p	2 , 5	6 , 11

Calculating Mixed NE in 2X2 games

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p	8 , **10**	5 , **4**
1-p	2 , **5**	6 , **11**

If Col plays L: $u_2 = 10p + 5(1-p) = 5 + 5p$

If Col plays R: $u_2 = 4p + 11(1-p) = 11 - 7p$

Must be equal!

Calculating Mixed NE in 2X2 games

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If Col plays L: $u_2 = 10p + 5(1-p) = 5 + 5p$

If Col plays R: $u_2 = 4p + 11(1-p) = 11 - 7p$

Must be equal!

$$5 + 5p = 11 - 7p \Rightarrow 12p = 6 \Rightarrow p = 0.5$$

Calculating Mixed NE in 2X2 games

- Only works if MNE exists! (see previous slide)
- Key idea: in equilibrium, each player is **indifferent** between her actions
  - Row player: need to play so that col player is indifferent $\Rightarrow \ p = 0.5$
  - Col player: need to play so that row player is indifferent

	q	1-q
p	**8** , 10	**5** , 4
1-p	**2** , 5	**6** , 11

If Row plays T: $u_1 = 8q + 5(1-q) = 5 + 3q$

If Row plays B: $u_1 = 2q + 6(1-q) = 6 - 4q$

Must be equal!

Calculating Mixed NE in 2X2 games

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- Key idea: in equilibrium, each player is **indifferent** between her actions
  - Row player: need to play so that col player is indifferent $\Rightarrow \; p = 0.5$
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If Row plays T: $u_1 = 8q + 5(1 - q) = 5 + 3q$

If Row plays B: $u_1 = 2q + 6(1 - q) = 6 - 4q$

Must be equal!

$$5 + 3q = 6 - 4q \Rightarrow 7q = 1 \Rightarrow q = 1/7$$

Calculating Mixed NE in 2X2 games

- Only works if MNE exists! (see previous slide)
- Key idea: in equilibrium, each player is **indifferent** between her actions
  - Row player: need to play so that col player is indifferent $\Rightarrow \quad p = 0.5$
  - Col player: need to play so that row player is indifferent $\Rightarrow \quad q = 1/7$

	q	1-q
p	8 , 10	5 , 4
1-p	2 , 5	6 , 11

What if you get $p \leq 0$ or $p \geq 1$?

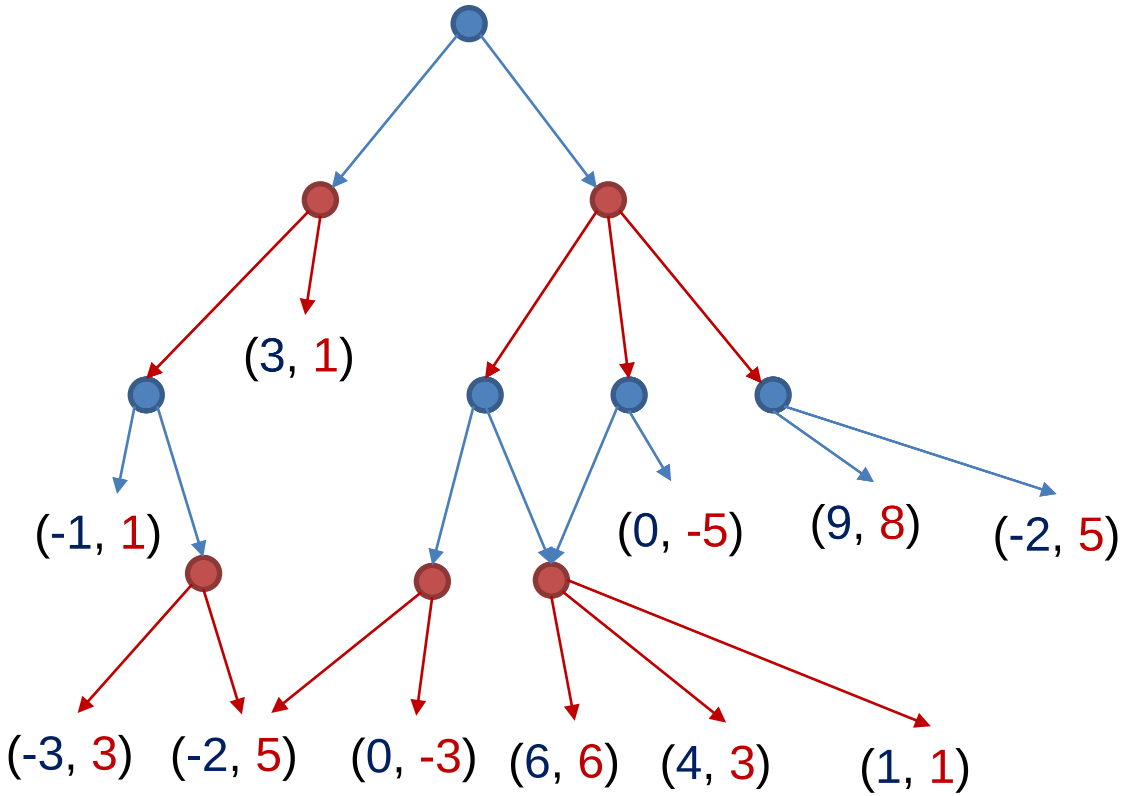
Tree-form games

Player 1

Player 2

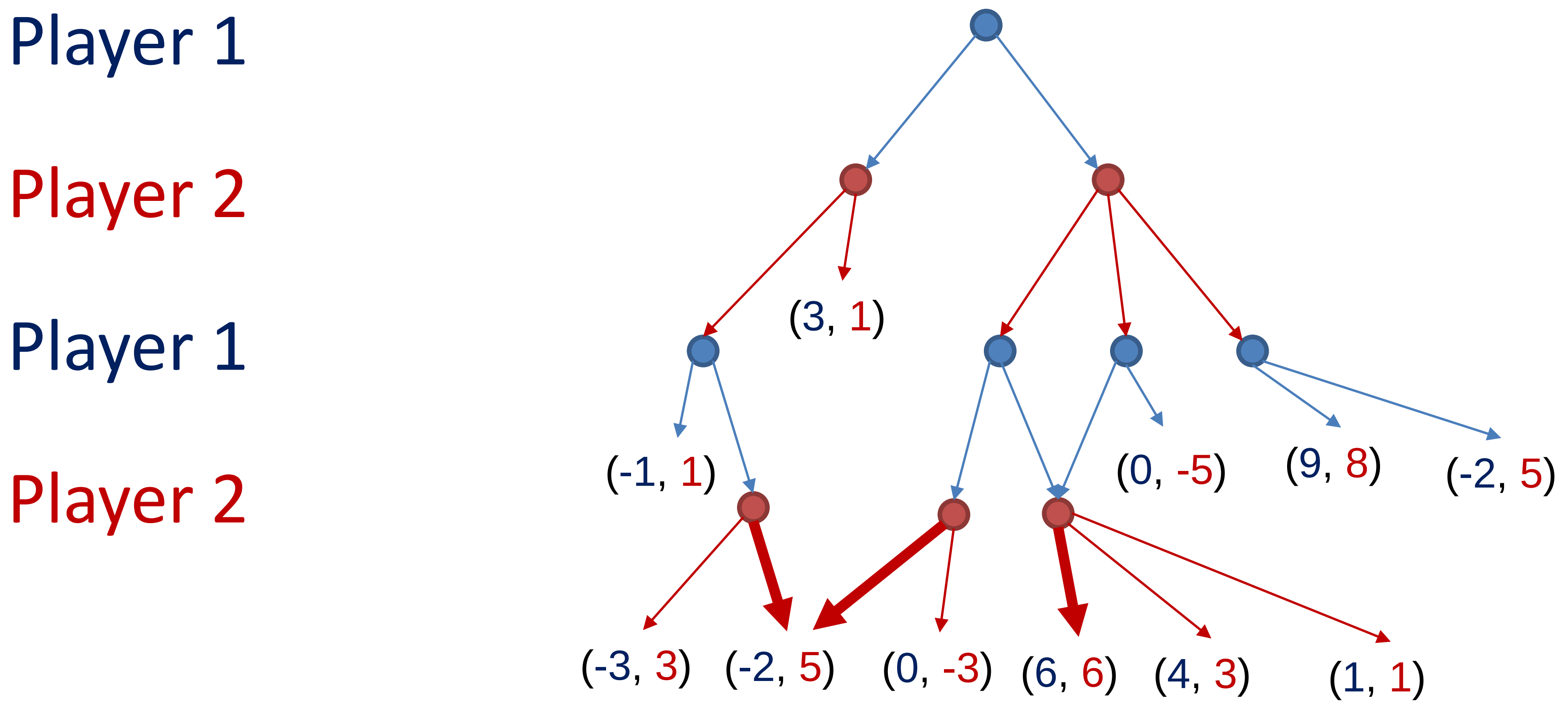
Player 1

Player 2



Theorem: in a generic game, there is a unique equilibrium
Proof: backward induction. □

Tree-form games



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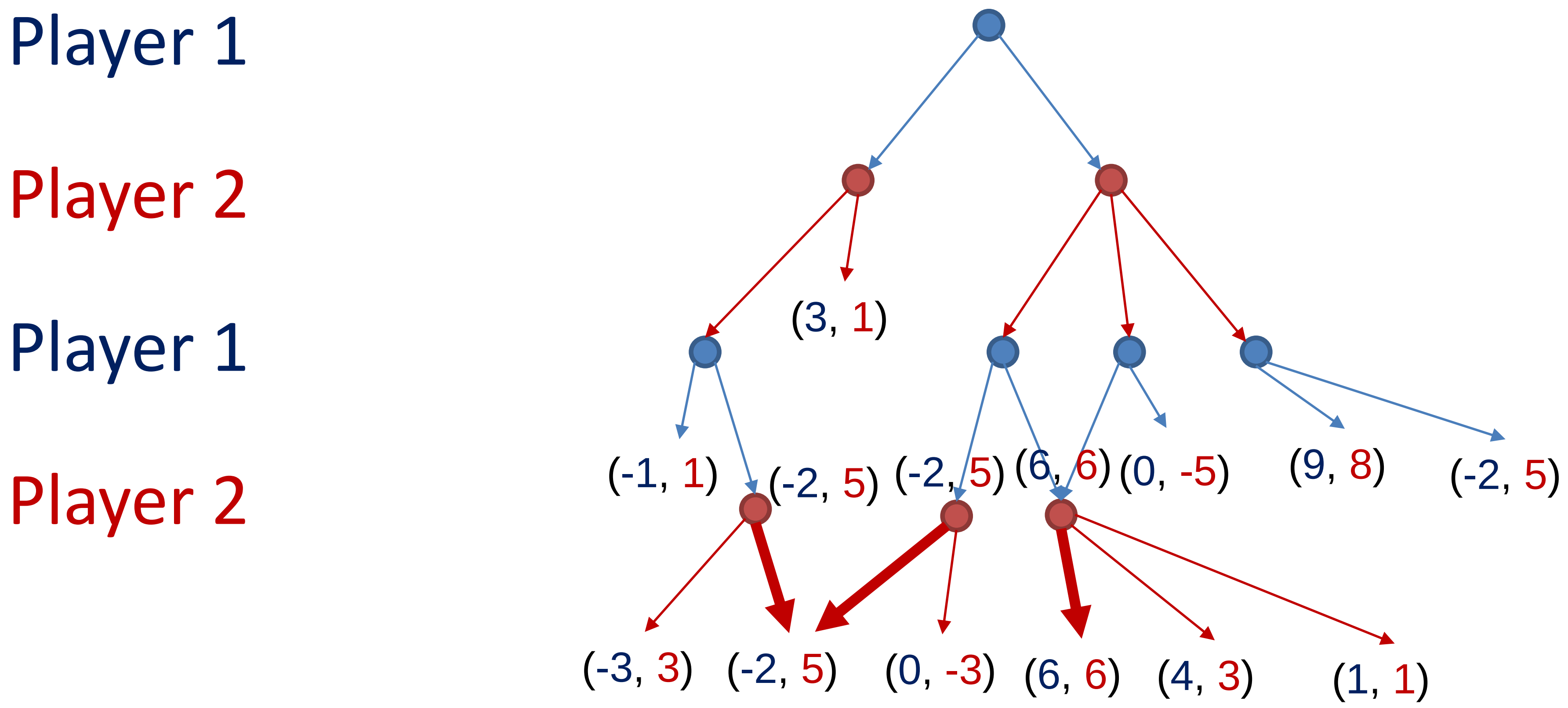
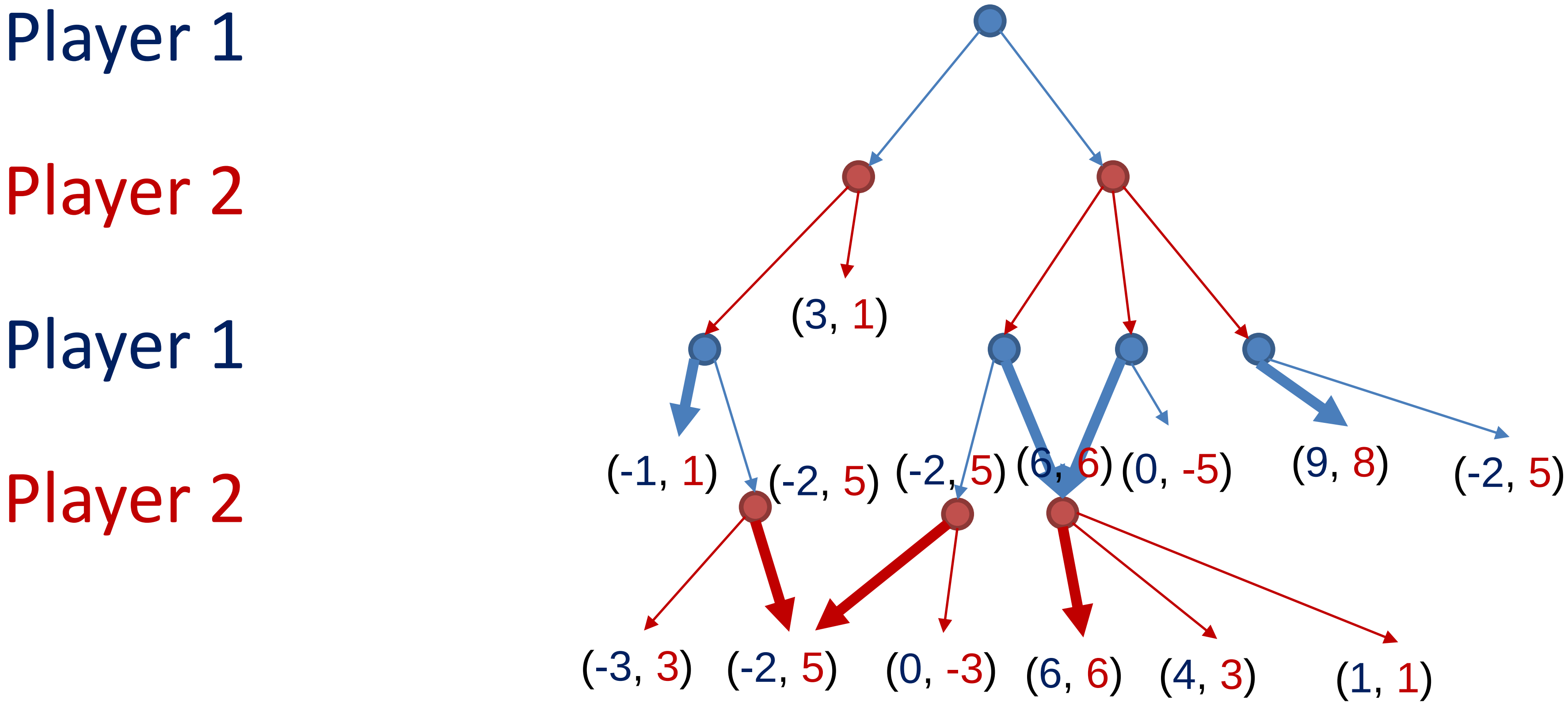


<u>Theorem</u>: in a generic game, there is a unique equilibrium

Proof: backward induction. □

Tree-form games



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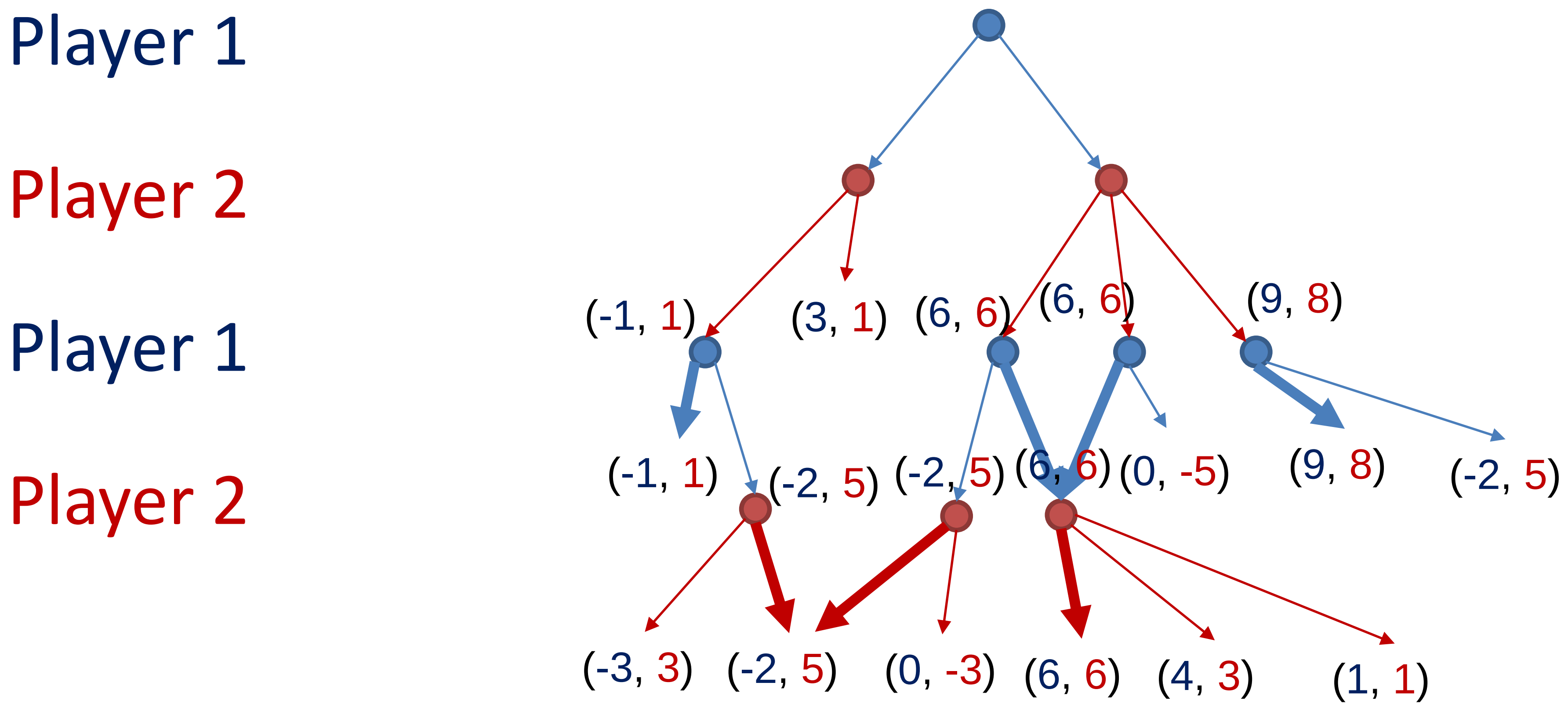

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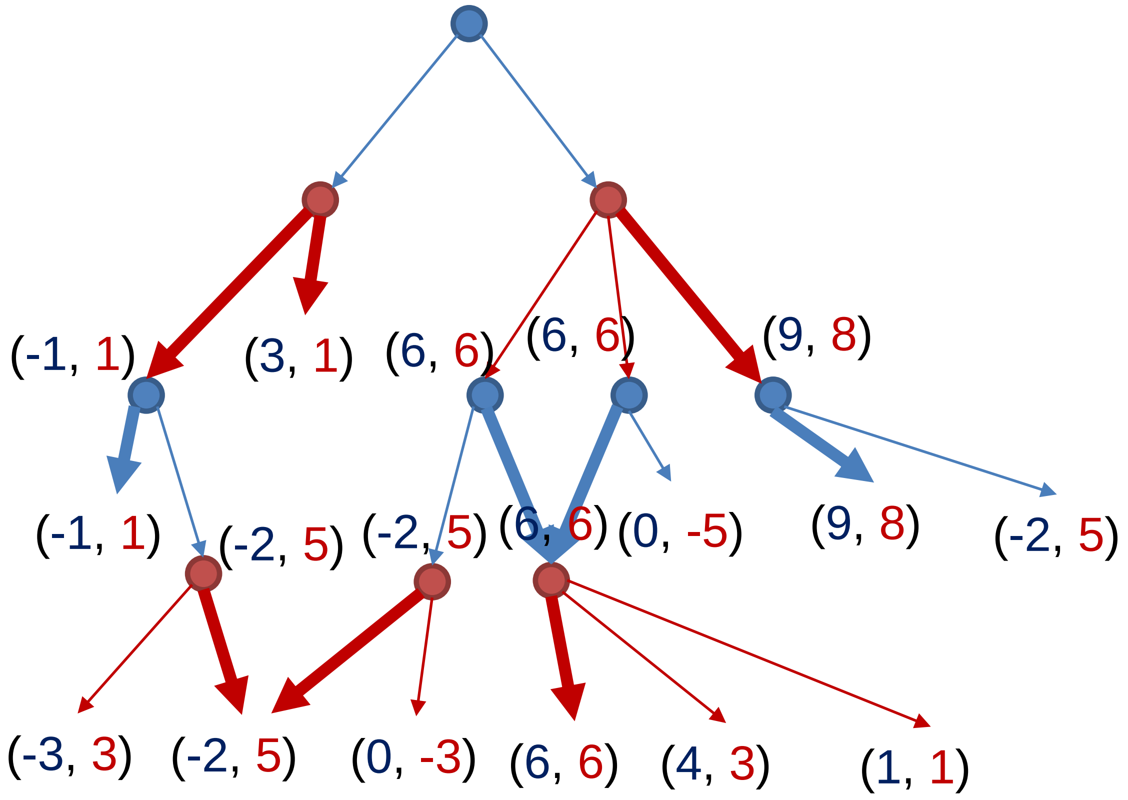
Tree-form games

Player 1

Player 2

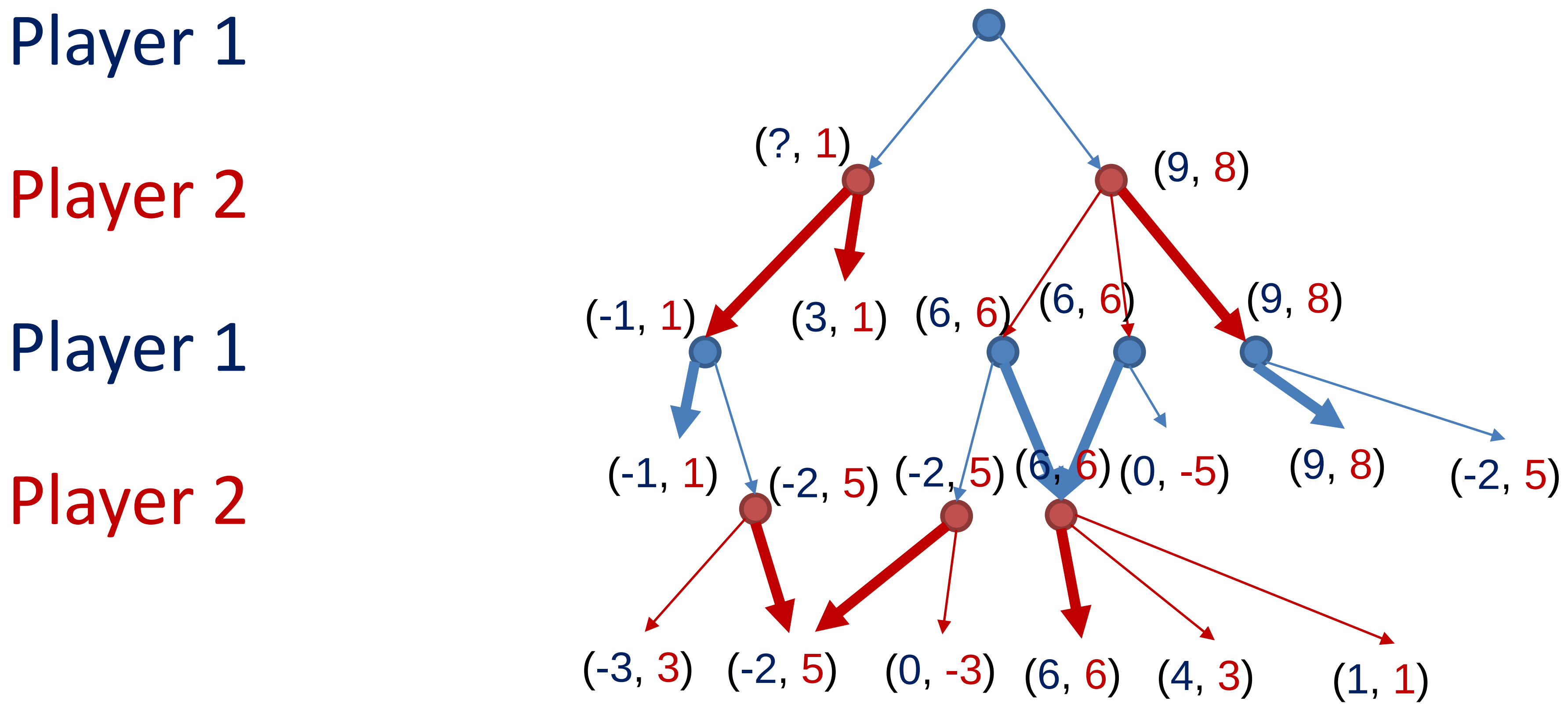
Player 1

Player 2



<u>Theorem</u>: in a generic game, there is a unique equilibrium
Proof: backward induction. $\square$

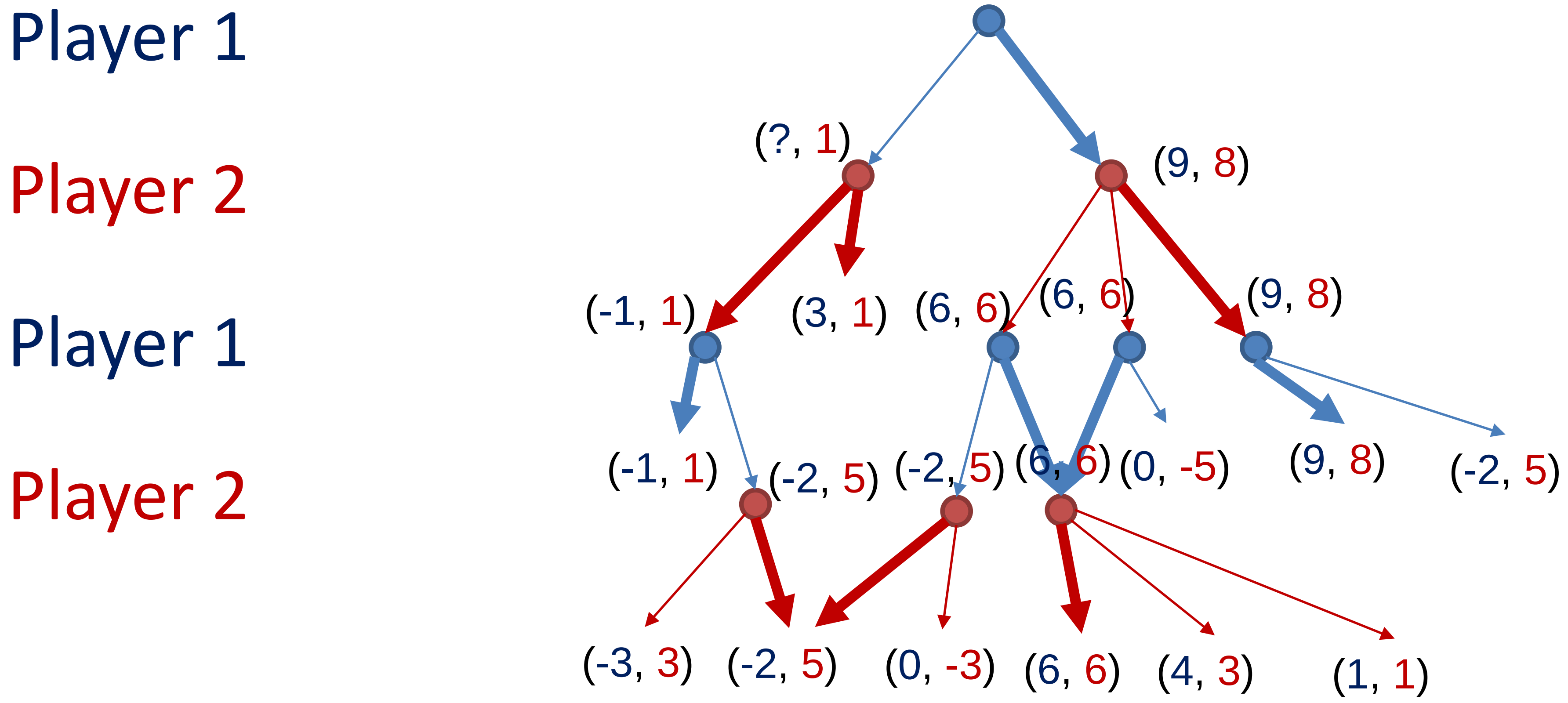
Tree-form games



<u>Theorem</u>: in a generic game, there is a unique equilibrium

Proof: backward induction. □

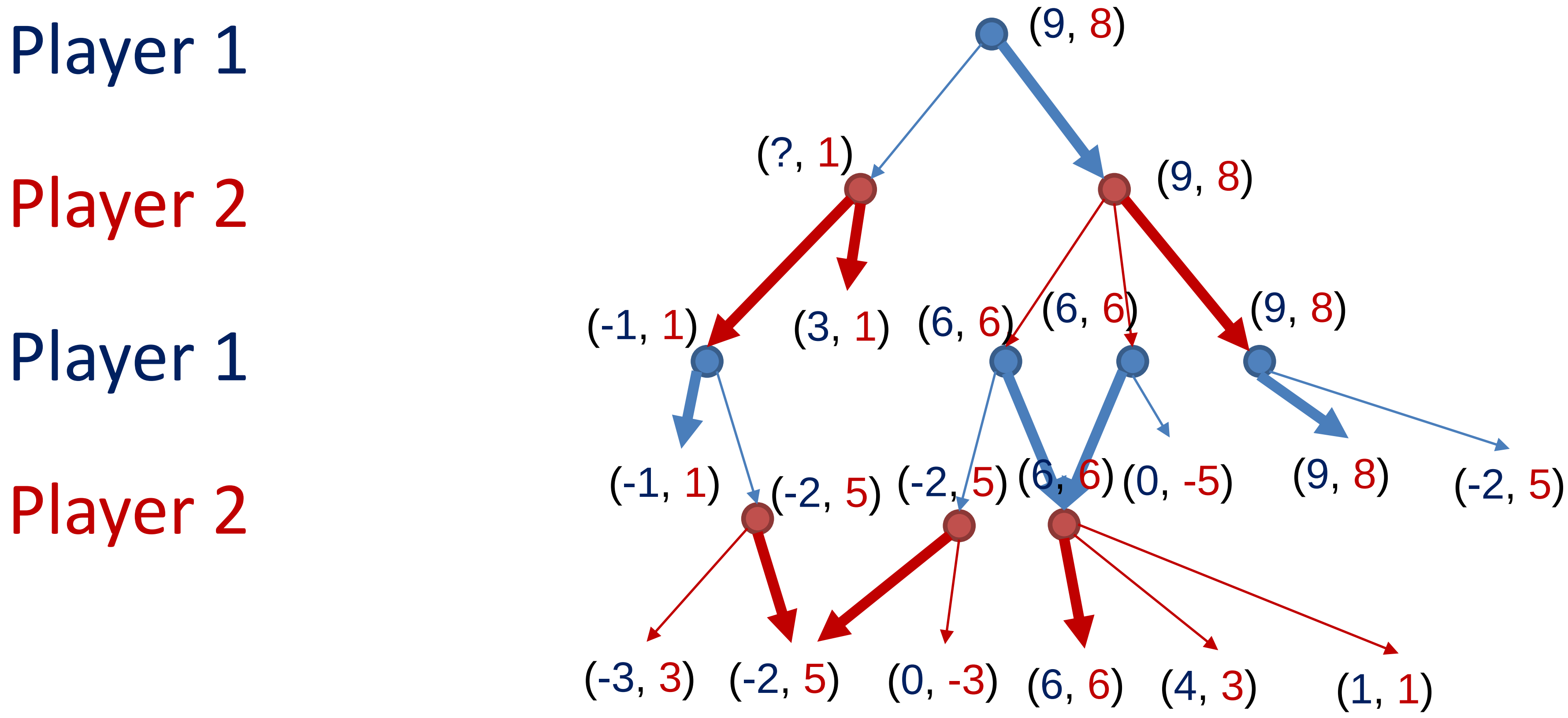
Tree-form games



<u>Theorem</u>: in a generic game, there is a unique equilibrium

Proof: backward induction. □

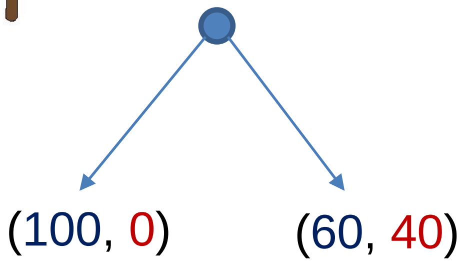
Tree-form games



<u>Theorem</u>: in a generic game, there is a unique (?) equilibrium
Proof: backward induction. □

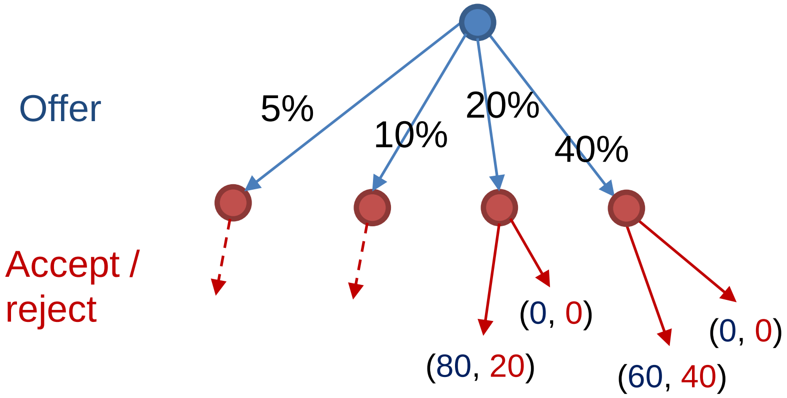
Tree-form games

- Chess
  - Backgammon?

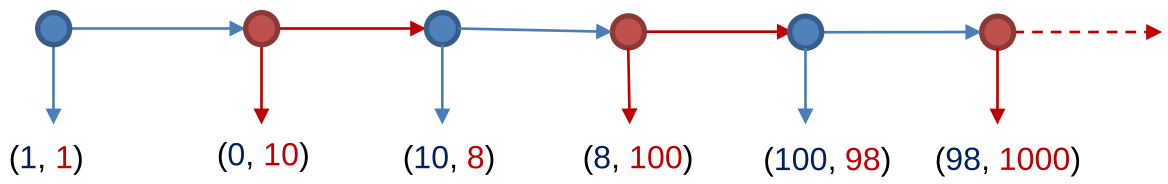


- The dictator game

- Ultimatum game



Centipede game



Do people always play according to equilibrium?
Dominant strategies?
Other solution concepts?

Can **you** predict how people play?
https://cpc-18.com/

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Graphs

קורסים קודמים?
אלגוריתמים?
NP Complete

Omer Ben-Porat (slides are adopted from Prof. Reshef Meir)

Graph Theory

- A graph is a pair (V, E)
- Where V is a set of vertices
- E is a set of edges
- $e = \{v, u\}$ in an undirected graph
- $e = (v, u)$ in a directed graph
- A path is a set of edges $\{(a_i, a_{i+1})\}_{i=1..n}$
- A cycle is a closed path ($a_{n+1} = a_1$)
- A path is simple if no vertex appears twice

More definitions

$E(x) \coloneqq \{y \mid (x, y) \in E\}$ (outgoing edges)

$degree(x) \coloneqq |E(x)|$

$P_{xy} \coloneqq \{all\ x - y\ simple\ paths\}$

Directed Acyclic Graph (DAG) has no cycles

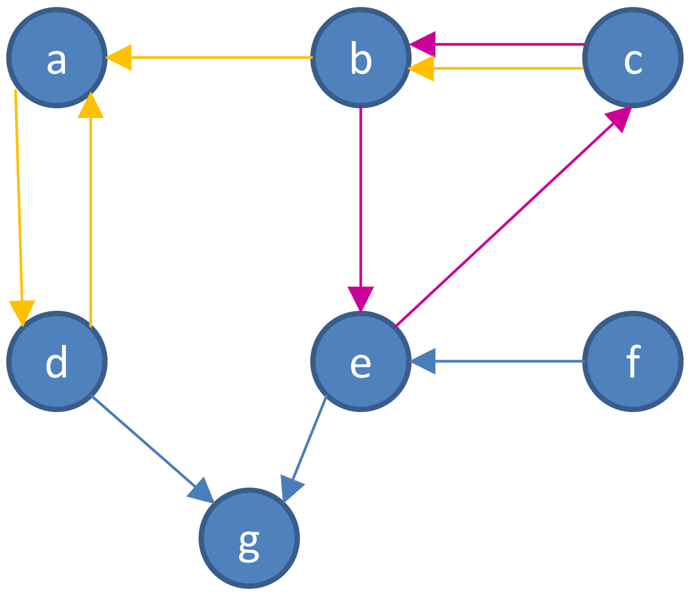
A <u>strongly connected component</u> is a set $U \subseteq V$ with a path between every pair $x, y \in U$

Example

Simple path (f,e,c,b,a)

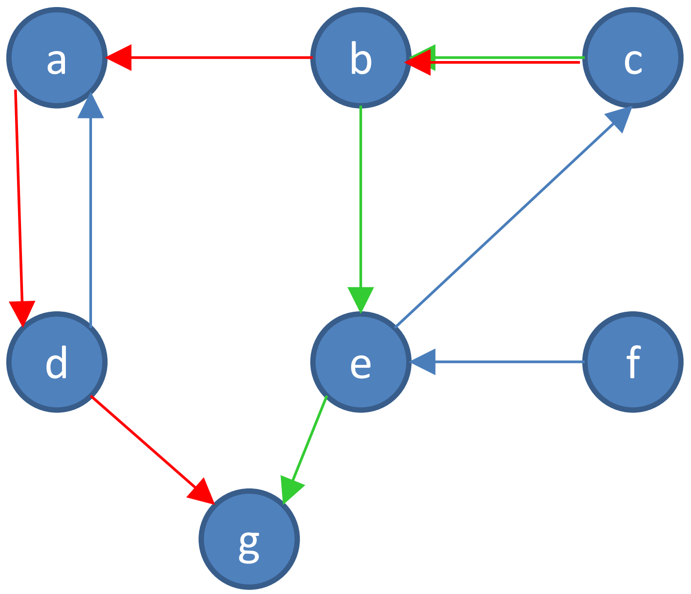
Cycle (e,c,b,e)

Path (c,b,a,d,a)

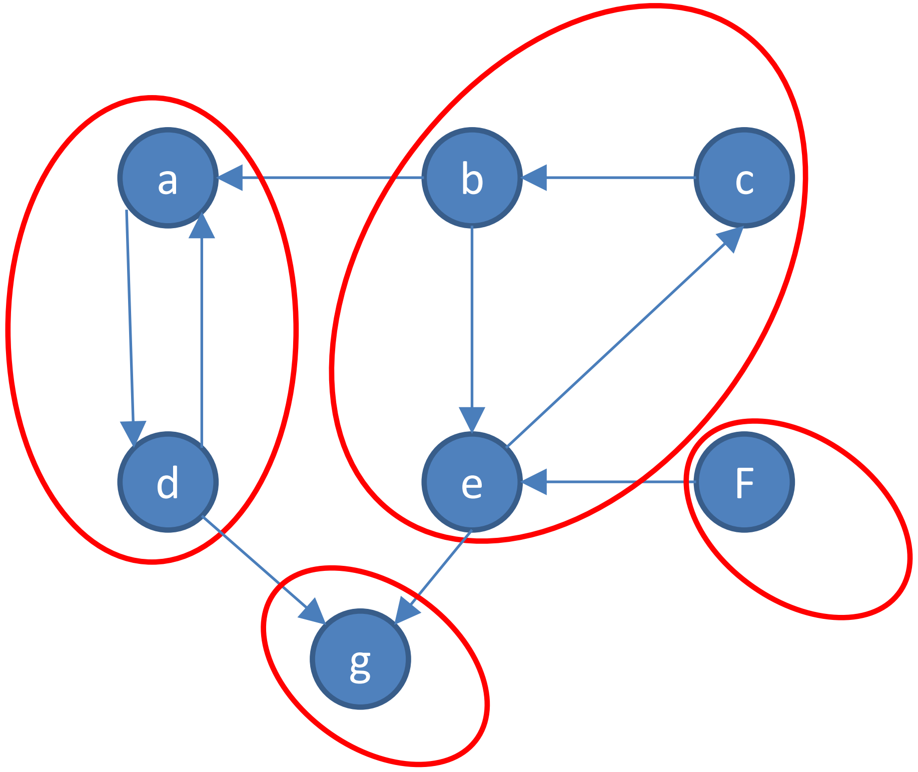


Example

$$P_{c,g} = \{(c,b,e,g),(c,b,a,d,g)\}$$



Strongly Connected Components (SCC)



Weighted graphs

$G = (V, E, w)$ where $w: E \to \mathbb{R}$ and

$$w(S) = \sum_{e \in S} w(e)$$

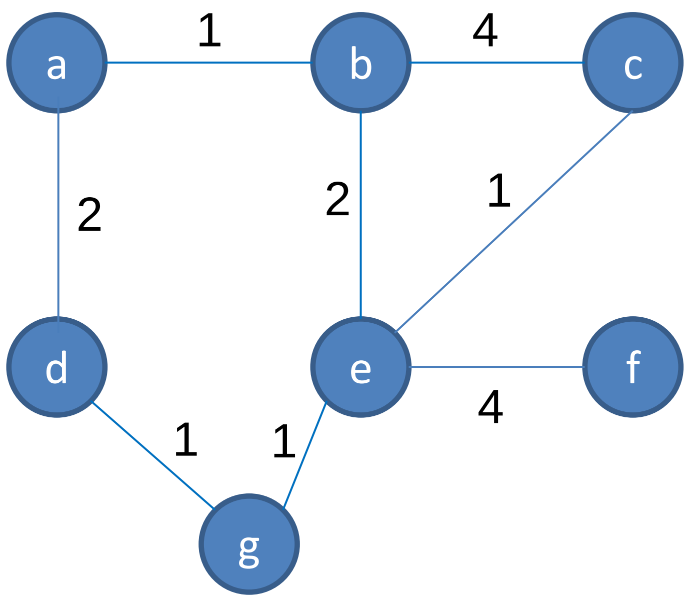
When w is a length:

$d(x, y) \coloneqq \min_{p \in P_{xy}} w(p)$ ($|p|$ when unweighted)

$diam(G) \coloneqq \max_{x,y \in V} d(x, y)$

Example

$$d(b, c) = 3$$
$$diam(G) = ?$$

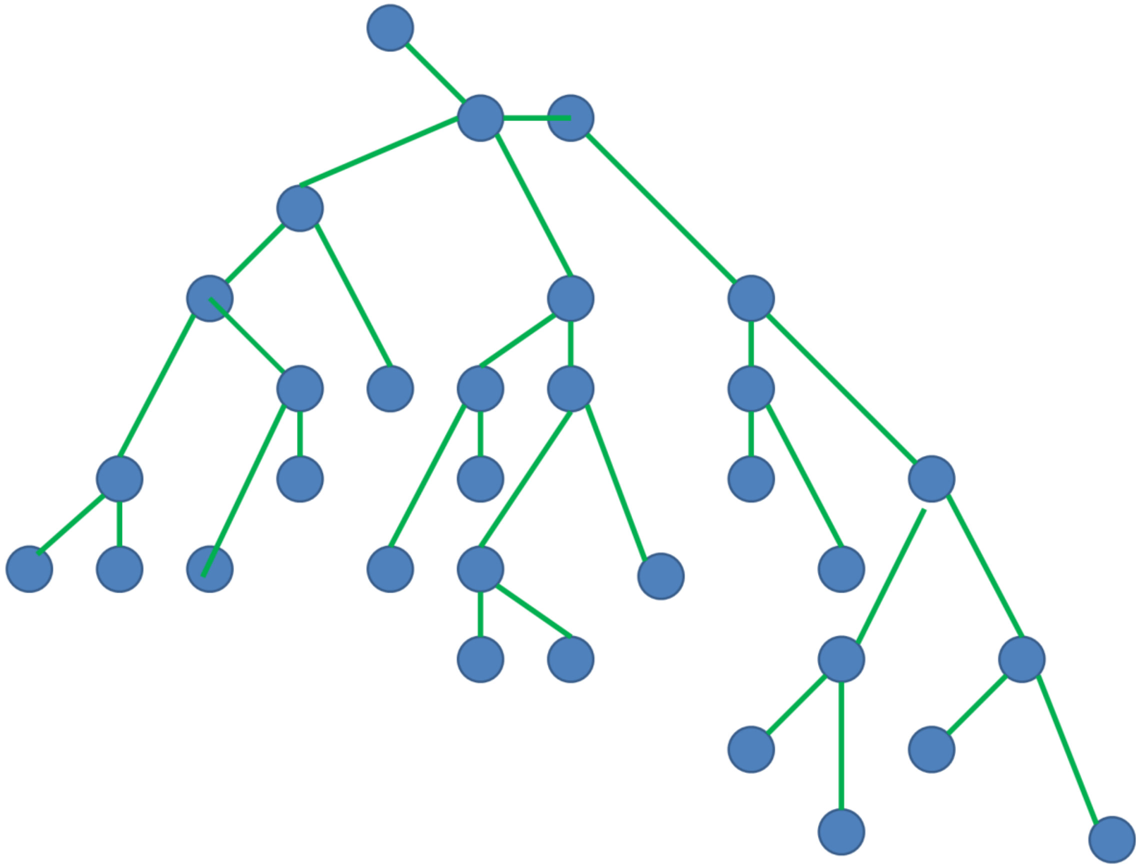


Algorithms

- Finding all SCC: topological sort (DFS algorithm)
- Finding shortest path from x to y: BFS algorithm
  - In a weighted graph: Dijkstra's algorithm

Trees

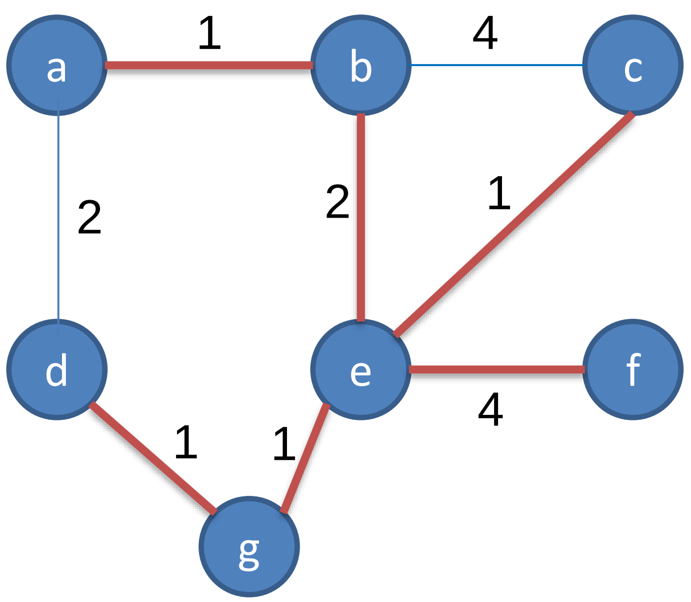
- A tree is an undirected graph in which any two vertices are connected by exactly one path
- Equivalently, connected acyclic undirected graph.
- A forest is an undirected graph in which any two vertices are connected by at most one path
- Equivalently, a disjoint union of trees.



Spanning Trees

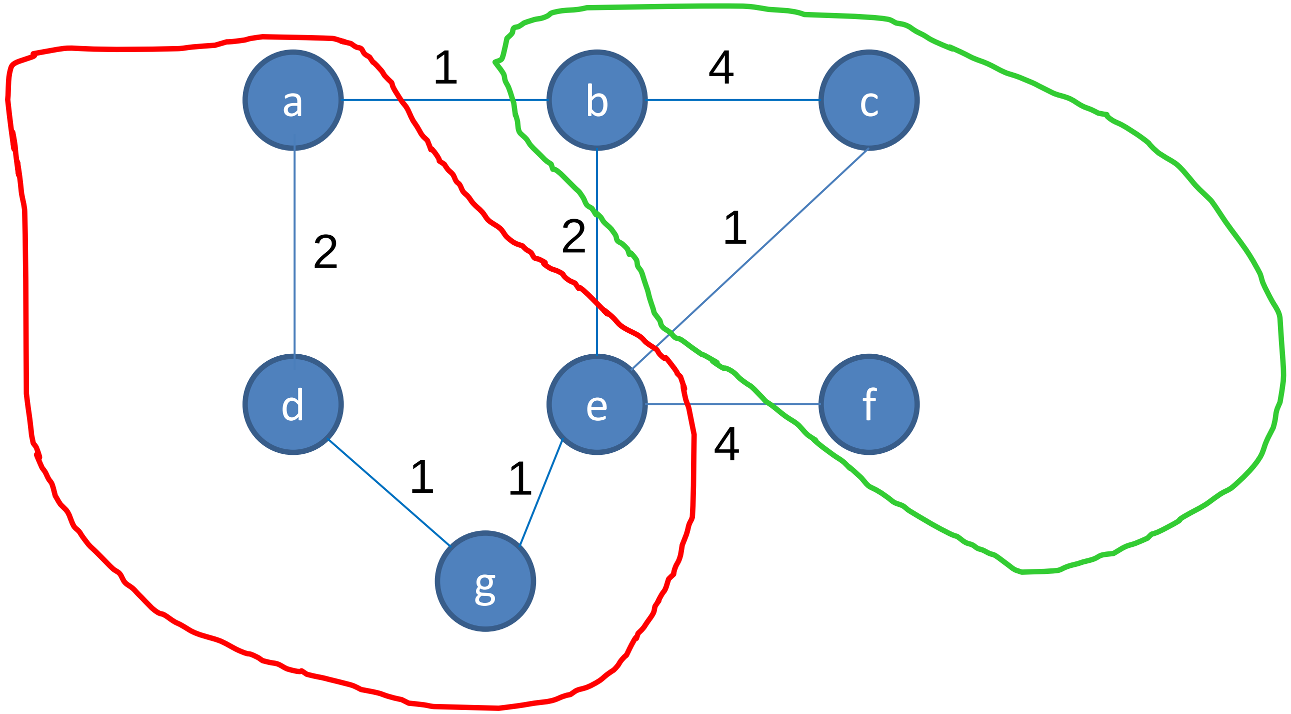
$T = (V', E')$ is a spanning tree of $G = (V, E)$ if it is a tree and $V' = V$

(Kruskal's algorithm for minimum spanning tree)



Graph Cuts

$C_V = (A, B)$ is a cut of G if $A \uplus B = V$

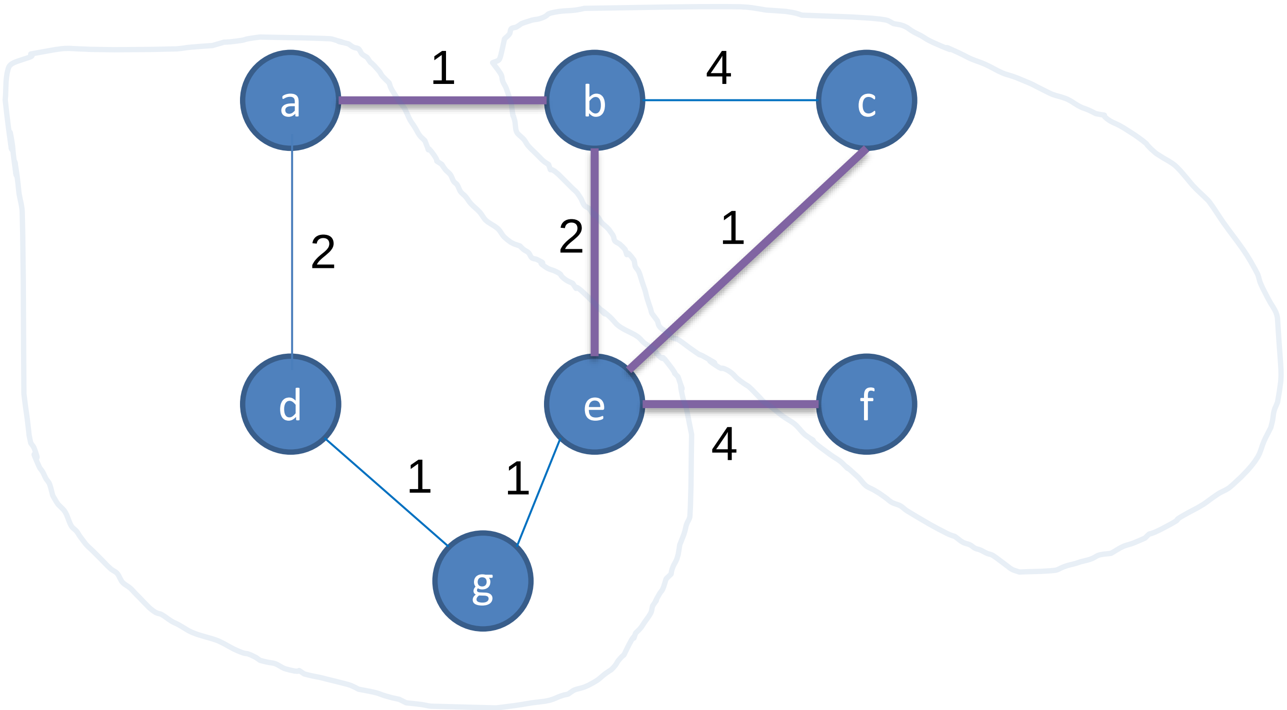


Graph Cuts (2)

$C_V = (A, B)$ is a cut of G if $A \uplus B = V$

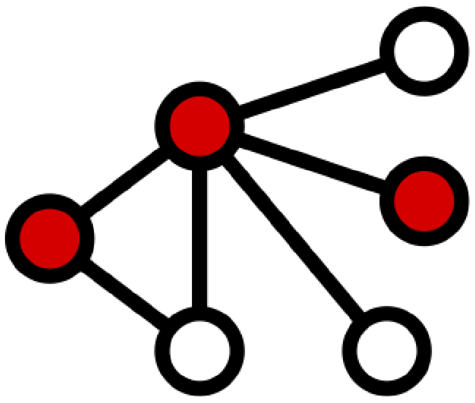
Also denote $C_E := \{(a,b) \in E | a \in A, b \in B\}$

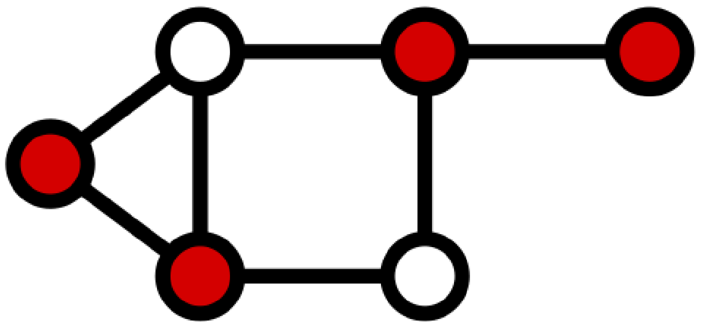
The size of cut C is $|C_E|$ (what is the weight?)



Vertex Cover

- An undirected Graph $G = (V, E)$
- A set $V' \subseteq V$ is a vertex cover of G if for every edge $\{u, v\} \in E$, either $u \in V'$ or $v \in V'$





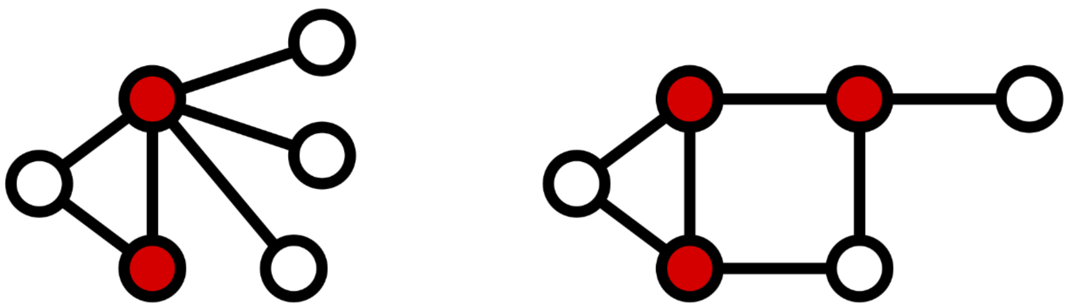
Minimum Vertex Cover

- The problem of finding a smallest vertex cover in a given graph
- As a decision problem:

Does G have a vertex cover of size at most k?

- As an optimization problem:

Smallest number k such that G has a vertex cover of size k



As Integer Linear Programming

Minimize $\sum_{v \in V} x_v$

Subject to

$x_u + x_v \geq 1 \; \forall \{u, v\} \in E$

$x_v \in \{0,1\} \; \forall v \in V$

- Runtime?
- Injecting costs?
- Approximation?

Electronic Commerce
096211

Lecture 1 – Introduction

Omer Ben-Porat

Course outline

- History – up to 2000
- The Internet has created **new kinds of markets**, and altered traditional markets in various ways. Our goal is to understand the sociological, economic, computational and ethical **challenges and tools** at the base of modern electronic commerce systems.

General information

- Teacher: Omer Ben-Porat
  - Office hours: TBD
  - TAs: Or (lead)
- Grading:
  - 3-4 programming exercises: 15-20%
    - May submit in pairs
    - Mandatory part, competitive part
    - May contain simple theoretical questions
    - Potentially huge bonus for winners!
  - Exam: 80-85%   (passing is mandatory)
  - Weekly exercises (2 bonus points)
- Prerequisites – Probability, Algorithms
- Changes from previous versions of the course


KYC

How to succeed in the course?

- Understand the (weekly) context
- Attend classes (online or offline)
  – Do not rely on slides
- Active participation in TA sections (“tirgul”)
- Solve weekly exercises
- Invest thought and effort in HW!
- Study towards the exam

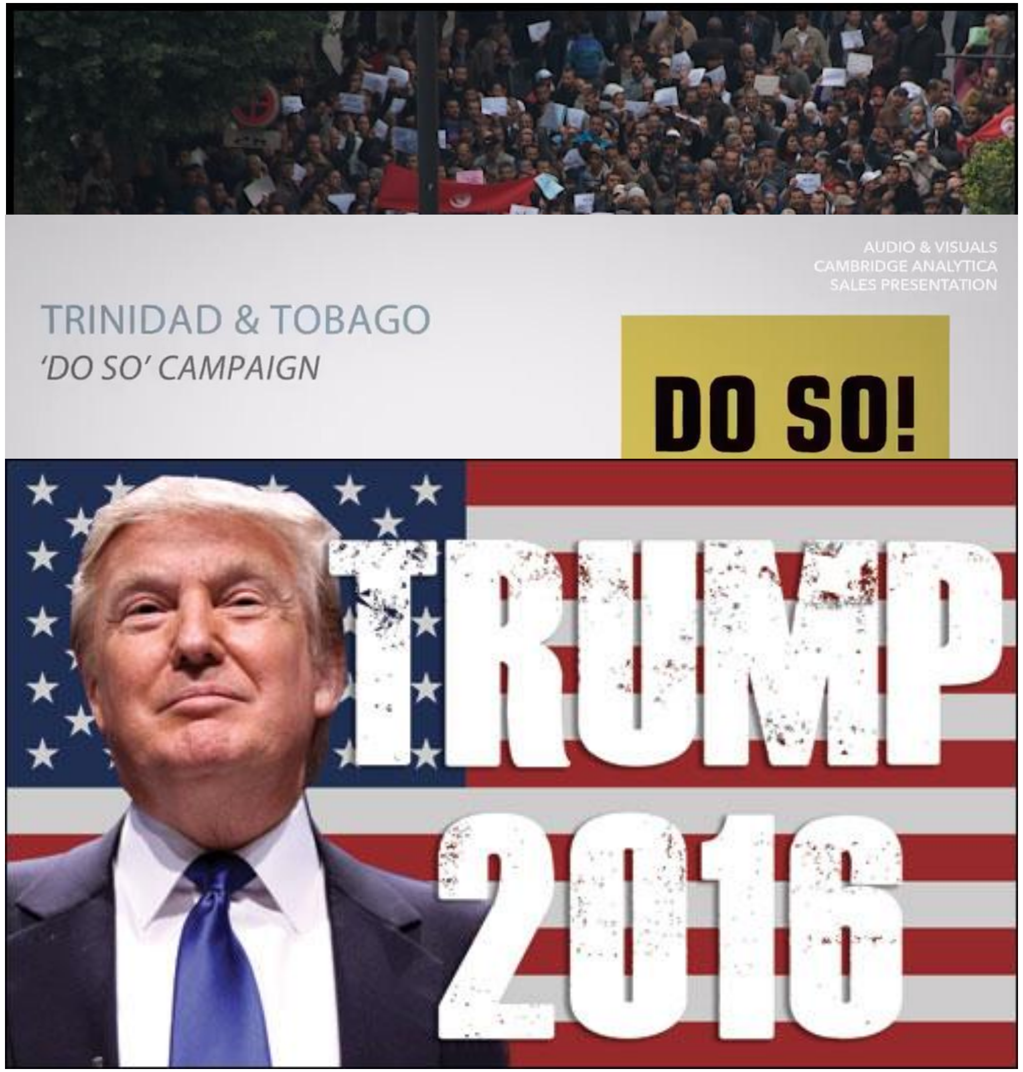
Course outline

- The Internet has created new kinds of markets, and altered traditional markets in various ways. Our goal is to understand the sociological, economic, computational and ethical challenges and tools at the base of modern electronic ommerce systems.

- Examples!

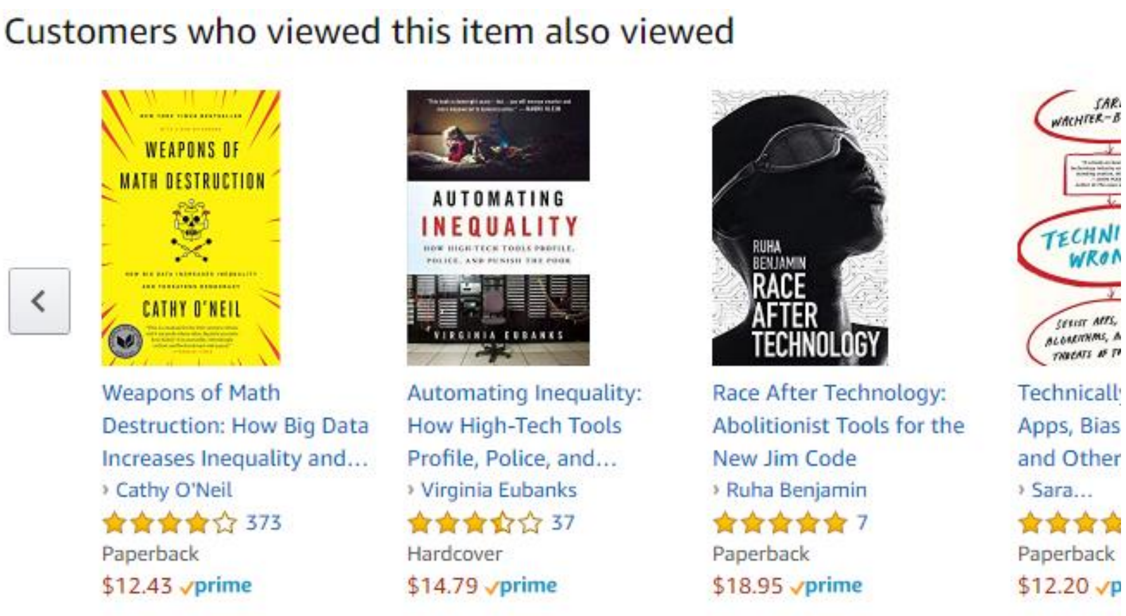
Textbook: Networks, Crowds, and Markets: Reasoning About a Highly Connected World by David Easley and Jon Kleinberg, Cambridge University Press (2010)

Social Interaction in Networks



- How (dis-)information is spread
- How to influence people?
- How to target the right audience?

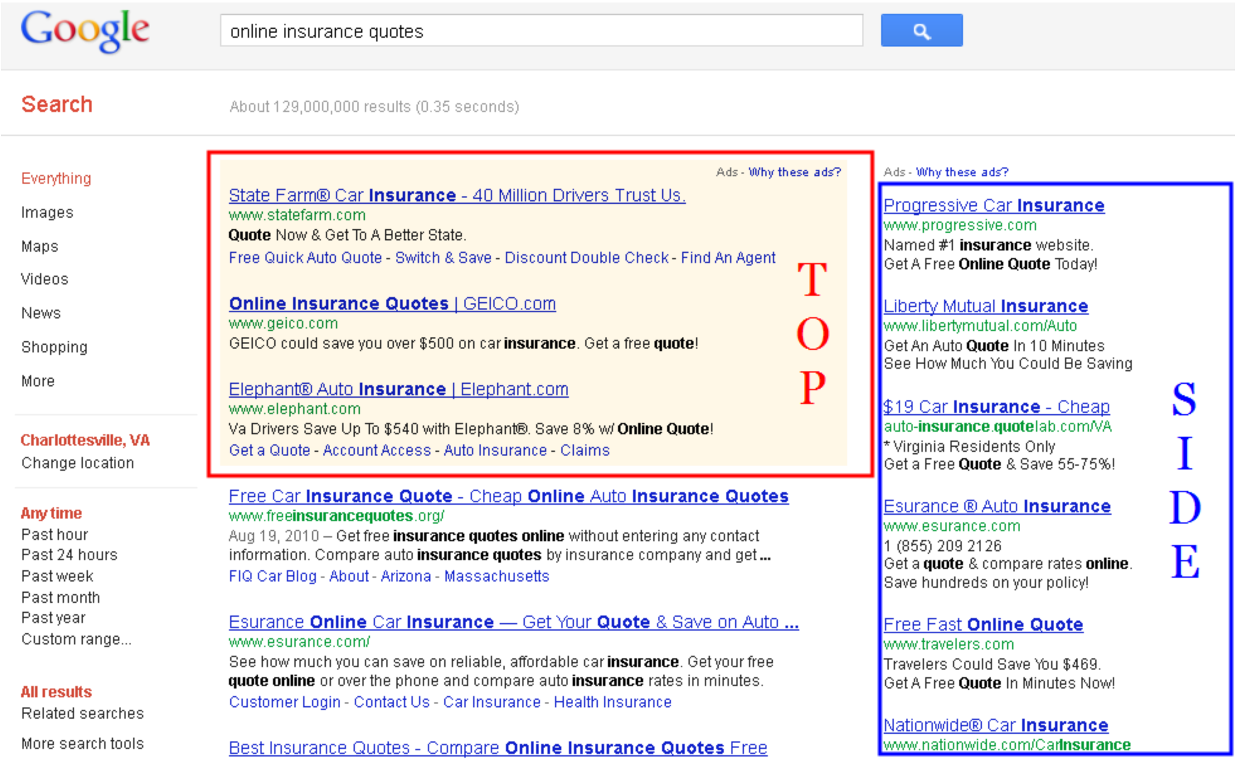
Online Markets and Platforms



amazon

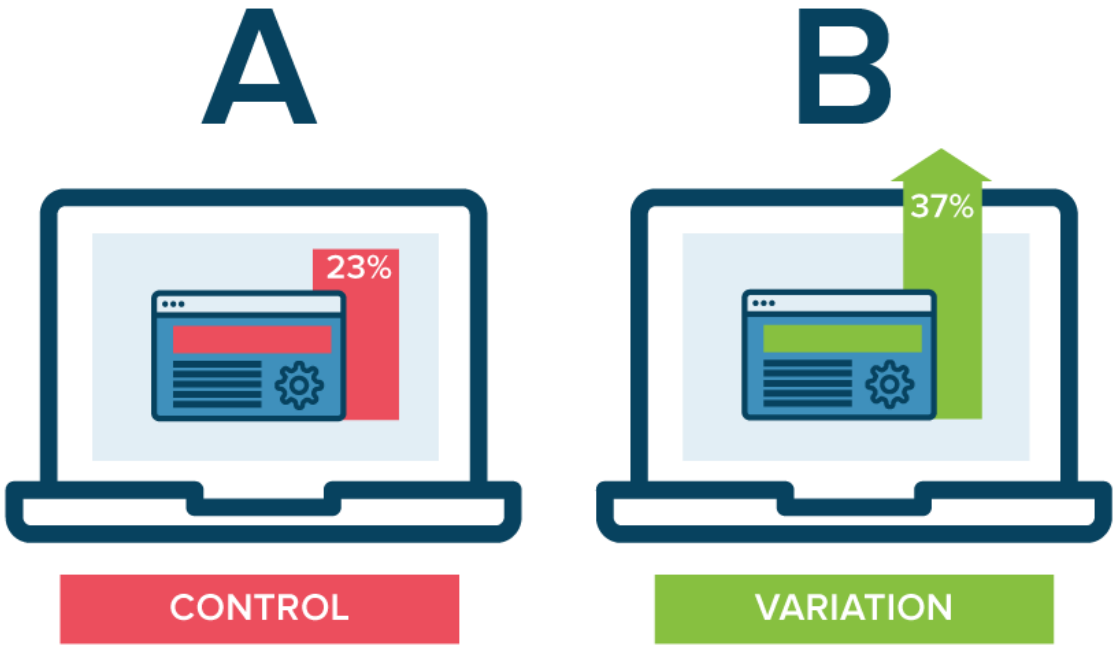
- Platform’s side:
  - How to recommend relevant items?
  - How to promote products?
- Seller’s side:
  - What to sell?
  - How to set prices?
  - How to promote products?

The Internet’s Oxygen - Auctions



- Platform:
  - How to sell estate?
  - How to set prices
- Customers:
  - Which queries to focus on?
  - How much to bid?

Online Decision Making

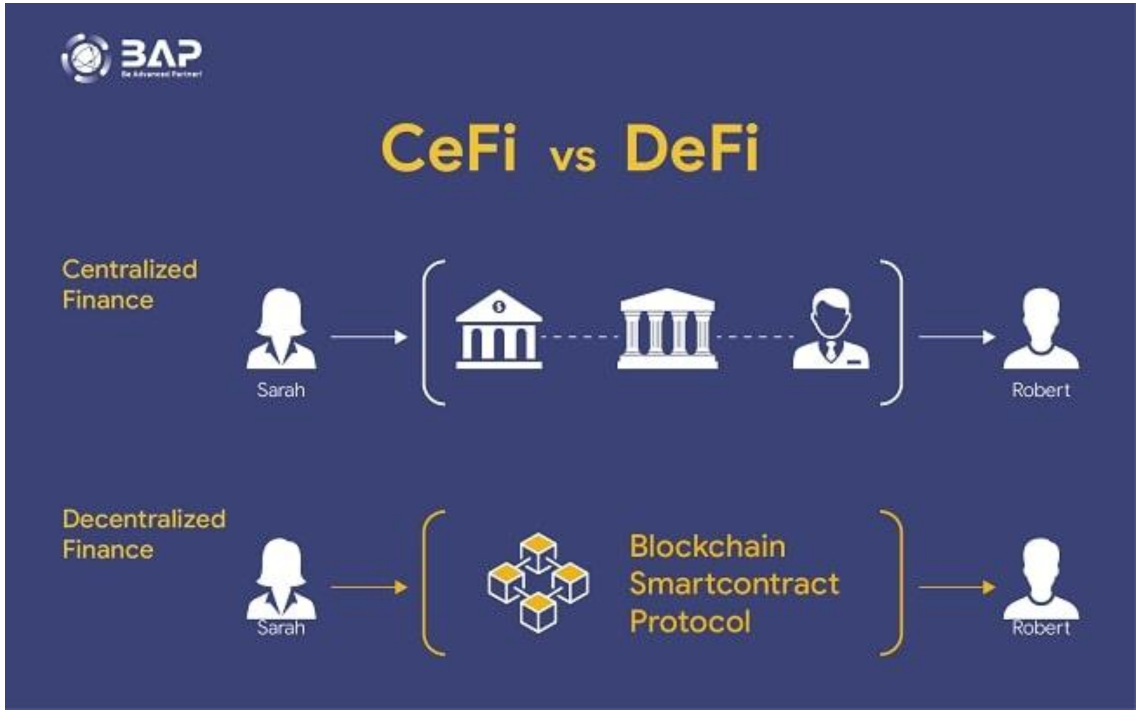


- Which ad to display?
- Does the treatment helps?
- Is the new pricing scheme better?

Prediction markets

Intrade elections futures as of July 12, 2008 source: Intrade.com

Cryptocurrencies, DeFi, NFT, …



- How to avoid 3rd party in transactions?
- How to handle a distributed ledger?

Guest Lectures!

Snapchat

Hi, I'm Cortana.

WiX

fiverr

amazon alexa

NETFLIX

Booking.com

similarweb

Medium

WHAT SHOULD I READ NEXT?

Spotify

LinkedIn

facebook

TikTok

movielens

airbnb

riskified

"The growth of the Internet will **slow drastically**… …most people have nothing to say to each other! By 2005 or so, it will become clear that **the Internet's impact on the economy has been no greater than the fax machine's**."

--Paul Krugman, 1998

(won the Nobel prize in Economics, 2008)

Course Structure

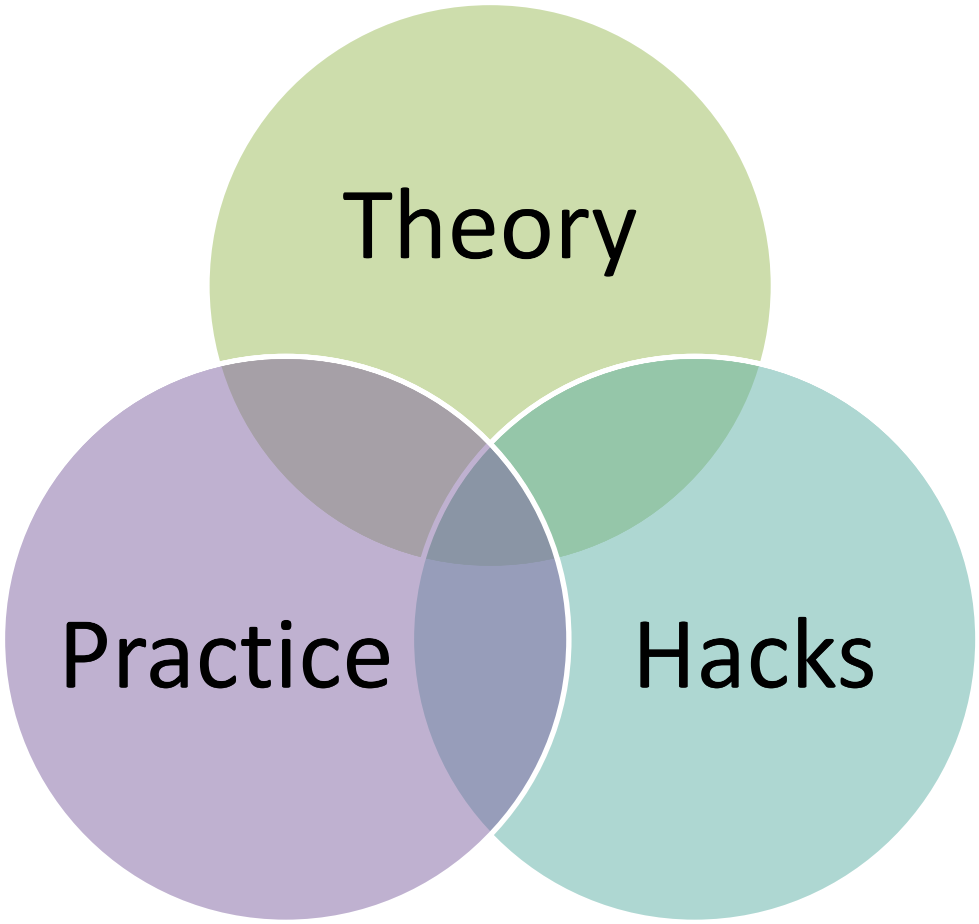
Online Societies
- Web ranking
- Diffusion
- Structure of the web
- Power laws
- Networks

Advertising
- Auctions
- Matching markets
- Sponsored search

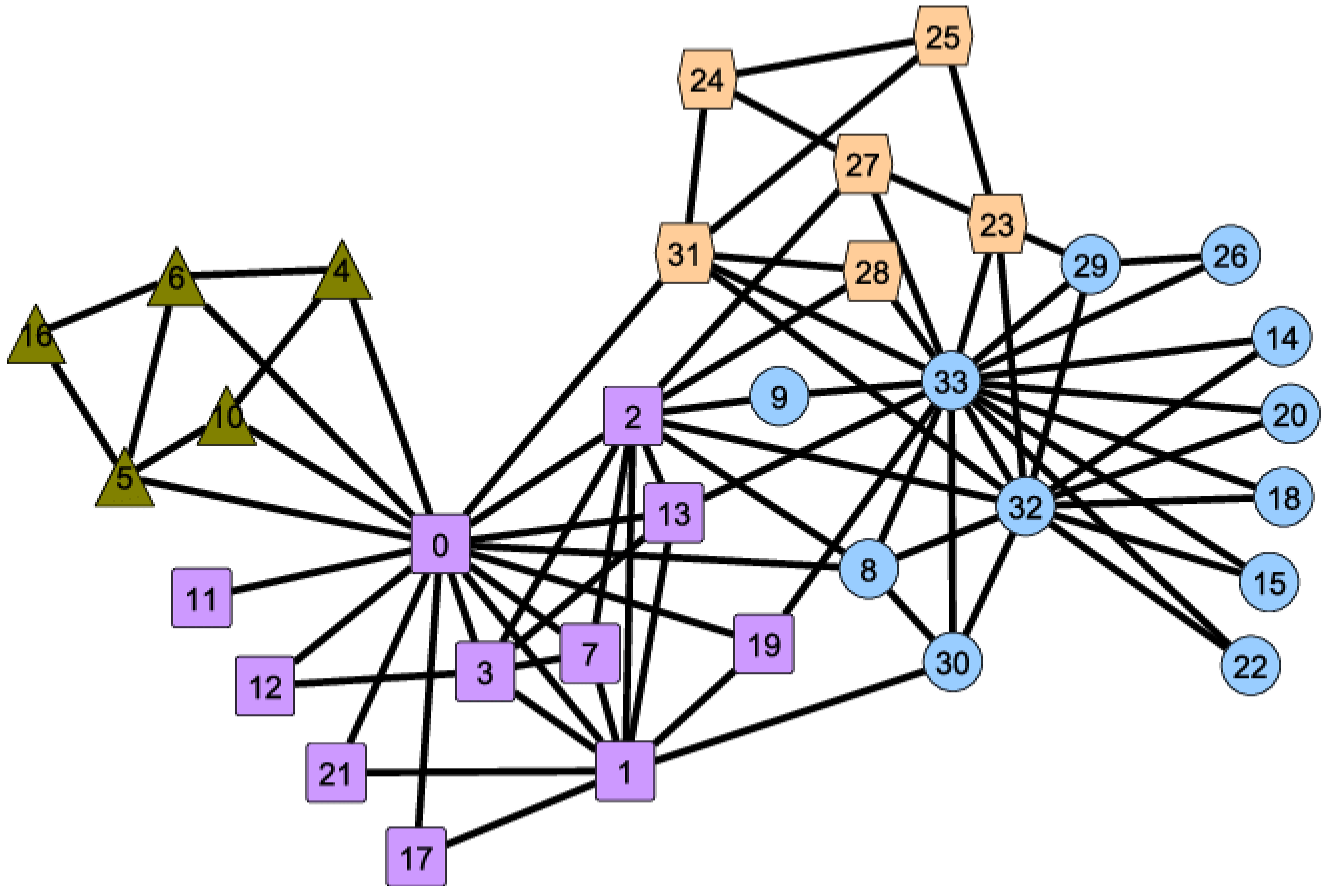
Decision Making
- Recommendation systems
- Multi-armed bandits

Advance
- Information in markets
- Externalities
- Streaming
- Blockchain

Tools and Techniques
- Game theory
- Algorithms
- Graph theory
- Optimization


Theory
Practice
Hacks

Karate Club



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Electronic Commerce
096211

Networks

Omer Ben-Porat (slides are adopted from Prof. Reshef Meir)

Course Structure

Online Societies
- Web ranking
- Diffusion
- Structure of the web
- Power laws
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Advertising
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Outline

- Introduction – from graphs to networks
- Triadic closure
- Centrality measures
- The “small world” phenomenon
- Homophily
- Affiliation

Outline

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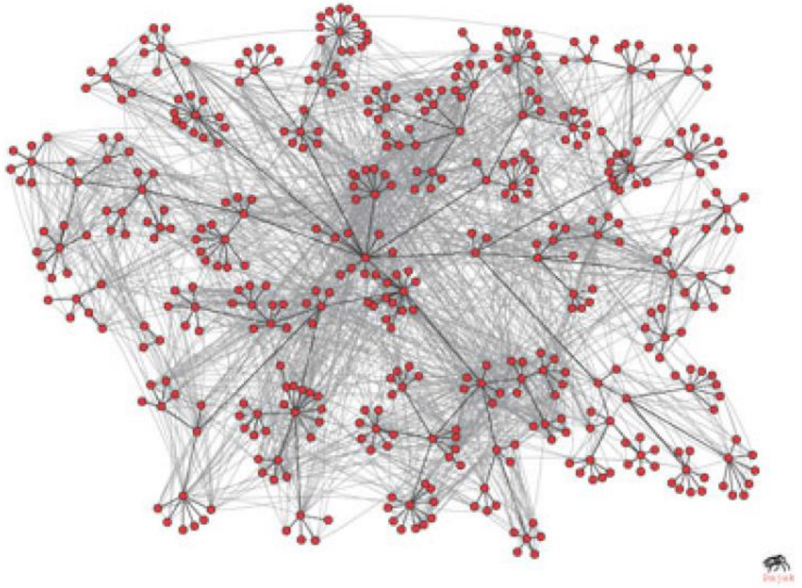
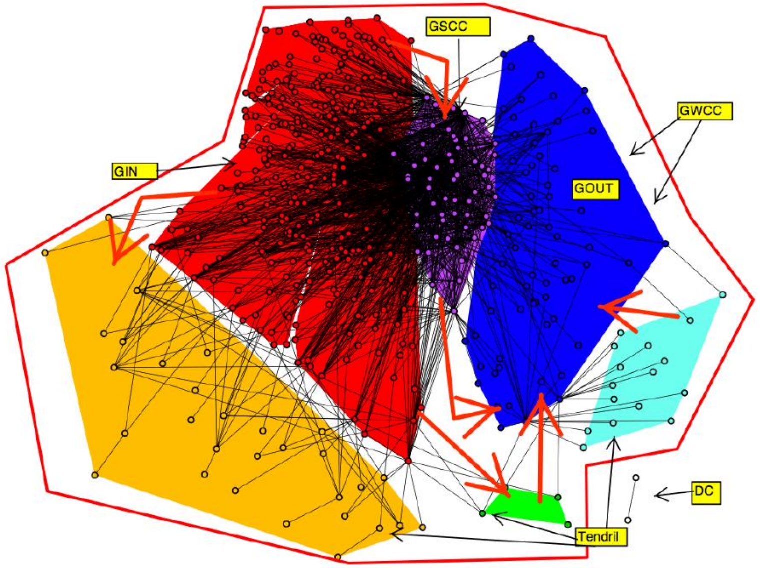
Why Study Networks?

1. Questions about the network itself
   – “who are the most collaborating authors?”
2. Approximating another network that we cannot measured
   – Distance in Facebook is a proxy to some real “friendship distance”
3. We care about general phenomena
   – Collaboration graphs, Facebook and IM are all examples of the same social phenomenon

Communication / Transactions



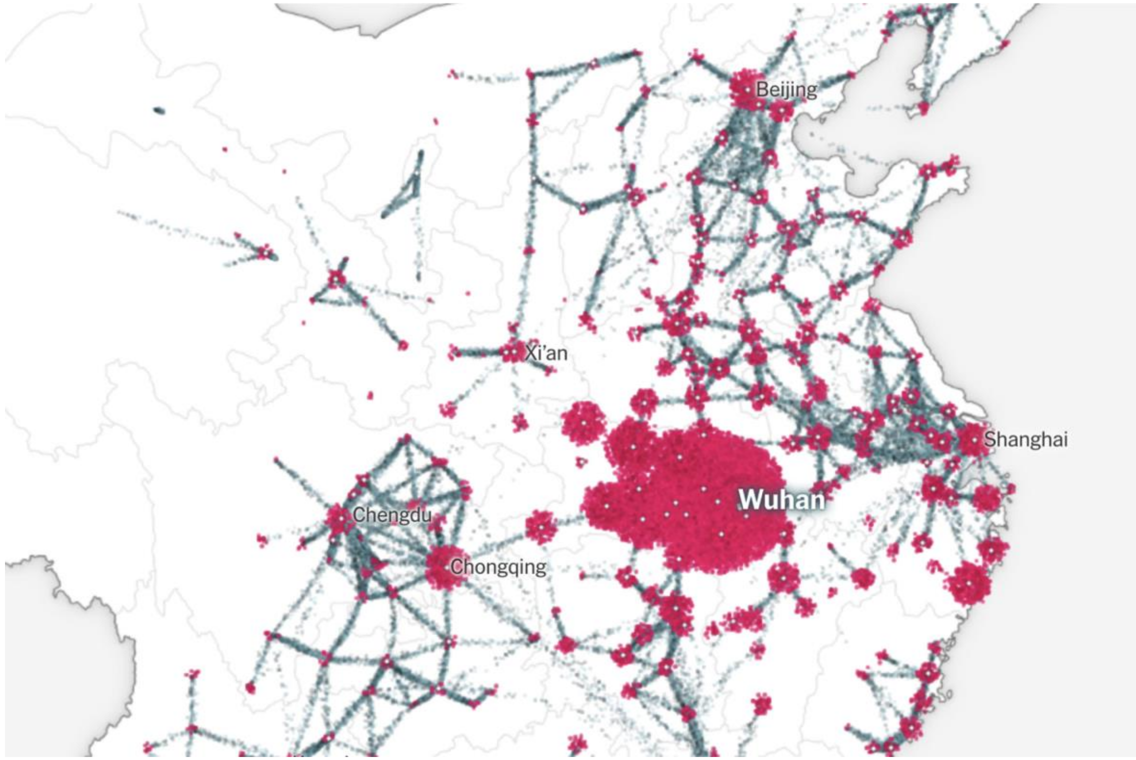
Emails inside HP



Loans among banks

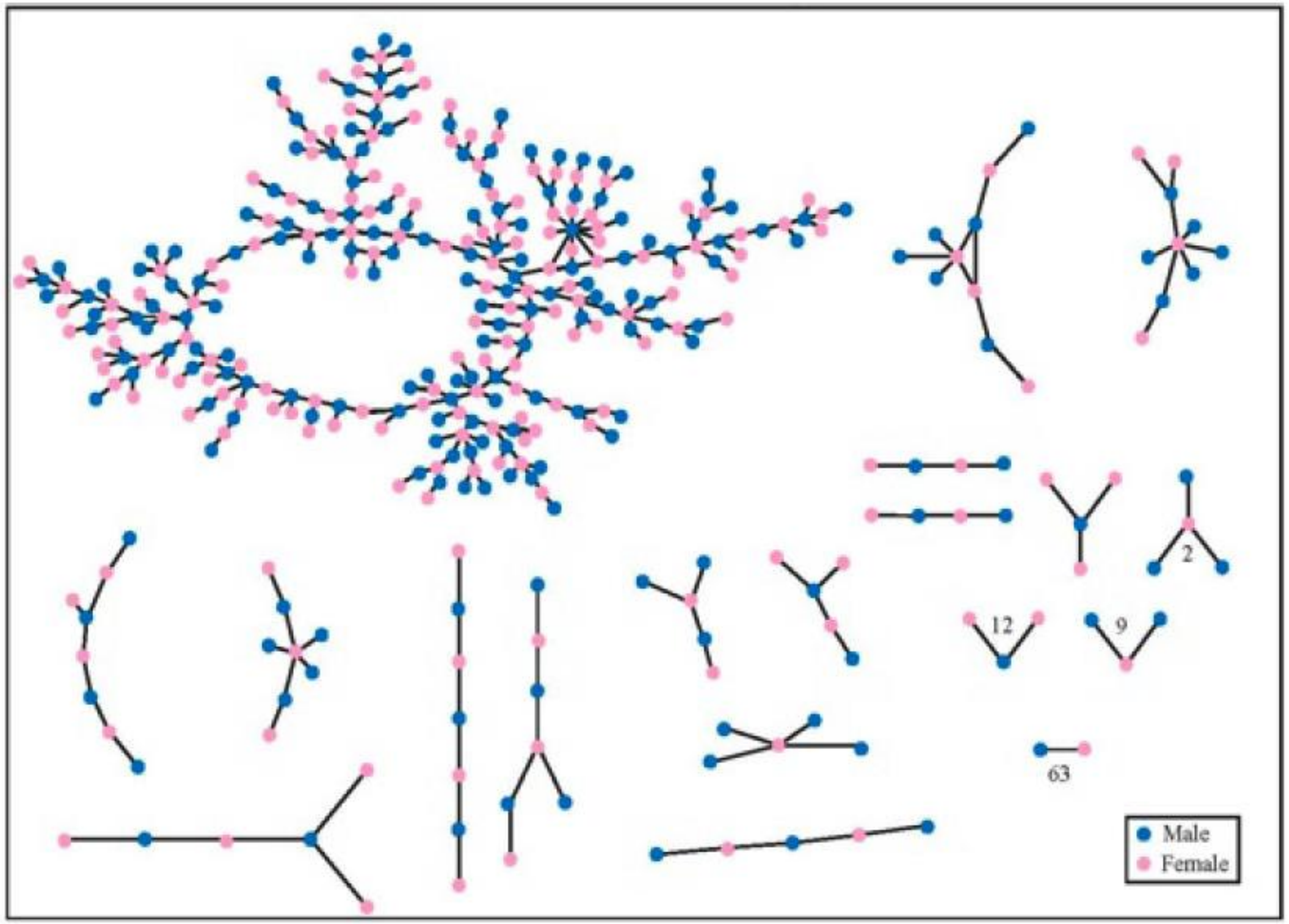
Which questions can we ask?

Covid19 Spread

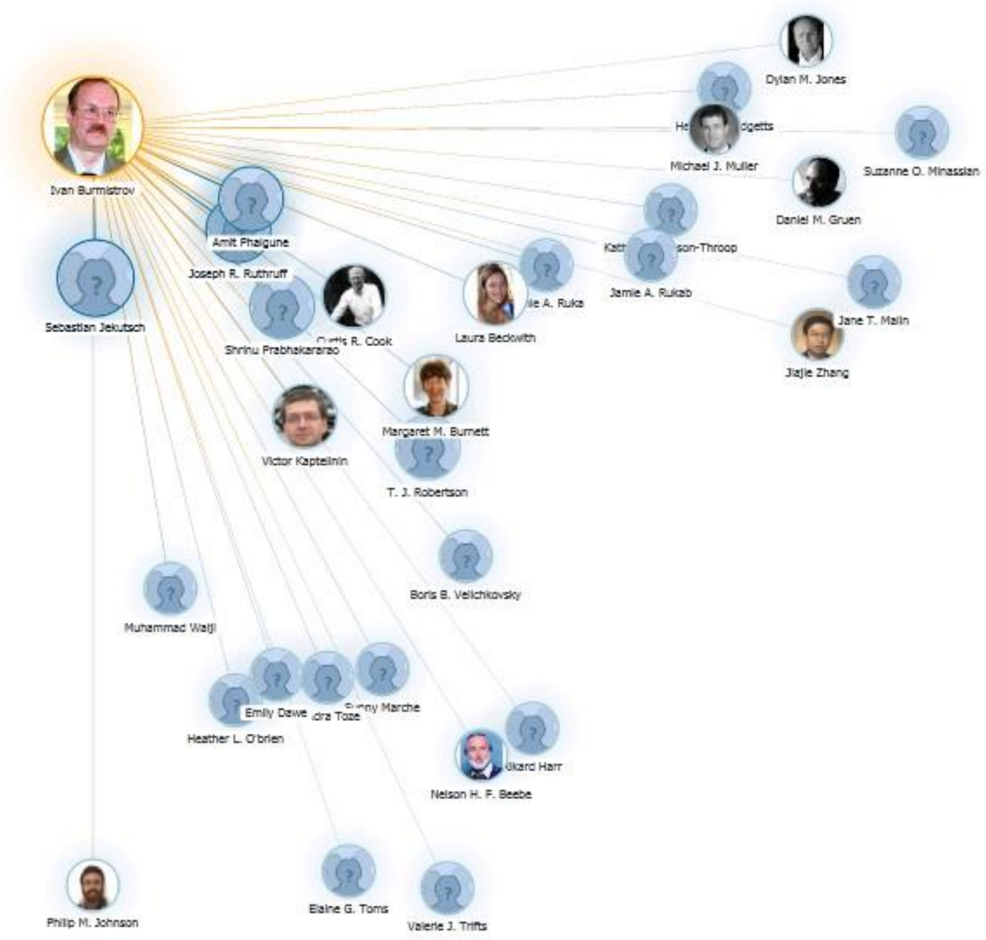


Which questions can we ask?

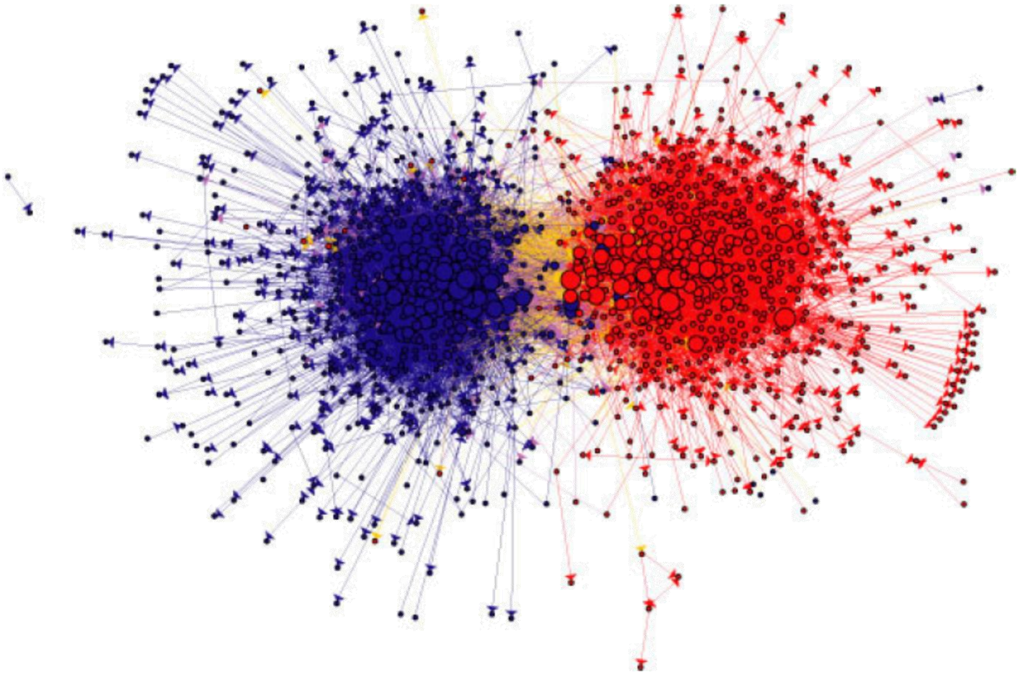
Sociology and Ramsey's theorem



Links and References

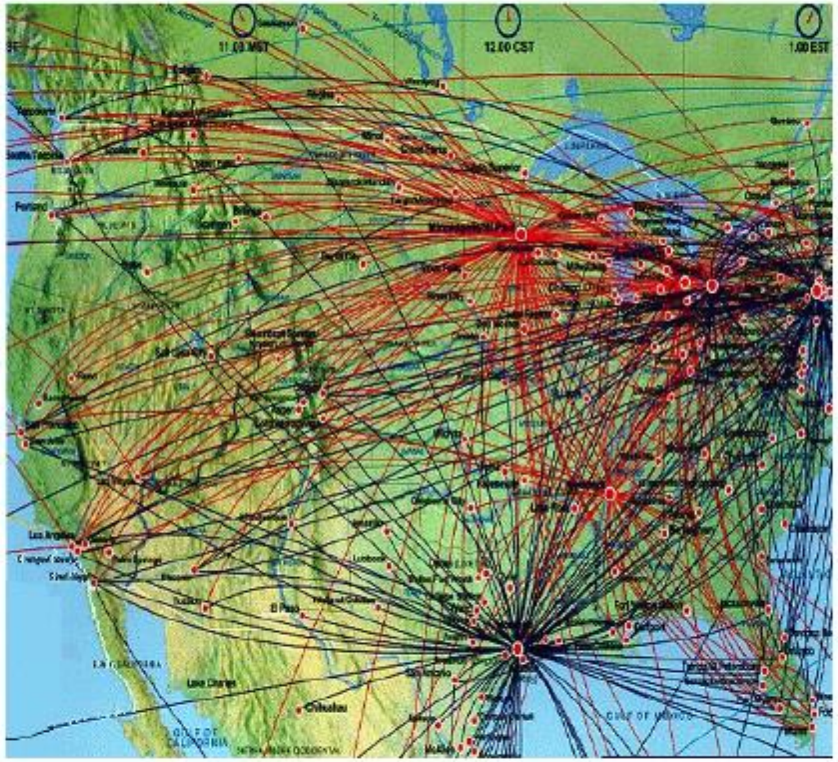


Academic citation



Political blogs

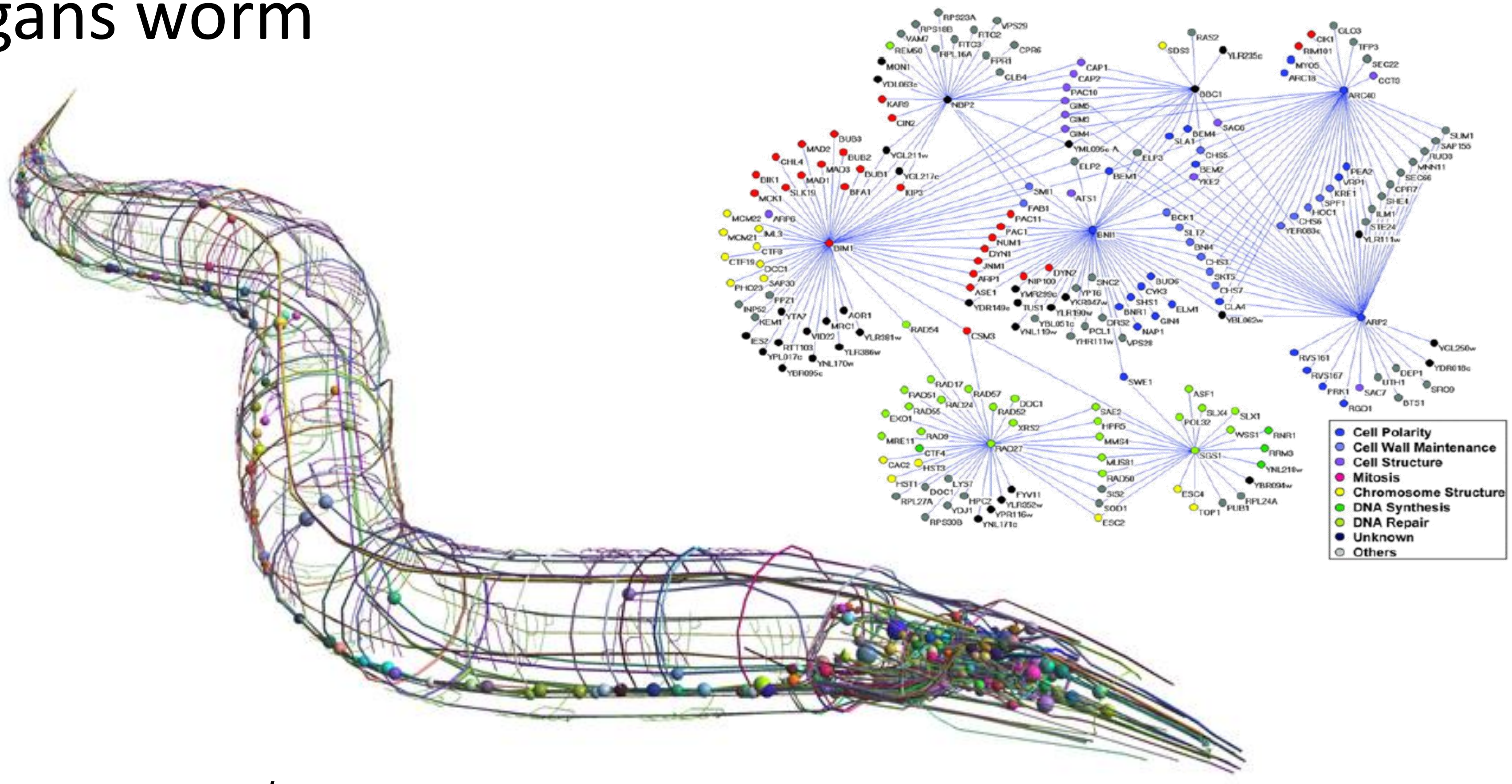
Transportation



Biological Networks

- Neurons of the C. elegans worm

Genetic interaction network



Source: http://browser.openworm.org/

Outline

- Introduction – from graphs to networks
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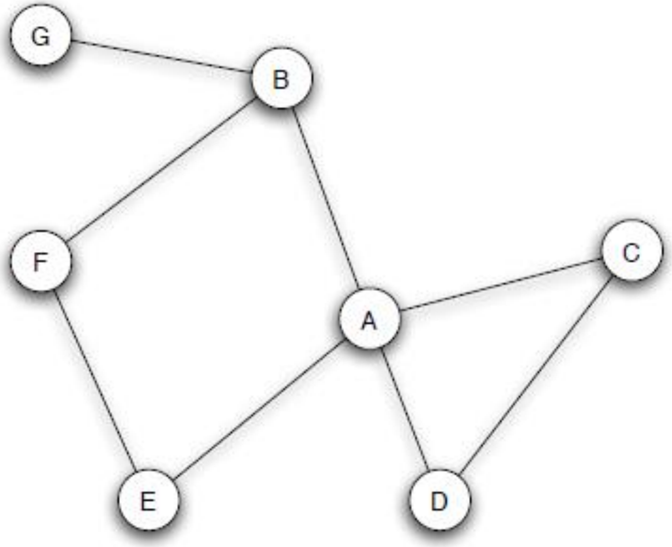
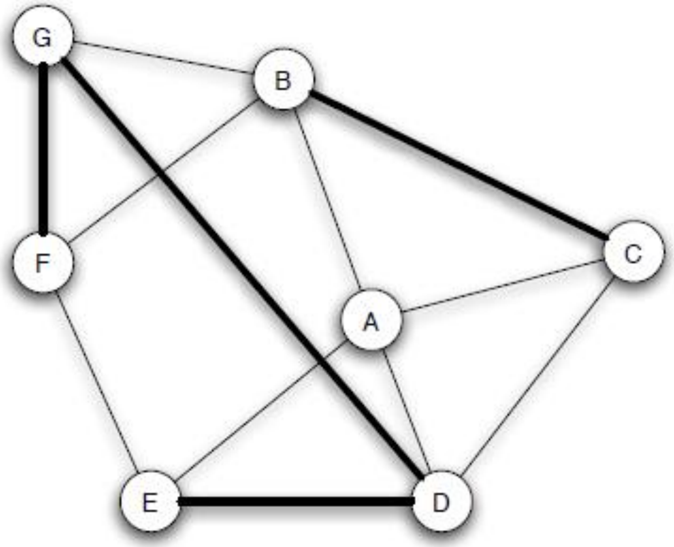
Triadic Closure

- Graphs are static, networks are dynamic
- What are the mechanisms by which edges form?
  - Depends on the type of the network!
- If two people in a social network have a friend in common, then there is an increased likelihood that they will become friends themselves at some point in the future
  - Opportunity to meet
  - Basis for trust
  - Incentive, latent stress

Triadic Closure



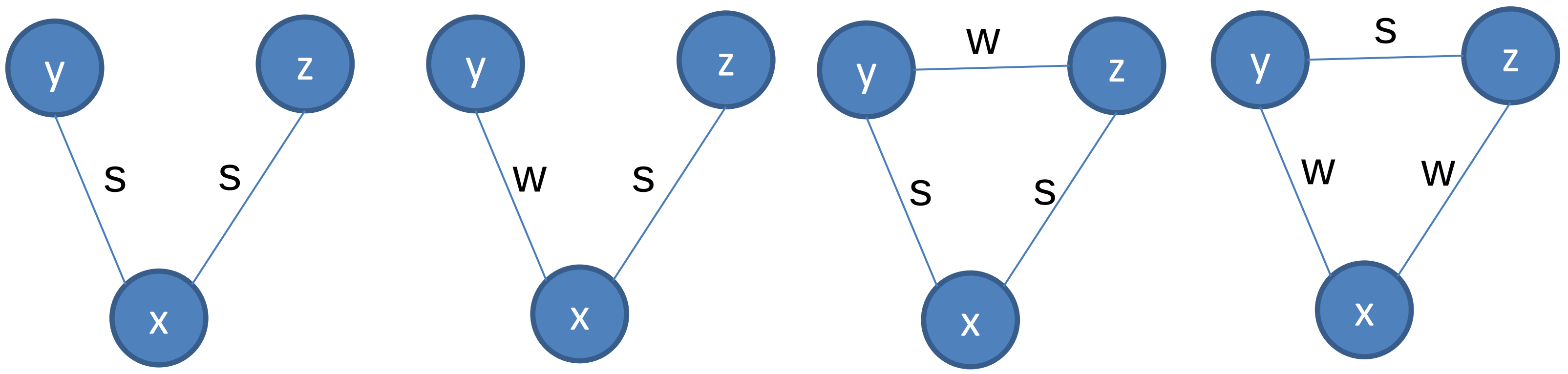
(a) *Before new edges form.*



(b) *After new edges form.*

Strong Triadic Closure

- Suppose there are "weak" and "strong" edges $G = (V, E_w, E_s)$
- $x \in V$ satisfies STC if every two "strong neighbors" are connected: $\forall y, z \in E_s(x), \{y, z\} \in E$



Strong Triadic Closure

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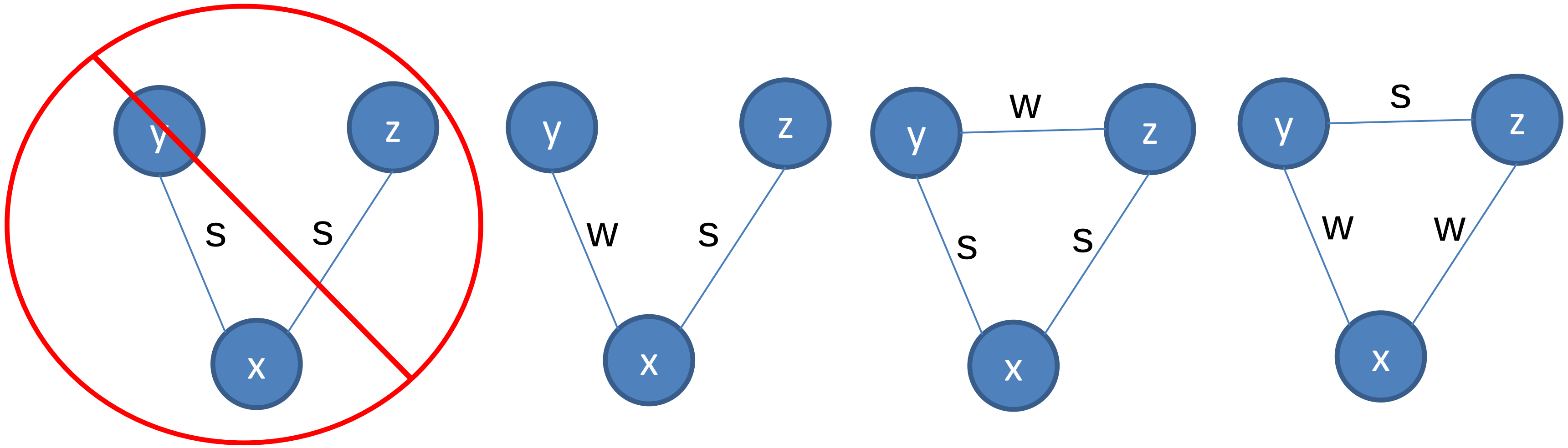

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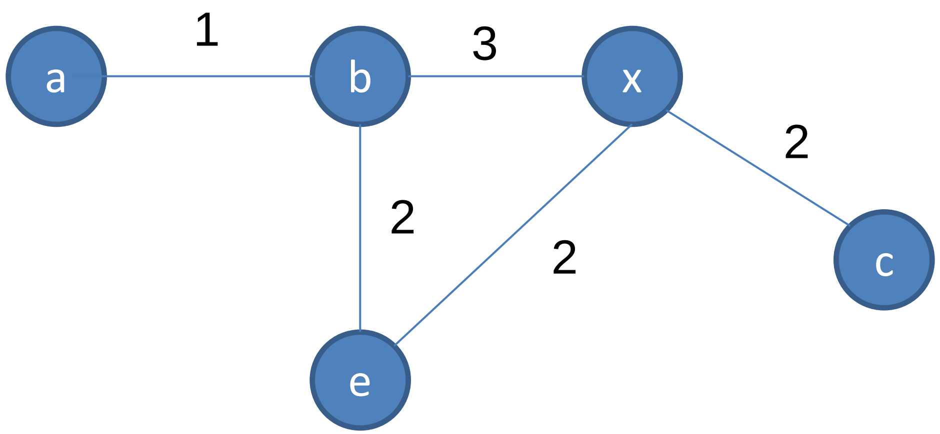
Centrality Measures

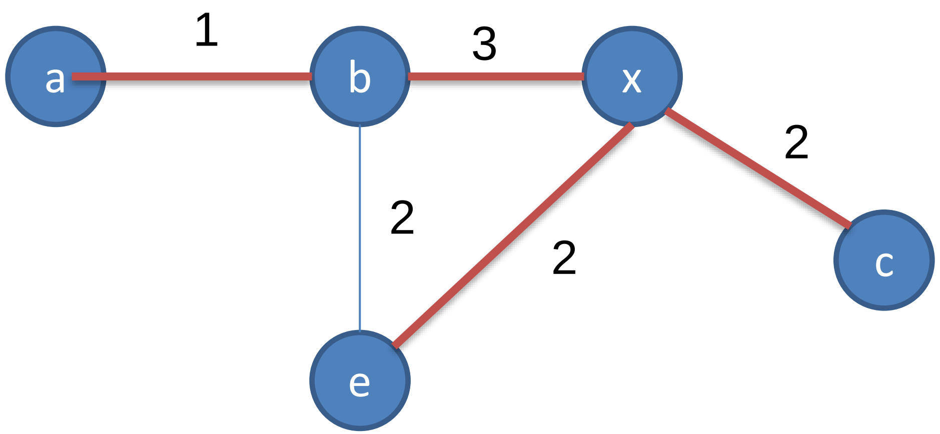
- Degree
- Closeness $CC(x) := \frac{n-1}{\sum_y d(x,y)}$
- Harmonic $HC(x) := \frac{1}{n-1} \sum_y \frac{1}{d(x,y)}$
- Betweenness $BC(x) := \frac{1}{\binom{n-1}{2}} \sum_{s,t \neq x} \frac{\sigma_{st}(x)}{\sigma_{st}}$ where

σ_{st}=#all $s - t$ shortest paths, and $\sigma_{st}(x)$ = #all $s - t$ shortest paths via x

Centrality: Example



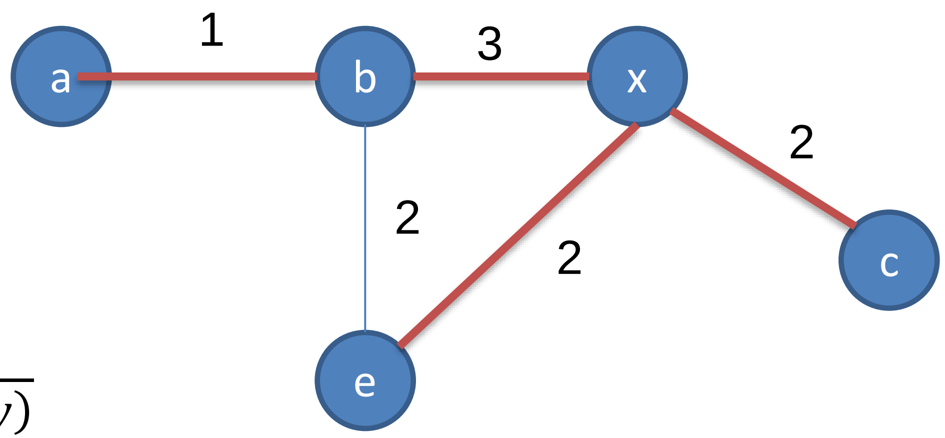
Centrality: Example



$$CC(x) \coloneqq \frac{n-1}{\sum_y d(x,y)}$$

$$CC(x) = \frac{4}{d(x,b)+d(x,c)+d(x,e)+d(x,a)} = \frac{4}{3+2+2+4} = \frac{4}{11}$$

Centrality: Example



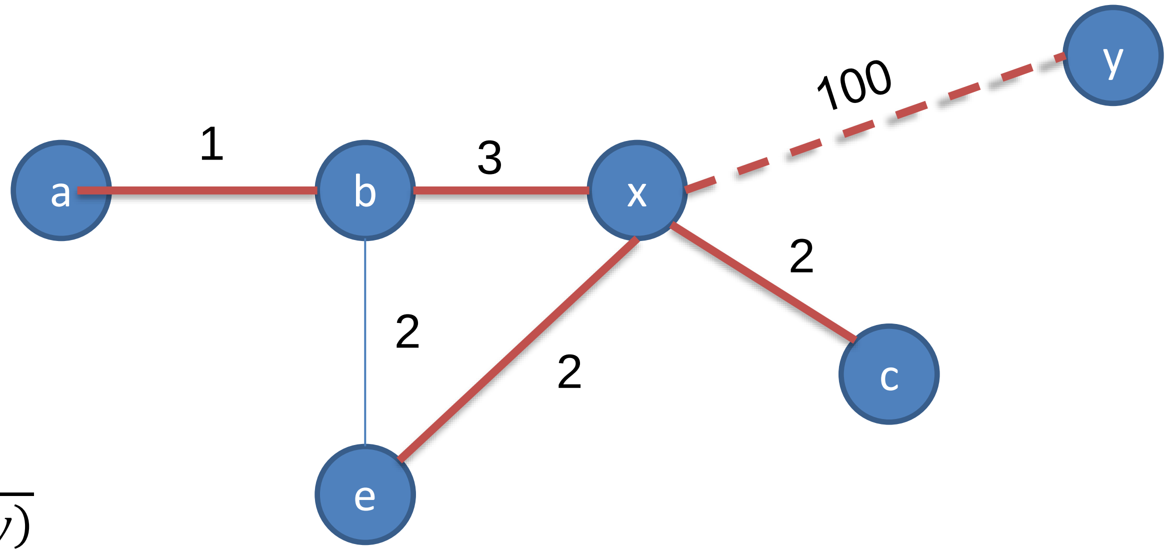
$$CC(x) := \frac{n-1}{\sum_y d(x,y)}$$

$$HC(x) := \frac{1}{n-1}\sum_y \frac{1}{d(x,y)}$$

$$CC(x) = \frac{4}{d(x,b)+d(x,c)+d(x,e)+d(x,a)} = \frac{4}{3+2+2+4} = \frac{4}{11}$$

$$HC(x) = \frac{1}{4}\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}\right) = \frac{1}{4}\cdot\frac{4+6+6+3}{12} = \frac{1}{4}\cdot\frac{19}{12} = \frac{19}{48}$$

Centrality: Example



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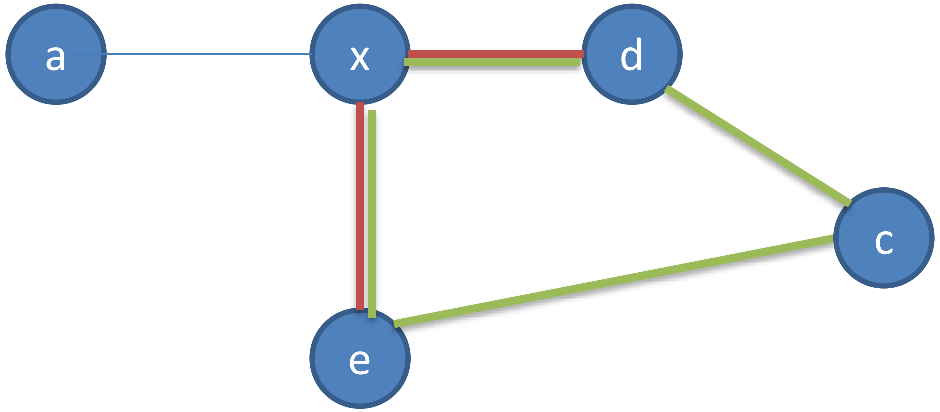
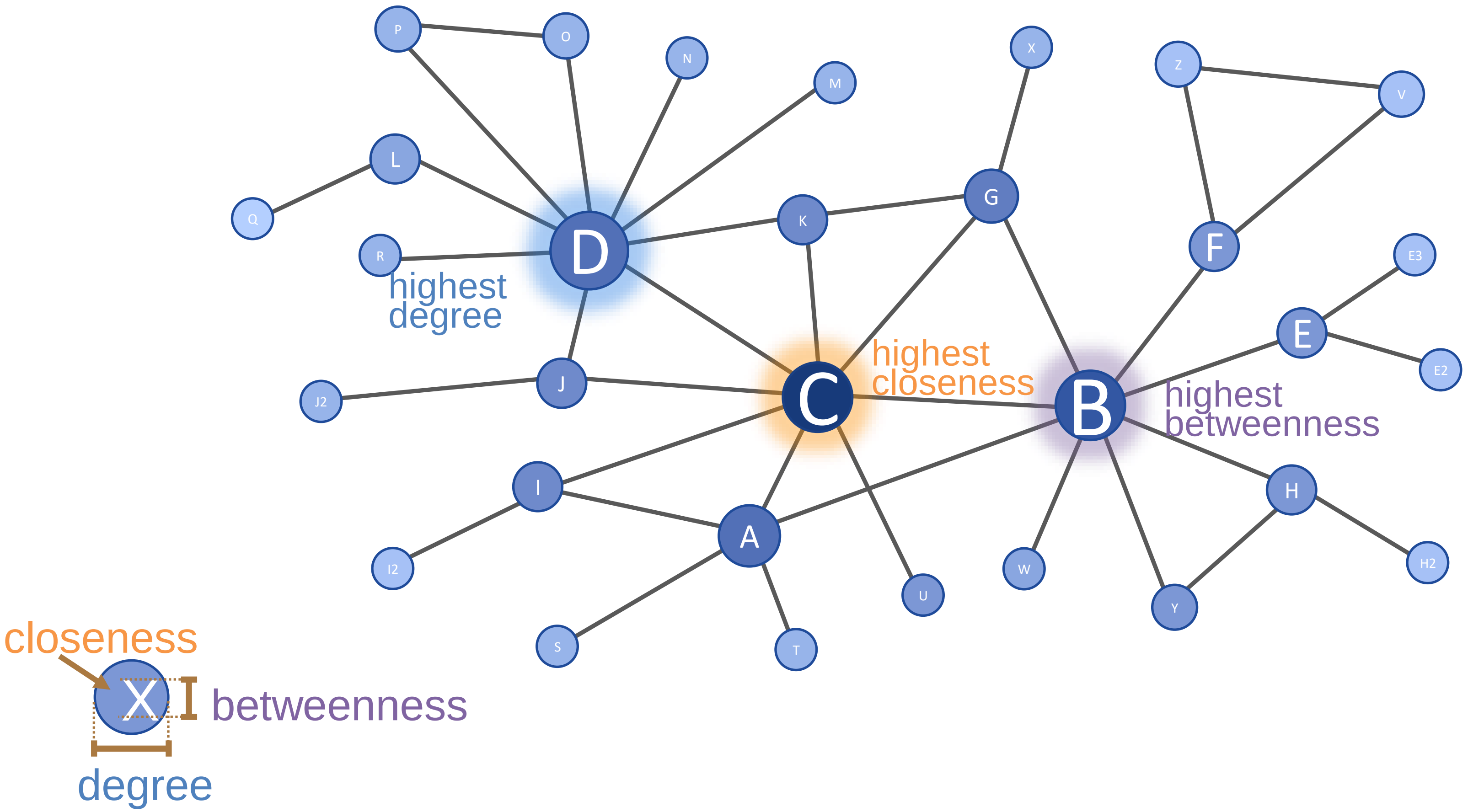
Centrality: Example



$$BC(x) := \frac{1}{\binom{n-1}{2}} \sum_{s,t \neq x} \frac{\sigma_{st}(x)}{\sigma_{st}}$$

$$BC(x) = \frac{1}{6}\left(\frac{\sigma_{ae}(x)}{\sigma_{ae}} + \frac{\sigma_{ad}(x)}{\sigma_{ad}} + \frac{\sigma_{ac}(x)}{\sigma_{ac}} + \frac{\sigma_{ed}(x)}{\sigma_{ed}} + \frac{\sigma_{ec}(x)}{\sigma_{ec}} + \frac{\sigma_{dc}(x)}{\sigma_{dc}}\right) =$$
$$\frac{1}{6}\left(\frac{1}{1} + \frac{1}{1} + \frac{2}{2} + \frac{1}{2} + \frac{0}{1} + \frac{0}{1}\right) = \frac{1}{6}\left(3\,\frac{1}{2}\right) = \frac{7}{12}$$

Which Node is the Most Important?



(Slides by Oskar Skibski)

Which Centrality Should Be Used?

Harmonic

Decay

Stress

Eccentricity

Closeness

Betweenness

k-Step Reach

Degree

Flow Betweenness

Katz

Eigenvector

Attachment

PageRank

Beta measure

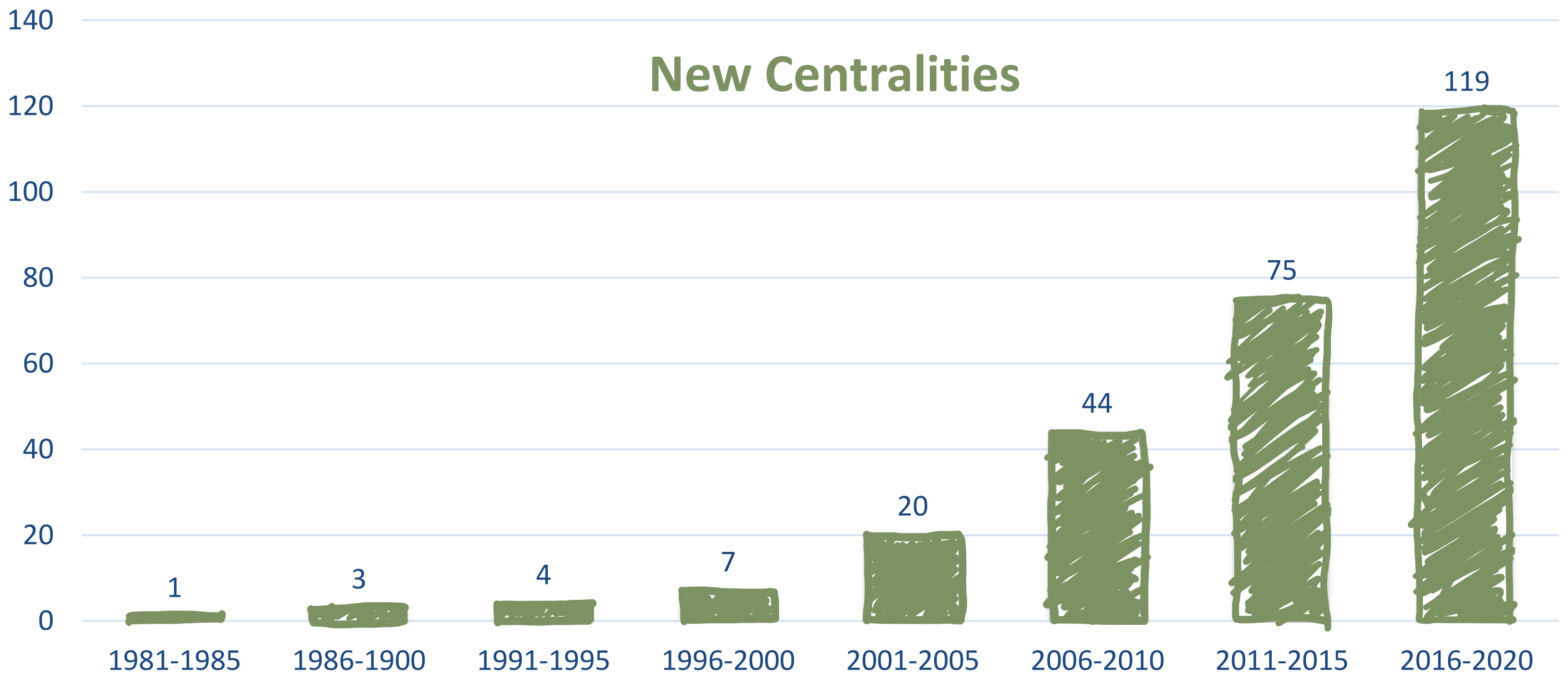
Connectivity

Bonacich

Katz prestige

(Slides by Oskar Skibski)

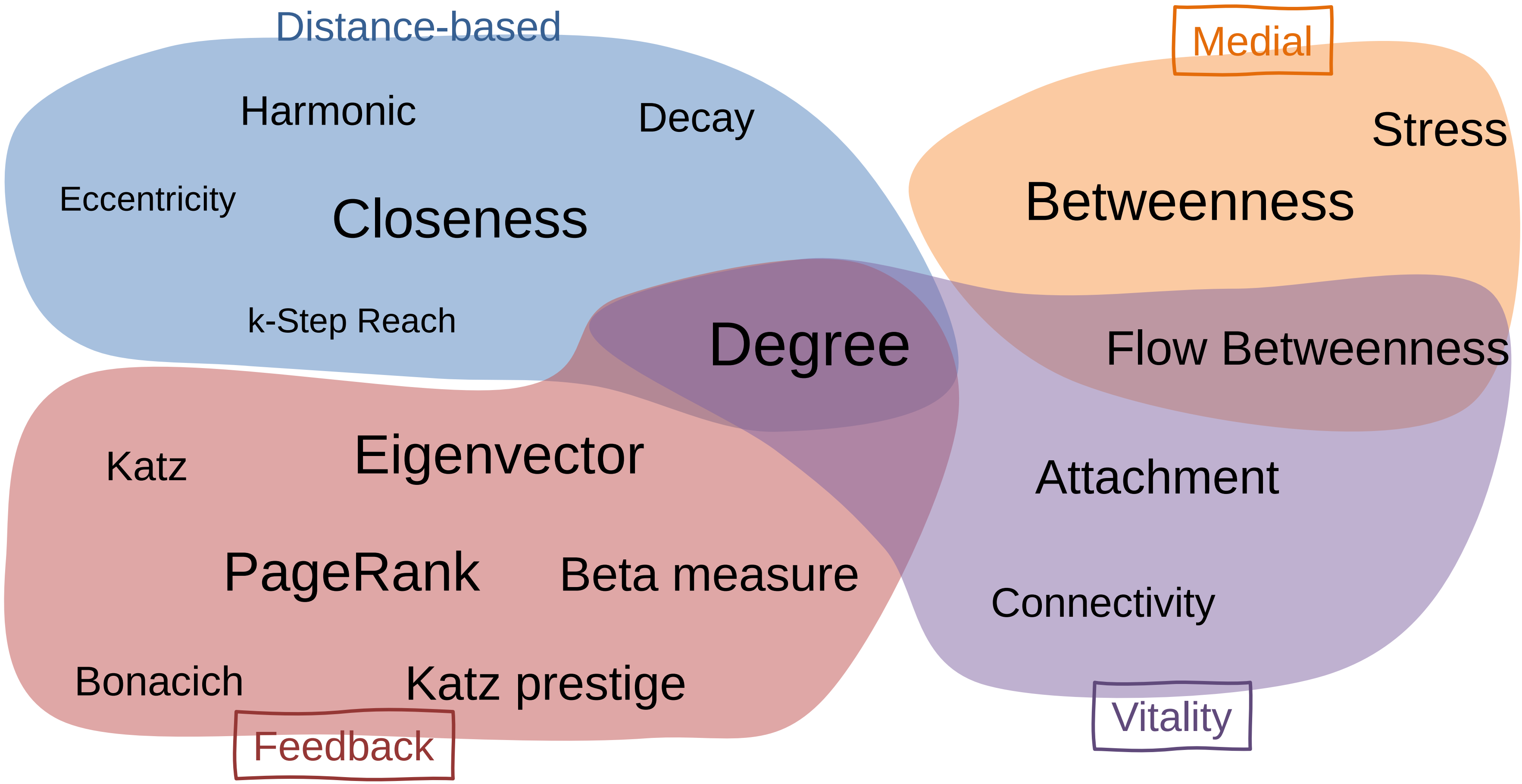
Which Centrality Should Be Used?



Source: http://www.centiserver.org/

(Slides by Oskar Skibski)

Which Centrality Should Be Used?



Demo: https://centrality.mimuw.edu.pl/centralities

(Slides by Oskar Skibski)

Centrality Axioms

- Locality: The centrality of a node will not change if other components are removed (Closeness? Betweenness?)
- Star maximization: The centrality of a node is maximized when it is a center of a star
- Monotonicity: Adding an edge does not decrease the centrality of any node
- …

Demo: https://centrality.mimuw.edu.pl/centralities

How to Use These Definitions and Models?

- Collect data on a social phenomenon
- Measure and quantify the behavior
  – E.g. count links, correlations between measures, etc.
  – Don’t invent your own measures before trying other
- Compare observation to what we would expect from a theoretical model or hypothesis
- We will see many examples

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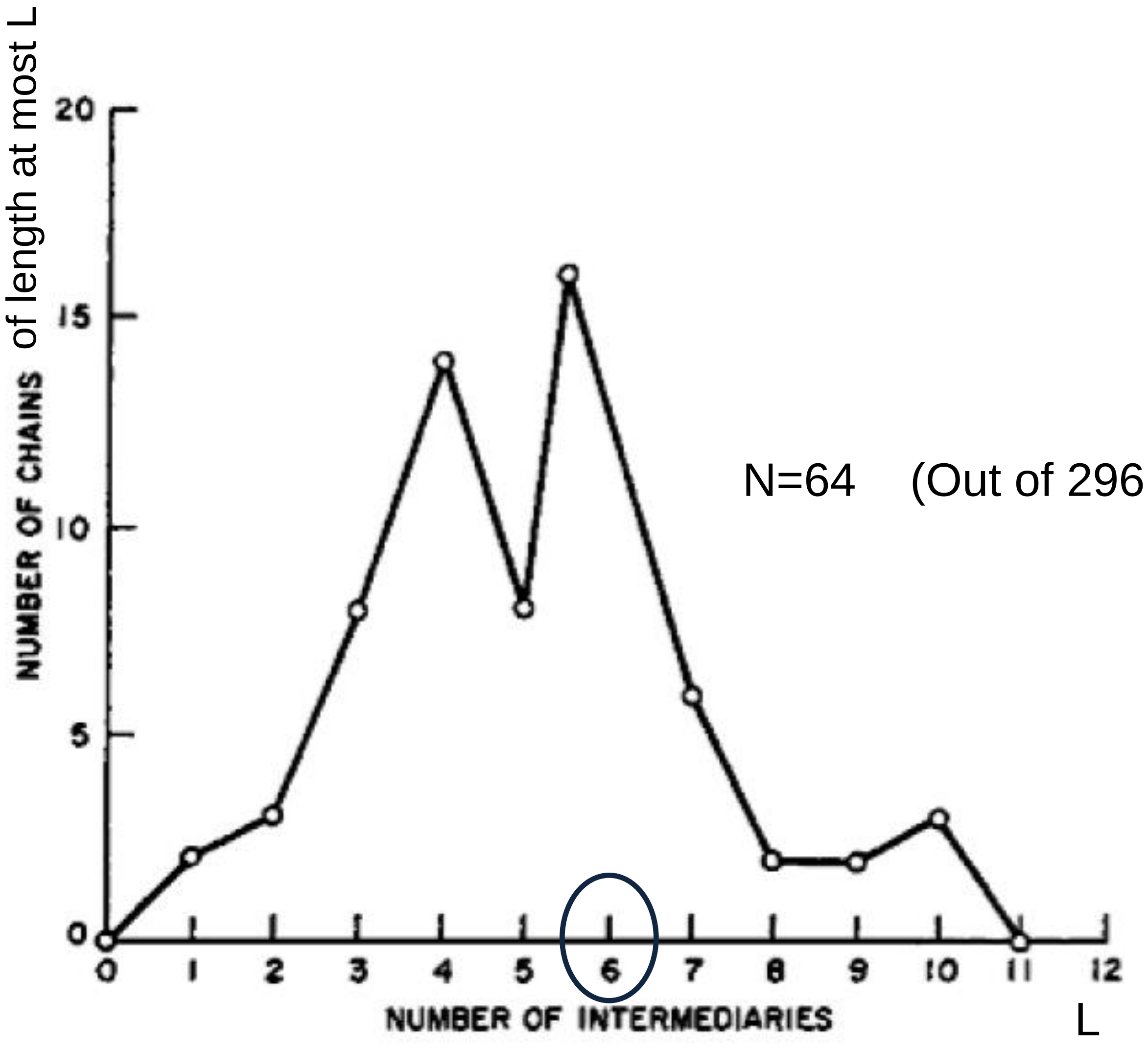
Milgram’s Small-World Experiment

- In the 1960’s, Stanley Milgram gave letter to 296 random people
- Their instructions:
  - “The letter has to get to a particular person” (a stockbroker in Boston)
  - “You may only send the letter to someone you know personally, with these instructions”

Milgram's Results

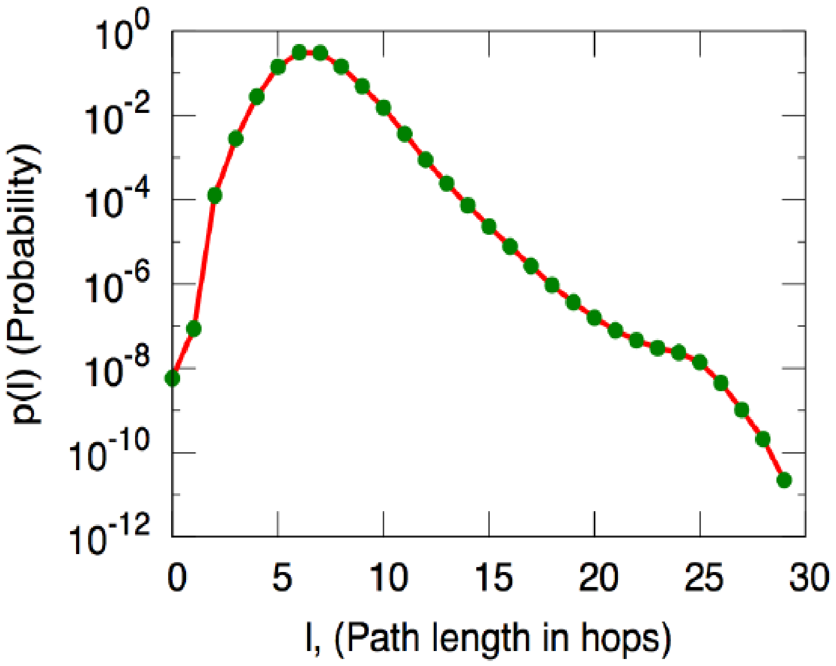


N=64 (Out of 296!)

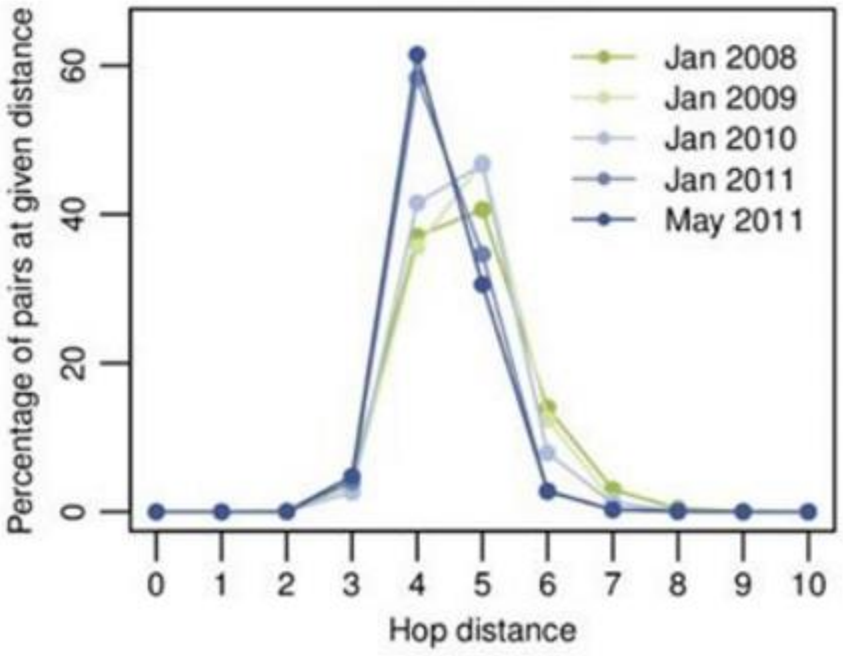
Problems?

6-Degrees of Separation?

Microsoft IM during 1 month



Facebook (in US)



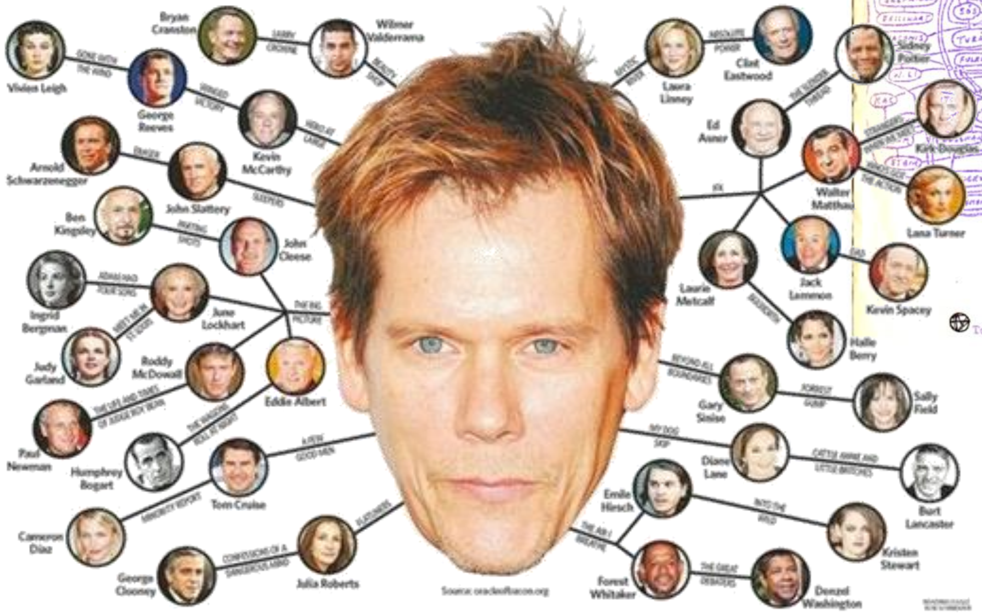
https://www.facebook.com/notes/facebook-data-science/anatomy-of-facebook/10150388519243859/

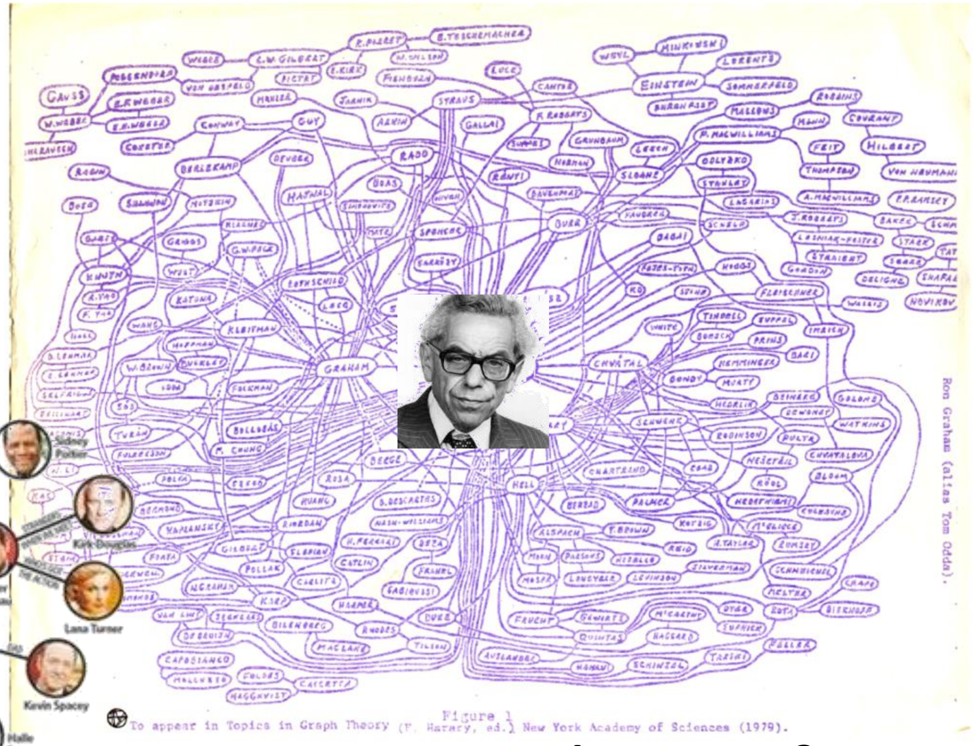
Why is the World Small?

- Main reason: “weak” links
- We will see theoretical and empirical evidence

Giant Components

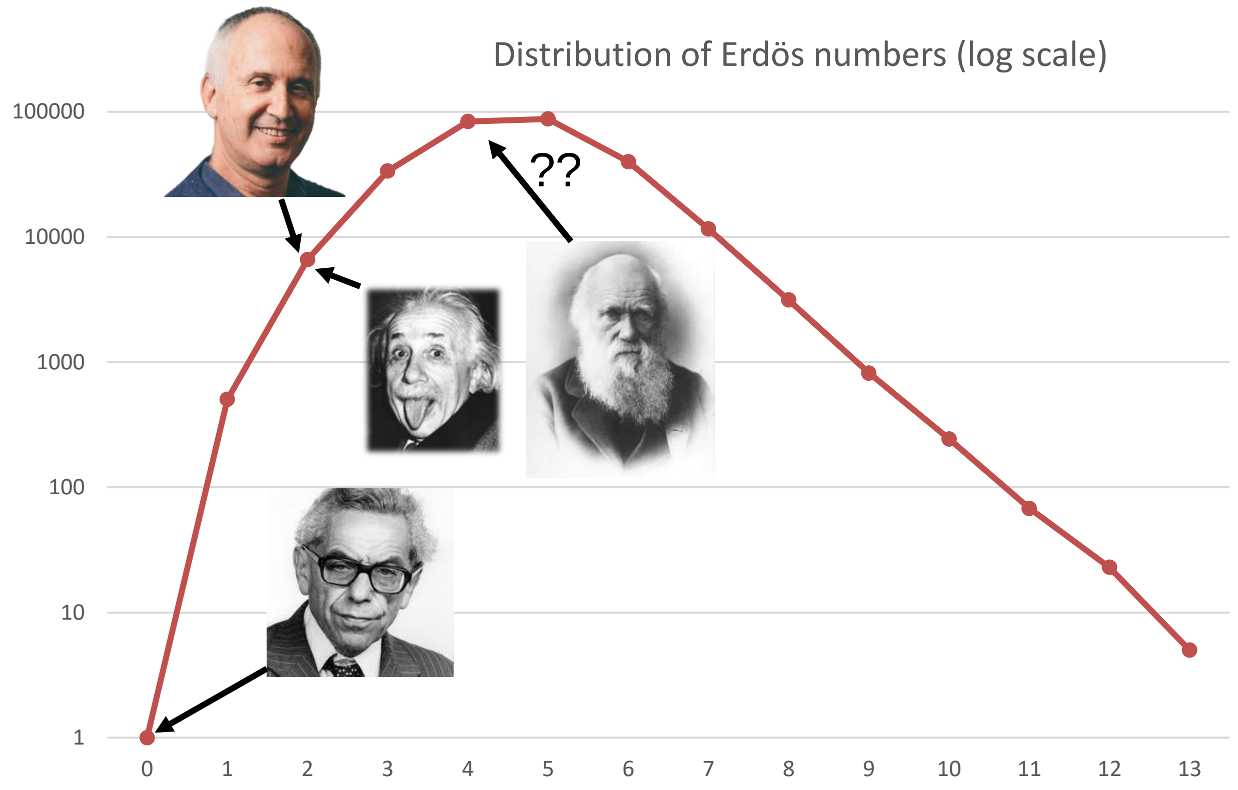
- Co-starring in movies





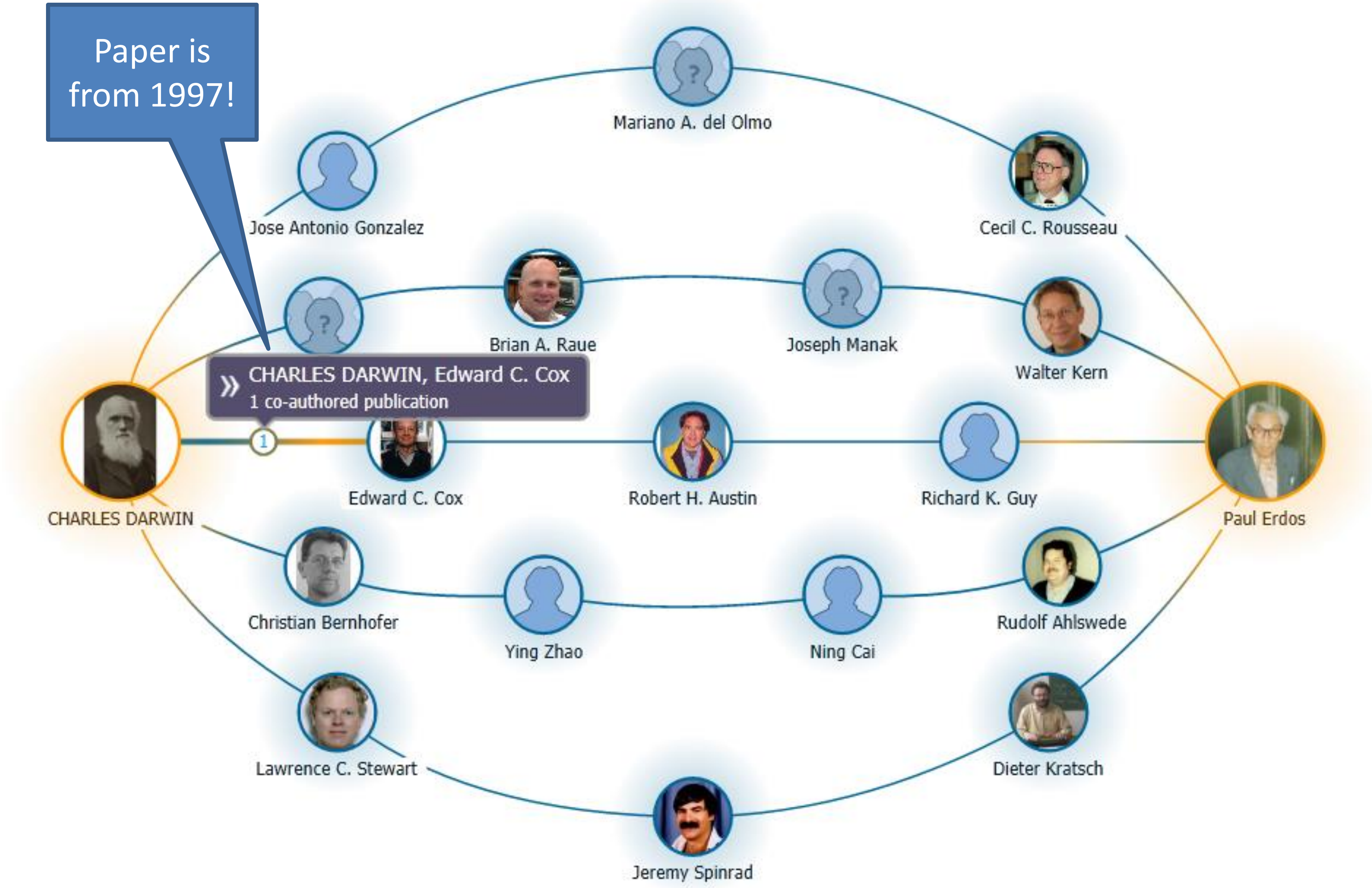
- Co-authors of math papers

Distribution of Erdös Numbers (log scale)



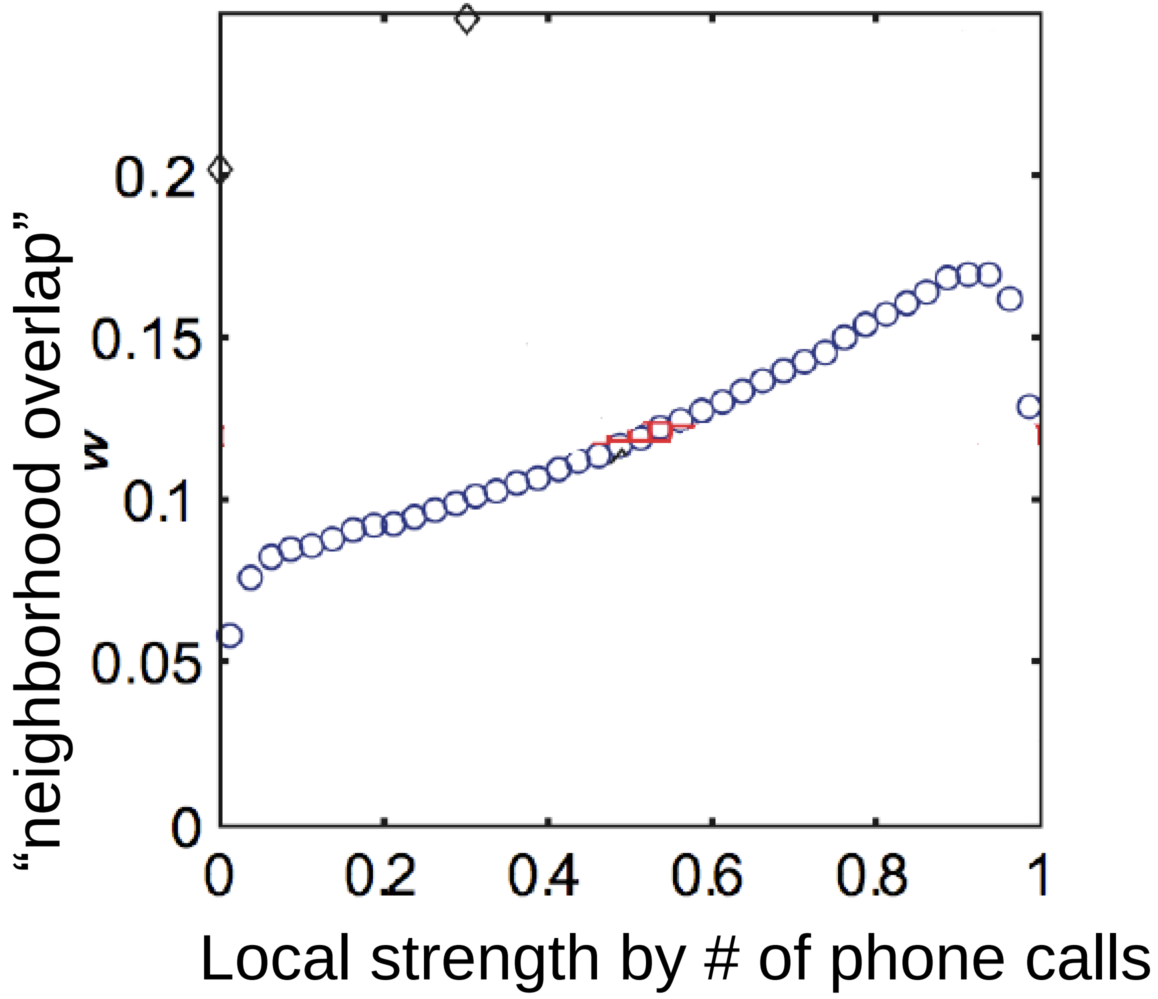
Source: https://wwwp.oakland.edu/enp/trivia/

https://www.csauthors.net/



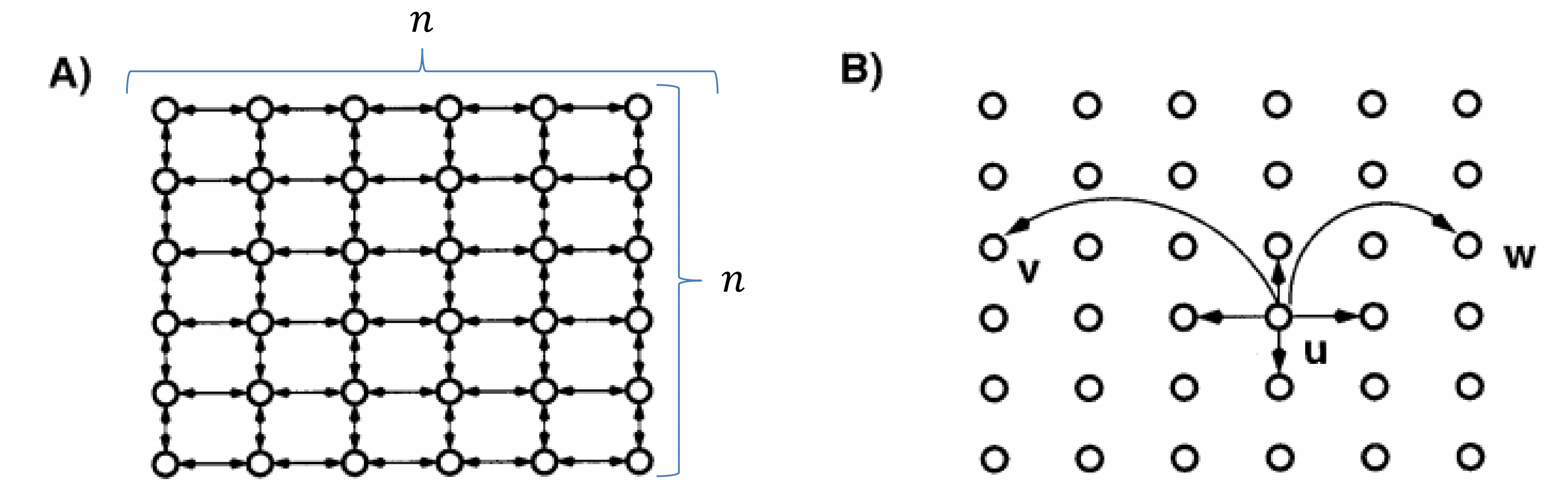
Paper is from 1997!
Mariano A. del Olmo
Jose Antonio Gonzalez
Cecil C. Rousseau
Brian A. Raue
Joseph Manak
Walter Kern
CHARLES DARWIN, Edward C. Cox
1 co-authored publication
1
CHARLES DARWIN
Edward C. Cox
Robert H. Austin
Richard K. Guy
Paul Erdos
Christian Bernhofer
Ying Zhao
Ning Cai
Rudolf Ahlswede
Lawrence C. Stewart
Dieter Kratsch
Jeremy Spinrad

JP Onnela's Data (PNAS 2007)



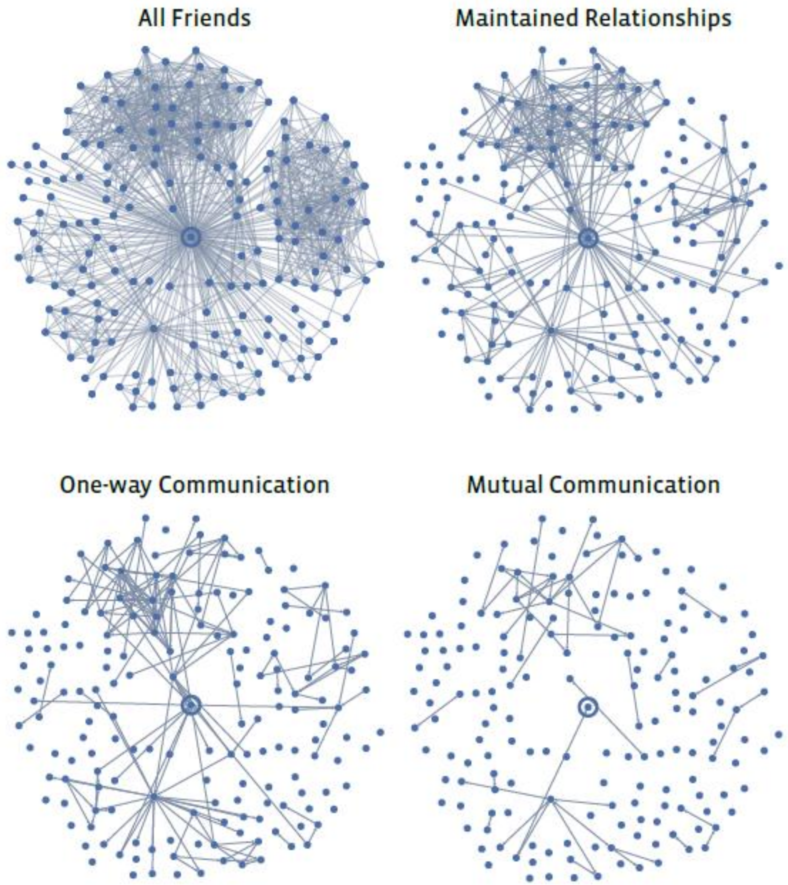
Kleinberg's "Small World" Model

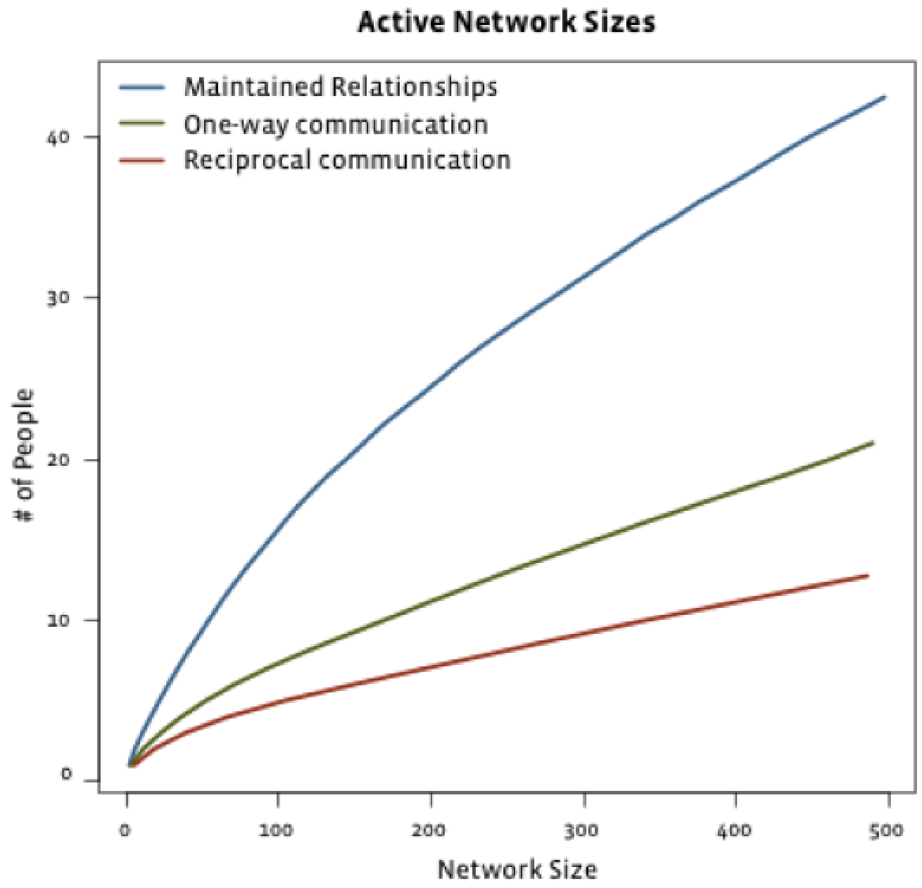
- <u>Theorem</u>: The expected distance between two nodes is $\sim\log(n)^2$



Kleinberg, Jon. "The small-world phenomenon: An algorithmic perspective." *STOC* 2000.

Facebook Links



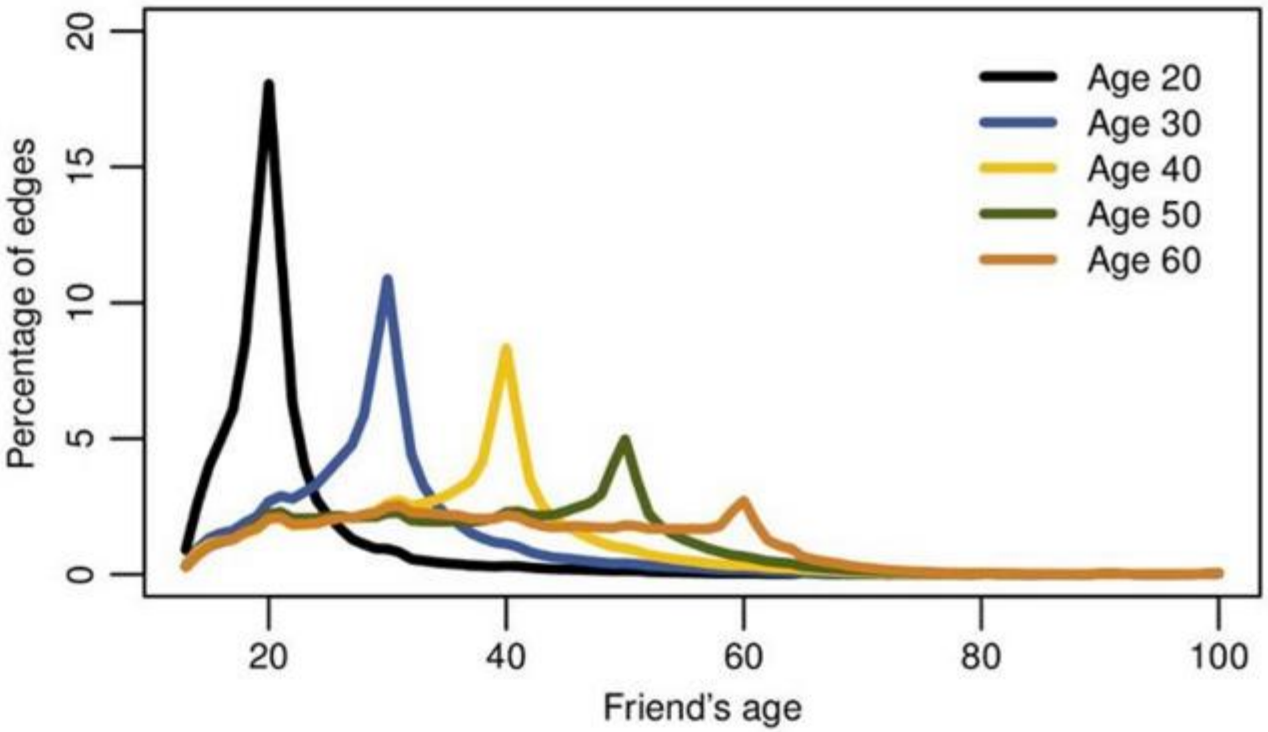


Outline

- Introduction – from graphs to networks
- Triadic closure
- Centrality measures
- The “small world” phenomenon
- Homophily
- Affiliation

What Do I Know About Your Friends?

- Their friends are your friends
- They are like you in other ways too:



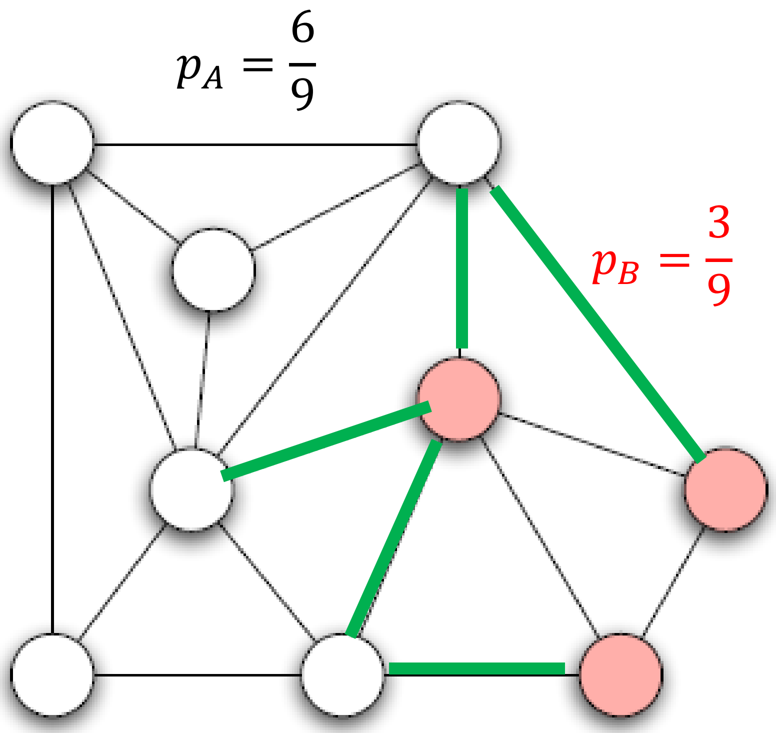
Homophily

- Let $G = (A, B, E)$ be a graph (unweighted, undirected) with two types of nodes.
- Measuring Homophily?
- $H(G) := \frac{number\ of\ A{-}B\ edges}{Exp.number\ of\ A{-}B\ edges} = \frac{|\{(a,b) \in E, a \in A, b \in B\}|}{pr(e\ is\ A{-}B)|E|} =$
$\frac{|\{(a,b) \in E, a \in A, b \in B\}|}{\frac{2|A||B|}{|A \cup B|^2}|E|} = \frac{|\{(a,b) \in E, a \in A, b \in B\}|}{2 \cdot p_A \cdot p_B \cdot |E|}$
- *(expected=random assignment of types, fixed graph)*

Measuring Homophily

$$H(G) = \frac{|\{(a,b) \in E, a \in A, b \in B\}|}{2 \cdot p_A \cdot p_B \cdot |E|}$$

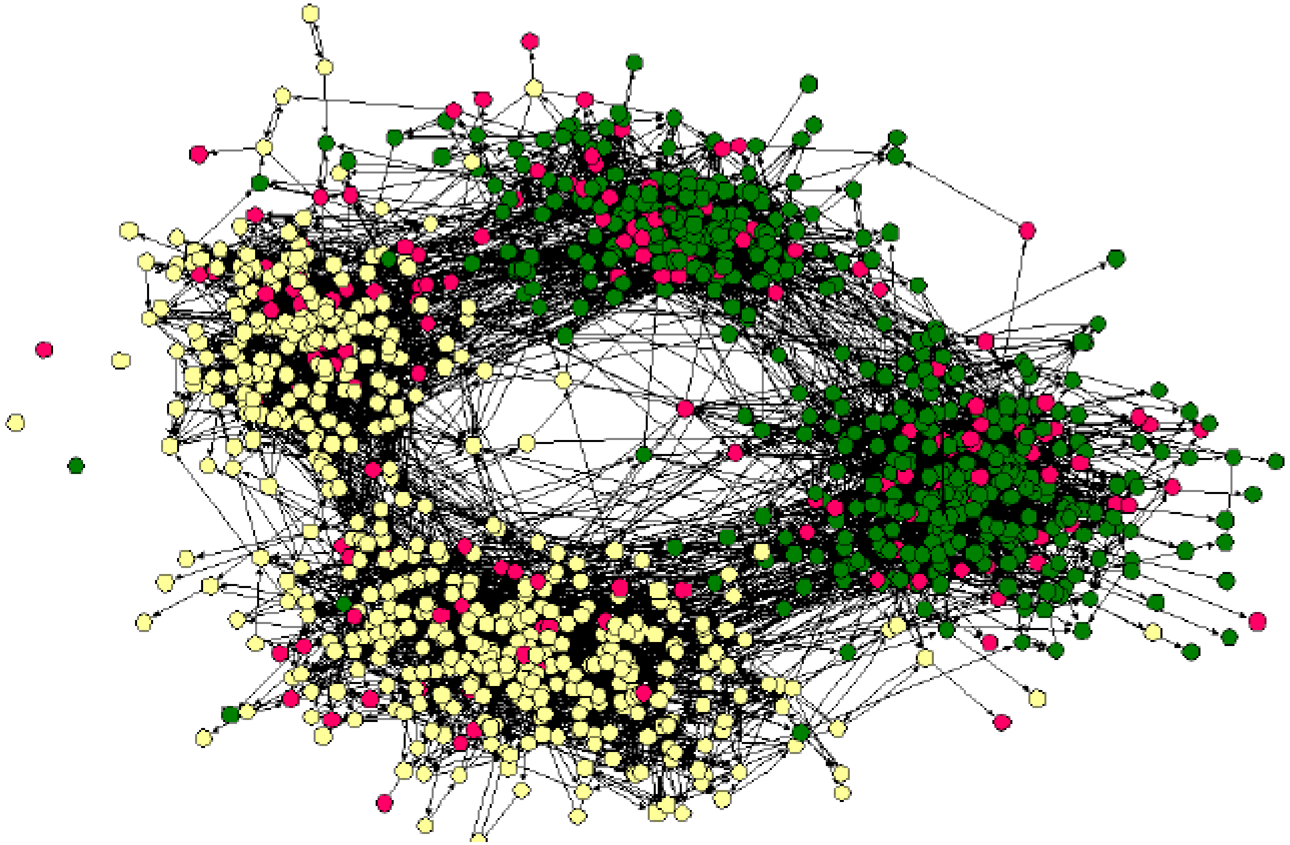
$$= \frac{5}{2 \cdot \frac{6}{9} \cdot \frac{3}{9} \cdot 18} = \frac{5}{8}$$

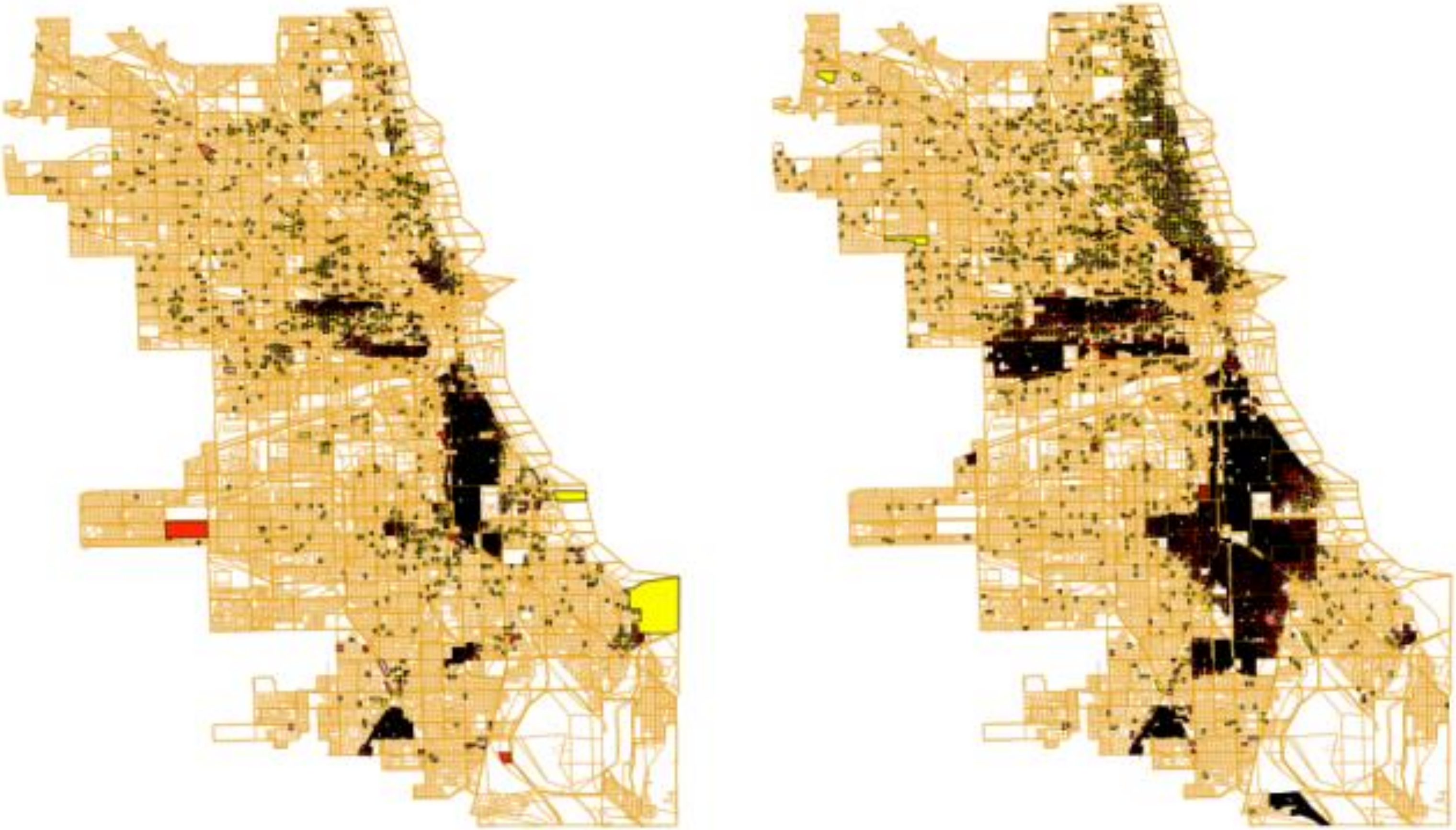


$H(G) < 1$ means there is homophily in G

$H(G) > 1$ means there is anti-homophily in G

Double Homophily

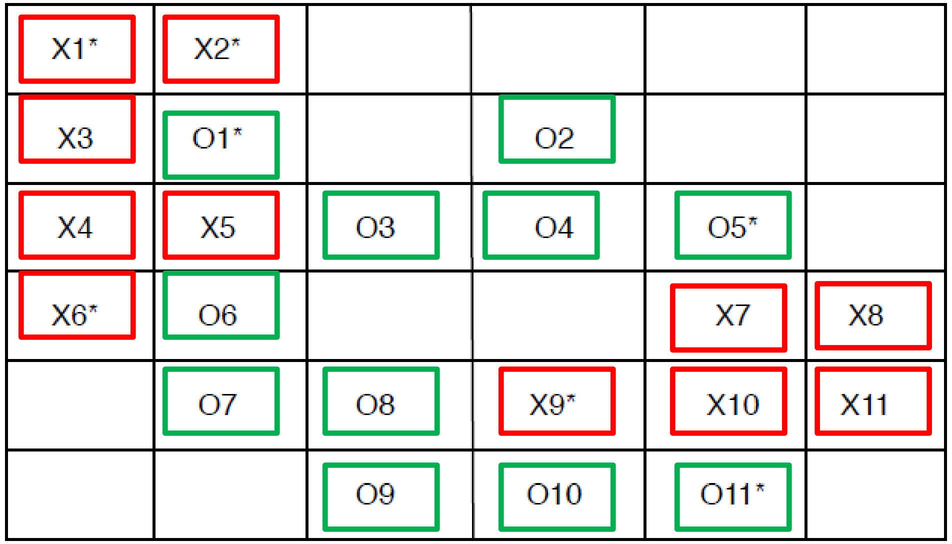




(a) *Chicago, 1940*

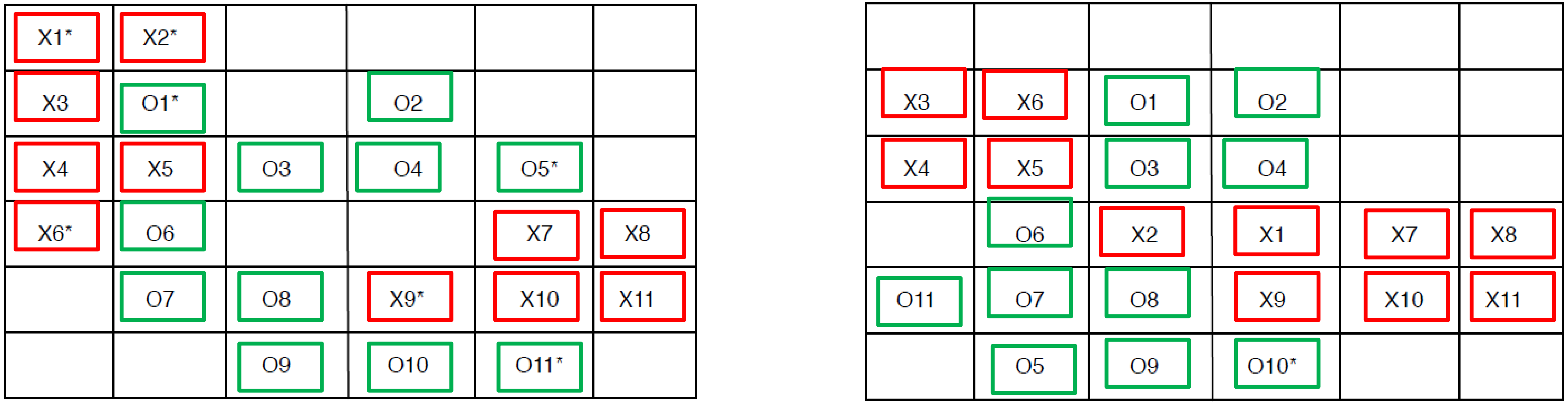
(b) *Chicago, 1960*

The Schelling Segregation and Homophily Model ($t = 3$)



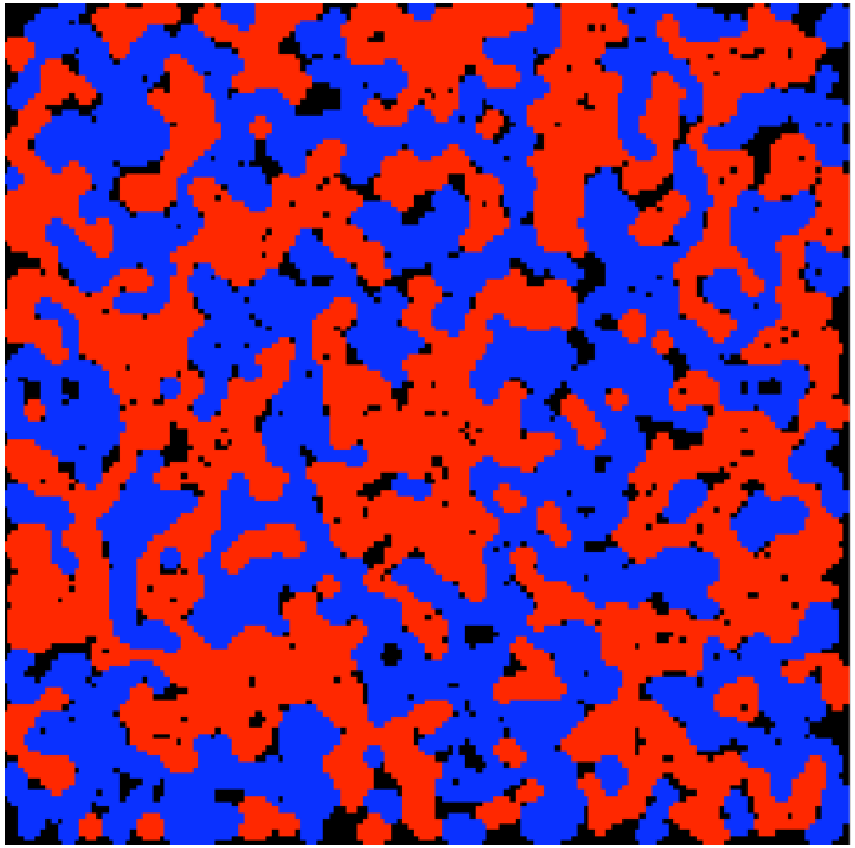
(a) *An initial configuration.*

The Schelling Segregation and Homophily Model ($t = 3$)

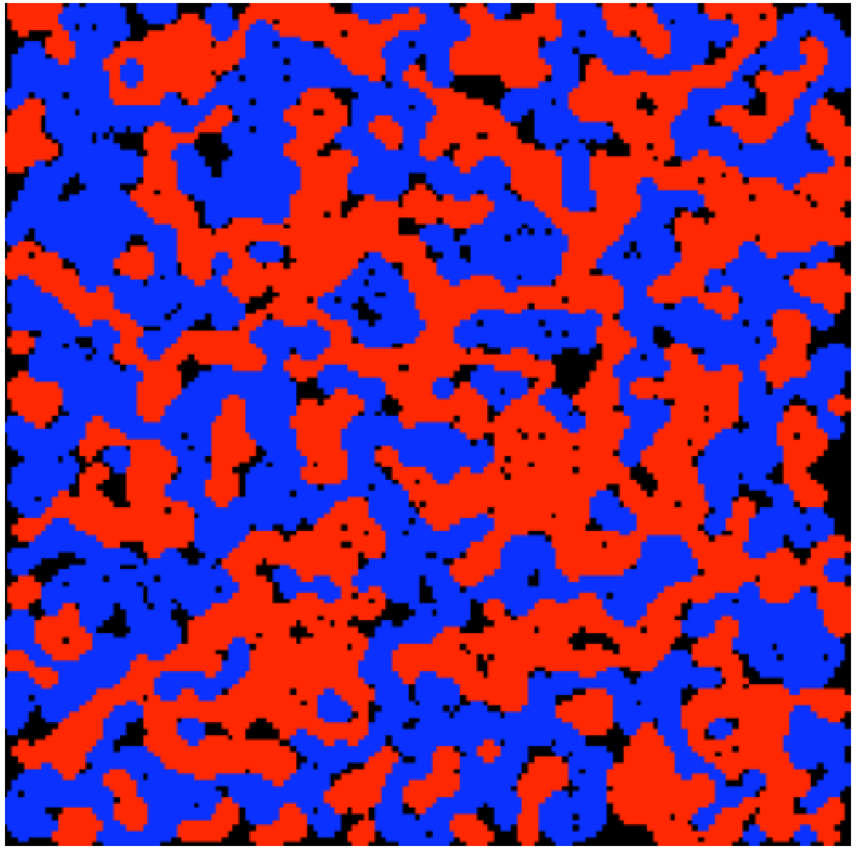


(a) *An initial configuration.* (b) *After one round of movement.*

The Schelling Segregation ($t = 3$)



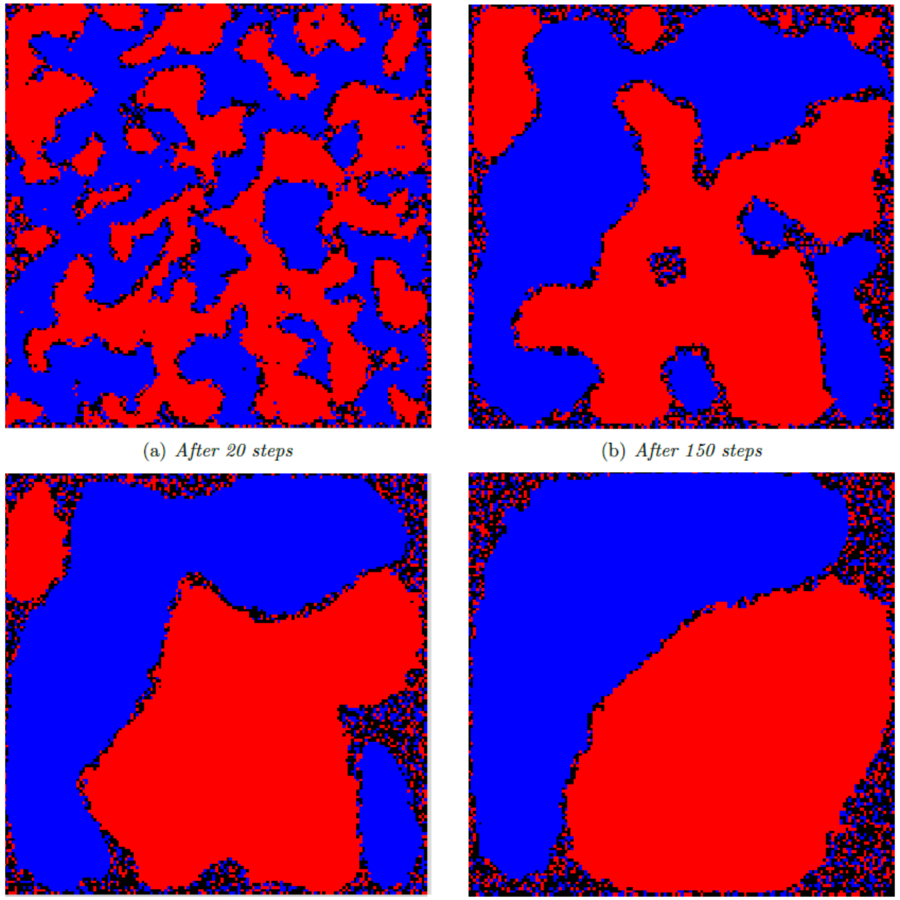
(a) *A simulation with threshold 3.*



(b) *Another simulation with threshold 3.*

A 150-by-150 grid with 10,000 agents of each type

The Schelling Segregation ($t = 4$)



(a) *After 20 steps* (b) *After 150 steps*

(c) *After 350 steps* (d) *After 800 steps*

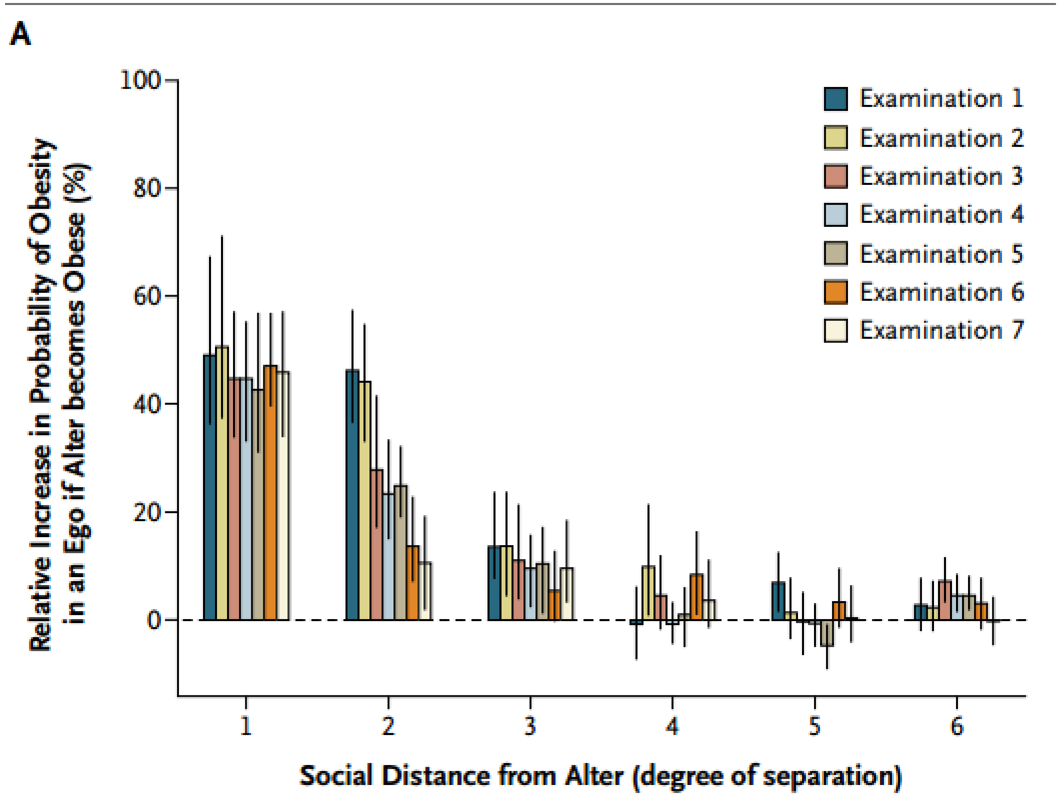
A 150-by-150 grid with 10,000 agents of each type

Chicken and Egg

- Do similar people become friends?
- Or do friends become similar?

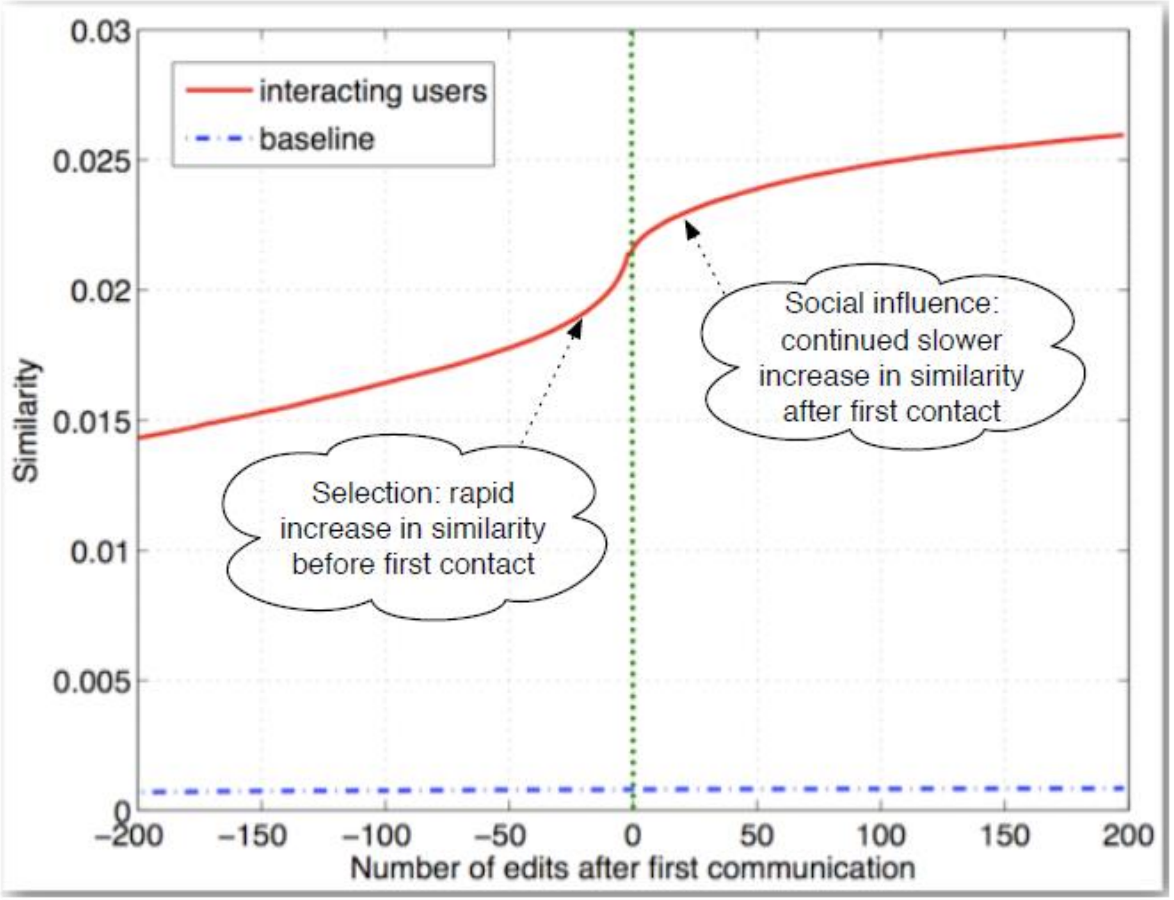
- In some cases, both.

Obesity and Social Distance



Christakis, Nicholas A., and James H. Fowler. "The spread of obesity in a large social network over 32 years." *New England journal of medicine* 357.4 (2007): 370-379.

Homophily and Social Influence in Wikipedia Edits



Outline

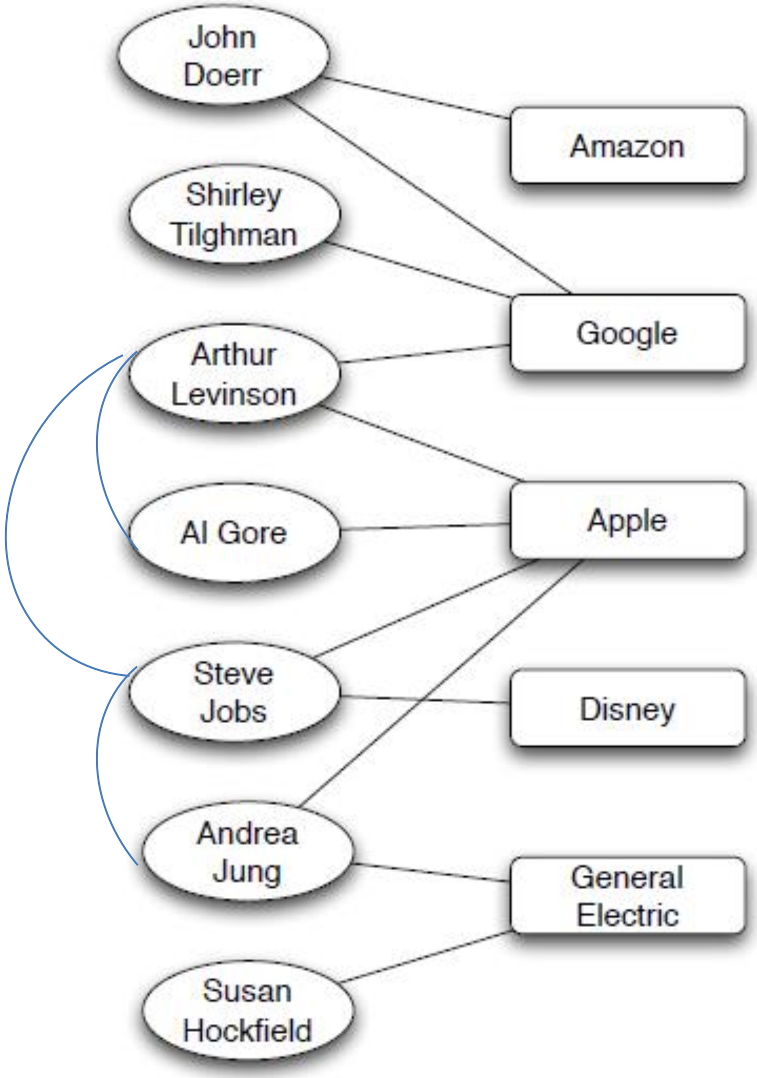
- Introduction – from graphs to networks
- Triadic closure
- Centrality measures
- The “small world” phenomenon
- Homophily
- Affiliation

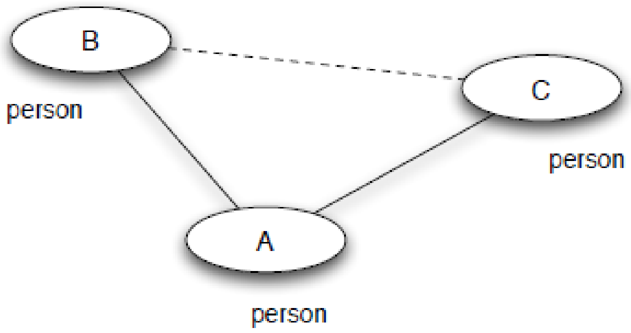
Affiliation

- Until now, the “context” was unknown / not modeled
- If the context affects the social connections, let’s include it in the network

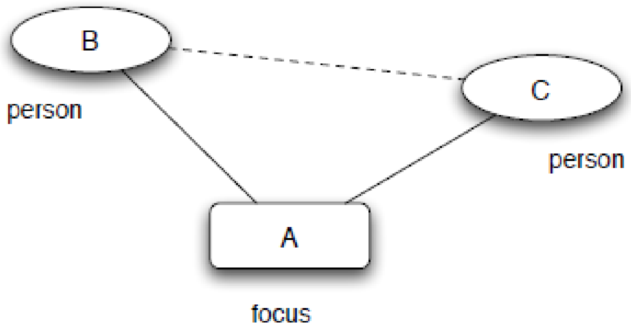
- Blue edges = social links
- Black edges = “membership” links

- The square nodes can be a concrete group (like a company or school) or any property or habit (“plays WoW”, “has HIV”, “Asian”, etc.)

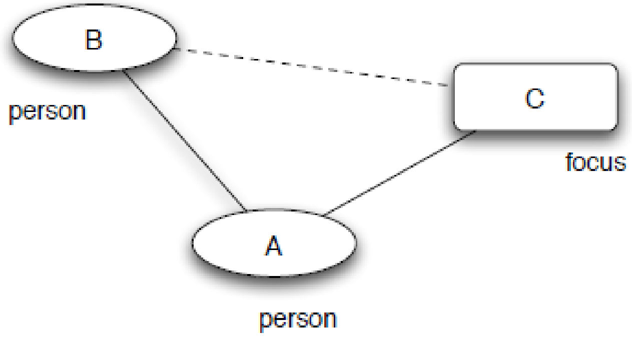




(a) *Triadic closure*



(b) *Focal closure*



(c) *Membership closure*

Electronic Commerce 096211

Diffusion / contagion in networks

Omer Ben Porat (some of the slides are adopted from Prof. Reshef Meir)

Course Structure

Online Societies
- Web ranking
- Diffusion
- Structure of the web
- Power laws
- Networks

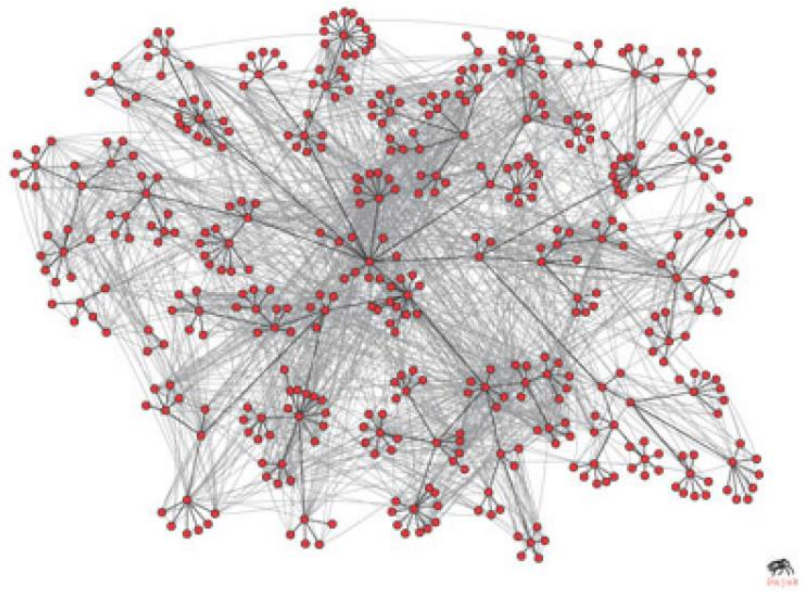
Advertising
- Auctions
- Matching markets
- Sponsored search

Decision Making
- Recommendation systems
- Multi-armed bandits

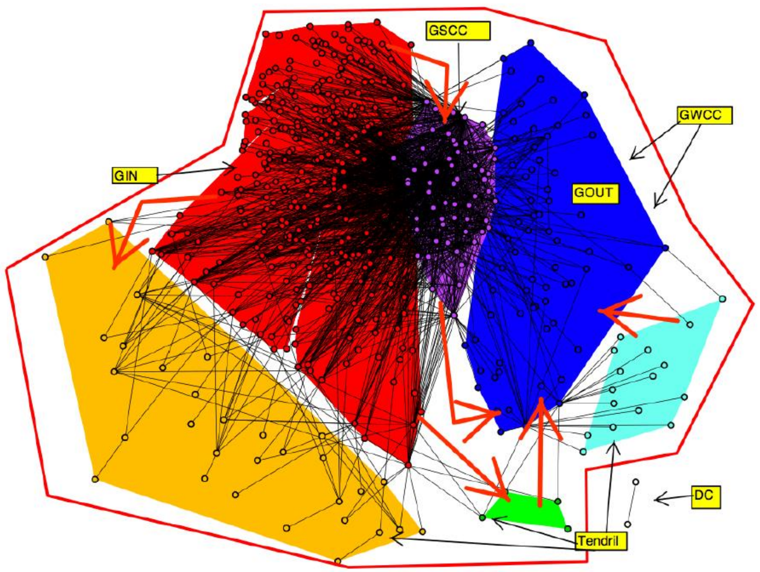
Advance
- Information in markets
- Externalities
- Streaming
- Blockchain

Tools and Techniques
- Game theory
- Algorithms
- Graph theory
- Optimization

Recall: From Networks to Graphs



Emails inside HP



Loans among banks

This class

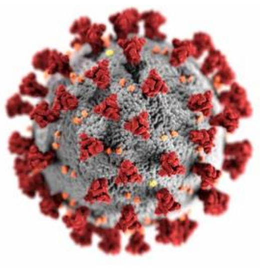
- SIR (No networks)
- Diffusion Models
  - The Linear Threshold model
  - The Independent Cascade model
- Influence Maximization - proofs

This class

- SIR (No networks)
- Diffusion Models
  - The Linear Threshold model
  - The Independent Cascade model
- Influence Maximization - proofs

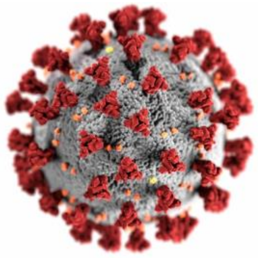
Susceptible Infected Removed/Recovered

$$Time = 0$$



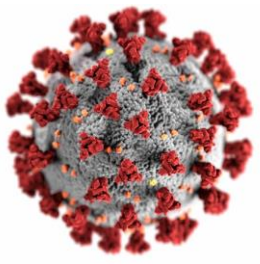
Susceptible | Infected | Removed/Recovered

$$Time = 1$$



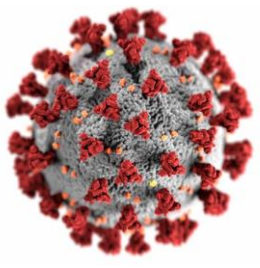
Susceptible **Infected** **Removed/Recovered**

$Time = 1$



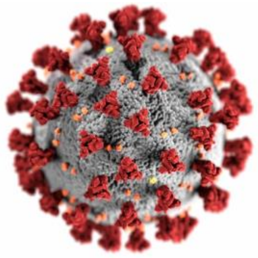
Susceptible Infected Removed/Recovered

$Time = 1$

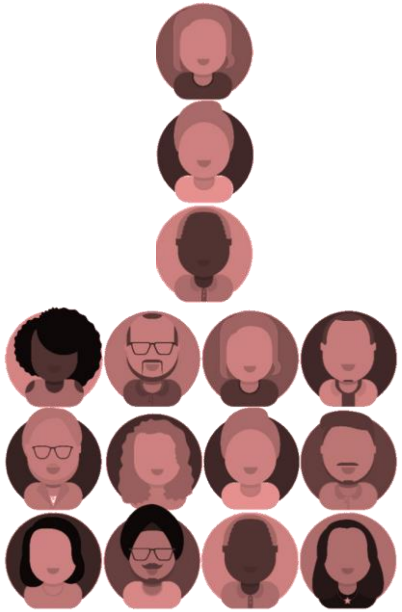


Susceptible **Infected** **Removed/Recovered**

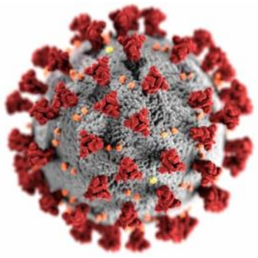
$Time = 2$



Susceptible **Infected** **Removed/Recovered**

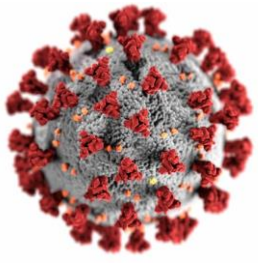


$Time = 2$



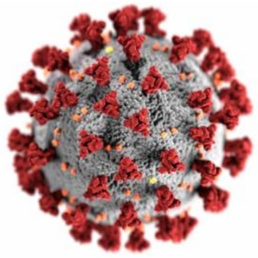
Susceptible **Infected** **Removed/Recovered**

$Time = 2$



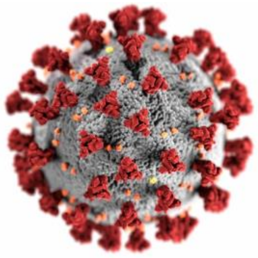
Susceptible Infected Removed/Recovered

$$Time = 3$$



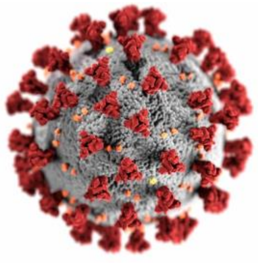
Susceptible **Infected** **Removed/Recovered**

$Time = 3$



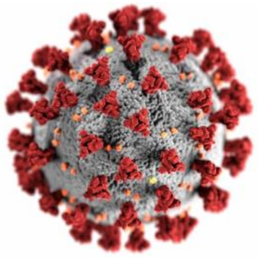
Susceptible | Infected | Removed/Recovered

$$Time = 3$$



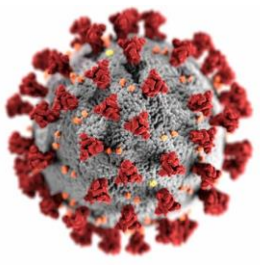
Susceptible Infected Removed/Recovered

$Time = 4$



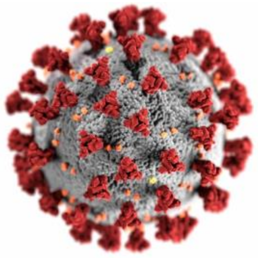
Susceptible Infected Removed/Recovered

$$Time = 4$$



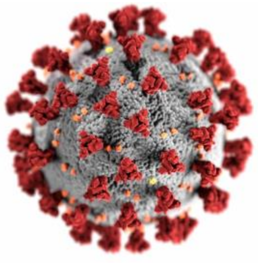
Susceptible | Infected | Removed/Recovered

$Time = 5$



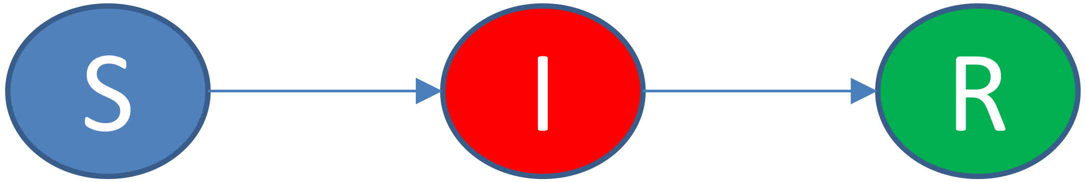
Susceptible **Infected** **Removed/Recovered**

$Time = \infty$



The SIR model

- Every person can be in one of three states:
  - Susceptible - healthy, may contract the disease
  - Infected - may infect others
  - Removed/Recovered - dead or immune



- No network structure! (equivalently, a complete graph)
- Extensions: SEIR, SIS

SIR Example

Every day:

$$S(t+1) = S(t) + \Delta S(t)$$
$$I(t+1) = I(t) + \Delta I(t)$$
$$R(t+1) = R(t) + \Delta R(t)$$

Some susceptible people are infected:

$$\Delta S(t) = -\beta \cdot I(t) \cdot S(t)$$

Some infected people recover:

$$\Delta R(t) = \gamma \cdot I(t)$$
$$\Delta I(t) = -\Delta S(t) - \Delta R(t) = \beta \cdot I(t) \cdot S(t) - \gamma \cdot I(t)$$

Model parameters:

- β: effective contact rate
- γ: rate of removal

SIR Example

Every day:

$$S(t+1) = S(t) + \Delta S(t)$$
$$I(t+1) = I(t) + \Delta I(t)$$
$$R(t+1) = R(t) + \Delta R(t)$$

Some susceptible people are infected:

$$\Delta S(t) = -\boldsymbol{\beta} \cdot I(t) \cdot S(t)$$

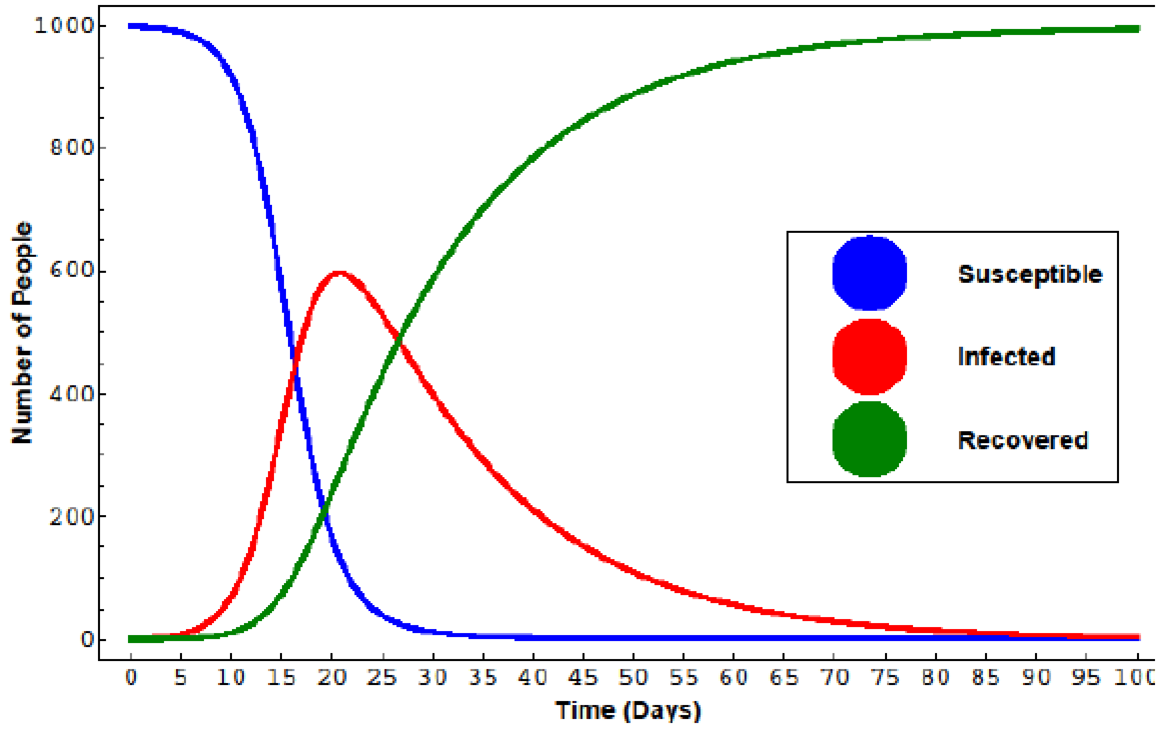
Some infected people recover:

$$\Delta R(t) = \boldsymbol{\gamma} \cdot I(t)$$
$$\Delta I(t) = -\Delta S(t) - \Delta R(t) = \boldsymbol{\beta} \cdot I(t) \cdot S(t) - \boldsymbol{\gamma} \cdot I(t)$$

Model parameters:

- $\boldsymbol{\beta}$: effective contact rate
- $\boldsymbol{\gamma}$: rate of removal

Predicting COVID-19 spread



Epidemic

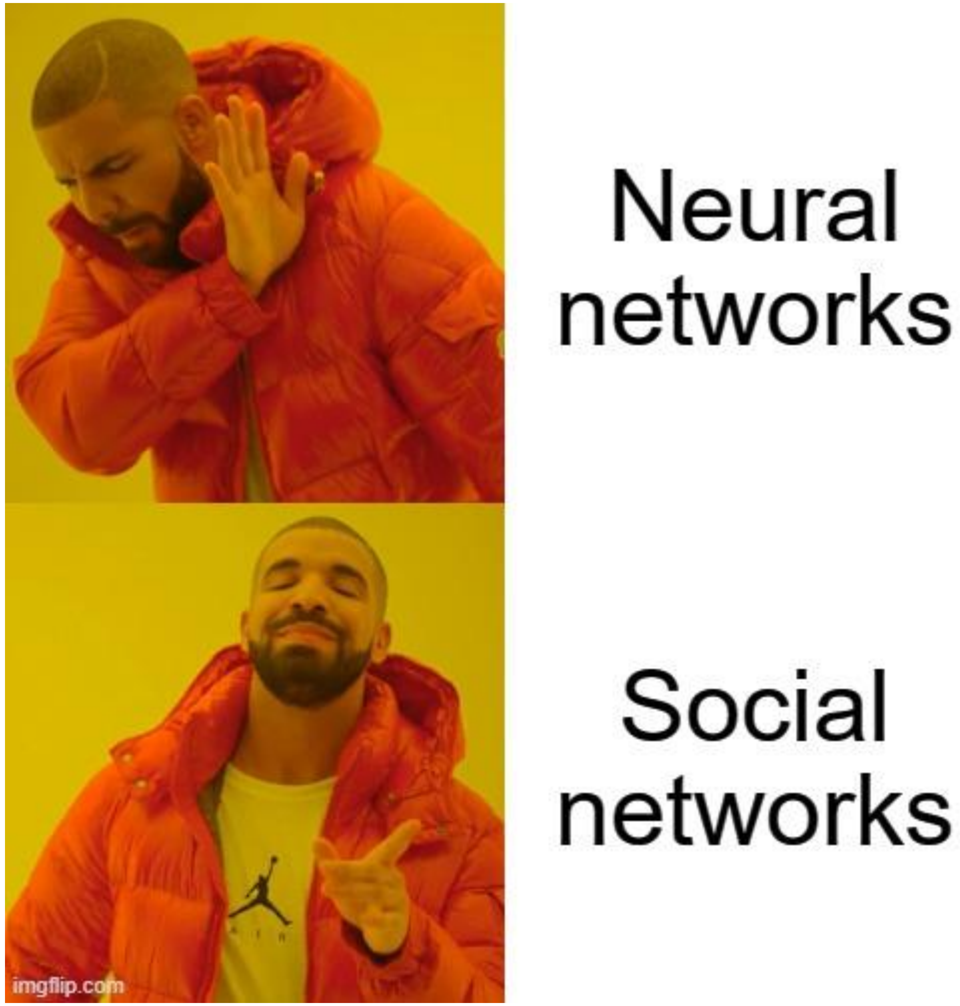
- Conditions for epidemic?
- Recall that $\Delta I(t) = \beta \cdot I(t) \cdot S(t) - \gamma \cdot I(t)$
- When t is small, $S(t) \approx 1$
- Thus, the epidemic cannot happen if $\Delta I(t) < 0$, or

$$\Delta I(t) \approx \beta \cdot I(t) - \gamma \cdot I(t) < 0 \Leftrightarrow \frac{\beta}{\gamma} \equiv R_0 < 1$$

  – 1 is the *epidemic threshold*

- In the graph extension of SIR, the *epidemic threshold* is $\frac{1}{\lambda_{1,A}}$
  – $\lambda_{1,A}$ is the spectral radius of the graph=dominant eigen value
  – (No proof here)

Predicting COVID-19 spread



https://www.zdnet.com/article/google-deepminds-effort-on-covid-19-coronavirus-rests-on-the-shoulders-of-giants/

https://www.nytimes.com/2020/03/13/science/coronavirus-social-networks-data.html

This class

- SIR (No networks)
- Diffusion Models
  - The Linear Threshold model
  - The Independent Cascade model
- Influence Maximization - proofs

Diffusion

- We will discuss the model of Kempe, Kleinberg and Tardos [2003] (link)
- Model 1: Linear threshold model
- Model 2: Independent cascade model
- Extensions: incomplete information, learning to influence, injecting features,…
  - See more details in Chapter 19 in the book

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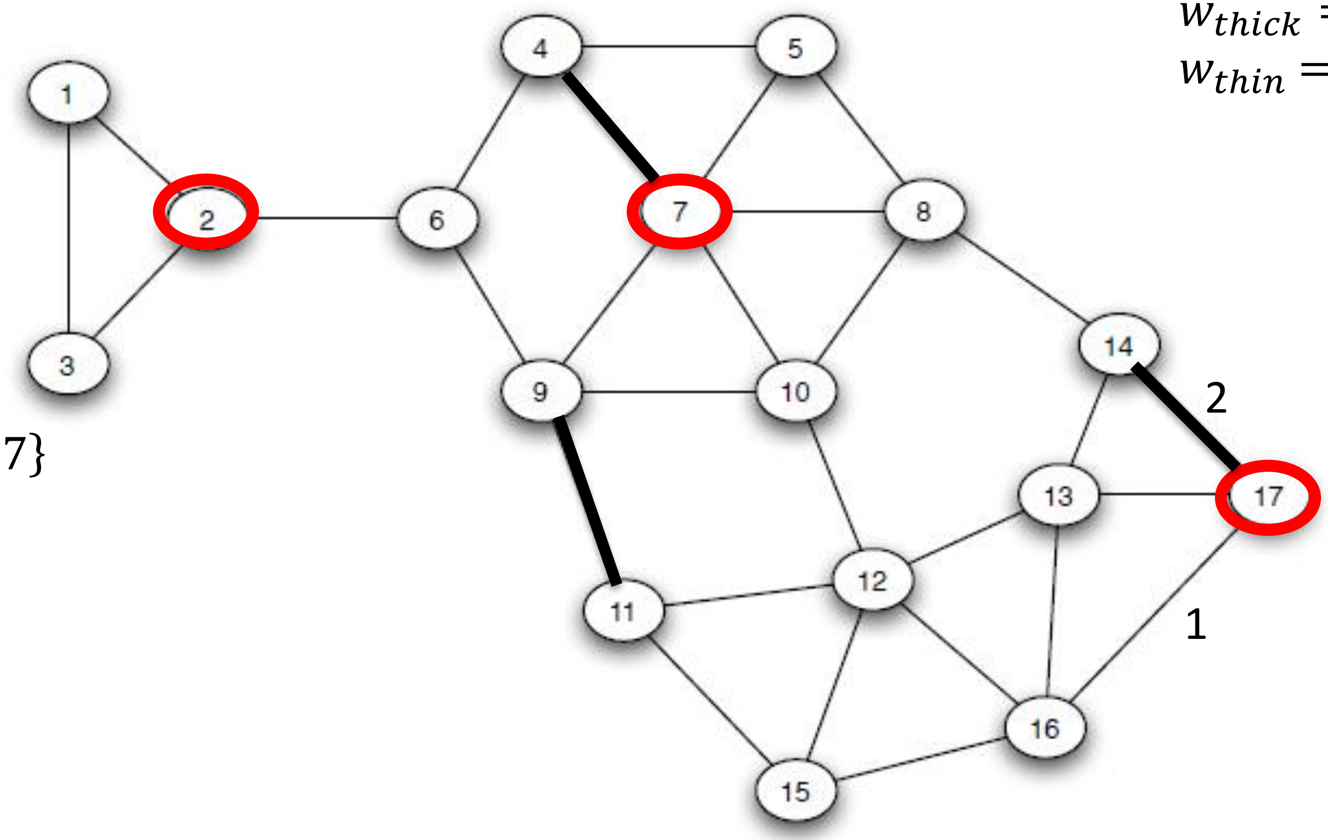
Model 1: Linear threshold model

- Each edge has a weight $w_{u,v}$
- Each node has a threshold θ_v
- Initially, some set of "infected" nodes $\mathrm{I}^0 \subset V$
- In every step $t = 1,2, \ldots$
  - Each node watches all her infected neighbors
  - If $\sum_{u \in I^{t-1}} w_{u,v} \geq \theta_v$, then v becomes active: $v \in I^t$
  - All active nodes remain infected <u>forever</u>

Linear threshold model: example

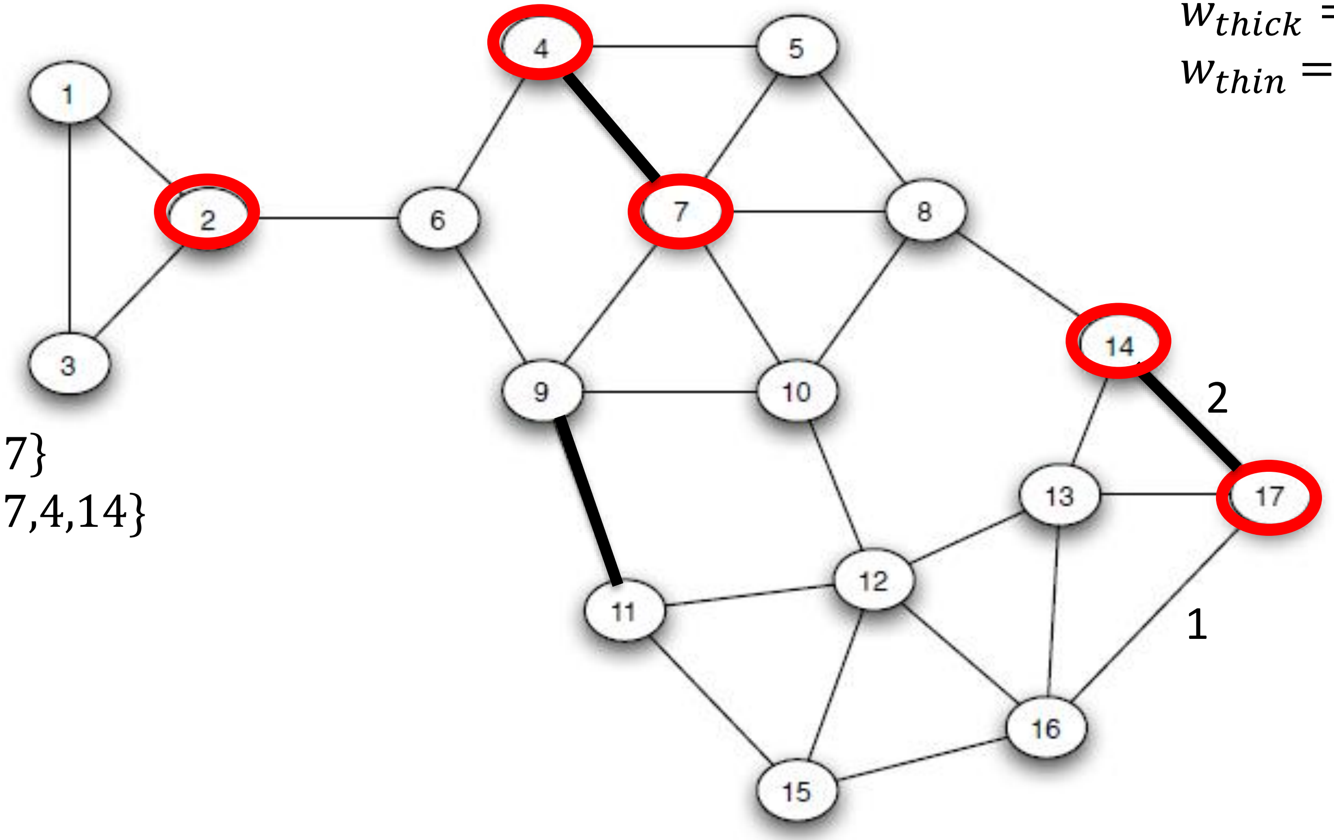
$\theta_v = 2$ for all v
$w_{thick} = 2$
$w_{thin} = 1$

$I^0 = \{2,7,17\}$



Linear threshold model: example

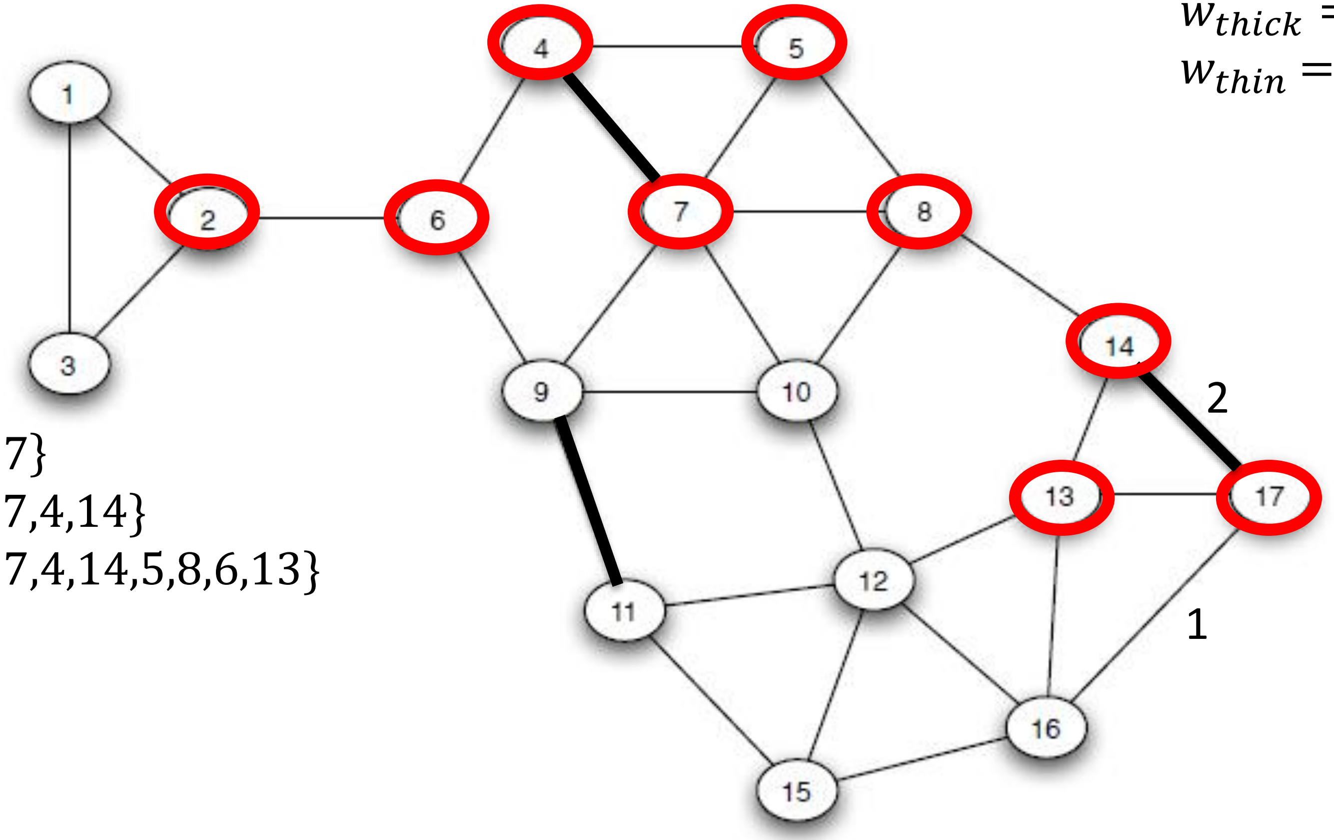
$\theta_v \;= 2$ for all v
$w_{thick} = 2$
$w_{thin} = 1$



$I^0 = \{2,7,17\}$
$I^1 = \{2,7,17,4,14\}$

Linear threshold model: example

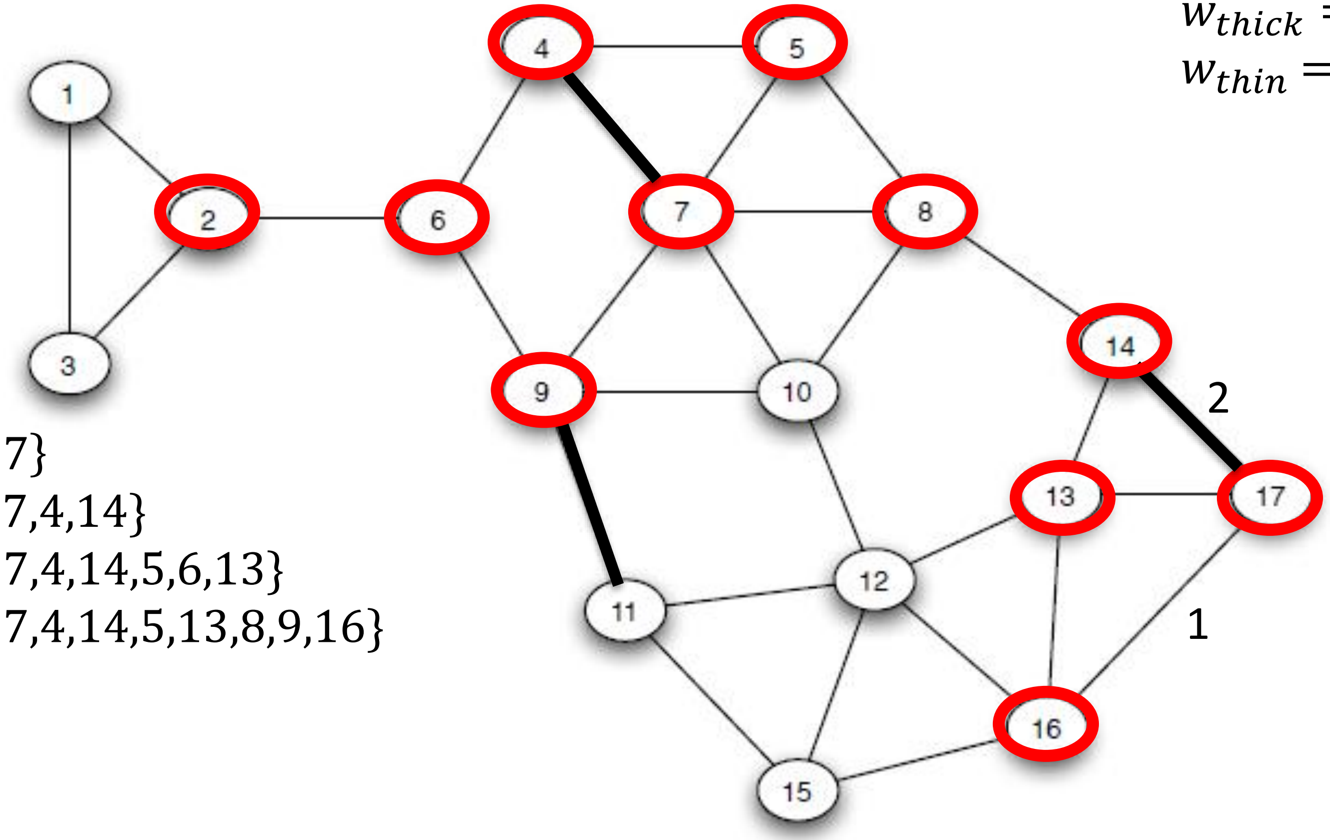
$\theta_v \ = 2$ for all v
$w_{thick} = 2$
$w_{thin} = 1$



$I^0 = \{2,7,17\}$
$I^1 = \{2,7,17,4,14\}$
$I^2 = \{2,7,17,4,14,5,8,6,13\}$

Linear threshold model: example

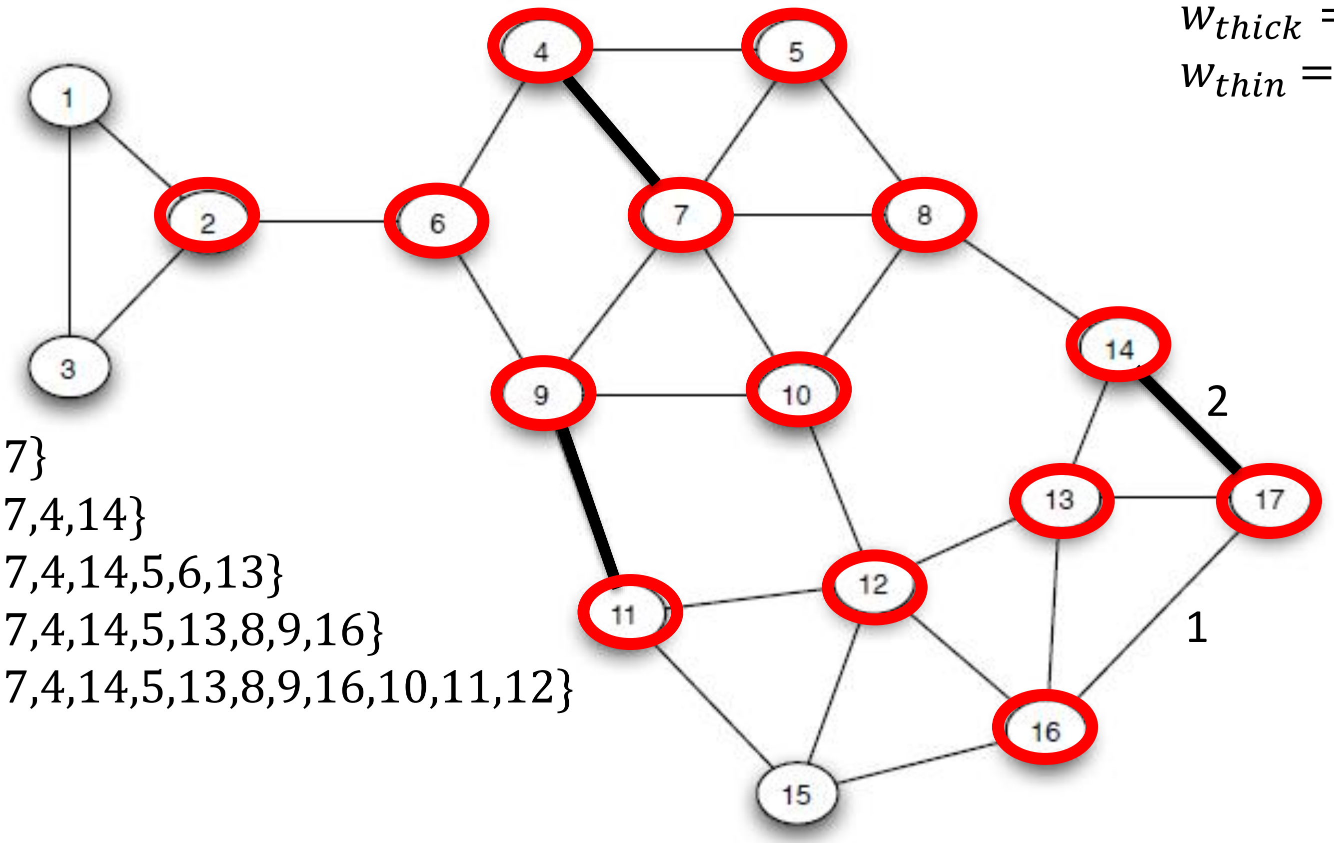
$\theta_v = 2$ for all v
$w_{thick} = 2$
$w_{thin} = 1$



$I^0 = \{2,7,17\}$
$I^1 = \{2,7,17,4,14\}$
$I^2 = \{2,7,17,4,14,5,6,13\}$
$I^3 = \{2,7,17,4,14,5,13,8,9,16\}$

Linear threshold model: example

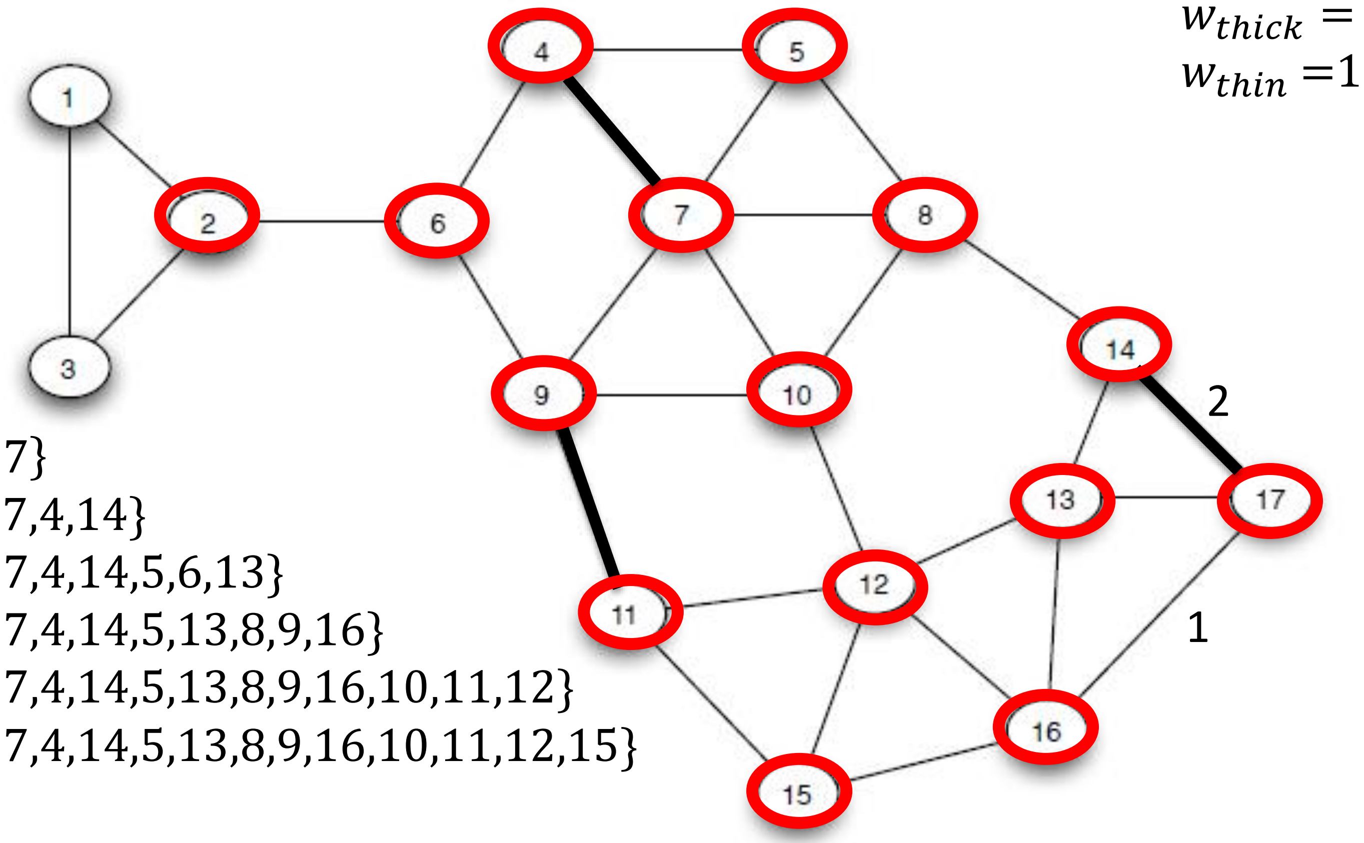
$\theta_v = 2$ for all v
$w_{thick} = 2$
$w_{thin} = 1$



$I^0 = \{2,7,17\}$
$I^1 = \{2,7,17,4,14\}$
$I^2 = \{2,7,17,4,14,5,6,13\}$
$I^3 = \{2,7,17,4,14,5,13,8,9,16\}$
$I^4 = \{2,7,17,4,14,5,13,8,9,16,10,11,12\}$

Linear threshold model: example

$\theta_v \ = 2$ for all v
$w_{thick} \ = 2$
$w_{thin} = 1$



$I^0 = \{2,7,17\}$
$I^1 = \{2,7,17,4,14\}$
$I^2 = \{2,7,17,4,14,5,6,13\}$
$I^3 = \{2,7,17,4,14,5,13,8,9,16\}$
$I^4 = \{2,7,17,4,14,5,13,8,9,16,10,11,12\}$
$I^5 = \{2,7,17,4,14,5,13,8,9,16,10,11,12,15\}$

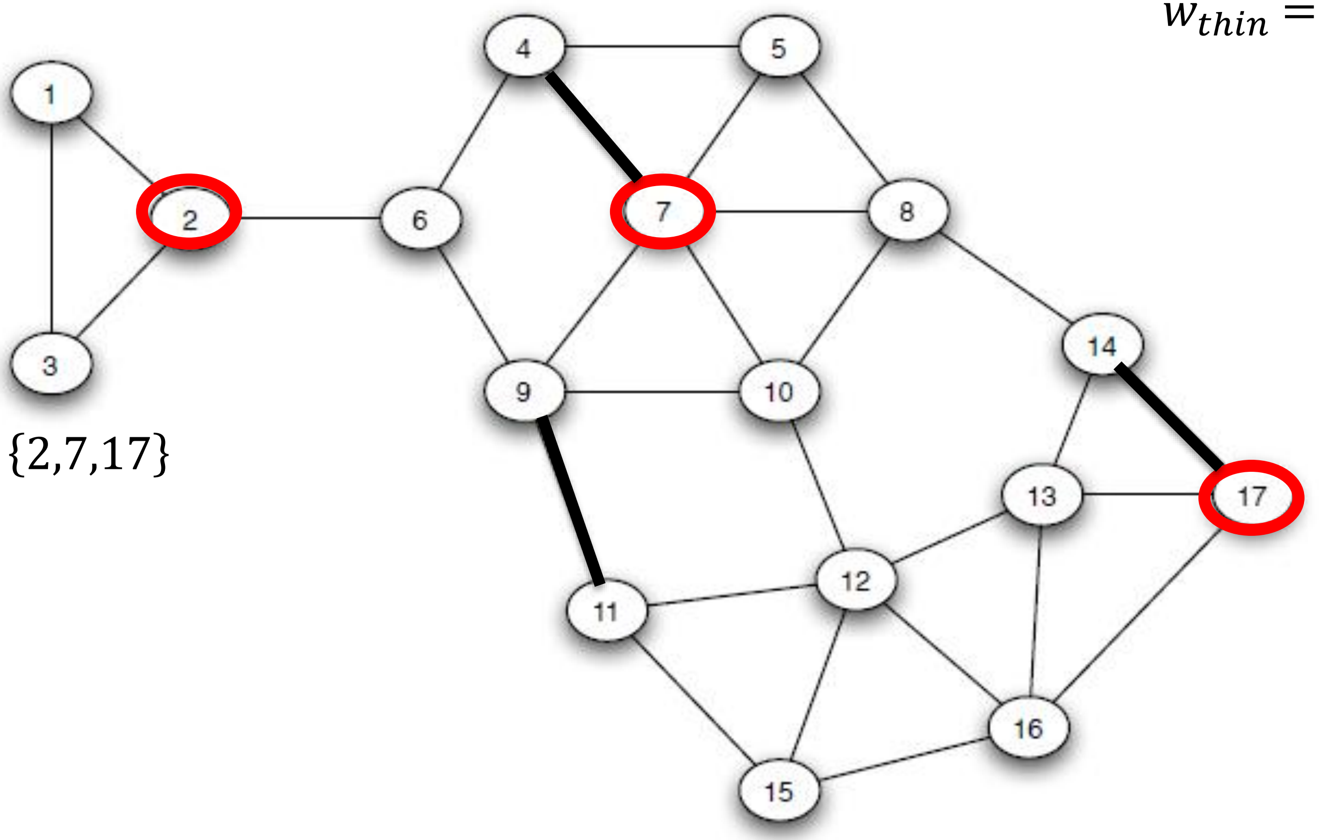
Model 2: Independent cascade model

- Each edge has a weight $w_{u,v}$
- No thresholds
- Initially, some set of "infected" nodes $NI^0 = I^0 \subset V$
- In every step $t = 1,2, \ldots$
  - NI^{t-1} are the new nodes infected at time $t - 1$
  - For each $v \in S^{t-1}$ and $u \in NI^{t-1}$, u infects v w.p. $w_{u,v}$
    - If infection occurs, v is added to NI^t
  - $I^t = I^{t-1} \cup NI^{t-1}$
  - $S^t = V \setminus I^t$

1. Random!
2. Edge $w_{u,v}$ is attempted once

Independent cascade model: example

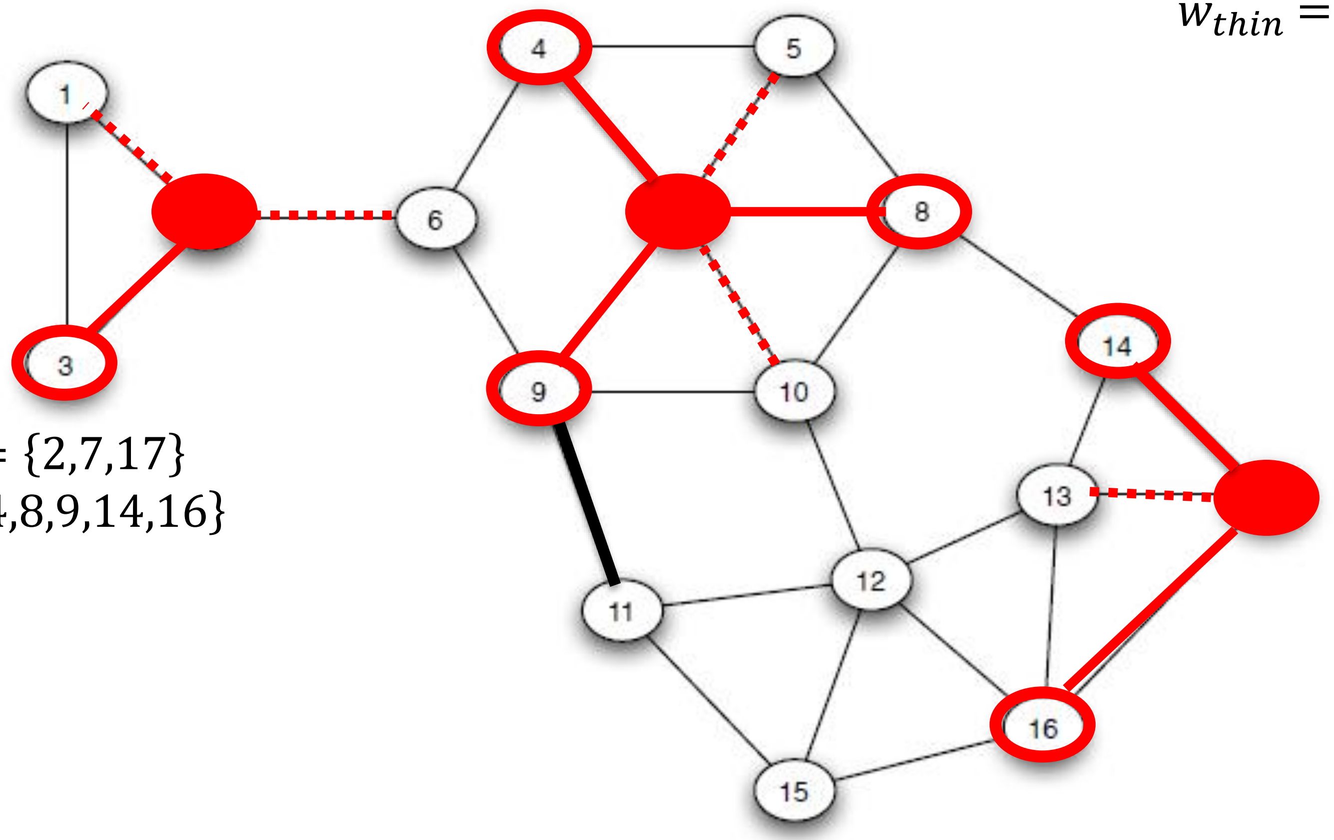
$w_{thick} = 1$
$w_{thin} = 0.5$



$NI^0 = I^0 = \{2,7,17\}$

Independent cascade model: example

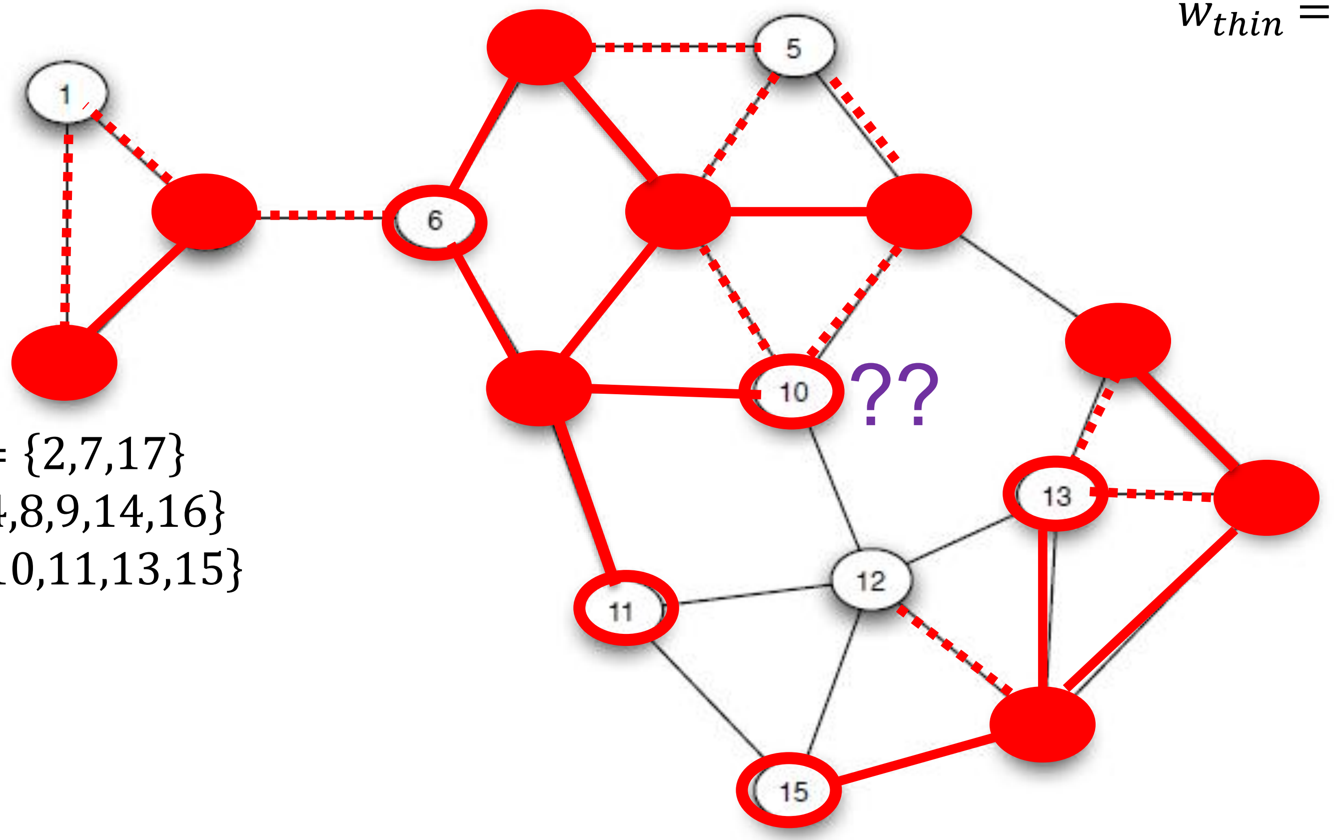
$w_{thick} = 1$
$w_{thin} = 0.5$



$NI^0 = I^0 = \{2,7,17\}$
$NI^1 = \{3,4,8,9,14,16\}$

Independent cascade model: example

$w_{thick} = 1$
$w_{thin} = 0.5$



$NI^0 = I^0 = \{2,7,17\}$
$NI^1 = \{3,4,8,9,14,16\}$
$NI^2 = \{6,10,11,13,15\}$

Independent cascade model: example

$w_{thick} = 1$
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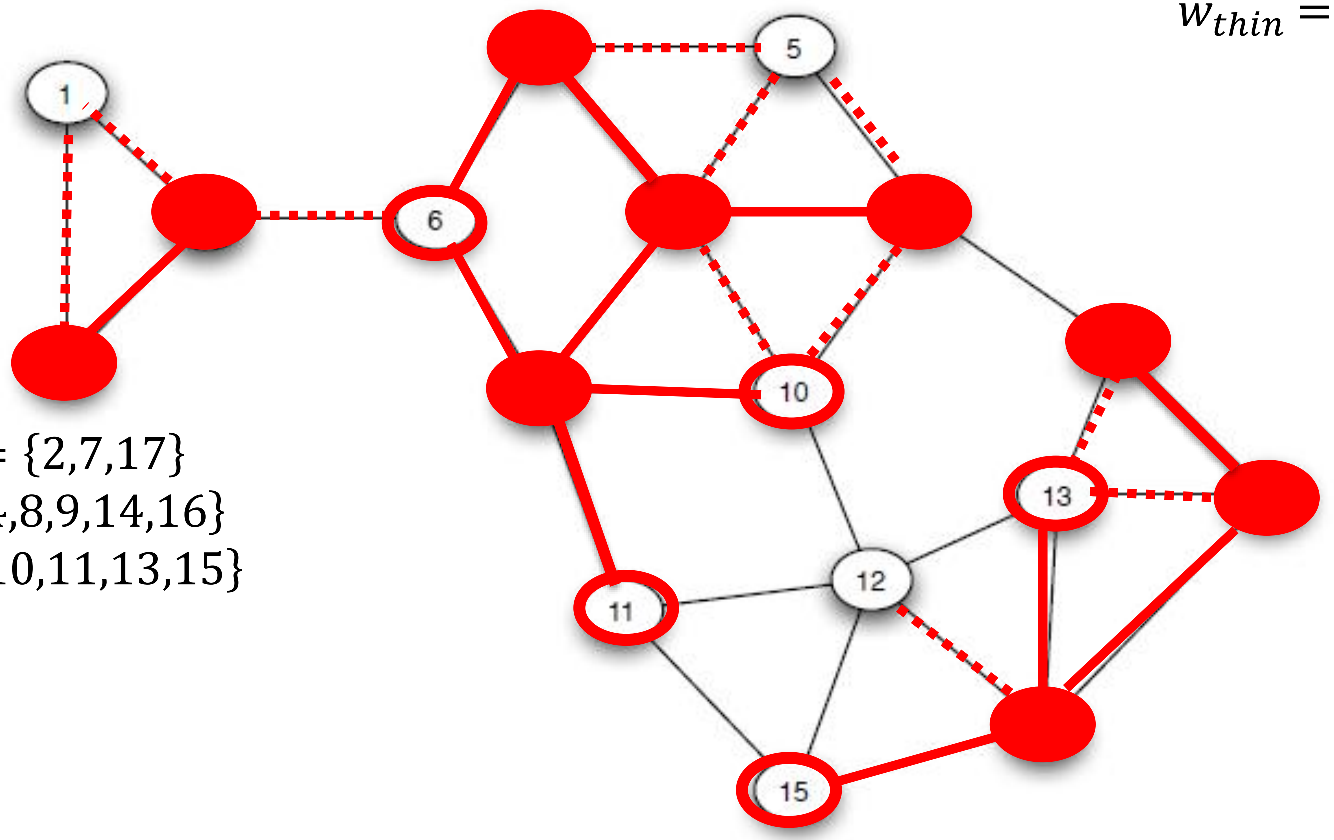

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Independent cascade model: example

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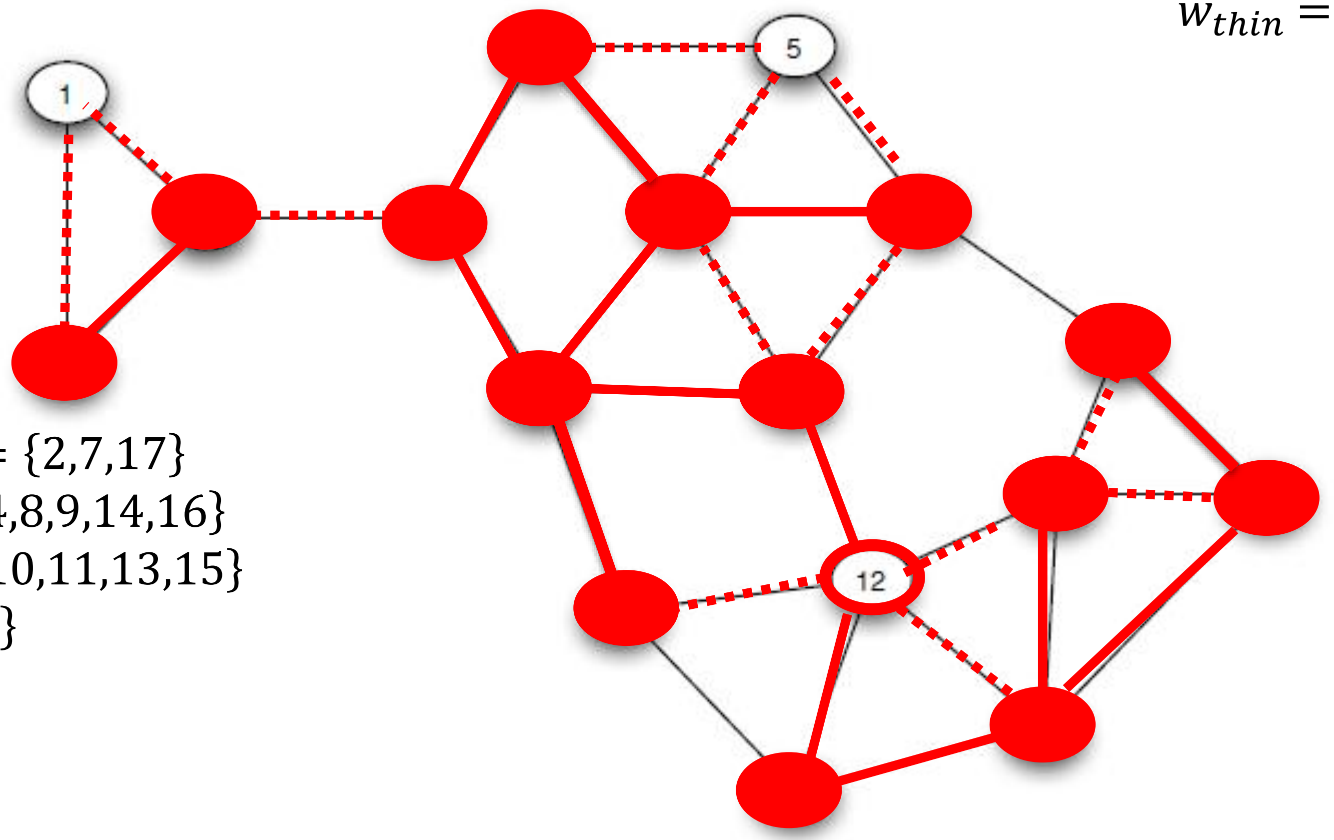
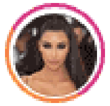


$NI^0 = I^0 = \{2,7,17\}$
$NI^1 = \{3,4,8,9,14,16\}$
$NI^2 = \{6,10,11,13,15\}$
$NI^3 = \{12\}$

FROM HERE ON, WE FOCUS ON THE INDEPENDENT CASCADE MODEL



kimkardashian • Follow

kimkardashian #CorrectiveAd I guess you saw the attention my last #morningsickness post received. The FDA has told Duchesnay, Inc., that my last post about Diclegis (doxylamine succinate and pyridoxine HCl) was incomplete because it did not include any risk information or important limitations of use for Diclegis. A link to this information accompanied the post, but this didn't meet FDA requirements. So, I'm re-posting and sharing this important information about Diclegis. For US Residents Only.
Diclegis is a prescription medicine used to treat nausea and vomiting of pregnancy in women who have not improved with change in diet or other non-medicine treatments.
Limitation of Use: Diclegis has not been studied in women with hyperemesis

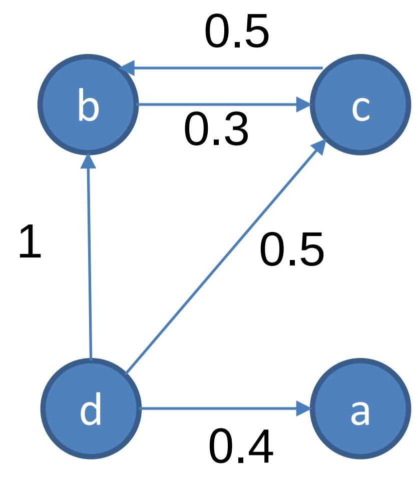
410,682 likes

AUGUST 31, 2015

Influence

- **Influence cone** $IC(S)$: the **expected** number of nodes getting a message (eventually), if initial the seed set is S

- $IC(\{a\}) = 1$
- $IC(\{c\}) = 1 + 0.5 = 1.5$
- $IC(\{b\}) = 1 + 0.3 = 1.3$
- $IC(\{a, b\}) = 1.3 + 1 = 2.3$
- $IC(\{d\}) =?$



Influence

- **Influence cone** $IC(S)$: the **expected** number of nodes getting a message (eventually), if initial the seed set is S

$$IC(S) = \sum_{x \in V} Pr(x|S)$$

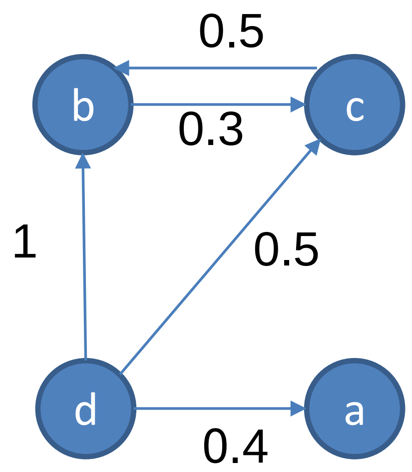
- $IC(\{d\}) = 1 + Pr(a|d) + Pr(b|d) + Pr(c|d)$

$= 1 + 0.4 + 1 + (1 - Pr(\neg c|d))$

$= 1 + 0.4 + 1 + \big(1 - (1 - 0.5)(1 - 0.3)\big)$

$= 2.4 + 0.65 = 3.05$

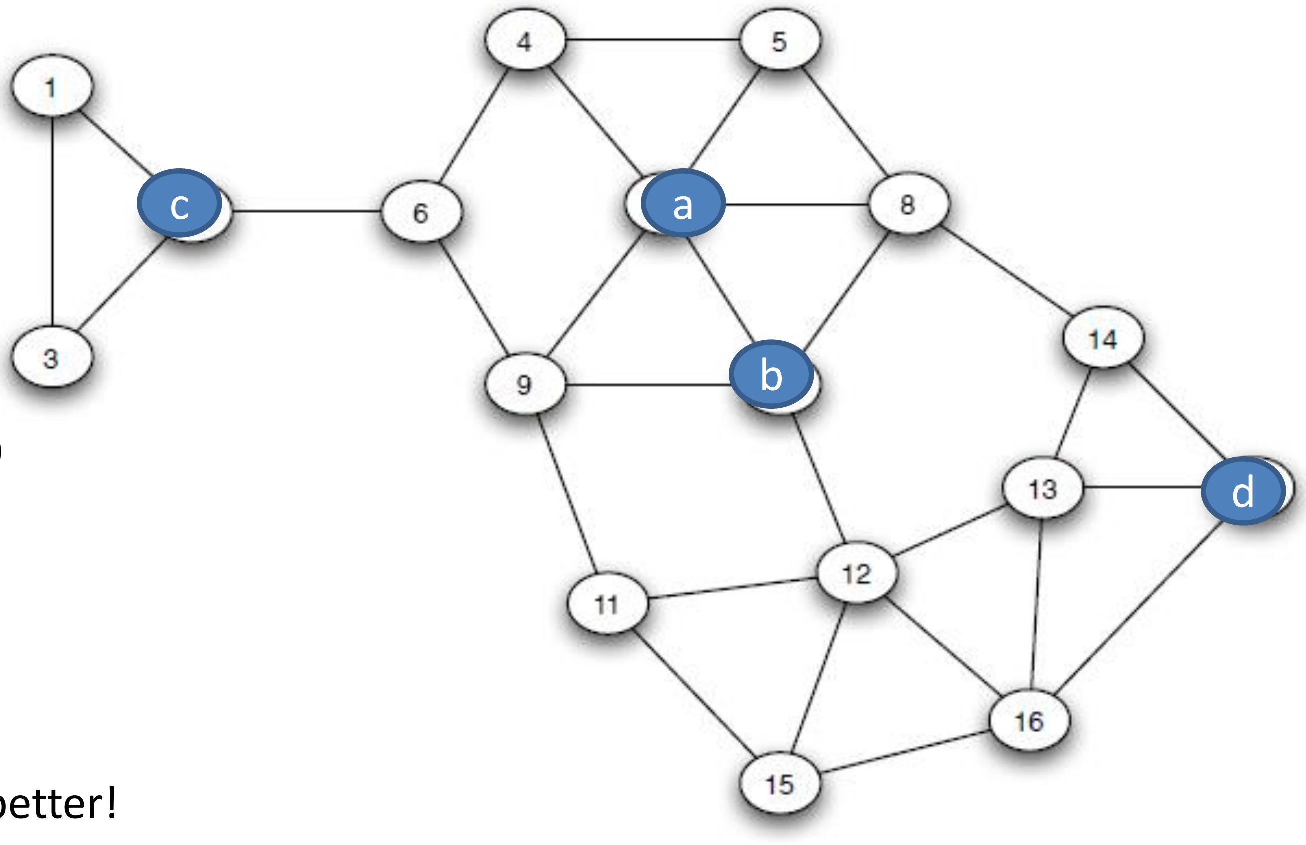
- $IC(\{a, d\}) = 3.65 \le IC(\{a\}) + IC(\{d\}) = 4.05$



Influence Maximization

- Problem definition: Given $G = (V, E, W)$ and $k \in \mathbb{N}$, find a set $S^\star$ such that
$$IC(S^\star) = \max_{S \subseteq V, |S| = k} IC(S)$$
- Combinatorial optimization!

Which 3 seeds would you choose?



(numbers are made up)

$IC(\{a\}) = 5.5$

$IC(\{b\}) = 5$

$IC(\{c\}) = 2.5$

$IC(\{d\}) = 4$

$IC(\{a, b, d\}) = 8.5$

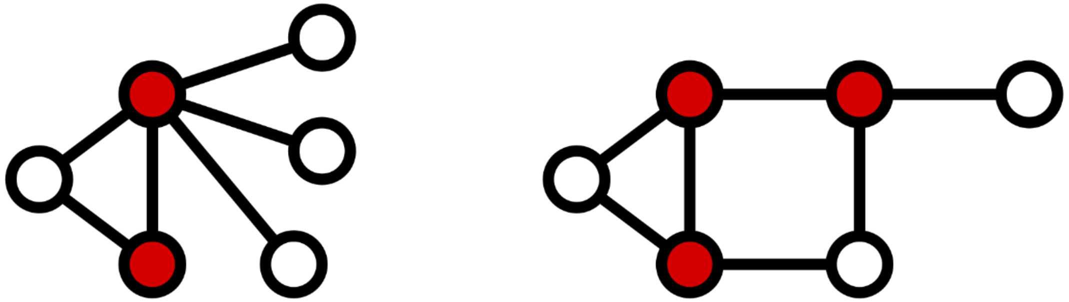
$IC(\{a, c, d\}) = 10 \Rightarrow$ better!

Optimal algorithm?

Minimum Vertex Cover (Week 1)

- An undirected Graph $G = (V, E)$
- A set $V' \subseteq V$ is a vertex cover of G if for every edge $\{u, v\} \in E$, either $u \in V'$ or $v \in V'$
- Decision problem:

Does G have a vertex cover of size at most k?



Hard to Maximize Influence

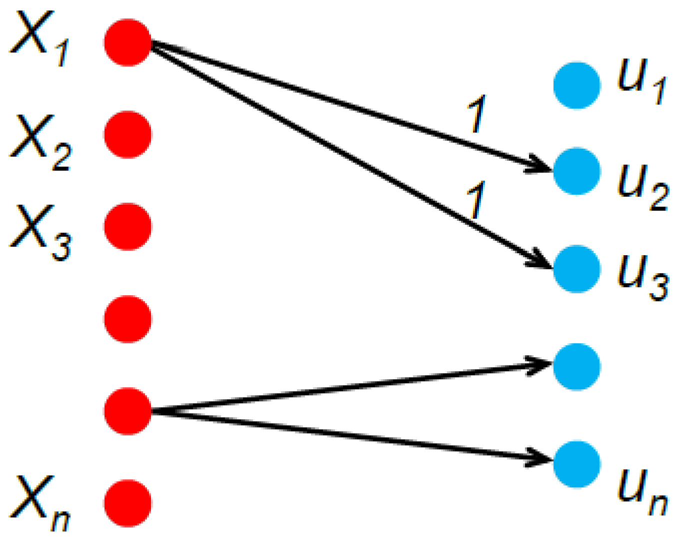
- Well known fact: minimum vertex cover (MVC) is hard to solve
  - Formally: the problem is NP-Complete
  - Informally: no known polynomial algorithm in the general case
- Can we show that influence maximization (IM) is at least as hard?
- Assume we have a black box that solves IM. Can we use it to solve MVC?

Hard to Maximize Influence

- Theorem (formal): IM is NP-Complete
- Theorem (highly informal): no known polynomial algorithm
- Proof outline:
  - Assume we get an MVC instance, $G = (V, E), k$
  - We construct an IM instance $G' = (V', E', W)$
  - Show that:

$$G \text{ has a vertex cover of size } k$$
$$\Leftrightarrow$$
$$G' \text{ there is a set } S \text{ of size } k \text{ with } IC(S) = k + |V|$$

Proof Outline



$N(v_1) = \{v_2, v_3\}$

- $u_i, X_i \in V'$ are copies of $v_i \in V$
- $(X_i, u_j) \in E'$ iff $(v_i, v_j) \in E$
- All weights are 1 (deterministic activation)
- Optimal solution only picks nodes from the left side
- From here – easy…

vertex cover of size k in $G \Leftrightarrow$

a set S of size k with $IC(S) = k + |V|$ in G'

A Greedy Algorithm for IM

Hill Climbing (HC) [input: $G = (V, E, W), k$]
Initialize $S \leftarrow \emptyset$
For $t = 1,2, \dots, k$
    for every $v \in V \setminus S_{t-1}$, compute
        $mv(S_{t-1}, v) := IC(S_{t-1} \cup \{v\}) - IC(S_{t-1})$
    find an element v_t that maximizes $mv(S_{t-1}, v_t)$
    $S_t \leftarrow S_{t-1} \cup \{v_t\}$ // add element v_t to S
Return S_k

- Theorem: Let S_k denote the output of HC on G, k. It holds that

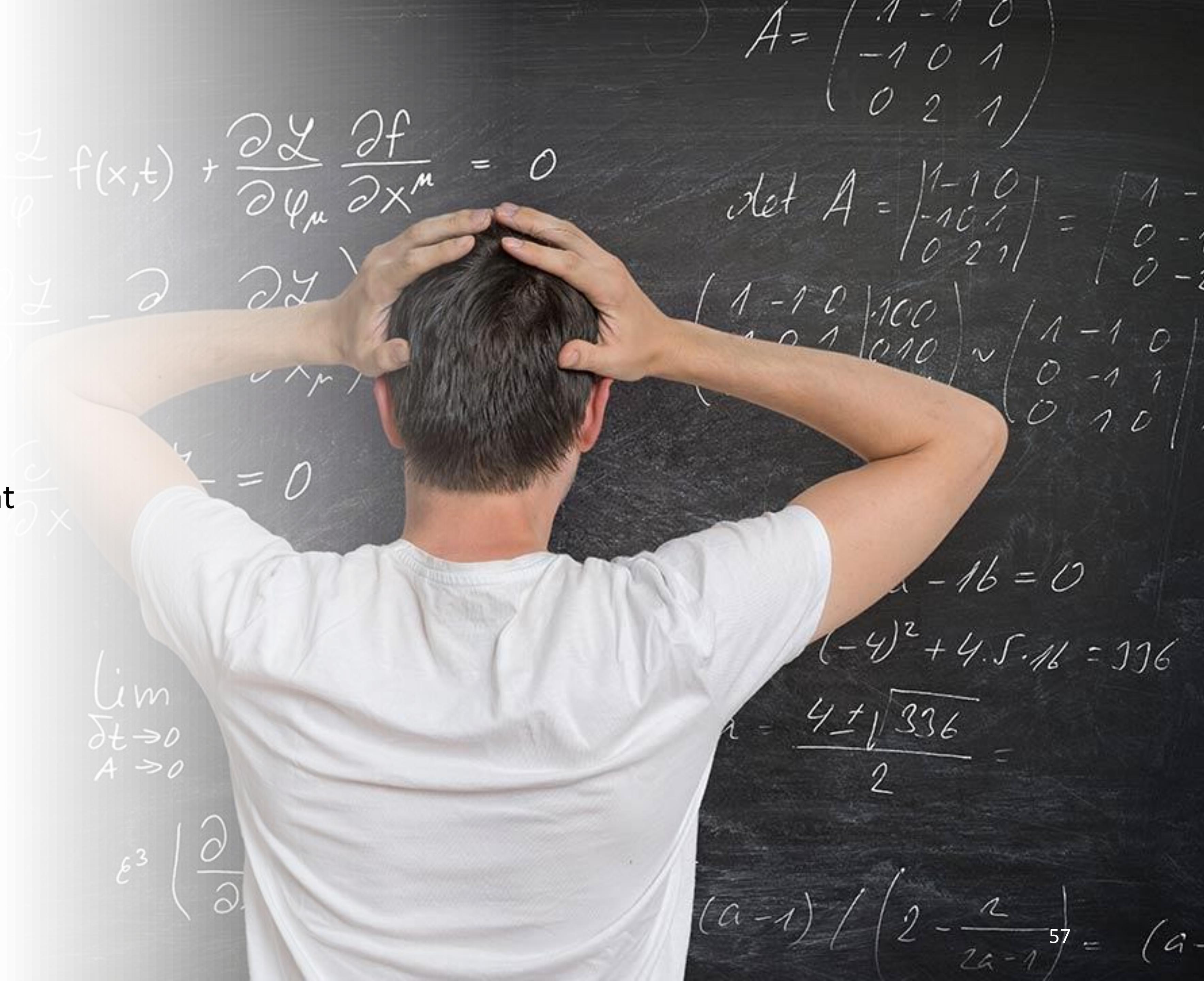
$$IC(S_k) \geq \left(1 - \frac{1}{e}\right) IC\left(S_{OPT(G,k)}\right)$$

This class

- Homophily – from last week
- SIR (No networks)
- Diffusion Models
  - The Linear Threshold model
  - The Independent Cascade model
- Influence Maximization - proofs

Theorem: Let S_k denote the output of HC on G, k. It holds that

$$IC(S_k) \geq \left(1 - \frac{1}{e}\right) IC\left(S_{OPT(G,k)}\right)$$



Greedy Algorithm – General structure

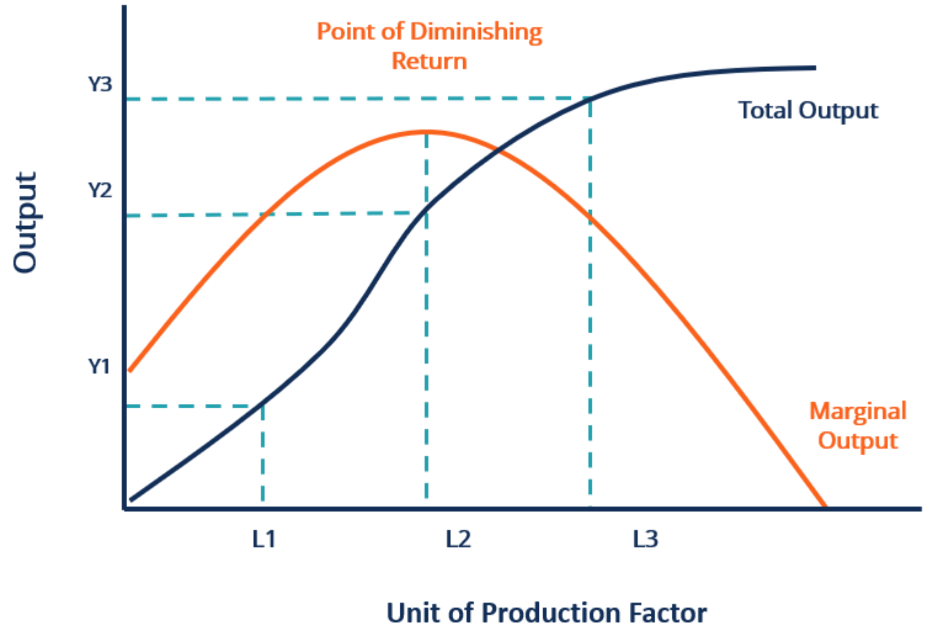
Hill Climbing (HC) [input: integer k, function $\boldsymbol{f}$, $\boldsymbol{f}: 2^{\Omega} \to \mathbb{R}$ for universe Ω]
Initialize $S \leftarrow \emptyset$
For $t = 1,2,\dots,k$
  for every $v \in \Omega \setminus S_{t-1}$, compute
    $mv(S_{t-1}, v) \coloneqq \boldsymbol{f}(S_{t-1} \cup \{v\}) - \boldsymbol{f}(S_{t-1})$
  find an element v_t that maximizes $mv(S_{t-1}, v_t)$
  $S_t \leftarrow S_{t-1} \cup \{v_t\}$
Return S_k

- Can we guarantee anything about the quality of the solution?

Outline (1)

- Definition: Let Ω be a finite set, and let $f: 2^{\Omega} \to \mathbb{R}$. We say that f is submodular if for all $S \subseteq T \subseteq \Omega$ and $u \in \Omega$ it holds that

$$f(S \cup \{u\}) - f(S) \geq f(T \cup \{u\}) - f(T).$$



Outline (2)

- Theorem: Let f is a submodular function, and let S_k denote the output of HC on f, k. It holds that

$$f(S_k) \geq \left(1 - \frac{1}{e}\right) f\left(S_{OPT(G,k)}\right).$$

Outline (3)

- Definition: Given $G = (V, E)$, $I \subseteq E$ and $u \in V$, we denote by X_u^I the *reachable set* of vertices through the edges in E.
- For $I \subseteq E$ and $S \subseteq V$, let $A_I(S) = \cup_{s \in S} X_s^I$ be the set of nodes reachable from S through the edges in I.
- Let D be distribution over 2^E such that

$$\Pr_D[I] = \prod_{e \in I} w_e \prod_{e \notin I} (1 - w_e).$$

  – Intuition: Sampling the active edges independently of the seed set.

Outline (4)

- For $I \subseteq E$ and $S \subseteq V$, let $A_I(S) = \cup_{s \in S} X_s^I$ be the set of nodes reachable from S through the edges in I.
- Let D be distribution over 2^E such that

$$\Pr_D[I] = \prod_{e \in I} w_e \prod_{e \notin I} (1 - w_e).$$

- Claim:

$$IC(S) = \sum_{x \in V} \Pr(x \; is \; active | S) = \mathbb{E}_D[|A_I(S)|]$$

Outline (5)

- <u>Claim</u>:

$$IC(S) = \sum_{x \in V} \Pr(x \; is \; active | S) = \mathbb{E}_D[|A_I(S)|]$$

- <u>Claim</u>: If $c_1, \dots, c_n$ is non-negative and $f_1, \dots, f_n$ are submodular functions, then $\tilde{f} = \sum_i c_i f_i$ is submodular.
- <u>Claim</u>: IC is submodular. To show this, given the previous claim, it is enough to show that $A_I(S)$ for arbitrary S and I.

Electronic Commerce 096211

Power laws and heavy tail distributions

Omer Ben Porat (some of the slides are adopted from Prof. Reshef Meir)

Course Structure

Online Societies
- Web ranking
- Diffusion
- Structure of the web
- Power laws
- Networks

Advertising
- Auctions
- Matching markets
- Sponsored search

Decision Making
- Recommendation systems
- Multi-armed bandits

Advance
- Information in markets
- Externalities
- Streaming
- Blockchain

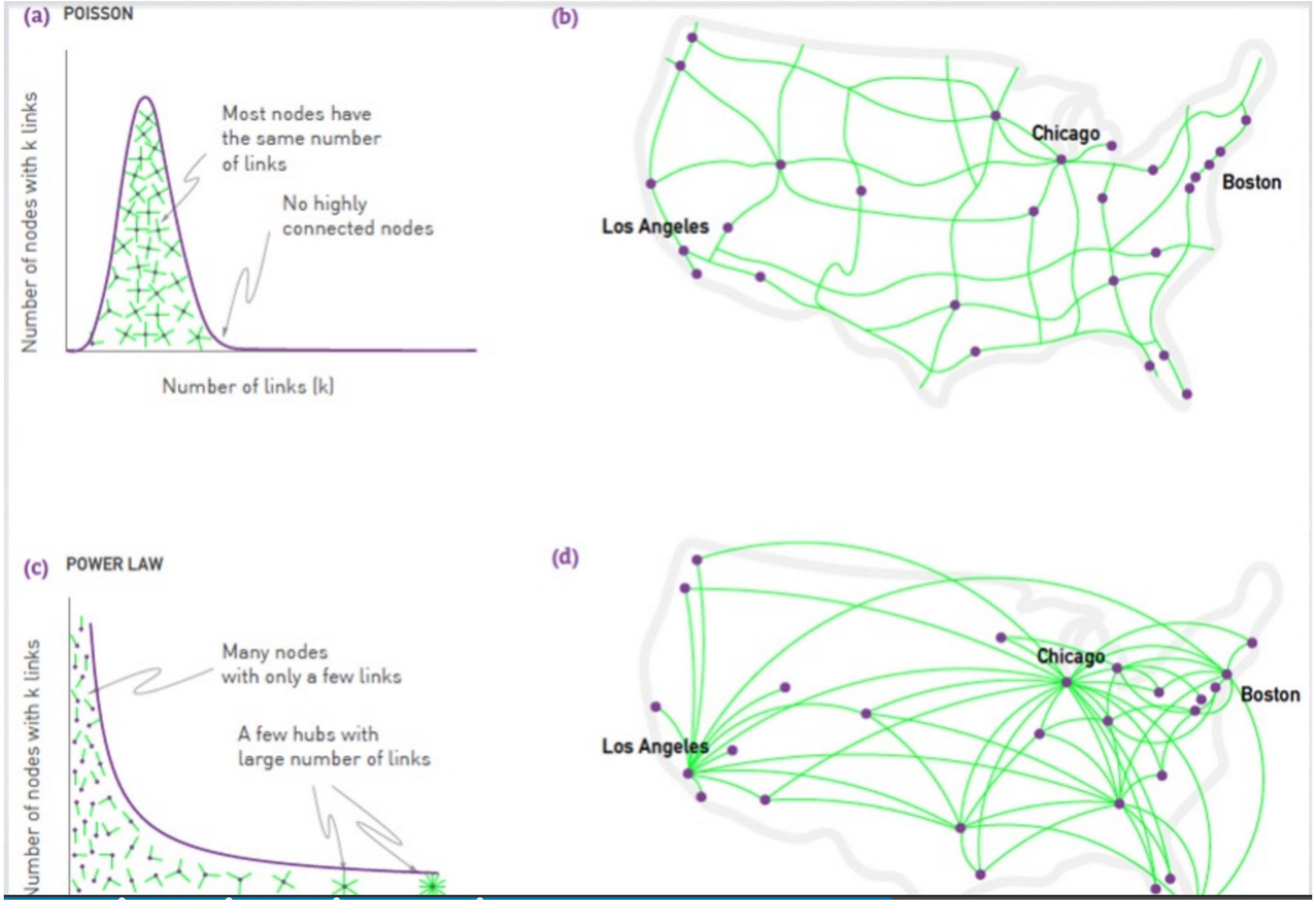
Tools and Techniques
- Game theory
- Algorithms
- Graph theory
- Optimization

Outline

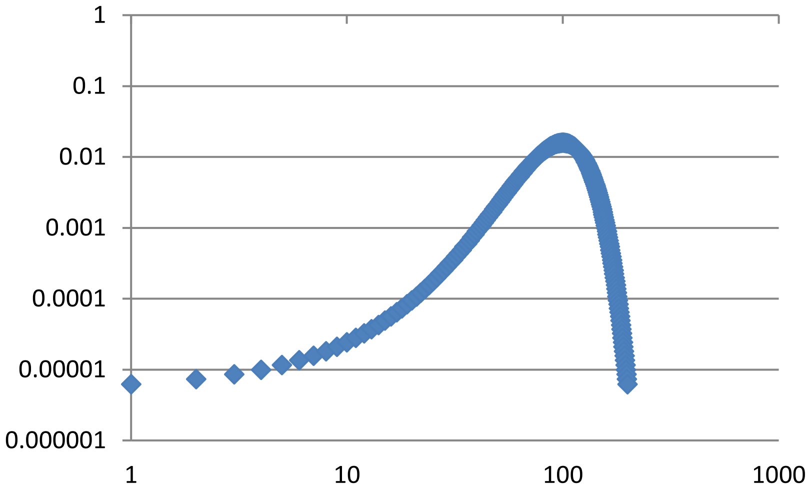
- **Motivation and examples**
- Isn’t everything Normal?
- Mathematical representation
- Parameter estimation
- Rich getting richer – experiment
- Business considerations
- Page linking process – derivation of power law

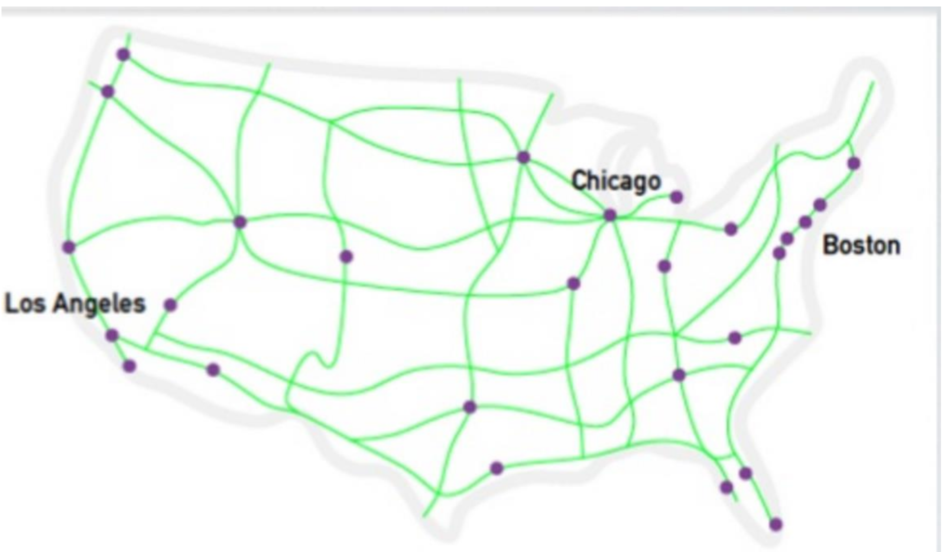
Also:
- Tips for HW1
- Yoav’s talk

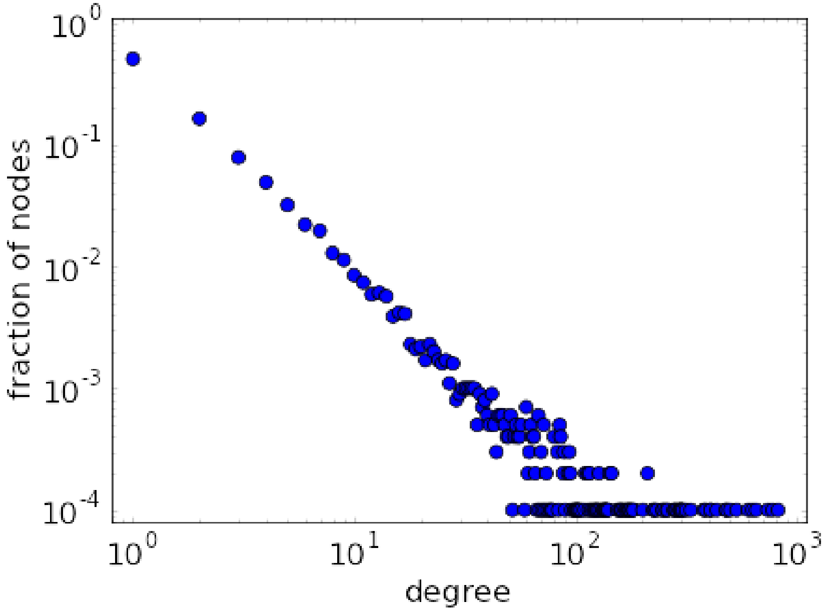
Source: http://www.mediafactory.org.au/molly-hill/2013/09/19/week-8-reading-barabasi-8020-rule/

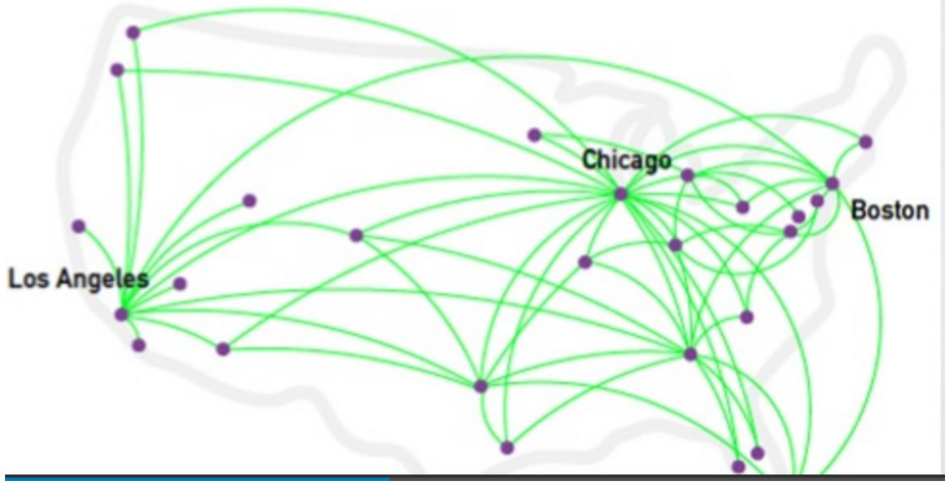


Source: http://www.mediafactory.org.au/molly-hill/2013/09/19/week-8-reading-barabasi-8020-rule/



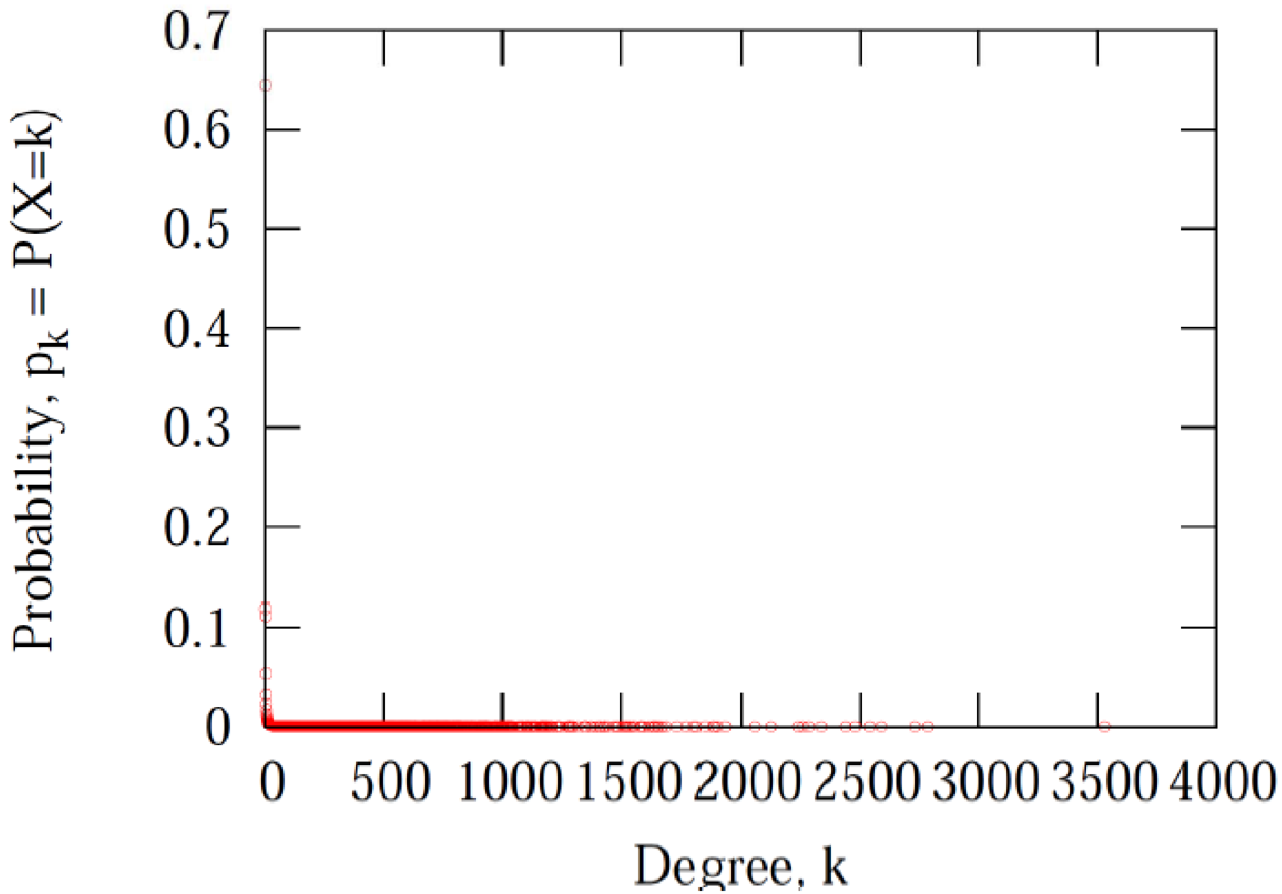






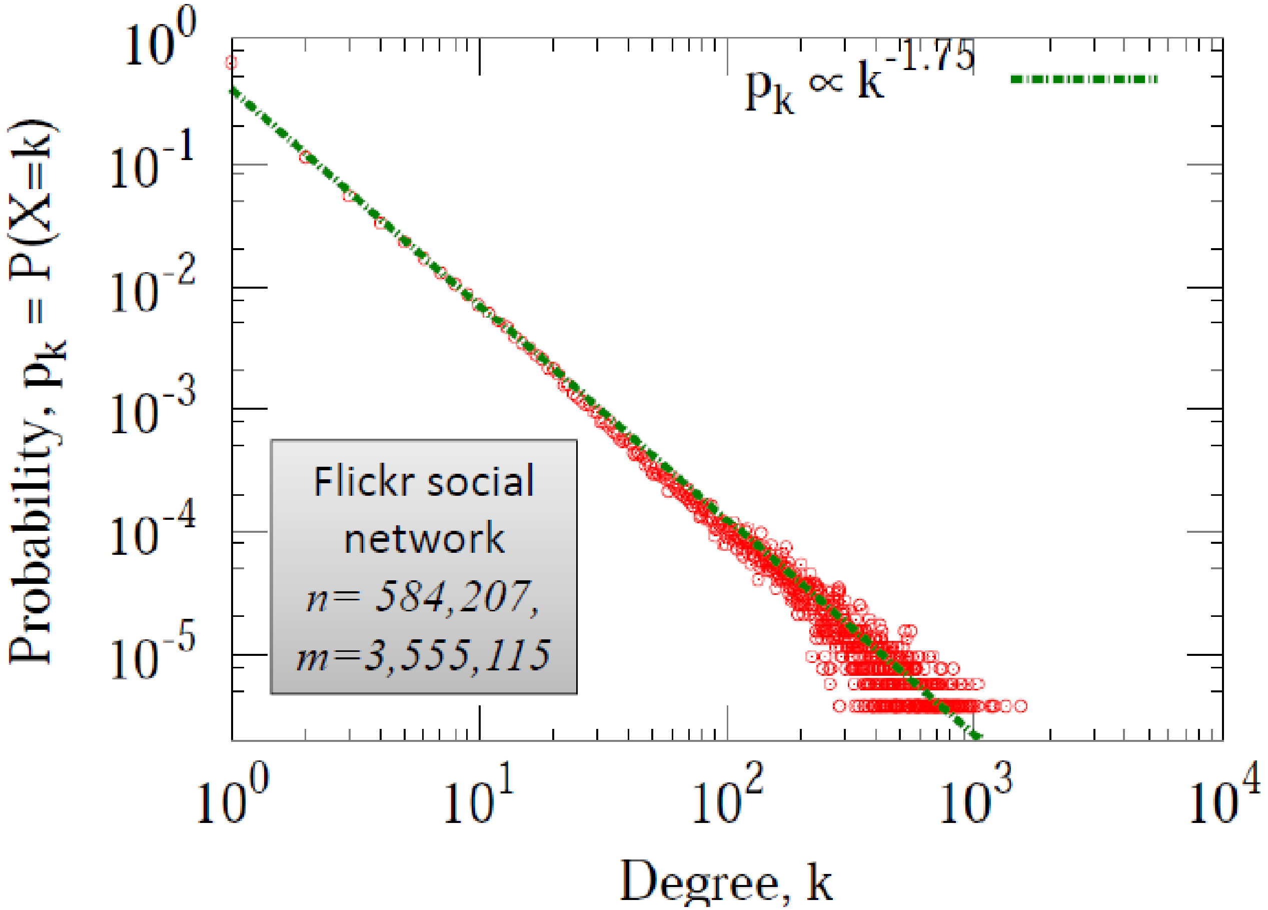
Source: http://www.mediafactory.org.au/molly-hill/2013/09/19/week-8-reading-barabasi-8020-rule/

Degree Frequency in Flickr

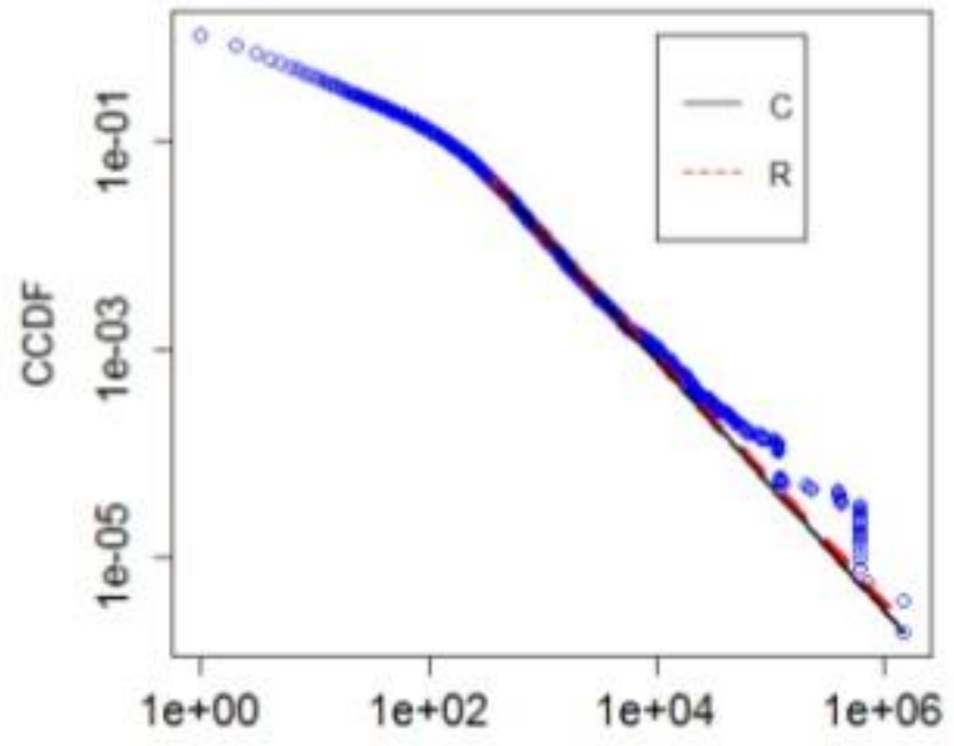


Flickr social network
n= 584,207,
m=3,555,115

Degree Frequency in Flickr

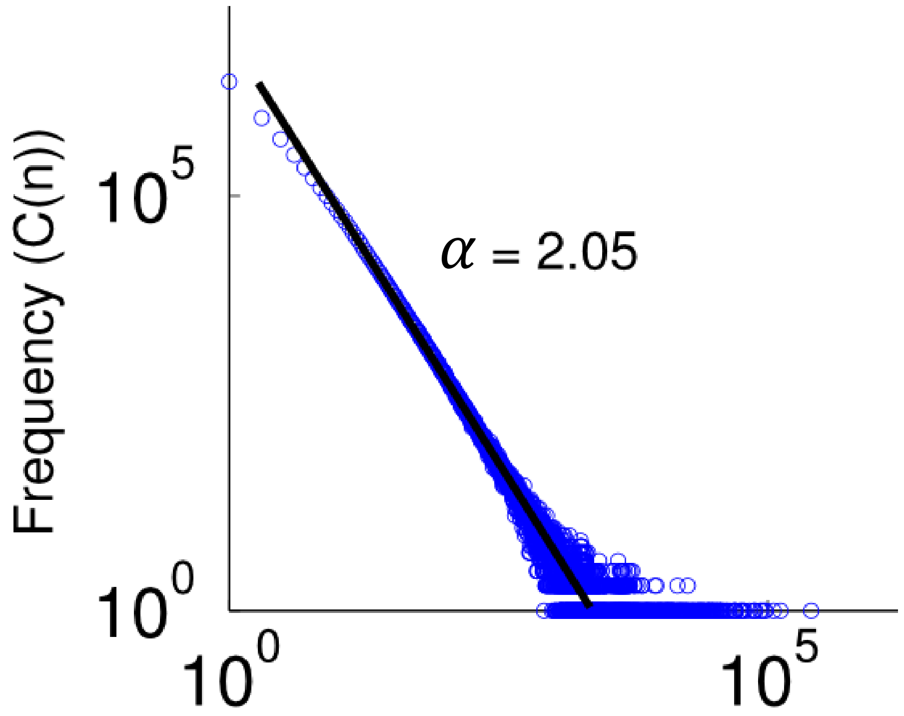


Source:
https://www.slideshare.net/ConorFeeney2/power-law-distributions-for-twitter-data



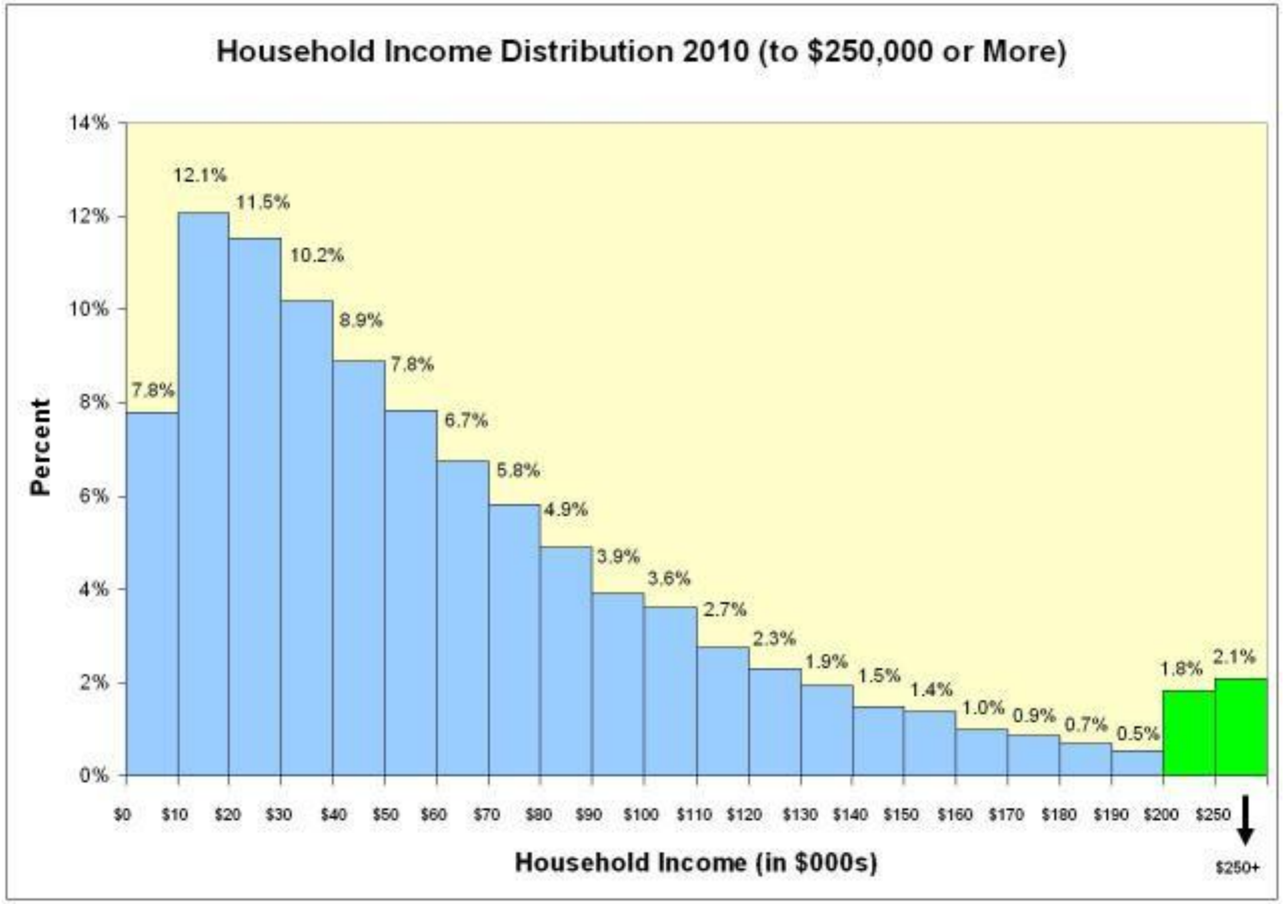
Number of Tweeter followers

Source:
https://en.wikiversity.org/wiki/User:Sergeyd/web_structure_chapter



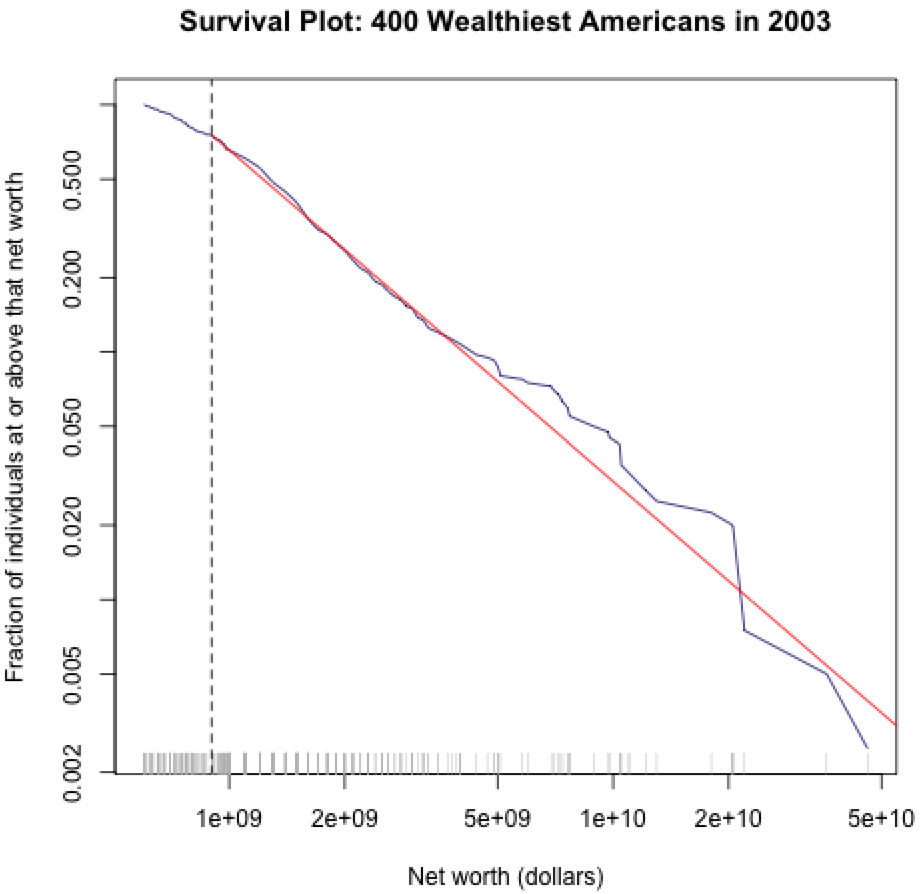
In-degree (n) of articles
(English Wikipedia)

US Income



Source: U.S. Census Bureau, Current Population Survey, 2011 Annual Social and Economic Supplement

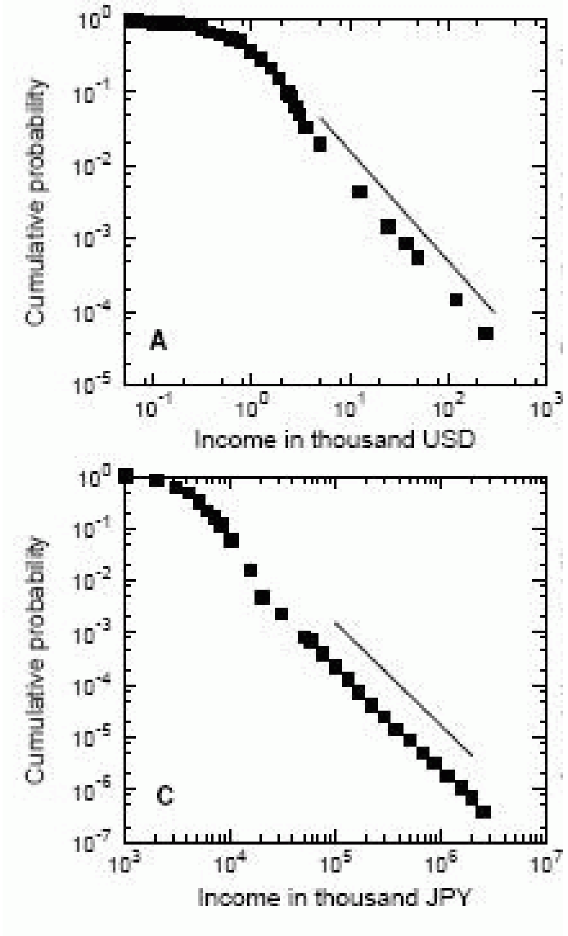
http://www.census.gov/hhes/www/cpstables/032011/hhinc/new06_000.htm

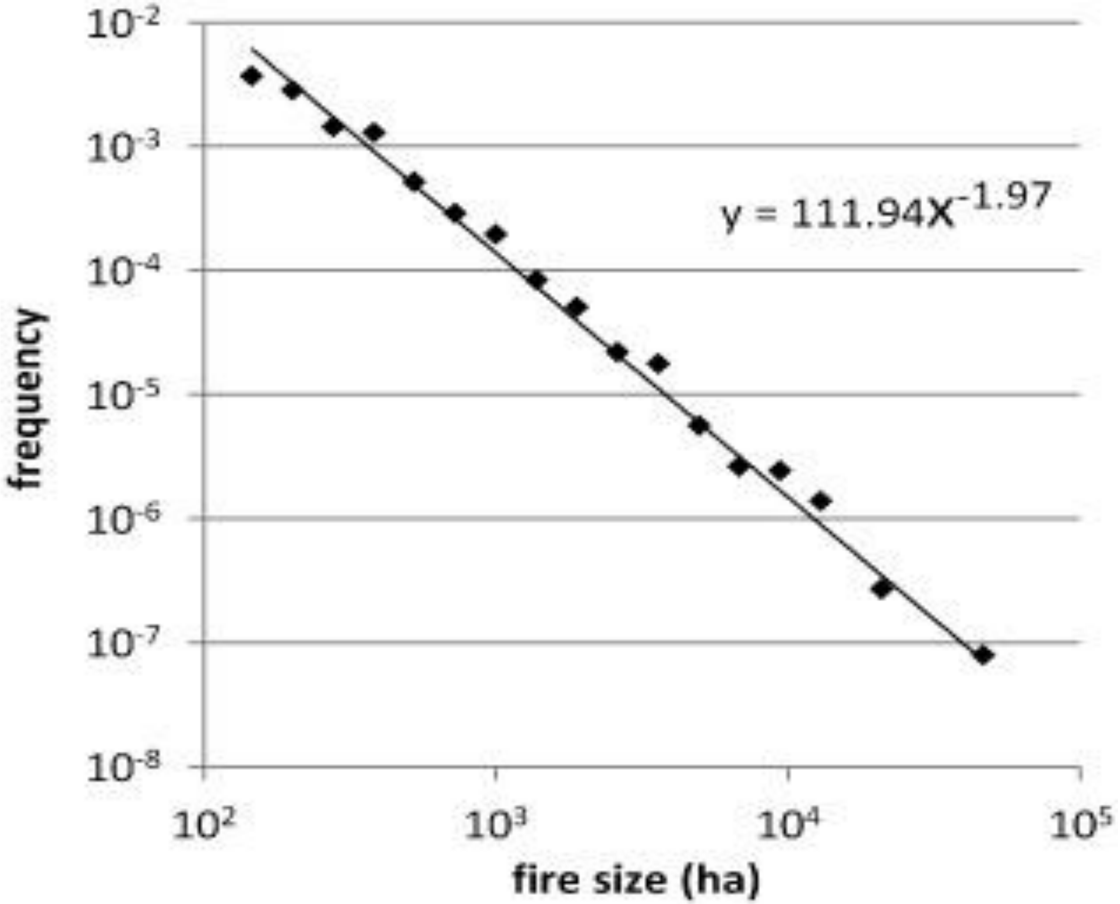


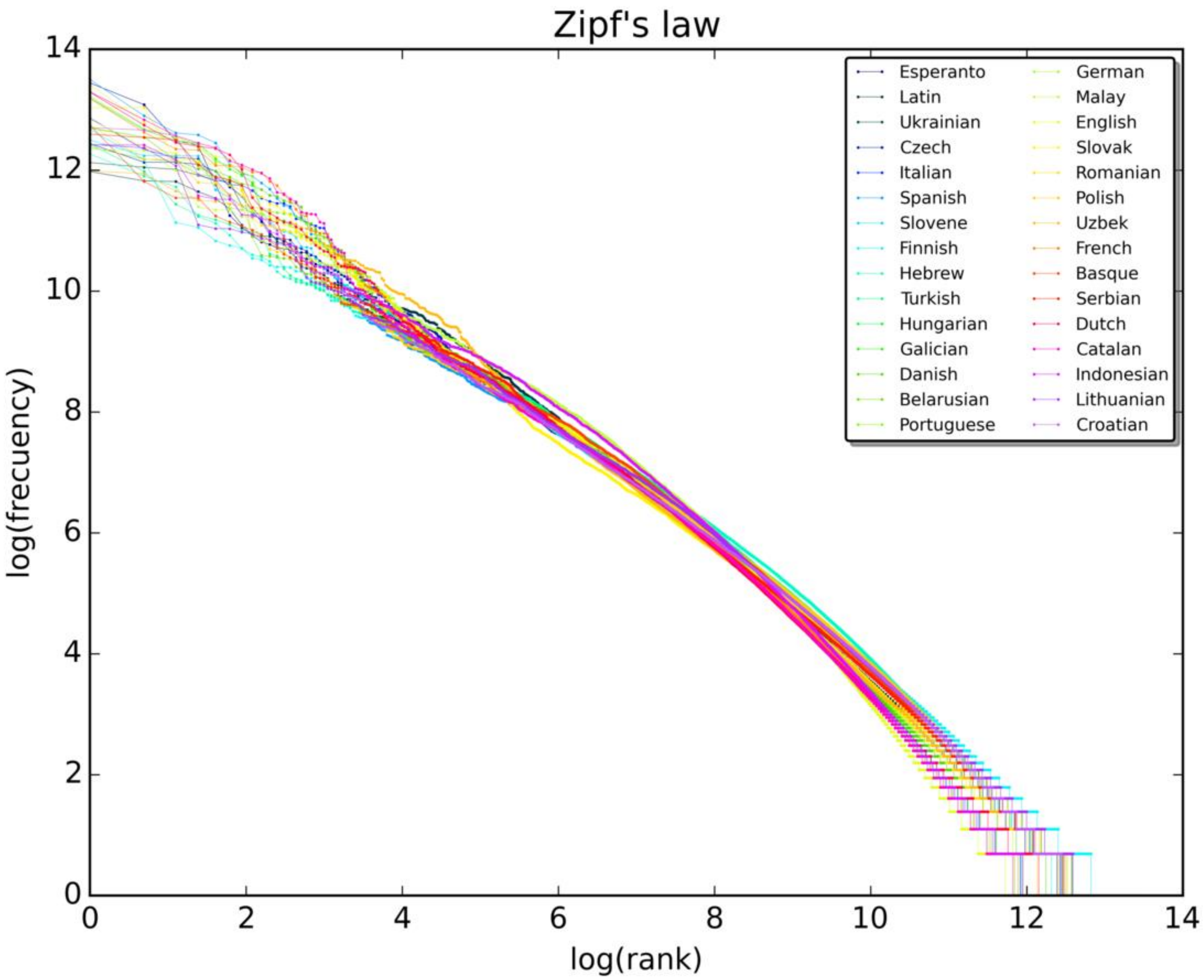
More Power Laws

Forest fires in Brazil

Income in US and Japan

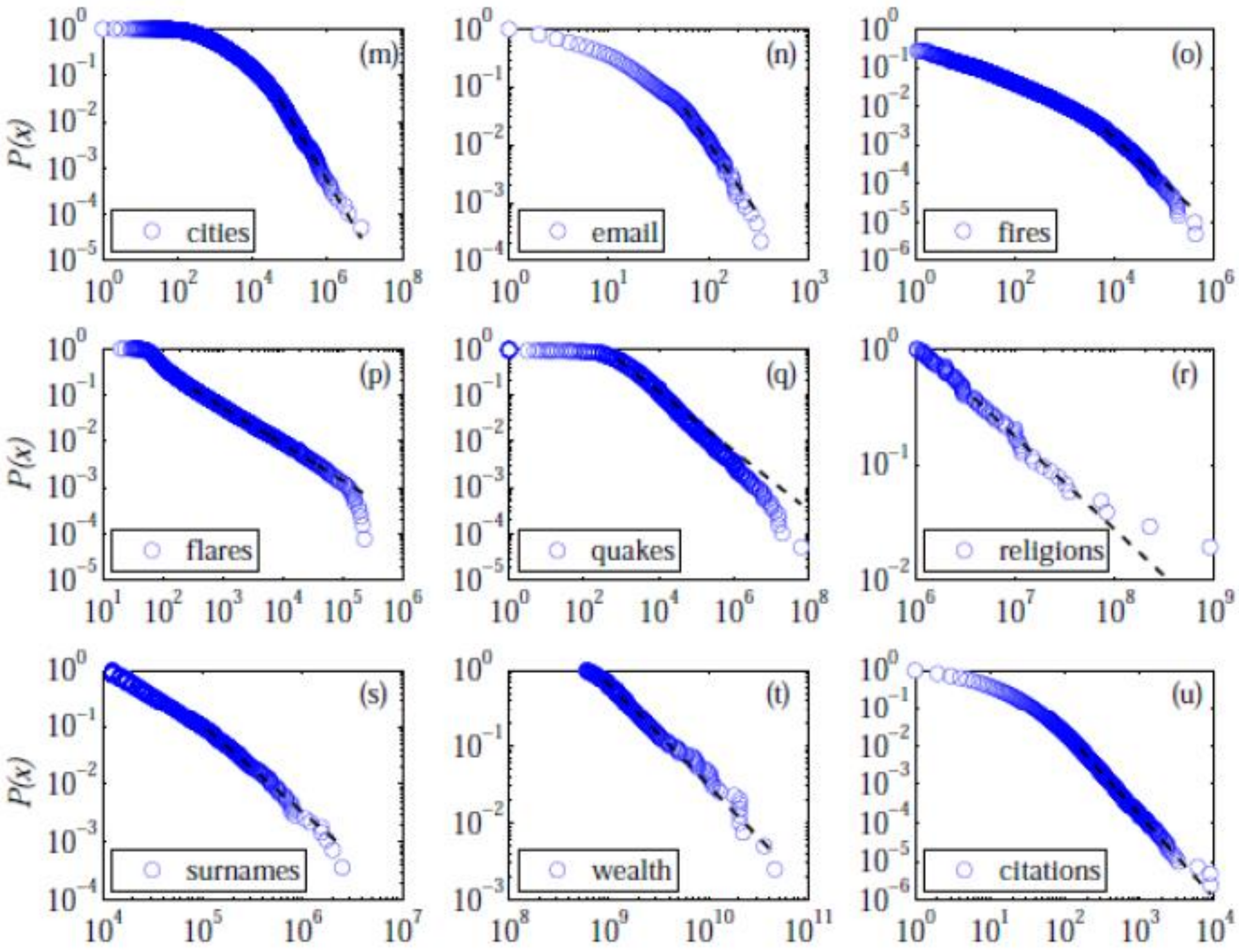






A plot of the rank versus frequency for the first 10 million words in 30 Wikipedias (dumps from October 2015) in a log-log scale.

And More…



(m)
cities
(n)
email
(o)
fires
(p)
flares
(q)
quakes
(r)
religions
(s)
surnames
(t)
wealth
(u)
citations
P(x)

Outline

- Motivation and examples
- **Isn’t everything Normal?**
- Mathematical representation
- Parameter estimation
- Rich getting richer – experiment
- Business considerations
- Page linking process – derivation of power law

Isn’t Everything Poisson/normal Dist.?

- מספר האטומים שמתפרקים בפרק זמן נתון בחומר רדיואקטיבי.
- מספר המכוניות שעוברות דרך נקודה מסוימת בכביש בפרק זמן מסוים.
- מספר שיחות הטלפון במרכז תמיכה בדקה.
- מספר הטאצ'דאונים בסופרבול האמריקאי.
- מספר המוטציות במקטע DNA לאחר חשיפה מסוימת לקרינה.
- מספר עצי האלון ביחידת שטח של יער.
- מספר הקוצים על חוטר של ורד.
- מספר הדוורים הננשכים על ידי כלבים במשך יום עבודה.

https://he.wikipedia.org/wiki/%D7%94%D7%AA%D7%A4%D7%9C%D7%92%D7%95%D7%AA_%D7%A4%D7%95%D7%90%D7%A1%D7%95%D7%9F

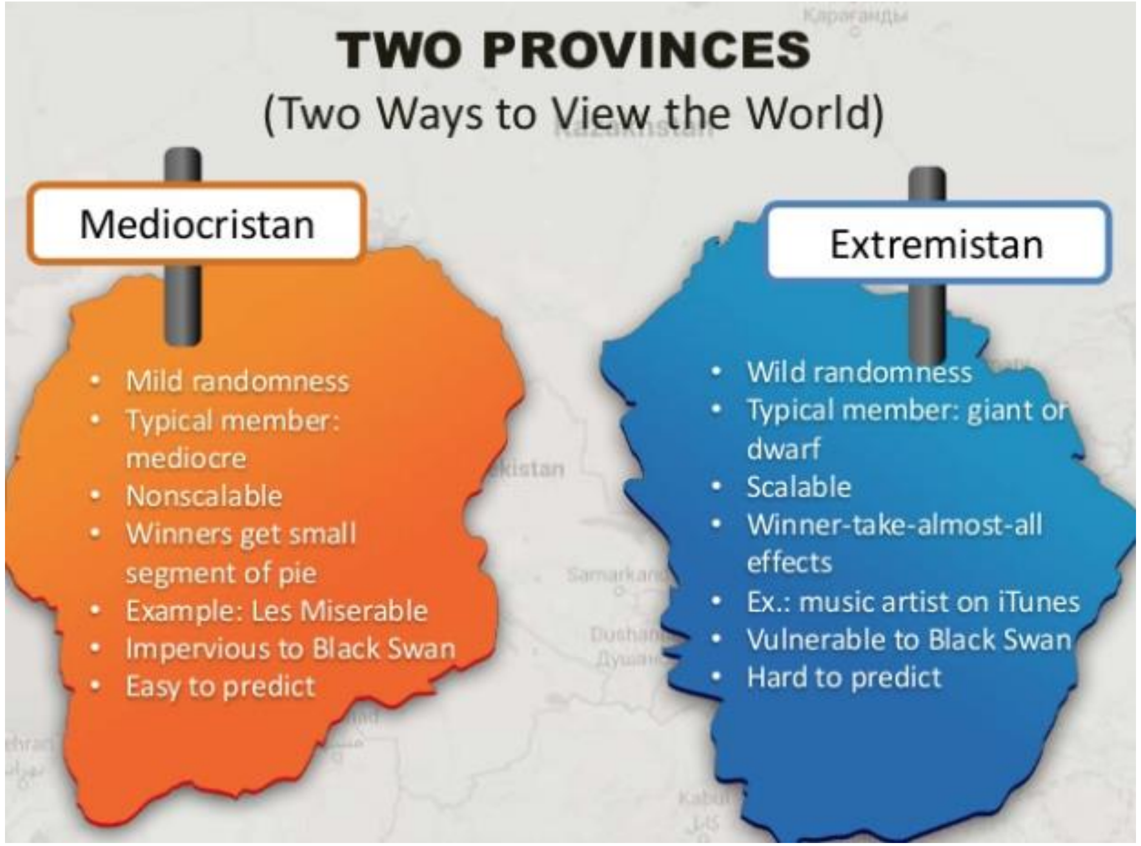
Why Power Laws?

- Feedback
  - Rich people earn more money
  - Large pandemics engulf more people
  - Huge fires are harder to put off

- Where do we find feedback in networks?

A Thought Experiment

- 1000 people in a stadium
- Measure height, weight
- Find the tallest, heaviest person in the world, add them. Change?
  - Up to 0.6%
- Now wealth. Add BG. Change?
  - More than 99.9%



Example (Simplified)

Mediocristan

Height > 1.87 $\approx 1/6.3$

Height > 1.97 $\approx 1/44$

Height > 2.07 $\approx 1/740$

Height > 2.27 $\approx 1/350000$

Height > 2.47 $\approx 1/(1 \cdot 10^9)$

Height > 2.67 $\approx 1/(8 \cdot 10^{20})$

Height > 2.77 $\approx 1/(2 \cdot 10^{30})$

Extremistan

Wealth > 1M $\approx 1/62.5$

Wealth > 2M $\approx 1/250$

Wealth > 4M $\approx 1/1000$

Wealth > 8M $\approx 1/4000$

Wealth > 16M $\approx 1/16{,}000$

Wealth > 32M $\approx 1/64{,}000$

Wealth > 320M $\approx 1/6{,}400{,}000$

Example (Simplified)

Mediocristan?

Wealth > 1M $\approx 1/62.5$

Wealth > 2M $\approx 1/127{,}000$

Wealth > 4M $\approx 1/14{,}000{,}000{,}000$

Wealth > 8M $\approx 1/886 \cdot 10^{15}$

Wealth > 16M $\approx 1/16 \cdot 10^{33}$

Wealth > 32M $\approx 1/72 \cdot 10^{255}$

Wealth > 320M $\approx 1/??$

Extremistan

Wealth > 1M $\approx 1/62.5$

Wealth > 2M $\approx 1/250$

Wealth > 4M $\approx 1/1000$

Wealth > 8M $\approx 1/4000$

Wealth > 16M $\approx 1/16{,}000$

Wealth > 32M $\approx 1/64{,}000$

Wealth > 320M $\approx 1/6{,}400{,}000$

Outline

- Motivation and examples
- Isn’t everything Normal?
- **Mathematical representation**
- Parameter estimation
- Rich getting richer – experiment
- Business considerations
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Math

- Normal, Binomial, Poisson (etc.) distributions are decaying **exponentially**: (for some const. α)

$$\Pr(X > k) \leq e^{-\alpha k}$$

  – Instances much larger than the mean are very rare!

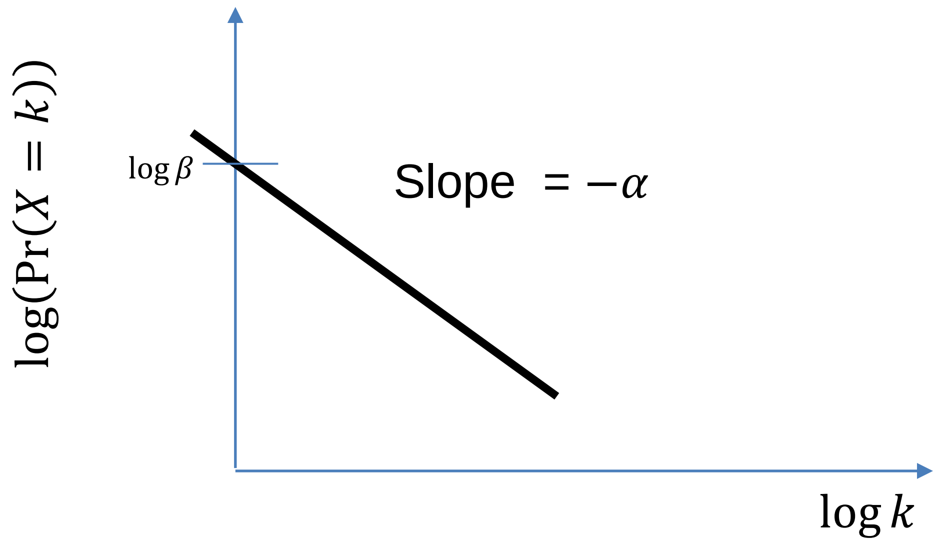
- **heavy tail** distribution: tail decays **polynomially**
  – We can still find huge instances
- Pareto distribution is a special case where

$$\Pr(X = k) \cong k^{-\alpha} \cdot \ \beta \text{ for } \alpha > 1\,, \beta > 0 \text{ (alt., } X_{\min})$$

More Math

- Moving to a log-log graph:

$$\log(\Pr(X = k)) = \log(k^{-\alpha} \cdot \beta) = \log \beta - \alpha \cdot (\log k)$$



Expectation? (1)

- Assume $X \sim Pareto(\alpha, \beta)$ such that $X_{\min} = 1$
- CDF: $F_X(x) = \int_1^x f(x)dx = \int_1^x \beta x^{-\alpha} dx$

$$F_X(\infty) = 1 \qquad \Rightarrow \qquad \int_1^\infty \beta x^{-\alpha} dx = 1$$

$$\Rightarrow \frac{1}{\beta} = \int_1^\infty x^{-\alpha} dx = \frac{1}{-\alpha + 1}\left[x^{-\alpha+1}\right]\Big|_1^\infty \overset{\alpha > 1}{\widehat{=}} \frac{1}{-\alpha + 1}[0 - 1]$$

$$\Rightarrow \beta = \alpha - 1$$

Expectation? (2)

$\Rightarrow \beta = \alpha - 1$

- Next,

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_{1}^{\infty} x \cdot \beta x^{-\alpha} dx = \frac{\beta}{-\alpha + 2} [x^{-\alpha+2}] \Big|_{1}^{\infty}$$

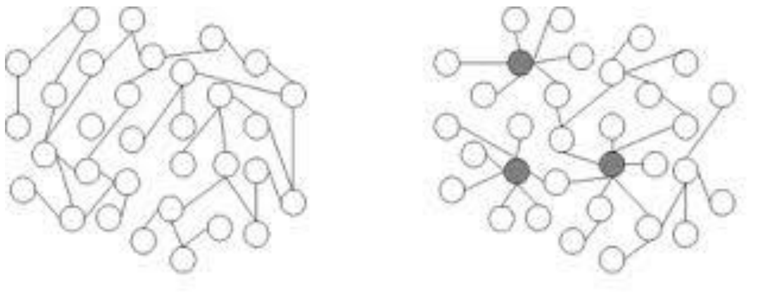
- If $\alpha \leq 2$, we get $\mathbb{E}(X) = \infty$
- Otherwise,

$$\mathbb{E}(X) = \cdots = \frac{\beta}{-\alpha + 2} [x^{-\alpha+2}] \Big|_{1}^{\infty} = \frac{-\beta}{-\alpha + 2} = \frac{\alpha - 1}{\alpha - 2}$$

Even More Math

- The mean of a Power law with parameter α is $\frac{\alpha-1}{\alpha-2}$
  - Only well defined if $\alpha > 2$
- The variance is $\sim \frac{(\alpha-1)}{(\alpha-2)^2(\alpha-3)}$
  - Only well defined if $\alpha > 3$
  - Low α -> heavy tail
- The median is $\sim 2^{\frac{1}{\alpha}}$
  - If $\alpha = 3$, mean=2 and median≈ 1.26
  - If $\alpha = 2.1$, mean=11 and median ≈ 1.39
- β does not matter (scale)

Scale-free Networks



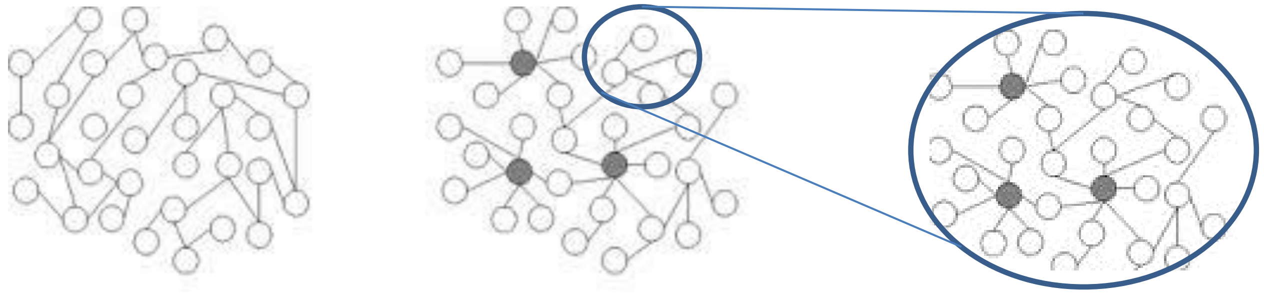
(a) Random network (b) Scale-free network

Degree distribution is ~Binomial

Degree distribution is ~Power law

$$\underset{50}{\overset{100}{\frac{\Pr(c\cdot k_1)}{\Pr(c\cdot k_2)}}} = \frac{(c\cdot k_1)^{-\alpha}\beta}{(c\cdot k_2)^{-\alpha}\beta} = \frac{c^{-\alpha}(k_1)^{-\alpha}\beta}{c^{-\alpha}(k_2)^{-\alpha}\beta} = \frac{(k_1)^{-\alpha}\beta}{(k_2)^{-\alpha}\beta} = \underset{5}{\overset{10}{\frac{\Pr(\,k_1)}{\Pr(\,k_2)}}}$$

Scale-free Networks



(a) Random network (b) Scale-free network

Degree distribution is ~Binomial

Degree distribution is ~Power law

Local “hubs” much more popular than nearby nodes

$$\frac{\Pr(c \cdot k_1)}{\Pr(c \cdot k_2)} = \frac{(c \cdot k_1)^{-\alpha}\beta}{(c \cdot k_2)^{-\alpha}\beta} = \frac{c^{-\alpha}(k_1)^{-\alpha}\beta}{c^{-\alpha}(k_2)^{-\alpha}\beta} = \frac{(k_1)^{-\alpha}\beta}{(k_2)^{-\alpha}\beta} = \frac{\Pr(k_1)}{\Pr(k_2)}$$

Scale-free Distribution

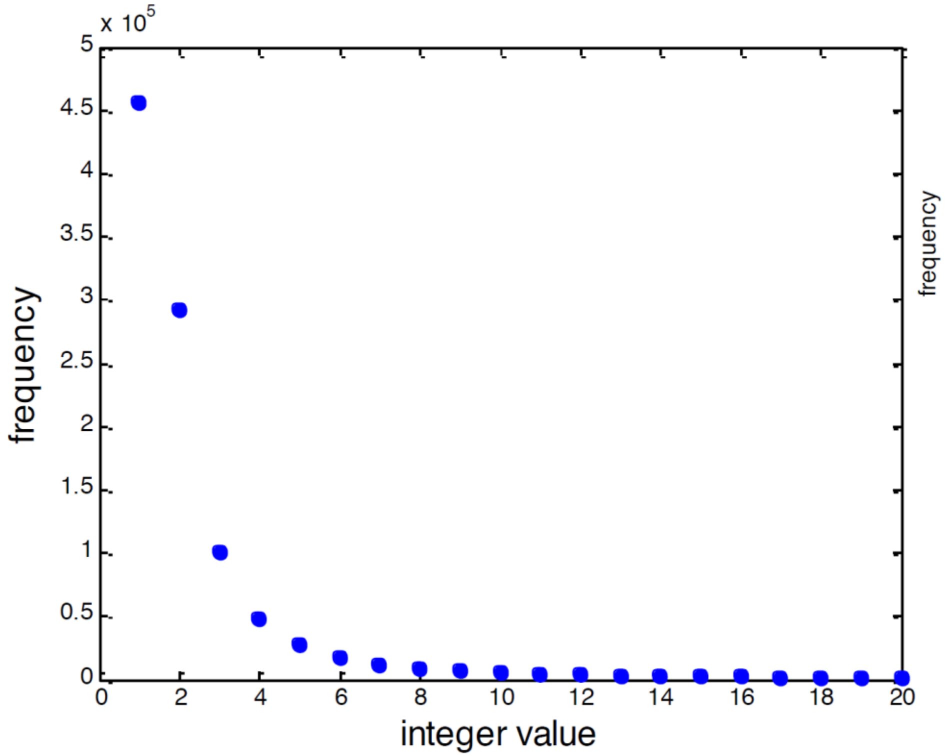
~~**Mediocristan**~~		**Extremistan**	
~~Wealth > 1M~~	~~$\approx 1/62.5$~~	Wealth > 1M	$\approx 1/62.5$
~~Wealth > 2M~~	~~$\approx 1/127,000$~~	Wealth > 2M	$\approx 1/250$
~~Wealth > 4M~~	~~$\approx 1/14,000,000,000$~~	Wealth > 4M	$\approx 1/1000$
~~Wealth > 8M~~	~~$\approx 1/886 \cdot 10^{15}$~~	Wealth > 8M	$\approx 1/4000$
~~Wealth > 16M~~	~~$\approx 1/16 \cdot 10^{33}$~~	Wealth > 16M	$\approx 1/16,000$
~~Wealth > 32M~~	~~$\approx 1/72 \cdot 10^{255}$~~	Wealth > 32M	$\approx 1/64,000$
~~Wealth > 320M~~	~~$\approx 1/??$~~	Wealth > 320M	$\approx 1/6,400,000$

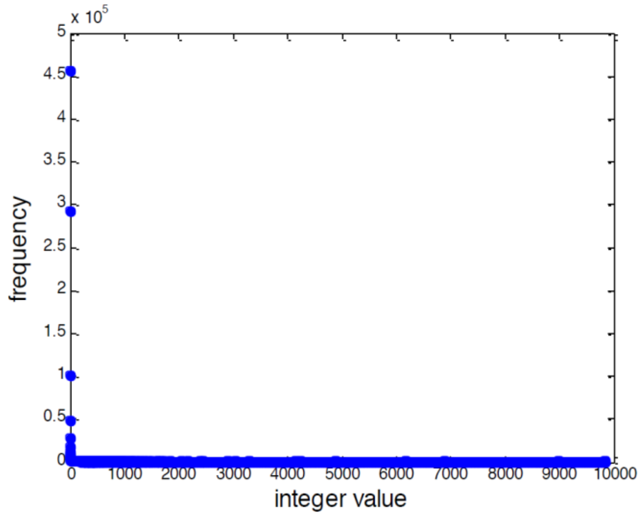
Outline

- Motivation and examples
- Isn’t everything Normal?
- Mathematical representation
- **Parameter estimation**
- Rich getting richer – experiment
- Business considerations
- Page linking process – derivation of power law

Estimating α from Data

- Generate 1M samples from Pareto with $\alpha = 2.5$
- YOU get the data. How to estimate α?

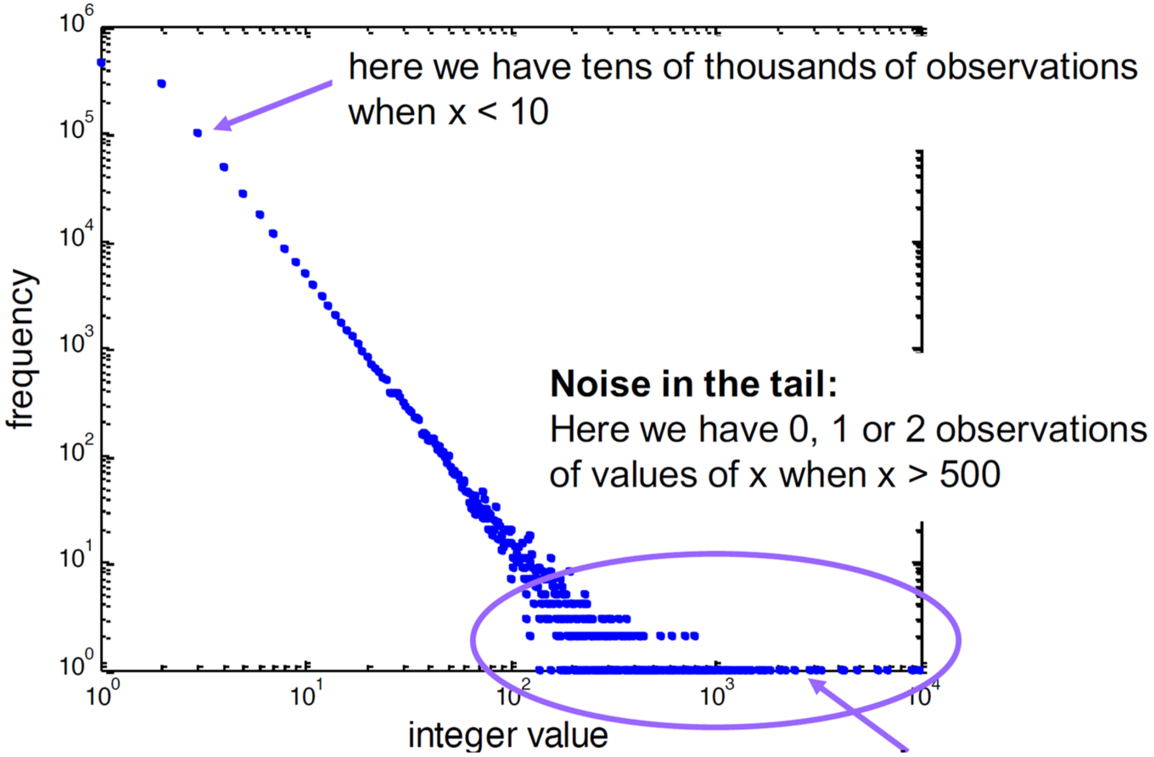




whole range

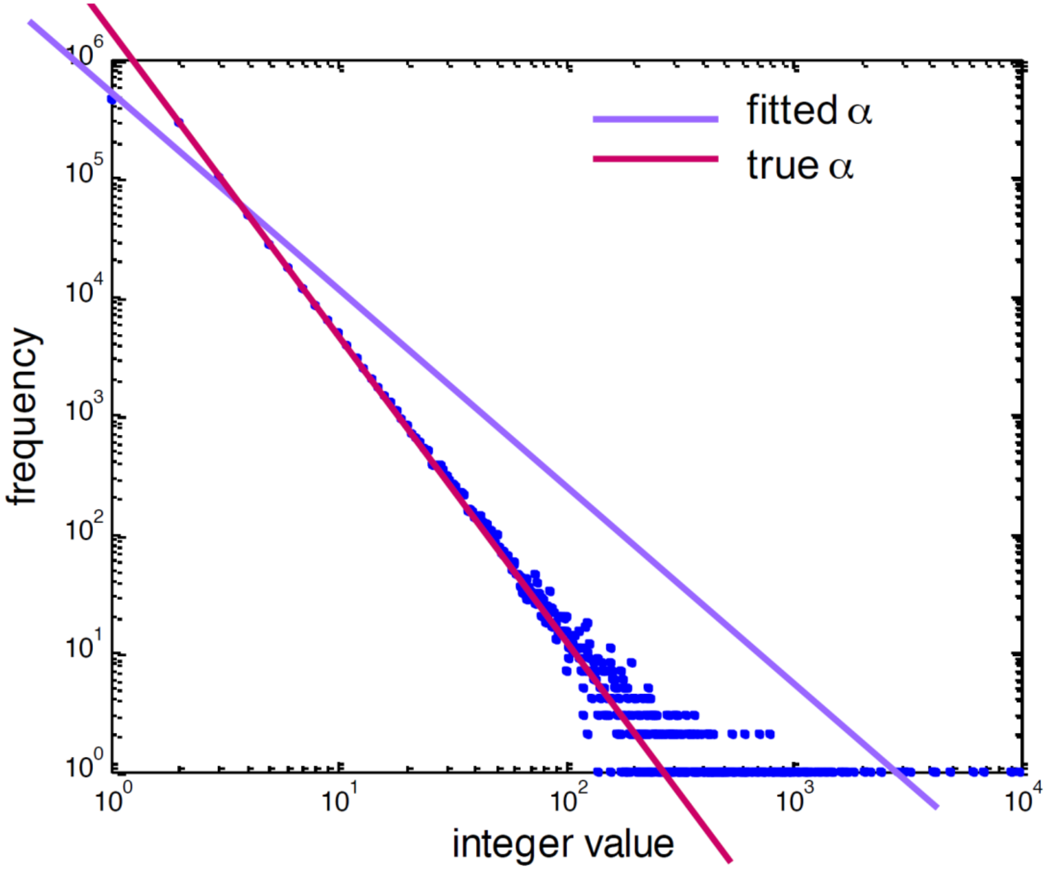
Estimating α from Data

- Generate 1M samples from Pareto with $\alpha = 2.5$
- YOU get the data. How to estimate α?



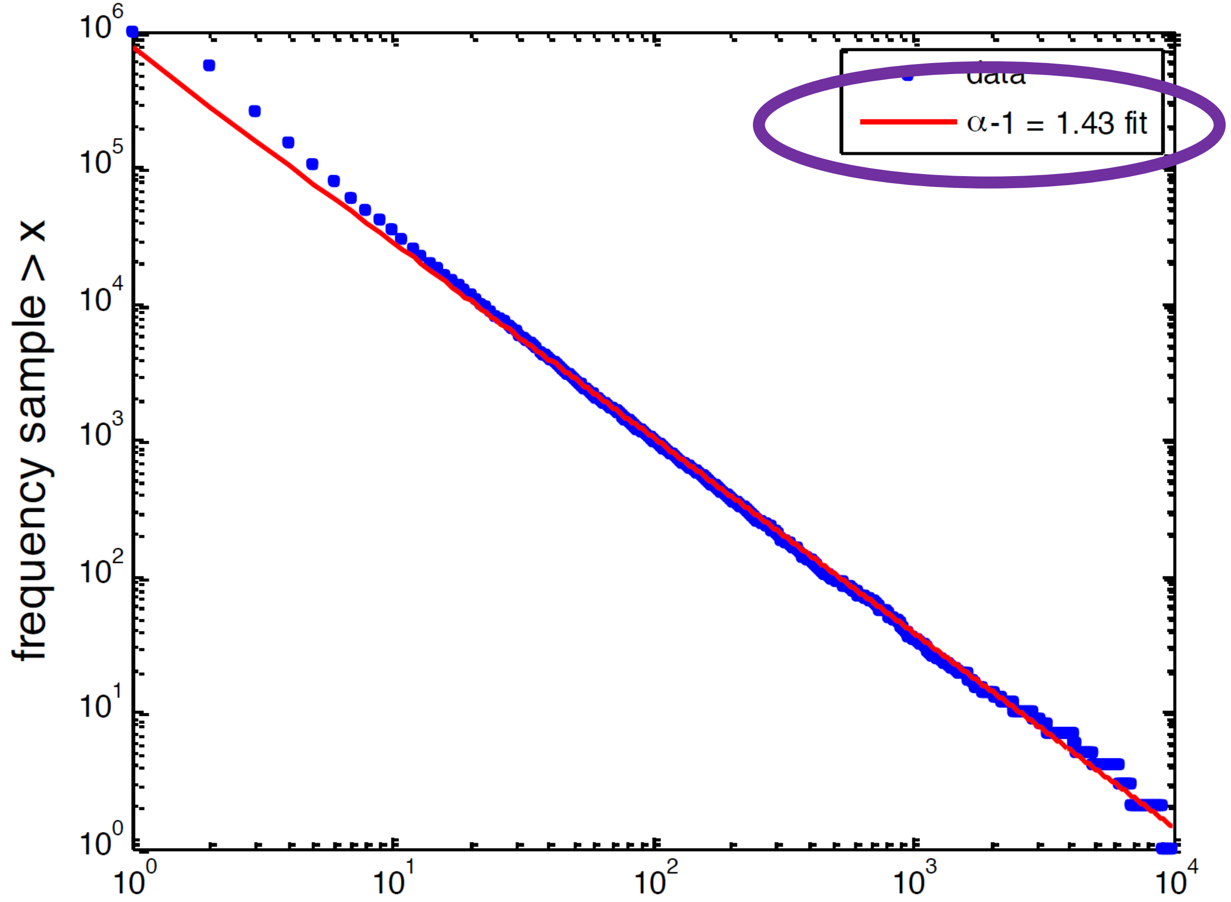
Estimating α from Data

1. BAD idea: fit a line on the log-log axis.



Estimating α from Data

2. OK idea: fit a line on the cumulative-log axis



Estimating α from Data

3. GOOD idea: maximum likelihood estimation (??)

Estimating α from Data

Assume $X \sim Pareto(\alpha, \beta)$ and such that $X_{\min} = 1$

$f_X(x) = (\alpha - 1)x^{-\alpha}$

$$\Rightarrow L(\alpha | x_1, x_2, \dots, x_n) = \prod_{i=1}^{n} f_X(x_i) = \prod_{i=1}^{n} (\alpha - 1)x_i^{-\alpha}$$

$$\Rightarrow l(\alpha | x_1, x_2, \dots, x_n) \coloneqq \ln L(\alpha | x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \ln[(\alpha - 1)x_i^{-\alpha}]$$

$$= n \ln(\alpha - 1) + \sum_{i=1}^{n} \ln[x_i^{-\alpha}] = n \ln(\alpha - 1) - \alpha \sum_{i=1}^{n} \ln[x_i]$$

$$\Rightarrow \frac{dl(\alpha)}{d\alpha} = \frac{n}{\alpha - 1} - \sum_{i=1}^{n} \ln[x_i] = 0 \Rightarrow \alpha = 1 + n \cdot \left(\sum_{i=1}^{n} \ln[x_i] \right)^{-1}$$

Estimating α from Data

3. GOOD idea: maximum likelihood estimation

$$\alpha = 1 + n \cdot \left(\sum_{i=1}^{n} \ln \left[\frac{x_i}{x_{min}} \right] \right)^{-1}$$

For the generated sample, yields $\alpha = 2.503$!

Examples of Estimated α

	x_{min}	exponent α
frequency of use of words	1	2.20
number of citations to papers	100	3.04
number of hits on web sites	1	2.40
copies of books sold in the US	2 000 000	3.51
telephone calls received	10	2.22
magnitude of earthquakes	3.8	3.04
diameter of moon craters	0.01	3.14
intensity of solar flares	200	1.83
intensity of wars	3	1.80
net worth of Americans	$600m	2.09
frequency of family names	10 000	1.94
population of US cities	40 000	2.30

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- Business considerations
- Page linking process – derivation of power law

Experimental Study of Inequality and Unpredictability in an Artificial Cultural Market

Matthew J. Salganik,[1,2*] Peter Sheridan Dodds,[2*] Duncan J. Watts[1,2,3*]

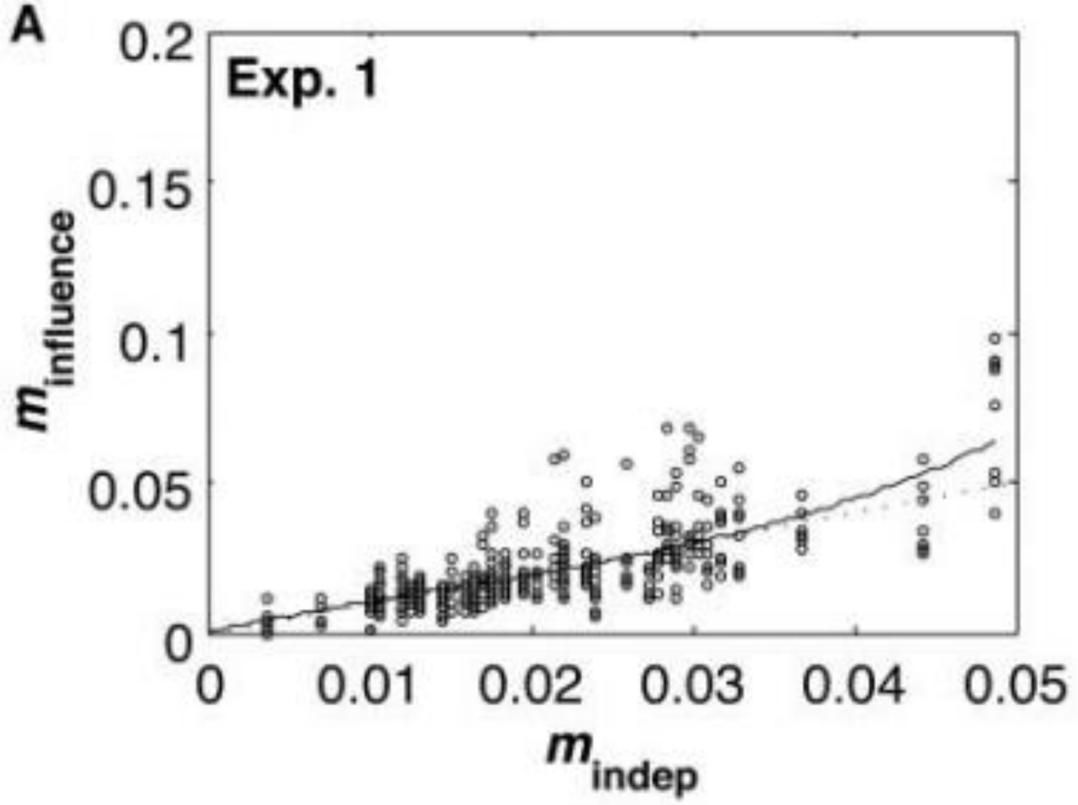
Hit songs, books, and movies are many times more successful than average, suggesting that "the best" alternatives are qualitatively different from "the rest"; yet experts routinely fail to predict which products will succeed. We investigated this paradox experimentally, by creating an artificial "music market" in which 14,341 participants downloaded previously unknown songs either with or without knowledge of previous participants' choices. Increasing the strength of social influence increased both inequality and unpredictability of success. Success was also only partly determined by quality: The best songs rarely did poorly, and the worst rarely did well, but any other result was possible.

Experiment

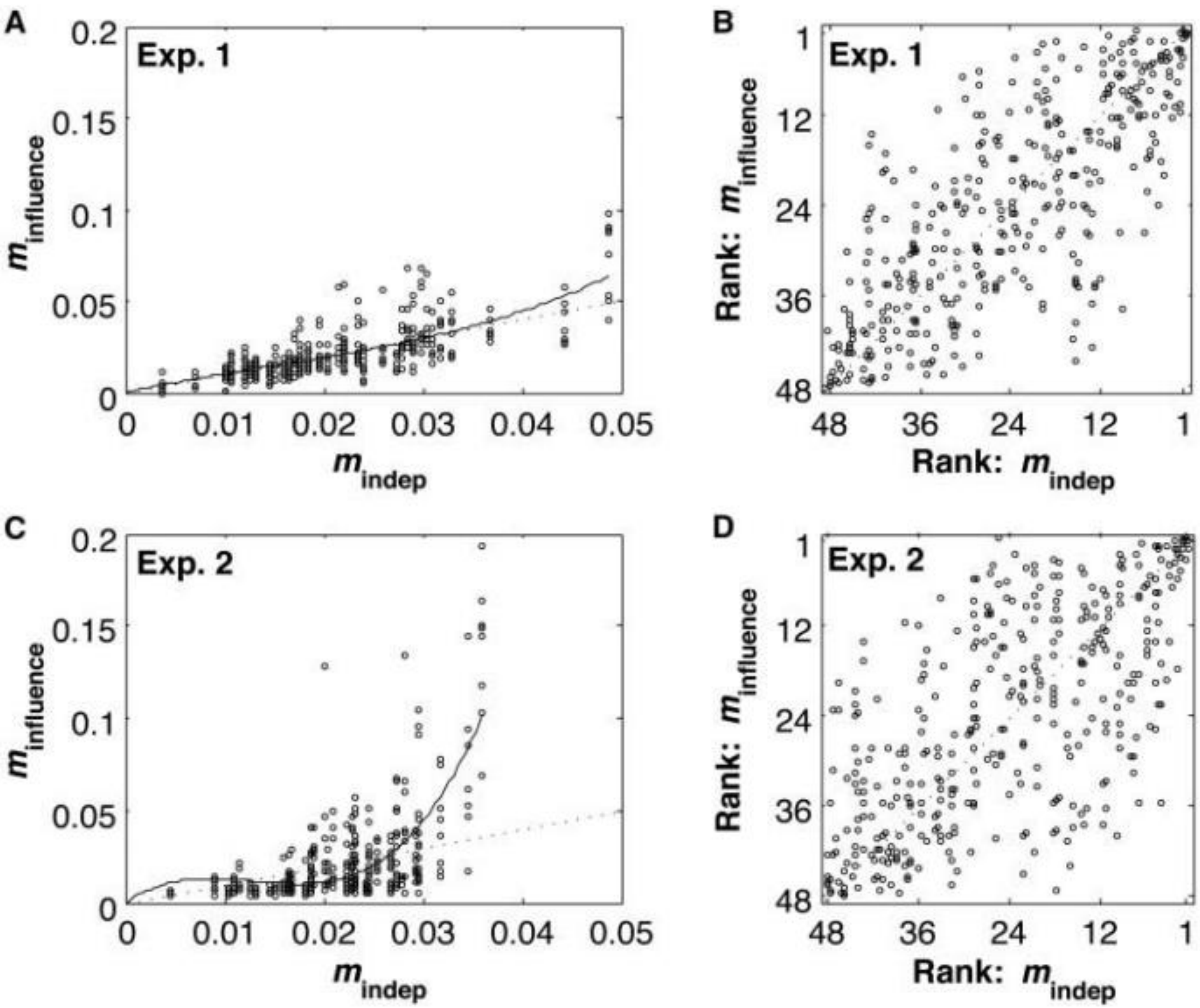
- 14341 participants, 48 songs of unknown bands
  - Participants listen to songs and can download them
- Exp 1: the songs are arranged **randomly** in 16 X 3 grid
  - Social group: the number of downloads per song appear next to it
  - Independent group: no download counts
- Exp 2: the songs are arranged in one column
  - Social group: number of downloads, descending order of current popularity
  - Independent group: no downloads counts, random order
- Social group experiments are executed 8 times
- Market share (downloads) of song i: $d_i / \sum_{k=1}^{48} d_k$

Analyzing the results

- Exp1: random 16X3 grid every time
- Every song has eight $m_{\text{influence}}$ values, and one m_{indep}



Source: Matthew Salganik, Peter Dodds, and Duncan Watts. Experimental study of inequality and unpredictability in an artificial cultural market. Science, 311:854{856, 2006.



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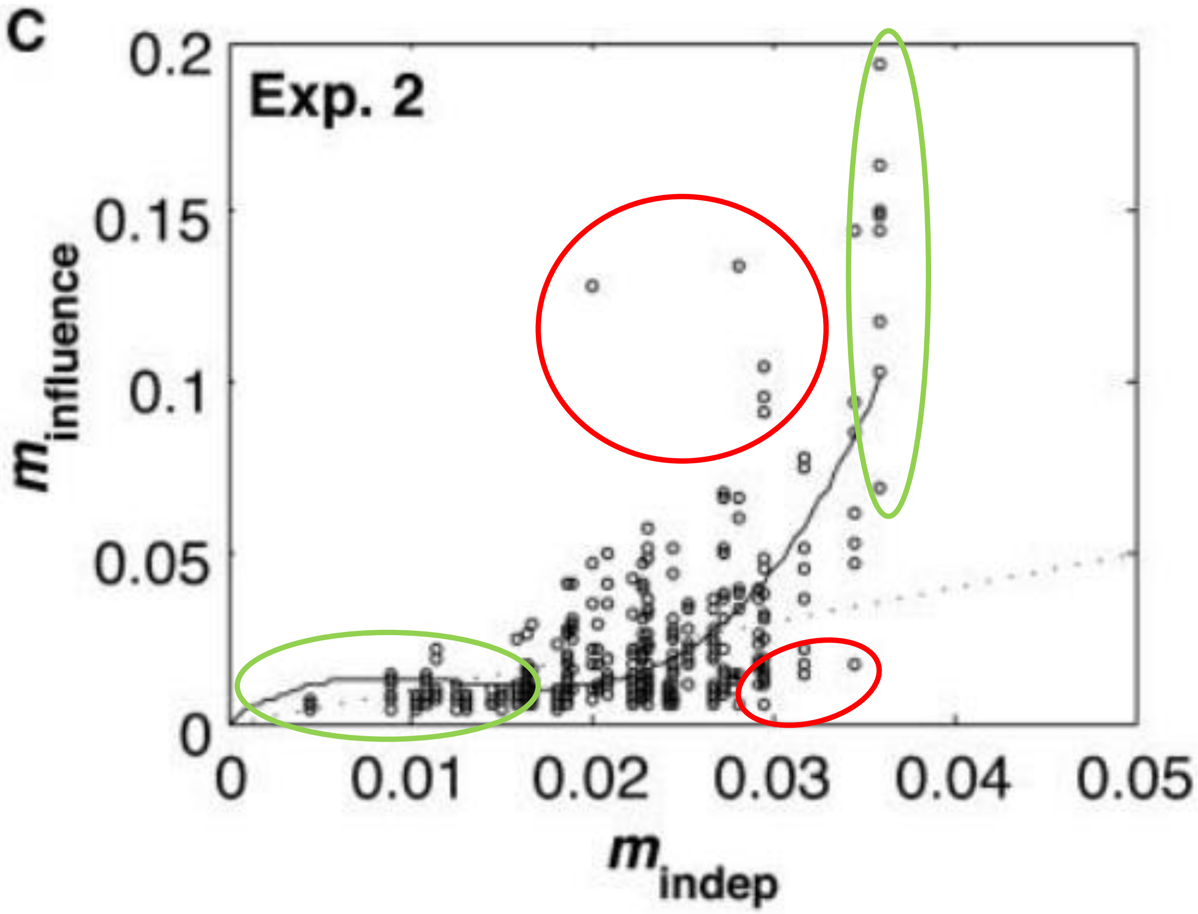

Source: Matthew Salganik, Peter Dodds, and Duncan Watts. Experimental study of inequality and unpredictability in an artificial cultural market. Science, 311:854{856, 2006.

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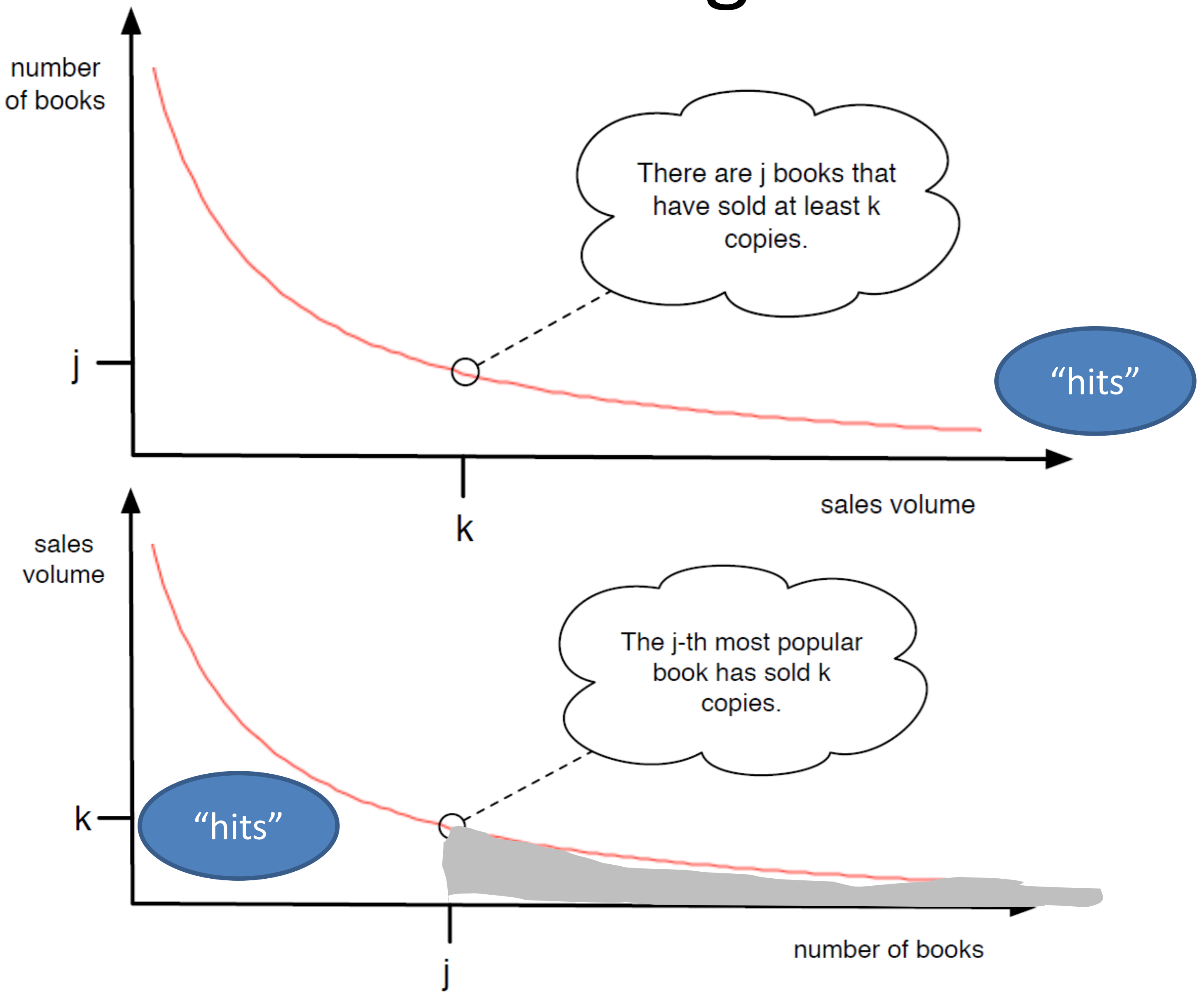
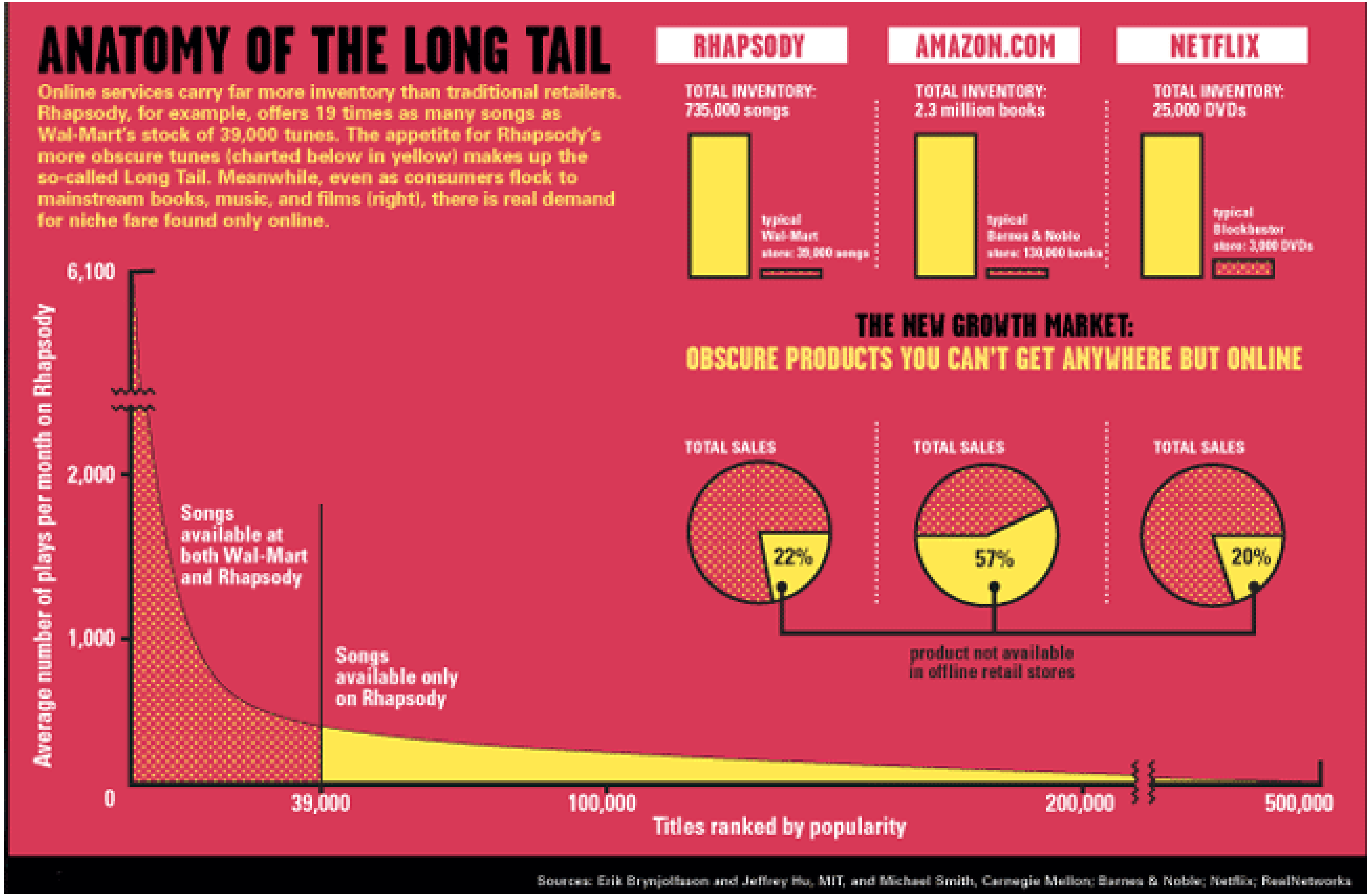
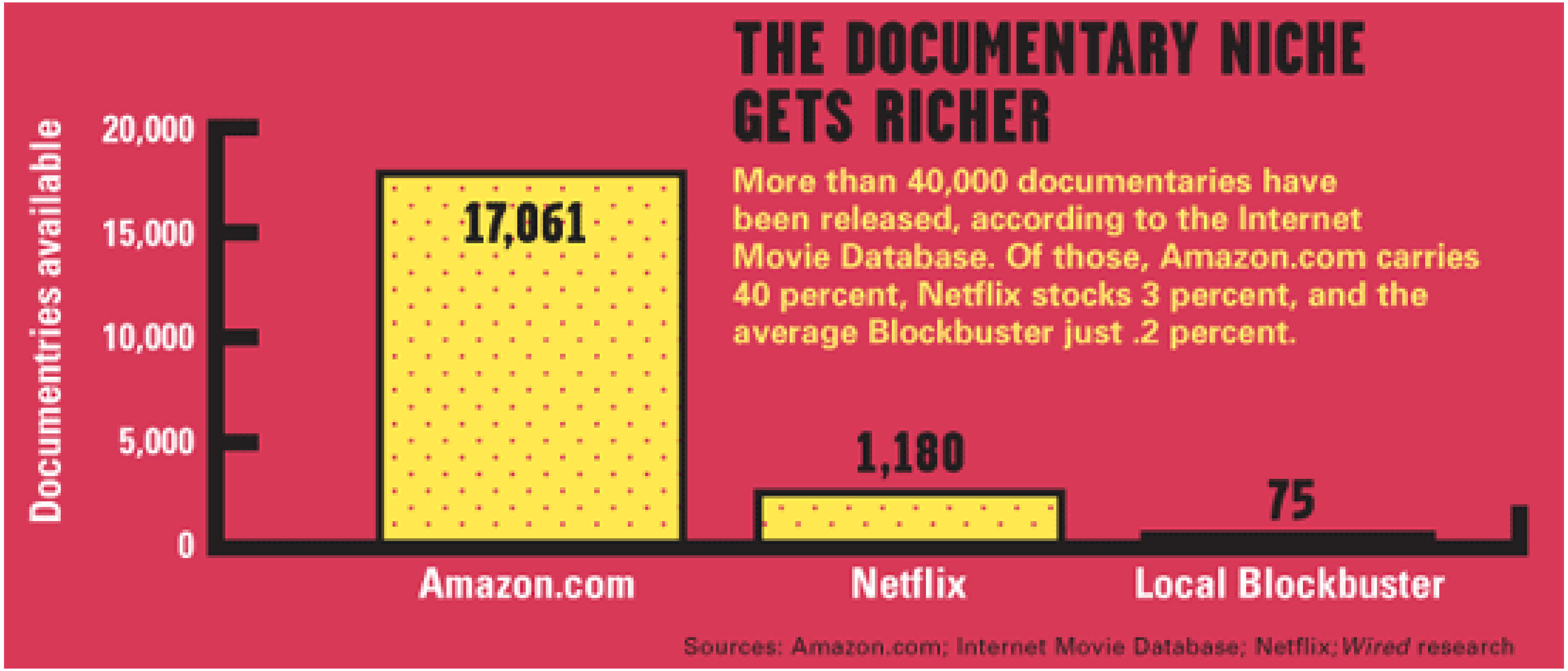
The long tail



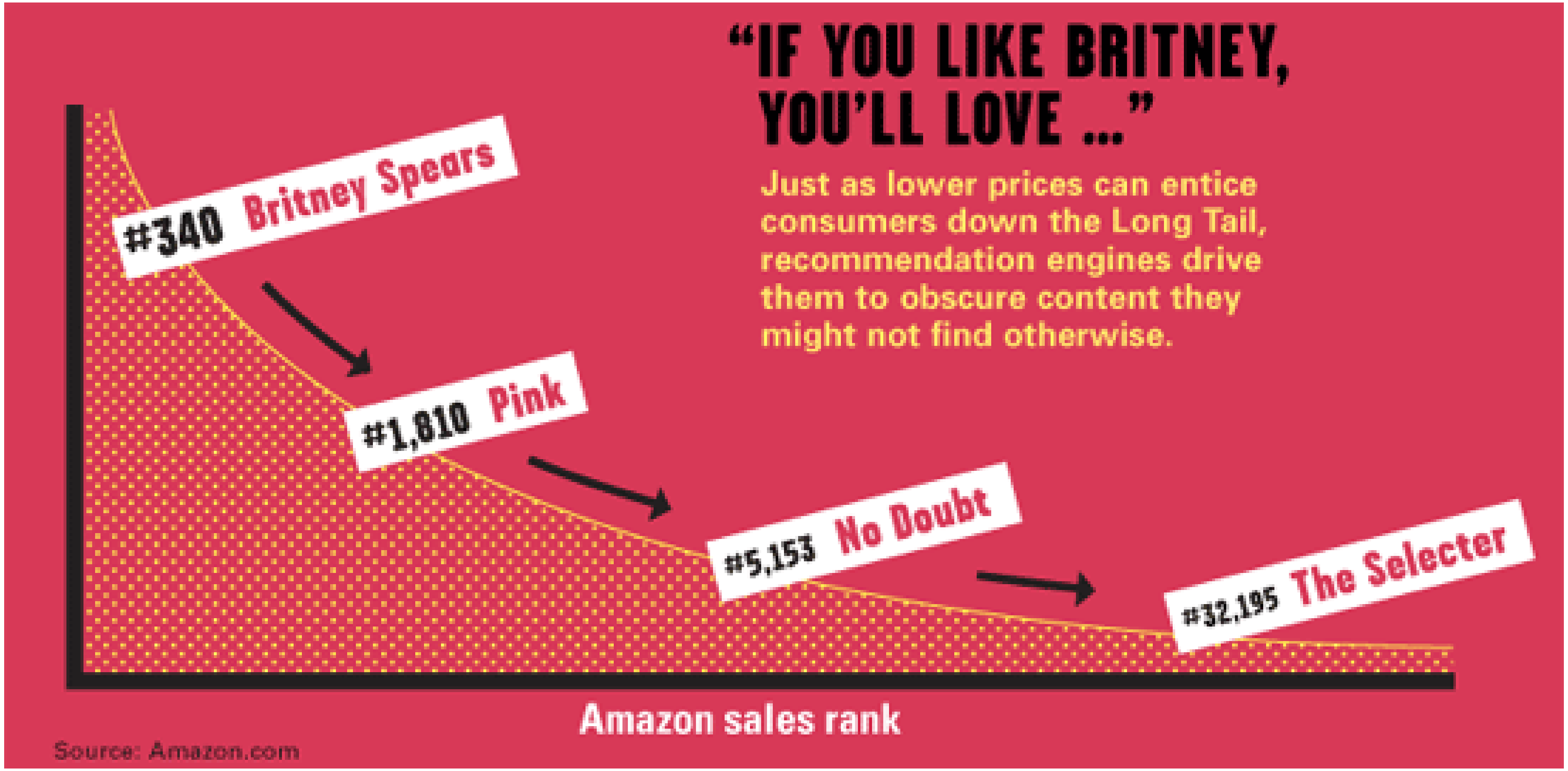


ANATOMY OF THE LONG TAIL
Online services carry far more inventory than traditional retailers. Rhapsody, for example, offers 19 times as many songs as Wal-Mart's stock of 39,000 tunes. The appetite for Rhapsody's more obscure tunes (charted below in yellow) makes up the so-called Long Tail. Meanwhile, even as consumers flock to mainstream books, music, and films (right), there is real demand for niche fare found only online.
RHAPSODY
TOTAL INVENTORY: 735,000 songs
typical Wal-Mart store: 39,000 songs
AMAZON.COM
TOTAL INVENTORY: 2.3 million books
typical Barnes & Noble store: 130,000 books
NETFLIX
TOTAL INVENTORY: 25,000 DVDs
typical Blockbuster store: 3,000 DVDs
THE NEW GROWTH MARKET:
OBSCURE PRODUCTS YOU CAN'T GET ANYWHERE BUT ONLINE
TOTAL SALES
22%
TOTAL SALES
57%
TOTAL SALES
20%
product not available in offline retail stores
Average number of plays per month on Rhapsody
6,100
2,000
1,000
0
Songs available at both Wal-Mart and Rhapsody
Songs available only on Rhapsody
39,000
100,000
200,000
500,000
Titles ranked by popularity
Sources: Erik Brynjolfsson and Jeffrey Hu, MIT, and Michael Smith, Carnegie Mellon; Barnes & Noble; Netflix; RealNetworks

Playing the Long Tail Game



Rec. Engines Push Heavy Tail to be Heavier



שאלה ממבחן

סעיף 3 (חמש נקודות)

אבאזון הינה חנות אינטרנטית עם מליוני ספרים. נמספר את הספרים כך שהספר המבוקש ביותר, כלומר זה שממנו נמכרו הכי הרבה עותקים הוא מס' 1, השני המבוקש ביותר הוא מס' 2, וכן הלאה. כל לקוחה שנכנסת לחנות קונה ספר אחד בדיוק, כאשר ההסתברות שהיא תרצה את הספר ה i היא בקירוב $\frac{1}{i^2} \cdot \frac{6}{\pi^2}$.

המנהלת של אבאזון רוצה לפתוח דוכן בבית הסטודנט עם היצע קטן של ספרים. הניחו כי כל לקוחה מעוניינת לרכוש ספר אחד בדיוק. כמו כן, הניחו כי כל לקוחה שמגיעה לדוכן החליטה כבר לפני הגעתה באיזה ספר היא מעוניינת, מבין כלל הספרים המוצעים למכירה <u>בחנות האינטרנטית</u> – $1,2,3,\ldots$, לפי אותה התפלגות של החנות האינטרנטית.

בהיותה ערה לחוקי הפופולריות ול rich-getting-richer, מנהלת אבאזון סבורה כי ניתן להחזיק רק מספר קטן של ספרים שונים בדוכן. מה מספר הספרים השונים המינימלי שהמנהלת צריכה להחזיק בדוכן, כך שלקוחה הראשונה שתגיע לדוכן תמצא את הספר שהיא בחרה מראש בהסתברות של 90% לפחות? יש לכתוב את מספר הספרים השונים בתוך המלבן מטה.

Outline

- Motivation and examples
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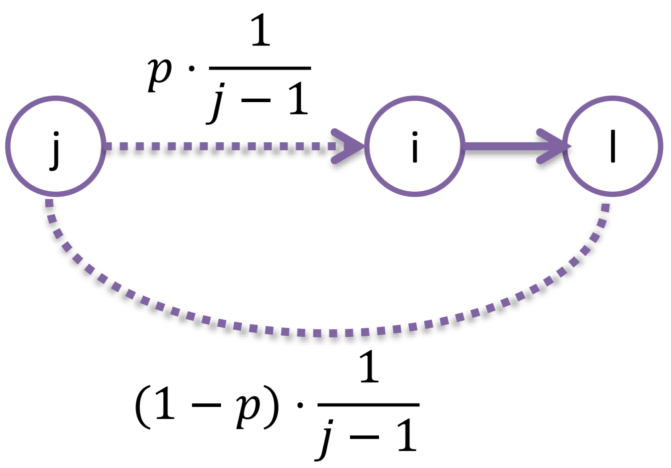
- **Page linking process – derivation of power law**

Feedback Process

- Pages are created in order, $1,2, \ldots, N$
- When page j is created, it produces a link to an earlier page:
  - W.p. p, page j selects a page i u.a.r. and creates a link to it
  - W.p. $1-p$, page j selects a page i u.a.r. and creates a link to the page i points to



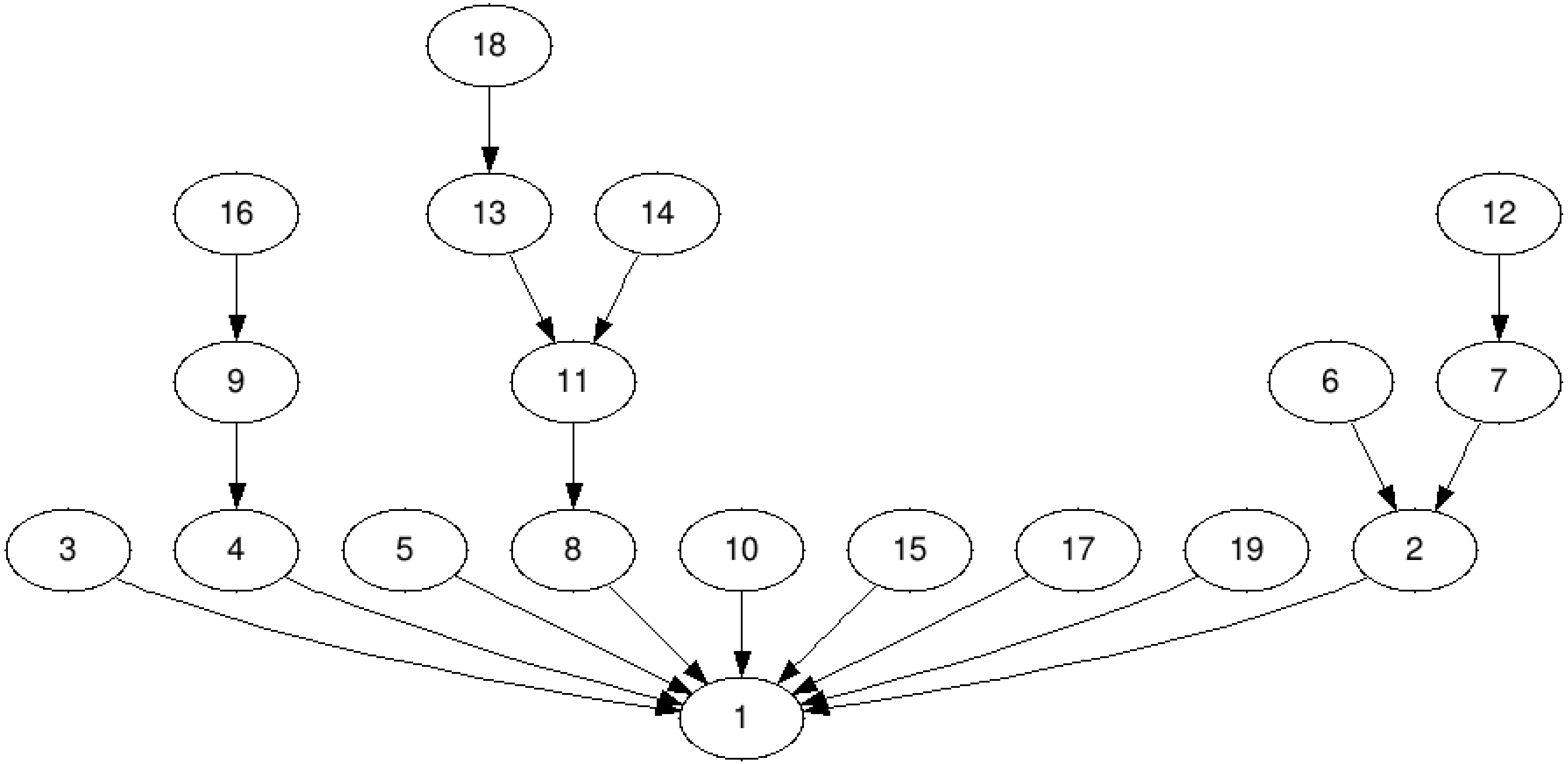
- Distribution of in-degree?

סימולטור רשתות חברתיות וחוקי חזקה

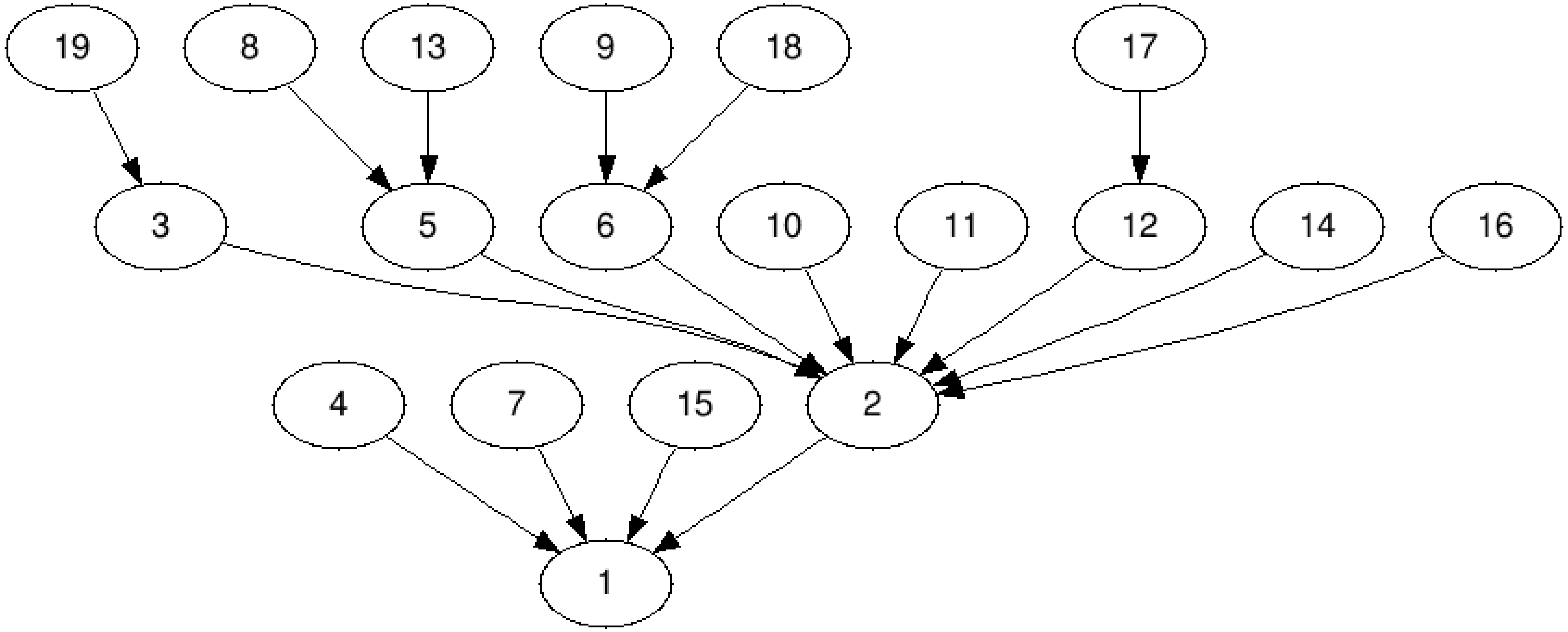
- הקוד שמייצר את הגרף נמצא בקישור הזה
- אתם יכולים לשחק עם גודל הגרף N ועם ההסתברות p ולראות איך הם משפיעים על ההתפלגות.
- בחלון בצד ימין יופיעו הלוג של הריצה
- כדי לראות את הגרף, העתיקו את כל הרשימה לחלון בקישור הבא (שימו לב שלא ניתן להציג יותר מכמה עשרות קודקודים)

- (תודה לפרופ' רשף מאיר על הכנת החומרים)

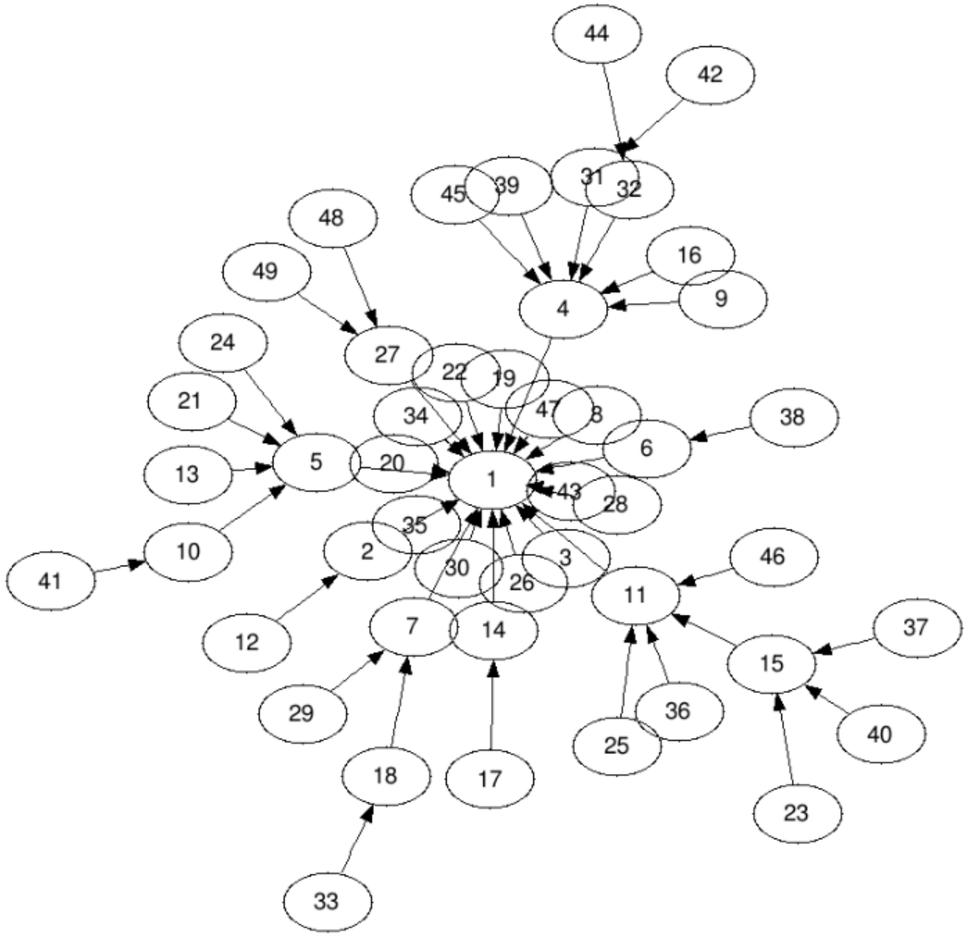
Example 1



Example 2



Example 3



44
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23
33

Analysis

- **Theorem**: The in-degree distribution is power law with
$$\alpha = 1 + \frac{1}{1-p}$$
- Let $X_l(t)$ be the rank of page l at time t
  - $X_j(t) = 0$ for $t \leq j$
- Observation: the second stage (w.p. 1-p,…) is equivalent to selecting i w.p.
$$\frac{X_i(j-1)}{j-1}$$
- True since
  - W.p. $1-p$, page j selects a page i u.a.r. and creates a link to the page i points to

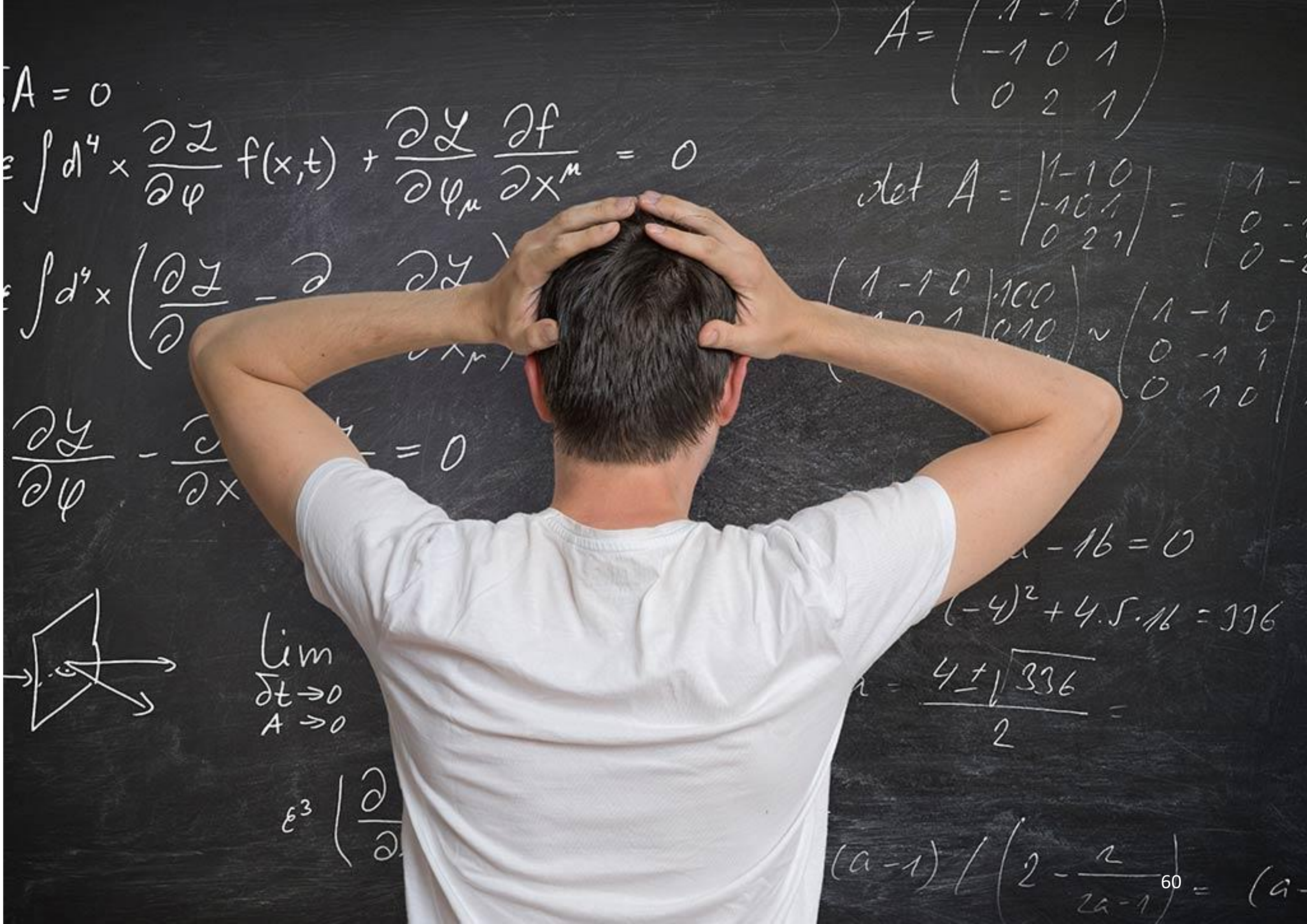
Analysis

- Observation: for $t+1 \geq j$, the probability of $t+1$ linking to j is

$$\Pr[t+1 \text{ links to } j] = \frac{p}{t} + \frac{qX_j(t)}{t}$$

$q = (1-p)$

- This is a random process!
- From here on, the proof gets complicated ☺
- Deterministic approximation
  – A few years later it was shown to be accurate



Analysis

- Deterministic approximation
  - Instead of discrete time steps, continuous ones
  - Instead of discrete jumps (added links), continuous ones
  - Still finitely many pages!
- New model: using x and not X
- For $j = 1, \dots N, t = [0, N]$

$$\frac{dx_j}{dt} = \Delta x_j(t) = \frac{p}{t} + \frac{q x_j(t)}{t}$$

- Differential equation!☹

Analysis

- New model:

$$\frac{dx_j}{dt} = \Delta x_j(t) = \frac{p}{t} + \frac{q x_j(t)}{t}$$

- Solving,

$$x_j(t) = \begin{cases} \frac{p}{q}\left[\left(\frac{t}{j}\right)^q - 1\right] & \text{if } t > j \\ 0 & \text{if } t \le j \end{cases}$$

- Not too hard, see book materials in Moodle

Analysis

- Solution: $x_j(t) = \frac{p}{q}\left[\left(\frac{t}{j}\right)^q - 1\right]$
- Number of nodes with in-degree greater than k at time N? $\{j: x_j(N) \geq k\}$
- The solution is every j such that

$$x_j(N) \geq k \text{ iff j} \leq N\left[\frac{q}{p} \cdot k + 1\right]^{-1/q}$$

- And the proportion is given by

$$1 - F(k) \equiv \Pr_{j\sim uni[N]}[x_j(N) \geq k] = \frac{1}{N} N \left[\frac{q}{p} \cdot k + 1\right]^{-1/q} = \left[\frac{q}{p} \cdot k + 1\right]^{-1/q}$$

Analysis

- Therefore,

$$f(k) \equiv \Pr_{j\sim uni[N]}\left[x_j(N) = k\right] = \frac{d}{dk}F(k) =- -\frac{1}{q}\frac{q}{p}\left[\frac{q}{p}\cdot k + 1\right]^{-1-\frac{1}{q}} = \frac{1}{p}\left[\frac{q}{p}\cdot k + 1\right]^{-1-\frac{1}{q}}$$
$$\propto k^{-1-\frac{1}{q}}$$

- Recall power law distributions have probability $k^{-\alpha}$
- What is α?

$$-\alpha = -1 - \frac{1}{q} =\Leftrightarrow \alpha = 1 + \frac{1}{q} = 1 + \frac{1}{1-p}$$

- Intuition: $p \to 1$ is like choosing at random
- Intuition: $p \to 0$ is herding. The power is close to 2.

Electronic Commerce 096211

Web Ranking, H&A, PageRank

Omer Ben Porat (some of the slides are adopted from Prof. Reshef Meir)

Course Structure

Online Societies
- Web ranking
- Diffusion
- Structure of the web
- Power laws
- Networks

Advertising
- Auctions
- Matching markets
- Sponsored search

Decision Making
- Recommendation systems
- Multi-armed bandits

Advance
- Information in markets
- Externalities
- Streaming
- Blockchain

Tools and Techniques
- Game theory
- Algorithms
- Graph theory
- Optimization

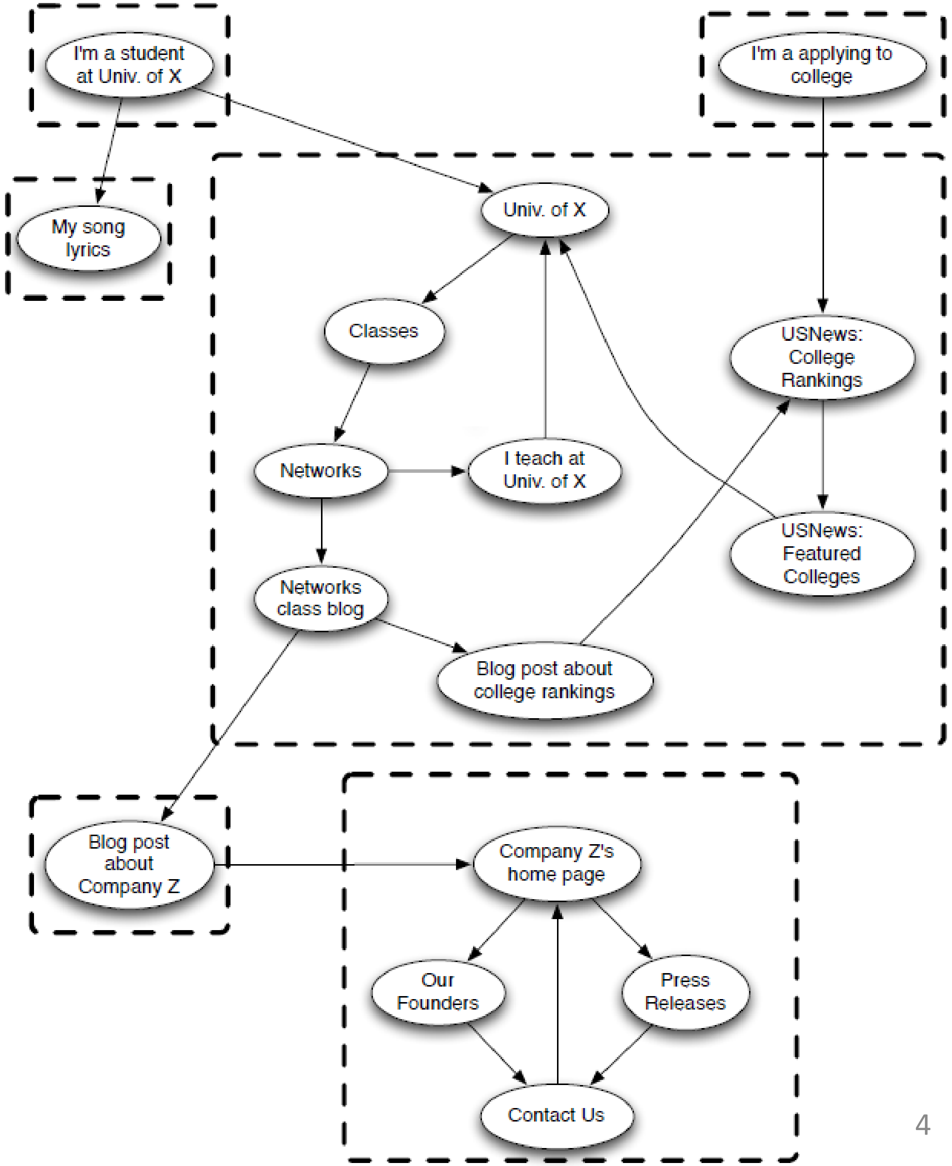
Outline

- Main question today:

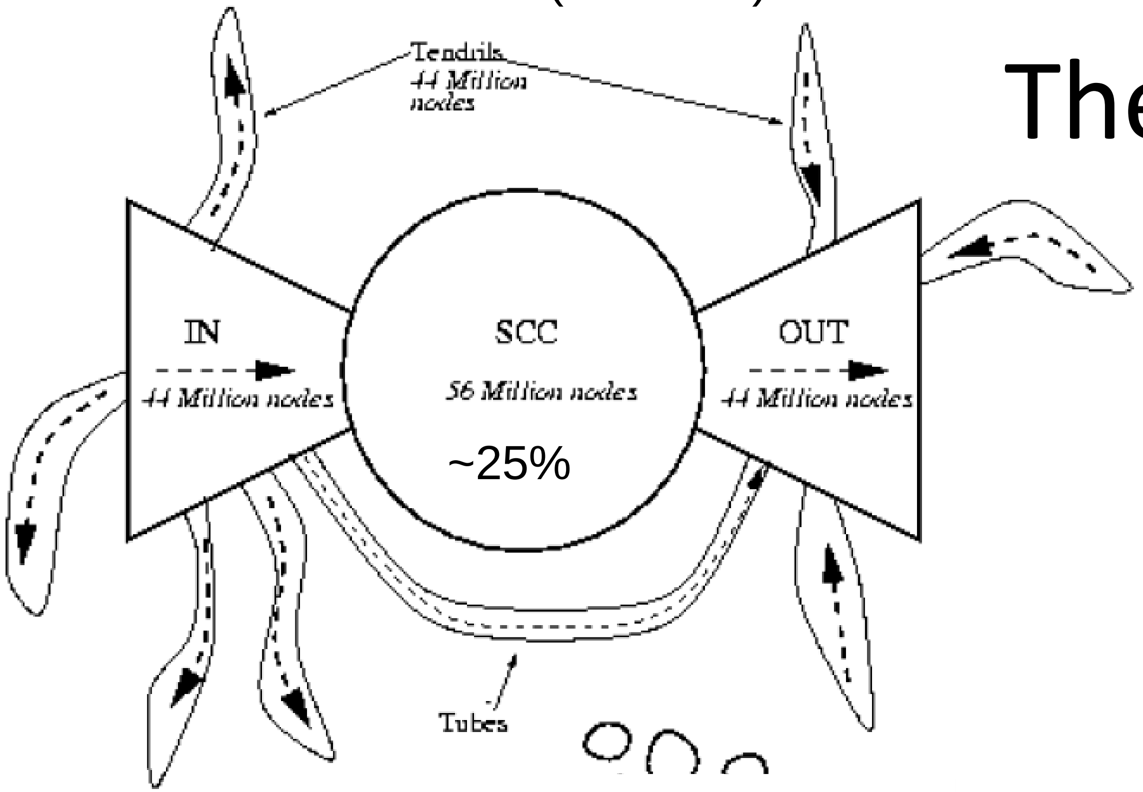
  How to determine if a webpage is important?
- Structure of the web
- Centrality measures in the Web
  - Hubs and Authorities
  - PageRank

Directed graphs

- The Web is a directed graph
- Recall some definitions:
  - Path
  - Strongly connected components
- What is the structure of the actual Web?
- Is there a "giant component"?

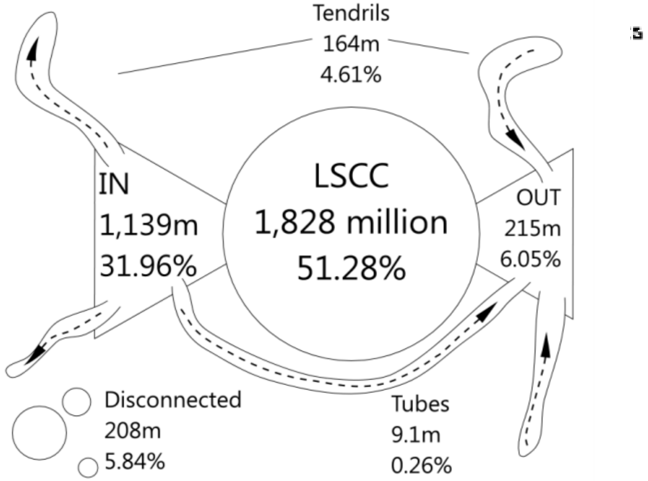


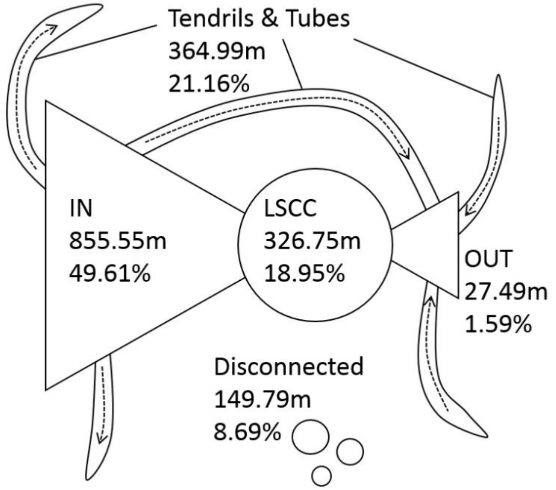
Year 2000 (AltaVista)

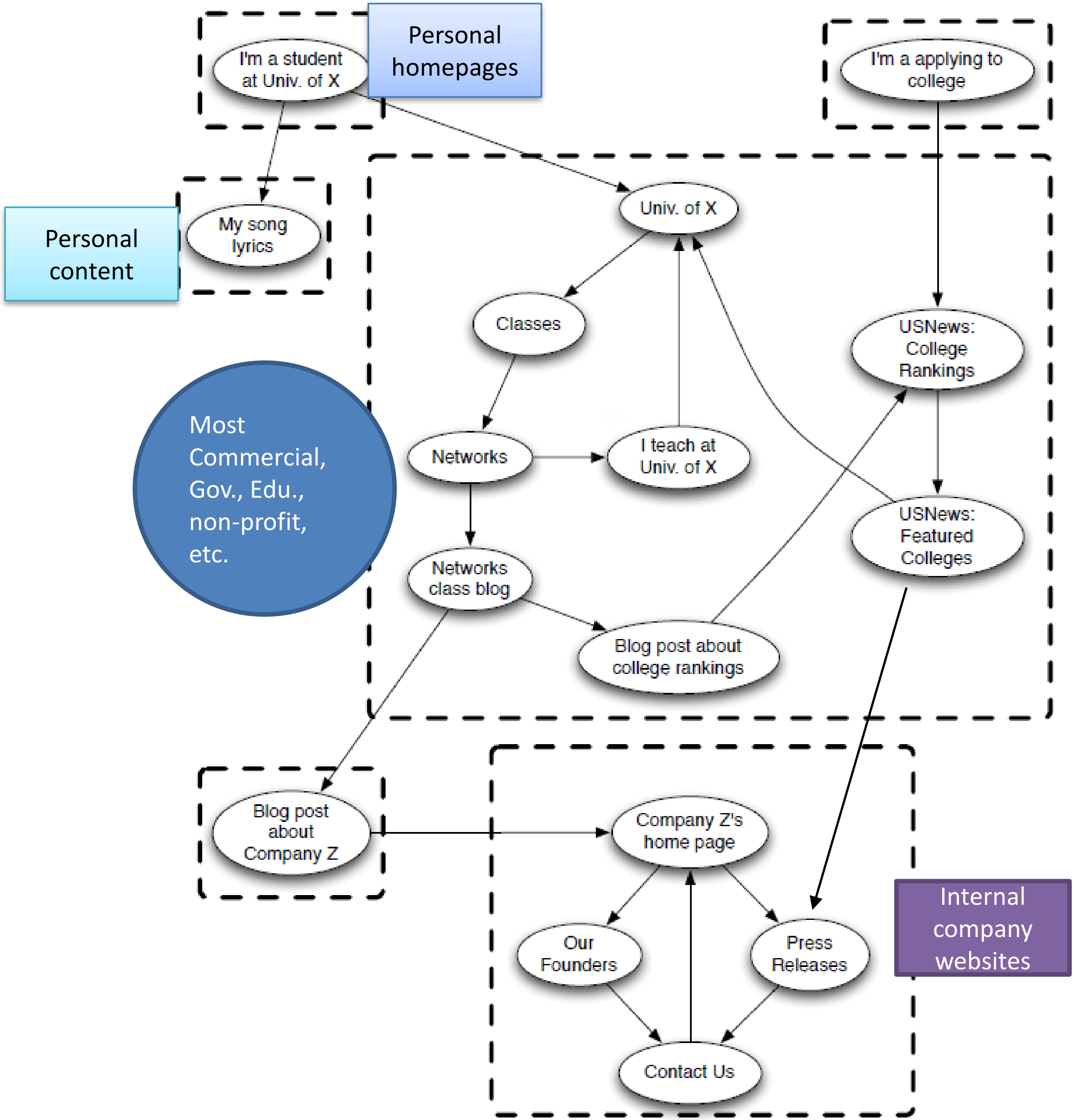


The Bow-Tie

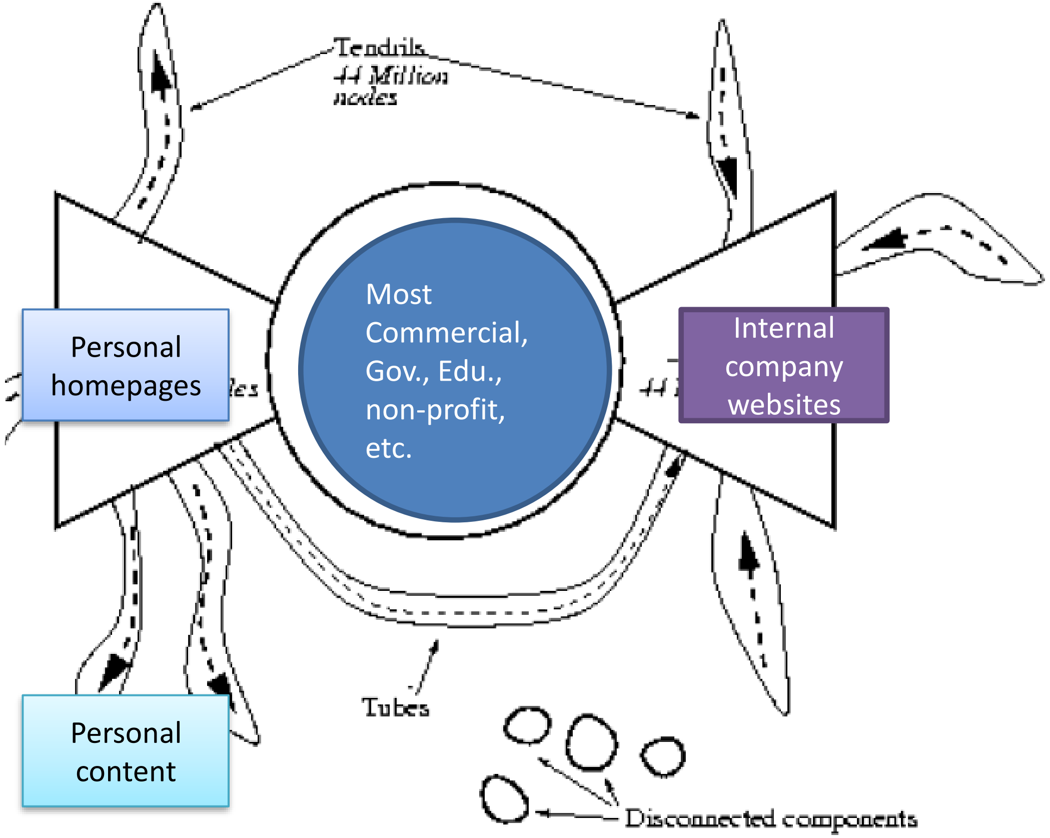
In 2014:







Personal homepages
I'm a student at Univ. of X
I'm a applying to college
Personal content
My song lyrics
Univ. of X
Classes
USNews: College Rankings
Most Commercial, Gov., Edu., non-profit, etc.
Networks
I teach at Univ. of X
USNews: Featured Colleges
Networks class blog
Blog post about college rankings
Blog post about Company Z
Company Z's home page
Our Founders
Press Releases
Contact Us
Internal company websites



Tendrils
44 Million nodes
Personal homepages
Most Commercial, Gov., Edu., non-profit, etc.
Internal company websites
Personal content
Tubes
Disconnected components

HOW TO SEARCH FOR INFORMATION?

Search and Reputation

- Information search - an old problem
  - Patents
  - Legal decisions
  - Scientific papers
  - Newspaper articles
- Main problem: a keyword may not be informative enough
  - Synonymy: “Green onion”
  - Polysemy: “Jaguar”

“scallion”

?

?

Web search

- Recall the WWW properties:
  - More content
  - Authors are not (all) professionals
    - Do not adhere to style and classification
    - May even be adversarial

    (competitors, conspiracy theorists)
  - Searchers are not (all) professionals
    - No special training
  - The meaning changes rapidly
- **Idea**: use the network structure

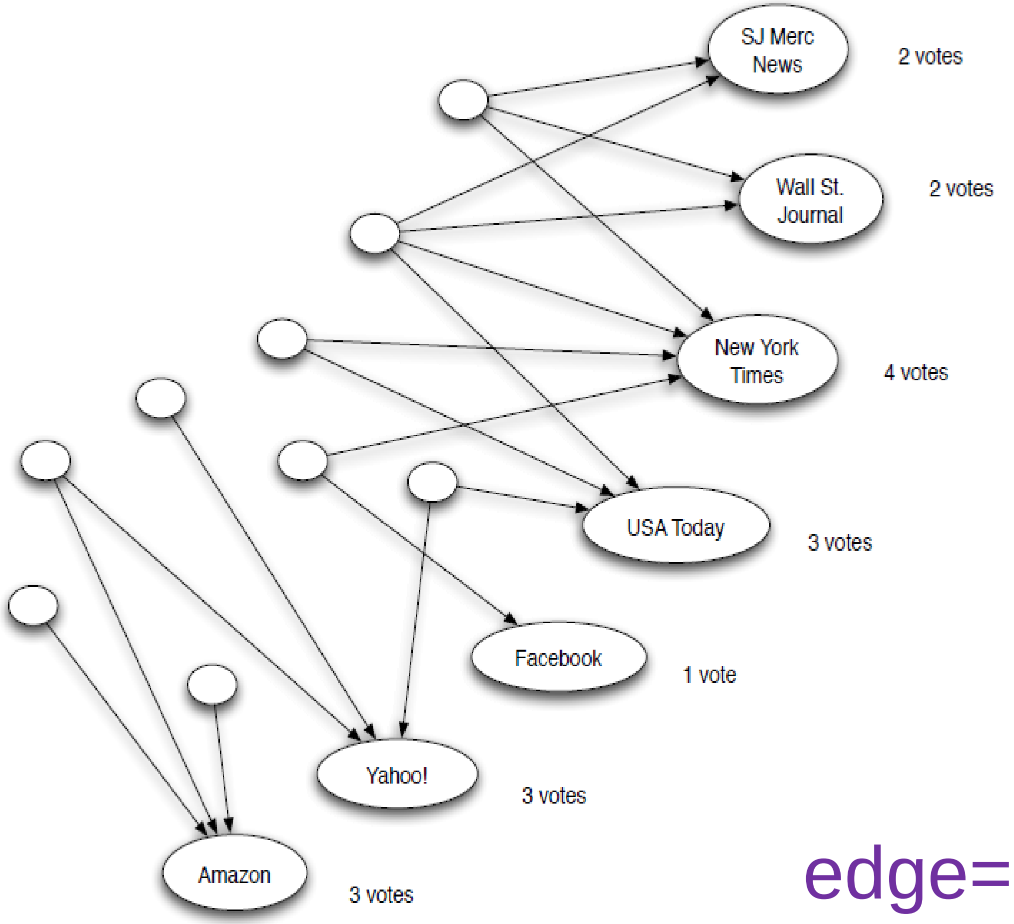
Measuring and Ranking Importance

- First: separate the question of “relevant pages” from “important pages”
  - i.e., we will ignore synonyms and polysemy
- Suppose we already filtered all relevant pages
- Which ones are the most important?
- Rank by order of importance
  - Approach #1: Hubs and Authorities
  - Approach #2: PageRank

Hubs and Authorities

רשויות ורכזים

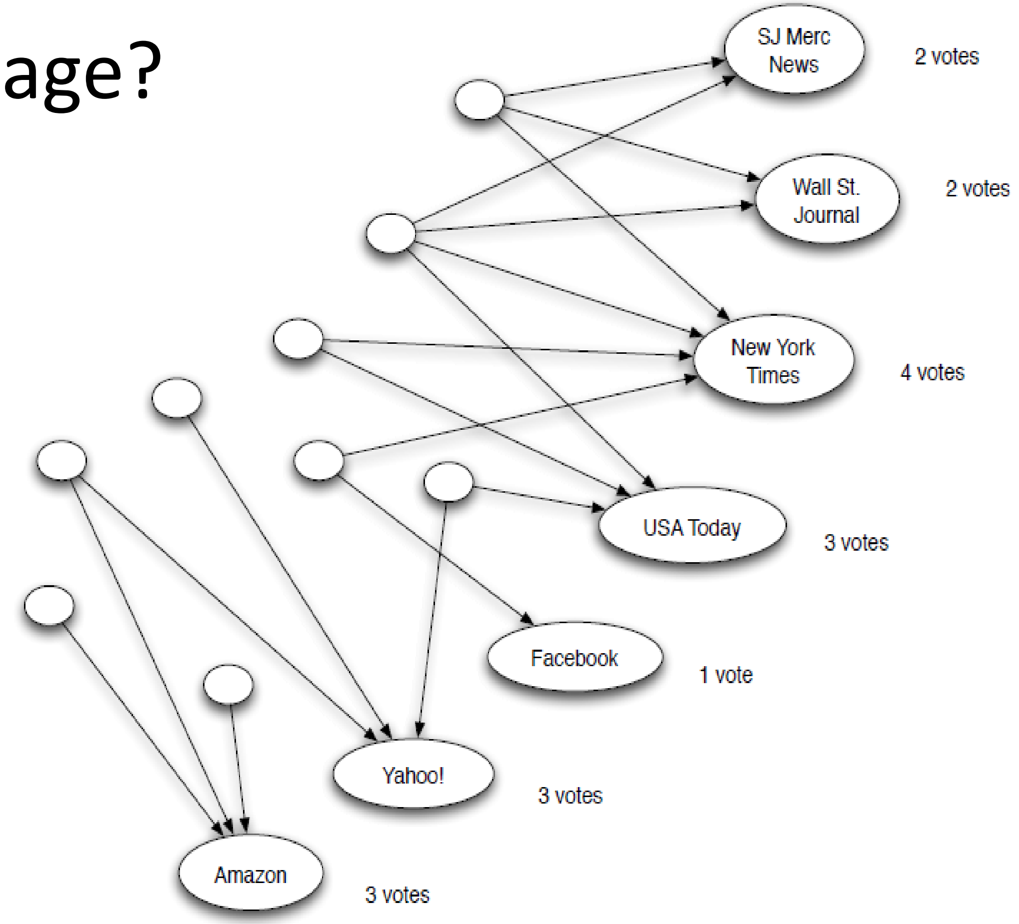
Information



edge=link

Information

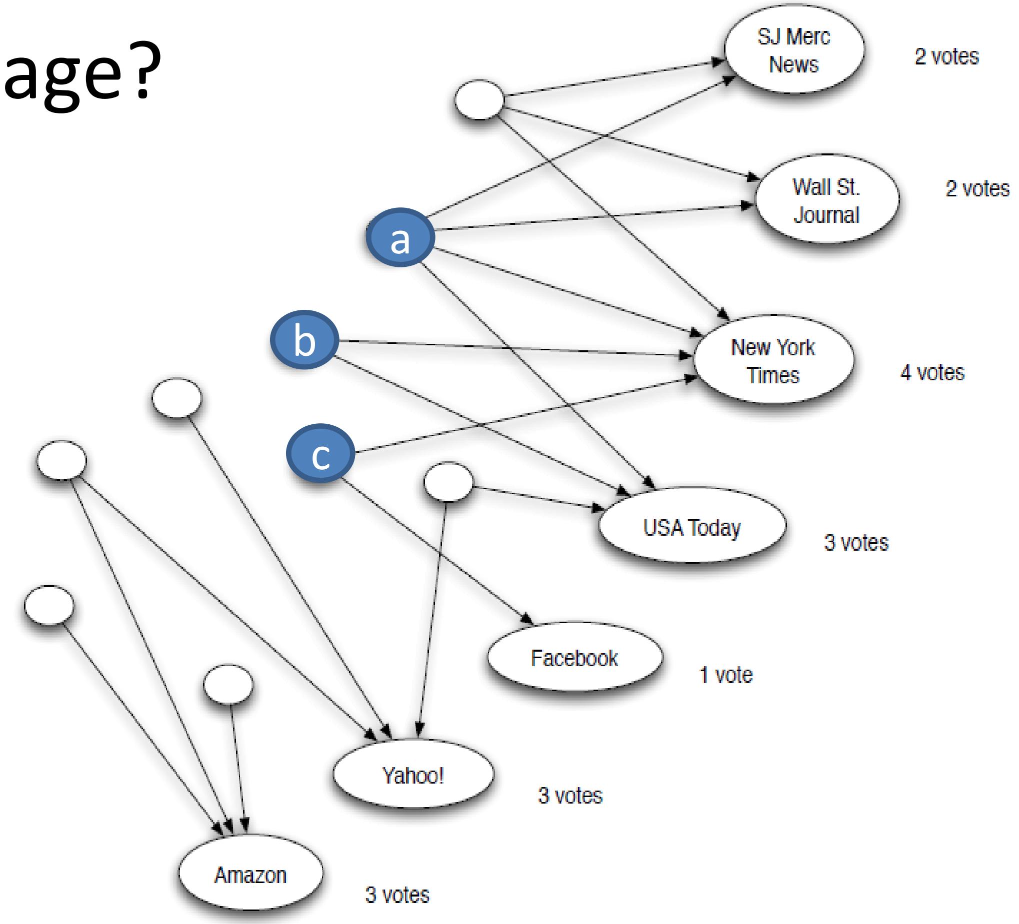
- What is an important webpage?
- Attempt:
  – use in-degree
  – "count votes"



Information

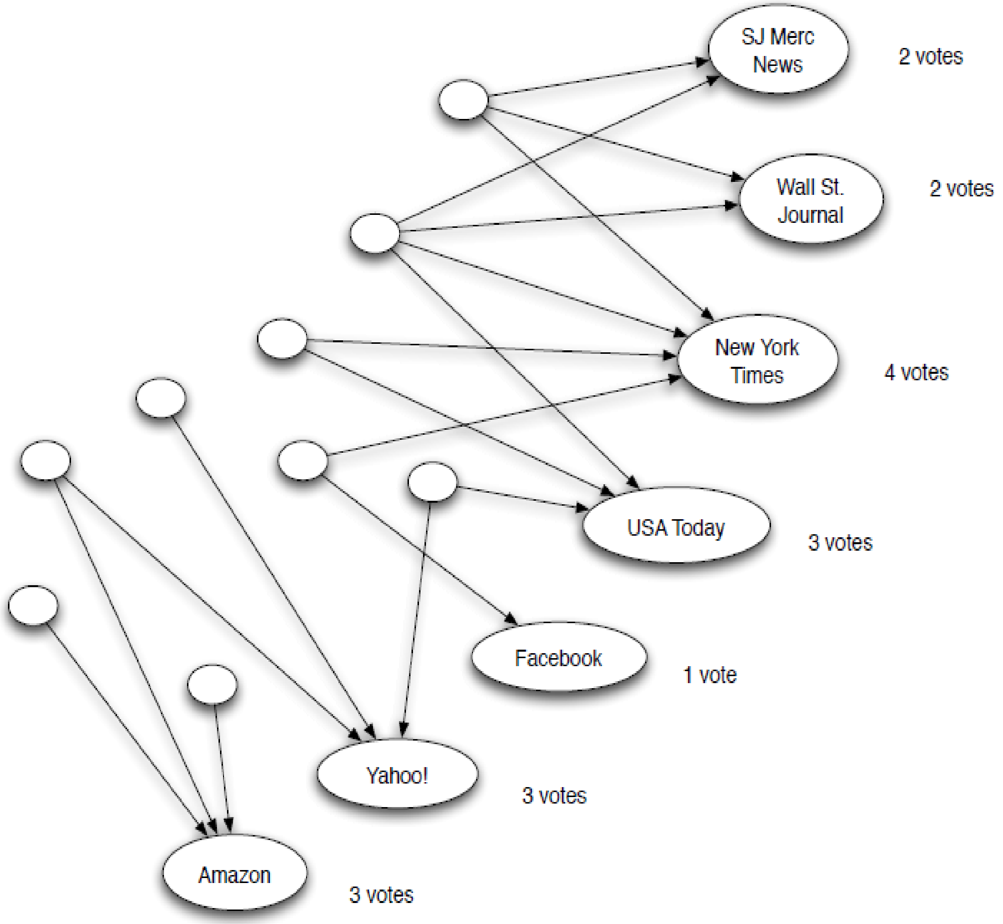
- What is an important webpage?
- Attempt:
  – use in-degree
  – "count votes"

- Should all "votes" count the same?
  – a is better than b
  – b is better than c



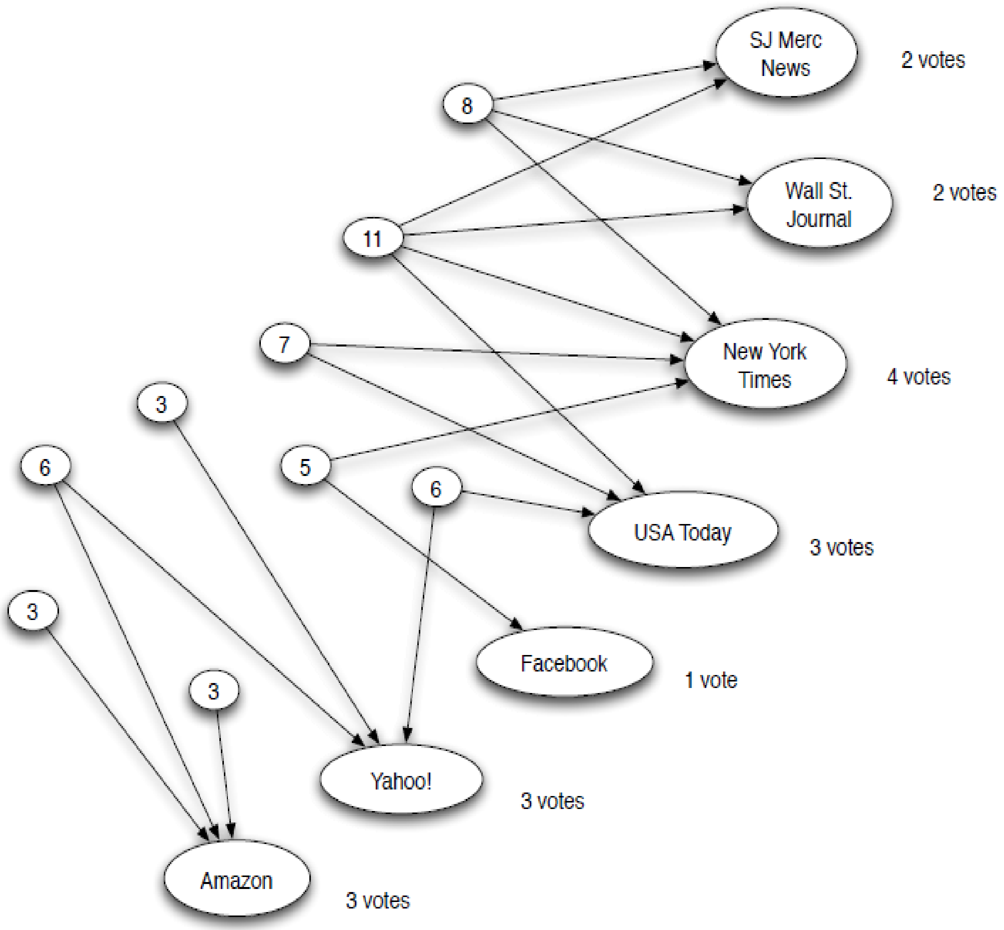
Hubs and Authorities

- BETTER: define
  - "Authority": a page that is linked by many different hubs
  - "Hub": a page that points to many other pages



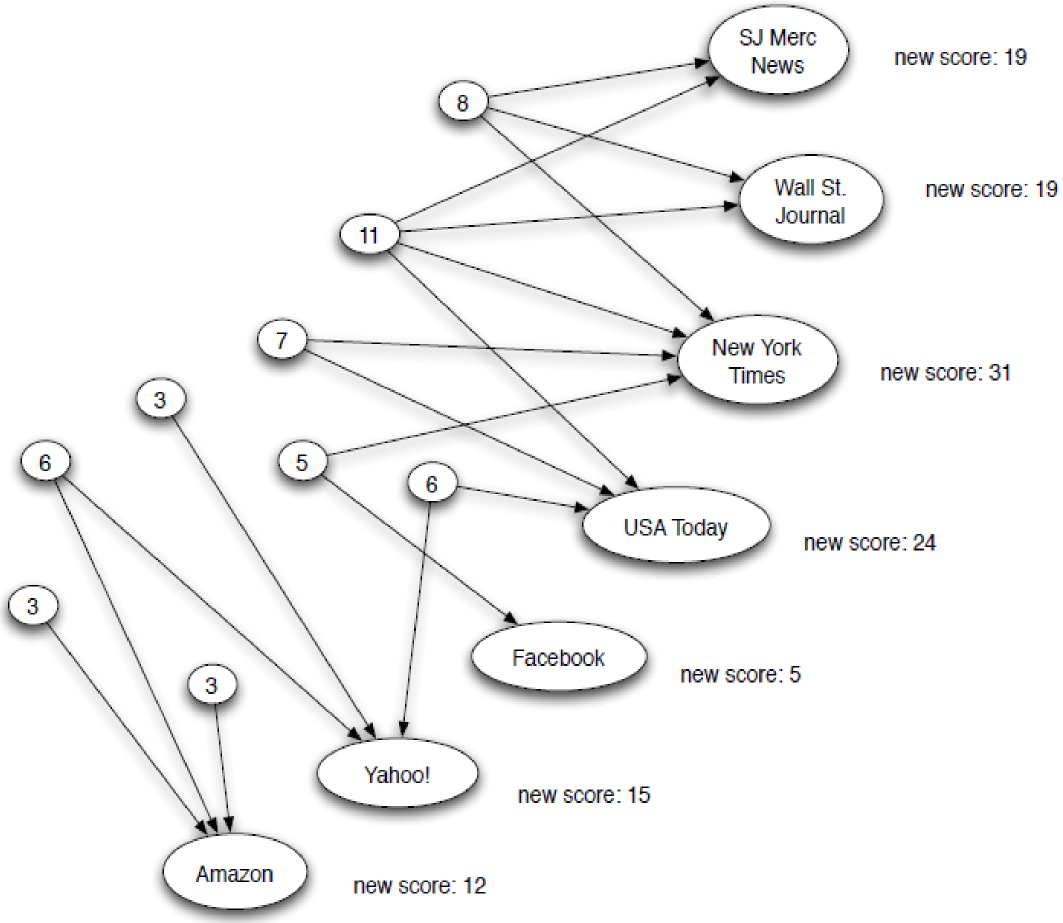
Hubs and Authorities

- BETTER: define
  - "Authority"
  - "Hub"
- Assume we have authority scores



Hubs and Authorities

- BETTER: define
  - "Authority"
  - "Hub"
- Assume we have authority scores
- Use them to compute hub scores



Hubs and Authorities

- Let $In(p)$ be all pages pointing to page p
- Let $Out(p)$ be all pages that page p points to
- The Hubs and Authorities (HITS) scores:

$$auth(p) = \sum_{q \in In(p)} hub(q)$$

$$hub(p) = \sum_{q \in Out(p)} auth(q)$$

- Recursive!
- HOW?

History [edit]

In journals [edit]

Many methods have been used to rank the importance of scientific journals. One such method is Garfield's impact factor. Journals such as *Science* and *Nature* are filled with numerous citations, making these magazines have very high impact factors. Thus, when comparing two more obscure journals which have received roughly the same number of citations but one of these journals has received many citations from *Science* and *Nature*, this journal needs be ranked higher. In other words, it is better to receive citations from an important journal than from an unimportant one.[2]

On the Web [edit]

This phenomenon also occurs in the Internet. Counting the number of links to a page can give us a general estimate of its prominence on the Web, but a page with very few incoming links may also be prominent, if two of these links come from the home pages of sites like Yahoo!, Google, or MSN. Because these sites are of very high importance but are also search engines, a page can be ranked much higher than its actual relevance.

Algorithm: Hubs and Authorities

- Initialization: $\forall p, hub^0(p) = 1$
- For $t = 1 \ldots$

$$auth^t(p) = \frac{1}{Z} \sum_{q \in In(p)} hub^{t-1}(q)$$

$$hub^t(p) = \frac{1}{Z'} \sum_{q \in Out(p)} auth^t(q)$$

Normalize at the end of the step

- Theorem 1: This process converges
- Theorem 2: For "real" networks, there are unique $hub, auth$ scores that satisfy these equations -> initialization does not matter
- Proof!

Proof Outline

1. Update Rule as Matrix Multiplication
2. Unwinding the k step
3. Multiplication in Terms of Eigenvectors
4. Convergence
5. Cheats

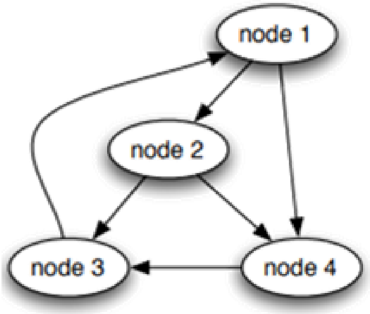
Proof Outline

1. **Update Rule as Matrix Multiplication**
2. Unwinding the k step
3. Multiplication in Terms of Eigenvectors
4. Convergence
5. Cheats

1. Update Rule as Matrix Multiplication

- Adjacency matrix M:

$$M_{i,j} = \begin{cases} 1 & i\ links\ j \\ 0 & otherwise \end{cases}$$



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Following,

$$\boldsymbol{h} = M\boldsymbol{a}$$
$$\boldsymbol{a} = M^T\boldsymbol{h}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 2 \\ 4 \end{bmatrix}$$

$$M\boldsymbol{a} = \boldsymbol{h}$$

$$auth(p) = \sum_{q \in In(p)} hub(q)$$
$$hub(p) = \sum_{q \in Out(p)} auth(q)$$

Proof Outline

1. Update Rule as Matrix Multiplication
2. **Unwinding the k step**
3. Multiplication in Terms of Eigenvectors
4. Convergence
5. Cheats

2. Unwinding the k step

- $\boldsymbol{a}^{(1)} = M^T \boldsymbol{h}^{(0)}$
- $\boldsymbol{h}^{(1)} = M\boldsymbol{a}^{(1)} = MM^T \boldsymbol{h}^{(0)}$
- $\boldsymbol{a}^{(2)} = M^T \boldsymbol{h}^{(1)} = M^T M \boldsymbol{a}^{(1)} = M^T M M^T \boldsymbol{h}^{(0)}$
- $\boldsymbol{h}^{(2)} = M\boldsymbol{a}^2 = MM^T MM^T \boldsymbol{h}^0 = (MM^T)^2 \boldsymbol{h}^{(0)}$
- $\boldsymbol{a}^{(3)} = M^T \boldsymbol{h}^2 = M^T M \boldsymbol{a}^1 = M^T (MM^T)^2 \boldsymbol{h}^{(0)}$
- $\boldsymbol{h}^{(3)} = M\boldsymbol{a}^3 = (MM^T)^3 \boldsymbol{h}^{(0)}$

- $\Rightarrow \quad \boldsymbol{h}^{(k)} = (MM^T)^k \boldsymbol{h}^{(0)}, \ \ \boldsymbol{a}^{(k)} = M^T (MM^T)^{k-1} \boldsymbol{h}^{(0)}$

$$auth^t(p) = \sum_{q \in In(p)} hub^{t-1}(q)$$

$$hub^t(p) = \sum_{q \in Out(p)} auth^t(q)$$

$(no\ scaling)$

$\boldsymbol{a} = M^T \boldsymbol{h}$
$\boldsymbol{h} = M\boldsymbol{a}$

Proof Outline

1. Update Rule as Matrix Multiplication
2. Unwinding the k step
3. **Multiplication in Terms of Eigenvectors**
4. Convergence
5. Cheats

3.1 Multiplication in Terms of Eigenvectors

- From here on we focus on the hub scores (auth is similar)
- Recall from prev.: $\boldsymbol{h}^{(k)} = (MM^T)^k \boldsymbol{h}^{(0)}$
- Definition: $\boldsymbol{x}$ is an eigenvector of a matrix A with an eigenvalue c if

$$A\boldsymbol{x} = c\boldsymbol{x}$$

- We'll show that $\boldsymbol{h}^{(k)}$ converges <u>to the direction</u> of an eigenvector of MM^T

3.2 Multiplication in Terms of Eigenvectors

- Recall that $(MM^T)^T = (M^T)^T M^T = MM^T \Rightarrow MM^T$ is symmetric!
- **<u>Theorem</u>**: Any $n \times n$ symmetric matrix has n eigenvectors that are unit vectors and all are mutually orthogonal (form a basis)
- Let $\boldsymbol{z}_1, \dots \boldsymbol{z}_n$ be the eigenvectors of MM^T with eigenvalues $c_1, \dots c_n$
  - WLOG $|c_1| \geq |c_2| \geq \cdots |c_n|$
  - $MM^T \boldsymbol{z}_i = c_i \boldsymbol{z}_i$
- ASSUME FOR NOW: $|c_1| > |c_2|$ (Cheat!)

3.3 Multiplication in Terms of Eigenvectors

- We can represent any vector as a linear combination of $(\boldsymbol{z}_i)_i$:

$$\boldsymbol{x} = \sum_{i=1}^{n} p_i \boldsymbol{z}_i$$

- Picking $\boldsymbol{x} = \boldsymbol{h}^{(0)}$ and since $MM^T \boldsymbol{z}_i = c_i \boldsymbol{z}_i$,

$$\boldsymbol{h}^{(1)} = (MM^T)\boldsymbol{h}^{(0)} = (MM^T)\sum_{i=1}^{n} p_i \boldsymbol{z}_i = \sum_{i=1}^{n} p_i (MM^T)\boldsymbol{z}_i = \sum_{i=1}^{n} p_i c_i \boldsymbol{z}_i$$

Recall $A\boldsymbol{x} = c\boldsymbol{x}$ if eigenvector

Proof Outline

1. Update Rule as Matrix Multiplication
2. Unwinding the k step
3. Multiplication in Terms of Eigenvectors
4. **Convergence**
5. Cheats

4.1 Convergence

$$(MM^T)\boldsymbol{h}^{(0)} = (MM^T)\sum_{i=1}^{n} p_i\boldsymbol{z}_i = \sum_{i=1}^{n} p_i(MM^T)\boldsymbol{z}_i = \sum_{i=1}^{n} p_i c_i \boldsymbol{z}_i$$

- Multiplying both sides by $(MM^T)^{k-1}$ and remembering that $MM^T\boldsymbol{z}_i = c_i\boldsymbol{z}_i$,

$$(MM^T)^k\boldsymbol{h}^{(0)} = \sum_{i=1}^{n} p_i c_i^k \boldsymbol{z}_i$$

- But recall that $\boldsymbol{h}^{(k)} = (MM^T)^k\boldsymbol{h}^{(0)}$

4.2 Convergence

- Hence

$$\boldsymbol{h}^{(k)} = (MM^T)^k \boldsymbol{h}^{(0)} = \sum_{i=1}^{n} p_i c_i^k \boldsymbol{z}_i$$

- Dividing both sides by c_1^k,

$$\frac{\boldsymbol{h}^{(k)}}{c_1^k} = p_1 \boldsymbol{z}_1 + \sum_{i=2}^{n} p_i \left(\frac{c_i}{c_1}\right)^k \boldsymbol{z}_i$$

- Or equivalently,

$$\boldsymbol{h}^{(k)} = c_1^k \left(p_1 \boldsymbol{z}_1 + \sum_{i=2}^{n} p_i \left(\frac{c_i}{c_1}\right)^k \boldsymbol{z}_i \right)$$

- Since we assume that $|c_1| > |c_2|$, we get that $\lim_{k\to\infty} \frac{(\boldsymbol{h})^k}{c_1^k} = p_1 \boldsymbol{z}_1$ Why $p_1 \neq 0$? Cheat!

Proof Outline

1. Update Rule as Matrix Multiplication
2. Unwinding the k step
3. Multiplication in Terms of Eigenvectors
4. Convergence
5. **Cheats**

5.1 Cheats

- Cheat #1: $\boldsymbol{h}^{(0)} = (1,1,\dots,1) = \sum_{i=1}^{n} p_i \boldsymbol{z}_i$ with $p_1 \neq 0$
  1. Notice that $\boldsymbol{h}^{(0)} \cdot \boldsymbol{z}_1 = \sum_{i=1}^{n} p_i \boldsymbol{z}_i\, \boldsymbol{z}_1 = p_1 \|\boldsymbol{z}_1\|^2 = p_1$
  2. There must exists some <u>positive</u> vector $\widetilde{\boldsymbol{x}} = \sum_{i=1}^{n} \widetilde{p}_i \boldsymbol{z}_i$ that isn't orthogonal to $\boldsymbol{z}_1$, thus $\widetilde{\boldsymbol{x}} \cdot \boldsymbol{z}_1 = \widetilde{p_1} \neq 0$
  3. <u>Exercise</u>: show that $c_1 > 0$ (symmetric matrix, dominant eigenvalue)
  4. Starting from $\widetilde{\boldsymbol{x}}$, $\lim_{k\to\infty} \frac{\left(MM^T\right)^k \widetilde{\boldsymbol{x}}}{c_1^k} = \widetilde{p_1} \boldsymbol{z}_1$
  5. Since $\frac{\left(MM^T\right)^k \widetilde{\boldsymbol{x}}}{c_1^k}$ involves non-negative numbers, $\widetilde{p_1} \boldsymbol{z}_1$ has non-negative coordinates; and at least one must be positive
  6. Finally, no positive vector can be orthogonal to $\boldsymbol{z}_1$ since it has at least one positive entry

5.2 Cheats

- Cheat #2: $|c_1| > |c_2|$
- This might not hold. BUT,
  1. If the graph is strongly connected, it does (Perron–Frobenius theorem)
  2. Even if $|c_1| = |c_2|$, the process will still converges (although not to a unique vector – depends on initialization)

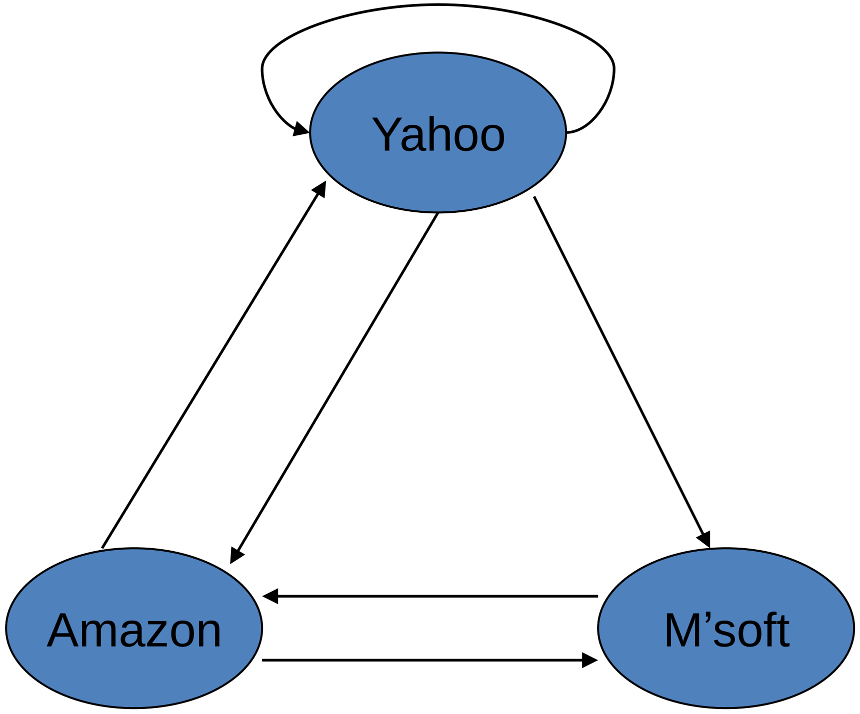
Toy Example

$$M = \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}, \quad MM^T = \begin{matrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{matrix}$$

(7 iterations to convergence)

a(yahoo)	=	1	5	24	114	. . .	$1+\sqrt{3}$
a(amazon)	=	1	4	18	84	. . .	2
a(m'soft)	=	1	5	24	114	. . .	$1+\sqrt{3}$

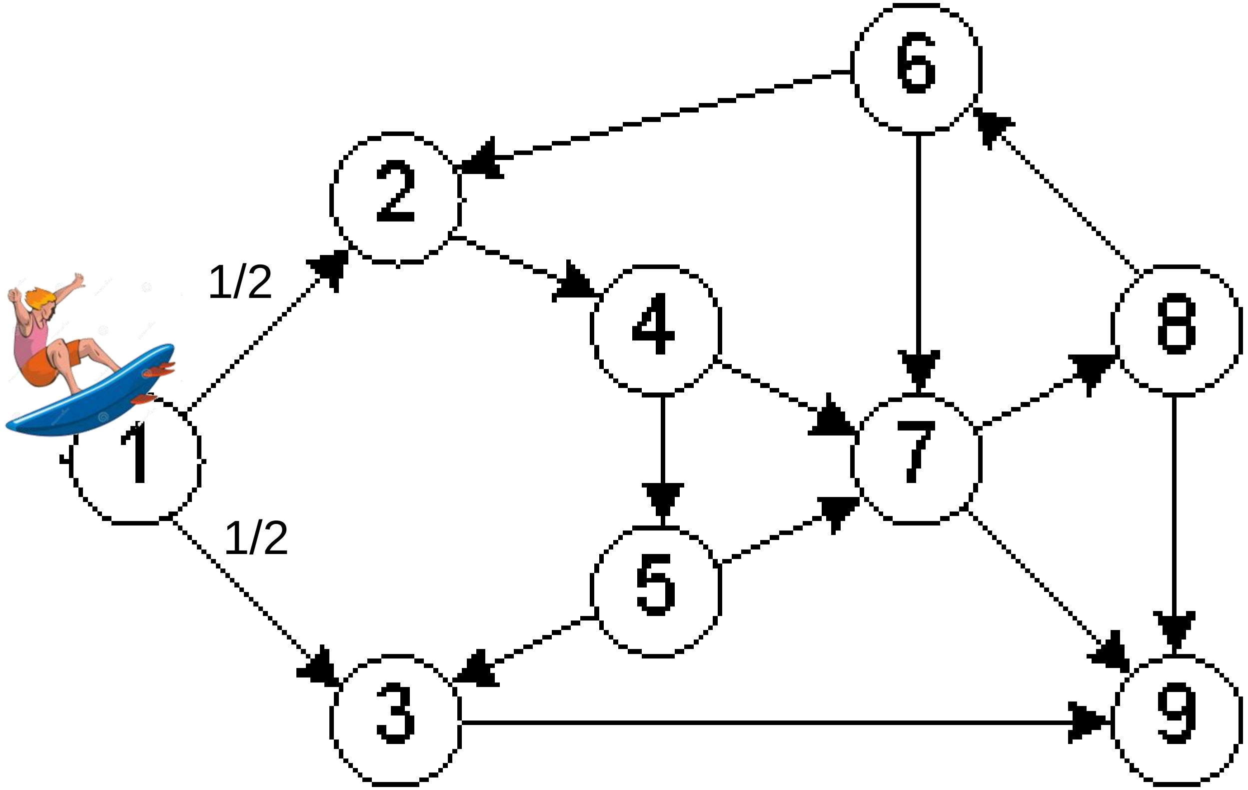
h(yahoo)	=	1	6	28	132	. . .	1.000
h(amazon)	=	1	4	20	96	. . .	0.735
h(m'soft)	=	1	2	8	36	. . .	0.268



PageRank

Which web pages are most important?

- The random surfer model



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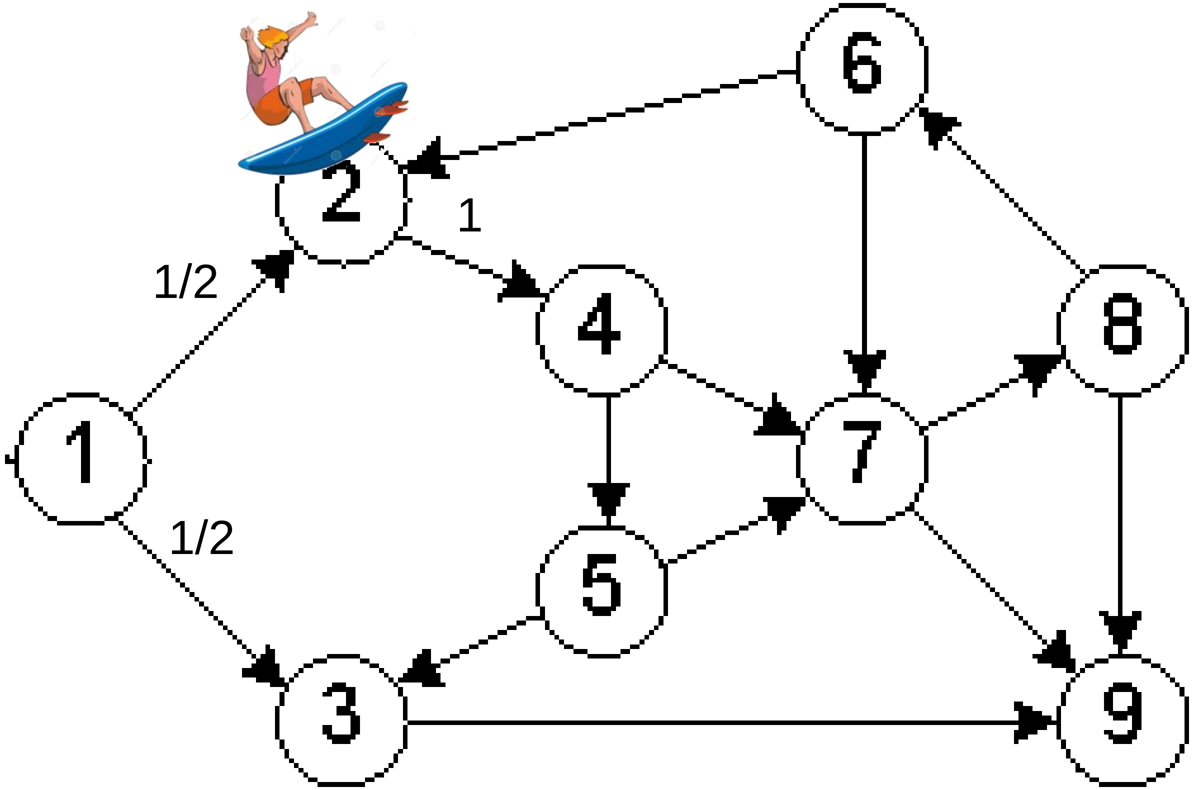

Which web pages are most important?

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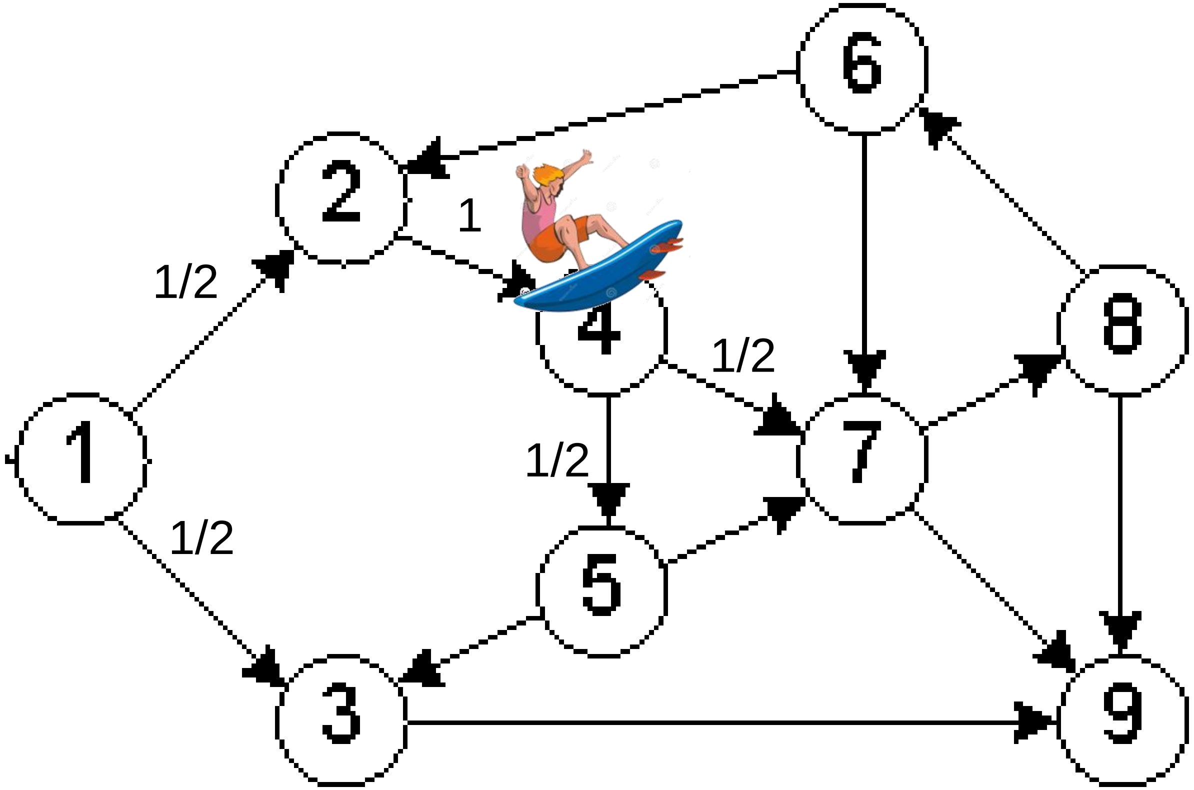
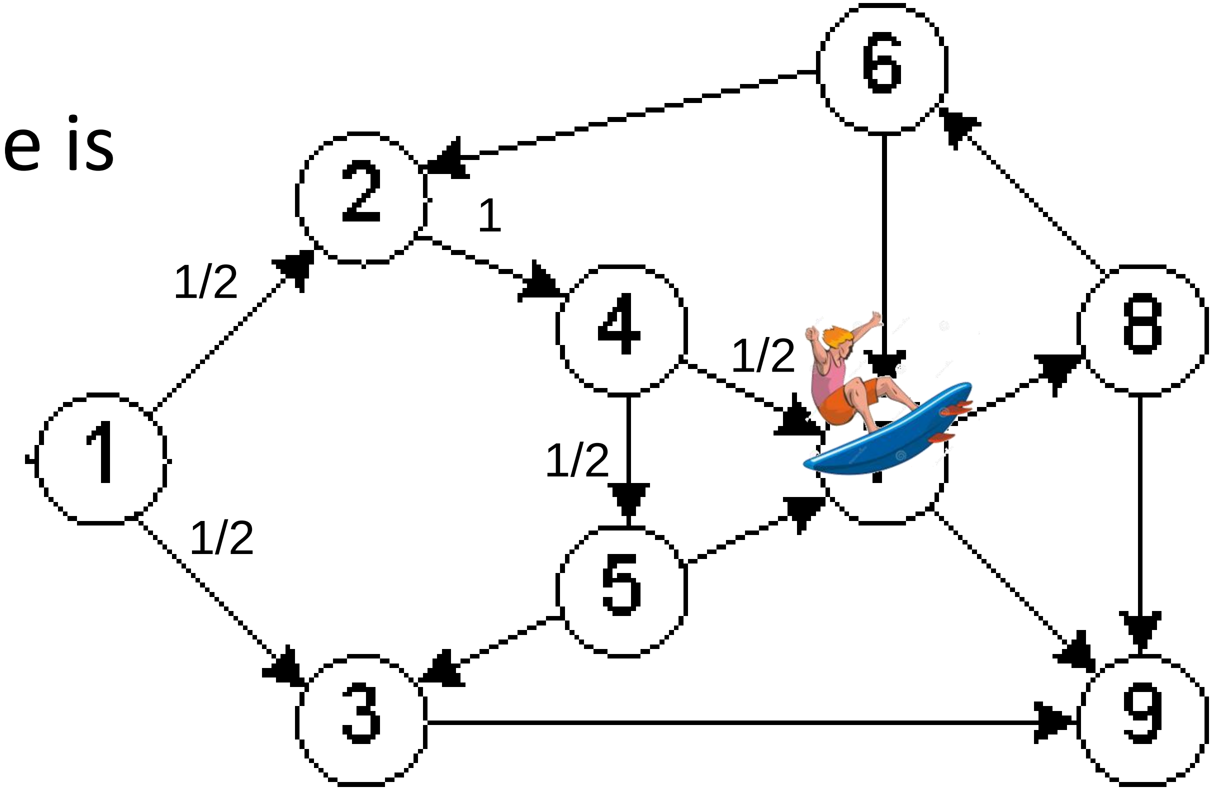


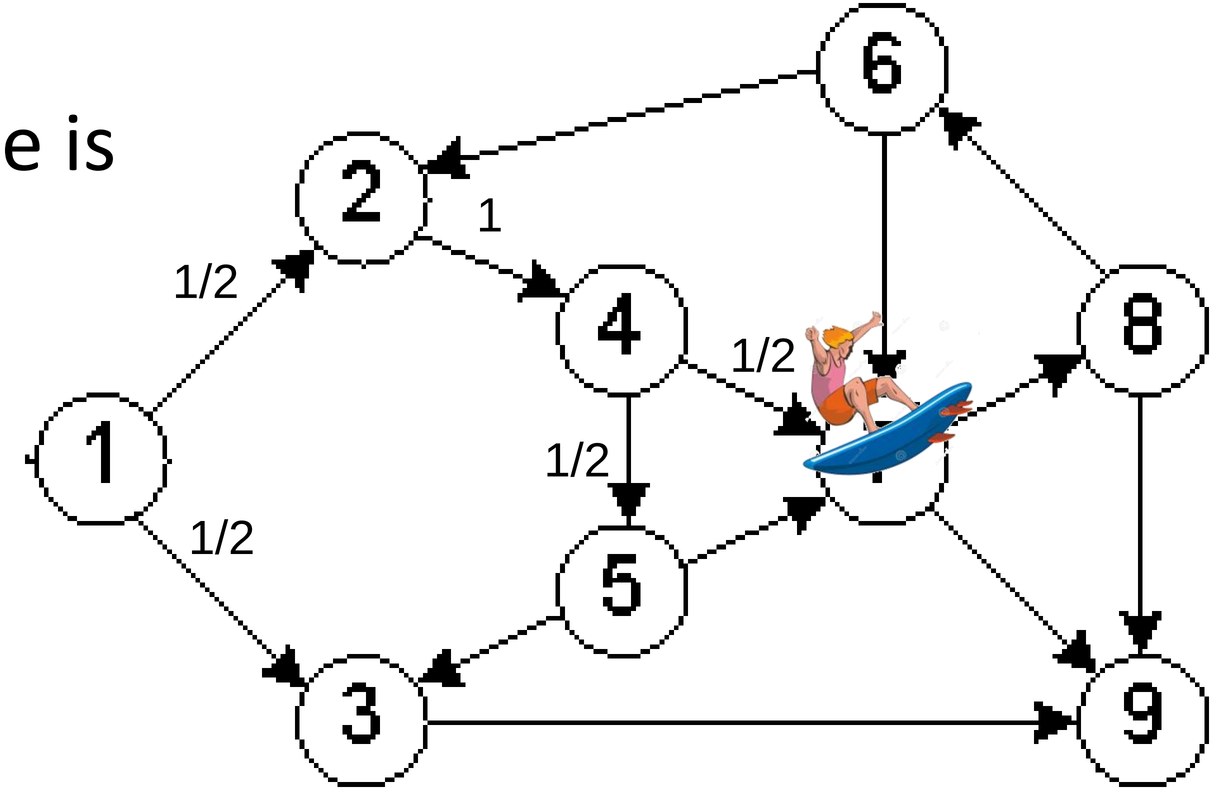
Which web pages are most important?

- The random surfer model
- How much time is spent in each node?



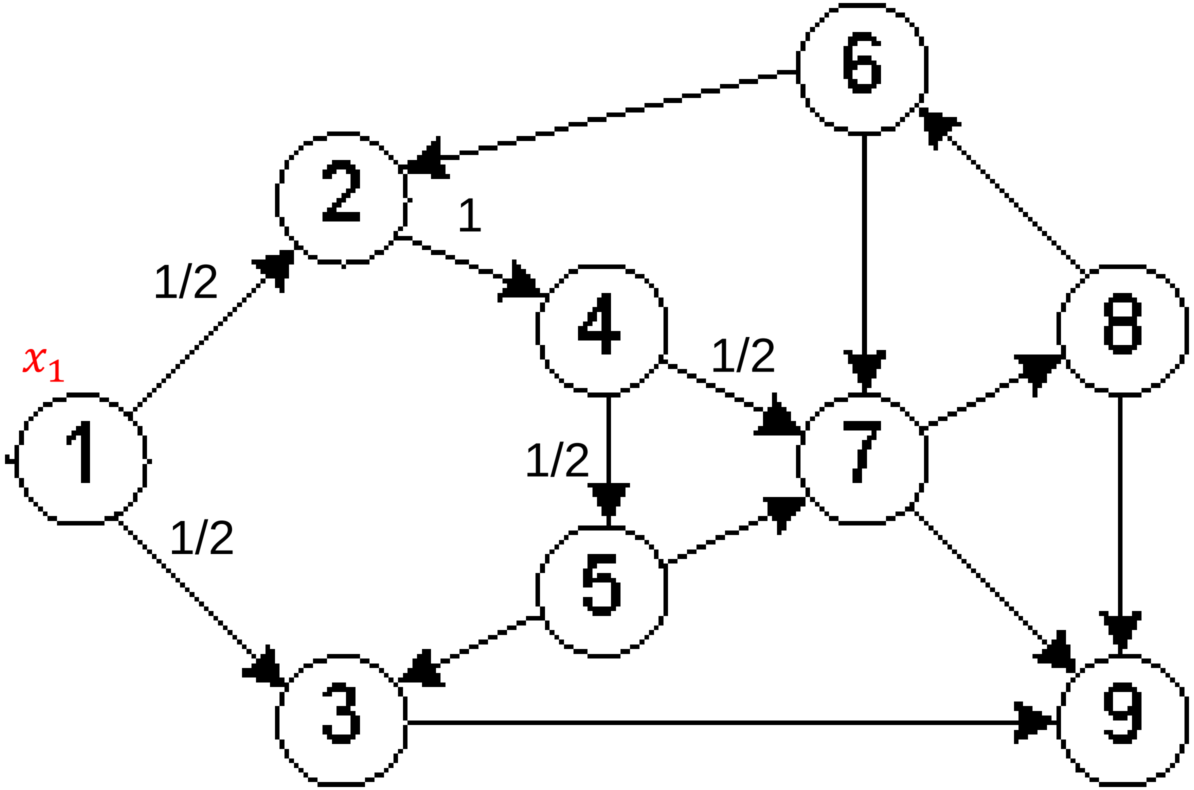
Which web pages are most important?

- The random surfer model
- How much time is spent in each node?
- Stationary distribution



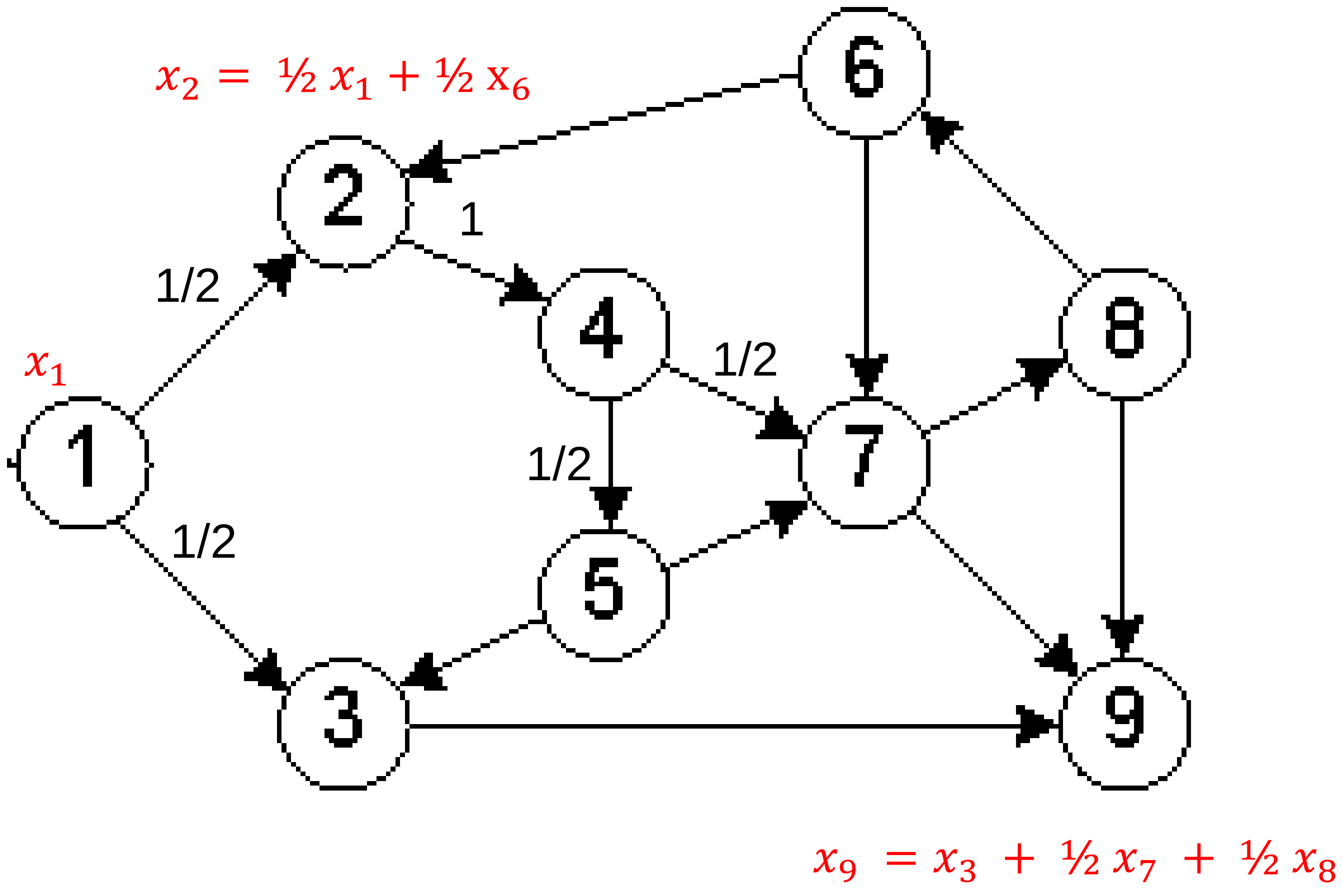
Which web pages are most important?

- Equivalent: A node is important if it is pointed by important nodes



Which web pages are most important?

- Equivalent: A node is important if it is pointed by important nodes



The PageRank Citation Ranking: Bringing Order to the Web

Page, L., Brin, S., Motwani, R., & Winograd, T.

January 29, 1998

Abstract

The importance of a Web page is an inherently subjective matter, which depends on the readers interests, knowledge and attitudes. But there is still much that can be said objectively about the relative importance of Web pages. This paper describes PageRank, a method for rating Web pages objectively and mechanically, effectively measuring the human interest and attention devoted to them.

We compare PageRank to an idealized random Web surfer. We show how to efficiently compute PageRank for large numbers of pages. And, we show how to apply PageRank to search and to user navigation.

Web Page	PageRank (average is 1.0)
Download Netscape Software	11589.00
http://www.w3.org/	10717.70
Welcome to Netscape	8673.51
Point: It's What You're Searching For	7930.92
Web-Counter Home Page	7254.97
The Blue Ribbon Campaign for Online Free Speech	7010.39
CERN Welcome	6562.49
Yahoo!	6561.80
Welcome to Netscape	6203.47
Wusage 4.1: A Usage Statistics System For Web Servers	5963.27
The World Wide Web Consortium (W3C)	5672.21
Lycos, Inc. Home Page	4683.31
Starting Point	4501.98
Welcome to Magellan!	3866.82
Oracle Corporation	3587.63

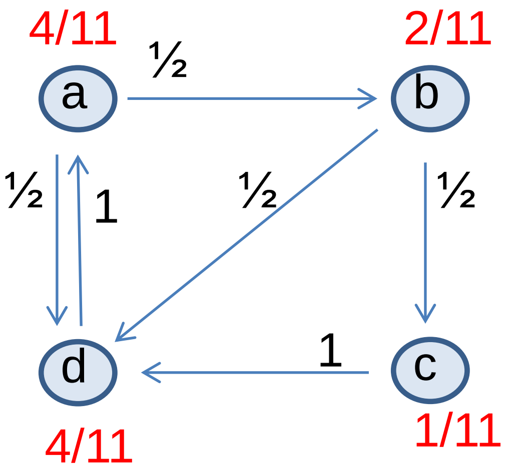
Table 1: Top 15 Page Ranks: July 1996

Stationary Distribution

- Let M be a matrix such that

$$M(a,b) = \Pr(b \to a) = \begin{cases} \dfrac{1}{\deg(b)} & if\ edge \\ 0 & otherwise \end{cases}$$

- Find $\boldsymbol{x}^\star$ such that $M\boldsymbol{x}^\star = \boldsymbol{x}^\star$
- How to find such a vector?
- Answer: Power iteration!
  - Problem?



$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ \\ \end{array} \overset{M}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix}} \overset{\boldsymbol{x}^\star}{\frac{1}{11}\begin{pmatrix} 4 \\ 2 \\ 1 \\ 4 \end{pmatrix}} = \overset{\boldsymbol{x}^\star}{\frac{1}{11}\begin{pmatrix} 4 \\ 2 \\ 1 \\ 4 \end{pmatrix}}$$

$$\begin{array}{cccc} \hline 1 & 1 & 1 & 1 \end{array}$$

PageRank - Algorithm

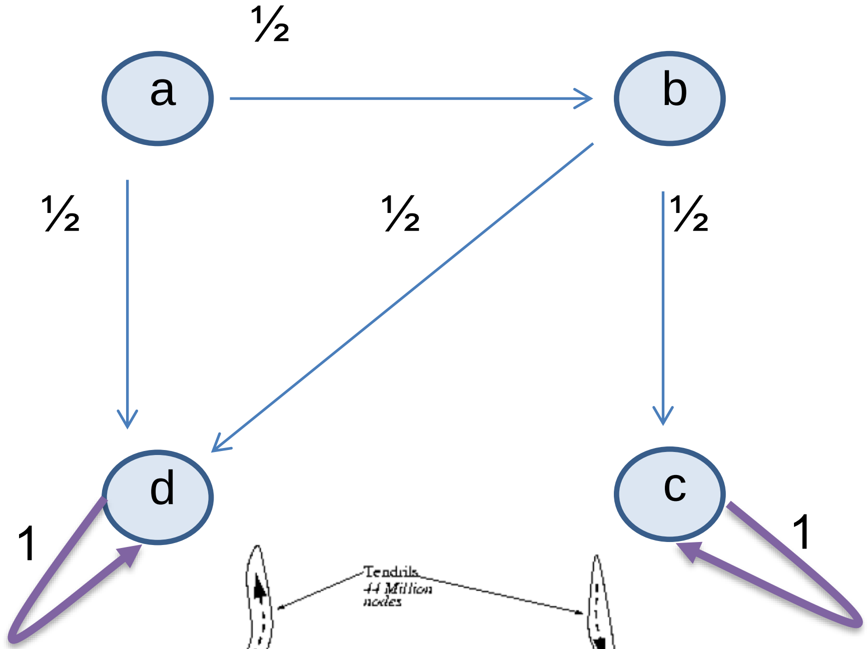
- Let M be a matrix such that

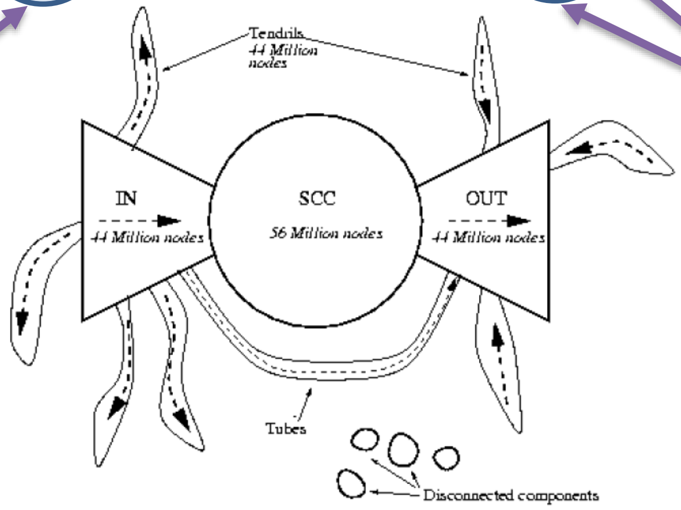
$$M(a,b) = \Pr(b \to a) = \begin{cases} \dfrac{1}{\deg(b)} & if\ edge \\ 0 & otherwise \end{cases}$$

- Set $\boldsymbol{x}^{(0)} = (1,1,\dots,1)$
- For $t = 0,1,2,\dots$
  - $\boldsymbol{x}^{(t+1)} = M\boldsymbol{x}^{(t)}$
  - If $\boldsymbol{x}^{(t+1)} \approx \boldsymbol{x}^{(t)}$, return $\boldsymbol{x}^{(t+1)}$

Dead Ends

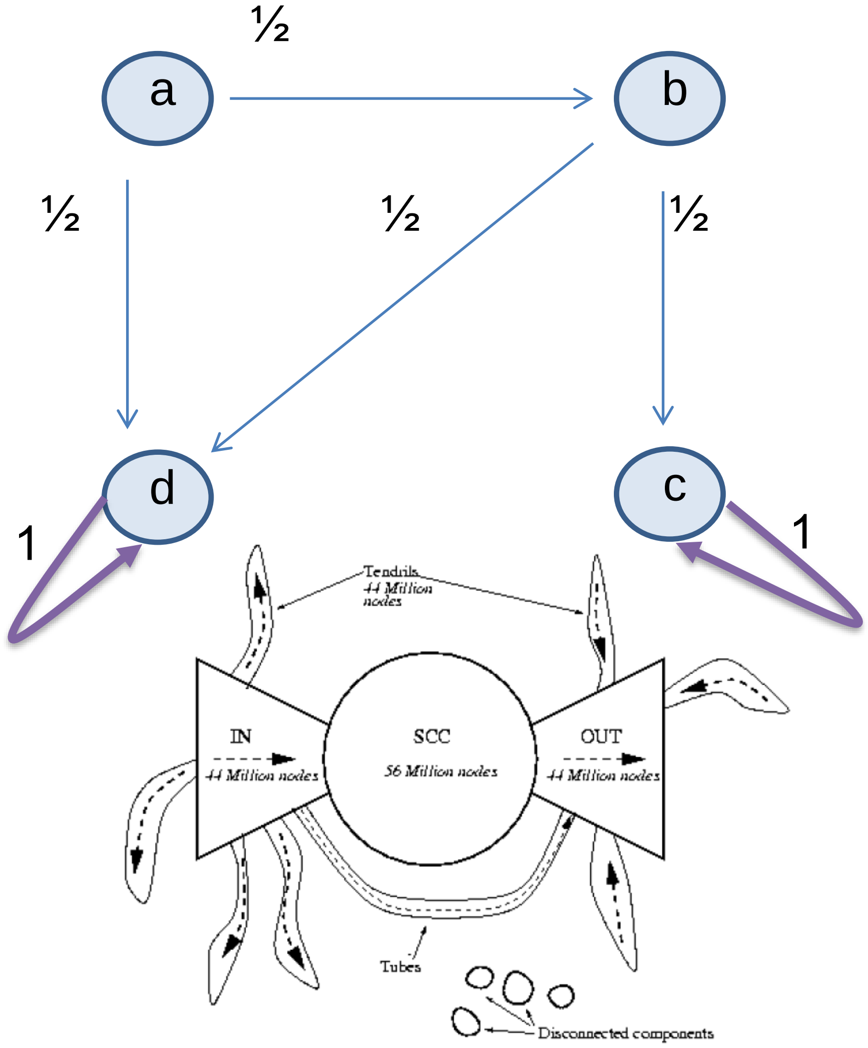
- What if there are nodes with no out links?





Dead Ends

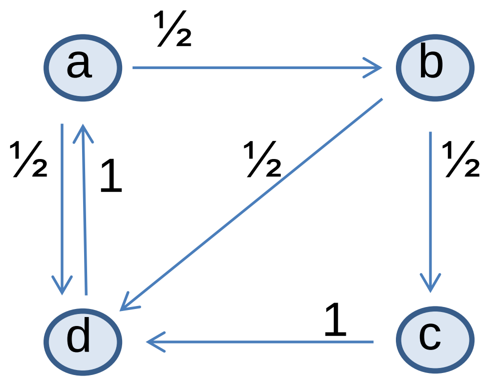
- What if there are nodes with no out links?
- Solution: add a “damping factor” (=teleport ☺)



Stationary distributions

- Solution: add a "damping factor" α to the random surf ☺
  - W.p. $\alpha < 1$ follow a link u.a.r.
  - W.p. $1 - \alpha$ jump to a random node
- $M_\alpha(a, b) = \alpha \cdot M(a, b) + (1 - \alpha) \cdot \frac{1}{n} \cdot 1_{n \times n}$

Example



$$M_\alpha(a,b) = \alpha \cdot M\ (a,b) + (1-\alpha)\cdot\frac{1}{n}\cdot 1_{n\times n}$$

$$\begin{array}{c} M \\ 0.8\begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix} \\ \hline 1 \quad 1 \quad 1 \quad 1 \end{array} + 0.2\cdot\frac{1}{4}\begin{array}{c} 1_{n\times n} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} M_{0.8} \\ \begin{pmatrix} .05 & .05 & .05 & .85 \\ .45 & .05 & .05 & .05 \\ .05 & .45 & .05 & .05 \\ .45 & .45 & .85 & .05 \end{pmatrix} \\ \hline 1 \quad 1 \quad 1 \quad 1 \end{array}$$

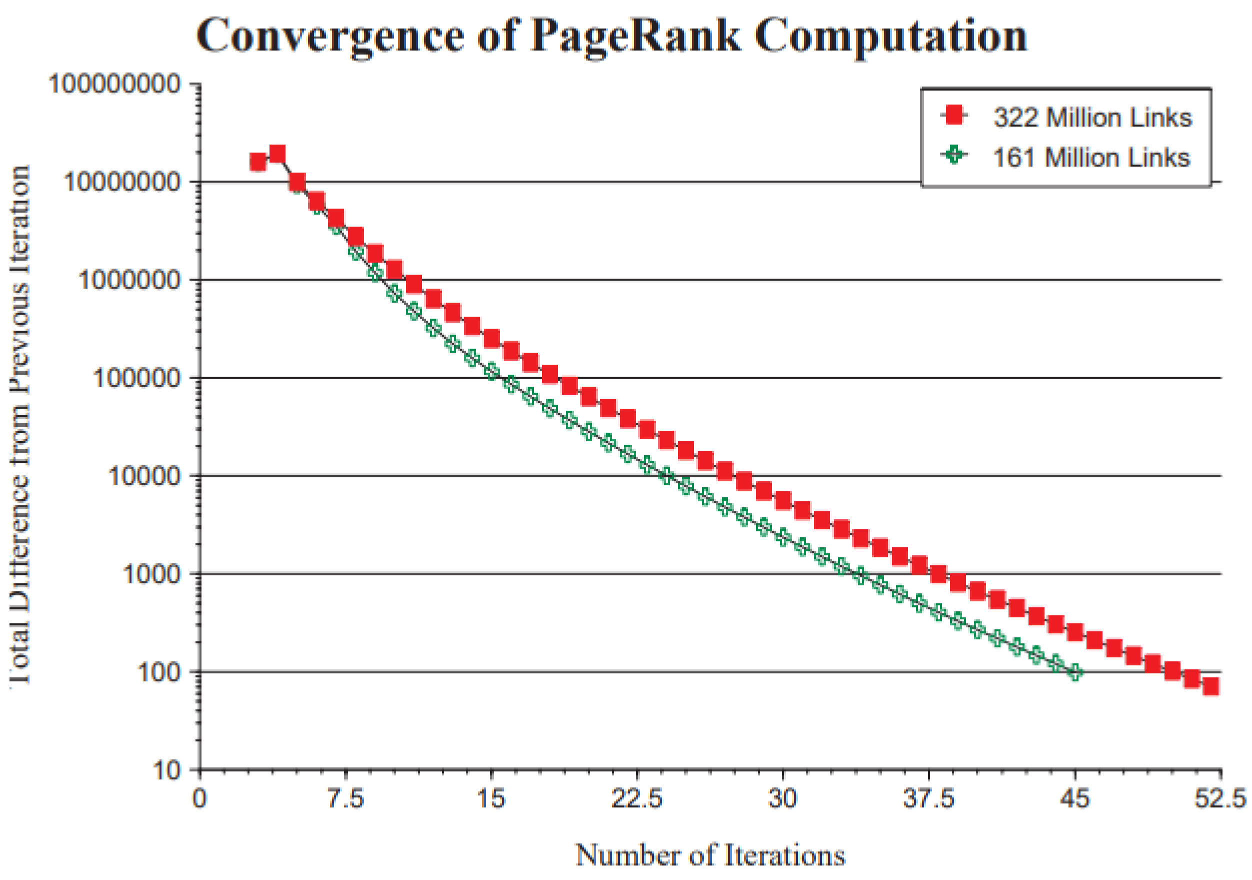
Computation

- There are several methods to compute $\boldsymbol{x}^{\star}$
  - A random walk on the matrix/graph
    - Highly inefficient
  - Solve linear equations
    - $(M - I)x = 0$
  - **The Power Method**

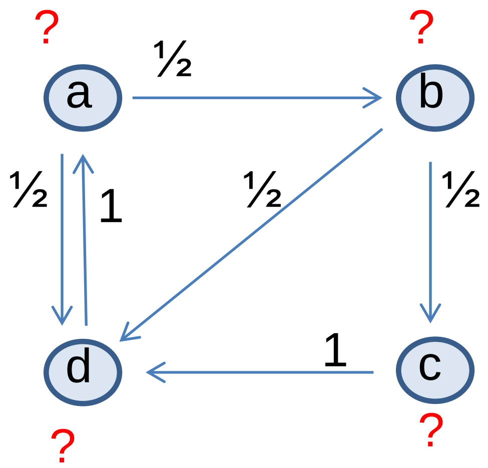
PageRank with the Power Method

- Recall that in HITS the matrix MM^T was symmetric
  - Not the case here!
- Instead, we use the Perron–Frobenius theorem
- Perron-Frobenius theorem: If a matrix is ***positive*** then
  - Its leading eigenvector is strictly positive (in particular real)
  - There are no other positive eigenvectors
  - $|c_1| > |c_j|$ for all $j > 1$
- (Holds even for non-negative matrices that are associated with a strongly connected graph)
- Exercise: show that if M is stochastic, $\lambda_1 = 1$
- From here, as with HITS

Time till convergence

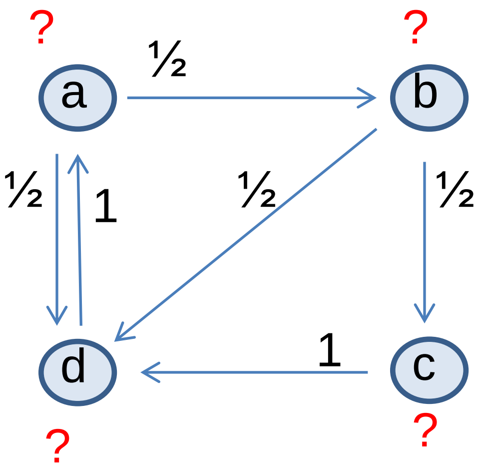


Hand Calculation



$$
\begin{array}{c}
 \\ a \\ b \\ c \\ d
\end{array}
\overset{M}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix}}
\overset{\boldsymbol{x}^t}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}
=
\overset{\boldsymbol{x}^{t+1}}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}
$$

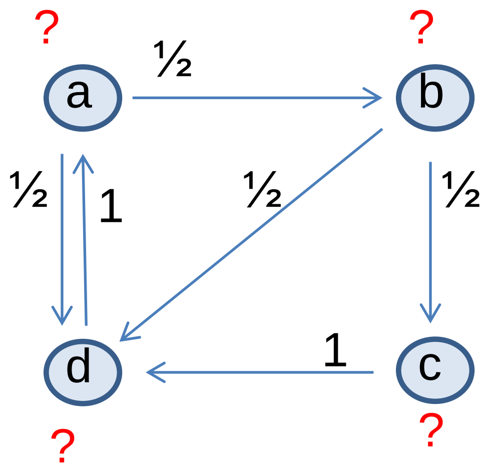
Hand Calculation



$$
\begin{array}{cc}
 & M \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix}
\end{array}
\begin{array}{c} \boldsymbol{x}^{t} \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{array}
=
\begin{array}{c} \boldsymbol{x}^{t+1} \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{array}
$$

0a+0b+0c+1d=a

Hand Calculation



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\begin{array}{cc}
 & M \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix}
\end{array}
\begin{array}{c} \boldsymbol{x}^t \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{array}
=
\begin{array}{c} \boldsymbol{x}^{t+1} \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{array}
$$

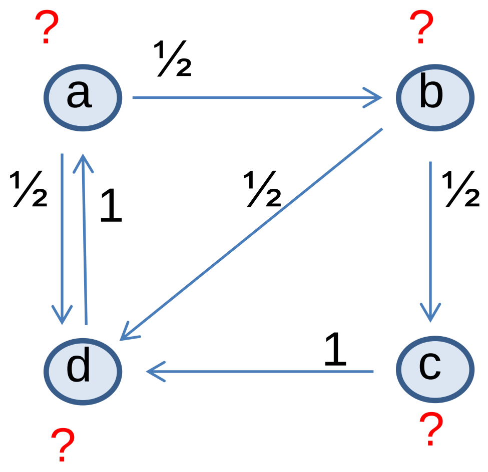
0a+0b+0c+1d=a

0.5a+0b+0c+0d=b

0a+0.5b+0c+0d=c

0.5a+0.5b+1c+0d=d

Hand Calculation



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\ \overset{M}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix}} \overset{\boldsymbol{x}^t}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}} = \overset{\boldsymbol{x}^{t+1}}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}$$

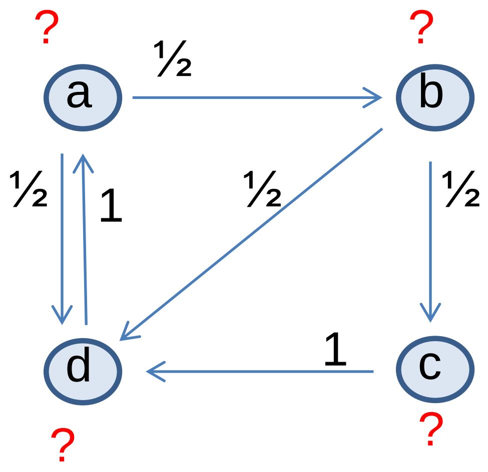
d=a

0.5a=b

0.5b=c

0.5a+0.5b+1c=d

Hand Calculation



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \overset{M}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix}} \overset{\boldsymbol{x}^t}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}} = \overset{\boldsymbol{x}^{t+1}}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}$$

d=a

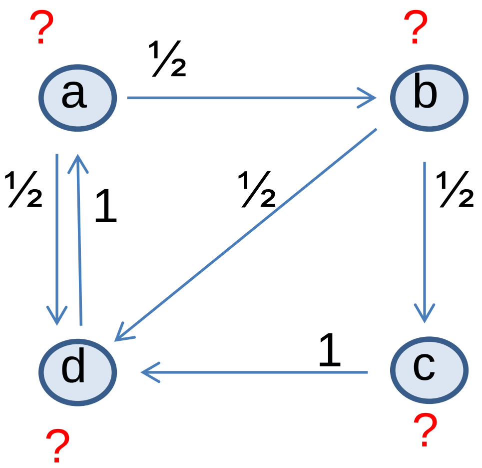
0.5a=b = 1

0.5b=c

0.5a+0.5b+1c=d

Fix the value of one variable

Hand Calculation



$$\begin{array}{c} \\ a\\ b\\ c\\ d \end{array}\overset{M}{\begin{pmatrix} 0 & 0 & 0 & 1\\ .5 & 0 & 0 & 0\\ 0 & .5 & 0 & 0\\ .5 & .5 & 1 & 0 \end{pmatrix}}\overset{\boldsymbol{x}^t}{\begin{pmatrix} a\\ b\\ c\\ d \end{pmatrix}} = \overset{\boldsymbol{x}^{t+1}}{\begin{pmatrix} a\\ b\\ c\\ d \end{pmatrix}}$$

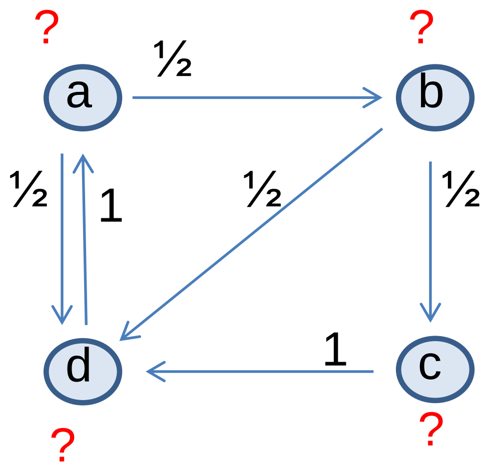
d=a = 2

0.5a=b = 1

0.5b=c = 0.5

0.5a+0.5b+1c=d

Hand Calculation



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} M \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \boldsymbol{x}^t \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{array} = \begin{array}{c} \boldsymbol{x}^{t+1} \\ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{array}$$

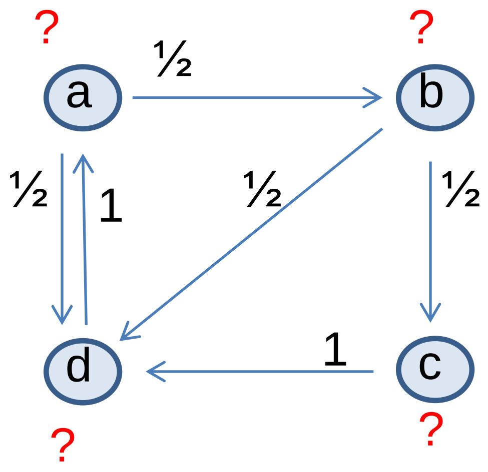
d=a = 2

0.5a=b = 1

0.5b=c = 0.5

1+0.5+0.5=d = 2

Hand Calculation



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \overset{M}{\begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix}} \overset{\boldsymbol{x}^t}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}} = \overset{\boldsymbol{x}^{t+1}}{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}}$$

d=a = 2

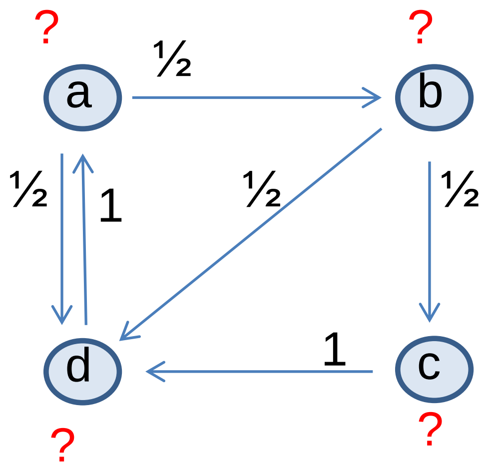
0.5a=b = 1

0.5b=c = 0.5

1+0.5+0.5=d = 2

Normalize: divide by
(2+1+0.5+2) = 5.5

Hand Calculation



$$\begin{array}{c c c c}
 & M & \boldsymbol{x}^{t} & & \boldsymbol{x}^{t+1} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\begin{pmatrix} 0 & 0 & 0 & 1 \\ .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 \\ .5 & .5 & 1 & 0 \end{pmatrix} &
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} & = &
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}
\end{array}$$

a = 4/11

b = 2/11

c = 1/11

d = 4/11

Normalize: divide by
(2+1+0.5+2) = 5.5

Challenges

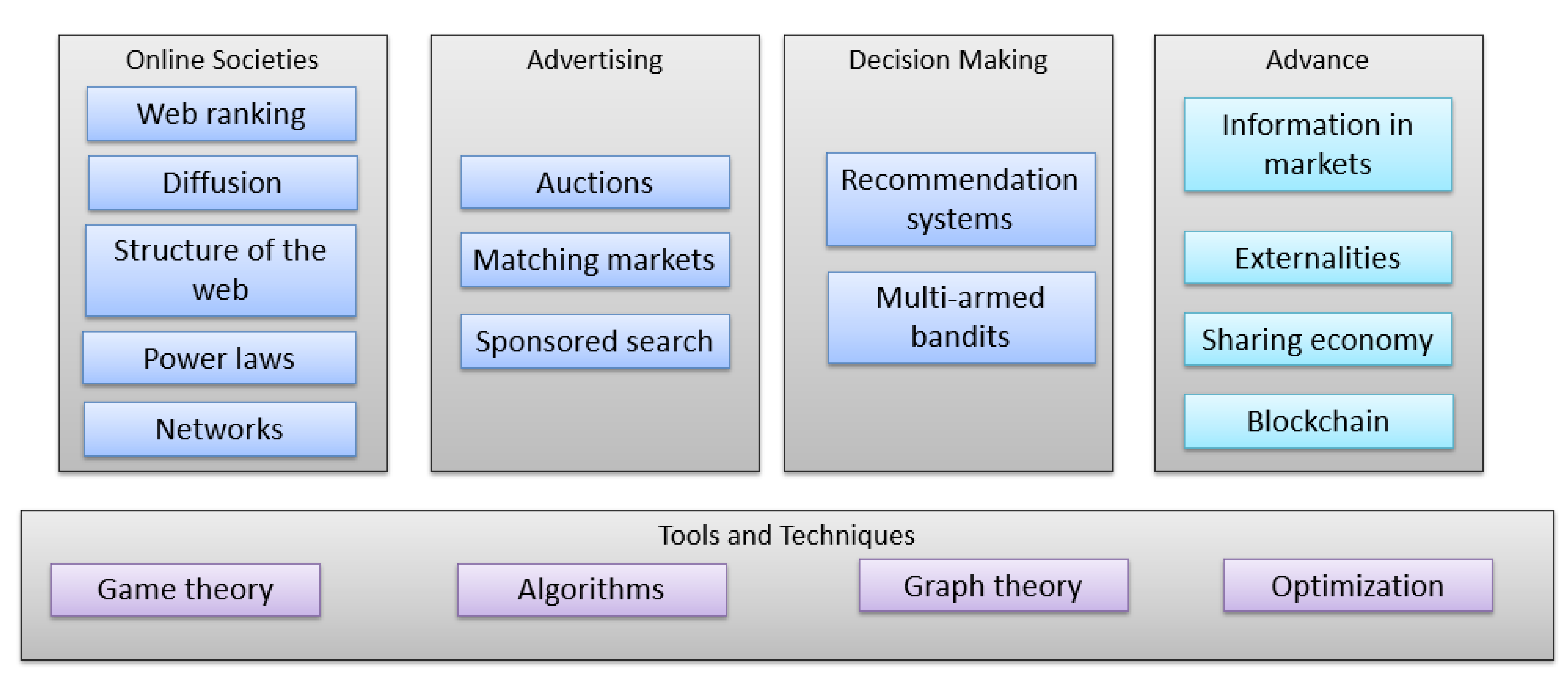
- Huge scale, efficient implementations
  - Changes over time
    - Long term vs. recent importance
  - Strategic behavior
    - Sham web sites to attract traffic
    - Links that improve ranking
    - “Dummy” websites to improve ranking

- NEXT CLASS: Auctions

Electronic Commerce 096211

Multi-Armed Bandits

Slides by Omer Ben Porat


Online Societies
Web ranking
Diffusion
Structure of the web
Power laws
Networks
Advertising
Auctions
Matching markets
Sponsored search
Decision Making
Recommendation systems
Multi-armed bandits
Advance
Information in markets
Externalities
Sharing economy
Blockchain
Tools and Techniques
Game theory
Algorithms
Graph theory
Optimization

Outline

- Motivation
- Multi-armed bandits 101
- Algorithms
  - Explore-then-exploit
  - ϵ-greedy
  - UCB
  - Thompson sampling
- Extensions – brief
  - Continuum of arms
  - Contextual bandits
  - Full feedback with adversarial costs

Explore vs exploit

Which is More Appealing (=CTR)

yahoo!

Mail News Finance

Murder victim found in adult entertainment venue

- www.npr.org/.../headless-body-in-**topless-bar**-headline-writer-d...

ɔ, who wrote what some consider one of the best headlines - 9 ביוני 2015

of all time, died Tuesday at the age of 74.

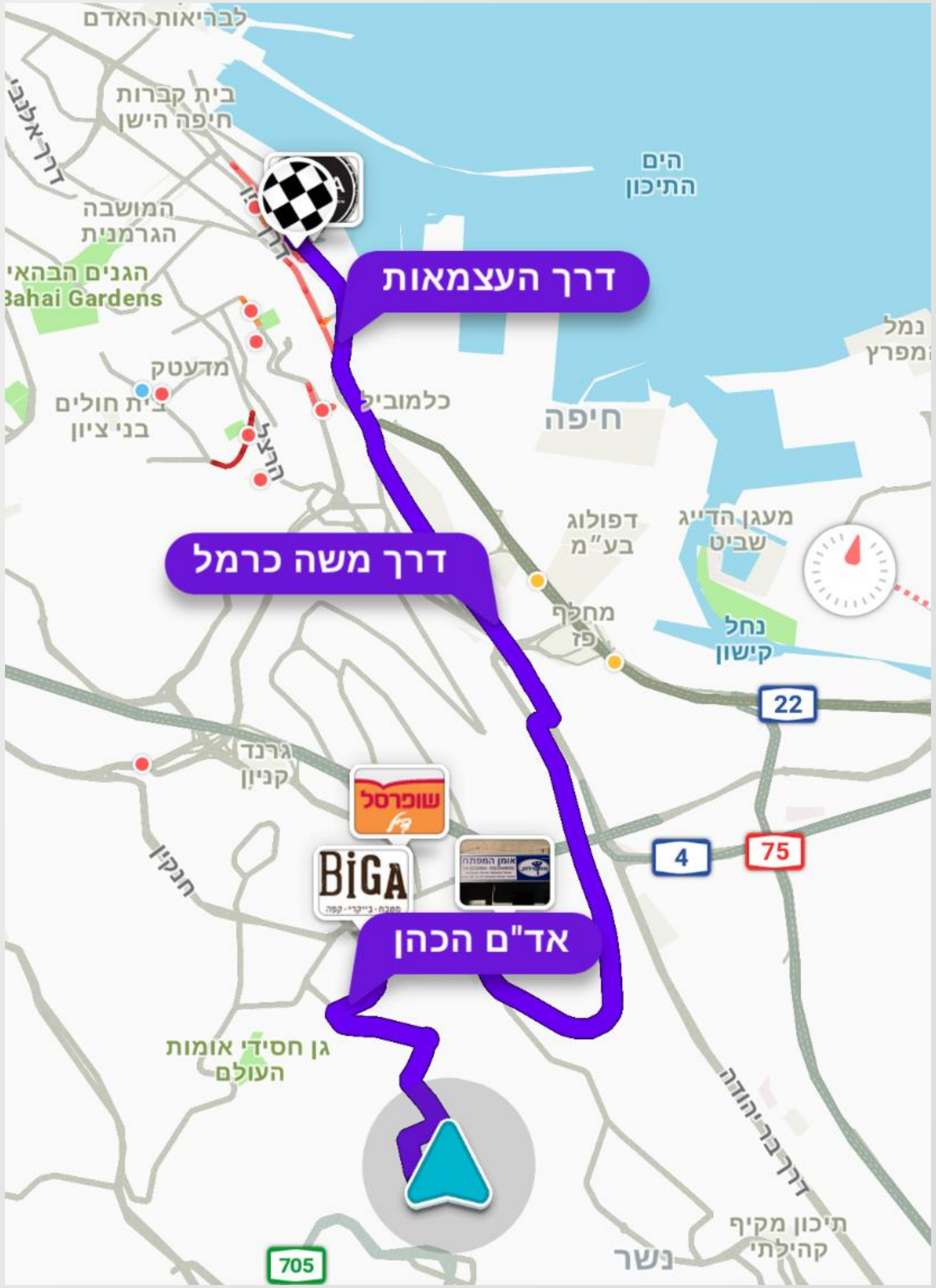
Headless Body found in Topless Bar

../headless-body-in-**topless-bar**-headline-writer-d...

ɪt some consider one of the best headlines - 9 ביוני 2015

of all time, died Tuesday at the age of 74.

בית קברות
חיפה הישן
הים
התיכון
המושבה
הגרמנית
מעלה השחרור
נמל
המפרץ
תחנת
הכוח
דרך נחל הגיבורים
מעגן הדייג
שביט
פארק
הקישון
מחלף
פז
22
דרך רופין
גרנד
קניון
75
mazda
קומוי
adactive
רוז
גן חסידי אומות
העולם
דרך בר יהודה
תיכון מקיף
קהילתי
נשר
705



לבריאות האדם
בית קברות
חיפה הישן
הים
התיכון
המושבה
הגרמנית
הגנים הבהאיים
Bahai Gardens
דרך העצמאות
נמל
המפרץ
מדעטק
בית חולים
בני ציון
כלמוביל
חיפה
דפולוג
בע"מ
מעגן הדייג
שביט
דרך משה כרמל
מחלף
פז
נחל
קישון
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גרנד
קניון
שופרסל
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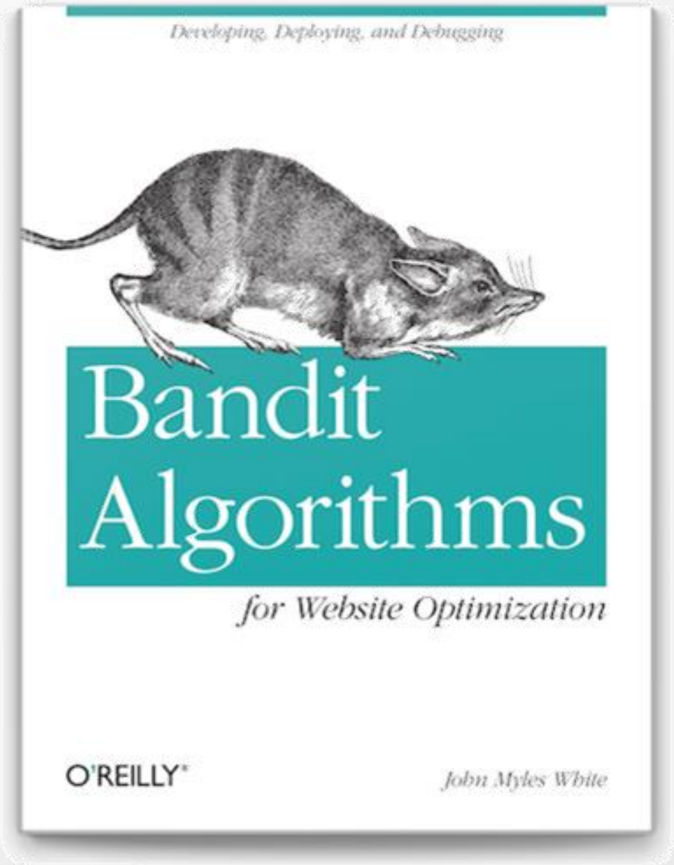
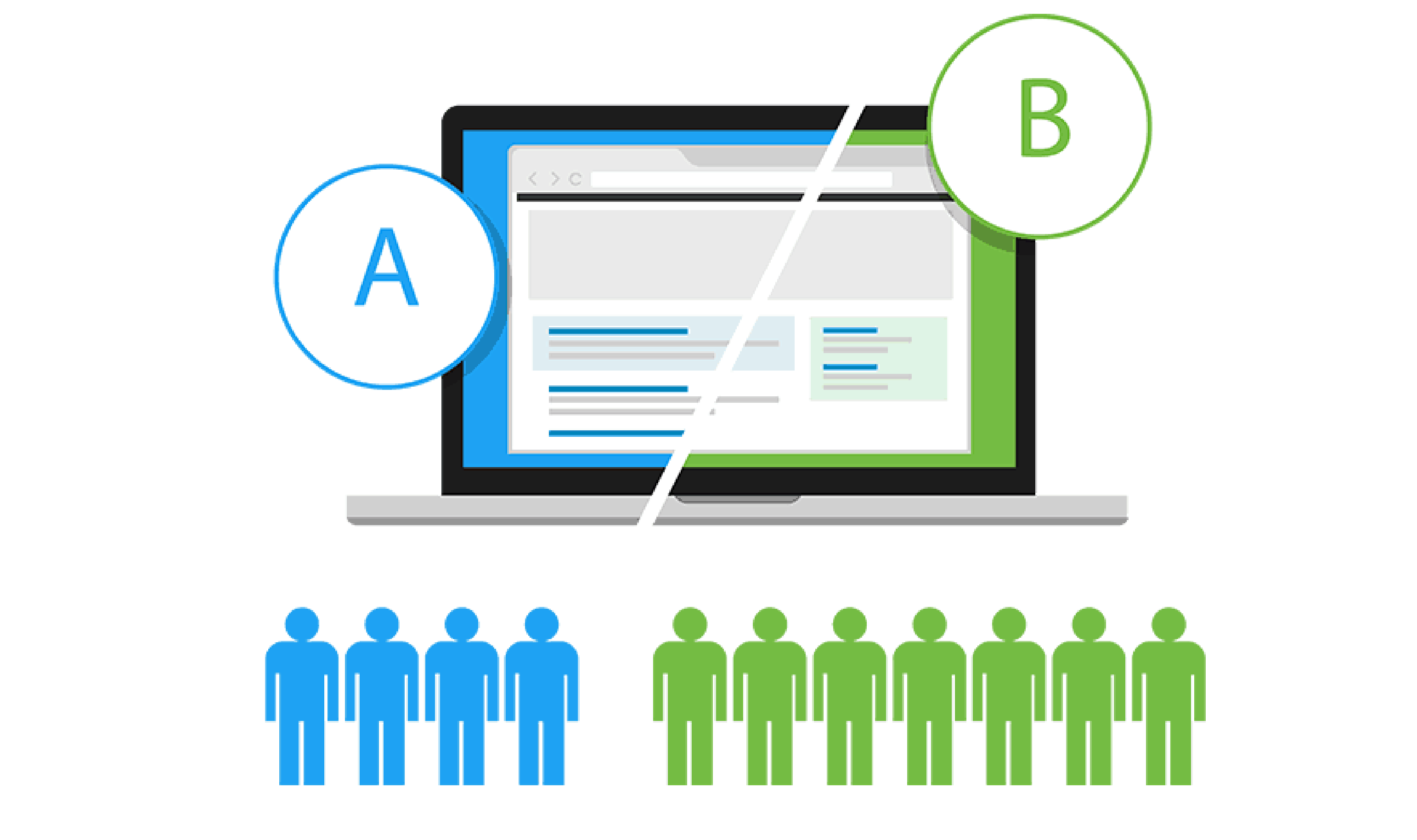
אד"ם הכהן
גן חסידי אומות
העולם
דרך בר יהודה
תיכון מקיף
קהילתי
נשר
705

Where to Place the Ad?

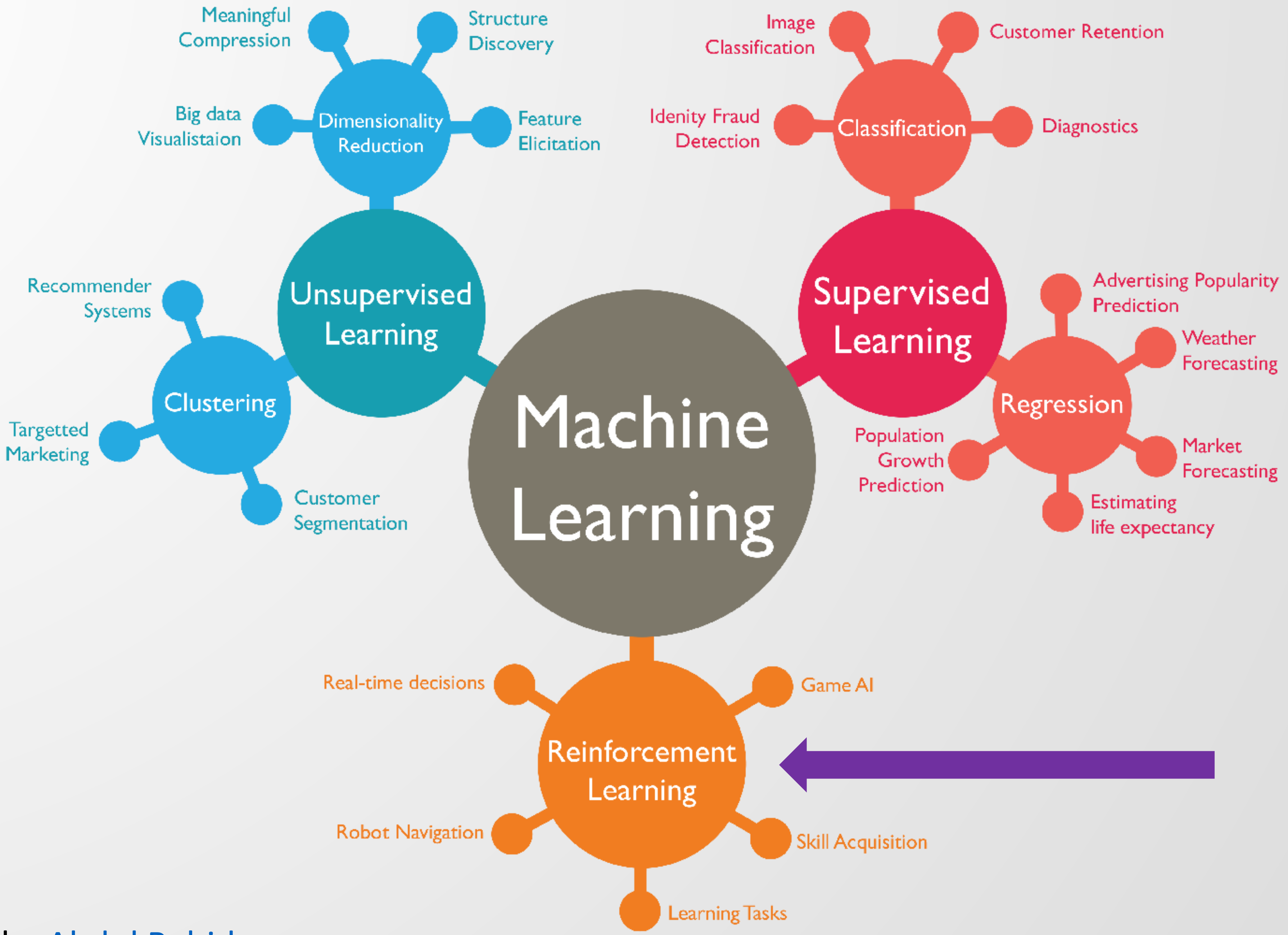



A
B

Originally considered by Allied scientists in World War II, it proved so intractable that, according to Peter Whittle, the problem was proposed to be dropped over Germany so that German scientists could also waste their time on it.[13]

Wikipedia, Multi-armed bandit

Application	Action (e.g.)	Reward (e.g.)
medical trials	which drug to prescribe	healthy/not.
web design	font color or page layout	#clicks.
web content	items/articles to emphasize	#clicks.
web search	search results given a query	#happy users.
advertisement	which ad to display	ad revenue.
recommender systems	which movie to watch	#recommendations followed.
sales	which products to offer at which prices	revenue.
procurement	which items to buy at which prices	#items procured.
auctions	which reserve price to use	revenue
crowdsourcing	which tasks to give to which workers, and at which prices	#completed tasks.
datacenters	server to route the job to	completion time.
Internet	which TCP settings to use	connection quality.
smart radios	radio frequency to use	#transmitted messages.
robot control	a "strategy" for a given task	completion time.

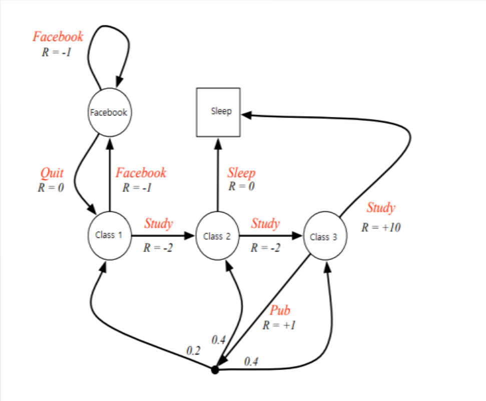


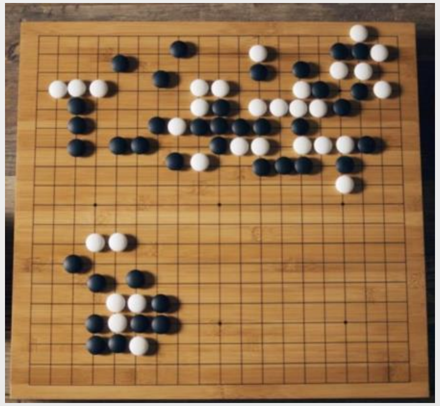
➤ Image by [Abdul Rahid]

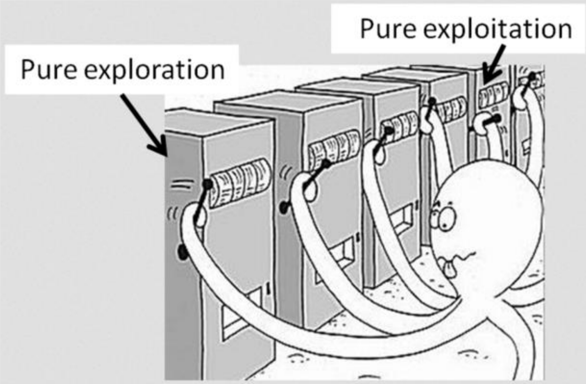
Terminology

- Simulator:
  - Optimal MDP policy, game playing
  - Goal: Learn how to maximize rewards

- Sequential/online:
  - Learn the environment on-the-fly
  - Goal: Collect "good" rewards as you go







*Online vs Offline Classification

- Offline
  - Get a set of samples
  - Pick a classifier (ERM?)
  - Reward=accuracy on the unseen distribution

- Online
  - Repeat:
    - Get a feature vector
    - Pick a classifier
    - Receive 1 if classified correctly, 0 o.w.
  - Reward=average round reward

*Online vs Offline Classification

- Offline
  - Sample $(x_i, y_i)_{i=1}^{m}$
  - Pick a classifier c
  - Reward: (e.g., accuracy)

$$\mathbb{E}_{x,y\sim\mathcal{D}}\left[\mathbb{I}_{y=c(x)}\right]$$

- Online
  - For $t = 1, \dots, T$
    - Receive x_t
    - Pick c_t
    - Obtain round reward $\mathbb{I}_{y_t=c_t(x_t)}$
  - Total reward:

$$\sum_{t=1}^{T} \mathbb{I}_{y_t=c_t(x_t)}$$

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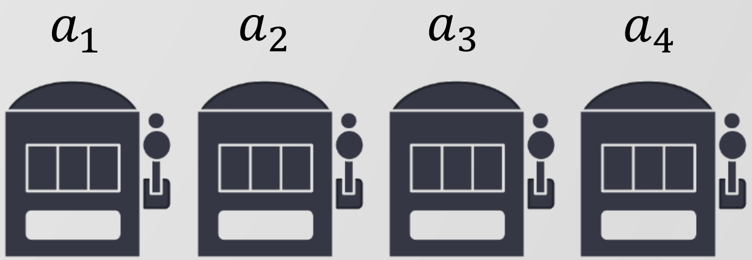
Basic Setting

- Parameters: K arms, T rounds (known); reward distribution D_a for each arm a (unknown)
- In each round $t = 1, \dots T$:
  1. Algorithm picks some arm a_t
  2. Reward $r_t \in [0,1]$ is sampled independently from D_{a_t}
  3. Algorithm collects reward r_t, observers nothing else

- Notations
  - Mean reward: $\mu(a) := \mathbb{E}[D_a]$
  - Best reward: $\mu^\star \coloneqq \max_{a \in A} \mu(a)$
  - Gap: $\Delta(a) \coloneqq \mu^\star - \mu(a)$



Regret

- Given a MAB algorithm, the cumulative regret is

$$R(T) = \mu^* \cdot T - \sum_{t=1}^{T} r_t(a_t)$$

- Recall that a_t depends on the algorithm and past rewards
  - $\Rightarrow R(T)$ is a random variable!
  - Expected regret $\mathbb{E}[R(T)]$ and not the realized one
- "Good" algorithms have very low average regret: $\frac{\mathbb{E}[R(T)]}{T} \to 0$
  - For instance, $O\left(T^{5/6}\right)$
  - But we can do a lot better…

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Explore-Then-Exploit

First Try: Explore-Then-Exploit

- Algorithm:
  1. Exploration phase: Try each arm N times
  2. Select an arm $\hat{a}$ with the highest average reward
  3. Exploitation phase: Play arm $\hat{a}$ in all remaining rounds

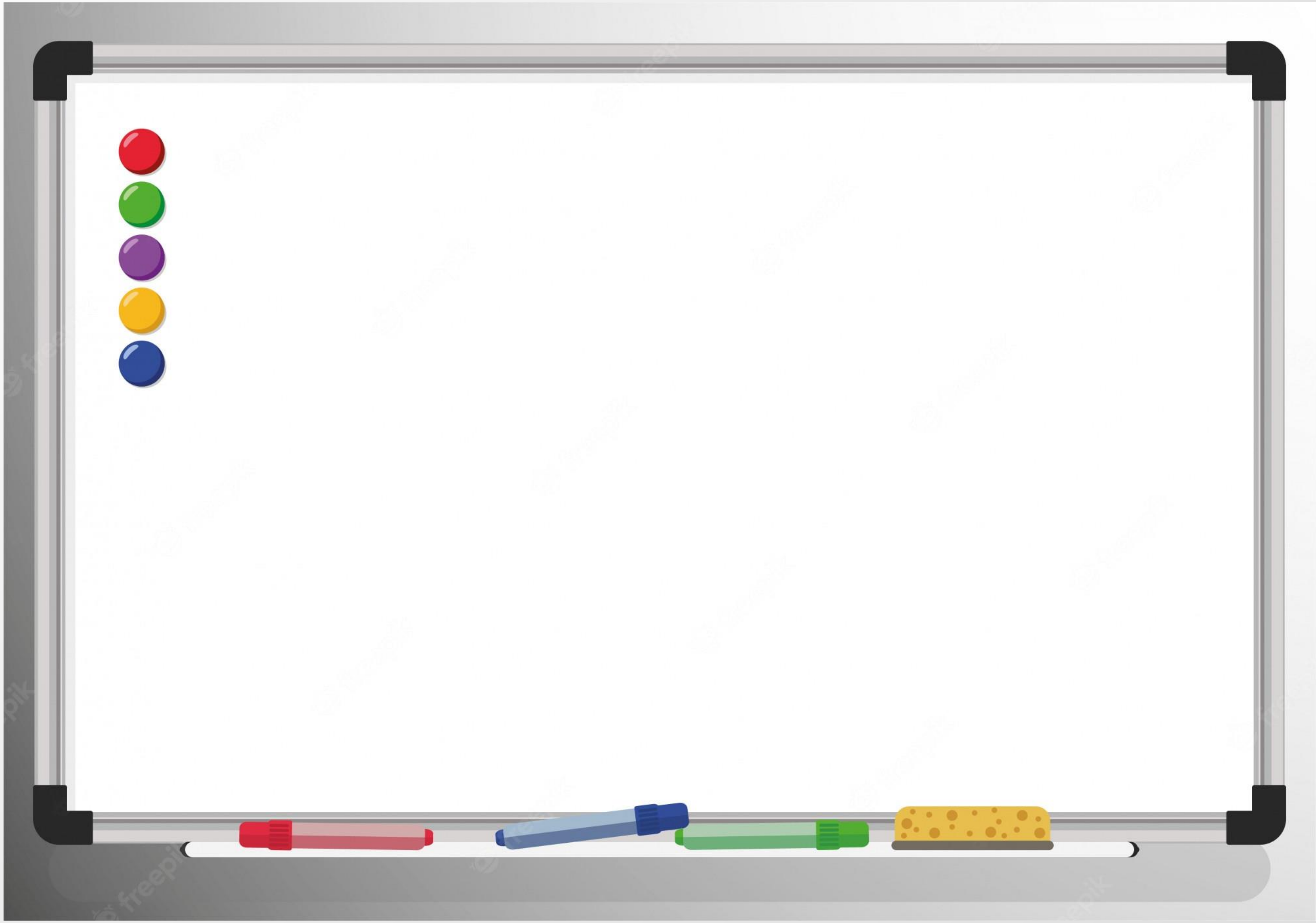
- Exploration: $N \cdot K$, exploitation: $T - N \cdot K$
- Tradeoff:
  - Small N sometime leads to picking "bad" arms
  - Large N wastes many rounds on exploration
  - Both cases lead to bad regret $\Rightarrow$ should be balanced

First Try: Explore-Then-Exploit

- Theorem: Let $N = \left(\frac{T}{K}\right)^{\frac{2}{3}} \cdot (\ln T)^{\frac{1}{3}}$. For every instance, using E-then-E yields
$$\mathbb{E}[R(T)] \le c \cdot T^{\frac{2}{3}} \cdot (K \cdot \ln T)^{\frac{1}{3}} = \tilde{O}\left(T^{\frac{2}{3}} K^{\frac{1}{3}}\right)$$
for some small constant c.
- The regret stems from two sources that our selection of N balances:*
$$\sum_{i=1}^{K} N\Delta(a_i) + (T - N \cdot K) \cdot \Delta(\hat{a})$$

Exploration ($\le N \cdot K$)

Exploitation



Probability Tools for Finite Sample

- Hoeffding's Inequality: Let $X_1, \dots X_n$ be i.i.d. variables with support $[0,1]$. Let $S_n = \sum_{i=1}^{n} X_n$. It holds that

$$\Pr[|S_n - \mathbb{E}[S_n]| \geq \epsilon] \leq 2\exp\left(-\frac{2\epsilon^2}{n}\right)$$

$$\mathrm{P}(|S_n - \mathrm{E}[S_n]| \geq t) \leq 2\exp\left(-\frac{2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

- Say we sampled N times arm a, and got an empirical average of $\bar{\mu}(a)$. How to use it?
- Notice that $\bar{\mu}(a) = S_n/N$, $\mu(a) = \mathbb{E}[S_n]/N$. Set $\epsilon = \alpha N$

$$\Pr[|S_n - \mathbb{E}[S_n]| \geq \epsilon] = \Pr[|S_n - \mathbb{E}[S_n]| \geq \alpha N] = \Pr\big[|\big(\bar{\mu}(a) - \mu(a)\big)N| \geq \alpha N\big] = \Pr[|\bar{\mu}(a) - \mu(a)| \geq \alpha]$$

- On the other hand,

$$\Pr[|S_n - \mathbb{E}[S_n]| \geq \epsilon] \leq 2\exp\left(-\frac{2\epsilon^2}{N}\right) = 2\exp\left(-\frac{2\alpha^2 N^2}{N}\right) = 2\exp(-2\alpha^2 N)$$

- Thus

$$\Pr[|\bar{\mu}(a) - \mu(a)| \geq \alpha] \leq 2\exp(-2\alpha^2 N)$$

- Equivalently,

$$\Pr[|\bar{\mu}(a) - \mu(a)| \leq \alpha] \geq 1 - 2\exp(-2\alpha^2 N)$$

- Set $\alpha = \sqrt{\frac{2\ln T}{N}}$. We get

$$\Pr\left[|\bar{\mu}(a) - \mu(a)| \leq \underbrace{\sqrt{\frac{2\ln T}{N}}}_{Radius}\right] \geq 1 - 2\exp\left(-2\frac{2\ln T}{N}N\right) = 1 - 2\exp(\ln T^{-4}) = 1 - \frac{2}{T^4}$$

Proof (1)

- Confidence interval:

$$\Pr\left[|\bar{\mu}(a) - \mu(a)| \le \underbrace{\sqrt{2\ln T/N}}_{Rad}\right] \ge 1 - \frac{2}{T^4}$$

- Equivalently, w.h.p. $\bar{\mu}(a) \in [\mu(a) - Rad, \mu(a) + Rad]$
  - What you see is what you get!
- Recall: The regret stems from two sources that our selection of N balances:

$$\underbrace{N \cdot K}_{\text{exploration}*} + \underbrace{(T - N \cdot K) \cdot \Delta(\hat{a})}_{\text{exploitation}}$$

- If N is large enough, Rad is small and so is $\Delta(\hat{a})$*
  - BUT, pay too much for exploration
- If N is small, exploration cost is small
  - BUT, Rad is large and so is $\Delta(\hat{a}) \rightarrow$ pay too much for exploitation of wrong arms

Proof (2.1)

- Recall: $Rad = \sqrt{\frac{2\ln T}{N}}$
- Let X denote the *clean event*: every empirical average $\bar{\mu}(a)$ is inside its Rad
  - $\Pr(X) \geq 1 - \frac{2K}{T^4}$ *(union bound)*

$$\mathbb{E}[R(T)]$$

$$= \Pr[X]\mathbb{E}[R(T)|X] + \Pr[\bar{X}]\,\mathbb{E}[R(T)|\bar{X}]) \overbrace{\leq}^{(1)} \Pr[X]\mathbb{E}[R(T)|X] + \frac{2K}{T^4}\mathbb{E}[R(T)|\bar{X}]$$

$$\overbrace{\leq}^{(2)} \Pr[X]\mathbb{E}[R(T)|X] + 1 \overbrace{\leq}^{(3)} \mathbb{E}[R(T)|X] + 1$$

- Consequently, we focus on the clean event only!

Proof (2.2)

- Consequently, we focus on the clean event only!
- If $\hat{a} = a^\star$ (line 2 of the algorithm), then we are done since $\Delta(\hat{a}) = \Delta(a^*) = 0$
- Otherwise, if $\hat{a} \neq a^\star$ then (recall clean event)
$$\mu(\hat{a}) + Rad > \bar{\mu}(\hat{a}) \geq \bar{\mu}(a^\star) \geq \mu(a^\star) - Rad$$
- Rearranging,
$$\Delta(\hat{a}) \coloneqq \mu(a^\star) - \mu(\hat{a}) \leq 2Rad = \sqrt{\frac{8 \ln T}{N}}$$

Proof (3.1)

- So $\mathbb{E}[R(T)|X]$ is upper bounded by

$$\underbrace{N \cdot K}_{\text{exploration}} + \underbrace{(T - N \cdot K) \cdot \overbrace{\sqrt{\frac{8 \ln T}{N}}}^{\Delta(\hat{a})}}_{\text{exploitation}}$$

- Set $N = \left(\frac{T}{K}\right)^{\frac{2}{3}} \cdot (\ln T)^{\frac{1}{3}}$

Proof (3.2)

- Set $N = \left(\frac{T}{K}\right)^{\frac{2}{3}} \cdot (\ln T)^{\frac{1}{3}}$
- Exploration:

$$N \cdot K = \left(\frac{T}{K}\right)^{\frac{2}{3}} \cdot (\ln T)^{\frac{1}{3}} \cdot K = \tilde{O}\left(T^{\frac{2}{3}}K^{\frac{1}{3}}\right)$$

- Exploitation:

$$(T - N \cdot K) \cdot \sqrt{\frac{8\ln T}{N}} \le T\sqrt{\frac{8\ln T}{N}} = \tilde{O}\left(TN^{-\frac{1}{2}}\right) = \tilde{O}\left(T\left(\frac{T}{K}\right)^{-\frac{1}{2}\cdot\frac{2}{3}}\right)\tilde{O}\left(T\left(\frac{T}{K}\right)^{-\frac{1}{3}}\right) = \tilde{O}\left(T^{\frac{2}{3}}K^{\frac{1}{3}}\right)$$

- All together,

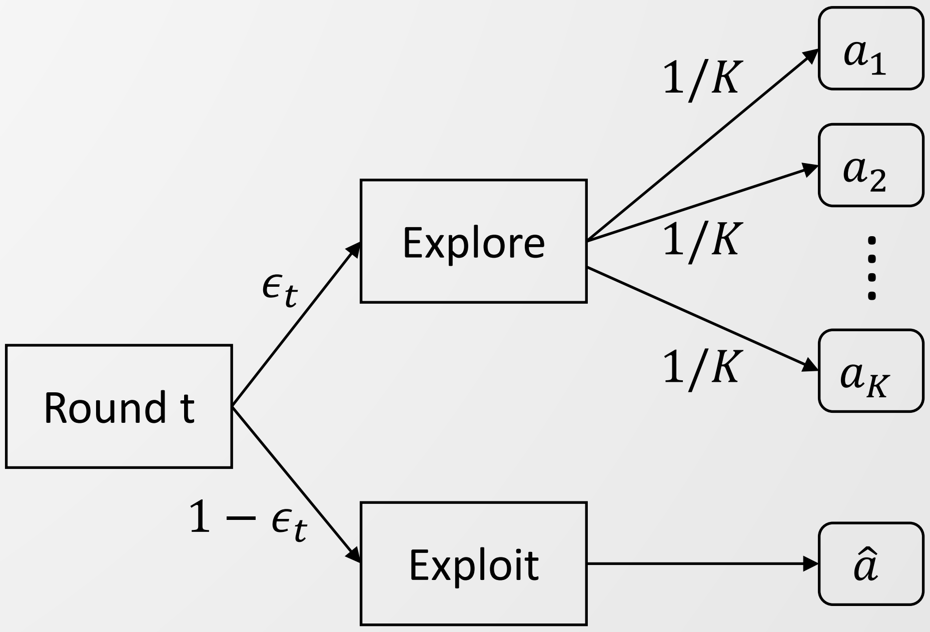
$$\mathbb{E}[R(T)] \le \mathbb{E}[R(T)|X] + 1 \le \tilde{O}\left(T^{\frac{2}{3}}K^{\frac{1}{3}}\right)$$

$\blacksquare$

ϵ-Greedy

Second Try: ϵ-Greedy

- Algorithm:



- Observation: If ϵ_t is constant, $R(T)$ is linear in T!
- Theorem: For the best selection of ϵ_t, it holds that $R(T) \approx \tilde{O}\left(T^{\frac{2}{3}}\right)$

UCB - Upper Confidence Bound

State of the art…

Third Try: Upper Confidence Bound (UCB)

- Optimism in the face of uncertainty!
- Main statistical tool (relies on Hoeffding Inequality)

$$\Pr[|\bar{\mu}_t(a) - \mu(a)| \le Rad_t(a)] \ge 1 - \frac{2}{T^4},$$

where $Rad_t(a) = \sqrt{\frac{2\ln T}{n_t(a)}}$, and $n_t(a)$ is the # times arm a got selected

- In other words, w.h.p. we have

$$\mu(a) \in [\bar{\mu}_t(a) - Rad_t(a), \underbrace{\bar{\mu}_t(a) + Rad_t(a)}_{UCB_t(a)}]$$

Optimism!

Third Try: Upper Confidence Bound (UCB)

- Optimism in the face of uncertainty!
- Algorithm:
  1. Try each arm once (K rounds)
  2. For each round $t = K + 1 \dots T$ do
     - Pick an arm a that maximizes $UCB_t(a)$

$$UCB_t(a) = \bar{\mu}_t(a) + Rad_t(a)$$

- An arm can have a high $UCB_t(a)$ for two reasons (or combination):
  - High average reward $\bar{\mu}_t(a)$
  - The confidence radius $Rad_t(a)$ is large $\Rightarrow$ hasn't been explored much

Third Try: Upper Confidence Bound (UCB)

1. Using the statistical bounds, we get that (applying twice the union bound)

$$\Pr[\forall a, t : |\bar{\mu}_t(a) - \mu(a)| \le Rad_t(a)] \ge 1 - \frac{2}{T^2}$$

2. If $a_t \neq a^\star$, then

$$\mu(a_t) + 2Rad_t(a_t) \ge \bar{\mu}_t(a_t) + Rad_t(a_t) = UCB_t(a_t) \ge UCB_t(a^\star) \ge \mu(a^\star)$$

3. Consequently, $\mu(a^\star) - \mu(a_t) = \Delta(a_t) \le 2Rad_t(a_t) = 2\sqrt{{}^{2\ln T}/_{n_t(a_t)}}$

4. We have

$$R(T) = \mu^* \cdot T - \sum_{t=1}^{T} \mu(a_t) = \sum_{a \in A} n_T(a)\left(\mu(a^\star) - \mu(a)\right)$$

$$= \sum_{a \in A} n_T(a)\,\Delta(a) \le \sum_{a \in A} 2\sqrt{2n_T(a)\ln T} = \sqrt{8\ln T}\sum_{a \in A}\sqrt{n_T(a)}$$

$$\le \sqrt{8TK\ln T}$$

Jensen's Inequality: if $a_1, \dots, a_n$ are real numbers and f is concave, then $\sum_{i=1}^{n} f(a_i) \le f\left(\sum_{i=1}^{n} a_i\right)$

Also: Instance dependent bounds, $R(T) \le \frac{K \ln T}{\Delta}$ for $\Delta = \min \Delta(a)$

Thompson Sampling

State of the art…

Bayesian Priors

- Bernoulli: For each arm i, there is a true (unknown) CTR θ_i
  - 1 if clicked, 0 otherwise
- For each arm, we construct a distribution probability $f_i(\theta_i = t)$

$$\overbrace{\Pr(\theta_i = t|evidance)}^{\text{Posterior}} = \frac{\overbrace{\Pr(evidance|\theta_i = t)}^{\text{likelihood}}\overbrace{\Pr(\theta_i = t)}^{\text{Prior}}}{\underbrace{\Pr(evidance)}_{\text{Const.}}}$$

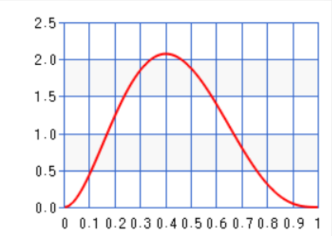
Conjugate Prior

$$\overbrace{\Pr(\theta_i = t|evidance)}^{\text{Posterior}} = \frac{\overbrace{\Pr(evidance|\theta_i = t)}^{\text{likelihood}}\overbrace{\Pr(\theta_i = t)}^{\text{Prior}}}{\underbrace{\Pr(evidance)}_{\text{Const.}}}$$

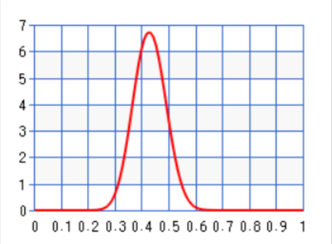
- Assume uniform distribution, $\theta \sim Uni(0,1)$
- Conditioning on θ, the number of clicks we got from N impressions follows $Binomial(N, \theta)$
  - For instance, $\Pr(12\ clicks, 794\ impressions|\theta) = \binom{794}{12}\theta^{12}(1-\theta)^{794-12}$
- Then, the posterior distribution is

$$Beta(\text{clicks} + 1, \text{impressions} - \text{clicks} + 1)$$

Beta(3,4)



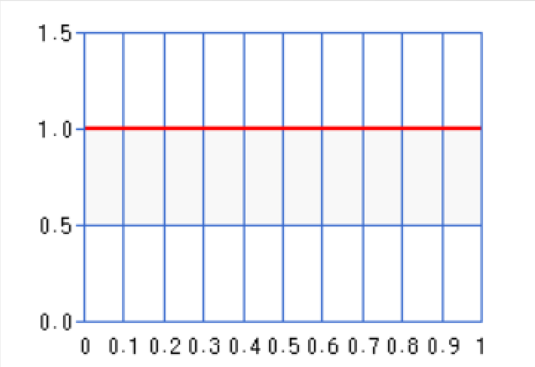
Beta(30,40)



Example

Round	Hits	Misses	Posterior
0	0	0	$Beta(1,1)$

Beta(1,1)



Example

Round	Hits	Misses	Posterior
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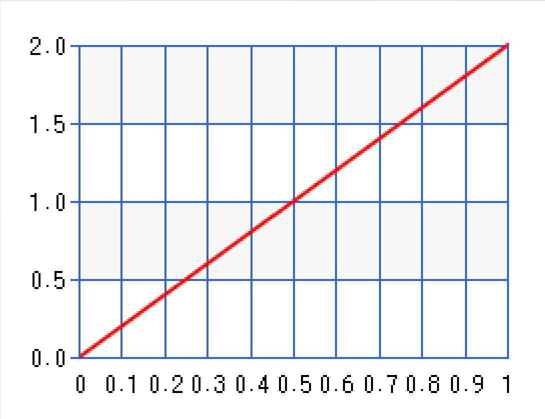
1	1	0	$Beta(2,1)$

Beta(2,1)



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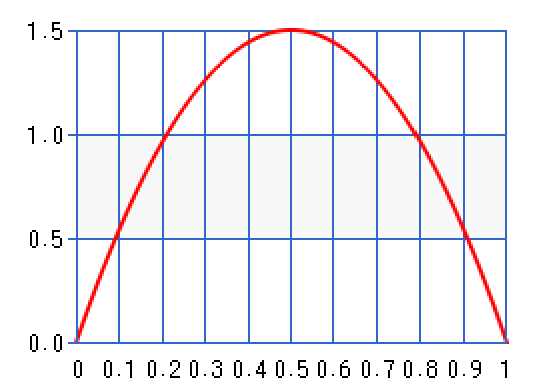
Beta(2,2)



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0	0	0	$Beta(1,1)$
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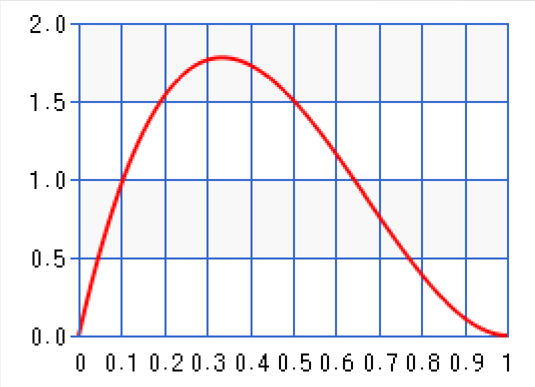
3	1	2	$Beta(2,3)$

Beta(2,3)



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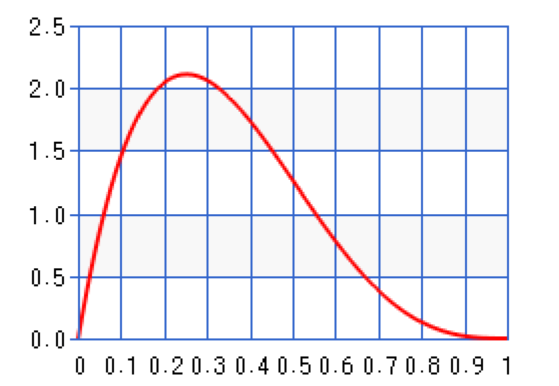
Beta(2,4)



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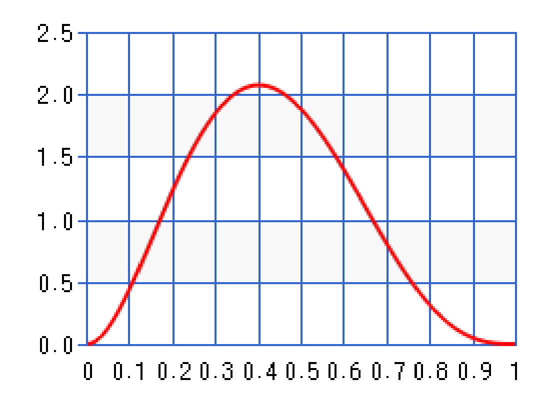
5	2	3	$Beta(3,4)$

Beta(3,4)



- Algorithm demo

Fourth Try: Thompson Sampling

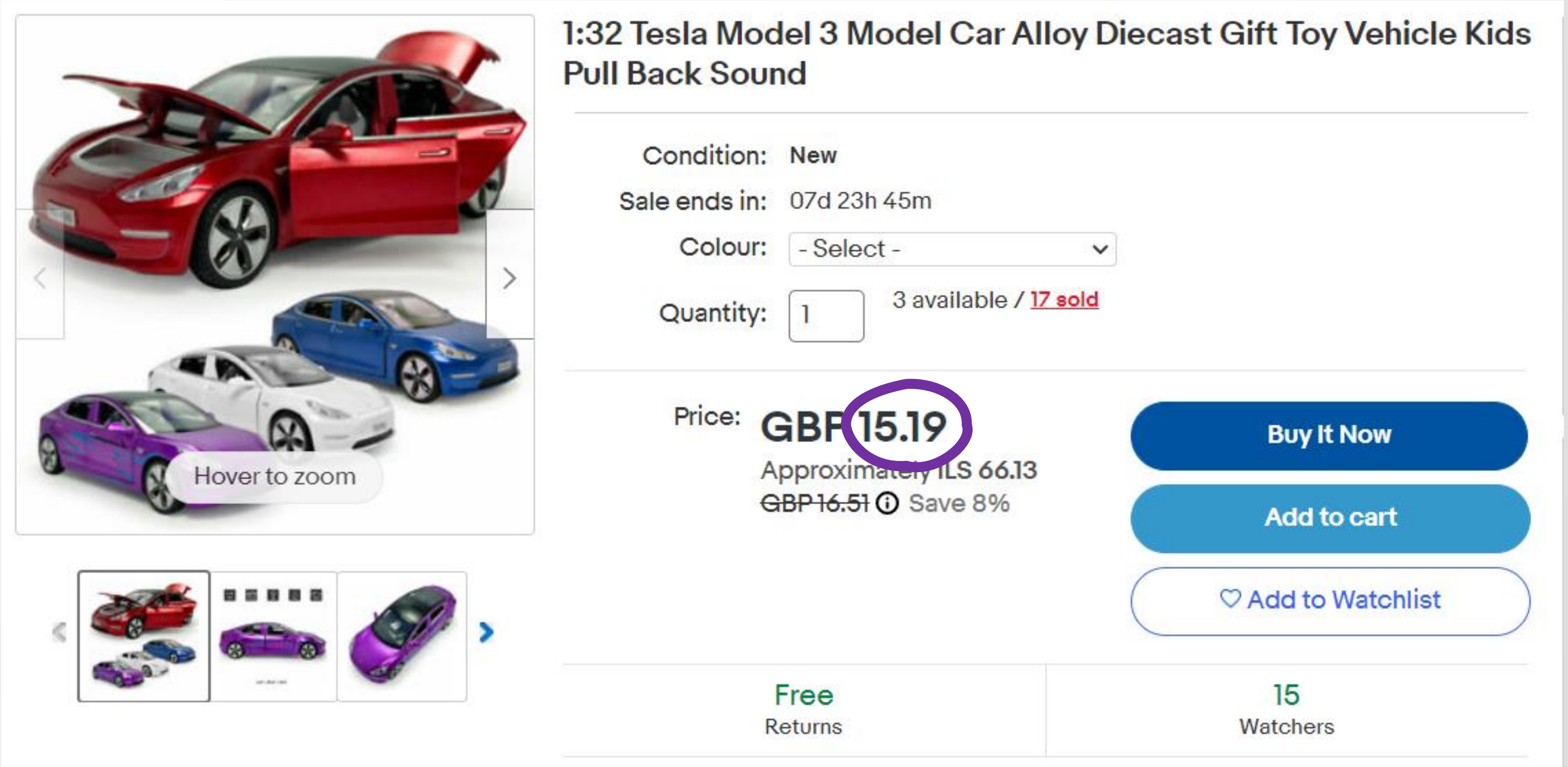
- Algorithm:
  1. For each round $t = 1 \dots T$ do
     - Sample from the posterior distribution of each arm
     - Pick the arm with the highest sample
     - Update the posterior of that arm

- An arm can have a high sample if
  - It has a high empirical average
  - It has a high variance – was not selected too much

Outline

- Motivation
- Multi-armed bandits 101
- Algorithms
  - Explore-then-exploit
  - ϵ-greedy
  - UCB
  - Thompson sampling
- Extensions – brief
  - Continuum of arms
  - Contextual bandits
  - Full feedback with adversarial costs

Continuum-Armed Bandits



Continuum-Armed Bandits

- Arms correspond to feature vectors
  - Dynamic pricing, documents,…
- Today: Arms correspond to points in the interval X
- Reward: $\mu_t(x) \sim Ber\big(\mathrm{M}(x)\big)$, where $\mathrm{M}: X \to [0,1]$,
- Without structural assumptions ⇒ Hopeless
  - Needle in a haystack
  - Require auxiliary information on similarity between arms
- Lipchitz condition:

$$\forall x, y \in X: \quad |\mathrm{M}(x) - \mathrm{M}(y)| \leq L|x - y|$$

Continuum-Armed Bandits

- Naïve solution: Uniform discretization
  - Fixed discretization error in every round → linear regret
- The Zooming algorithm: Adaptive discretization
  - Covers all arms with confidence radii
  - Maintain a set of active arms
  - Runs UCB on the set of active arms
  - If an arm is not covered, actives it (→ covered)

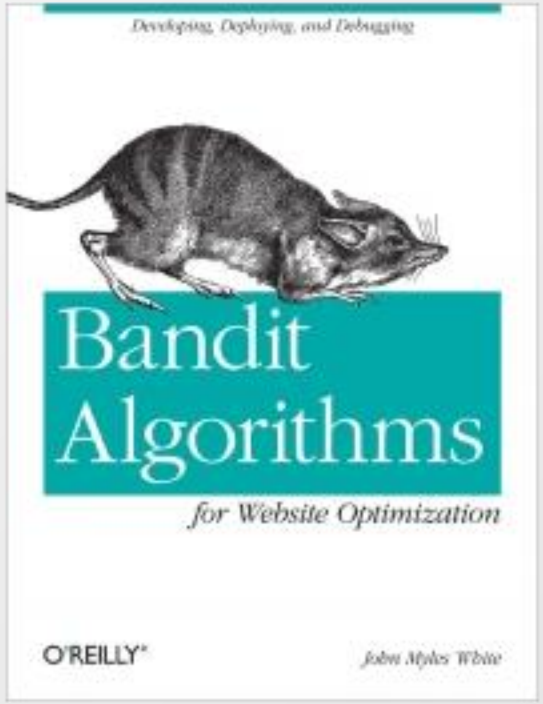
- Achieves regret $\leq O\left(T^{\frac{2}{3}}(L\ln T)^{\frac{1}{3}}\right)$

Example

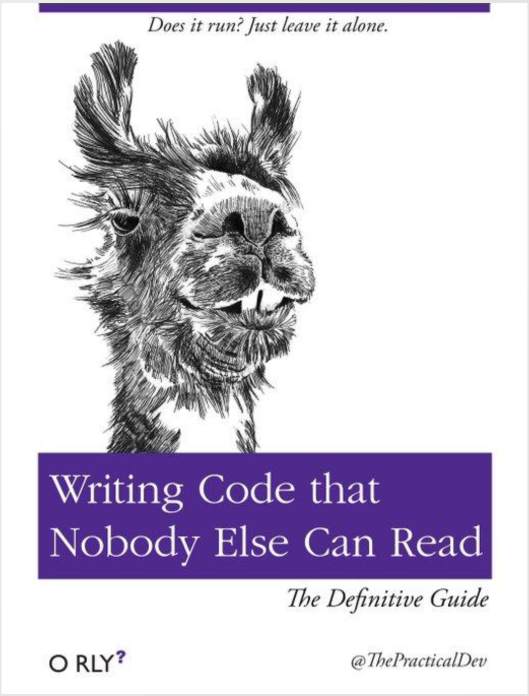
Contextual Multi-Armed Bandits

- Select an arm based on context!

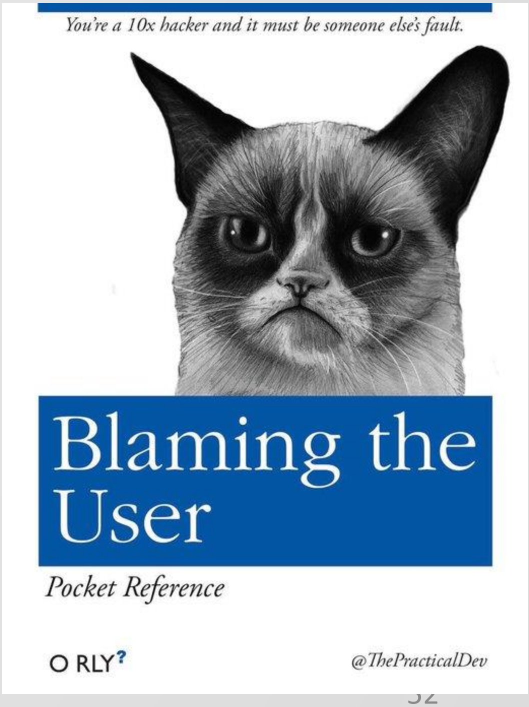
Data Scientist



Data Analyst



Developer



Contextual Bandits

- In each round $t \in [T]$:
  1. Algorithm observes u_t and a set A_t of arms with their feature vectors $x_{t,a}$
     - These vectors summarize information from both user and arm→context
  2. Algorithm chooses $a_t \in \mathrm{A}_t$
  3. Algorithm receives reward r_{t,a_t}, observers nothing else
     - Assuming $\mathbb{E}\left(r_{t,a} \middle| x_{t,a}\right) = x_{t,a}^T \cdot \mu_{a_t}$ for every arm a, where $\mu_a \in \mathbb{R}^d$ is an unknown vector
- Let $a_t^\star$ be the best arm in round t
- Regret:

$$\mathbb{E}\left[\sum_{t=1}^{T} r_{t,a_t^\star}\right] - \mathbb{E}\left[\sum_{t=1}^{T} r_{t,a_t}\right]$$

LinUCB algorithm: $\tilde{\mathrm{O}}\left(\sqrt{dT}\right)$ works well in practice even for scenarios without linearity

[link to paper]

Full Feedback and Adversarial Costs

- IID assumption is "too easy"
  - Rewards can arbitrarily change over time
  - Models time shifts, seasonality, diminishing returns, etc.
- All rewards are visible, not only that of the chosen arm (Full feedback)
- Parameters: K arms, T rounds (known)
- In each round $t \in [T]$:
  1. Adversary chooses rewards $r_t(a) \geq 0$ for each arm
  2. Algorithm picks some arm a_t
  3. Algorithm get reward $r_t(a_t)$ for the chosen arm
  4. The rewards of all arms, $r_t(a): a \in [K]$ are revealed
- Regret

$$\max_a \sum_{t=1}^{T} r_t(a) - \sum_{t=1}^{T} r_t(a_t)$$

Hedge algorithm: $O\left(\sqrt{T \ln K}\right)$

Conclusion

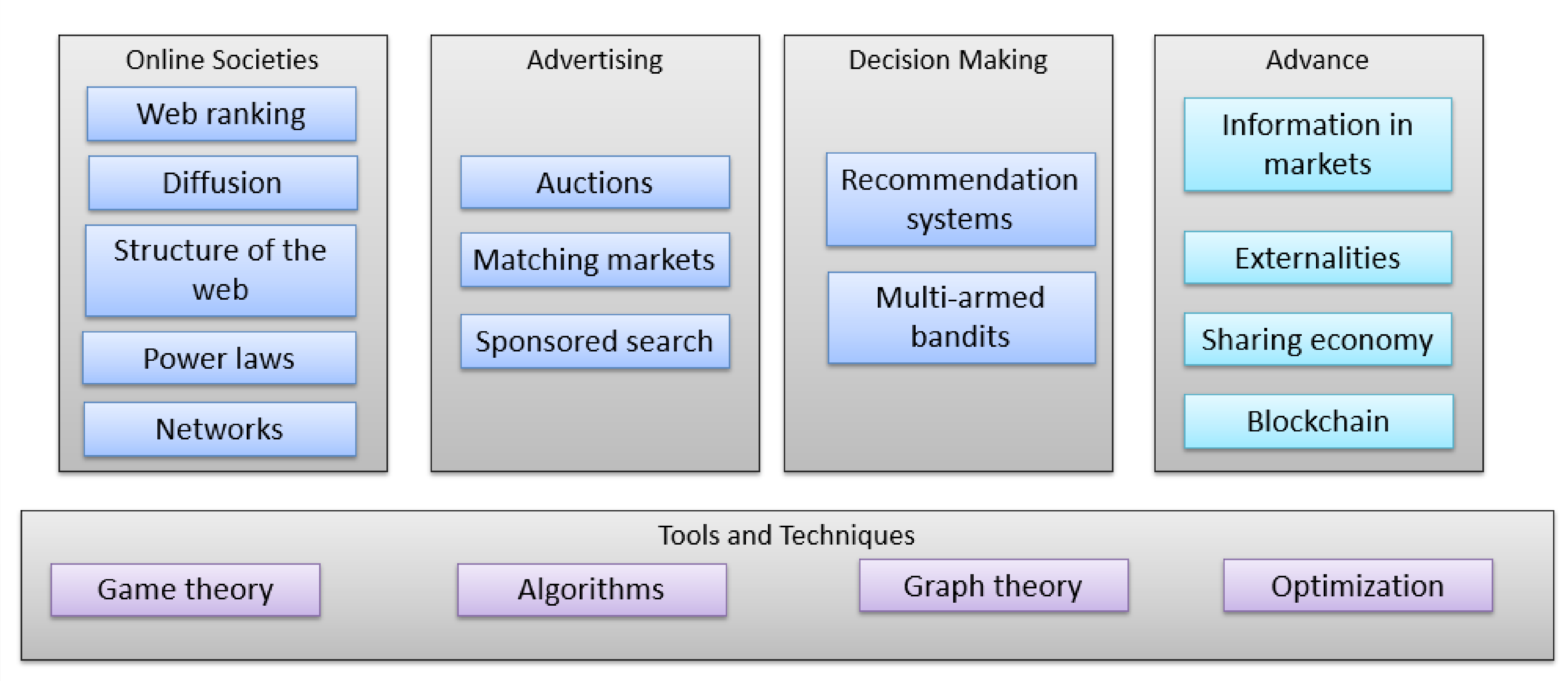
- A power algorithmic toolbox, well-studied, with many variants and extensions
- Main idea: Balance exploration and exploitation
- Algorithms are usually easy to implement
  - Implementing for large-scale applications runs into several engineering challenges

- Useful resources: [Introduction to Multi-Armed Bandits, Alex Slivkins]

Electronic Commerce
096211

Multi-Armed Bandits

Slides by Omer Ben-Porat



Online Societies
Web ranking
Diffusion
Structure of the web
Power laws
Networks
Advertising
Auctions
Matching markets
Sponsored search
Decision Making
Recommendation systems
Multi-armed bandits
Advance
Information in markets
Externalities
Sharing economy
Blockchain
Tools and Techniques
Game theory
Algorithms
Graph theory
Optimization


DOUBLE
DOUBLE DIAMOND
TRIPLE DIAMOND

Outline

- Motivation
- Multi-armed bandits 101 (recap)
- Lower Bound
- Algorithms
  - UCB (+analysis)
  - Thompson sampling
- Extensions – brief
  - Continuum of arms
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Which is More Appealing (=CTR)

Mail News Finance

Murder victim found in adult entertainment venue
- www.npr.org/.../headless-body-in-**topless-bar**-headline-writer-d...
9 ביוני 2015 -), who wrote what some consider one of the best headlines
of all time, died Tuesday at the age of 74.

Headless Body found in Topless Bar
../headless-body-in-**topless-bar**-headline-writer-d...
9 ביוני 2015 - at some consider one of the best headlines
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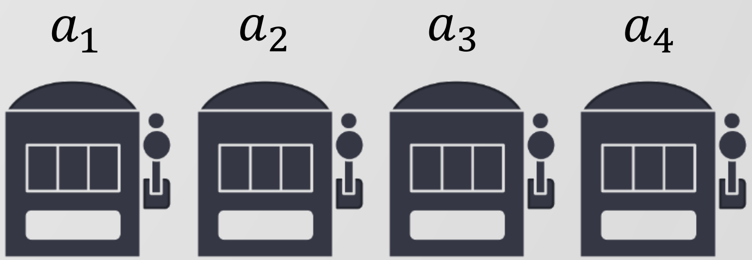
Basic Setting

- Parameters: K arms, T rounds (known); reward distribution D_a for each arm a (unknown)
- In each round $t = 1, \ldots T$:
  1. Algorithm picks some arm a_t
  2. Reward $r_t \in [0,1]$ is sampled independently from D_{a_t}
  3. Algorithm collects reward r_t, observers nothing else

- Notations
  - Mean reward: $\mu(a) := \mathbb{E}[D_a]$
  - Best reward: $\mu^\star \coloneqq \max_{a \in A} \mu(a)$
  - Gap: $\Delta(a) \coloneqq \mu^\star - \mu(a)$



Regret

- Given a MAB algorithm, the cumulative regret is

$$R(T) = \mu^* \cdot T - \sum_{t=1}^{T} r_t(a_t)$$

- Recall that a_t depends on the algorithm and past rewards
  - $\Rightarrow R(T)$ is a random variable!
  - Expected regret $\mathbb{E}[R(T)]$ and not the realized one
- "Good" algorithms have very low average regret: $\frac{\mathbb{E}[R(T)]}{T} \to 0$
  - But we can do a lot better…

First Try: Explore-Then-Exploit

- Algorithm:
  1. Exploration phase: Try each arm N times
  2. Select an arm $\hat{a}$ with the highest average reward
  3. Exploitation phase: Play arm $\hat{a}$ in all remaining rounds

- Exploration: $N \cdot K$, exploitation: $T - N \cdot K$
- Tradeoff:
  - Small N sometime leads to picking "bad" arms
  - Large N wastes many rounds on exploration
  - Both cases lead to bad regret $\Rightarrow$ should be balanced

First Try: Explore-Then-Exploit

- Theorem: Let $N = \left(\frac{T}{K}\right)^{\frac{2}{3}} \cdot (\ln T)^{\frac{1}{3}}$. For every instance, using E-then-E yields

$$\mathbb{E}[R(T)] \leq c \cdot T^{\frac{2}{3}} \cdot (K \cdot \ln T)^{\frac{1}{3}} = \tilde{O}\left(T^{\frac{2}{3}} K^{\frac{1}{3}}\right)$$

  for some small constant c.

- Proof outline:
  - Clean event technique (Tutorial)
  - W.h.p., $\hat{a}$ is either the best arm or almost the best

- Question: Is this a good algorithm?

Lower Bound

- Theorem: For any algorithm, there exists an instance for which
$$\mathbb{E}[R(T)] \geq \Omega\left(\sqrt{KT}\right)$$
- *An* instance, not *all* instances!
- Proof idea (informal)
- Consider the possible instances $I_1, \ldots, I_K$, where
$$I_j = \begin{cases} \mu(i) = \frac{1}{2} & \text{if } i \neq j \\ \mu(i) = \frac{1}{2} + \epsilon & i = j \end{cases}$$
- Any algorithm that identifies the best arm should pull $\frac{1}{\epsilon^2}$ times at least 1/3 of the arms to guess correctly w.h.p.
- Spends too much time on exploring sub-optimal arms, or loses a factor of ϵ

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- In other words, w.h.p. we have

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Optimism!

Upper Confidence Bound (UCB)

- Optimism in the face of uncertainty!
- Algorithm:
  1. Try each arm once (K rounds)
  2. For each round $t = K + 1 \dots T$ do
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$$UCB_t(a) = \bar{\mu}_t(a) + Rad_t(a)$$

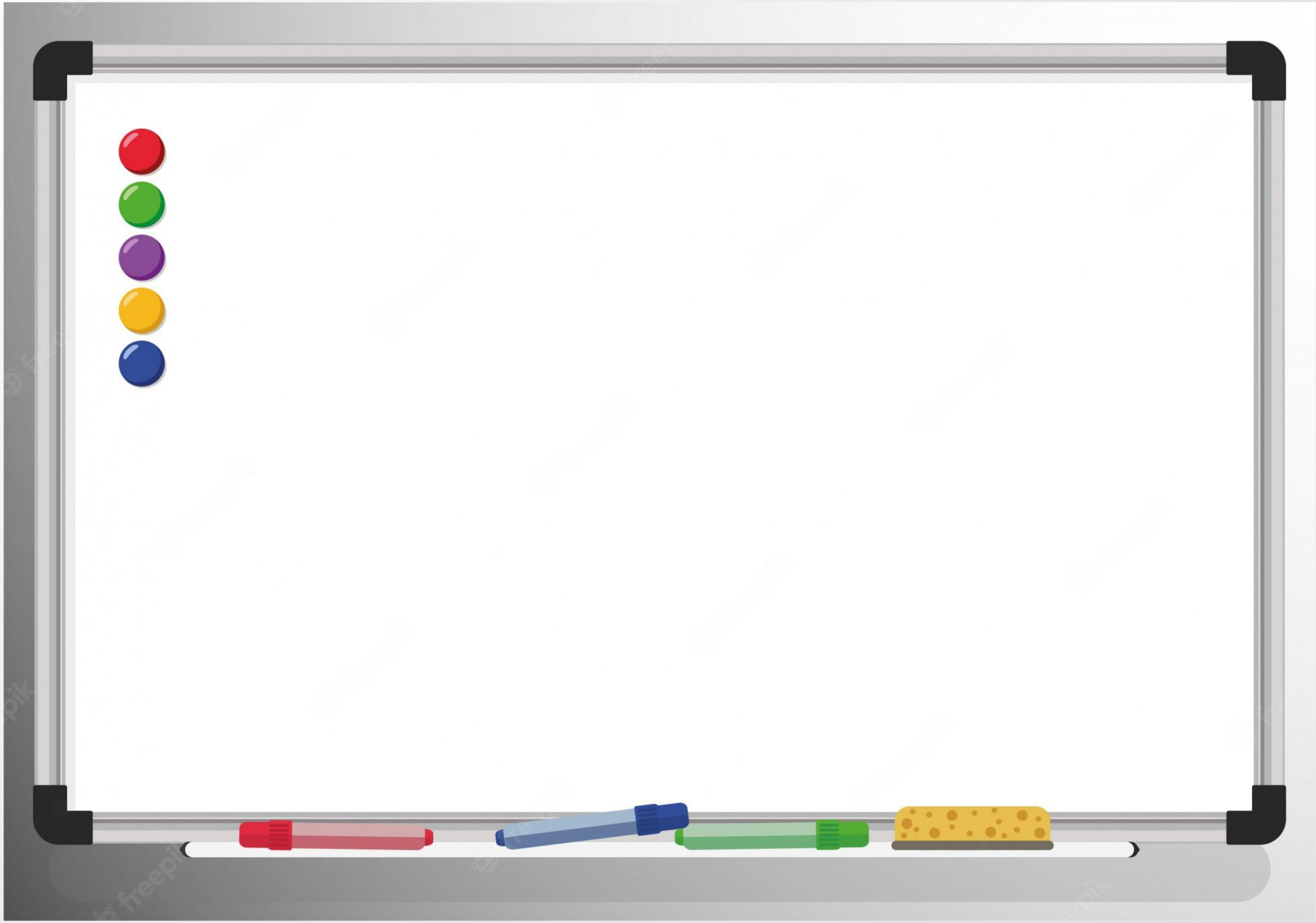
- An arm can have a high $UCB_t(a)$ for two reasons (or combination):
  - High average reward $\bar{\mu}_t(a)$
  - The confidence radius $Rad_t(a)$ is large $\Rightarrow$ hasn't been explored much

UCB: Formal Guarantees

- For every instance, using UCB yields

$$\mathbb{E}[R(T)] \leq c \cdot \sqrt{8TK \ln T} = \tilde{O}\left(\sqrt{KT}\right)$$

- Recall the lower bound $\Omega\left(\sqrt{KT}\right)$, E-then-E $\tilde{O}\left(T^{\frac{2}{3}}K^{\frac{1}{3}}\right)$
- Why is the proof different/harder?
  - In E-then-E, no temporal treatment needed
  - Here, adaptive: Success in round t depends on rounds $1,2,..t-1$
  - But, the “clean event” technique is still helpful
  - $Rad_t(a) = \sqrt{\frac{2 \ln T}{n_t(a)}}$



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$$= \sum_{a\in A} n_T(a)\,\Delta(a) \le \sum_{a\in A} 2\sqrt{2n_T(a)\ln T} = K\sqrt{8\ln T}\,\frac{1}{K}\sum_{a\in A}\sqrt{n_T(a)} \le K\sqrt{8\ln T}\sqrt{\frac{1}{K}\sum_{a\in A} n_T(a)}$$

$$\le \sqrt{8TK\ln T}$$

Also: Instance dependent bounds, $R(T) \le \frac{K\ln T}{\Delta}$ for $\Delta = \min \Delta(a)$

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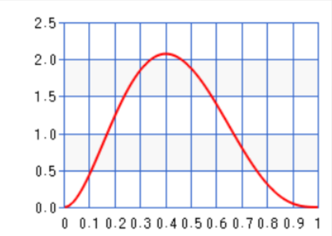

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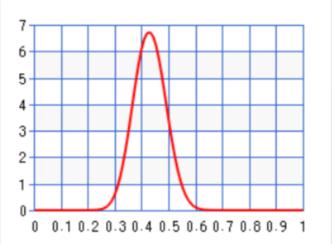

Tricks and Magic

- Reminder: $f_{\alpha,\beta}(\theta) = \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}$, where $B(\alpha,\beta)$ is the Beta function
- $f(\theta|evidance) = \frac{\overbrace{P(evidance|\theta)}^{Bin\ (N,\theta)}\ \overbrace{f(\theta)}^{Beta(\alpha,\beta)}}{P(evidance)} = \frac{\binom{n}{k}\theta^k(1-\theta)^{n-k}f_{\alpha,\beta}(\theta)}{P(evidance)}$

$$= \frac{\binom{n}{k}\theta^k(1-\theta)^{n-k}\theta^{\alpha-1}(1-\theta)^{\beta-1}}{P(evidance)B(\alpha,\beta)} = \frac{\theta^{k+\alpha-1}(1-\theta)^{n-k+\beta-1}}{C}$$

- Since $f(\theta|evidance)$ is a density function, the normalization factor C is the beta function

$$C = B(\underbrace{k}_{Hits} + \alpha, \underbrace{n-k}_{Misses} + \beta)$$

Example

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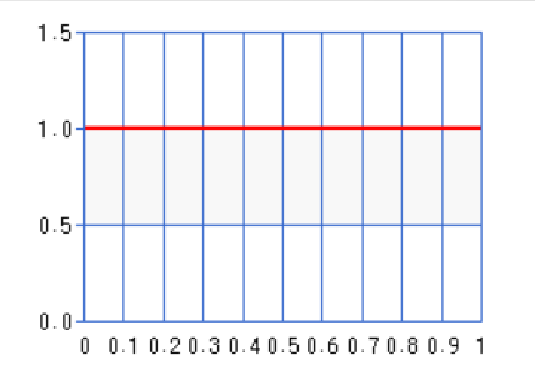
Beta(1,1)



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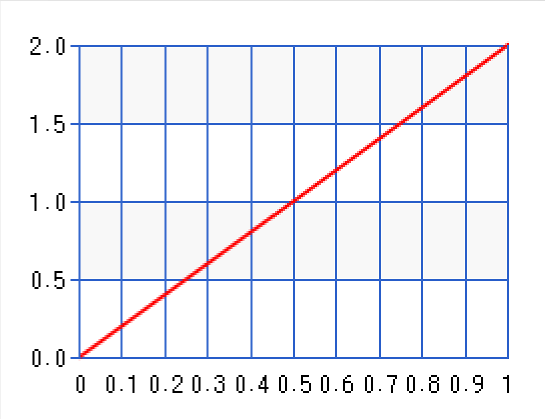

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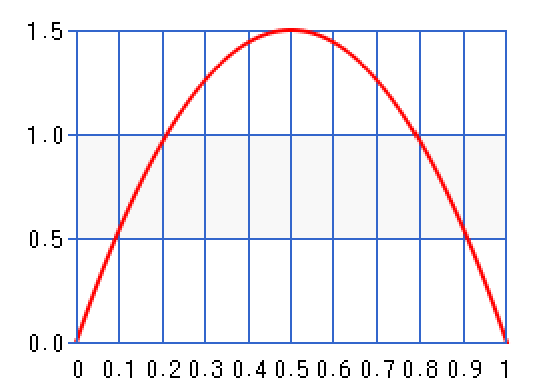
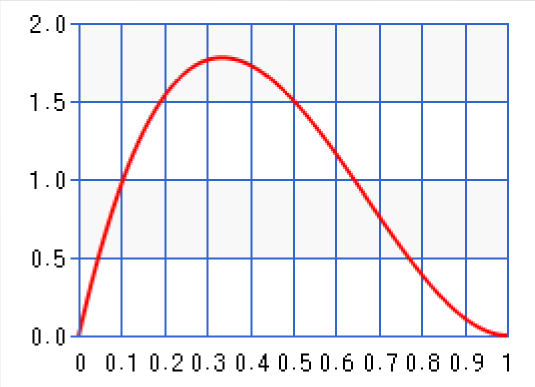


Example

Round	Hits	Misses	Posterior
0	0	0	$Beta(1,1)$
1	1	0	$Beta(2,1)$
2	1	1	$Beta(2,2)$
3	1	2	$Beta(2,3)$

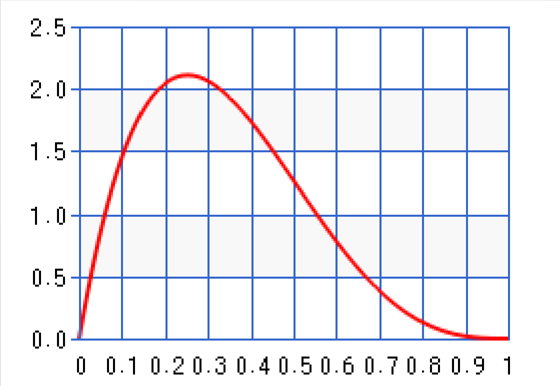
Beta(2,3)



Example

Round	Hits	Misses	Posterior
0	0	0	$Beta(1,1)$
1	1	0	$Beta(2,1)$
2	1	1	$Beta(2,2)$
3	1	2	$Beta(2,3)$
4	1	3	$Beta(2,4)$

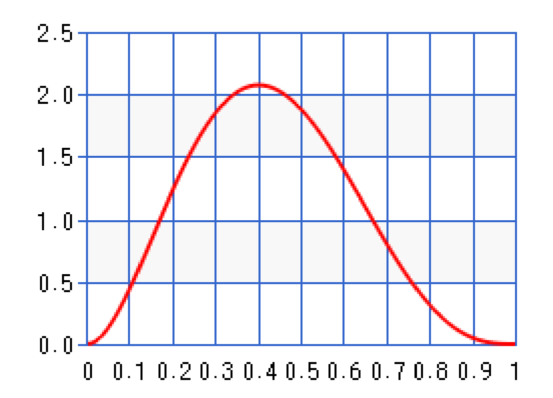
Beta(2,4)



Example

Round	Hits	Misses	Posterior
0	0	0	$Beta(1,1)$
1	1	0	$Beta(2,1)$
2	1	1	$Beta(2,2)$
3	1	2	$Beta(2,3)$
4	1	3	$Beta(2,4)$
5	2	3	$Beta(3,4)$

Beta(3,4)



Thompson Sampling

- Algorithm:
  1. For each round $t = 1 \dots T$ do
     - Sample from the posterior distribution of each arm
     - Pick the arm with the highest sample
     - Update the posterior of that arm

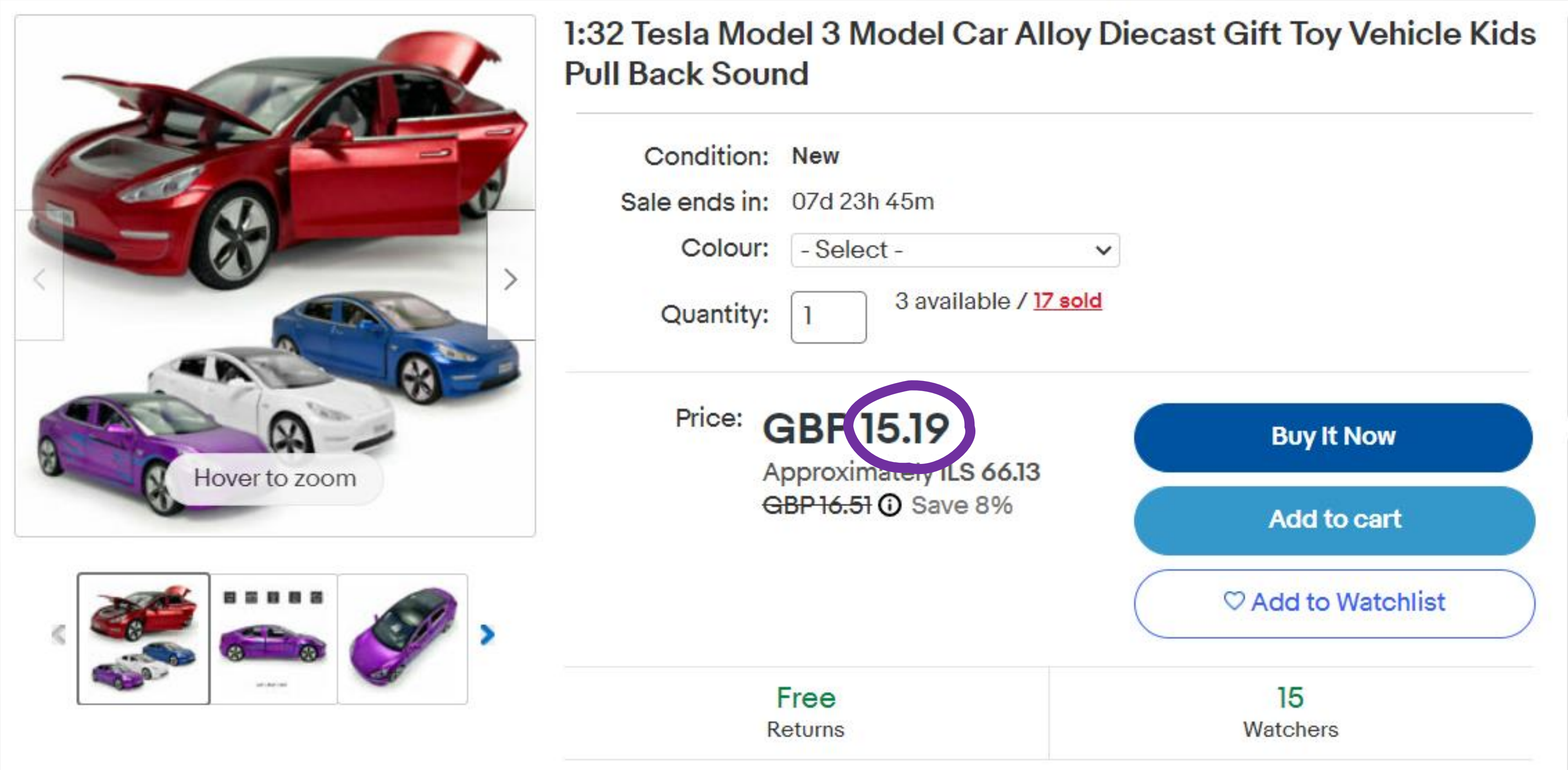
- An arm can have a high sample if
  - It has a high empirical average
  - It has a high variance – was not selected too much

- Algorithm demo

Outline

- Motivation
- Multi-armed bandits 101
- Algorithms
  - Explore-then-exploit
  - ϵ-greedy
  - UCB
  - Thompson sampling
- Extensions – brief
  - Continuum of arms
  - Contextual bandits
  - Full feedback with adversarial costs

Continuum-Armed Bandits



Continuum-Armed Bandits

- Arms correspond to feature vectors
  - Dynamic pricing, documents,…
- Today: Arms correspond to points in the interval X
- In every round, the algorithm picks $a_t \in X$
- Reward: For some $\mathrm{M}: X \to [0,1], r_t(x) \sim Ber\big(\mathrm{M}(x)\big)$, where
  - $\mu(x) \coloneqq \mathbb{E}[r_t(x)] = M(x)$
- Without structural assumptions ⇒ Hopeless (needle in a haystack)
  - $R(T) = \max_{x \in X} \mu(x) \cdot T - \sum_{t=1}^{T} r_t(a_t)$
- Lipchitz condition:

$$\forall x, y \in X: \quad |\mathrm{M}(x) - \mathrm{M}(y)| \leq L|x - y|$$

Continuum-Armed Bandits

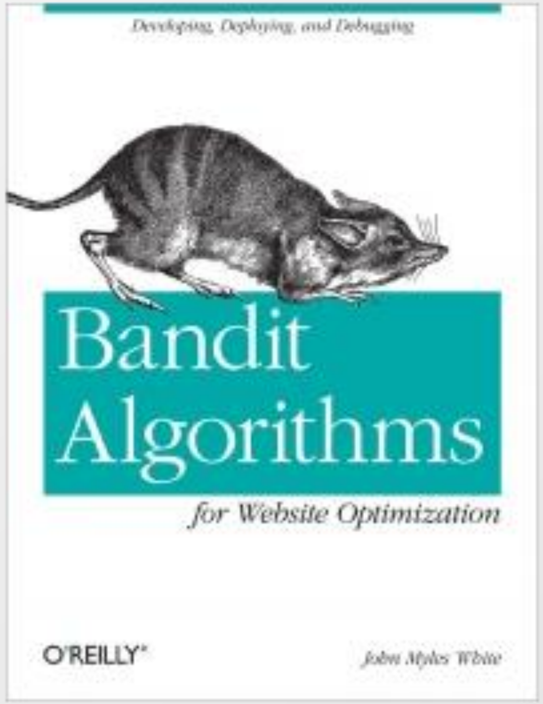
- Naïve solution: Uniform discretization
  - Fixed discretization error in every round → linear regret
- The Zooming algorithm: Adaptive discretization
  - Covers all arms with confidence radii
  - Maintain a set of active arms
  - Runs UCB on the set of active arms
  - If an arm is not covered, actives it (→ covered)
- Achieves regret $\leq O\left(T^{\frac{2}{3}}(L \ln T)^{\frac{1}{3}}\right)$

Example

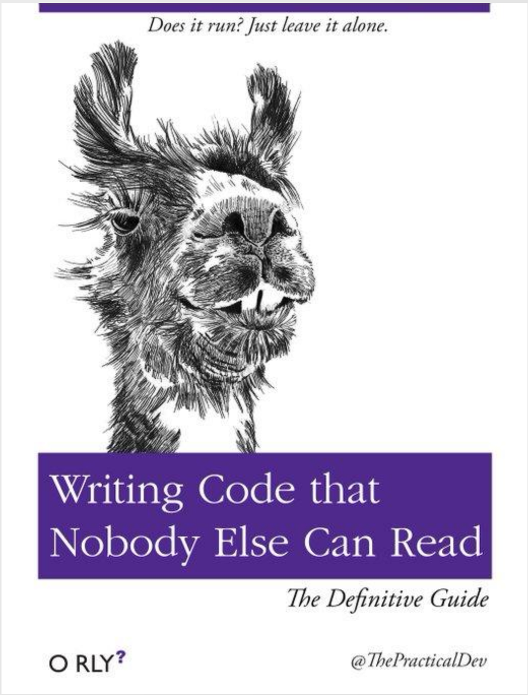
Contextual Multi-Armed Bandits

- Select an arm based on context!

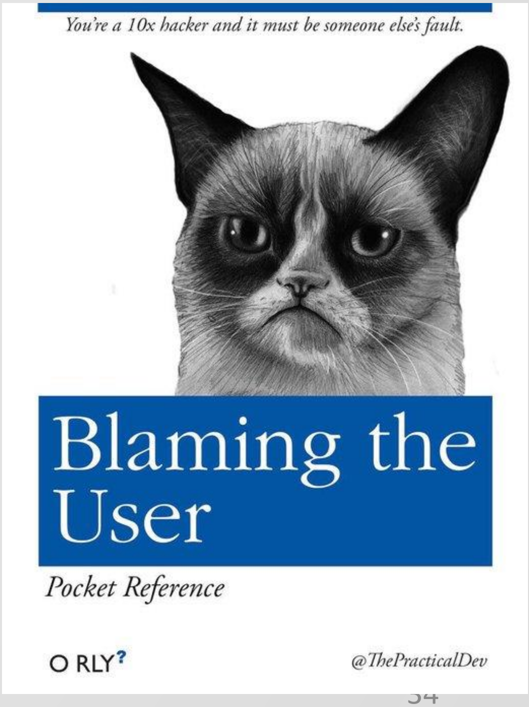
Data Scientist



Data Analyst



Developer



Contextual Bandits

- In each round $t \in [T]$:
    1. Algorithm observes u_t and a set A_t of arms with their feature vectors $x_{t,a}$
        - These vectors summarize information from both user and arm→context
    2. Algorithm chooses $a_t \in A_t$
    3. Algorithm receives reward r_{t,a_t}, observers nothing else
        - Assuming $\mathbb{E}\left(r_{t,a} \middle| x_{t,a}\right) = x_{t,a}^T \cdot \mu_{a_t}$ for every arm a, where $\mu_a \in \mathbb{R}^d$ is an unknown vector
- Let $a_t^\star$ be the best arm in round t
- Regret:

$$\mathbb{E}\left[\sum_{t=1}^{T} r_{t,a_t^\star}\right] - \mathbb{E}\left[\sum_{t=1}^{T} r_{t,a_t}\right]$$

LinUCB algorithm (2010): $\tilde{\mathrm{O}}\left(\sqrt{dT}\right)$ works well in practice even for scenarios without linearity

link to paper

Full Feedback and Adversarial Costs

- IID assumption is "too easy"
  - Rewards can arbitrarily change over time
  - Models time shifts, seasonality, diminishing returns, etc.
- All rewards are visible, not only that of the chosen arm (full feedback)
- Parameters: K arms, T rounds (known)
- In each round $t \in [T]$:
  1. Adversary chooses rewards $r_t(a) \geq 0$ for each arm
  2. Algorithm picks some arm a_t
  3. Algorithm get reward $r_t(a_t)$ for the chosen arm
  4. The rewards of all arms, $r_t(a)$: $a \in [K]$ are revealed
- Regret:

$$\max_a \sum_{t=1}^{T} r_t(a) - \sum_{t=1}^{T} r_t(a_t)$$

Hedge algorithm: $O\left(\sqrt{T \ln K}\right)$

Conclusion

- A power algorithmic toolbox, well-studied, with many variants and extensions
- Main idea: Balance exploration and exploitation
- Algorithms are usually easy to implement
  - Implementing for large-scale applications runs into several engineering challenges

- Useful resources: [Introduction to Multi-Armed Bandits, Alex Slivkins]

Electronic Commerce 096211

Introduction to Recommendation System

Omer Ben Porat (some of the slides are adopted from Prof. Carmel Domshlak)

Course Structure

Online Societies
- Web ranking
- Diffusion
- Structure of the web
- Power laws
- Networks

Advertising
- Auctions
- Matching markets
- Sponsored search

Decision Making
- Recommendation systems
- Multi-armed bandits

Advance
- Information in markets
- Externalities
- Sharing economy
- Blockchain

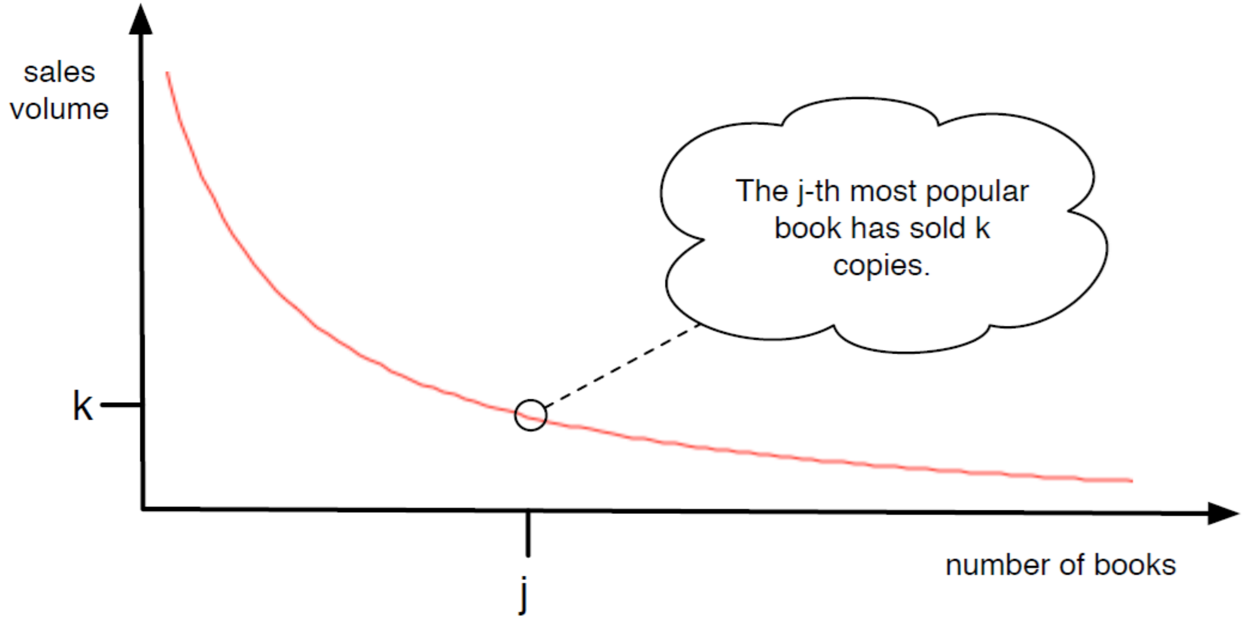
Tools and Techniques
- Game theory
- Algorithms
- Graph theory
- Optimization

Outline

- Motivation
- Setup
  - General
  - Netflix prize
  - Ideas
  - Machine learning workflow (in a nutshell)
- Algorithmic approaches

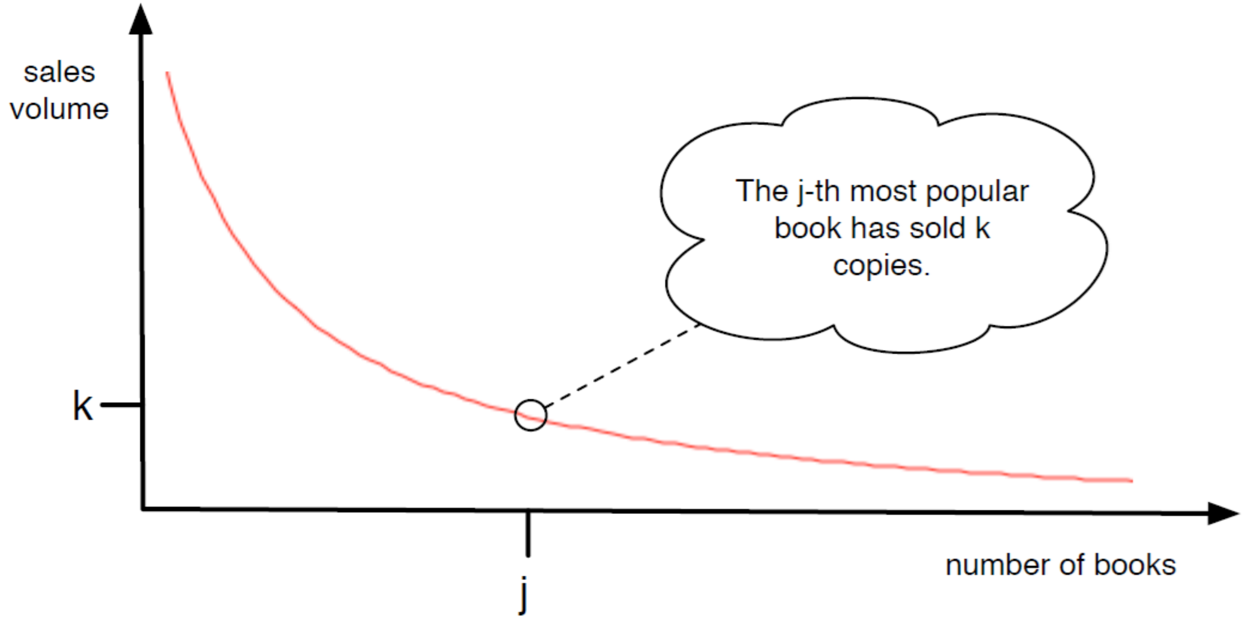
Making Business of Niche Products

- The Long Tale (of popularity distribution)
- Efficient navigation is a game changer
  - search engines
  - recommender systems



Making Business of Niche Products

- The Long Tale (of popularity distribution)
- Efficient navigation is a game changer
  - search engines
  - **recommender systems**



How does Netflix recommend movies?

- Netflix - before streaming
- DVD rental business
  - No rental stores, just wait for DVDs to arrive by mail
  - No late fees, but monthly subscription fee
- Streaming movies and TV programs from video servers
  - 25% of the Internet traffic
  - 23M users by 1998
- Why would it need a recommender system? Why not random?
  - rich getting richer, preferences

Setup: Input

- Millions of users U, tens of thousands items I (movie titles)
- History: Each data point $\langle u, i, r_{ui}, t_{ui} \rangle$ consists of:
  - u user ID
  - i movie title (index)
  - r_{ui} number of stars, 1-5
  - t_{ui} date of the rating
- Billions of data points ...
- Each user rated ≈ 200 movies, each movie was rated ≈ 5000 times
- We won’t be using additional information (actors, genre, directors..)

Movies

Users	1	2	3	4	5	6	7	8
1		5		2	4			
2	4		3	1			3	
3		5	4		5		4	
4						1	1	2
5	3		?		?	3		
6		?	2		4		?	

Setup: Output

- Millions of users U, tens of thousands items I (movie titles)
- Rating predictions: For each $(u, i) \in U \times I$, output $\hat{r}_{ui} \in \mathbb{R}$
  - Interested only in $\hat{r}_{ui}$ for which no r_{ui}
  - Interpretation of $\hat{r}_{ui}$= 4.15?

- **We'll deal with predictions, not decisions** (explain)
  - Decisions: k predictions (= movies) that the user did not watch yet (not really top k...)

Setup: Test

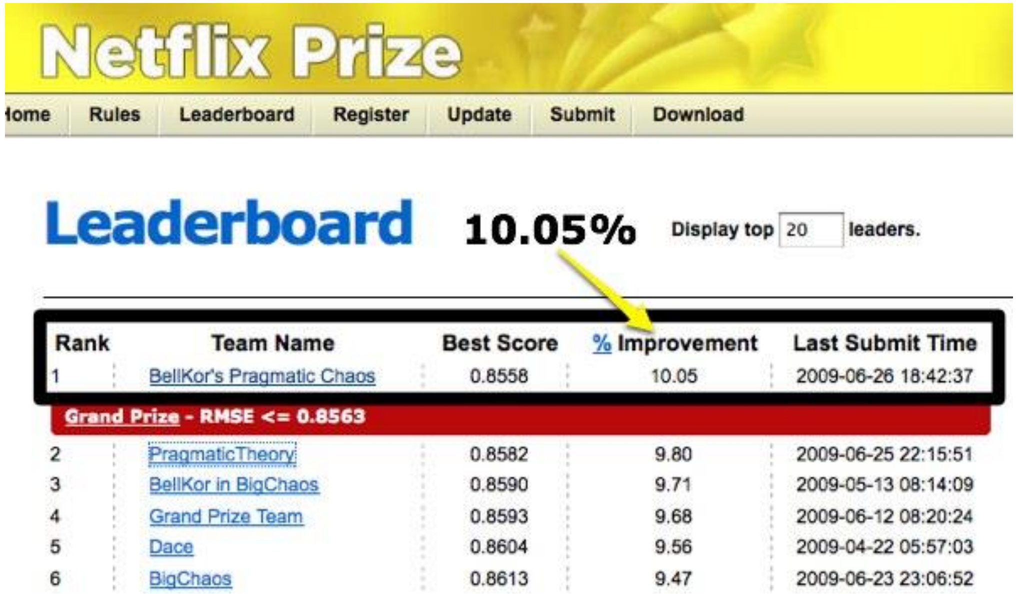
- Real test: Mind-reading ☺
- Proxy: Minimize Root Mean Squared Error (RMSE)
- With C user/item pairs (u, i) for which we have both $\hat{r}_{ui}$ and r_{ui},

$$RMSE = \sqrt{\sum_{(u,i)} \frac{(\hat{r}_{ui} - r_{ui})^2}{C}}$$

- Other tests are possible
- Relates to “output”, not to “final output”
- To sum: Some approximation

Netflix Prize Challenge

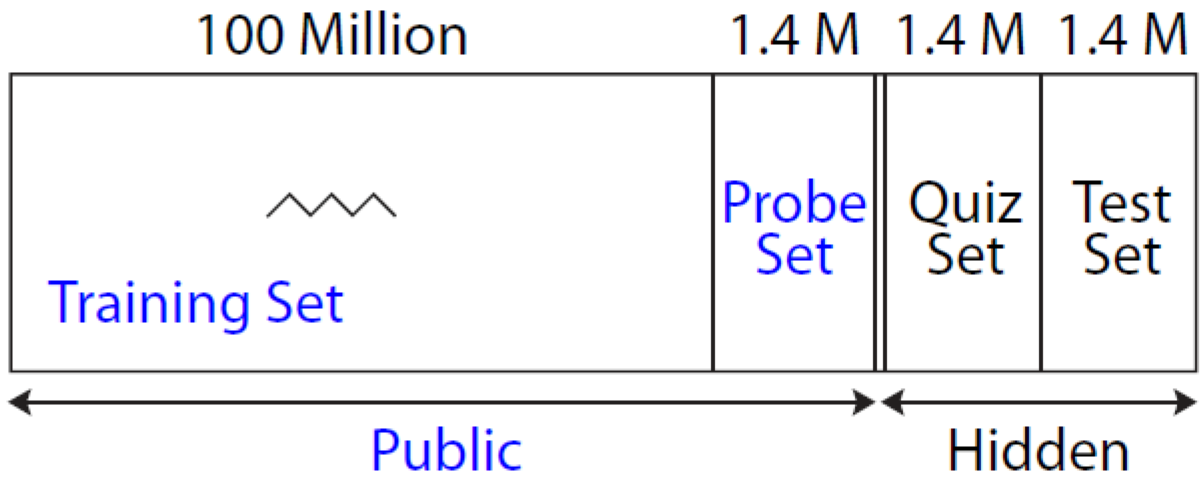
- Instead of hiring engineers and paying millions... Challenge!
- Oct 2006 to Sept 2009
- 1M$ to the winning team



Rank	Team Name	Best Score	% Improvement	Last Submit Time
1	BellKor's Pragmatic Chaos	0.8558	10.05	2009-06-26 18:42:37
Grand Prize - RMSE <= 0.8563				
2	PragmaticTheory	0.8582	9.80	2009-06-25 22:15:51
3	BellKor in BigChaos	0.8590	9.71	2009-05-13 08:14:09
4	Grand Prize Team	0.8593	9.68	2009-06-12 08:20:24
5	Dace	0.8604	9.56	2009-04-22 05:57:03
6	BigChaos	0.8613	9.47	2009-06-23 23:06:52

Netflix Prize Challenge

- Netflix's own system obtained an RMSE of 0.9514 on Quiz
- Reduction of RMSE by just 0.01 already changes some top-10 recommendation lists
- Goal: Improve RMSE by ≥ 10% (RMSE ≤ 0.8563 on Test)



Key Ideas

- Input matrix R (temporal info ignored)
- Main techniques:
  - Content-based filtering
    - If you like a comedy with Mr. Bean, then you will probably like another comedy with Mr. Bean
  - Collaborative filtering (CF)
    - If your ratings are like these of Mr. Bean, then you will probably like stuff that he likes

Movies

Users	1	2	3	4	5	6	7	8
1		5		2	4			
2	4		3	1			3	
3		5	4		5		4	
4						1	1	2
5	3		?		?	3		
6		?	2		4		?	

Key Ideas

- Input matrix R (temporal info ignored)
- Main techniques:
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Movies

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4						1	1	2
5	3		?		?	3		
6		?	2		4		?	

Naïve prediction

- Cosine similarity: $\cos(\boldsymbol{u}, \boldsymbol{v}) = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\|}$
  - Treating blanks as zeros
- Focus on user 1:
  - Similarity to user 2: $\frac{2}{\sqrt{25+4+16}\sqrt{16+9+1+9}} = 0.05$
  - Similarity to user 3: $\frac{25+20}{\sqrt{25+4+16}\sqrt{25+16+25+16}} = 0.74$
- Idea: complete missing values of user 1 with those of user 3
- We will do much better!

Movies

Users	1	2	3	4	5	6	7	8
1		5		2	4			
2	4		3	1			3	
3		5	4		5		4	
4						1	1	2
5	3		?		?	3		
6		?	2		4		?	

Collaborative Filtering (CF)

- Three main approaches:
  - Baseline predictors
  - Neighbourhood model
  - Latent Factor model

- Leading systems: combination of both models + bag of tricks
  - Implicit feedback
  - Temporal dynamics
  - ...

General Machine Learning Workflow

- Divide the data into two parts:
  - Training data
  - Test data
- Define a model, namely a way to compute $\hat{r}_{ui}$
- Use the training data to learn model parameters
- Use test data to evaluate the model

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{pmatrix}
5 & 4 & 4 & - & \color{red}{5} \\
- & 3 & 5 & \color{red}{3} & 4 \\
5 & 2 & - & \color{red}{2} & 3 \\
- & \color{red}{2} & 3 & 1 & 2 \\
4 & - & \color{red}{5} & 4 & 5 \\
\color{red}{5} & 3 & - & 3 & 5 \\
3 & \color{red}{2} & 3 & 2 & - \\
5 & \color{red}{3} & 4 & - & 5 \\
\color{red}{4} & 2 & 5 & 4 & - \\
\color{red}{5} & - & 5 & 3 & 4
\end{pmatrix}
\end{array}
$$

Outline

- Motivation
- Setup
  - General
  - Netflix prize
  - Ideas
  - Machine learning workflow (in a nutshell)
- **Algorithmic approaches**

Approaches

- Baseline predictors
  - Plain average
  - Individual bias
  - Biases through least squares
  - Regularization
  - Temporal models (in a nutshell)
- Neighborhood model
- Latent factors
  - Alternating projection
  - SVD

Plain Average

- For all (u, i) pairs,

$$\hat{r}_{ui} = r_{avg}$$

where r_{avg} is the average of all the ratings in R.

- Extremely lazy and inaccurate recommendation system

- How to evaluate?

$$\begin{array}{c|c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} & \begin{pmatrix} 5 & 4 & 4 & - & \color{red}{5} \\ - & 3 & 5 & \color{red}{3} & 4 \\ 5 & 2 & - & \color{red}{2} & 3 \\ - & \color{red}{2} & 3 & 1 & 2 \\ 4 & - & \color{red}{5} & 4 & 5 \\ \color{red}{5} & 3 & - & 3 & 5 \\ 3 & \color{red}{2} & 3 & 2 & - \\ 5 & \color{red}{3} & 4 & - & 5 \\ \color{red}{4} & 2 & 5 & 4 & - \\ \color{red}{5} & - & 5 & 3 & 4 \end{pmatrix} \end{array}$$

Individual Bias

- For all (u, i) pairs,

$$\hat{r}_{ui} = r_{avg} + b_u + b_i$$

where

  - b_u: deviation from r_{avg} of the average of only u's ratings, and
  - b_i: deviation from r_{avg} of the average of only ratings on item i
- Example
  - ”The Godfather” has $b_i = 1.2$, but Carmel is a harsh reviewer with $b_u = -0.5$
  - If $r_{avg} = 3.6$, predict that Carmel would rate ”The Godfather” with

$$3.6 - 0.5 + 1.2 = 4.3$$

- Much better, but might not minimize RMSE
  - The choice of bias parameters is rather arbitrary (why average?)

Biases through Least Squares

- Start with known ratings R, and

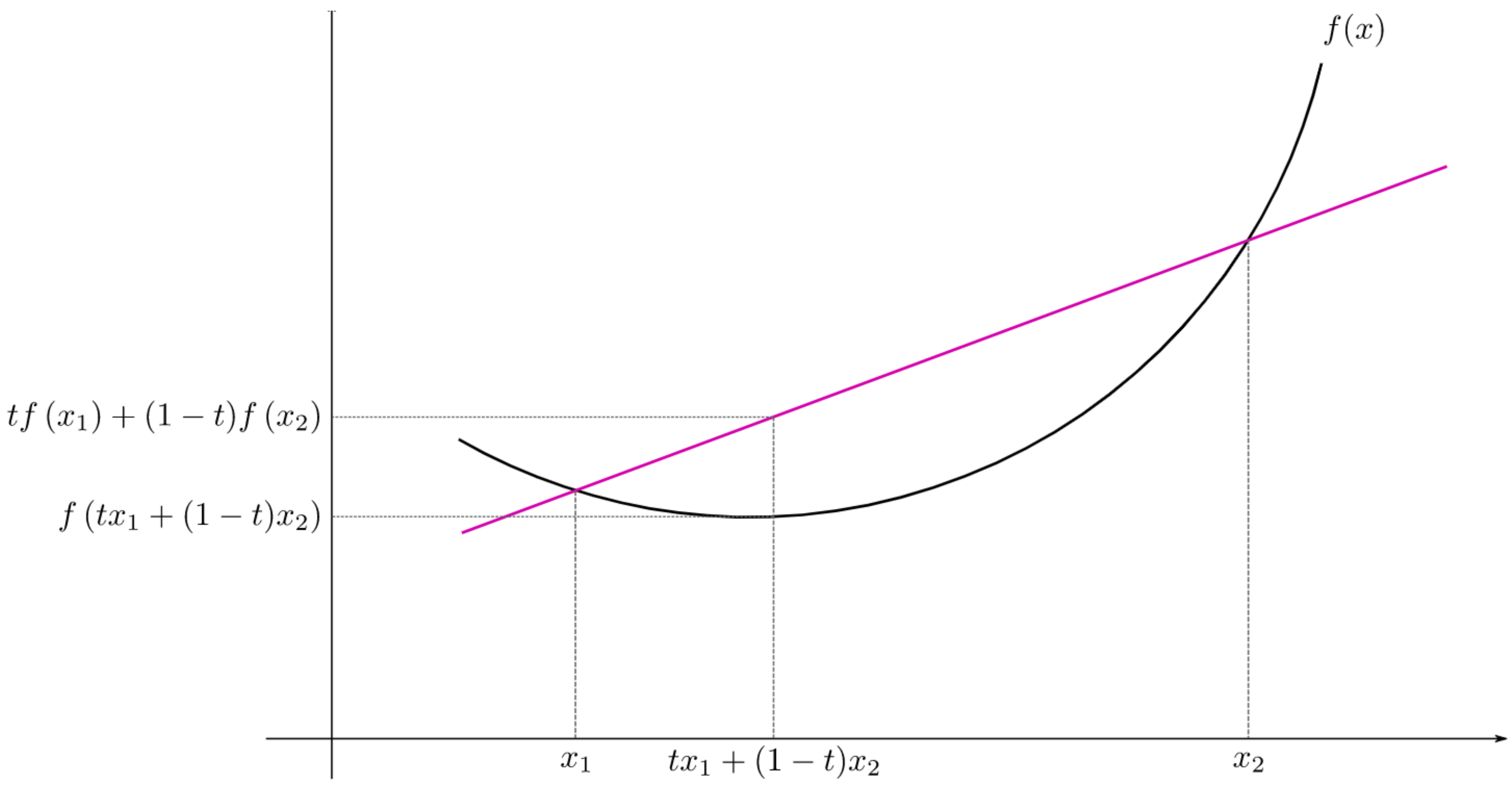
$$\min_{\{b_u, b_i\}} \sum_{(u,i)} \left(r_{avg} + b_u + b_i - r_{ui}\right)^2$$

- Minimizing square error = minimizing MSE = minimizing RMSE
- So what do we have here?
  - Minimize convex quadratic function with no constraints
  - N + M = #users + #movies parameters

$$RMSE = \sqrt{\sum_{(u,i)} \frac{(\hat{r}_{ui} - r_{ui})^2}{C}}$$

For all $0 \le t \le 1$ and all $x_1, x_2 \in X$:

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2)$$



For all $0 \leq t \leq 1$ and all $x_1, x_2 \in X$:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Example – one user (1) rates two movies (A and B)

- Training set: r_{1A} and r_{1B}
- Model parameters: $\{b_1\} \cup \{b_A, b_B\}$

$$\min_{\{b_u, b_i\}} \sum_{(u,i)} (\widehat{r}_{ui} - r_{ui})^2 = \|\mathbf{A}\mathrm{b} - \mathbf{c}\|_2^2$$

Which is here:

$$\min_{\{b_1, b_A, b_B\}} \left[(b_1 + b_B + r_{\text{avg}} - r_{1B})^2 + (b_1 + b_A + r_{\text{avg}} - r_{1A})^2 \right] =$$

$$\min_{\{b_1, b_A, b_B\}} \left\| \left(\underbrace{\boxed{\begin{matrix} 1 \\ 1 \end{matrix}}}_{\text{users}} \; \underbrace{\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}}_{\text{movies}} \right) \begin{pmatrix} b_1 \\ b_A \\ b_B \end{pmatrix} - \begin{pmatrix} r_{1A} - r_{\text{avg}} \\ r_{1B} - r_{\text{avg}} \end{pmatrix} \right\|_2^2$$

Minimizing RMSE

- Notice that

$$MSE(\boldsymbol{b}) = (\|A\boldsymbol{b} - \boldsymbol{c}\|_2)^2 = (A\boldsymbol{b} - \boldsymbol{c})^T(A\boldsymbol{b} - \boldsymbol{c})$$
$$= \boldsymbol{b}^T A^T A\boldsymbol{b} - 2\boldsymbol{b}^T A^T \boldsymbol{c} + \boldsymbol{c}^T \boldsymbol{c}$$

$\begin{pmatrix} b_1 & b_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ and $A^T A$ is symmetric!

- MSE is convex → minimized where first derivative is zero:

$$2(A^T A)\boldsymbol{b} - 2A^T\boldsymbol{c} = 0$$

- Solution $\boldsymbol{b}^\star$ to MSE = solution to the linear equations

$$(A^T A)\boldsymbol{b} = A^T\boldsymbol{c}$$

- Minimizing (convex) quadratic functions boils down to solving linear equations, because we take the derivative and set it to zero

Example – one user (1) rates two movies (A and B)

- Training set: r_{1A} and r_{1B}
- Model parameters: $\{b_1\} \cup \{b_A, b_B\}$

$$(\mathbf{A}^T\mathbf{A})\mathrm{b} = \mathbf{A}^T\mathbf{c}$$

is here

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_A \\ b_B \end{pmatrix} = \begin{pmatrix} r_{1A} + r_{1B} - 2r_{\text{avg}} \\ r_{1A} - r_{\text{avg}} \\ r_{1B} - r_{\text{avg}} \end{pmatrix}$$

Recall:

$$A^T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$c = \begin{pmatrix} r_{1A} - r_{avg} \\ r_{1B} - r_{avg} \end{pmatrix}$$

Overfitting and Regularization

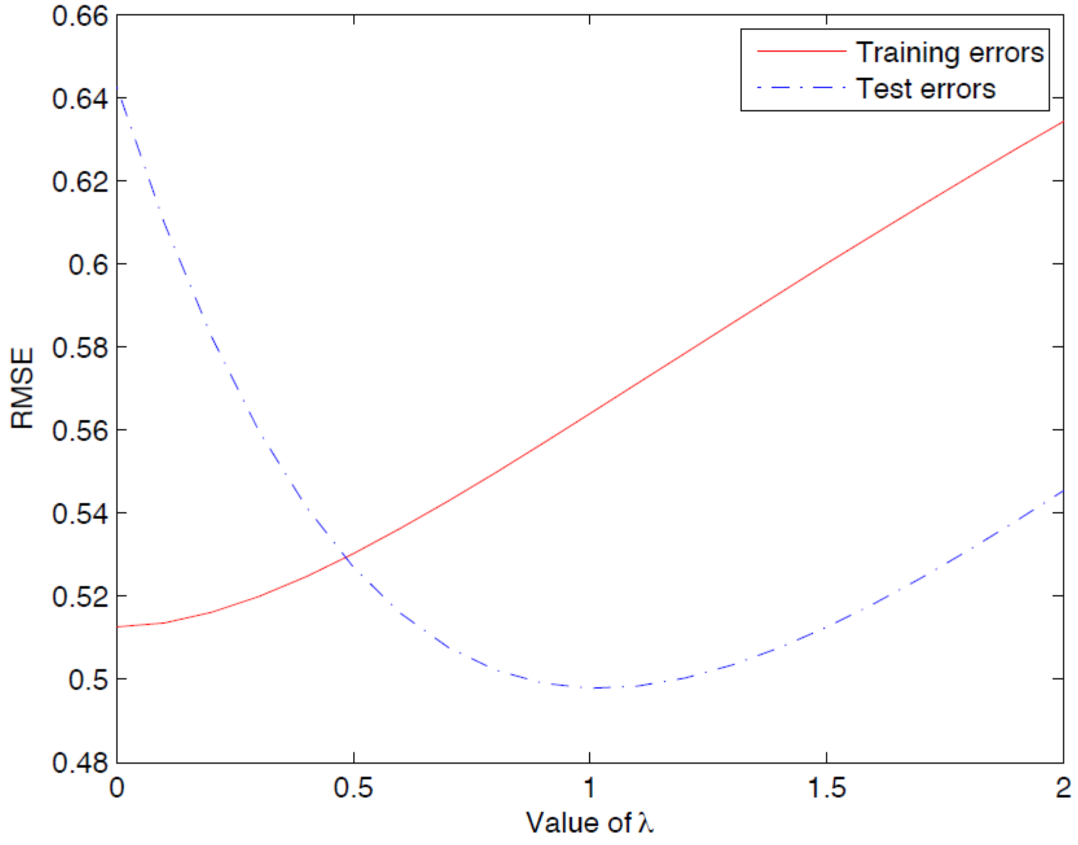
- Occam's Razor: Super-refined explanatory model that loses its predictive power
- Regularization (Balance Precision and Simplicity)

$$\min_{\{b_u, b_i\}} \sum_{(u,i)} (\widehat{r}_{ui} - r_{ui})^2 + \lambda \left(\sum_u b_u^2 + \sum_i b_i^2 \right)$$

- Note: remains a least squares problem!

Overfitting and Regularization (abstraction)

$$\min_{\{b_u, b_i\}} \sum_{(u,i)} (\widehat{r}_{ui} - r_{ui})^2 + \lambda \left(\sum_u b_u^2 + \sum_i b_i^2 \right)$$



Temporal Models (in a nutshell)

- Netflix prize's dataset contained data points over > 5 years
- Movies went in and out of fashion over that period
- Each user's taste also changes over time
- (*Won't dive into it*)

$$\min_{\{b_u, b_i\}} \sum_{(u,i)} (\hat{r}_{ui}(t) - r_{ui})^2 + \lambda \left(\sum_u b_u^2 + \sum_i b_i^2 \right)$$

$$\hat{r}_{ui}(t) = r_{\text{avg}} + b_u(t) + b_i(t)$$

Approaches

- Baseline predictors
- Neighborhood model
- Latent factors
  - Alternating projection
  - SVD

- So far: nothing beyond (all kinds of) averages over R
- Next: Movie-movie correlation

Neighbourhood method

- Similarity measure and weighted prediction
- Given a movie i, we want to pick its L nearest neighbours
  - You pick L
- Errors matrix $\tilde{R} = R - \hat{R}$
- Distance among movies is measured by a similarity metric d, e.g., cosine similarity between <u>columns</u> in $\tilde{R}$

$$d_{ij} = \frac{\tilde{\boldsymbol{r}}_i^T \tilde{\boldsymbol{r}}_j}{\|\tilde{\boldsymbol{r}}_i\|_2 \left\|\tilde{\boldsymbol{r}}_j\right\|_2} = \frac{\sum_u \tilde{r}_{ui} \tilde{r}_{uj}}{\sqrt{\sum_u (\tilde{r}_{ui})^2 \sum_u \left(\tilde{r}_{uj}\right)^2}}$$

Neighbourhood method

- For each movie i , the neighbours of i , $\mathcal{L}_i$, are L movies j with the highest $\left|d_{ij}\right|$
- Neighbourhood-based predictor:

$$\hat{r}_{ui}^N = \hat{r}_{ui} + \frac{\sum_{j \in \mathcal{L}_i} d_{ij} \tilde{r}_{uj}}{\sum_{j \in \mathcal{L}_i} |d_{ij}|}$$

Where $\tilde{r}_{uj}$ is the (u, j) element of $\tilde{R} = R - \hat{R}$

Neighbourhood method

- Summary:
  1. Train a baseline predictor by solving least squares
  2. Obtain the baseline prediction matrix $\hat{R}$, calculate $\tilde{R} = R - \hat{R}$
  3. Compute movie-movie similarity matrix D
  4. Pick a neighbourhood size L to construct a neighbourhood of movies $\mathcal{L}_i$ for each i
  5. Compute the baseline predictor plus neighbourhood predictor as

$$\hat{r}_{ui}^{N} = \hat{r}_{ui} + \frac{\sum_{j \in \mathcal{L}_i} d_{ij} \tilde{r}_{uj}}{\sum_{j \in \mathcal{L}_i} |d_{ij}|}$$

  to the final prediction for each pair (u, i)
  6. This gives us $\tilde{R}^{N}$

For instance, $\hat{r}_{ui} = (r_{avg} + b_u + b_i)$

Example

- 40 ratings of 10 users on 5 movies
- Rating matrix R
- 80% dense matrix, much denser than real systems' data
- Rating average $r_{avg} = 3.83$
- Black: train, red: test

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10
\end{array}
\begin{array}{c}
\begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{pmatrix}
5 & 4 & 4 & - & \color{red}{5} \\
- & 3 & 5 & \color{red}{3} & 4 \\
5 & 2 & - & \color{red}{2} & 3 \\
- & \color{red}{2} & 3 & 1 & 2 \\
4 & - & \color{red}{5} & 4 & 5 \\
\color{red}{5} & 3 & - & 3 & 5 \\
3 & \color{red}{2} & 3 & 2 & - \\
5 & \color{red}{3} & 4 & - & 5 \\
\color{red}{4} & 2 & 5 & 4 & - \\
\color{red}{5} & - & 5 & 3 & 4
\end{pmatrix}
\end{array}
$$

Example: Baseline predictor

- Minimize the sum of squares of all the elements in the vector $\boldsymbol{A}\boldsymbol{b} - \boldsymbol{c}$, which is

$$
\begin{array}{c c}
 & \begin{array}{ccccccccc} 1 & 2 & 3 & \dots & 10 & A & B & \dots & E \end{array} \\
\begin{array}{c} 1 \\ 2 \\ \vdots \\ 30 \end{array} & \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & & \vdots & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 \end{pmatrix}
\end{array}
\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{10} \\ b_A \\ b_B \\ \vdots \\ b_E \end{pmatrix}
-
\begin{pmatrix} r_{1A} - r_{\text{avg}} \\ r_{3A} - r_{\text{avg}} \\ \vdots \\ r_{10E} - r_{\text{avg}} \end{pmatrix}
$$

Example: Baseline predictor

- Minimize the sum of squares of all the elements in the vector $A\boldsymbol{b} - \boldsymbol{c}$
- Solution $\boldsymbol{b}^\star$ to the linear equations

$$(A^T A)\boldsymbol{b} = A^T \boldsymbol{c}$$

is

$$\mathbf{b}_u^* = [0.62, 0.42, -0.28, -1.78, 0.52, 0.49, -1.24, 0.45, 0.40, 0.23]^T$$

$$\mathbf{b}_i^* = [0.72, -1.20, 0.60, -0.60, 0.33]^T$$

Example: Baseline predictor

- Intuition from the training data and baseline predictor

$$\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 5 & 4 & 4 & - & ? \\ - & 3 & 5 & ? & 4 \\ 5 & 2 & - & ? & 3 \\ - & ? & 3 & 1 & 2 \\ 4 & - & ? & 4 & 5 \\ ? & 3 & - & 3 & 5 \\ 3 & ? & 3 & 2 & - \\ 5 & ? & 4 & - & 5 \\ ? & 2 & 5 & 4 & - \\ ? & - & 5 & 3 & 4 \end{pmatrix} \end{array}$$

- Users 1, 2, 5, 6 and 8 tend to give higher ratings
- Users 4 and 7 tend to give lower ratings
- Movies A and C tend to receive higher ratings
- Movies B and D tend to receive lower ratings.

$$\mathbf{b}_u^* = [0.62, 0.42, -0.28, -1.78, 0.52, 0.49, -1.24, 0.45, 0.40, 0.23]^T$$

$$\mathbf{b}_i^* = [0.72, -1.20, 0.60, -0.60, 0.33]^T$$

Example: Baseline predictor

Clipping predicted ratings to $[1, 5]$, r_{avg} and $\mathbf{b}^*$ give us

$$
\mathbf{R} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{pmatrix}
5 & 4 & 4 & - & 5 \\
- & 3 & 5 & 3 & 4 \\
5 & 2 & - & 2 & 3 \\
- & 2 & 3 & 1 & 2 \\
4 & - & 5 & 4 & 5 \\
5 & 3 & - & 3 & 5 \\
3 & 2 & 3 & 2 & - \\
5 & 3 & 4 & - & 5 \\
4 & 2 & 5 & 4 & - \\
5 & - & 5 & 3 & 4
\end{pmatrix} \end{array}
\qquad
\widehat{\mathbf{R}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array}
\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\
\begin{pmatrix}
5.00 & 3.09 & 4.9 & - & 4.62 \\
- & 2.89 & 4.69 & 3.49 & 4.42 \\
4.10 & 2.19 & - & 2.78 & 3.71 \\
- & 1 & 2.49 & 1.29 & 2.22 \\
4.90 & - & 4.79 & 3.58 & 4.51 \\
4.88 & 2.96 & - & 3.56 & 4.48 \\
3.15 & 1.23 & 3.03 & 1.82 & - \\
4.84 & 2.92 & 4.72 & - & 4.44 \\
4.84 & 2.92 & 4.72 & 3.51 & - \\
4.61 & - & 4.49 & 3.29 & 4.22
\end{pmatrix} \end{array}
$$

RMSE: 0.51 for train set; 0.64 for test set

Example: Neighborhood Model

$$\widetilde{\mathbf{R}} = \mathbf{R} - \widehat{\mathbf{R}} = \begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} & \begin{pmatrix} 0 & 0.91 & -0.90 & - & ? \\ - & 0.11 & 0.31 & ? & -0.42 \\ 0.9 & -0.19 & - & ? & -0.71 \\ - & ? & 0.51 & -0.29 & -0.22 \\ -0.90 & - & ? & 0.42 & 0.49 \\ ? & 0.04 & - & -0.56 & 0.52 \\ -0.15 & ? & -0.03 & 0.18 & - \\ 0.16 & ? & -0.72 & - & 0.56 \\ ? & -0.87 & 0.33 & 0.54 & - \\ ? & - & 0.51 & -0.29 & -0.22 \end{pmatrix} \end{array}$$

$$\begin{aligned} d_{BC} &= \frac{\widetilde{r}_{1B}\widetilde{r}_{1C} + \widetilde{r}_{2B}\widetilde{r}_{2C} + \widetilde{r}_{9B}\widetilde{r}_{9C}}{\sqrt{(\widetilde{r}_{1B}^2 + \widetilde{r}_{2B}^2 + \widetilde{r}_{9B}^2)(\widetilde{r}_{1C}^2 + \widetilde{r}_{2C}^2 + \widetilde{r}_{9C}^2)}} \\ &= \frac{(0.91)(-0.90) + (-0.11)(0.31) + (-0.87)(0.33)}{\sqrt{(0.91^2 + 0.11^2 + 0.87^2)(0.90^2 + 0.31^2 + 0.33^2)}} \\ &= -0.84 \end{aligned}$$

$$\mathbf{D} = \begin{array}{c c} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} - & -0.21 & -0.41 & -0.97 & -0.75 \\ -0.21 & - & -0.84 & -0.73 & 0.51 \\ -0.41 & -0.84 & - & -0.22 & -0.93 \\ -0.97 & -0.73 & -0.22 & - & -0.068 \\ -0.75 & 0.51 & -0.93 & 0.068 & - \end{pmatrix} \end{array}$$

Example: Neighborhood Model

- Computing $\hat{r}_{ui}^{N}$:
  - Find $L = 2$ movie neighbours with the largest absolute similarity values, $|d_{ik}|$ and $|d_{il}|$
  - Check if user u has rated both movies k and l
    - If so, use both, as in the formula below
    - If the user rated only one of them, then just use that movie
    - If the user rated neither, then do not use the neighbourhood method
  - Calculate predicted rating as

$$\hat{r}_{ui}^{N} = \left(r_{avg} + b_u + b_i\right) + \frac{d_{ik}\tilde{r}_{uk} + d_{il}\tilde{r}_{ul}}{|d_{ik}| + |d_{il}|}$$

Example: Neighborhood Model

Computing $\widehat{r}_{3D}$ using $\mathbf{D}$ (while $r_{3D} = 2$)

$$\widetilde{\mathbf{R}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 0.91 & -0.90 & - & ? \\ - & 0.11 & 0.31 & ? & -0.42 \\ 0.9 & -0.19 & - & ? & -0.71 \\ - & ? & 0.51 & -0.29 & -0.22 \\ -0.90 & - & ? & 0.42 & 0.49 \\ ? & 0.04 & - & -0.56 & 0.52 \\ -0.15 & ? & -0.03 & 0.18 & - \\ 0.16 & ? & -0.72 & - & 0.56 \\ ? & -0.87 & 0.33 & 0.54 & - \\ ? & - & 0.51 & -0.29 & -0.22 \end{pmatrix} \end{array}$$

$$\mathbf{D} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} - & -0.21 & -0.41 & -0.97 & -0.75 \\ -0.21 & - & -0.84 & -0.73 & 0.51 \\ -0.41 & -0.84 & - & -0.22 & -0.93 \\ -0.97 & -0.73 & -0.22 & - & -0.068 \\ -0.75 & 0.51 & -0.93 & 0.068 & - \end{pmatrix} \end{array}$$

$$\begin{aligned} \widehat{r}^{N}_{3D} &= \left(r_{avg} + b_3 + b_D\right) + \frac{d_{DA}\widetilde{r}_{3A} + d_{DB}\widetilde{r}_{3B}}{|d_{DA}| + |d_{DB}|} \\ &= 2.78 + \frac{(-0.97)(0.90) + (-0.73)(-0.19)}{0.97 + 0.73} \\ &= 2.35 \end{aligned}$$

Example: Neighborhood Model

$$\mathbf{R} = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 5 & 4 & 4 & - & 5 \\ - & 3 & 5 & 3 & 4 \\ 5 & 2 & - & 2 & 3 \\ - & 2 & 3 & 1 & 2 \\ 4 & - & 5 & 4 & 5 \\ 5 & 3 & - & 3 & 5 \\ 3 & 2 & 3 & 2 & - \\ 5 & 3 & 4 & - & 5 \\ 4 & 2 & 5 & 4 & - \\ 5 & - & 5 & 3 & 4 \end{pmatrix} \end{array}$$

$$\widehat{\mathbf{R}}^N = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 5.00 & 3.99 & 3.99 & - & 5.00 \\ - & 2.58 & 4.86 & 3.38 & 4.11 \\ 4.81 & 2.19 & - & 2.35 & 2.81 \\ - & 1 & 2.71 & 1.29 & 1.71 \\ 4.46 & - & 4.30 & 4.49 & 4.42 \\ 4.97 & 3.52 & - & 3.52 & 4.48 \\ 2.97 & 1.16 & 3.03 & 1.97 & - \\ 4.28 & 3.64 & 4.16 & - & 4.77 \\ 4.25 & 2.44 & 4.54 & 4.33 & - \\ 4.87 & - & 4.71 & 3.29 & 3.71 \end{pmatrix} \end{array}$$

RMSE with $\widehat{\mathbf{R}}$: 0.51 for train set; 0.64 for test set

RMSE with $\widehat{\mathbf{R}}^N$: 0.34 for train set; 0.54 for test set

Approaches

- Baseline predictors
- Neighborhood model
- Latent factors
  - Alternating projection
  - SVD

- So far: nothing beyond (all kinds of) averages over R
- Next: Movie-movie correlation

Latent factor method

- Typical data:
  - HUGE feedback matrix

  (Netflix Prize was $N = 480,000 \times M = 17,770$)
  - Very SPARSE
- Natural (?) assumption:
  - similarities among users and movies are not just a statistical facts
  - Induced by some low-dimensional structures hidden in the data

Latent factor method

- Taste of user u is captures by a K-dimensional vector $\boldsymbol{p}_u$
- Appeal of movie i is captured by a K -dimensional vector $\boldsymbol{q}_i$
- Prediction is $\hat{r}_{ui} = \boldsymbol{p}_u^T \boldsymbol{q}_i$
- Find $\{\boldsymbol{p}_u\}$ and $\{\boldsymbol{q}_i\}$ that minimize RMSE:

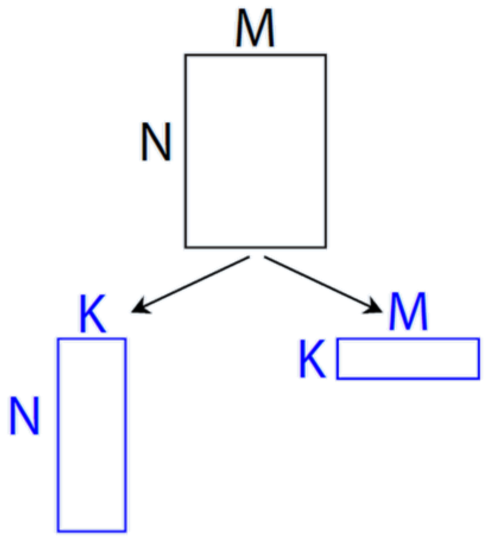
$$\min_{P,Q} \sum_{(u,i)} (r_{ui} - \boldsymbol{p}_u^T \boldsymbol{q}_i)^2$$

where P is $N \times K$ and Q is $K \times M$ (N users, M movies)

- Solve, get $P^\star$ and $Q^\star$, then $\widehat{R}^L = P^\star Q^\star$

K = 10? K = 100? K = 1000?

- How should we pick K?
  - Avoid overfitting!
  - Cross validation
  - Elbow method
- Netflix prize: $N = 480{,}000, M = 17{,}700$, winning model used $K = 200$



Alternating projection

- Is

$$\min_{P,Q} \sum_{(u,i)} (r_{ui} - \boldsymbol{p}_u^T \boldsymbol{q}_i)^2$$

a least squares problem?

- NO, both $\boldsymbol{p}_u$ and $\boldsymbol{q}_i$ are optimization variables
- YES if either $\boldsymbol{p}_u$ or $\boldsymbol{q}_i$ is fixed
- Solution: Alternating projection
  - Converging, EM-like algorithm

SVD

Low rank approximation

- We can always decompose a real matrix A into

$$A_{N\times M} = U_{N\times r}\ \Sigma_{r\times r}\ V_{r\times M}^T$$

where

- r is the rank of A
- U, Σ, V are unique
- U, V are column-orthonormal ($U^T U = V^T V = I$)
- U (V) is left (right) singular vectors: The columns are orthogonal eigenvectors of AA^T ($A^T A$)
- Σ is diagonal
  - Entries are singular values (=eigenvalues of AA^T)

Low rank approximation

- In our context: $R_{N\times M} = U_{N\times r}\Sigma_{r\times r}V_{r\times M}^{T}$, where r is the rank of R
  – U is the user to latent factors matrix
  – V is the movie to latent factors matrix

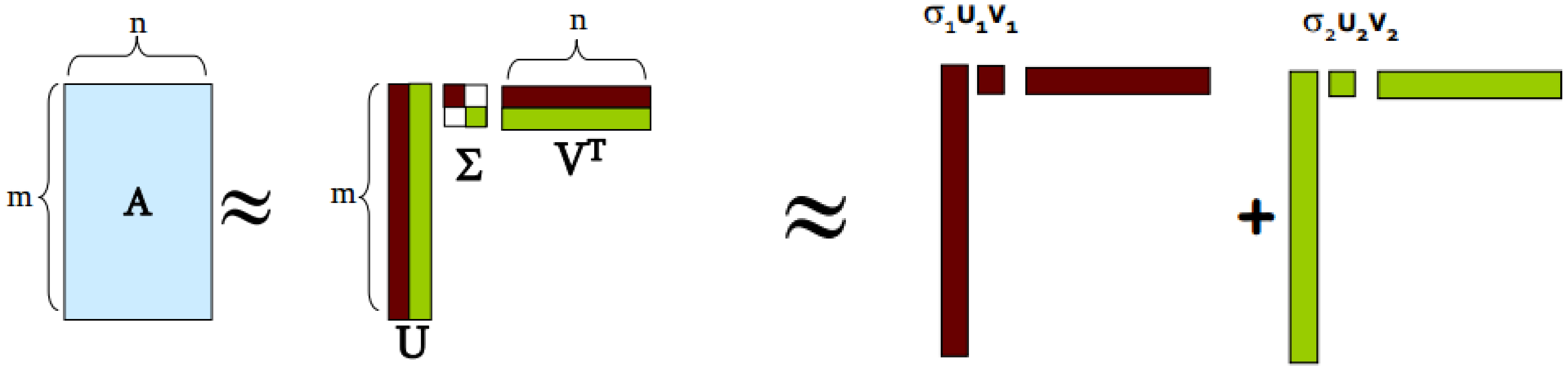
Interpretation

‘**movies**’, ‘**users**’ and ‘**concepts**’:

- *U*: user-to-concept similarity matrix
- *V*: movie-to-concept similarity matrix
- Σ: its diagonal elements:
  ‘strength’ of each concept

	Matrix	Alien	Star Wars	Casablanca	Titanic
Joe	1	1	1	0	0
Jim	3	3	3	0	0
John	4	4	4	0	0
Jack	5	5	5	0	0
Jill	0	2	0	4	4
Jenny	0	0	0	5	5
Jane	0	1	0	2	2

$$\mathbf{A} \approx \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^T$$



($\boldsymbol{u}_i, \boldsymbol{v}_i$ are vectors, σ_i is a scaler)

Example

$A = U \Sigma V^T$ - example: **Users to Movies**

$$
\begin{array}{c}
\begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

Example

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \cancel{\mathbf{1.3}} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

Example

More details

- **Q: How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\approx
\begin{bmatrix}
\mathbf{0.13} & 0.02 & \cancel{-0.01} \\
\mathbf{0.41} & 0.07 & \cancel{-0.03} \\
\mathbf{0.55} & 0.09 & \cancel{-0.04} \\
\mathbf{0.68} & 0.11 & \cancel{-0.05} \\
0.15 & \mathbf{-0.59} & \cancel{\mathbf{0.65}} \\
0.07 & \mathbf{-0.73} & \cancel{\mathbf{-0.67}} \\
0.07 & \mathbf{-0.29} & \cancel{\mathbf{0.32}}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \cancel{\mathbf{1.3}}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
\cancel{0.40} & \cancel{\mathbf{-0.80}} & \cancel{0.40} & \cancel{0.09} & \cancel{0.09}
\end{bmatrix}
$$

Example

More details

- Q: How exactly is dim. reduction done?
- A: Set smallest singular values to zero

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.92} & \mathbf{0.95} & \mathbf{0.92} & 0.01 & 0.01 \\ \mathbf{2.91} & \mathbf{3.01} & \mathbf{2.91} & -0.01 & -0.01 \\ \mathbf{3.90} & \mathbf{4.04} & \mathbf{3.90} & 0.01 & 0.01 \\ \mathbf{4.82} & \mathbf{5.00} & \mathbf{4.82} & 0.03 & 0.03 \\ 0.70 & \mathbf{0.53} & 0.70 & \mathbf{4.11} & \mathbf{4.11} \\ -0.69 & 1.34 & -0.69 & \mathbf{4.78} & \mathbf{4.78} \\ 0.32 & \mathbf{0.23} & 0.32 & \mathbf{2.01} & \mathbf{2.01} \end{bmatrix}$$

Frobenius norm:

$$\|M\|_F = \sqrt{\Sigma_{ij} M_{ij}^2}$$

$$\|A-B\|_F = \sqrt{\Sigma_{ij} (A_{ij}-B_{ij})^2}$$

is "small"

Example

Compactly, we have:

$q_{concept} = q\,V$

$R \approx U\Sigma V^T \Rightarrow RV \approx U\Sigma$

$$\begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ \cancel{0.40} & \cancel{\mathbf{-0.80}} & \cancel{0.40} & \cancel{0.09} & \cancel{0.09} \end{bmatrix}$$

E.g.:

$$q = \begin{array}{c} \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\ \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \times \begin{bmatrix} 0.56 & 0.12 \\ 0.59 & -0.02 \\ 0.56 & 0.12 \\ 0.09 & -0.69 \\ 0.09 & -0.69 \end{bmatrix} = \begin{bmatrix} 2.8 & 0.6 \end{bmatrix}$$

movie-to-concept similarities (V)

SciFi-concept

Example

- **Observation:** User ***d*** that rated (‘*Alien*’, ‘*Serenity*’) will be **similar** to user **q** that rated (‘*Matrix*’), although ***d*** and **q** have **zero ratings in common**!

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\mathbf{q}= & \begin{bmatrix} 0 & 4 & 5 & 0 & 0 \end{bmatrix} \\
\mathbf{d}= & \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\quad
\begin{array}{c}
\dashrightarrow \begin{bmatrix} 2.8 & 0.6 \end{bmatrix} \\
\dashrightarrow \begin{bmatrix} 5.2 & 0.4 \end{bmatrix}
\end{array}
$$

Zero ratings in common

Similarity ≠ 0

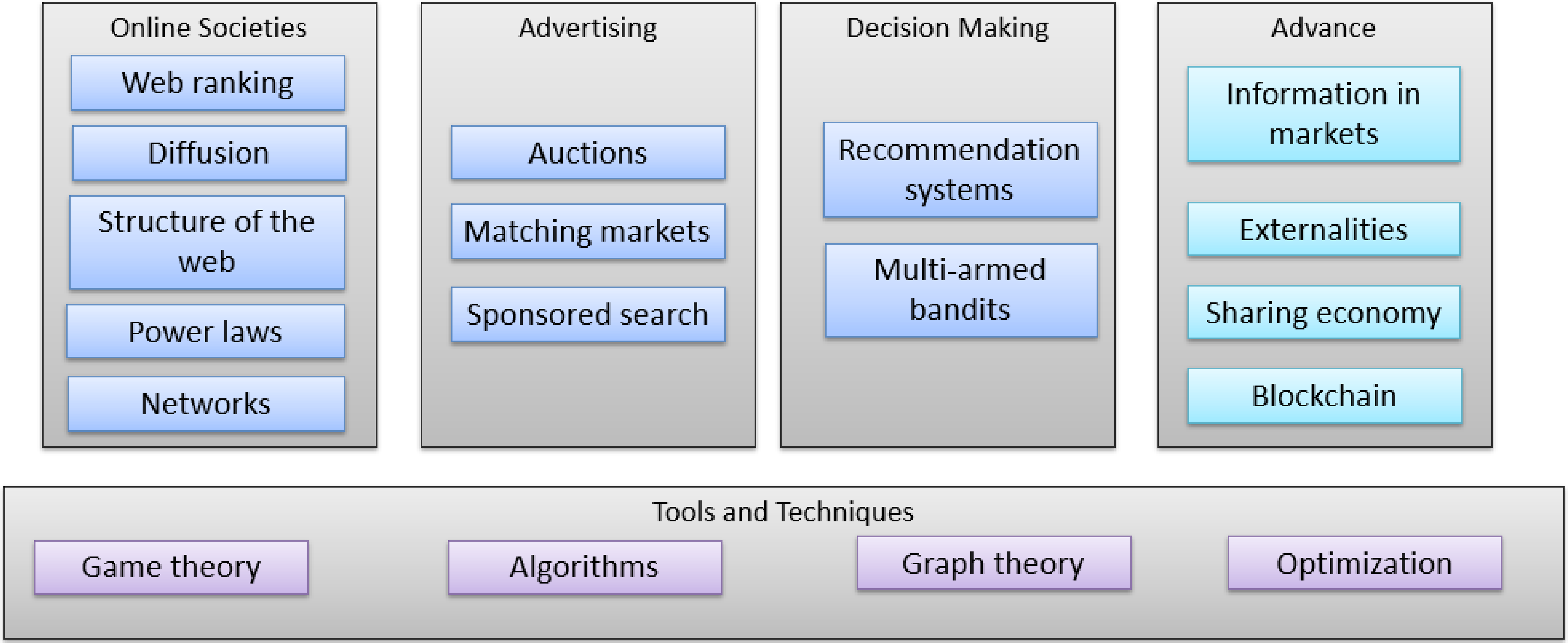
Summary

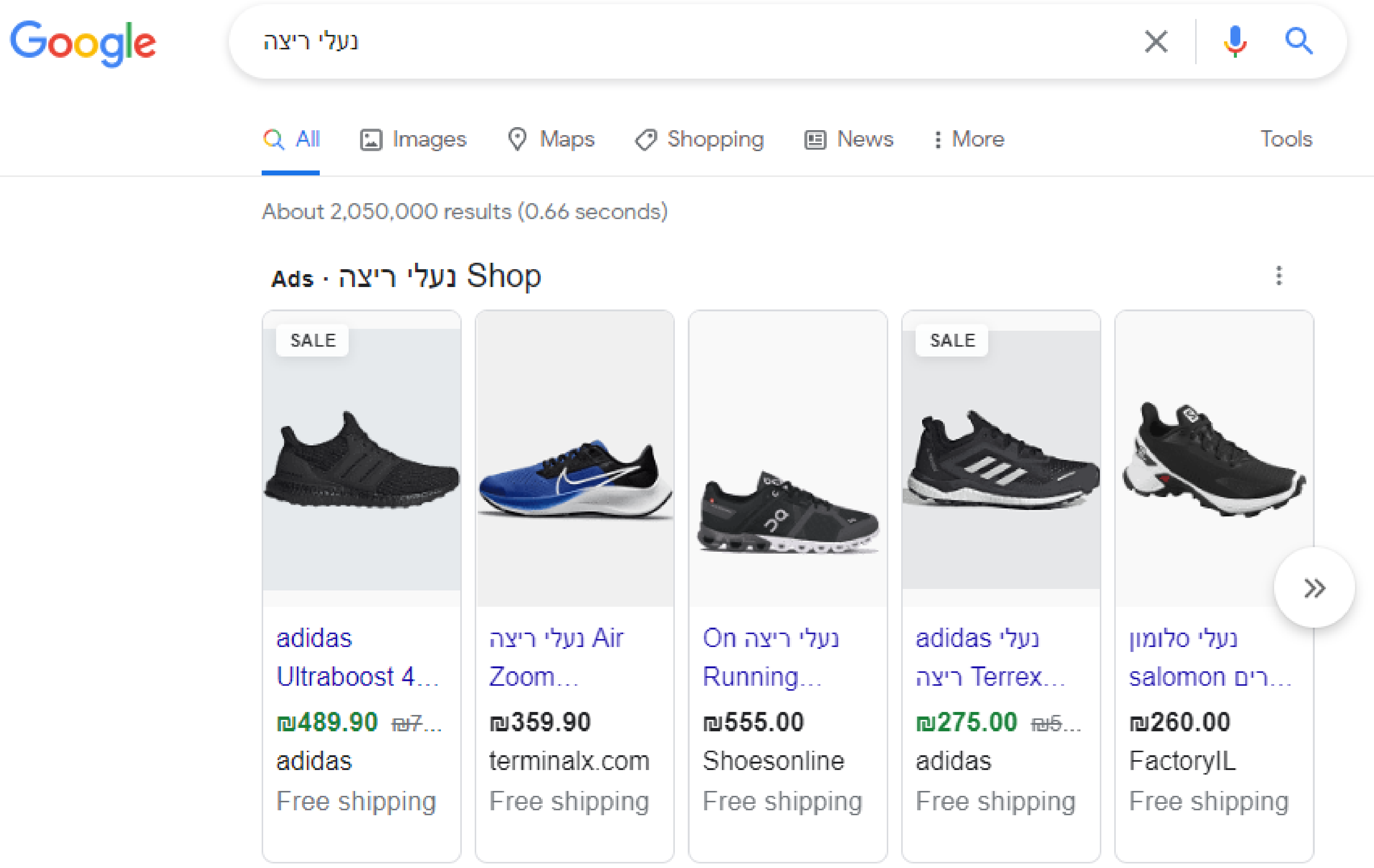
- Recommendation systems are everywhere!
- Today: introduction, algorithmic approaches
- BUT many more
  - Engineering?
  - Constraints?
  - Societal?
  - Incentives?
  - Ethical?

Electronic Commerce 096211

Auctions

Omer Ben Porat (some of the slides are adopted from Prof. Reshef Meir)

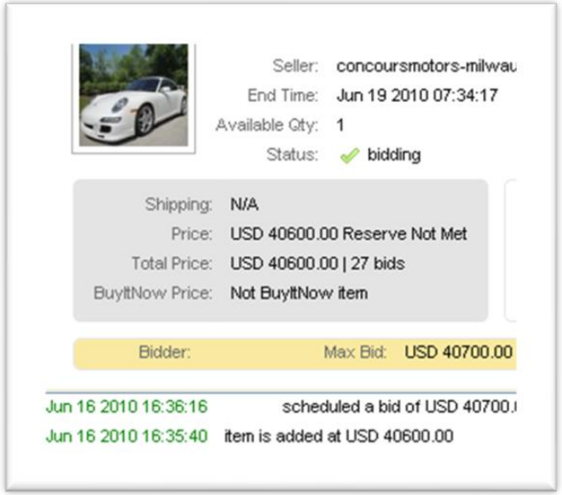

Online Societies
Web ranking
Diffusion
Structure of the web
Power laws
Networks
Advertising
Auctions
Matching markets
Sponsored search
Decision Making
Recommendation systems
Multi-armed bandits
Advance
Information in markets
Externalities
Sharing economy
Blockchain
Tools and Techniques
Game theory
Algorithms
Graph theory
Optimization


Google
נעלי ריצה
All
Images
Maps
Shopping
News
More
Tools
About 2,050,000 results (0.66 seconds)
Ads · נעלי ריצה Shop
SALE
adidas Ultraboost 4…
₪489.90 ₪7…
adidas
Free shipping
נעלי ריצה Air Zoom…
₪359.90
terminalx.com
Free shipping
On נעלי ריצה Running…
₪555.00
Shoesonline
Free shipping
SALE
adidas נעלי ריצה Terrex…
₪275.00 ₪5…
adidas
Free shipping
נעלי סלומון salomon רים…
₪260.00
FactoryIL
Free shipping

Where do we see auctions?

- Art auctions
- e-Bay
- Government auctions rights to use land, oil, etc.
- Google/ Facebook ads
- More…

- Today: we sell a single item!
- Games?



סיום מכרז תדרים דור 5

יחידה: מינהל כלכלה, דוברות
תאריך פרסום: 12.08.2020
נושא: שירותי תקשורת, תדרי רדיו, דור 5

ועדת המכרזים פרסמה זה עתה את תוצאות מכרז התדרים לדור 5, אשר יוביל את מדינת ישראל לחזית הטכנולוגיה העולמית, לצד המדינות המתקדמות בעולם

Auction Types

- Ascending price (English auction)

- Descending price (Dutch auction)
- Sealed bid:
  - First price
  - Second price
- (Ties?)
- Variations: multiple bids, second chance,…

Sealed bid auctions

- Each bidder i has a type $v_i \geq 0$
- The action of bidder i is a bid $b_i \geq 0$

- First price auction:
  - Highest bidder i^*wins the item
  - Pays her bid $p_{i^*} = b_{i^*}$
  - Gets utility of $u_{i^*} = v_{i^*} - p_{i^*}$
  - All other bidders get 0

Sealed bid auctions

- Each bidder i has a type $v_i \geq 0$
- The action of bidder i is a bid $b_i \geq 0$

- Second price auction:
  - Highest bidder i^* wins the item
  - Pays second highest bid $p_{i^*} = \max_{j \neq i^*} b_j$
  - Gets utility of $u_{i^*} = v_{i^*} - p_{i^*}$
  - All other bidders get 0

SECOND PRICE AUCTIONS

Truthful auctions

Theorem: It is a dominant strategy for any bidder i in a second-price auction to bid her true value $b_i = v_i$.

Proof

- W.l.o.g. agent 1 wins so b_1 is highest, pays b_2.
- Suppose some bidder i tries to lie. Four cases:
  - $i = 1$ is the winner, $b'_i > b_2$
  - $i = 1$ is the winner, $b'_i \leq b_2$
  - $i > 1$ loses, $b'_i < b_1$
  - $i > 1$ loses, $b'_i \geq b_1$

Proof

- W.l.o.g. agent 1 wins so b_1 is highest, pays b_2.
- Suppose some bidder i tries to lie. Four cases:
  - $i = 1$ is the winner, $b'_i > b_2$

    No change, $u'_i = u_i$
  - $i = 1$ is the winner, $b'_i \leq b_2$
  - $i > 1$ loses, $b'_i < b_1$

    No change, $u'_i = u_i = 0$
  - $i > 1$ loses, $b'_i \geq b_1$

Proof

- W.l.o.g. agent 1 wins so b_1 is highest, pays b_2.
- Suppose some bidder i tries to lie. Four cases:
  - $i = 1$ is the winner, $b'_i > b_2$

    No change, $u'_i = u_i$
  - $i = 1$ is the winner, $b'_i \leq b_2$

    $u_i = v_i - b_2 = b_1 - b_2 \geq 0$
    $u'_i = 0$
  - $i > 1$ loses, $b'_i < b_1$

    No change, $u'_i = u_i = 0$
  - $i > 1$ loses, $b'_i \geq b_1$

    $u_i = 0$
    $u'_i = v_i - b_1 = b_i - b_1 \leq 0$

$\square$

Multiple equilibria

- Second price auction
- 5 players, $v_i = i$
- PNE 1: $b_i = i$ (truthfulness)
- Another PNE? (Challenge)

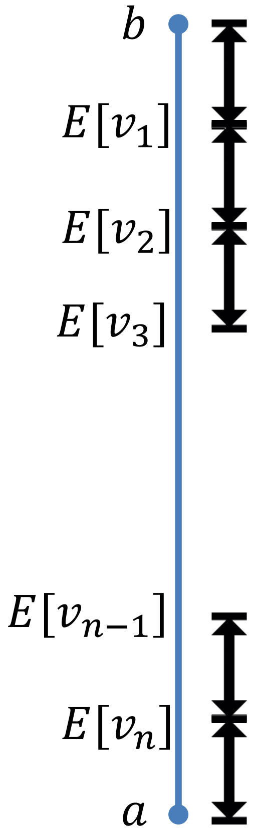
Revenue

- The seller's revenue is of key importance
- What is the revenue to the seller in 2nd price auction?

- The 2nd highest value: $v_1, v_2, v_3, v_4, \ldots, v_n$
- What is the expectation of v_2?
  - Depends on the distribution

Expected value of order statistics

– If all $v_i \sim U[a, b]$, then $E[v_k] = a + (b - a)\frac{n+1-k}{n+1}$

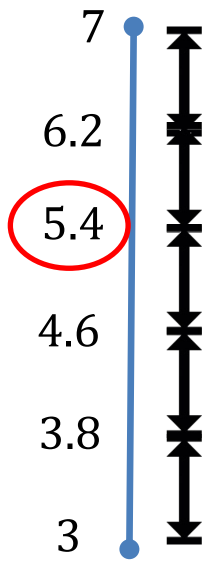


Expected value of order statistics

– If all $v_i \sim U[a, b]$, then $E[v_k] = a + (b - a)\frac{n+1-k}{n+1}$

– Example: if all $v_i \sim U[3, 7]$ and $n = 4$ then

- $E[v_1] = 3 + (7 - 3) \cdot \frac{4+1-1}{4+1} = 6.2$
- $E[v_2] = 3 + (7 - 3) \cdot \frac{3}{5} = 5.4$
- $E[v_3] = 3 + (7 - 3) \cdot \frac{2}{5} = 4.6$
- $E[v_4] = 3 + (7 - 3) \cdot \frac{1}{5} = 3.8$



Back to revenue

- What is the revenue to the seller in 2nd price auction?
  - The 2nd highest value: $v_1, v_2, v_3, v_4, \dots, v_n$
  - If all $v_i \sim U[a, b]$, then
    - $E[r] = E[v_2] = a + (b - a)\frac{n-1}{n+1}$

Over what do we optimize? How to analyze revenue?

HOW TO MAXIMIZE REVENUE?

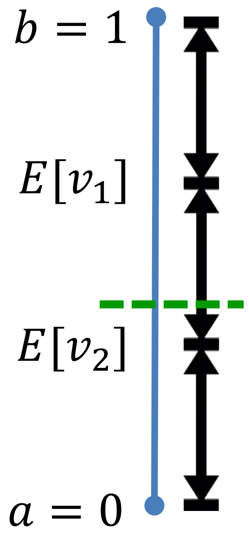
Revenue

- Can the revenue be increased?
  - Yes, by setting a reserve price Z

Revenue

- Can the revenue be increased?
  – Yes, by setting a reserve price Z
  – Example: 2 bidders with value $v_i \sim U[0,1]$

  $$-r = \begin{cases} \min(v_1, v_2) & \text{if} \min(v_1, v_2) \geq Z \\ Z & \text{if} \max(v_1, v_2) \geq \mathrm{Z} \geq \min(v_1, v_2) \\ 0 & \text{if} \min(v_1, v_2) < Z \end{cases}$$



- Socially optimal?

Optimal reserve price

$$r = \begin{cases} \min(v_1, v_2) & \text{if} \min(v_1, v_2) \geq Z \\ Z & \text{if} \max(v_1, v_2) \geq \mathrm{Z} \geq \min(v_1, v_2) \\ 0 & \text{if} \min(v_1, v_2) < Z \end{cases}$$

$$E_{v_1 v_2 \sim U[0,1]}[r] = \Pr(v_1 \geq Z, v_2 \geq Z) * E_{v_1 v_2 \sim U[Z,1]}[r] +$$
$$2\Pr(v_1 \geq Z, v_2 < Z) * E_{v_1 \sim U[Z,1], v_2 \sim U[0,Z]}[r] =$$
$$(1-Z)^2 * E_{v_1 v_2 \sim U[Z,1]}[\min(v_1, v_2)] +$$
$$2 * Z(1-Z) * E_{v_1 \sim U[Z,1], v_2 \sim U[0,Z]}[Z] =$$
$$(1-Z)^2 * \left(Z + \frac{1-Z}{3}\right) + 2 * Z(1-Z) * Z =$$
$$1\frac{1}{3} X^3 - 3X^2 + 2X$$

$a + \frac{(b-a)(n-1)}{n+1}$

Set X = 1-Z

By derivation, $E[r]$ is maximized for $X = 0.5$ (so $Z = 1 - X = 0.5$ too)

Increasing revenue

- Computing the optimal reserve price [Myerson'81]:
  - HUGH space of auctions!
  - Complicated
  - Requires to know the distribution of values

Theorem (informal): Assuming identically distributed valuations, adding one more bidder to the second/first price auction will generate more expected revenue than any reserve price!

- [Bulow and Klemperer '94]

FIRST PRICE AUCTIONS

First price auctions

- Not truthful
- What is a bidding strategy?
  - $s(value) = bid$
  - Examples: $b_i = v_i$, $b_i = 3v_i$, $b_i = \sqrt{v_i} + 3$, …

- What is an equilibrium?
  - Bayes-Nash equilibrium (BNE)
  - Same as Nash equilibrium, except that the actions are functions (from value to bid)

First price auction

- Claim 1: $s(v) = \frac{v}{2}$ is a BNE for 2 bidders with $v_i \sim U[0,1]$
  - Each bidder bids half her value

- (Expected revenue?)

Proof of Claim 1

- We prove for bidder 1
- Suppose bidder 2 indeed bids $b_2 = 0.5\ v_2$
  - Then $b_2 \sim U[0,0.5]$
- What is the utility for bidding b_1?
  - Bidder 1 wins if $b_2 < b_1$
    - Gains utility of $v_1 - b_1$
  - Bidder 1 loses if $b_2 > b_1$
    - Gains 0
  - We can ignore a tie (why?)

Proof of Claim 1 (cont.)

- What is the **expected** utility for bidding b_1?

$$E[u_1 \mid v_1, b_1] = \int_{b_2=0}^{0.5} [\![1\ wins]\!]\,(v_1 - b_1) 2 d b_2$$

$$= 2(v_1 - b_1) \int_{b_2=0}^{0.5} [\![b_2 < b_1]\!]\, d b_2$$

$$= 2(v_1 - b_1) \int_{b_2=0}^{b_1} 1\, d b_2$$

$$= 2(v_1 - b_1) b_2 \Big|_{b_2=0}^{b_1}$$

$$= 2 v_1 b_1 - 2(b_1)^2$$

Proof of Claim 1 (cont.)

- What is the **expected** utility for bidding b_1?

$$E[u_1 \mid v_1, b_1] = 2v_1 b_1 - 2(b_1)^2$$

- What is the best bid b_1?
  - By derivation:

$$0 = \frac{\partial E[u_1 | v_1, b_1]}{\partial b_1} = \frac{\partial(2v_1 b_1 - 2(b_1)^2)}{\partial b_1}$$

$$= 2v_1 - 4b_1$$

$$\Rightarrow b_1 = \frac{v_1}{2}$$

$\square$

First price auction

- Claim 1: $s(v) = \frac{v}{2}$ is a BNE for 2 bidders with $v_i \sim U[0,1]$

What about other distributions?

Revenue equivalence

- **<u>Theorem</u>**: Suppose bidders' values are i.i.d. from some distribution. The revenue for the seller in 1st price auction (in BNE) and in 2nd price auction (in DSE) is the same. (exercise)
- <u>Definition</u>: An auction is *reasonable* if it only allocates to bidders with a positive bid.
- The Revenue Equivalence Theorem (Myerson ['81]):

Suppose bidders' values are i.i.d. from some distribution. In any two *reasonable* auctions that use the same allocation, the payments and revenue in equilibrium are the same.

GENERAL REMARKS

Auction Goals

- Social welfare: Sell the item to whoever has the highest value
  - Governmental auctions
  - Long term partners: Google?
  - Second price is optimal!
  - Robust!

- Revenue maximization
  - The optimal auction depends on the value distribution (or data)

All-pay auctions

- Similar to a 1st price auction, except all bidders pay their bid even if losing
  - Sports and R&D competitions
  - Political lobbying
  - War (Colonel Blotto game)
- Easy to see no pure equilibrium exists
- Typically poor welfare (and revenue), unless players are asymmetric, or overly optimistic.

Dependent values

- Bidders’ values may be correlated
- For example, $v_i = v + X_i$, where X_i is random
  - Model 1: the true value of i is v_i
    - X_i differ due to personal taste
    - 2nd price is still truthful
    - Reserve prices not always optimal
  - Model 2: the true value of i is v
    - X_i differ due to wrong estimations
    - “the winner’s curse”
    - Bidding truthfully in 2nd price auction no longer a dominant strategy

מכרז בשדה דב

שטח במ"ר:	7,575	מחיר מינימום ב₪	215,321,877	מחיר שומה ב₪:	676,049,849

תוכנית	מגרש
תמל/3001	107

גוש	חלקה
6634	15
	149
	150
	169
	219
	221
	223
	242

	סכום הצעה ב₪	
1	1,457,777,888	הצעה זוכה
2	1,264,790,118	
3	1,245,835,405	
4	1,222,770,770	
5	1,108,175,000	
6	1,063,010,000	
7	952,555,555	
8	946,000,036	
9	940,556,556	
10	917,024,368	
11	888,987,555	
12	440,500,036	

More on auctions

- Combinatorial auctions
  - E.g. buying several land slots to build a large building
  - Generalizing “second price” : VCG
- Competing auctions
  - Multiple sellers on e-Bay
  - E-Bay vs. Alibaba
- Ad auctions
  - (later in the course)
- Which auctions are truthful?

References

- Myerson, Roger B. "Optimal auction design." *Mathematics of operations research* 6.1 (1981): 58-73.
- Bulow, Jeremy, and Paul Klemperer. *Auctions vs. negotiations*. No. w4608. National Bureau of Economic Research, 1994.
- Nisan, Noam et al. (eds.) *Algorithmic Game Theory*. Cambridge University Press. 2007.

Electronic Commerce
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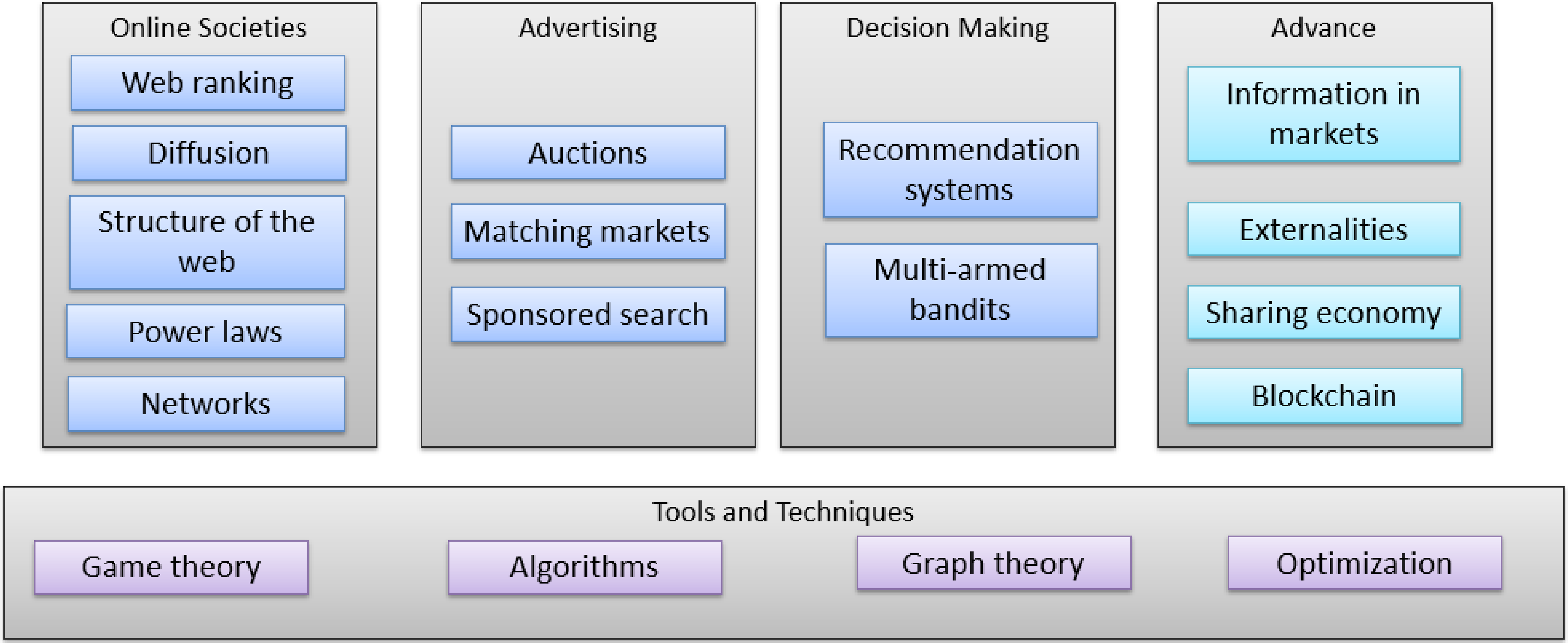
Matching Markets

Omer Ben Porat (some of the slides are adopted from Prof. Reshef Meir)


Online Societies
Web ranking
Diffusion
Structure of the web
Power laws
Networks
Advertising
Auctions
Matching markets
Sponsored search
Decision Making
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Multi-armed bandits
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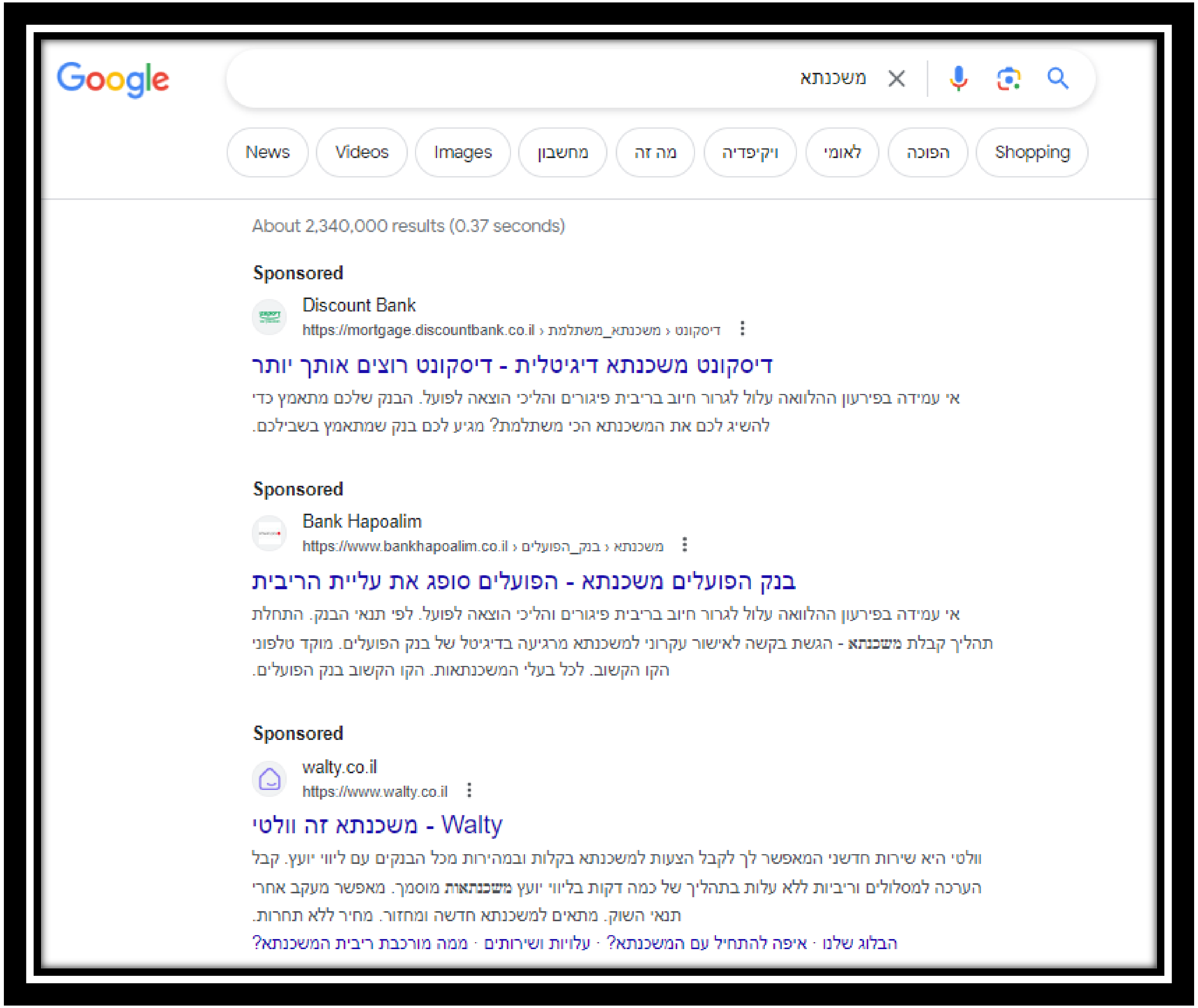
Externalities
Sharing economy
Blockchain
Tools and Techniques
Game theory
Algorithms
Graph theory
Optimization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Outline

- Matching in markets
- Social welfare driven!
- Market clearing prices: How prices are formed?
- We’ll see two powerful tools:
  - VCG
  - ~~deferred acceptance~~

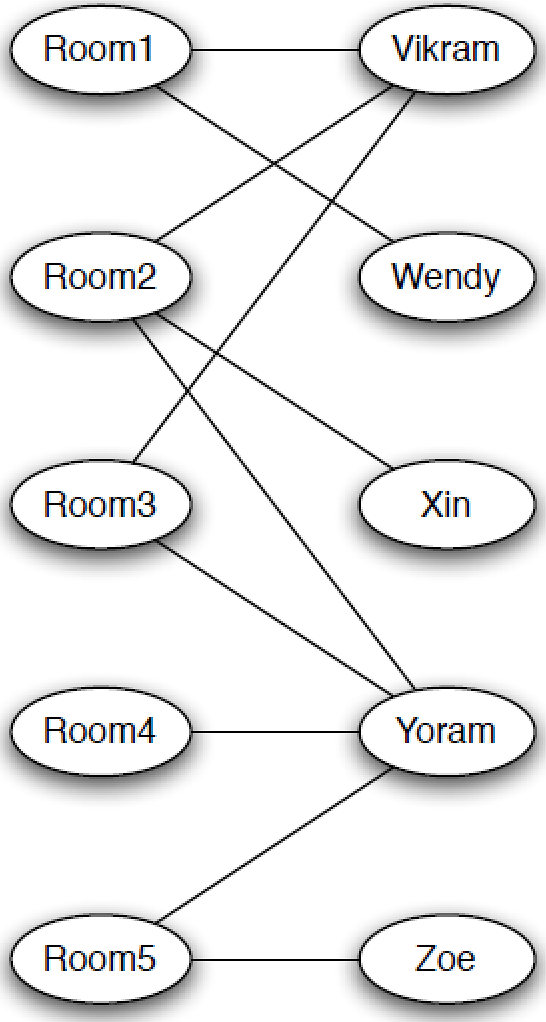
Markets

- We want to match buyers with items/sellers

1. Can we match everyone?
2. Can we price items such that everyone is happy?
3. Can we price items for strategic buyers?
4. Can we find a good match when both sides have preferences? (and no money)

Some graph theory

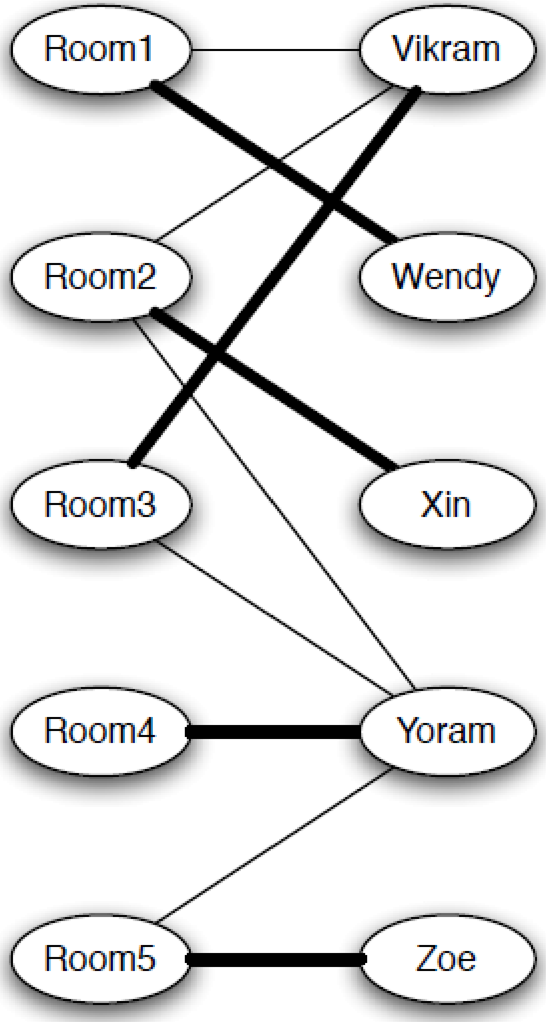
- Bipartite graph: (L, R, E)



Some graph theory

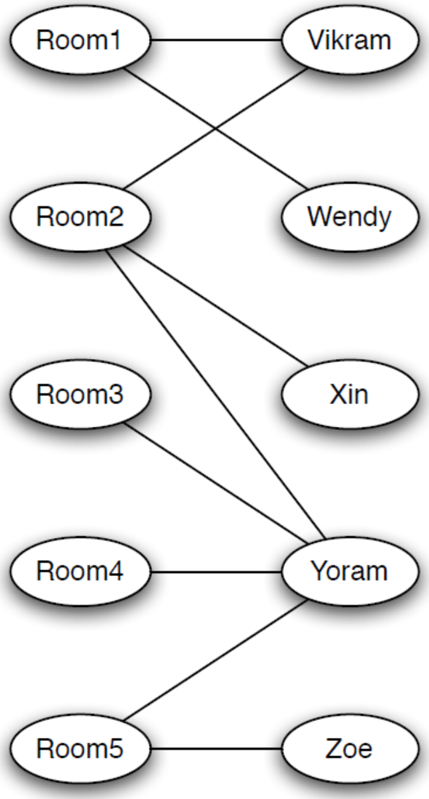
- Bipartite graph: (L, R, E)

- A perfect matching:
  - A subgraph where each L −node is connected to exactly one R −node
  - If $|L| < |R|$, perfect matching means matching all nodes in L

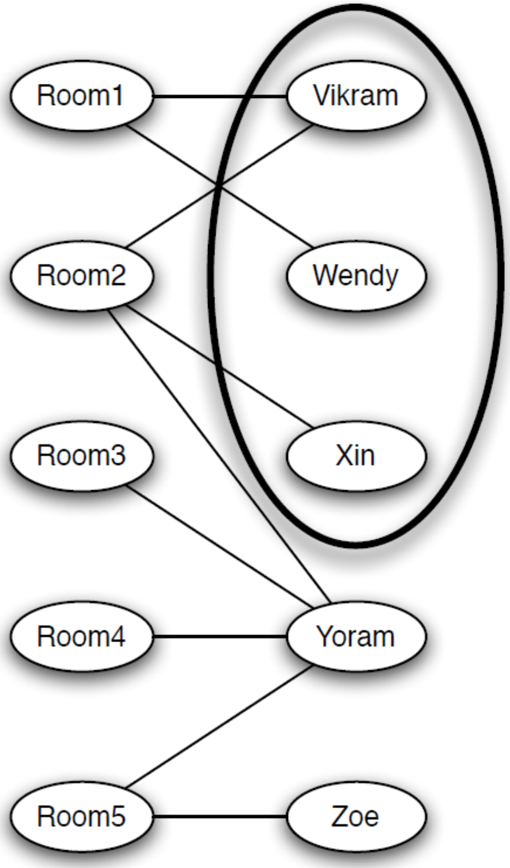


Perfect matchings

- If there is a perfect matching, we can simply describe it
- How can we prove such a matching does not exist?
- Constricted sets:
  - A subset $L' \subseteq L$ is constricted if
$$|L'| > |\underbrace{\{r \in R: \exists l \in L', (r, l) \in E\}}_{\Gamma(L')}|$$
  - Same for $R' \subseteq R$



(a) *Bipartite graph with no perfect matching*



(b) *A constricted set demonstrating there is no perfect matching*

- Impossible to find rooms for all of

  R' ={Vikram, Wendy, Xin}

Hall’s Marriage Theorem

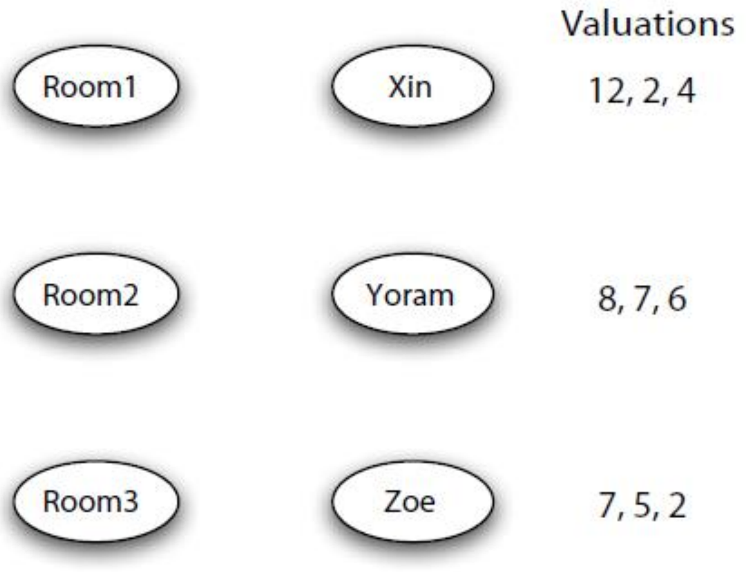
- **Theorem**: (L, R, E) has a full matching for L if and only if there are no constricted sets in L.
- Algorithm?

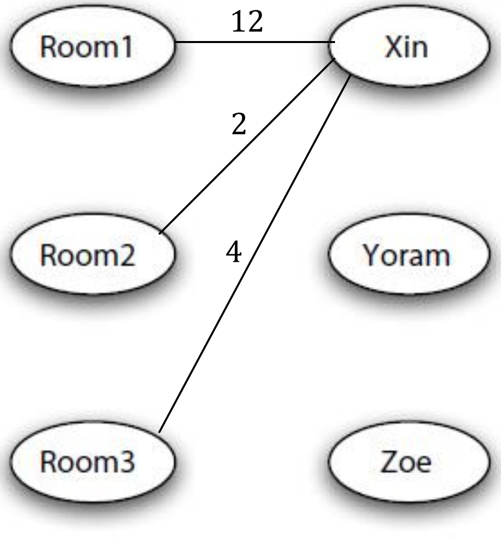
Weighted Preferences

- Hall’s theorem only applies when preferences are very simple
  - Accept / reject
- Students may want to express richer preferences
  - Can also be thought of as a **weighted** graph
- Recall: Social welfare!

Weighted Preferences

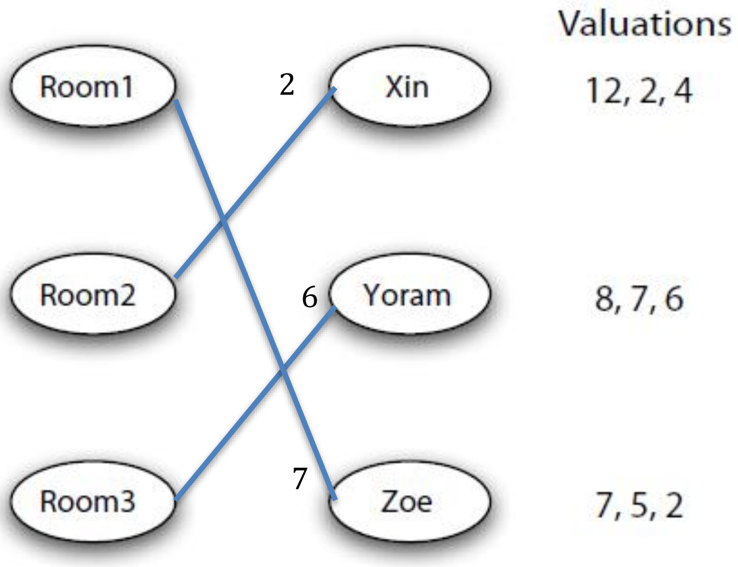
- Hall's theorem only applies when preferences are very simple
  - Accept / reject
- Students may want to express richer preferences
  - Can also be thought of as a **weighted** graph



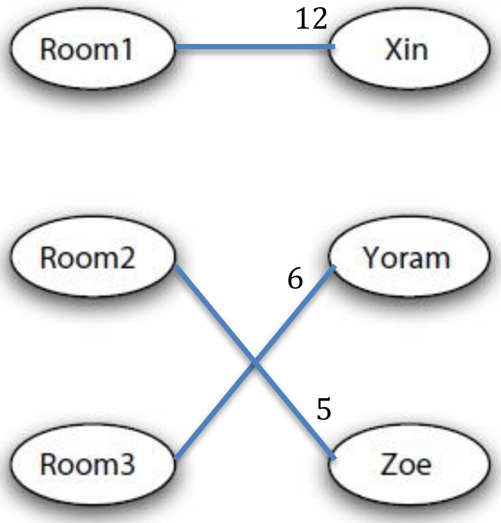


Optimal Matching

- There are many possible matchings
- The optimal matching has the maximum sum of valuations $(SW(M) := \sum_{(i,j)\in M} v_{ij})$
- How can the market find the best matching?



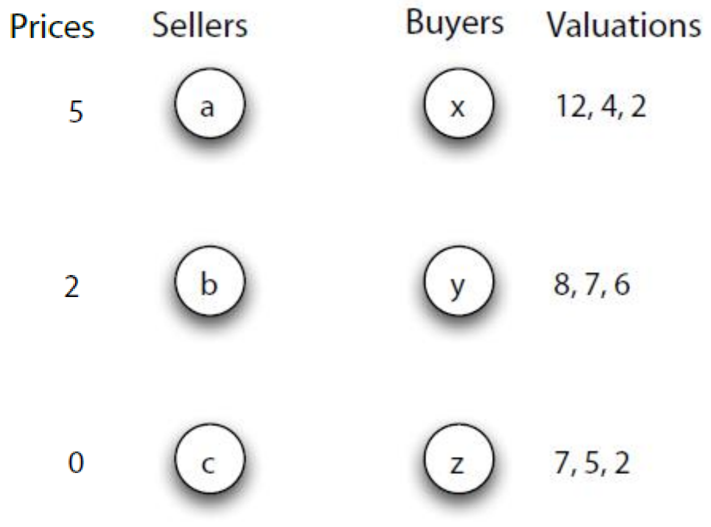
$SW(M_1) = 2 + 6 + 7 = 15$



$SW(M_2) = 12 + 6 + 5 = 23$

Matching Markets - Decentralized

- Each buyer $i \in [n]$ has value v_{ij} for house j
- If i gets house $j \in [n]$ for price p_j, her utility $u_i = v_{ij} - p_j$
- Seller j earns utility of p_j if has a buyer, else 0



(a) *Buyer Valuations*

"Yeah, I like this penthouse but it's too expensive"

- Assume that prices are fixed
- Social welfare as sum of valuations: $SW(M) = \sum_{(i,j)\in M} v_{ij}$
- Social welfare as sum of utilities:

$$SW(M,\boldsymbol{p}) = \sum_{i \in buyers} u_i(M,\boldsymbol{p}) + \sum_{j \in sellers} u_j(M,\boldsymbol{p})$$

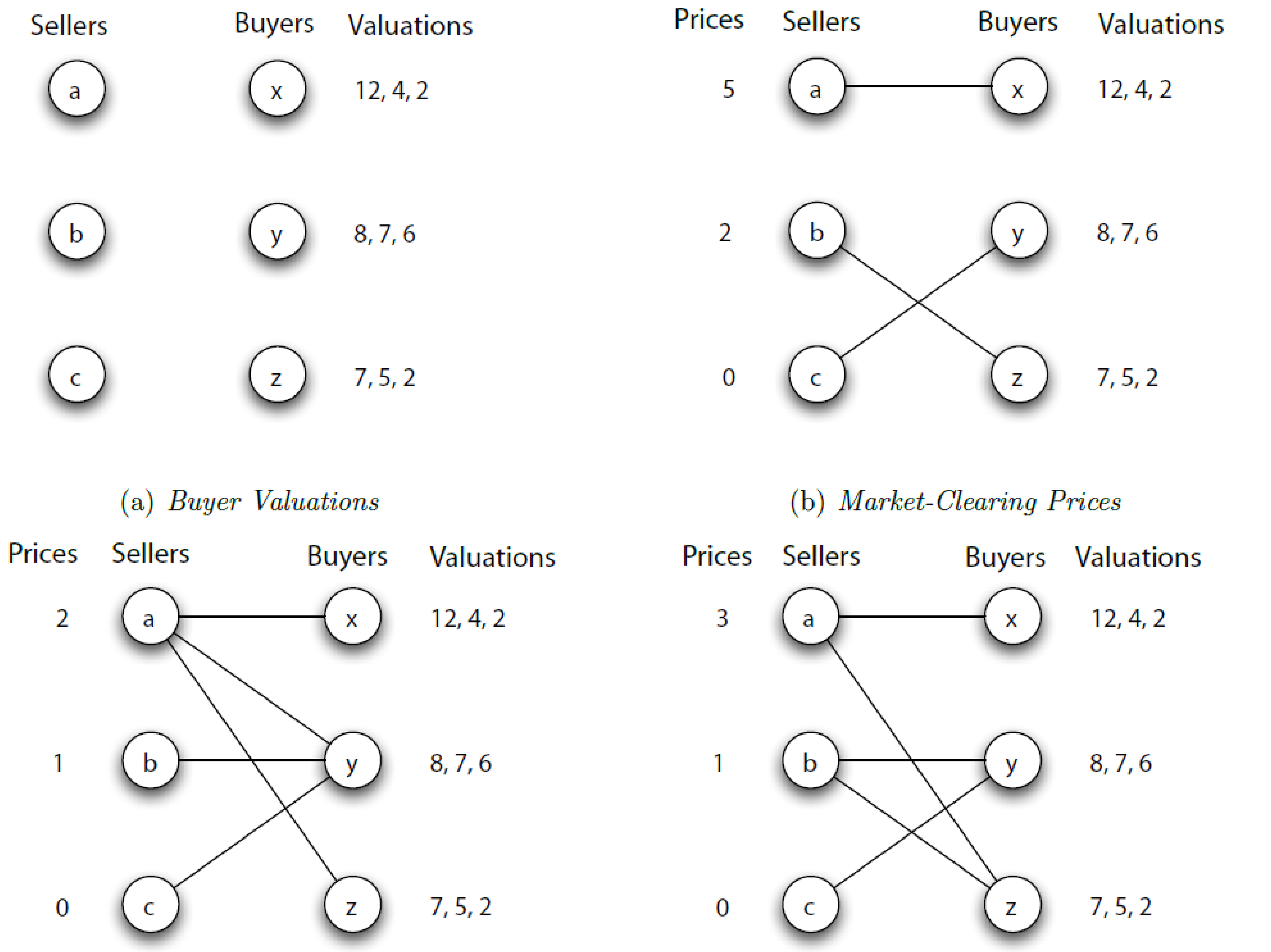
$$= \sum_{i \in buyers} (v_{i,M(i)} - p_i) + \sum_{j \in sellers} p_j$$

$$= \sum_{i \in buyers} v_{i,M(i)} - \sum_i p_i + \sum_j p_j$$

$$= \sum_{(i,j)\in M} v_{i,j}$$

Prices do not affect the social welfare

- Edges: only most preferred items!
- Recall perfect matching!



(a) *Buyer Valuations*

(b) *Market-Clearing Prices*

(c) *Prices that Don't Clear the Market*

(d) *Market-Clearing Prices (Tie-Breaking Required)*

Matching Markets and Clearing Prices

- Market proceeds as follows:
  - Each seller posts a price p_j
  - Each buyer i selects the house j that maximizes $v_{ij} - p_j$
  - If $v_{ij} - p_j < 0$ then i does not buy
- If each buyer selects a different house, we say the market is cleared
- What if a buyer is indifferent among several houses?
  - We get an unweighted graph $G_{\boldsymbol{p}}$ with edge (i, j) for all houses j that i prefers under prices $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$
- Prices $\boldsymbol{p}$ are market-clearing if the graph $G_{\boldsymbol{p}}$ has a perfect matching

Market clearing prices

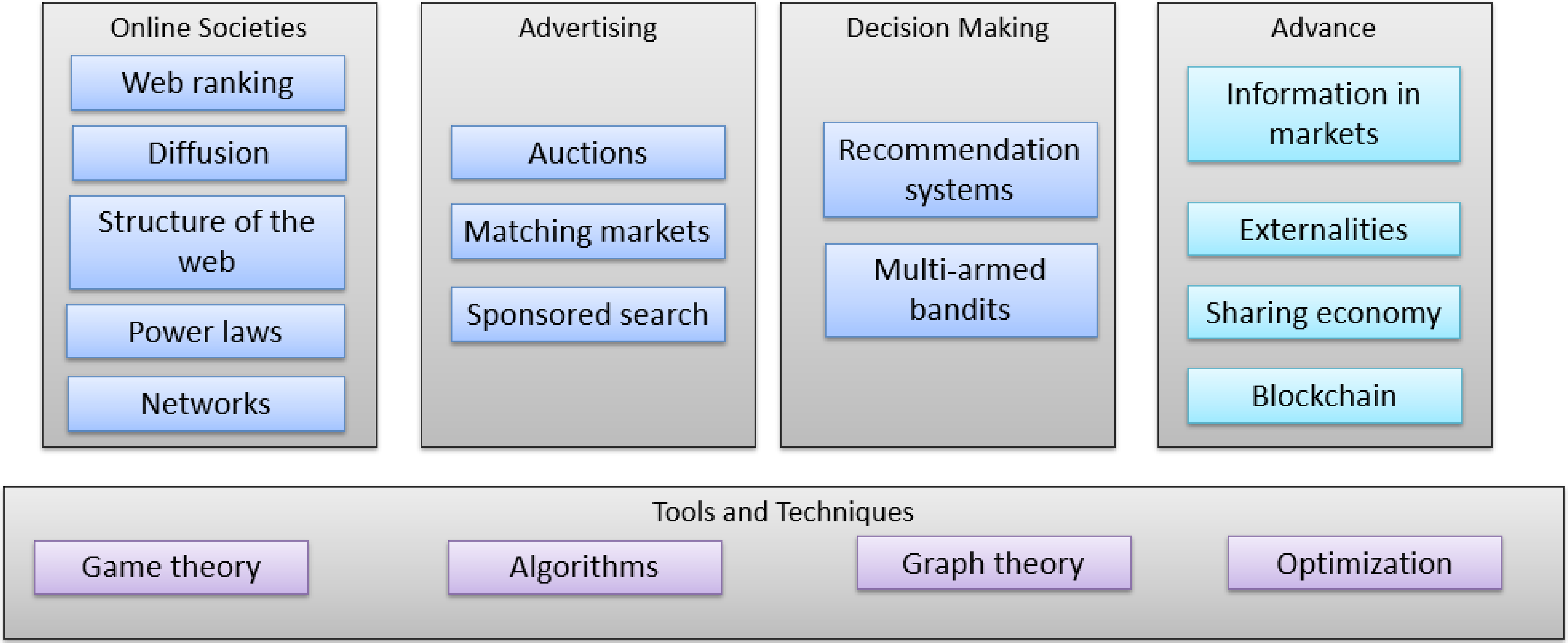
- **Theorem 1**: Market-clearing prices always exist
- We'll later see why, and how to find them. No need to know valuations!

- **Theorem 2:** Any matching attained from market-clearing prices is optimal (maximizes the social welfare)

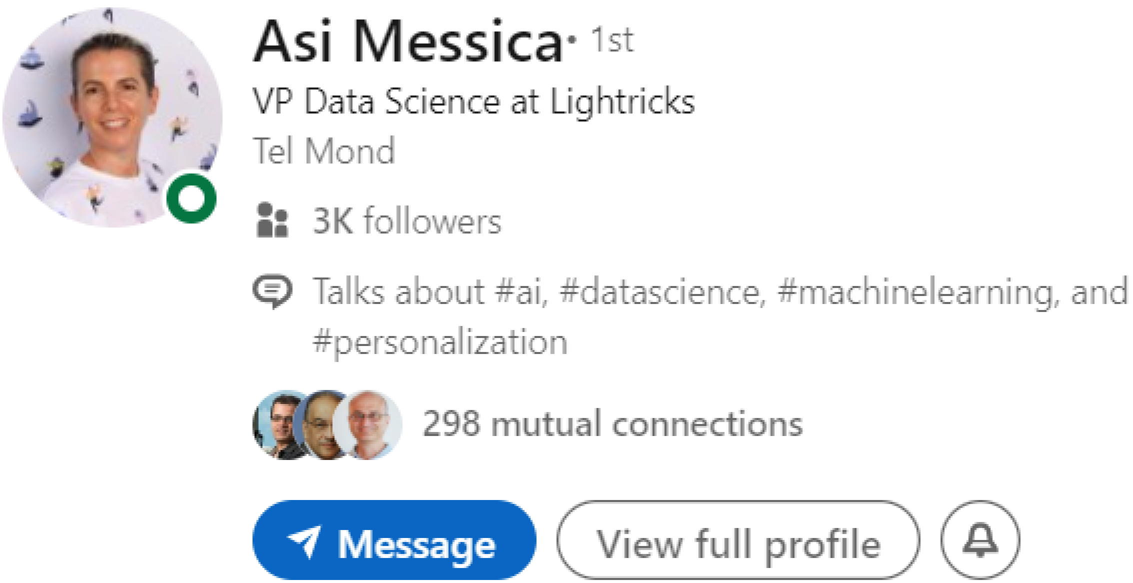
Electronic Commerce 096211

Matching Markets

Omer Ben Porat (some of the slides are adopted from Prof. Reshef Meir)


Online Societies
Web ranking
Diffusion
Structure of the web
Power laws
Networks
Advertising
Auctions
Matching markets
Sponsored search
Decision Making
Recommendation systems
Multi-armed bandits
Advance
Information in markets
Externalities
Sharing economy
Blockchain
Tools and Techniques
Game theory
Algorithms
Graph theory
Optimization

Today 1


Asi Messica· 1st
VP Data Science at Lightricks
Tel Mond
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Today 2

משאל להערכת ההוראה והקורס - אביב תשפ"ג - יוצא לדרך

1 attachment

OC ACADEMIC-STAFF-L - Te
Staff Members Mailing List on t

To: ACADEMIC-STAFF-L@LISTS Thu 6/15/2023 10:19 AM

אל להערכת ההוראה והקורס...
2 MB

לסגל ההוראה שלום רב,

למידה והוראה איכותיות הן מטרות משותפות לסגל, לסטודנטים.ות ולכלל הטכניון. כדי לקדם את איכות הלמידה וההוראה, לימודי הסמכה ביחד עם המרכז לקידום הלמידה וההוראה ואס"ט יוזמים מדי סמסטר את המשאל להערכת ההוראה והקורס. המשאל מתמקד בבחינת העמדות כלפי חווית הלמידה והתהליכים המעורבים בה, בעיקר בהיבטים להם השפעה ישירה על קהל הסטודנטים.ות. מידע הנאסף במסגרת המשאל מסייע לסגל ההוראה, לפקולטות ולטכניון להעריך ביתר בהירות את הקורסים, את ההוראה ואת הלמידה. עוד מסייע המשאל למקד ובמידת הצורך לשדרג את הכלים ואת שיטות ההוראה, ומאפשר לסטודנטים.ות לחלוק עם המרצה תובנות לגבי תהליך הלמידה ותוצריו.

תוצאות המשאל עבור סגל הוראה

יציאה

שם:בן-פורת עומר

Error

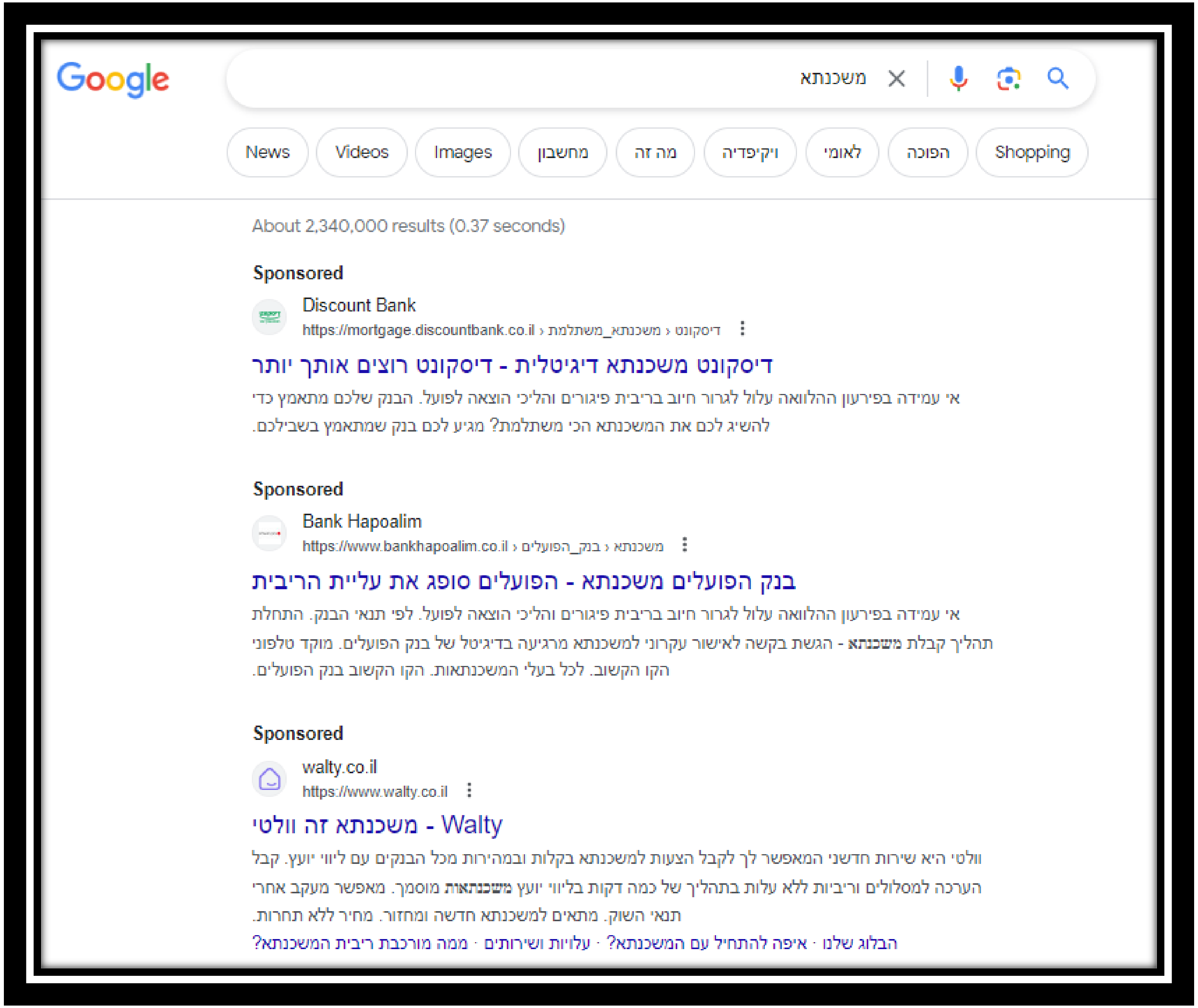
Server Error:

Program: " WMRNSDS8 "

Function: Send_Text

Html Text Length=32445

The Maximum alowed is 32317 only


Google
משכנתא
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מחשבון
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הפוכה
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About 2,340,000 results (0.37 seconds)
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https://mortgage.discountbank.co.il › משכנתא_משתלמת › דיסקונט
דיסקונט משכנתא דיגיטלית - דיסקונט רוצים אותך יותר
אי עמידה בפירעון ההלוואה עלול לגרור חיוב בריבית פיגורים והליכי הוצאה לפועל. הבנק שלכם מתאמץ כדי להשיג לכם את המשכנתא הכי משתלמת? מגיע לכם בנק שמתאמץ בשבילכם.
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הבלוג שלנו · איפה להתחיל עם המשכנתא? · עלויות ושירותים · ממה מורכבת ריבית המשכנתא?

Outline

- Matching in markets
- Social welfare driven!
- Market clearing prices: How prices are formed?
- We'll see two powerful tools:
  - VCG
  - ~~deferred acceptance~~

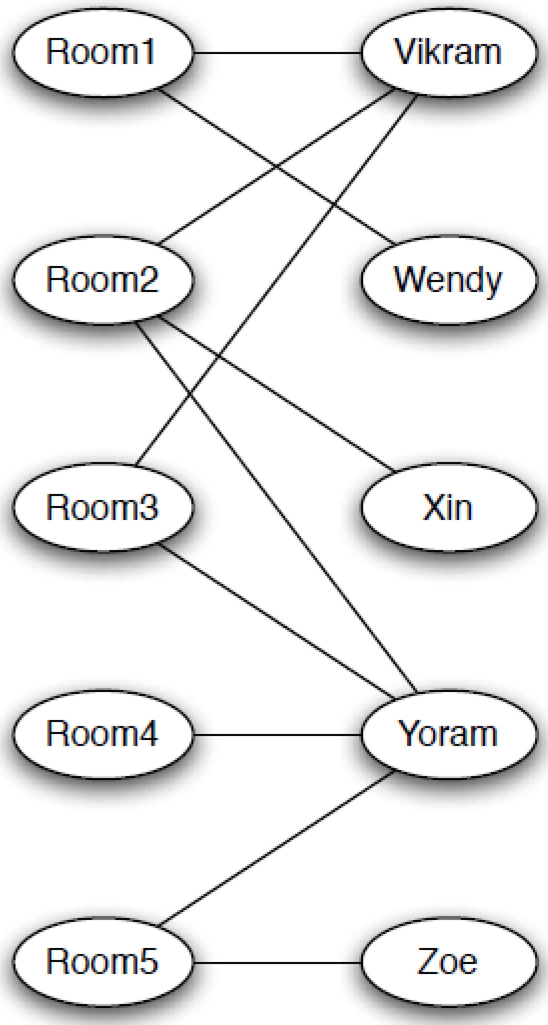
Markets

- We want to match buyers with items/sellers

1. Can we match everyone?
2. Can we price items such that everyone is happy?
3. Can we price items for strategic buyers?
4. Can we find a good match when both sides have preferences? (and no money)

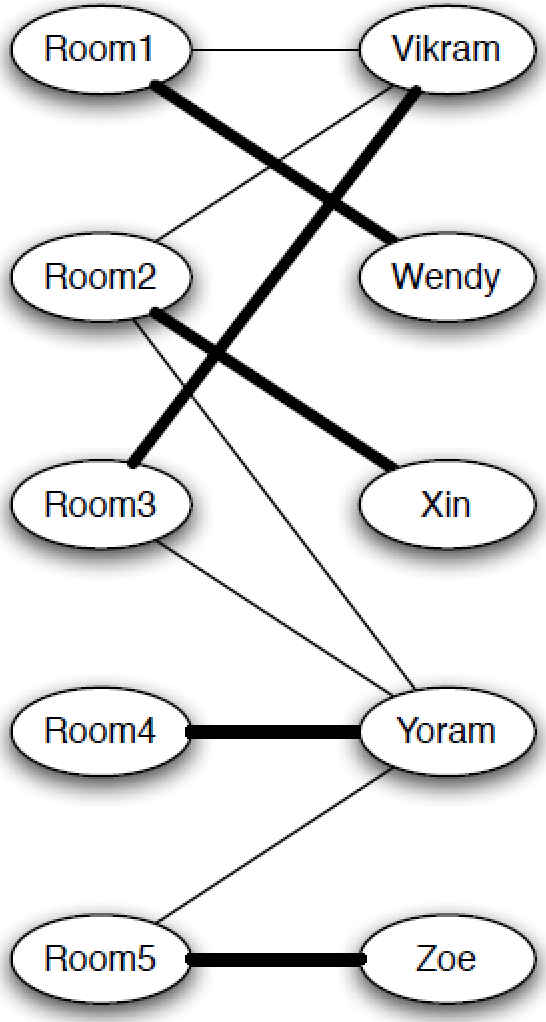
Some graph theory

- Bipartite graph: (L, R, E)



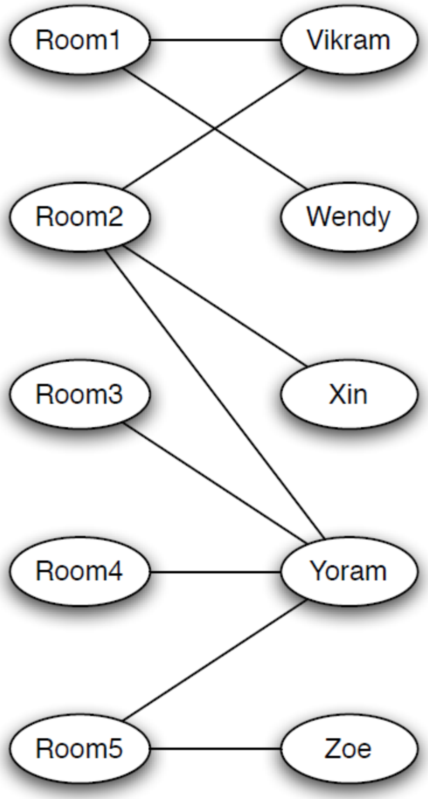
Some graph theory

- Bipartite graph: (L, R, E)
- A perfect matching:
  - A subgraph where each L −node is connected to exactly one R −node
  - If $|L| < |R|$, perfect matching means matching all nodes in L

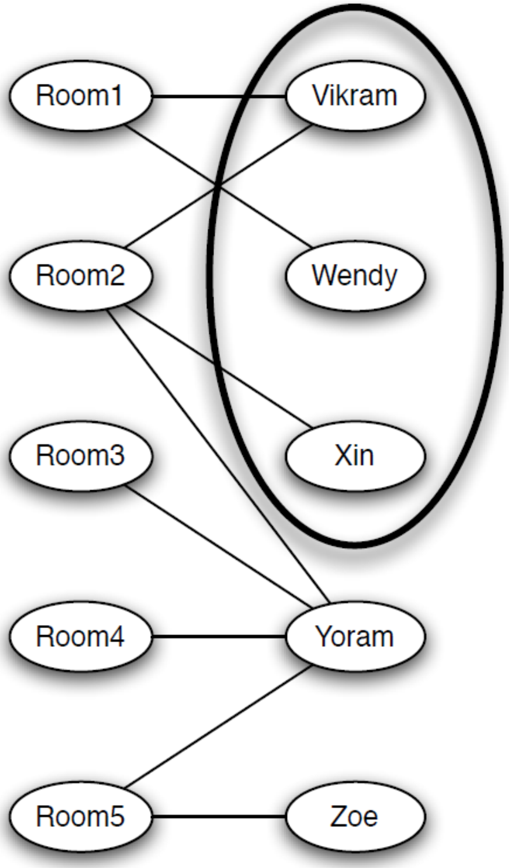


Perfect matchings

- If there is a perfect matching, we can simply describe it
- How can we prove such a matching does not exist?
- Constricted sets:
  - A subset $L' \subseteq L$ is constricted if
$$|L'| > |\underbrace{\{r \in R : \exists l \in L', (r,l) \in E\}}_{\Gamma(L')}|$$
  - Same for $R' \subseteq R$



(a) *Bipartite graph with no perfect matching*



(b) *A constricted set demonstrating there is no perfect matching*

- Impossible to find rooms for all of

  R' ={Vikram, Wendy, Xin}

Hall's Marriage Theorem

- **Theorem**: (L, R, E) has a full matching for L if and only if there are no constricted sets in L.
- Algorithm?

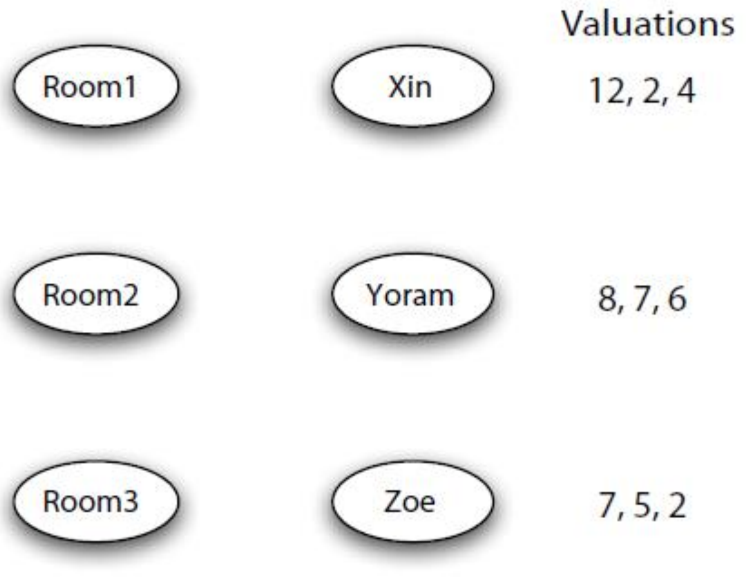
Weighted Preferences

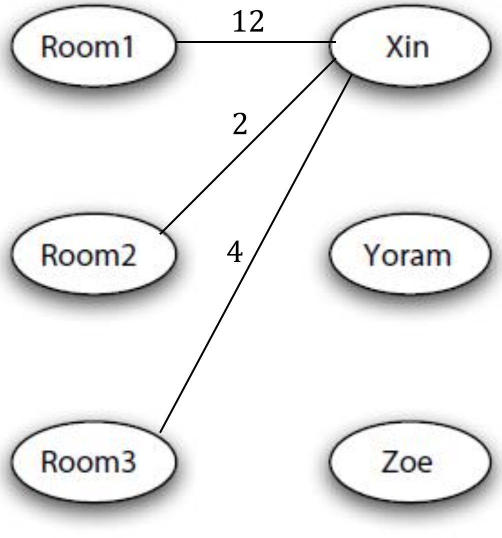
- Hall's theorem only applies when preferences are very simple
  - Accept / reject
- Students may want to express richer preferences
  - Can also be thought of as a **weighted** graph

- Recall: Social welfare!

Weighted Preferences

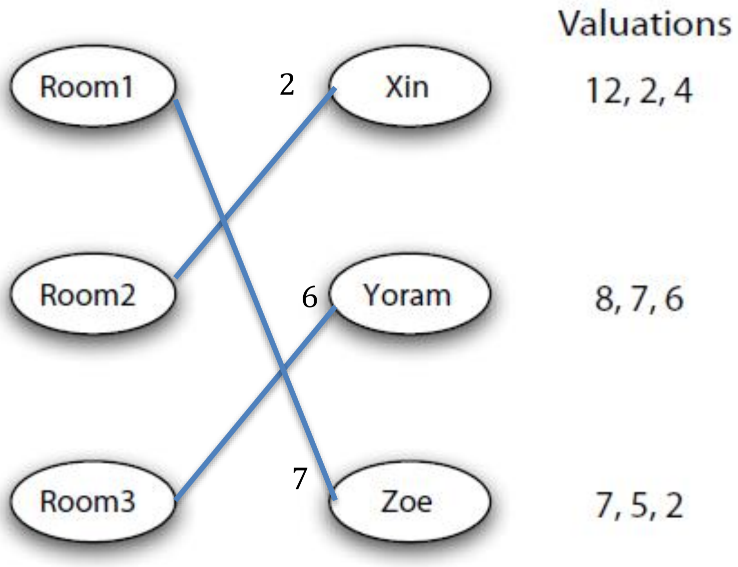
- Hall's theorem only applies when preferences are very simple
  - Accept / reject
- Students may want to express richer preferences
  - Can also be thought of as a **weighted** graph



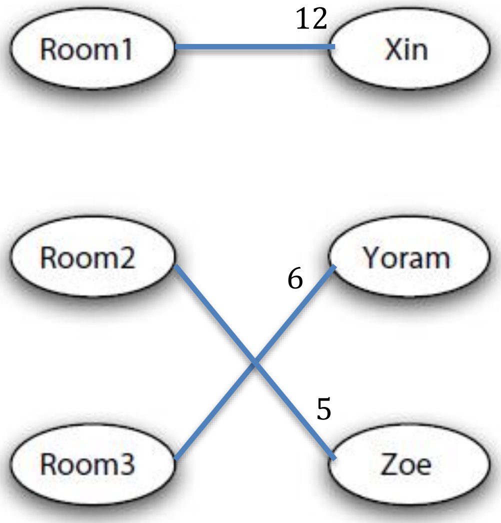


Optimal Matching

- There are many possible matchings
- The optimal matching has the maximum sum of valuations ($SW(M) := \sum_{(i,j)\in M} v_{ij}$)
- How can the market find the best matching?



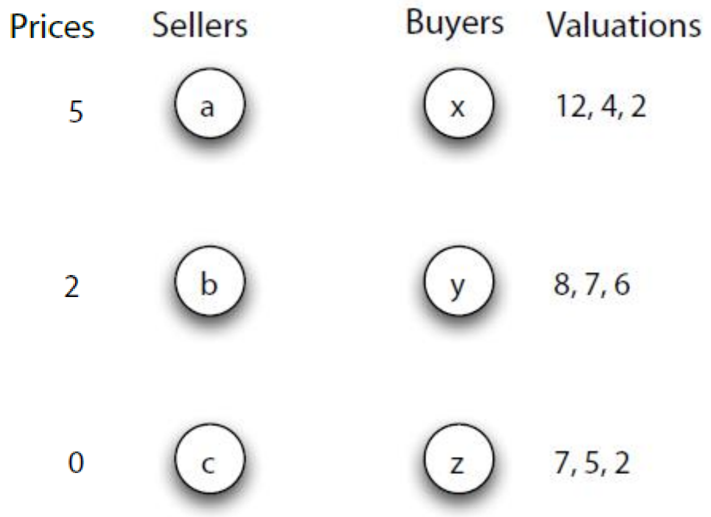
$SW(M_1) = 2 + 6 + 7 = 15$



$SW(M_2) = 12 + 6 + 5 = 23$

Matching Markets - Decentralized

- Each buyer $i \in [n]$ has value v_{ij} for house j
- If i gets house $j \in [n]$ for price p_j, her utility $u_i = v_{ij} - p_j$
- Seller j earns utility of p_j if has a buyer, else 0



(a) *Buyer Valuations*

"Yeah, I like this penthouse but it's too expensive"

- Assume that prices are fixed
- Social welfare as sum of valuations: $SW(M) = \sum_{(i,j)\in M} v_{ij}$
- Social welfare as sum of utilities:

$$SW(M, \boldsymbol{p}) = \sum_{i \in buyers} u_i(M, \boldsymbol{p}) + \sum_{j \in sellers} u_j(M, \boldsymbol{p})$$

$$= \sum_{i \in buyers} (v_{i,M(i)} - p_i) + \sum_{j \in sellers} p_j$$

$$= \sum_{i \in buyers} v_{i,M(i)} - \sum_i p_i + \sum_j p_j$$

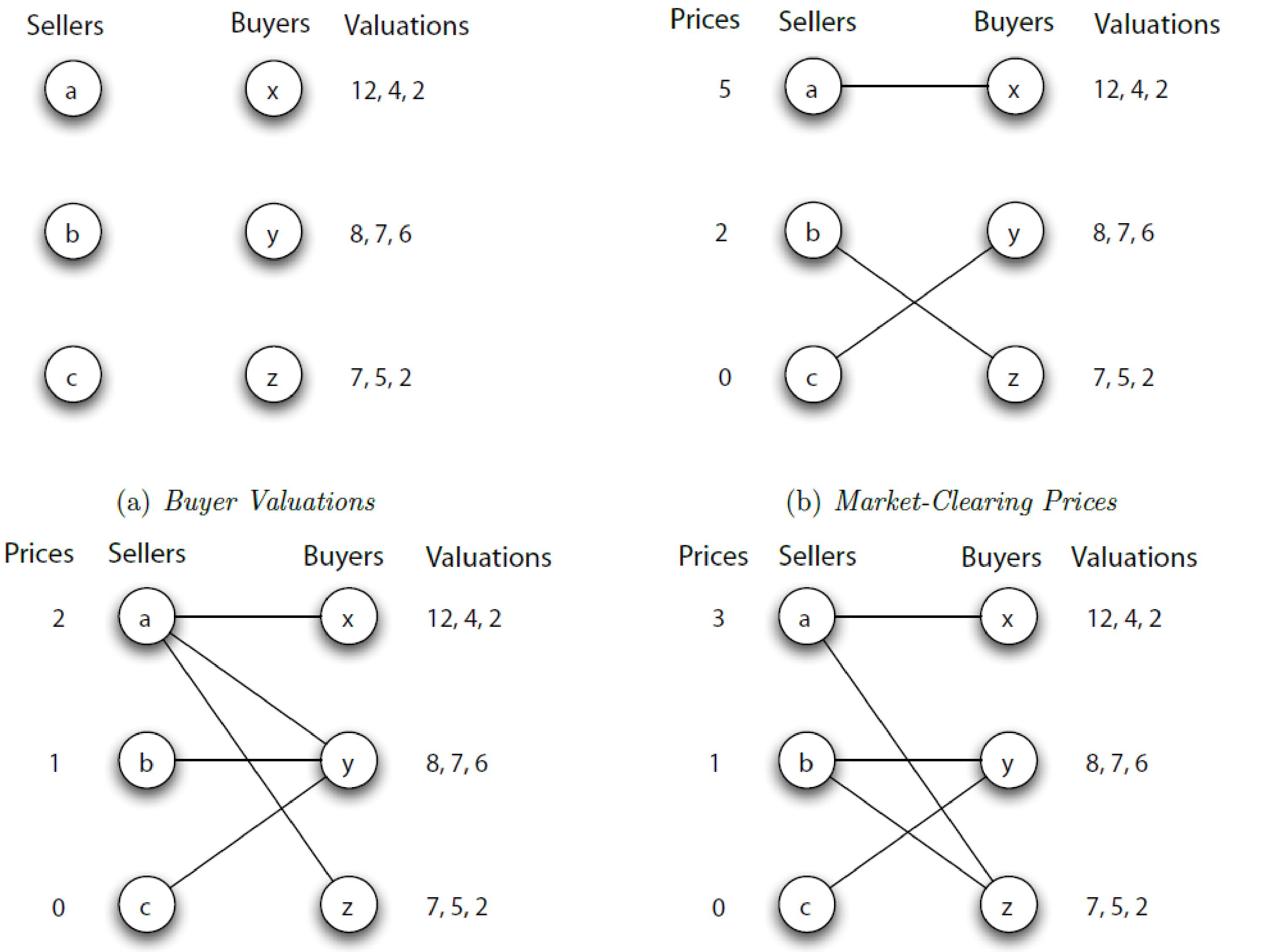
$$= \sum_{(i,j)\in M} v_{i,j}$$

Prices do not affect the social welfare

Matching Markets and Clearing Prices

- Market proceeds as follows:
  - Each seller posts a price p_j
  - Each buyer i selects the house j that maximizes $v_{ij} - p_j$
  - If $v_{ij} - p_j < 0$ then i does not buy
- If each buyer selects a different house, we say the market is cleared
- What if a buyer is indifferent among several houses?
  - We get an unweighted graph $G_{\boldsymbol{p}}$ with edge (i, j) for all houses j that i prefers under prices $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$
- Prices $\boldsymbol{p}$ are market-clearing if the graph $G_{\boldsymbol{p}}$ has a perfect matching

- Edges: only most preferred items!
- Recall perfect matching!



(a) *Buyer Valuations*

(b) *Market-Clearing Prices*

(c) *Prices that Don't Clear the Market*

(d) *Market-Clearing Prices (Tie-Breaking Required)*

Market clearing prices

- **Theorem 1**: Market-clearing prices always exist
- We'll later see why, and how to find them. No need to know valuations!

- **Theorem 2:** Any matching attained from market-clearing prices is optimal (maximizes the social welfare)

Market clearing prices

- **Theorem 2:** Any matching attained from a fixed set of market-clearing prices is optimal.
- Proof: Given market-clearing prices $\boldsymbol{p}$, M is maximizing

$$\sum_i u_i(M, \boldsymbol{p}) = \sum_i (v_{i,M(i)} - p_{M(i)})$$

$$= \sum_i v_{i,M(i)} - \sum_j p_j = SW(M) - constant \qquad \square$$

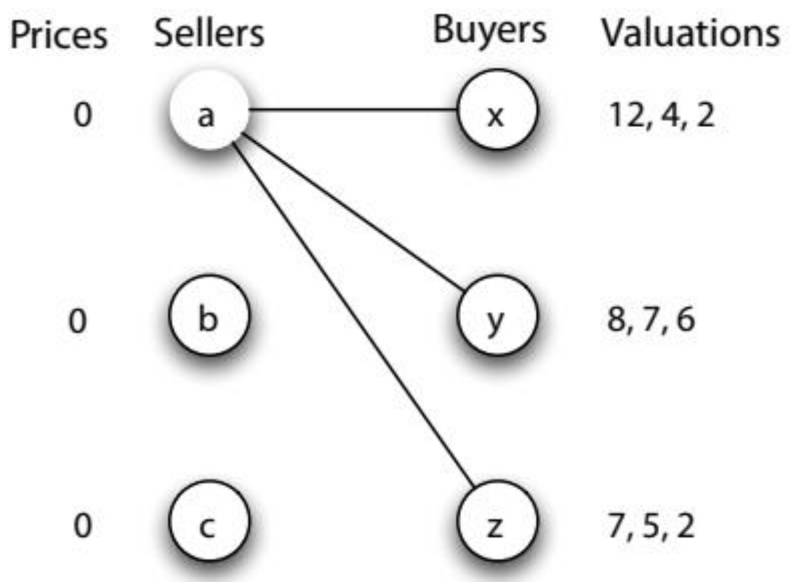
Market clearing prices

- <u>**Theorem 1**</u>: Market-clearing prices always exist.

- How to find market clearing prices?
  - Algorithm
  - Example
  - Proof

Finding market clearing prices

1. Set all prices to 0
2. Repeat:
   a) Find a minimal constricted set of buyers B
      - Otherwise, break and return
   b) All sellers j in $\Gamma(B)$ increase p_j by 1
   c) If all prices (of all sellers) are > 0, decrease all prices by 1



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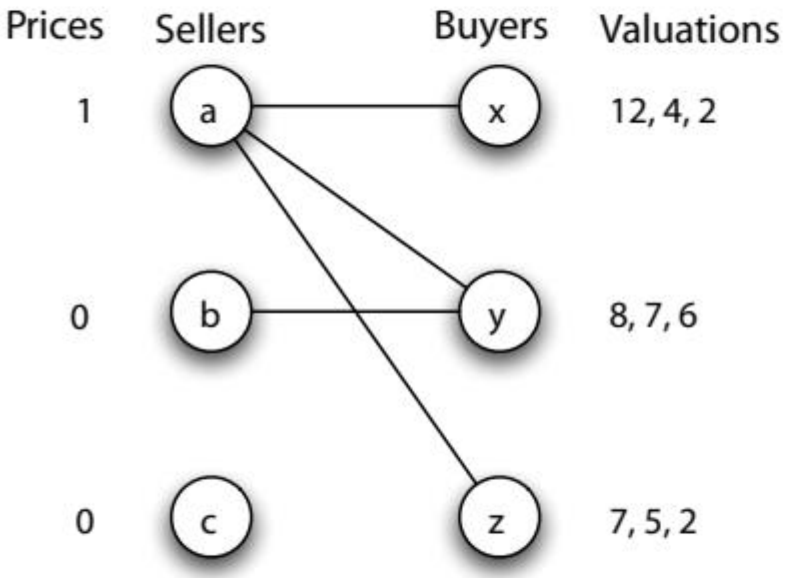

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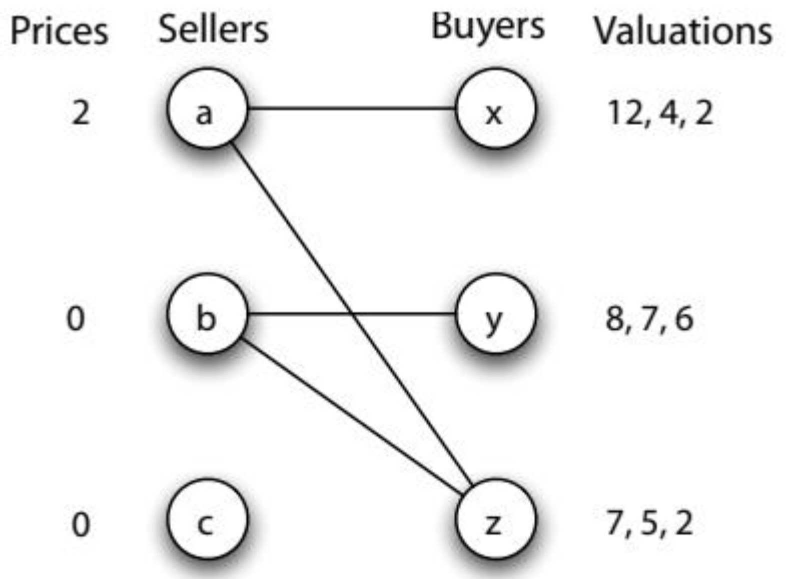

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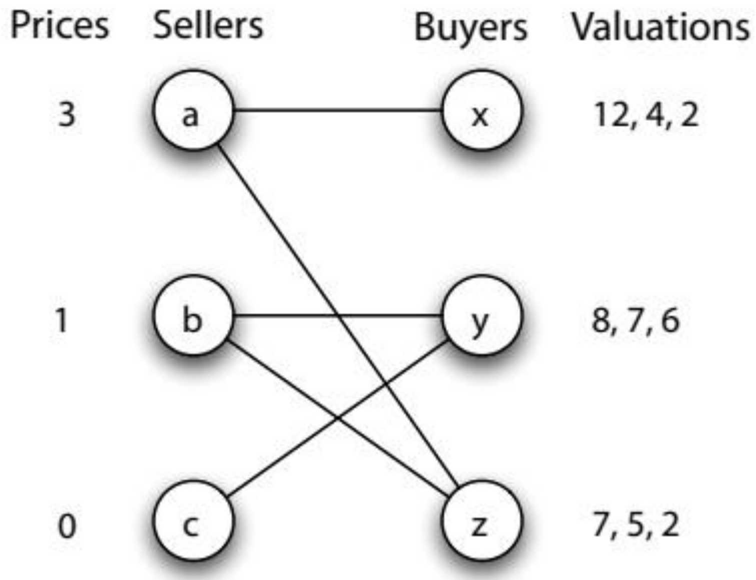


Convergence??

Proof of Theorem 1

- Define:

$$\phi_i(G_{\boldsymbol{p}}) \coloneqq \max_j (v_{ij} - p_j) = \max u_i$$

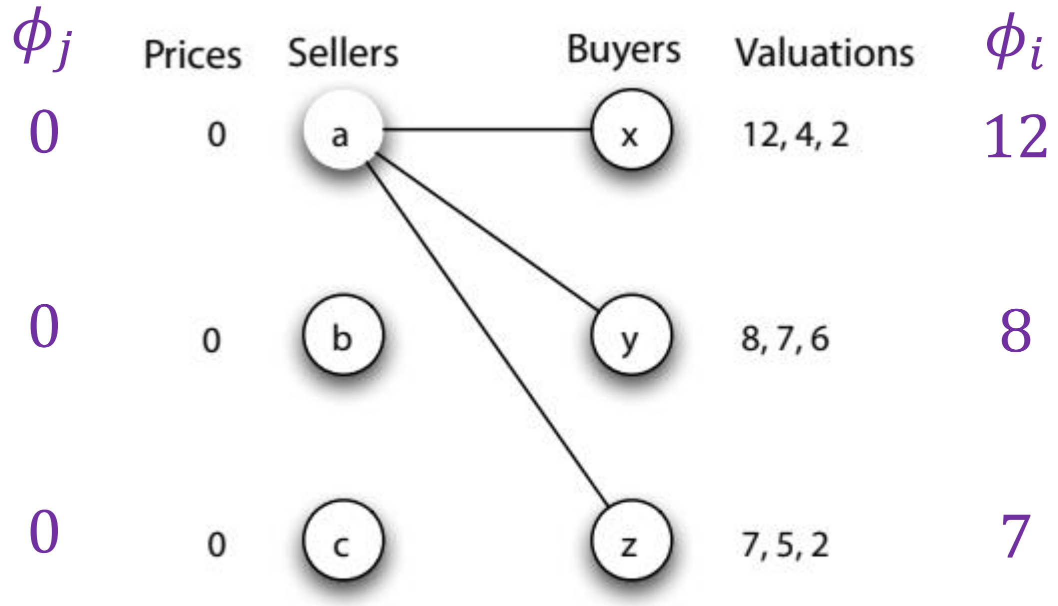
$$\phi_j(G_{\boldsymbol{p}}) \coloneqq p_j = \max u_j$$

$$\phi(\boldsymbol{p}) \coloneqq \sum_i \phi_i(G_{\boldsymbol{p}}) + \sum_j \phi_j(G_{\boldsymbol{p}})$$

- Claim 1: $\phi(\boldsymbol{p})$ is always non-negative
- Claim 2: $\phi(\boldsymbol{p})$ is always decreasing

Finding market clearing prices

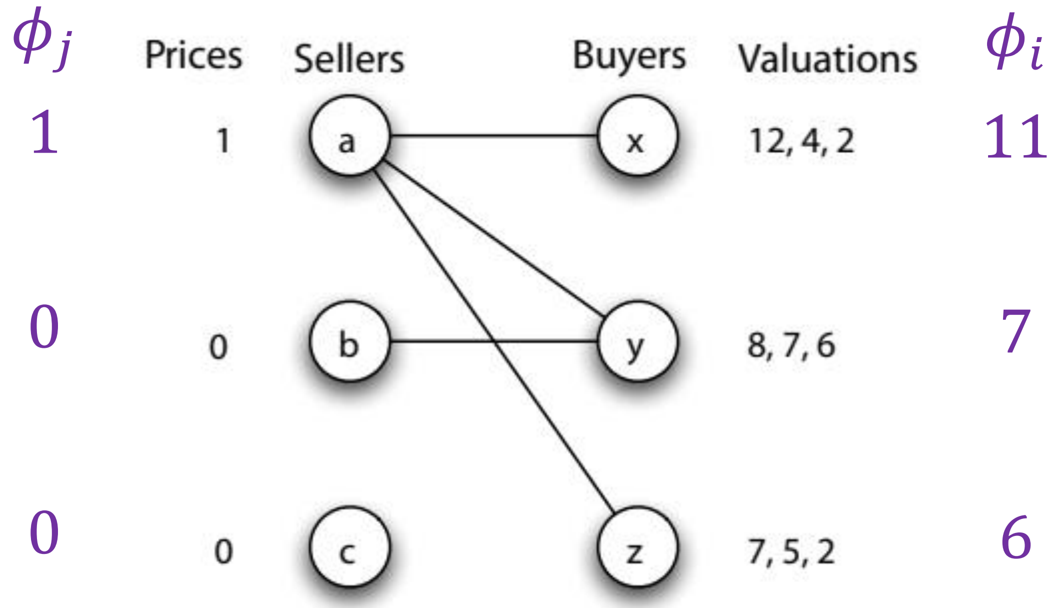
1. Set all prices to 0
2. Repeat:
   a) Find a minimal constricted set of buyers B
      - Otherwise, break and return
   b) All sellers j in $\Gamma(B)$ increase p_j by 1
   c) If all prices (of all sellers) are > 0, decrease all prices by 1



$\phi(\boldsymbol{p}) = 27$

Finding market clearing prices

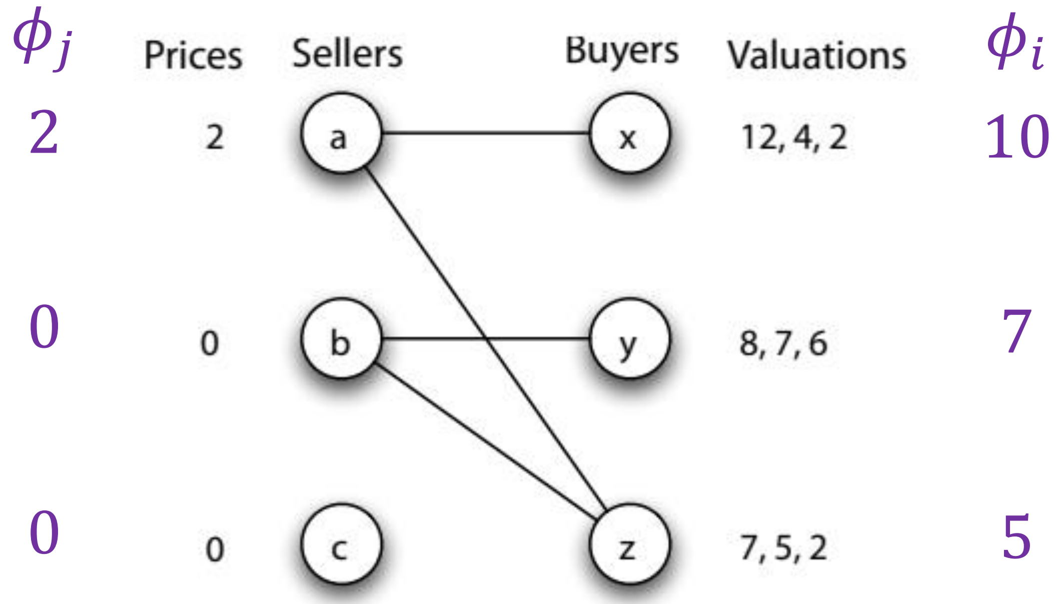
1. Set all prices to 0
2. Repeat:
   a) Find a minimal constricted set of buyers B
      - Otherwise, break and return
   b) All sellers j in $\Gamma(B)$ increase p_j by 1
   c) If all prices (of all sellers) are > 0, decrease all prices by 1



$\phi(\boldsymbol{p}) = 25$

Finding market clearing prices

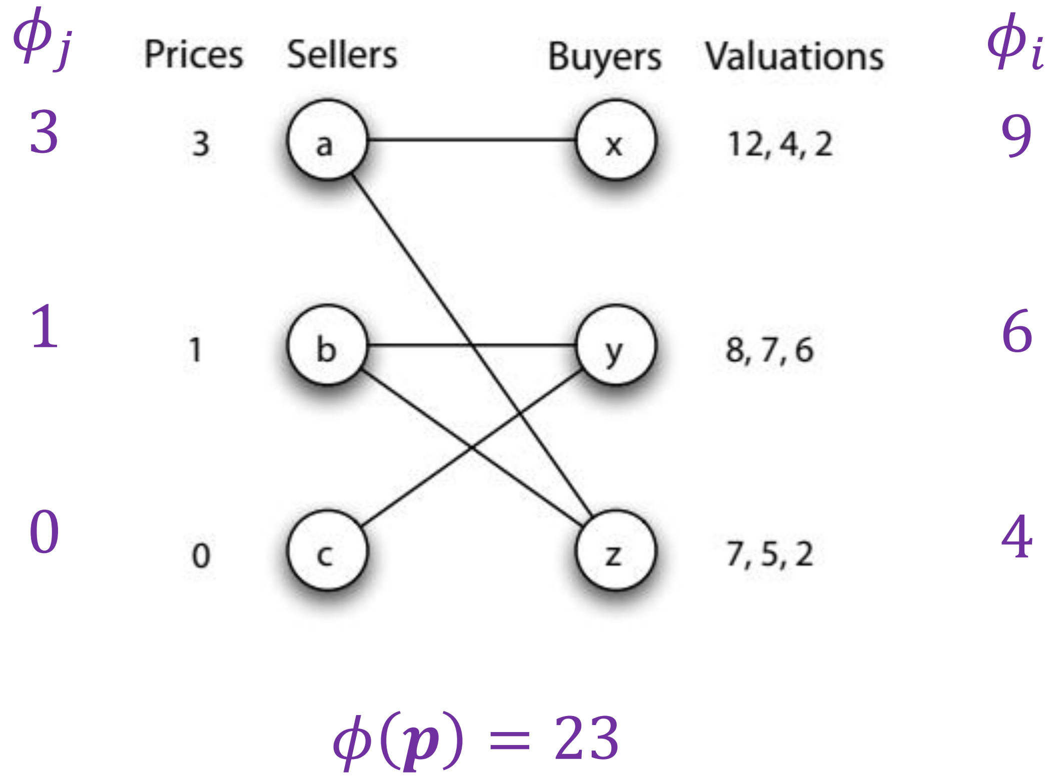
1. Set all prices to 0
2. Repeat:
   a) Find a minimal constricted set of buyers B
      - Otherwise, break and return
   b) All sellers j in $\Gamma(B)$ increase p_j by 1
   c) If all prices (of all sellers) are > 0, decrease all prices by 1



$\phi(\boldsymbol{p}) = 24$

Finding market clearing prices

1. Set all prices to 0
2. Repeat:
    a) Find a minimal constricted set of buyers B
        - Otherwise, break and return
    b) All sellers j in $\Gamma(B)$ increase p_j by 1
    c) If all prices (of all sellers) are > 0, decrease all prices by 1



Proof – decreasing Potential

1. Set all prices to 0
2. Repeat:
   a) Find a minimal constricted set of buyers B
      - Otherwise, break and return

   Case b

   b) All sellers j in $\Gamma(B)$ increase p_j by 1
   c) If all prices (of all sellers) are > 0, decrease all prices by 1

$$\phi(\boldsymbol{p}') = \sum_i \phi_i(G_{\boldsymbol{p}'}) + \sum_j \phi_j(G_{\boldsymbol{p}'})$$

$$= \sum_i \phi_i(G_{\boldsymbol{p}'}) + \sum_j \phi_j(G_{\boldsymbol{p}}) + |\Gamma(B)|$$

$$= \sum_i \phi_i(G_{\boldsymbol{p}}) - |B| + \sum_j \phi_j(G_{\boldsymbol{p}}) + |\Gamma(B)|$$

$$< \sum_i \phi_i(G_{\boldsymbol{p}}) + \sum_j \phi_j(G_{\boldsymbol{p}}) = \phi(\boldsymbol{p})$$

Constricted, $|B| > |\Gamma(B)|$

Recall: $\phi_i(G_{\boldsymbol{p}}) \coloneqq \max_j(v_{ij} - p_j)$, $\phi_j(G_{\boldsymbol{p}}) \coloneqq p_j$

Proof – decreasing Potential

1. Set all prices to 0
2. Repeat:
   a) Find a minimal constricted set of buyers B
      - Otherwise, break and return
   b) All sellers j in $\Gamma(B)$ increase p_j by 1
   c) If all prices (of all sellers) are > 0, decrease all prices by 1

Case c

$$\phi(\boldsymbol{p}') = \sum_i \phi_i(G_{\boldsymbol{p}'}) + \sum_j \phi_j(G_{\boldsymbol{p}'})$$

$$= \sum_i \phi_i(G_{\boldsymbol{p}'}) + \sum_j \phi_j(G_{\boldsymbol{p}}) - n$$

$$= \sum_i \phi_i(G_{\boldsymbol{p}}) + n + \sum_j \phi_j(G_{\boldsymbol{p}}) - n$$

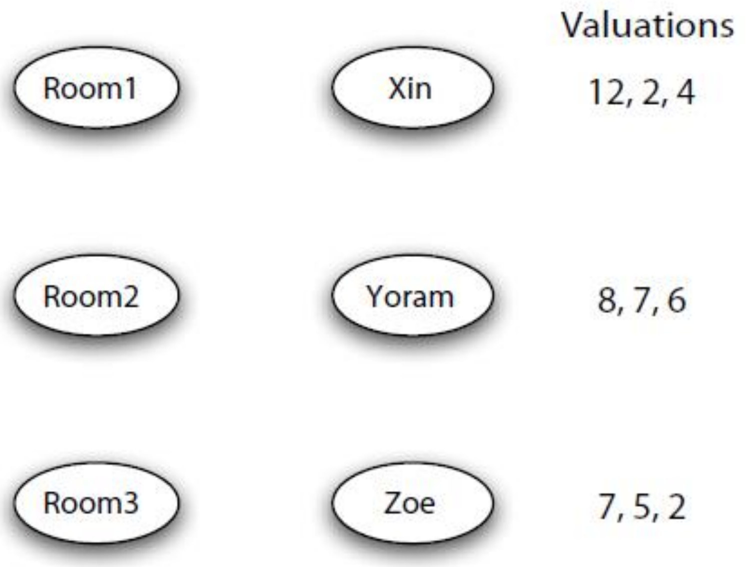
$$= \sum_i \phi_i(G_{\boldsymbol{p}}) + \sum_j \phi_j(G_{\boldsymbol{p}}) = \phi(\boldsymbol{p})$$

Convergence->No constricted set->Hall

Recall: $\phi_i(G_{\boldsymbol{p}}) \coloneqq \max_j (v_{ij} - p_j)$, $\phi_j(G_{\boldsymbol{p}}) \coloneqq p_j$

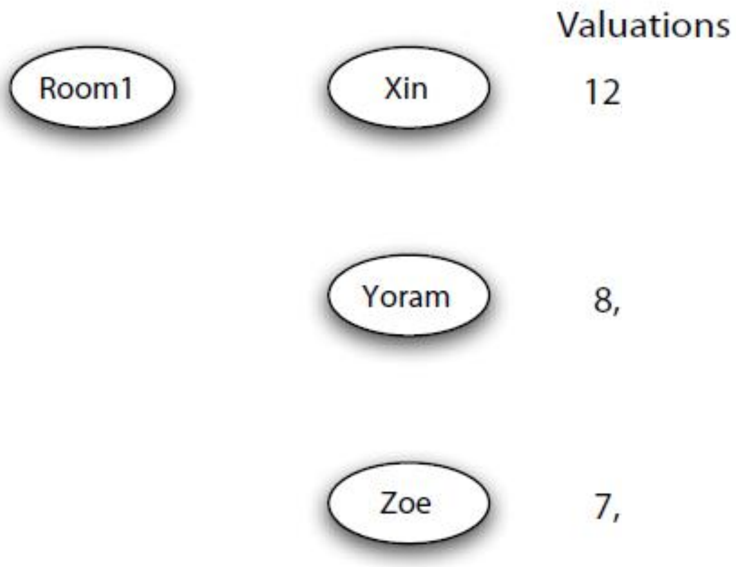
Matching as an auction

- Suppose all rooms are owned by the same seller, buyers are strategic
- Who should get each room?
- At what price?



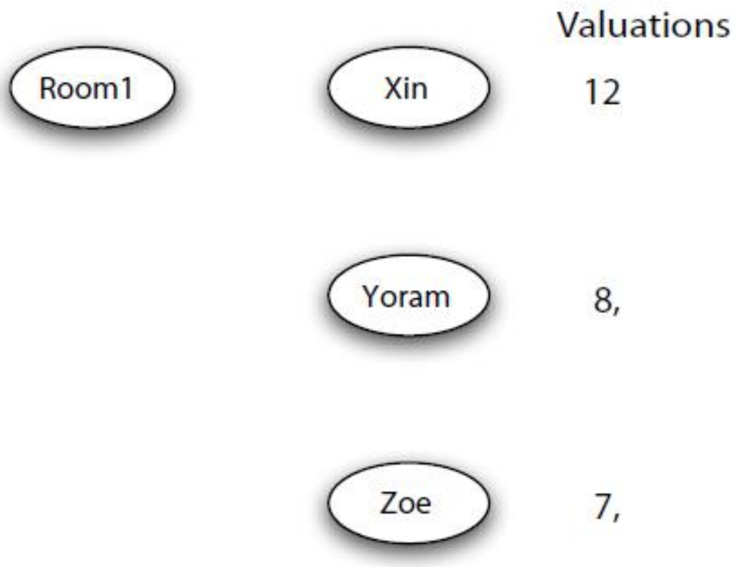
Matching as an auction

- Suppose all rooms are owned by the same seller, buyers are strategic
- Simple case: there is just one room



Matching as an auction

- Suppose all rooms are owned by the same seller, buyers are strategic
- Simple case: there is just one room
  - Use second price auction!



Matching as an auction

- Suppose all rooms are owned by the same seller, buyers are strategic
- How to generalize the second-price auction?
- Recall we need to define:
  - An allocation rule (who gets what room)
  - A payment rule (room prices)

- Interested in social welfare!
- Money is used “artificially”

VCG

- Vickrey-Clarke-Groves
- The VCG principle is that any player should pay the "damage" her actions inflict on other players
- Formally, denote by V_B^S the maximal social welfare of all players in B and all rooms in S
- Each agent i getting item j should pay

$$p_i^{VCG} = V_{B-i}^{S} - V_{B-i}^{S-j}$$

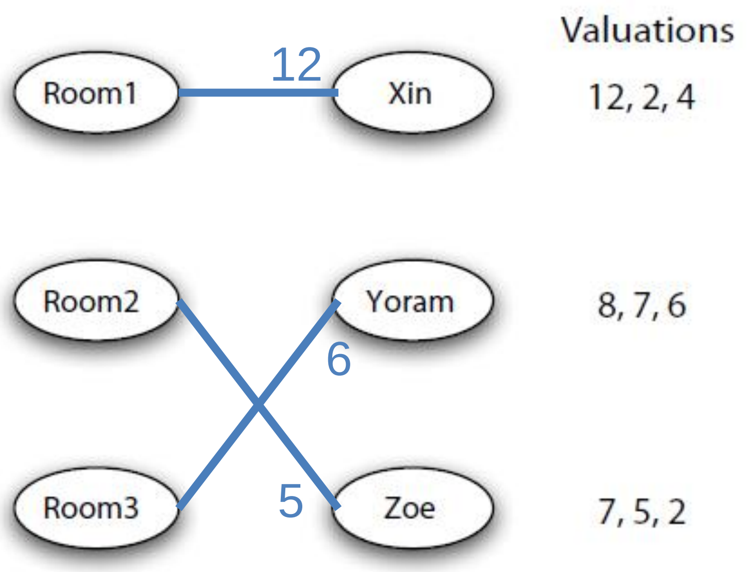
Optimal matching when i does not exist

Current social welfare, without i: $V_B^S - v_{ij}$

VCG example

- Start from the optimal allocation
- How much should Xin ($= X$) pay?

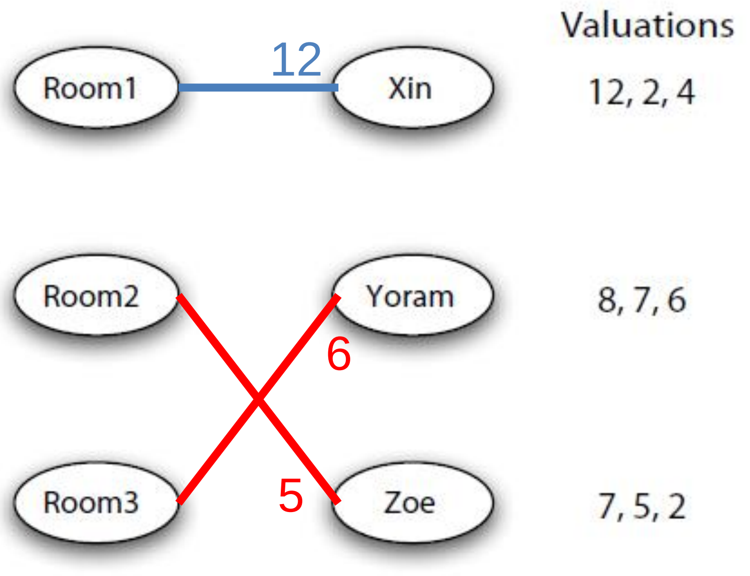
$$p_X^{VCG} = V_{B-X}^{S} - V_{B-X}^{S-j}$$



VCG example

- Start from the optimal allocation
- How much should Xin pay?

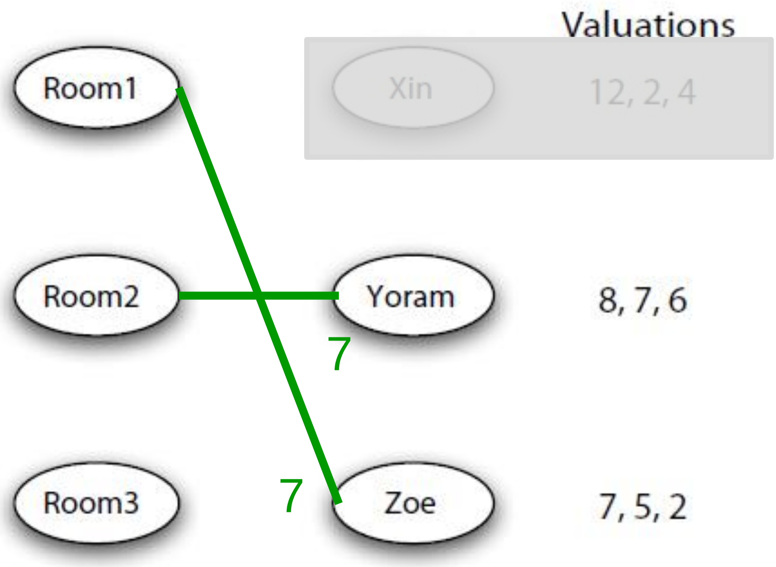
$$p_X^{VCG} = V_{B-x}^{S} - \overset{11}{V_{B-X}^{S-j}}$$



VCG example

- Start from the optimal allocation
- How much should Xin pay?

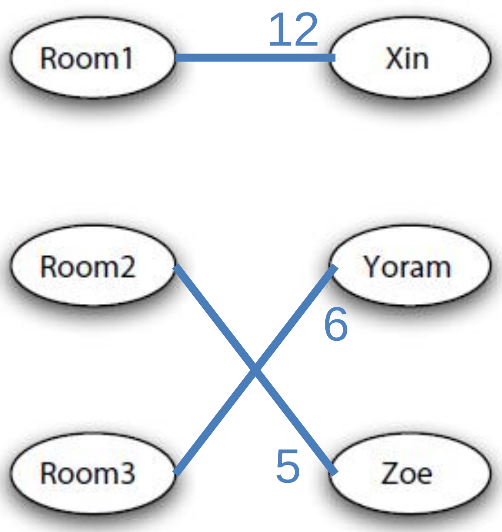
$$p_X^{VCG} = \underbrace{V_{B-X}^{S}}_{14} - \underbrace{V_{B-X}^{S-j}}_{11}$$



VCG example

- Start from the optimal allocation
- How much should Xin pay?

$$p_X^{VCG} = \underbrace{V_{B-X}^{S}}_{}{}^{\!\!14} - V_{B-X}^{S-j}{}^{\!\!11} = 3$$

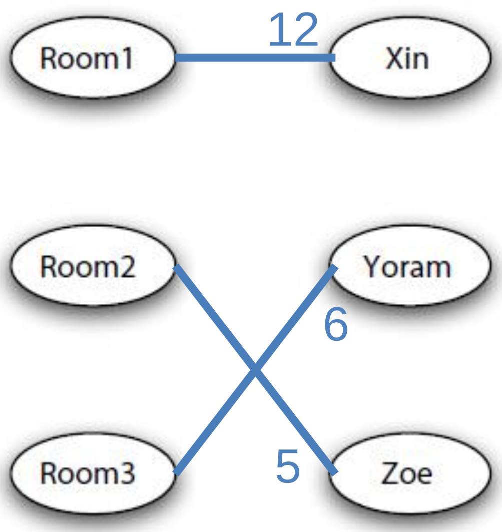


Valuations	prices
12, 2, 4	$p_X = 14 - 11 = 3$
8, 7, 6	
7, 5, 2	

VCG example

- Wait, we got the _______________ prices!

$$p_i^{VCG} = V_{B-i}^{S} - V_{B-i}^{S-j}$$



Valuations	prices
12, 2, 4	$p_X = 14 - 11 = 3$
8, 7, 6	$p_Y = 17 - 17 = 0$
7, 5, 2	$p_Z = 19 - 18 = 1$

The VCG auction

Theorem 3: The VCG auction yields optimal allocation and market clearing prices

Theorem 4: It is a dominant strategy for all bidders to report their true valuations

- VCG can be used in many other contexts
  - We will meet it again in the next classes

Advanced GTO2-week3 VCG

VCG IN GENERAL COMBINATORIAL AUCTIONS

General combinatorial auctions

- A set of items M
- Each bidder has a value function $v_i: 2^M \to \mathbb{R}_+$
  - A value for every subset of items
- A set Σ of all valid allocations

An **auction** gets $(v_1, v_2, \ldots, v_n)$ as input and returns:

1. An allocation $\sigma \in \Sigma$

   ($\sigma(i) \subseteq M$ are the items allocated to i)
2. Payments $p_1, p_2, \ldots, p_n$

Utility for i is $u_i = v_i\big(\sigma(i)\big) - p_i$

Reminder - VCG

- Returns the optimal allocation σ^*
- The VCG principle is that any player should pay the "damage" her actions inflict on other players
- Formally, denote by V_B^S the maximal social welfare of all bidders in B and all items S
- Thus each agent i should pay

$$p_i^{VCG} = V_{B-i}^{S} - V_{B-i}^{S-\sigma^*(i)}$$

Best allocation when i does not exist

Current social welfare, without i.
Note that:
$V_{B-i}^{S-\sigma^*(i)} + v_i\big(\sigma^*(i)\big) = V_B^S$

VCG is truthful

<u>Theorem</u>: it is a dominant strategy for any agent to bid her real values v_i

Value (social welfare) of the actual optimal allocation σ^*

Proof:

Utility for i when truthful:

$$u_i = v_i(\sigma^*(i)) - \left(V_{B-i}^{S} - V_{B-i}^{S-\sigma^*(i)}\right) = V_B^S - V_{B-i}^S$$

$$V_{B-i}^{S-\sigma^*(i)} + v_i(\sigma^*(i)) = V_B^S$$

Utility for i when reporting some v_i' :

$$u_i' = v_i(\sigma'(i)) - \left(V_{B-i}^{S} - V_{B-i}^{S-\sigma'(i)}\right) = V_{B'}^S - V_{B-i}^S$$

Value (social welfare) of the optimal allocation for the wrong values $(v_1, \dots, v_i', \dots v_n)$

Thus $V_B^S \geq V_{B'}^S$

And $u_i \geq u_i'$

$\square$

VCG

- In the general case, we have many agents and many items.
- Computing the optimal allocation is NP-complete!
- Can we approximate the optimal solution?
  - In the general case: not really
  - Plus, approximate mechanisms are not truthful!
- However, in ad auctions…

~~Deferred Acceptance~~

(We stop here)

More types of matchings

How is this related to E-Commerce?

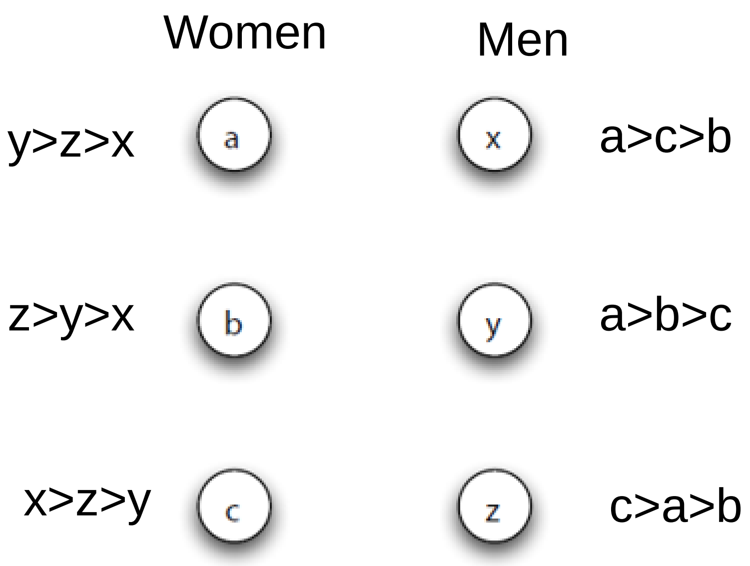
- Both sides have preferences
  - The “stable marriage” problem
  - Many-to-one matchings [Roth and Sotomayor’92]
  - Students to schools
  - “Deferred acceptance”
- On-line matching [Karp et al.’90]
  - Buyers/students arrive one by one
- Graph is not bi-partite
  - Matching roommates
  - kidney exchange [Roth et al.’05]

Not covered

Stable marriage

- Still a perfect matching problem
- No money
- Both sides have (ordinal) preferences

- Motivation:
  - Students and schools
  - Hospitals and residents
  - Israeli Mechina
  - Organ transplantation
  - …

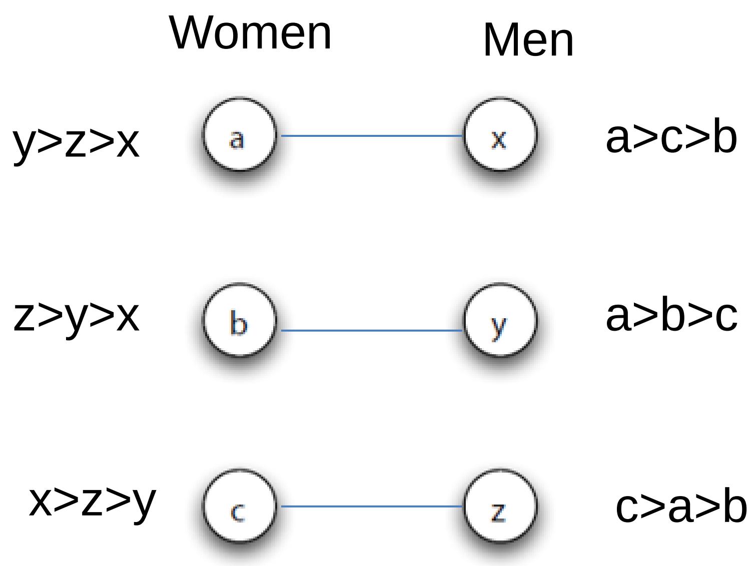


Stable marriage

- Still a perfect matching problem
- No money
- Both sides have (ordinal) preferences

A “blocking pair” is a man and a woman who prefer each other over their current match

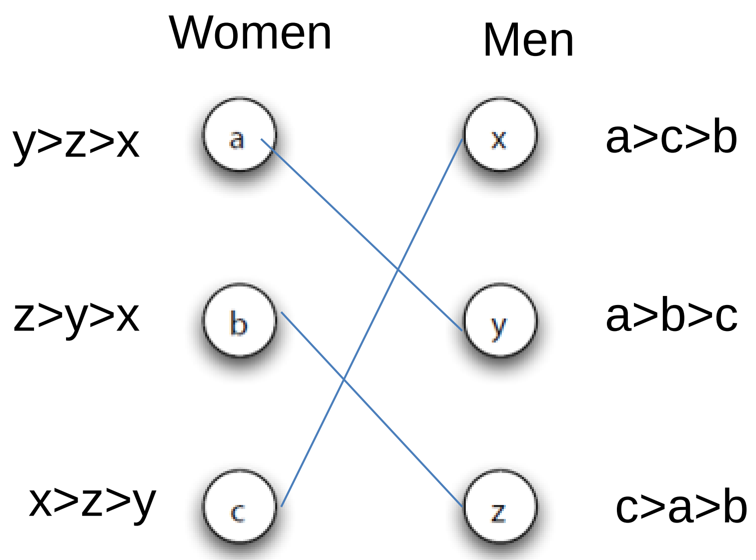
- {a,y} are a blocking pair



Stable marriage

A matching is **stable** if there are no blocking pairs

- A stable match:



Deferred Acceptance

- K schools, N students, $K \leq N$
- Each student i has ranking over schools
- Each school j has ranking over students, and a quota q_j
  - For $q_j = 1$, we get the stable marriage problem!

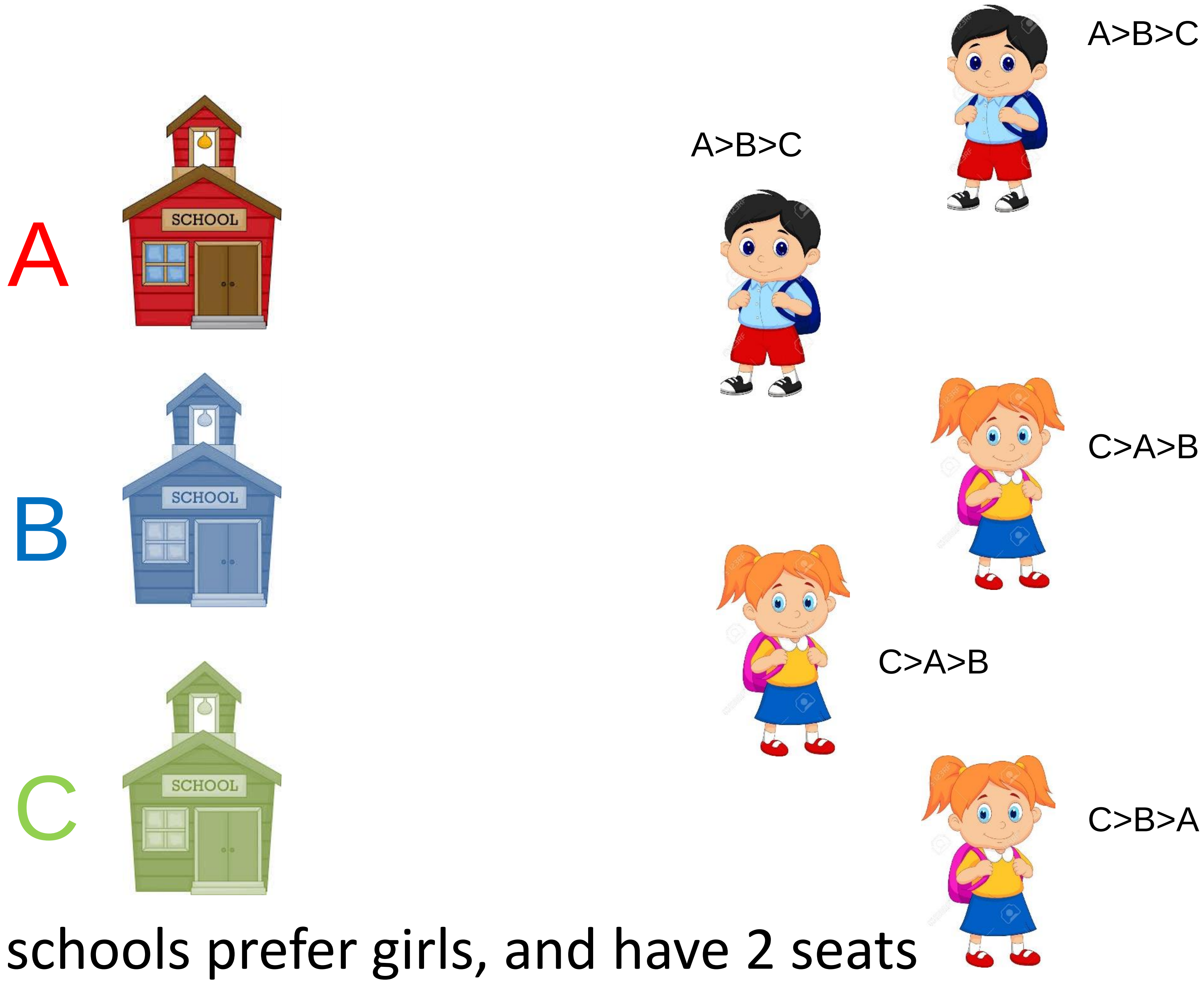
- Boston Matching

A "blocking pair" is a <u>student</u> and a <u>school</u> who prefer each other over their current match

A matching is **stable** if there are no blocking pairs

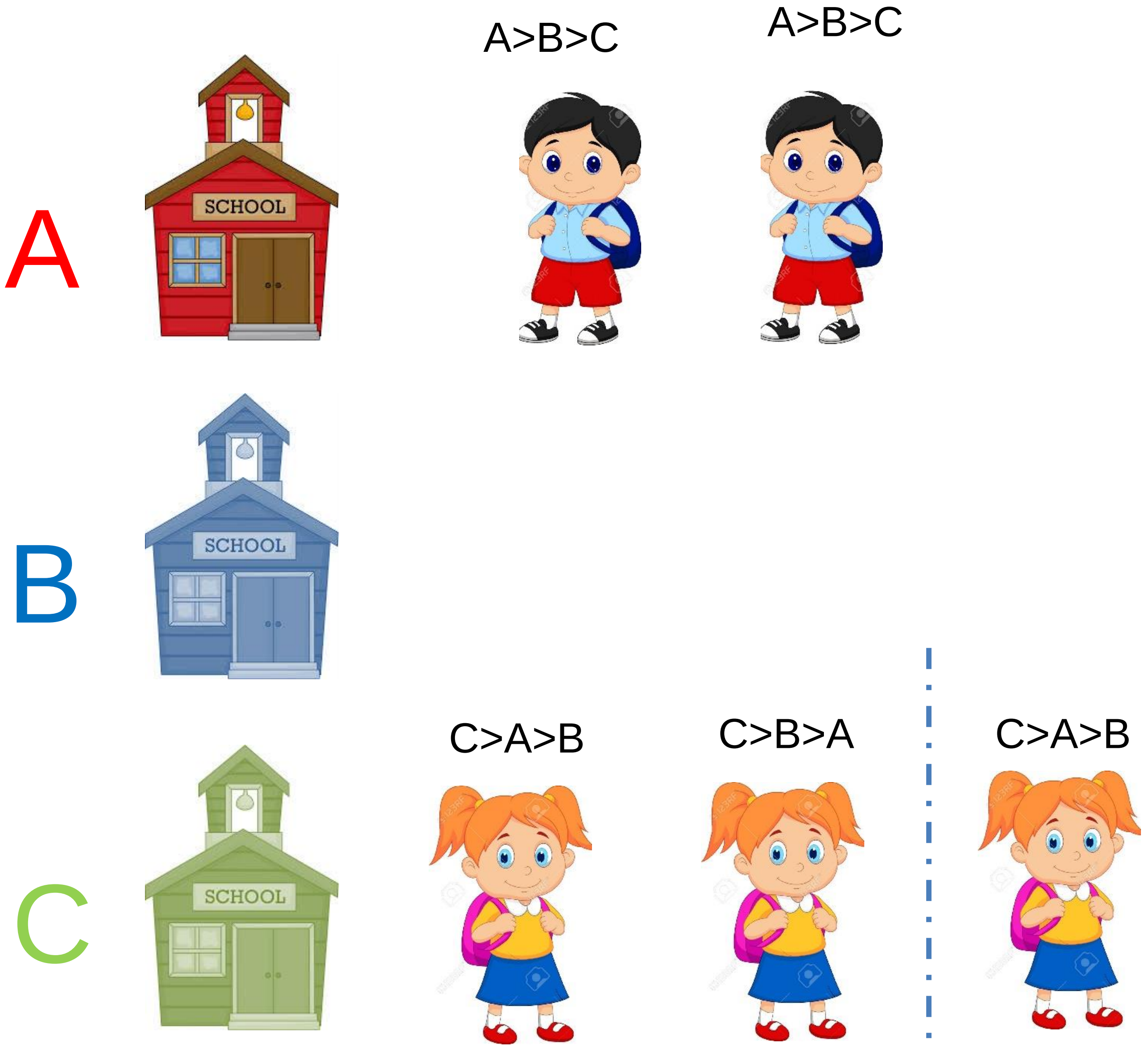
Deferred Acceptance

- The "Deferred Acceptance" mechanism [RS'92]
  - Generalize the Gale-Shapley algorithm
- In each round:
  - Each (rejected) student applies to her next-best school
  - Each school j <u>conditionally accepts</u> best q_j current applicants, <u>permanently rejects</u> all other applicants
- Until no student applies



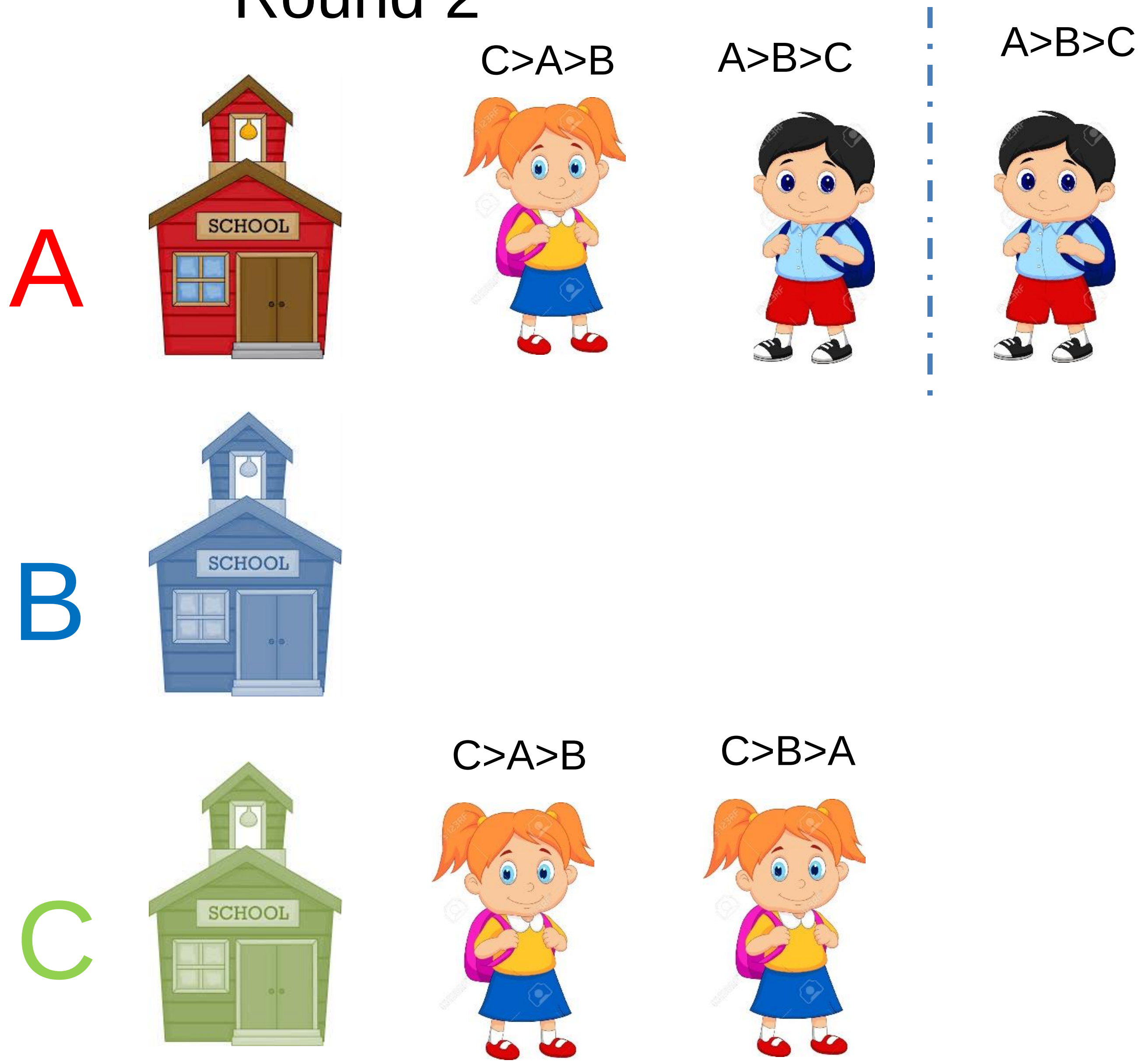
All schools prefer girls, and have 2 seats

Round 1



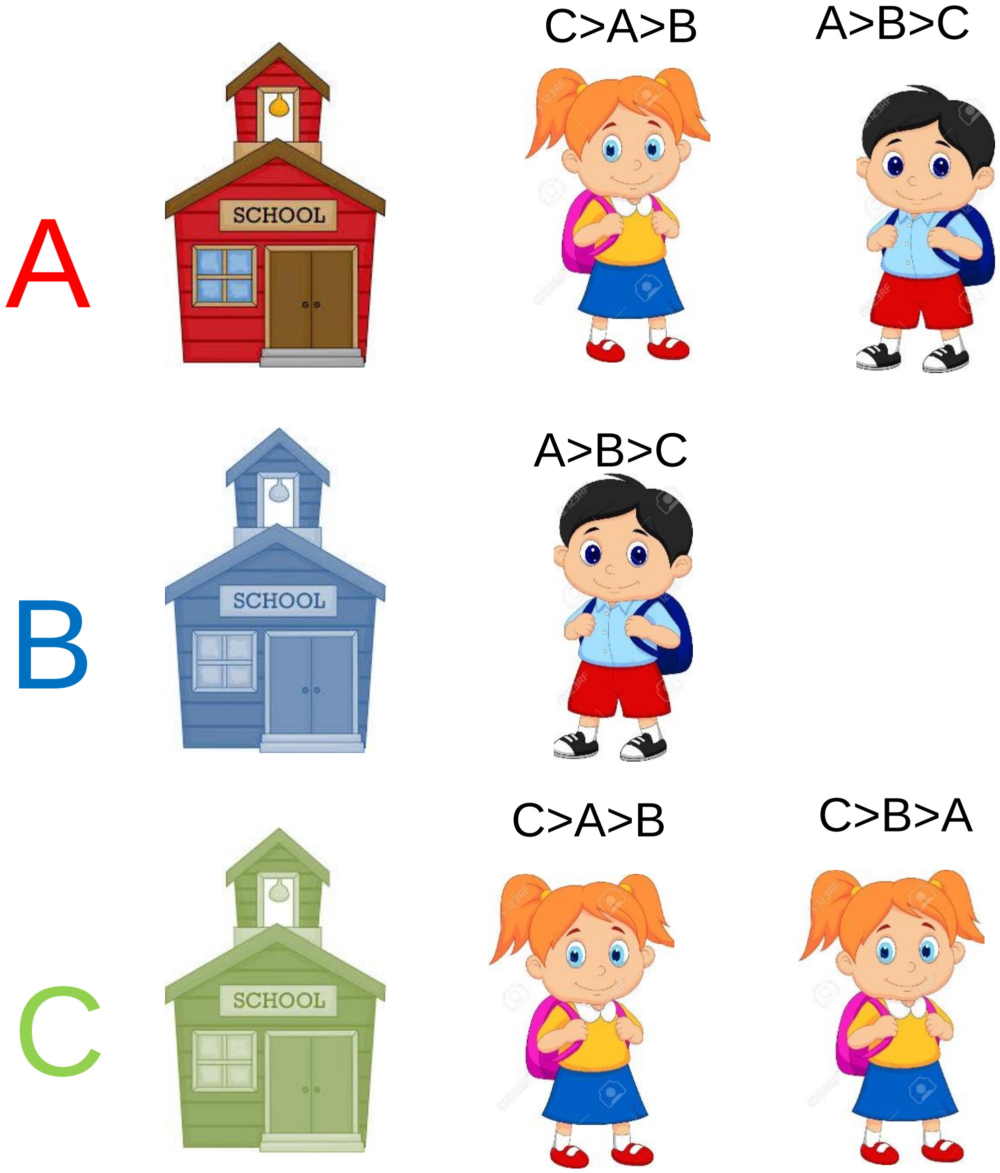
All schools prefer girls, and have 2 seats

Round 2



All schools prefer girls, and have 2 seats

Round 3



All schools prefer girls, and have 2 seats

Deferred Acceptance

- **Theorem:** The DA results in a stable match
  - No pair of (school, student) that prefer each other over their current match
  - Markets without stable matching mechanisms tend to unravel [Roth'08]
- **Theorem:** The DA mechanism is optimal for the students (up to tie breaking)

- **Theorem:** It is a dominant strategy for the students to follow the DA protocol

- What about the schools? (exercise)

References

- Roth, Alvin E., and Marilda A. Oliveira Sotomayor. *Two-sided matching: A study in game-theoretic modeling and analysis*. No. 18. Cambridge University Press, 1992.
- Derigs, Ulrich. "Solving non-bipartite matching problems via shortest path techniques." *Annals of Operations Research* 13.1 (1988): 225-261.
- Karp, Richard M., Umesh V. Vazirani, and Vijay V. Vazirani. "An optimal algorithm for on-line bipartite matching." *Proceedings of the twenty-second annual ACM symposium on Theory of computing*. ACM, 1990.
- Azar, Yossi, Joseph Naor, and Raphael Rom. "The competitiveness of on-line assignments." *Journal of Algorithms* 18.2 (1995): 221-237.
- Roth, Alvin E., Tayfun Sönmez, and M. Utku Ünver. "Pairwise kidney exchange." *Journal of Economic theory* 125.2 (2005): 151-188.
- Roth, Alvin E. "What have we learned from market design?." The Economic Journal 118.527 (2008): 285-310.

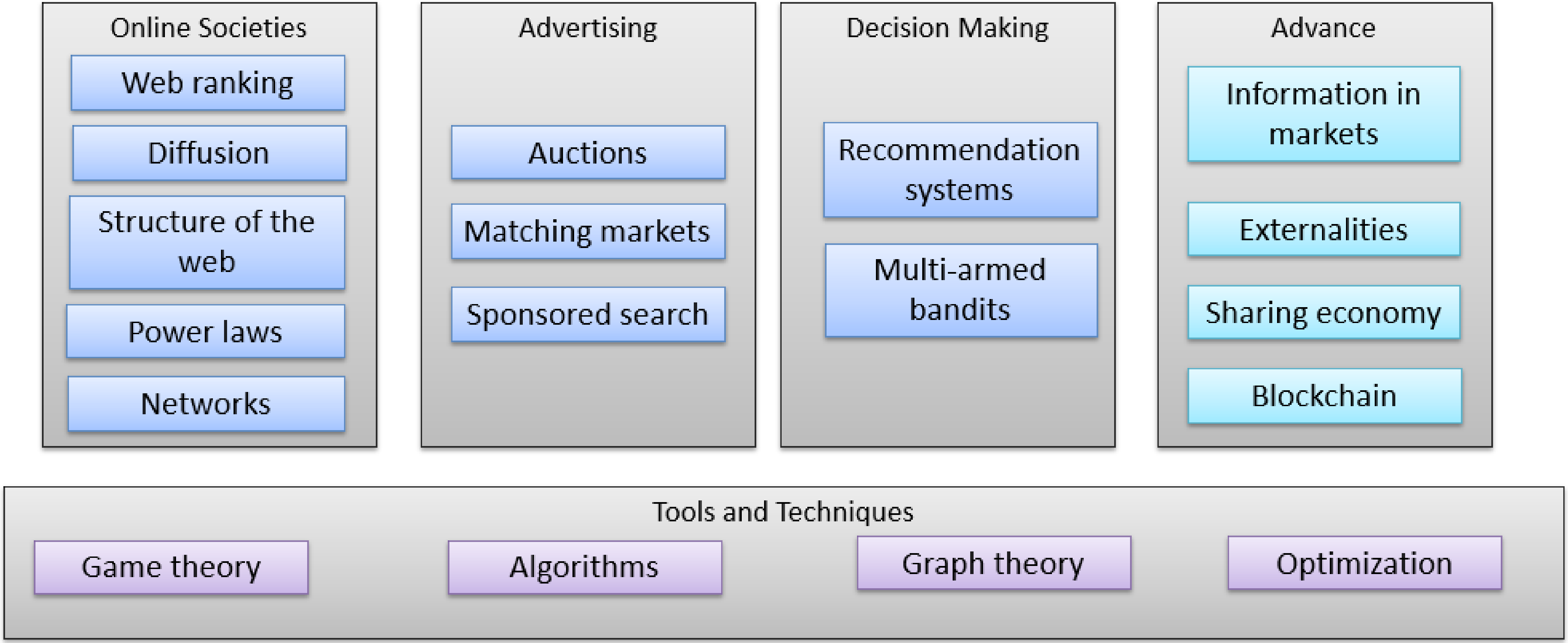
Electronic Commerce
096211

Ad auctions and Sponsored Search

Omer Ben Porat (some of the slides are adopted from Prof. Reshef Meir)

Updates

- HW 4 (yes)
- Tutorial 13 (Wed + video)



Online Societies
Web ranking
Diffusion
Structure of the web
Power laws
Networks
Advertising
Auctions
Matching markets
Sponsored search
Decision Making
Recommendation systems
Multi-armed bandits
Advance
Information in markets
Externalities
Sharing economy
Blockchain
Tools and Techniques
Game theory
Algorithms
Graph theory
Optimization

Outline

- VCG in general combinatorial auctions
- Ad auctions – motivation and model
- VCG for ad auctions
- GSP for ad auctions
- GSP “nice” equilibrium analysis
- Ad quality
- Practical issues

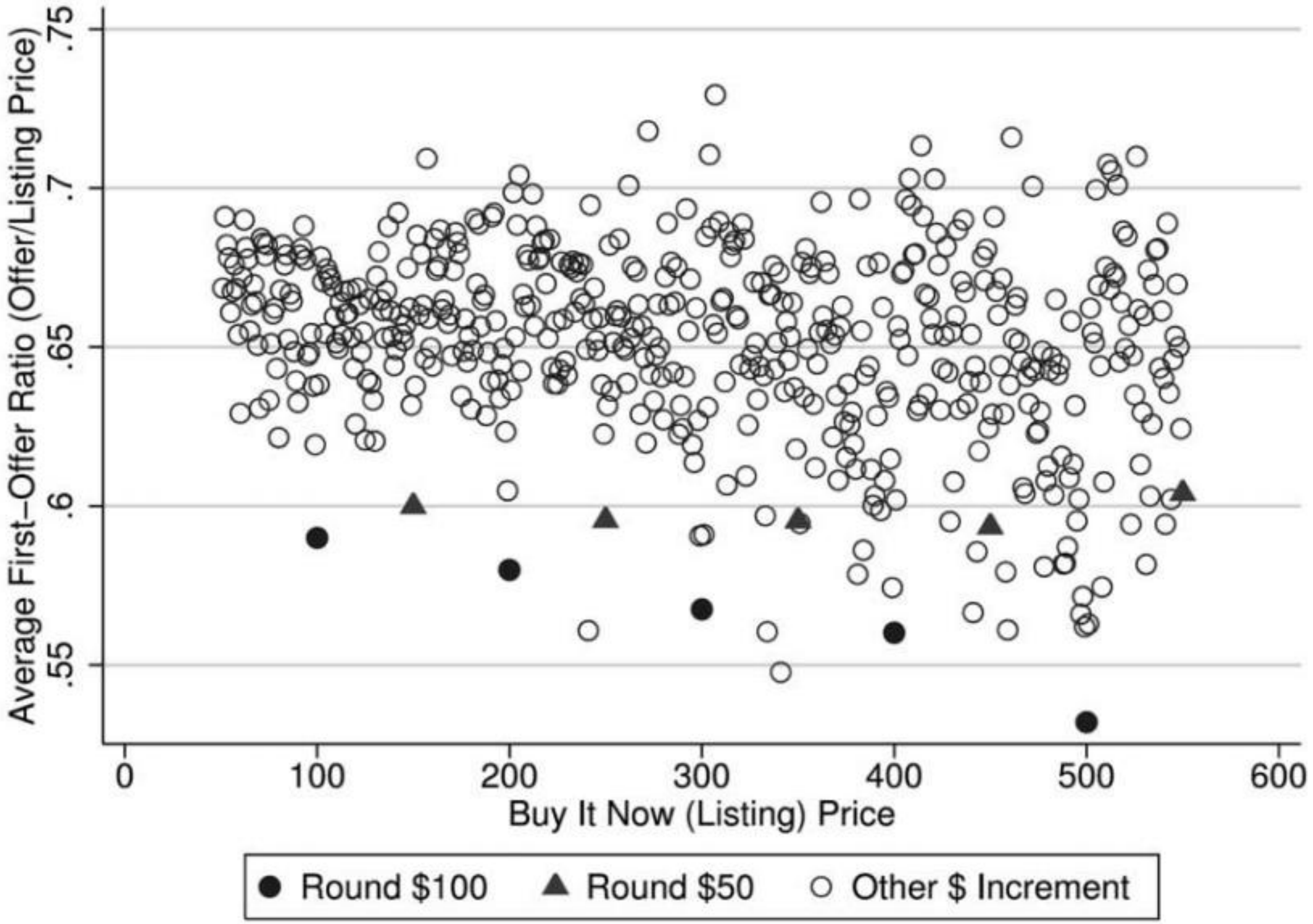


FIG. 2.—Average first offers by BIN price. This scatter plot presents average first offers, normalized by the BIN price to be between zero and one, grouped by unit intervals of the BIN price, defined by $(z-1,\ z]$. When the BIN price is on an interval rounded to a number ending in 00, it is represented by a circle; 50 numbers are represented by a triangle.

Source: https://faculty.haas.berkeley.edu/stadelis/round.pdf

VCG IN GENERAL COMBINATORIAL AUCTIONS

General combinatorial auctions

- A set of items M
- Each bidder has a value function $v_i: 2^M \to \mathbb{R}_+$
  - A value for every subset of items
- A set Σ of all valid allocations

An **auction** gets $(v_1, v_2, \dots, v_n)$ as input and returns:

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2. Payments $p_1, p_2, \dots, p_n$

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Reminder - VCG

- Returns the optimal allocation σ^*
- The VCG principle is that any player should pay the "damage" her actions inflict on other players
- Formally, denote by V_B^S the maximal social welfare of all bidders in B and all items S
- Thus each agent i should pay

$$p_i^{VCG} = V_{B-i}^{S} - V_{B-i}^{S-\sigma^*(i)}$$

Best allocation when i does not exist

Current social welfare, without i.
Note that:
$V_{B-i}^{S-\sigma^*(i)} + v_i\big(\sigma^*(i)\big) = V_B^S$

VCG is truthful

<u>Theorem</u>: it is a dominant strategy for any agent to bid her real values v_i

Value (social welfare) of the actual optimal allocation σ^*

Proof:

Utility for i when truthful:

$$u_i = \; v_i(\sigma^*(i)) - \left(V_{B-i}^{S} - V_{B-i}^{S-\sigma^*(i)}\right) = \boxed{V_B^S} - \boxed{V_{B-i}^S}$$

Utility for i when reporting some v_i' :

$$V_{B-i}^{S-\sigma^*(i)} + v_i(\sigma^*(i)) = V_B^S$$

$$u_i' = v_i(\sigma'(i)) - \left(V_{B-i}^{S} - V_{B-i}^{S-\sigma\prime(i)}\right) = \boxed{V_{B\prime}^S} - \boxed{V_{B-i}^S}$$

Thus $V_B^S \geq V_{B\prime}^S$

And $u_i \geq u_i'$

Value (social welfare) of the optimal allocation for the wrong values $(v_1, \dots, v_i', \dots v_n)$

$\square$

VCG

- In the general case, we have many agents and many items.
- Computing the optimal allocation is NP-complete!
- Can we approximate the optimal solution?
  - In the general case: not really
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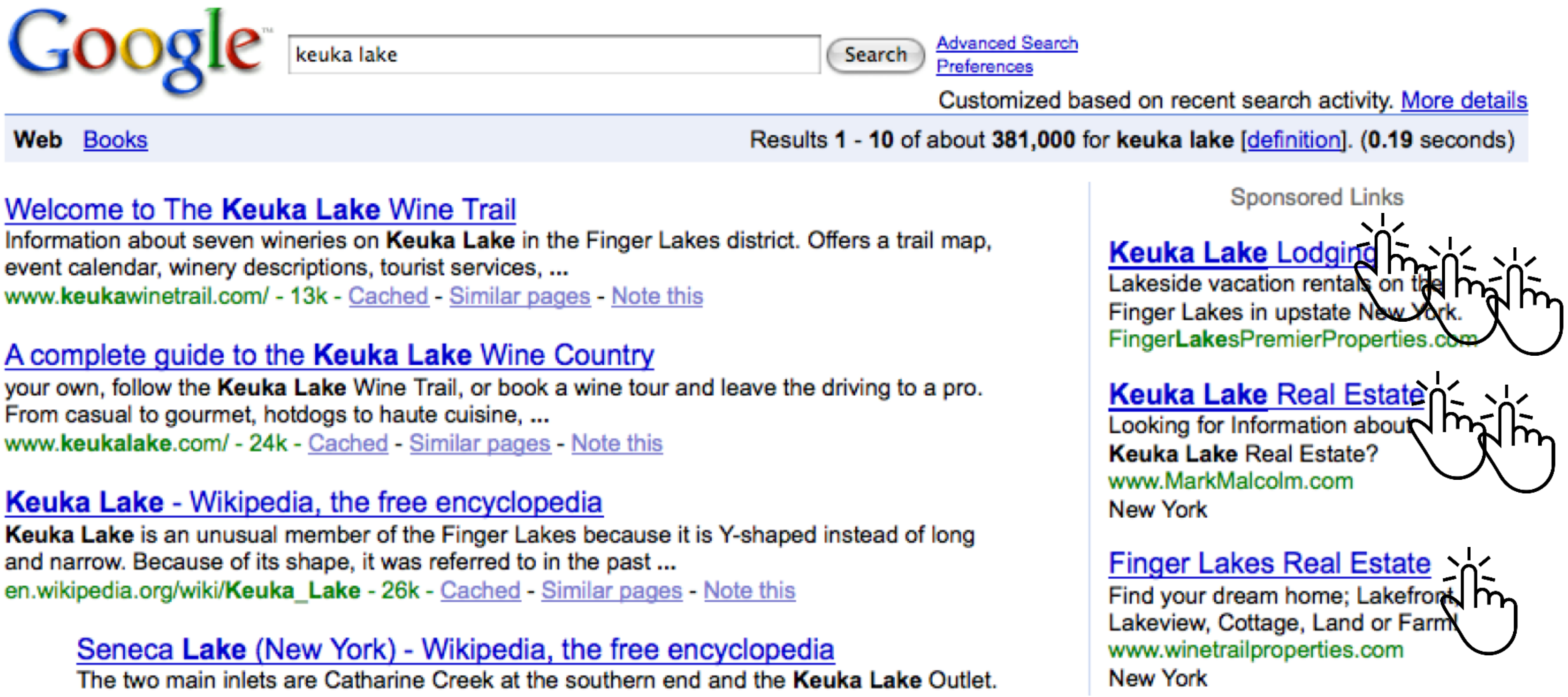
- However, in ad auctions…

AD AUCTIONS – MOTIVATION


Google
keuka lake
Search
Advanced Search
Preferences
Customized based on recent search activity. More details
Web Books
Results 1 - 10 of about 381,000 for keuka lake [definition]. (0.19 seconds)
Welcome to The Keuka Lake Wine Trail
Information about seven wineries on Keuka Lake in the Finger Lakes district. Offers a trail map, event calendar, winery descriptions, tourist services, ...
www.keukawinetrail.com/ - 13k - Cached - Similar pages - Note this
A complete guide to the Keuka Lake Wine Country
your own, follow the Keuka Lake Wine Trail, or book a wine tour and leave the driving to a pro. From casual to gourmet, hotdogs to haute cuisine, ...
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Keuka Lake - Wikipedia, the free encyclopedia
Keuka Lake is an unusual member of the Finger Lakes because it is Y-shaped instead of long and narrow. Because of its shape, it was referred to in the past ...
en.wikipedia.org/wiki/Keuka_Lake - 26k - Cached - Similar pages - Note this
Seneca Lake (New York) - Wikipedia, the free encyclopedia
The two main inlets are Catharine Creek at the southern end and the Keuka Lake Outlet.
Sponsored Links
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Lakeside vacation rentals on the Finger Lakes in upstate New York.
FingerLakesPremierProperties.com
Keuka Lake Real Estate
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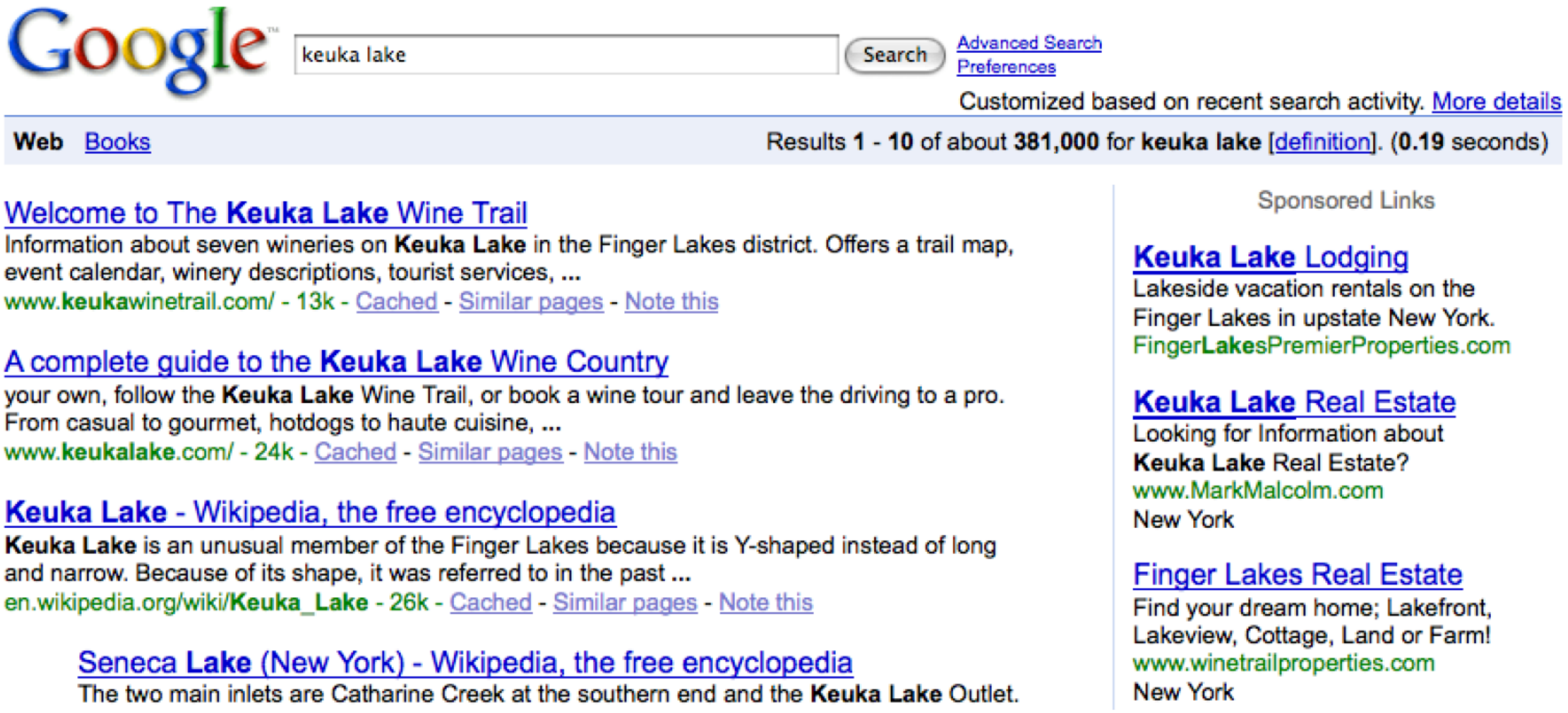
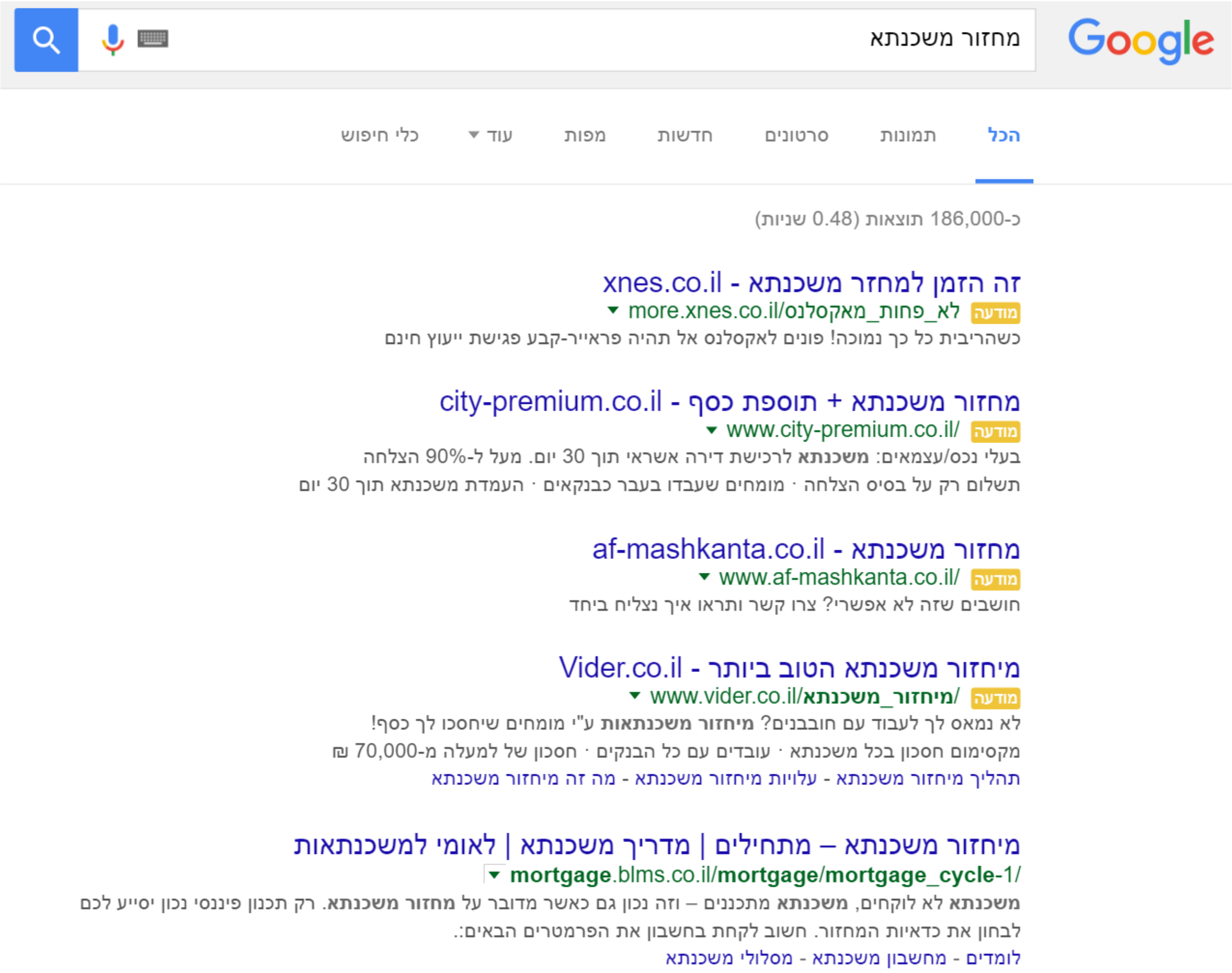
About 62,000 results (0.68 seconds)



Conversion: 1% X 200$ = 2$

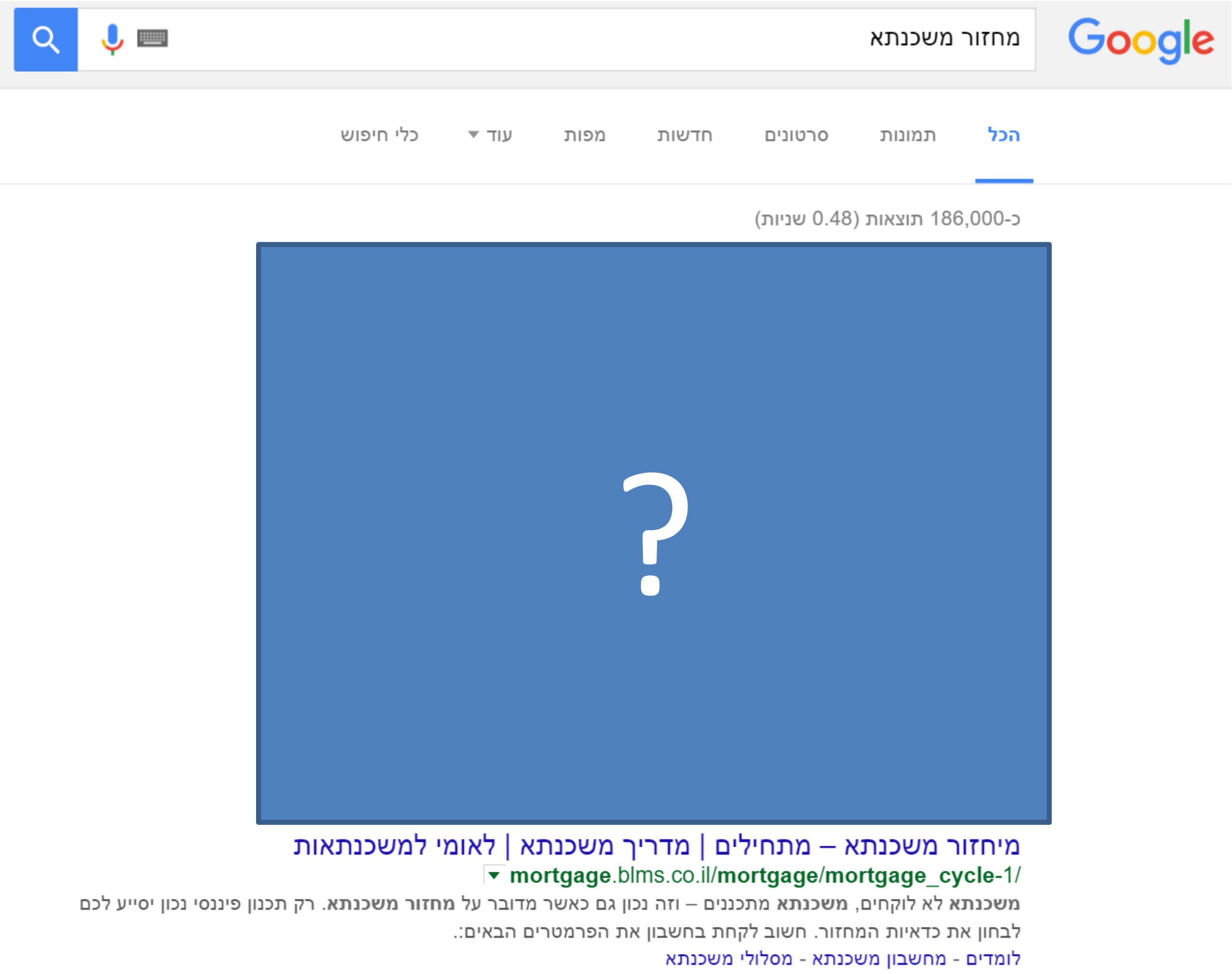
What is the worth of a click?

- Many advertisers
- Value can be a few cents, and up to 50$
- How to price ads?
- Use an auction!



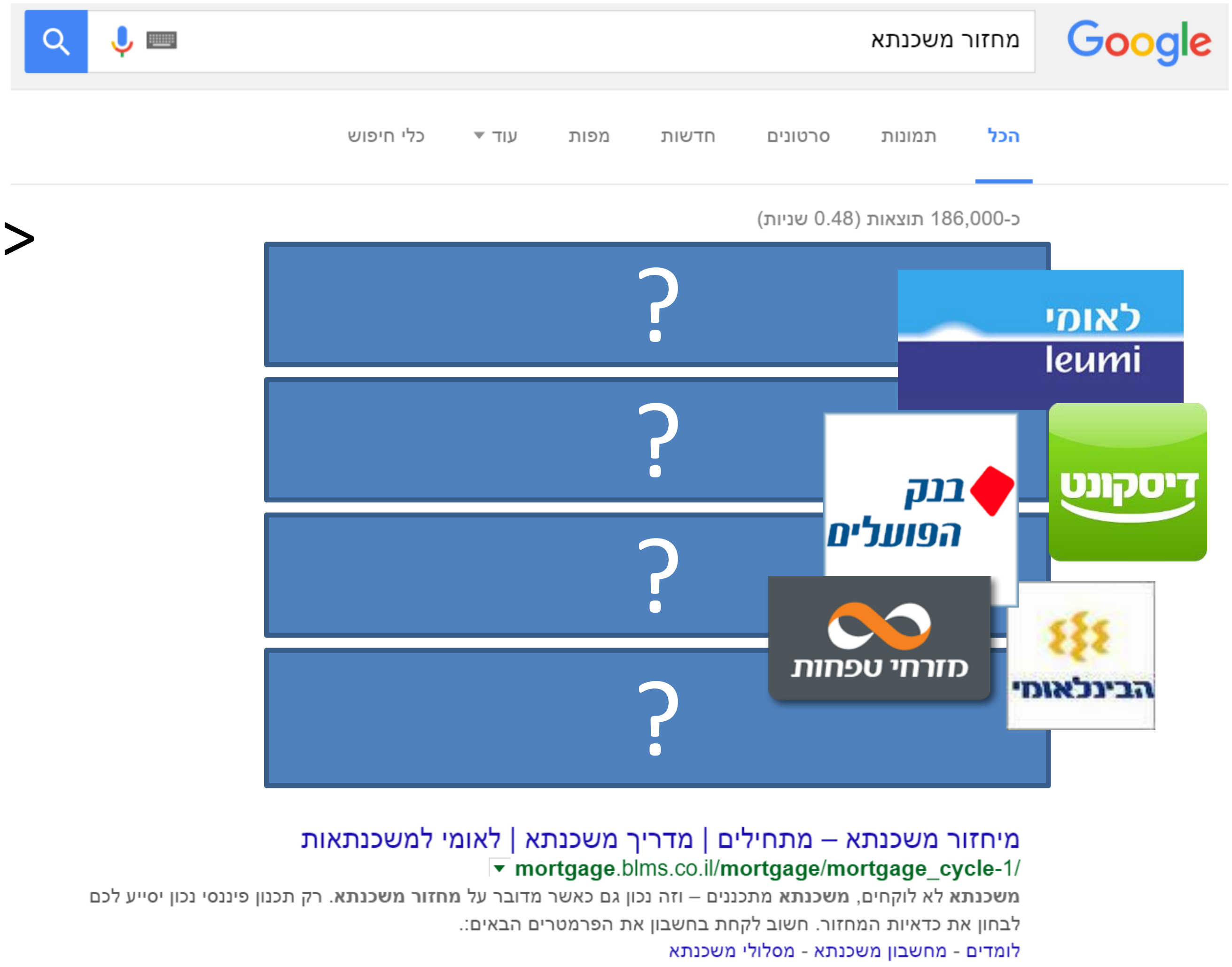
Auctioning ad slots

- Suppose we have a single slot to sell
- A good solution: Second rice auction



Auctioning ad slots

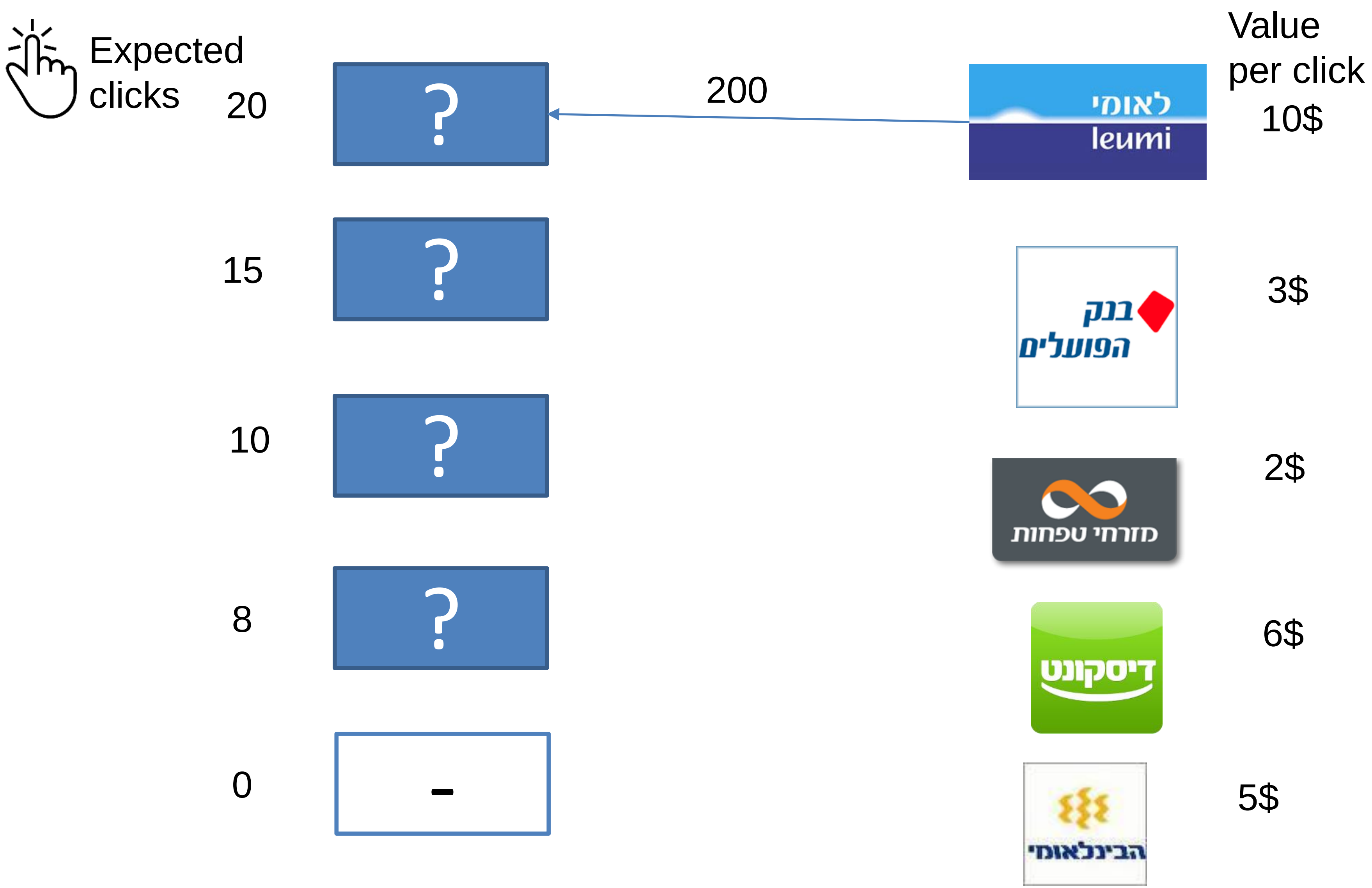
- Want to sell several slots
- Higher slots attract more clicks -> worth more money
- How to decide who gets what?



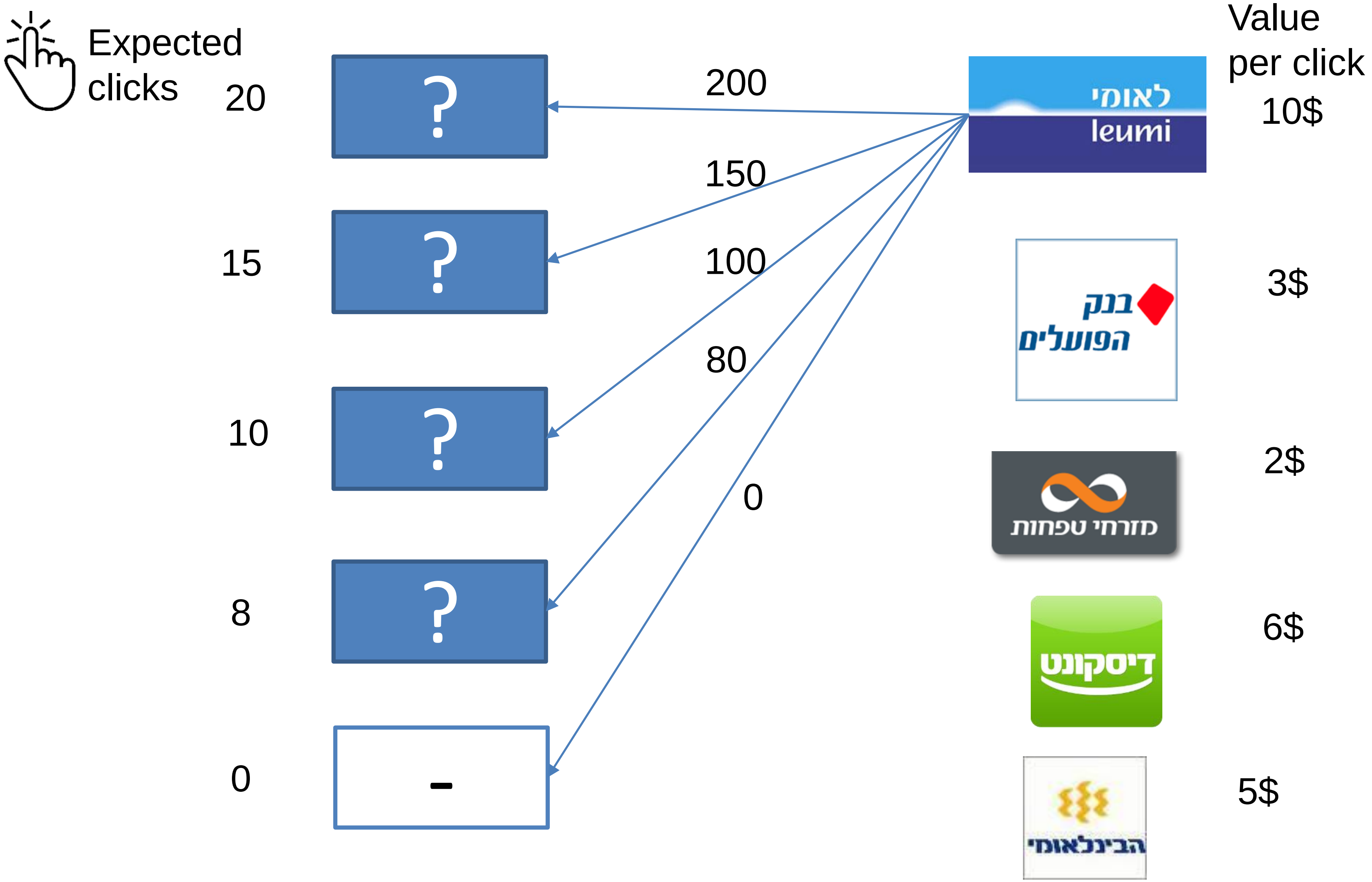
- Suppose all values are known

Expected clicks			Value per click
20	?	לאומי leumi	10$
15	?	בנק הפועלים	3$
10	?	מזרחי טפחות	2$
8	?	דיסקונט	6$
0	-	הבינלאומי	5$

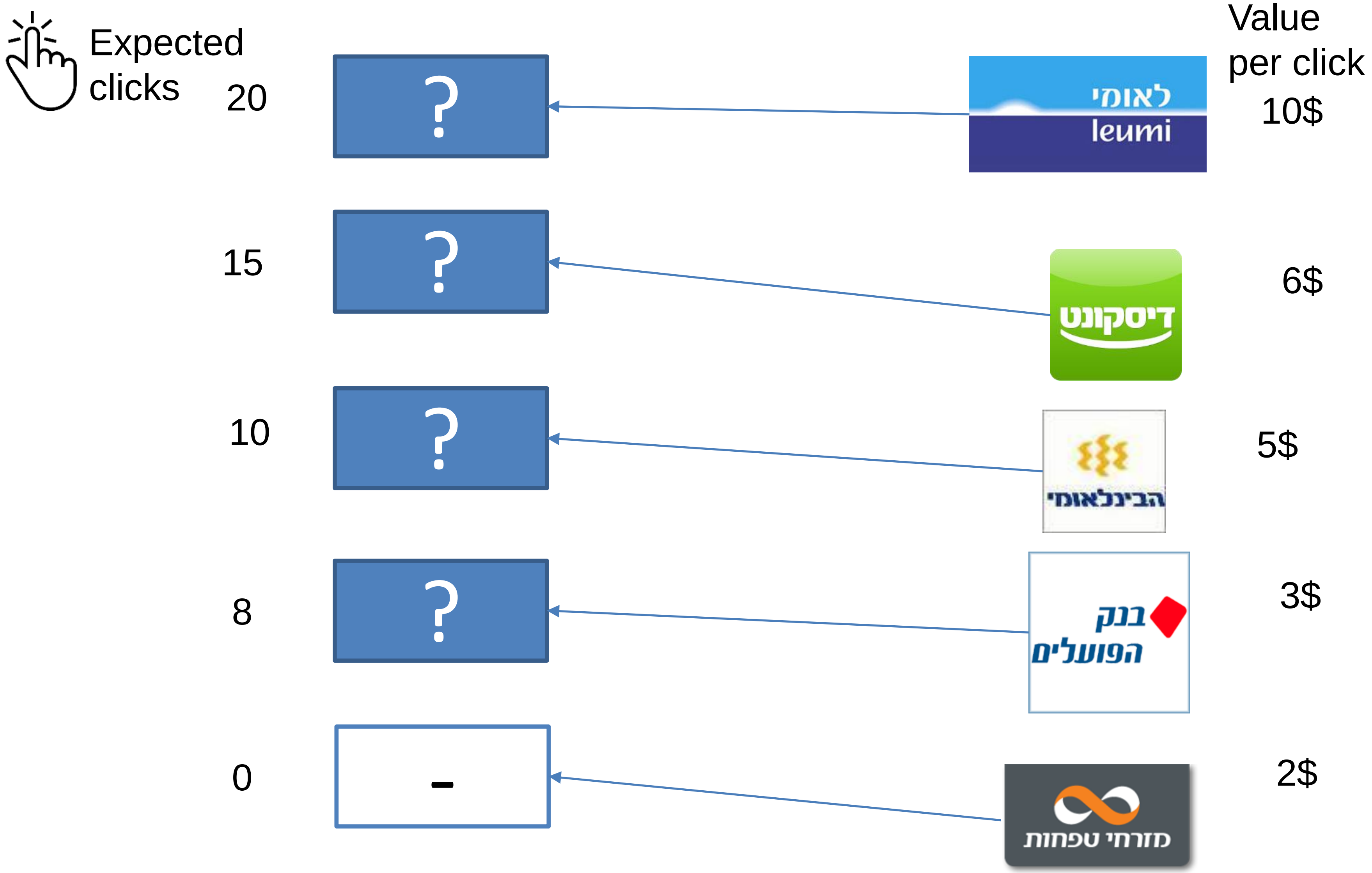
- Suppose all values are known



- Suppose all values are known
- This is a matching problem!
- How to maximize social welfare?



- Suppose all values are known
- This is a matching problem!
- How to maximize social welfare? Match highest value to best slot



- Problem: advertisers’ values are unknown.
- We need an auction for multiple slots
  - Want maximal social welfare + high revenue

Expected clicks	Slot		Value per click
20	?	leumi	?$
15	?		?$
10	?		?$
8	?		?$
0	-		?$

Formal model

- Each advertiser i has a **value per click** v_i
- Every slot has a click through rate (**CTR**) x_j
- Suppose i gets slot j for price-per-click p
  - Then i gets utility of $u_{ij} = x_j(v_i - p\,)$
- We (the platform) do not know $v_1 \ldots v_n$
- A **mechanism** collects bids, and returns an assignment + prices

Position Auctions

- There are at least 2 ways to generalize the Second Price auction for multiple slots:
- The VCG mechanism
  - Truthful (like second price)
  - Pricing more complicated
- The GSP mechanism
  - Very simple
  - Non-truthful
  - Need to study equilibrium

VCG FOR AD AUCTIONS

Reminder - VCG

- Vickrey-Clarke-Groves
- The VCG principle is that any player should pay the “damage” her actions inflict on other players
- Formally, denote by V_B^S the maximal social welfare of all bidders in B and all slots S
- Thus each agent i getting slot j should pay

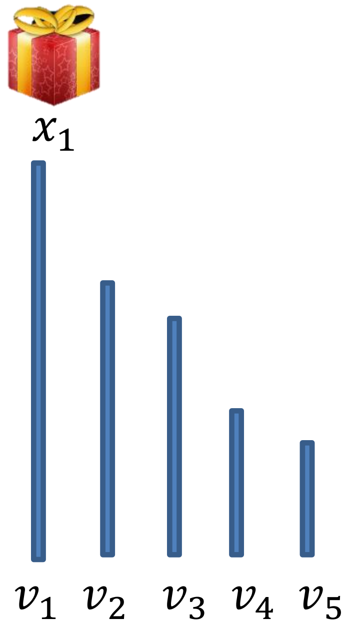
$$p_i^{VCG} = V_{B-i}^{S} - V_{B-i}^{S-j}$$

Best matching when i does not exist

Current social welfare, without i: $V_B^S - u_i$

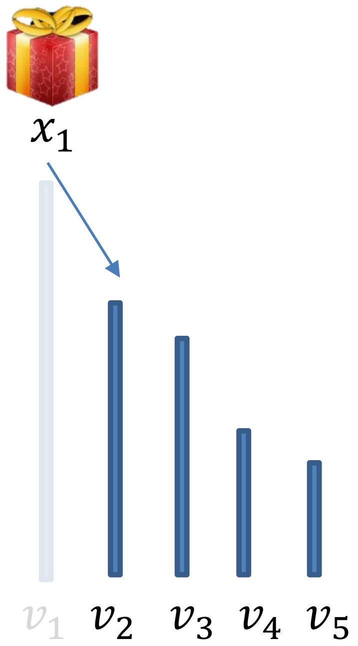
VCG for ad auctions

- Example: Second Price auction (one item)
- How much should player 1 (the winner) pay?



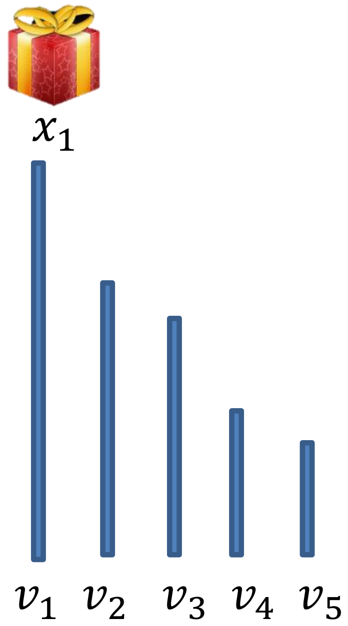
VCG for auctions

- 1 slot
- How much should **player 1** (the winner) pay?
- Welfare of all except player 1 is: $V_{B-1}^{S-1} = 0$
- Welfare if player 1 did not exist: $V_{B-1}^{S} = v_2 x_1$
- Player 1 should pay $v_2 x_1 - 0 = v_2 x_1$
- I.e., $p_1 = v_2 = v_2 \frac{x_1 - x_2}{x_1}$ per click



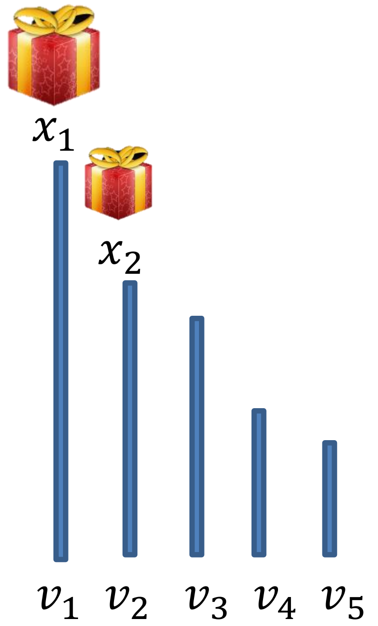
VCG for auctions

- 1 slot
- How much should **player $i > 1$** (loser) pay?
- Welfare of all except player i is: $v_1 x_1$
- Welfare of all, if player i did not exist: $v_1 x_1$
- Player i should pay $v_1 x_1 - v_1 x_1 = 0$



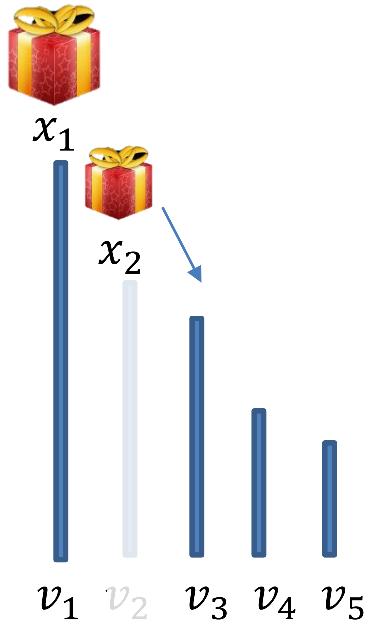
VCG for auctions

- 2 slots (Sort players by reported values, Give best slot to first player, etc.)
- How much should **player $i > 2$** pay?
- Welfare of all except player i is: $v_1 x_1 + v_2 x_2$
- Welfare of all, if player i did not exist: $v_1 x_1 + v_2 x_2$
- Player i should pay $v_1 x_1 + v_2 x_2 - (v_1 x_1 + v_2 x_2) = 0$



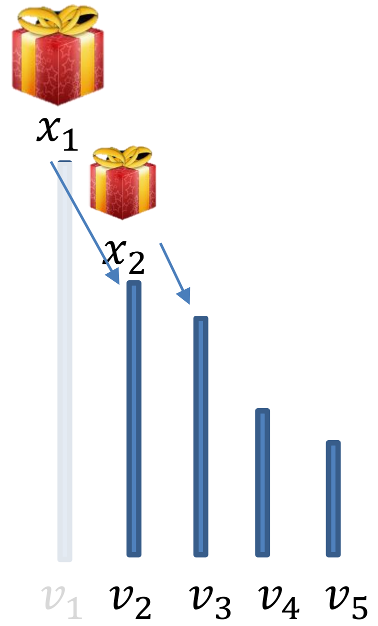
VCG for auctions

- 2 slots
- How much should **player 2** pay?
- Welfare of all except player 2 is: $v_1 x_1$
- Welfare of all, if player 2 did not exist: $v_1 x_1 + v_3 x_2$
- Player 2 should pay $v_1 x_1 + v_3 x_2$ - $v_1 x_1 = v_3 x_2$
- i.e., $p_2 = v_3 \frac{x_2 - x_3}{x_2}$ per click



VCG for auctions

- 2 slots
- How much should **player 1** pay?
- Welfare of all except player 1 is: $v_2 x_2$
- Welfare of all, if player 1 did not exist: $v_2 x_1 + v_3 x_2$
- Player 1 should pay:
- $v_2 x_1 + v_3 x_2$ - $v_2 x_2 = v_2(x_1 - x_2) + v_3(x_2 - x_3)$
- i.e., $p_1 = v_2 \frac{x_1 - x_2}{x_1} + v_3 \frac{x_2 - x_3}{x_1}$ per click



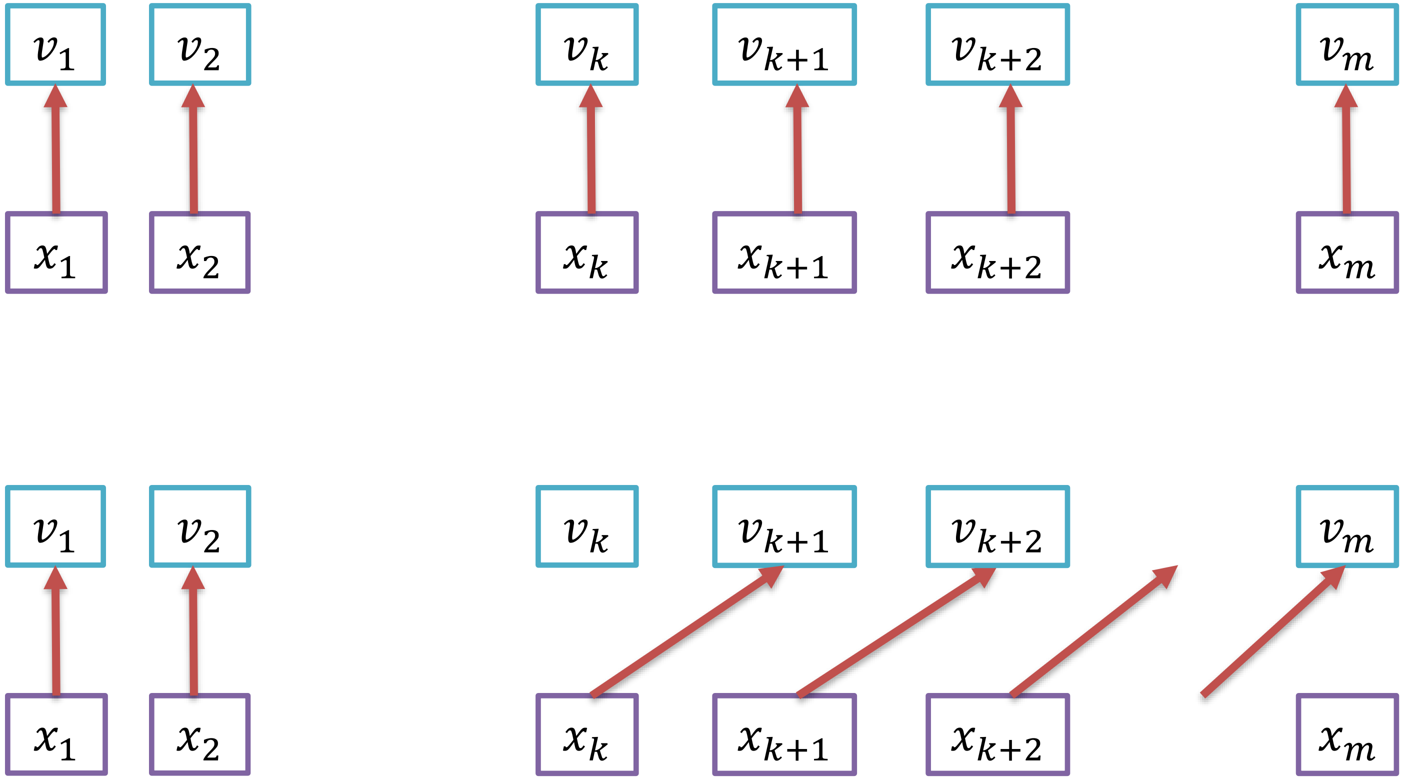
Question!

Expected clicks	
100	Slot 1
50	Slot 2

	Value per click
לאומי leumi	4
בנק הפועלים	3
מזרחי טפחות	0

What are the VCG prices?
What is the utility of each bidder?
What is the revenue for the platform?

Illustration



$$p_k = \frac{1}{x_k}\left(V_{B-k}^{S} - V_{B-k}^{S-k}\right)$$
$$= \frac{1}{x_k}\sum_{i=k}^{m} v_{i+1}(x_i - x_{i+1})$$

VCG for Positional Auctions

- Sort players by (reported) values
- Give best slot to first player, etc. (player k gets slot k)
- Player k should pay:

$$V_{B-k}^{S} - V_{B-k}^{S-k} = v_{k+1}(x_k - x_{k+1}) + v_{k+2}(x_{k+1} - x_{k+2}) + \cdots + v_{m+1}(x_m - x_{m+1})$$

- Cost per click is:

$$p_k = \frac{1}{x_k} \sum_{i=k \dots m} v_{i+1}\,(x_i - x_{i+1})$$

- Utility of k is:

$$u_k = x_k(v_k - p_k) = x_k v_k - \sum_{i=k \dots m} v_{i+1}(x_i - x_{i+1})$$

VCG summary

- VCG is truthful – all agents bid their true value
- Corollary: the assignment maximizes social welfare
- How about platform's revenue?

$$R^{VCG} = \sum_{k=1}^{m} x_k p_k = \sum_{k=1}^{m} \sum_{i=k}^{m} v_{i+1}(x_i - x_{i+1})$$

$$= \sum_{k=1}^{m} k \cdot v_{k+1}(x_k - x_{k+1})$$

“Generalized Second Price”

GSP FOR AD AUCTIONS

GSP

- GSP for m slots
  - Sort players by (reported) values b_i
  - Give best slot to first player, etc.
  - Thus player k gets slot k
  - Each player k pays $p_k = b_{k+1}$ per click
    - Next-highest bid
  - Utility of k is $u_k = x_k(v_k - b_{k+1})$

Real value

Bid (reported value)

GSP analysis

- For 1 slot, GSP is equivalent to second price auction
  - Thus truthful for $m = 1$
- Suppose there are 2 slots with $x_1 = 8, x_2 = 7$
- There are 3 agents with $v_1 = 5\,, v_2 = 4\,, v_3 = 1$
- If they are truthful, $u_1 = 8(5 - 4) = 8$
- If agent 1 reports $v_1' = 3$, gets slot 2, and:
  - $u_1' = x_2(v_1 - b_3) = 7(5 - 1) = 28 > u_1$
- Not truthful for $m > 1$ slots!

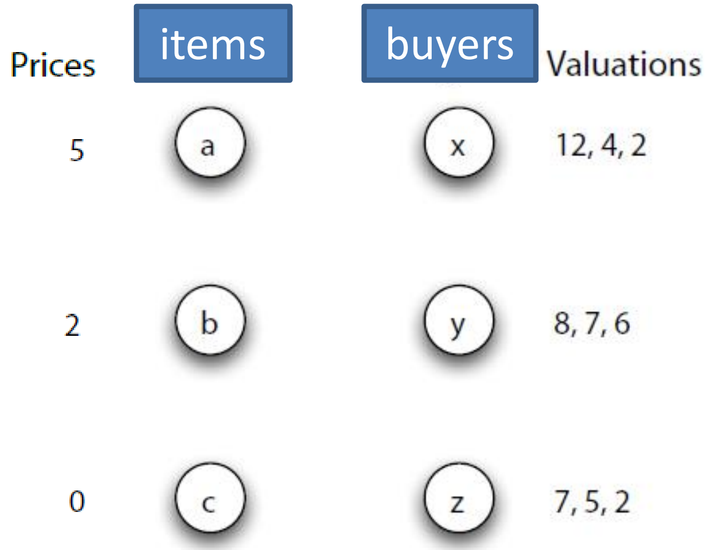
GSP analysis

- How should/would advertisers bid?
- GSP is a game!
  - Can analyze equilibria
- Problem: players do not know others' valuations, thus don't know the game
  - Like in 1st price auction
    - Equilibrium analysis was very complicated!
- We will assume complete information
  - (but the platform doesn't know the values)

GSP equilibria

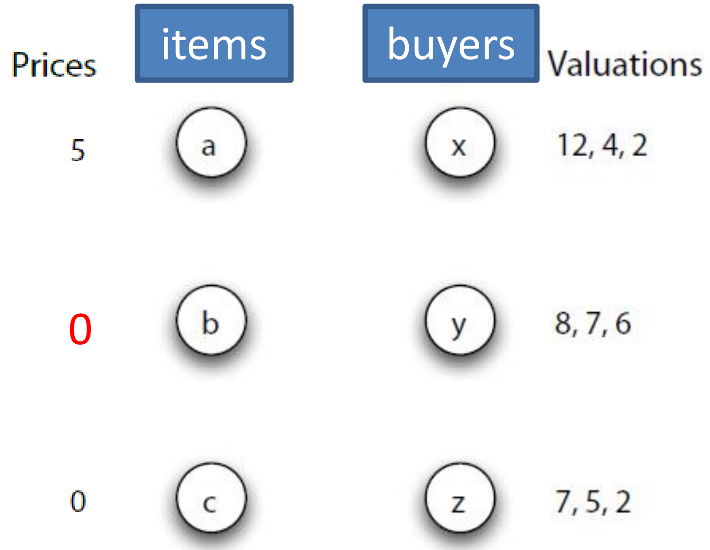
- GSP may have many (even pure) Nash eqs
- Some may be inefficient
  - A player with a high value bids low, gets bad slot
  - Social welfare is not optimal
  - Revenue may be lower or higher than in VCG
- But there is at least one “nice” pure Nash equilibrium
  - Order of bids = order of values (→ efficient)
  - Payments have a simple structure
  - Revenue always at least as in VCG

Recap: Market Clearing Prices



At these prices:
Buyer x prefers item a
Buyer y prefers item c
Buyer z prefers item b

Prices p are market clearing



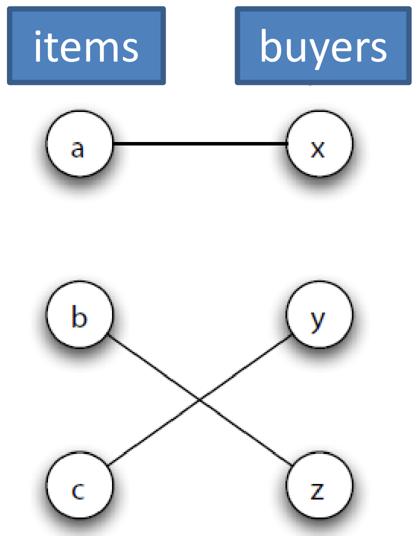
At these prices:
Buyer x prefers item a
Buyer y prefers item b
Buyer z prefers item b

{b} is called a **constricted set**

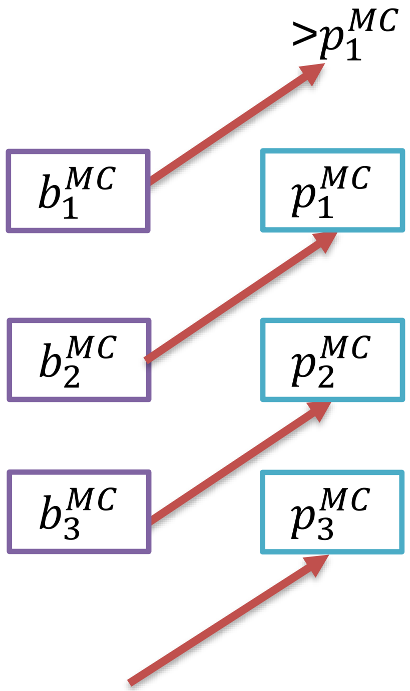
GSP "nice" equilibrium

- Suppose we had all the values $\boldsymbol{v} = v_1, \dots, v_n$
- We could compute market clearing prices $p_1^{MC}, \dots, p_m^{MC}$
- What would be agents' actions?
  - $b_i^{MC} = p_{i-1}^{MC}$ for any $i > 1$ (equiv., $p_i^{MC} = b_{i+1}^{MC}$)
  - b_1^{MC} can be anything $> b_2^{MC}$
- Recall that at MC prices, each bidder prefers her assigned slot:

$$\forall i, j, x_i(v_i - p_i) \geq x_j(v_i - p_j)$$



Illustration



$$b_i^{MC} = p_{i-1}^{MC}$$

$$p_i^{MC} = b_{i+1}^{MC}$$

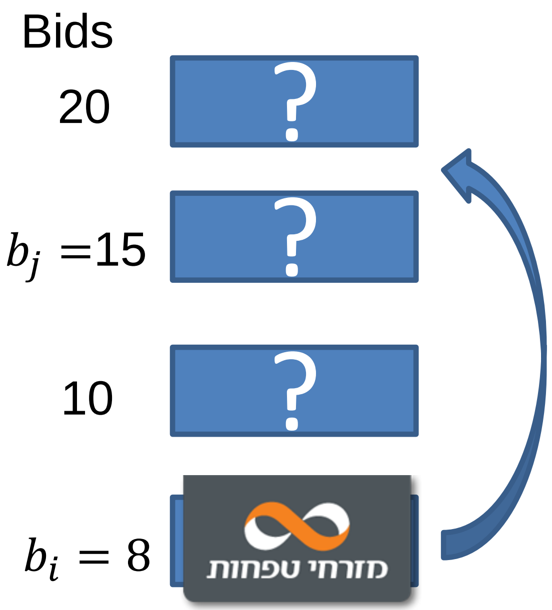
Goal: Explain why charging MC prices is equivalent to receiving the associated bids

<u>Claim</u> 1: $\boldsymbol{b}^{MC}$ is a Nash equilibrium for $\boldsymbol{v}$

- Let us consider a possible deviation of i
  1. Either bid higher to get a better slot
  2. Or bid lower to get a worse slot and pay less

Claim 1: $\boldsymbol{b}^{MC}$ is a Nash equilibrium for $\boldsymbol{v}$

- Let us consider a possible deviation of i
  1. **Either bid higher to get a better slot**
  2. Or bid lower to get a worse slot and pay less



- Suppose i bids $b_i' > b_j$ to get slot $j < i$
- Then i pays p_j, gets

$b_j^{MC} = p_{j-1}^{MC} \geq p_j^{MC}$

$$u_i' = x_j(v_i - b_j) \leq x_j(v_i - p_j)$$

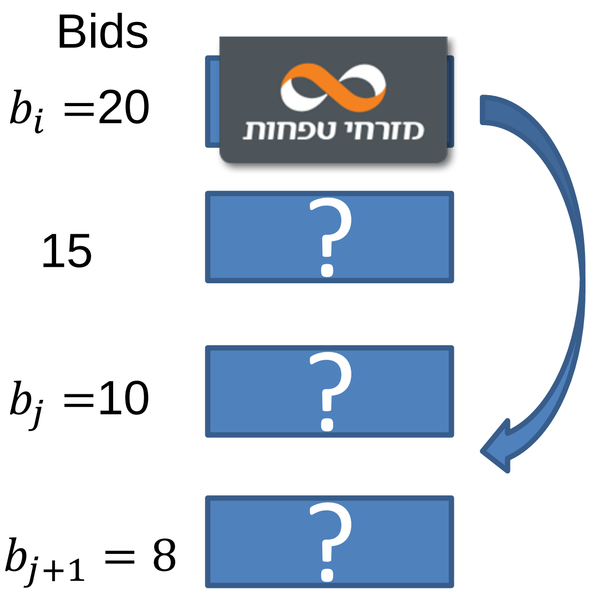
- But under MC prices

$$u_i = x_i(v_i - p_i) \geq x_j(v_i - p_j) \geq u_i'$$

- i does not gain!

Claim 1: $\boldsymbol{b}^{MC}$ is a Nash equilibrium for $\boldsymbol{v}$

- Let us consider a possible deviation of i
  1. Either bid higher to get a better slot
  2. **Or bid lower to get a worse slot and pay less**



- Suppose i bids $b_i' < b_j$ to get slot $j > i$
- Then i pays b_{j+1}, gets

$$u_i' = x_j(v_i - b_{j+1}) = x_j(v_i - p_j)$$

- But under MC prices

$$u_i = x_i(v_i - p_i) \geq x_j(v_i - p_j) = u_i'$$

- i does not gain (again)!

$\square$

GSP EQUILIBRIUM ANALYSIS

Value per click

v_1

v_2

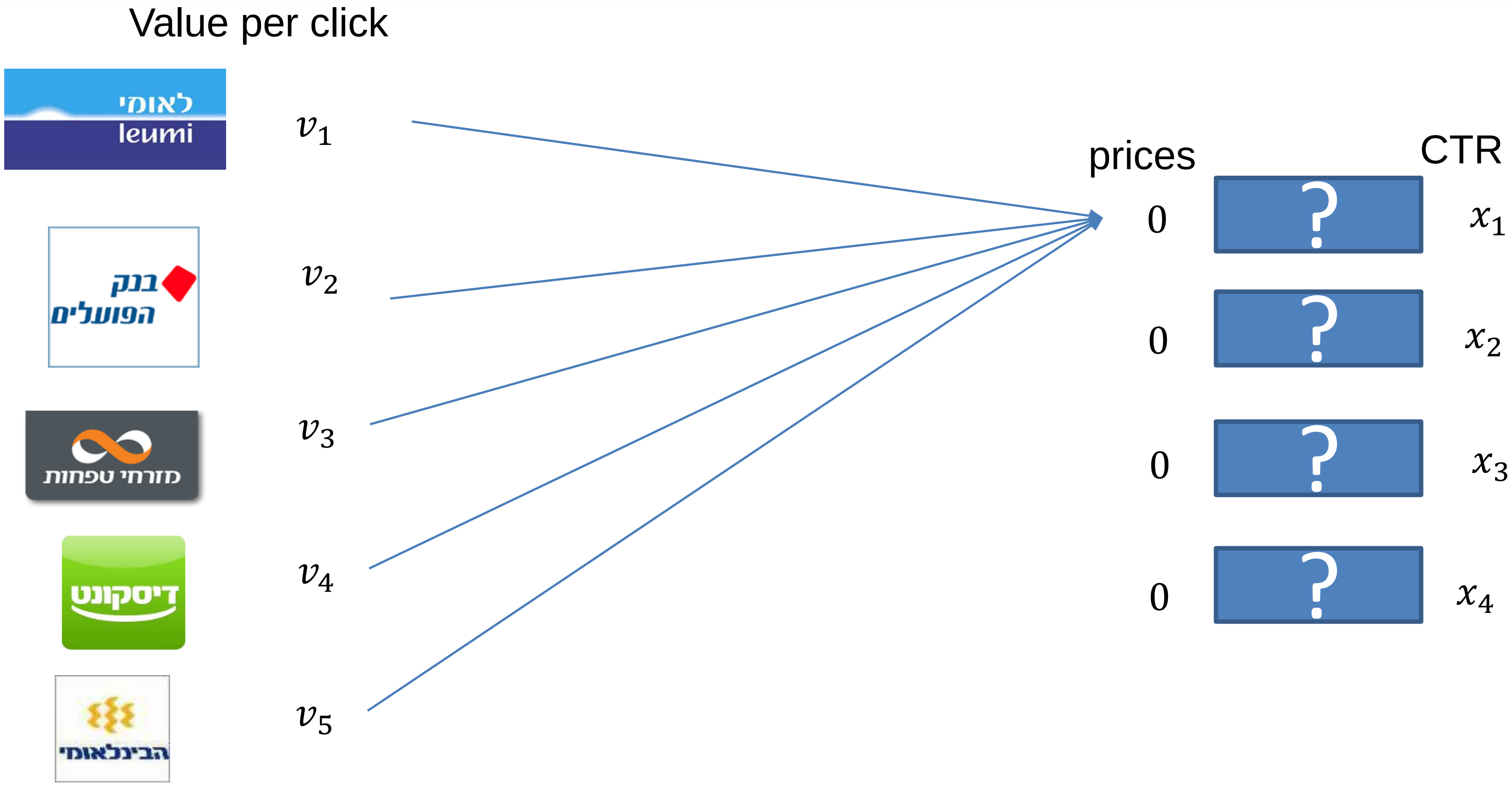
v_3

v_4

v_5

prices		CTR
0	?	x_1
0	?	x_2
0	?	x_3
0	?	x_4

- Initially everyone wants slot 1 (constricted)
- Increase price of slot {1} until only agent 1 remains
  - Increase the price until agent 2 is indifferent



- Initially everyone wants slot 1 (constricted)
- Increase price of slot {1} until only agent 1 remains
  - Agent 2 is indifferent

Value per click

v_1

v_2

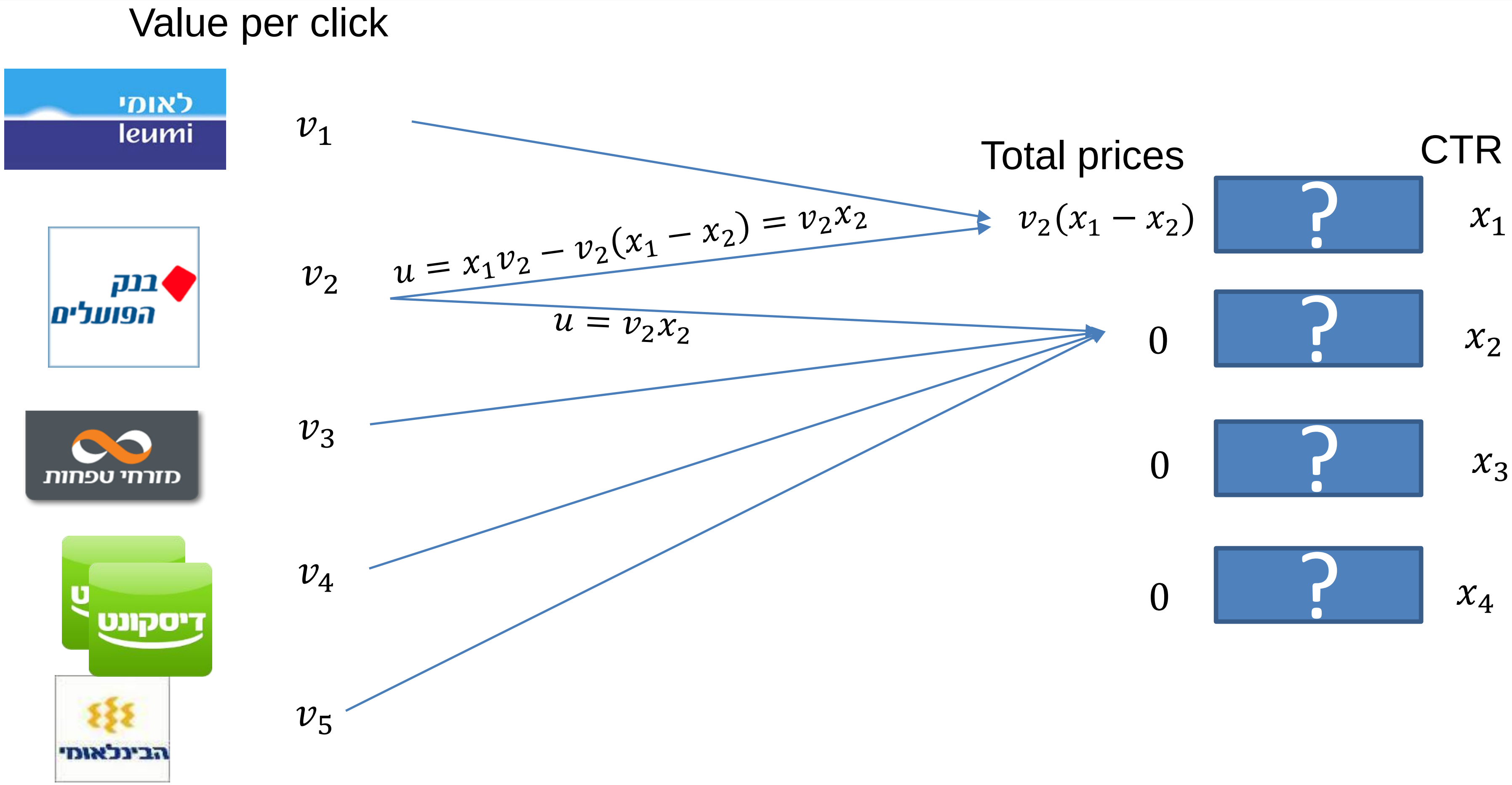
v_3

v_4

v_5

Total prices		CTR
$v_2(x_1 - x_2)$	?	x_1
0	?	x_2
0	?	x_3
0	?	x_4

- Now all want slot 1 **or** 2 (constricted)
- Increase price of slots {1,2} until only agents 1,2 remain



- Now everyone wants slot 1 or 2 (constricted)
- Increase price of slots {1,2} until only agents 1,2 remains
  - Agent 3 is indifferent between slots 2 and 3

Value per click

v_1

v_2

v_3

v_4

v_5

Total prices

CTR

$v_3(x_2 - x_3) + v_2(x_1 - x_2)$? x_1

$v_3(x_2 - x_3) +$ 0 ? x_2

0 ? x_3

0 ? x_4

$u = x_2 v_3 - v_3(x_2 - x_3) = v_3 x_3$

$u = v_3 x_3$

- Now everyone wants slot 1,2 or 3 (constricted)
- Increase price of slots {1,2,3} until
  - Agent 4 is indifferent between slots 3 and 4

Value per click

v_1

v_2

v_3

v_4

v_5

Total prices		CTR
$v_4(x_3 - x_4) + v_3(x_2 - x_3) + v_2(x_1 - x_2)$	?	x_1
$v_4(x_3 - x_4) + v_3(x_2 - x_3) + 0$	?	x_2
$v_4(x_3 - x_4) + 0$	?	x_3
0	?	x_4

- Now everyone wants slot 1…k (constricted)
- Increase price of slots $\{1,2,\dots,k\}$ until
  - Agent $k+1$ is indifferent between slots k and $k+1$

Value per click

Bank	Value per click	Total prices		CTR
לאומי leumi	v_1			
בנק הפועלים	v_2	$v_4(x_3-x_4)+v_3(x_2-x_3)+v_2(x_1-x_2)$	?	x_1
מזרחי טפחות	v_3	$v_4(x_3-x_4)+v_3(x_2-x_3)+$ 0	?	x_2
דיסקונט	v_4	$v_4(x_3-x_4)+$ 0	?	x_3
הבינלאומי	v_5	$\dots$ 0	?	x_4

(minimum)

$$b_{k+1}^{MC} = p_k^{MC} = \frac{1}{x_k}\sum_{i=k\dots m} v_{i+1}(x_i - x_{i+1}) = p_k^{VCG}$$

- Corollary: for (this) MC equilibrium, allocations and payments exactly as in VCG!
- Also the revenue

Value per click

v_1

v_2

v_3

v_4

v_5

Total prices		CTR
$v_4(x_3-x_4)+v_3(x_2-x_3)+v_2(x_1-x_2)$	?	x_1
$v_4(x_3-x_4)+v_3(x_2-x_3)+$ 0	?	x_2
$v_4(x_3-x_4)+$ 0	?	x_3
$\dots$ 0	?	x_4

$$b_{k+1}^{MC} = \quad p_k^{MC} = \frac{1}{x_k}\sum_{i=k\ldots m} v_{i+1}(x_i - x_{i+1}) \quad = p_k^{VCG}$$

- Computing **highest** MC payments:
  – Increase price of slot 1 until agent 1 is indifferent, and so on

Value per click

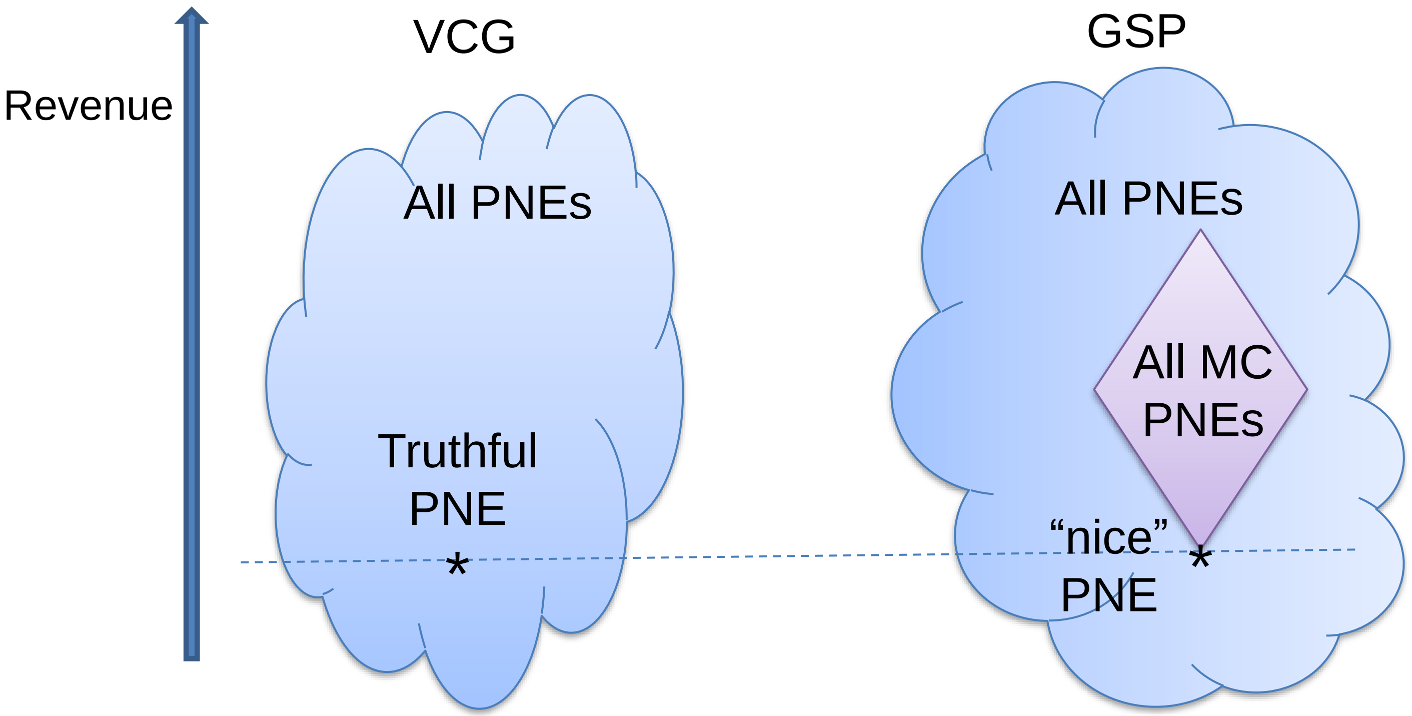
Value per click	Total prices		CTR
v_1			
v_2	$v_3(x_3 - x_4) + v_2(x_2 - x_3) + v_1(x_1 - x_2)$	?	x_1
	$v_3(x_3 - x_4) + v_2(x_2 - x_3) + 0$	?	x_2
v_3	$v_3(x_3 - x_4) + 0$	?	x_3
v_4	$\dots$ 0	?	x_4
v_5			

(maximum)

$$b_{k+1}^{MC} = p_k^{MC} = \frac{1}{x_k} \sum_{i=k \dots m} v_i(x_i - x_{i+1})$$

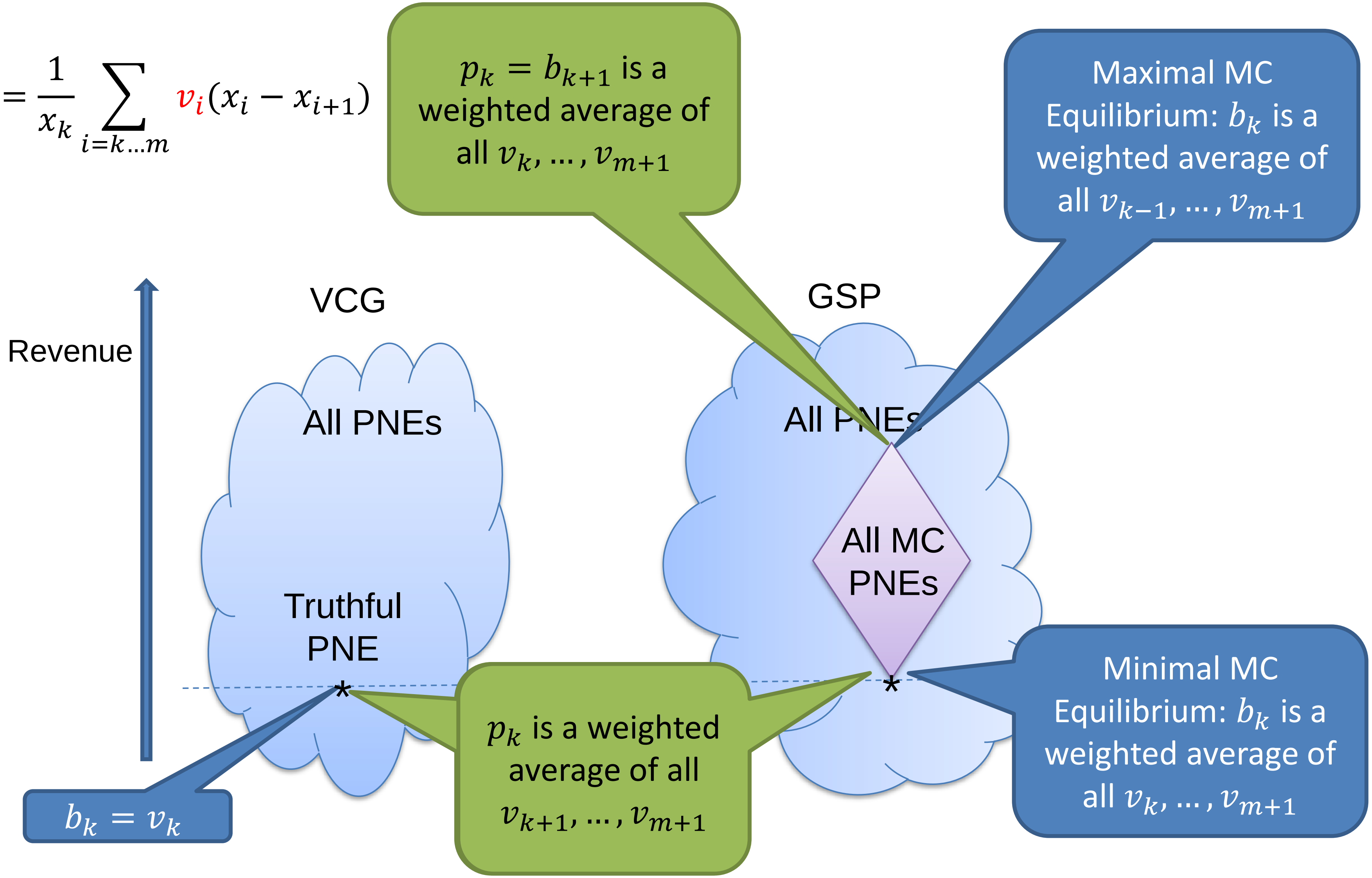
What are the MC bids?

- Corollary: for (this) MC equilibrium, allocations and payments exactly as in VCG!
- Also the revenue
- What about other equilibria?



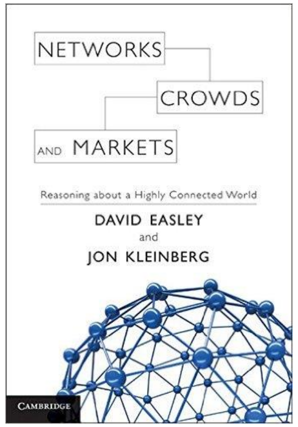
What are the MC bids?

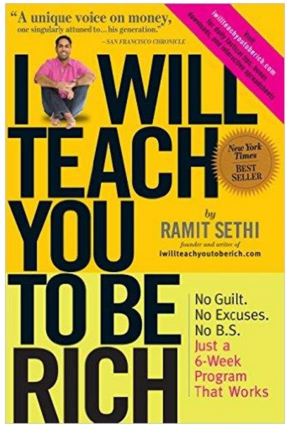
$$p_k^{MC} = \frac{1}{x_k} \sum_{i=k \dots m} v_i (x_i - x_{i+1})$$



Ad quality

- Thus far we assumed that the only thing affecting the number of clicks (CTR) is the position *j*
  - This was what Yahoo! etc. really used
  - Some ads will probably attract more traffic than others





  - The publisher prefers to post attractive ads

Ad quality

- In general, there may be a complicated relation between position, ad features, and clicks
  - Will require many parameters q_{ij}
- Google suggested a simpler model:
  - A single "quality score" q_i for each ad
  - The CTR for ad i at slot j is $x_j q_i$
  - The expected utility for i is

$$x_j q_i v_i - TP_i = x_j q_i (v_i - p_i)$$

- How to modify the GSP/VCG mechanism?

How does Google know this?

Total payment of i

Payment per click of i

VCG with ad quality

- Each player i bids her value v_i
- Sort players by decreasing $v_i^* = (v_i q_i)$
- Give best slot to first player, etc.
- Thus player k gets slot k
- Compute p_i^{VCG} using $v_1^* \dots v_n^*$ instead of $v_1 \dots v_n$
- Not harder than the basic case

GSP with ad quality

- Each player i bids b_i
- Sort players by decreasing $b_i^* = (b_i q_i)$
- Give best slot to first player, etc.
- Thus player k gets slot k
- Compute GSP payments using b^* instead of b_i
  - Each player k pays $TP_k = x_k b_{k+1}^* = x_k b_{k+1} q_{k+1}$ (in total)
  - Thus $p_k = \frac{TP_k}{x_k q_k} = b_{k+1} \frac{q_{k+1}}{q_k}$
- Utility of k is $u_k = x_k q_k (v_k - p_k) = x_k (v_k q_k - b_{k+1} q_{k+1})$
- The analysis is similar

Illustration

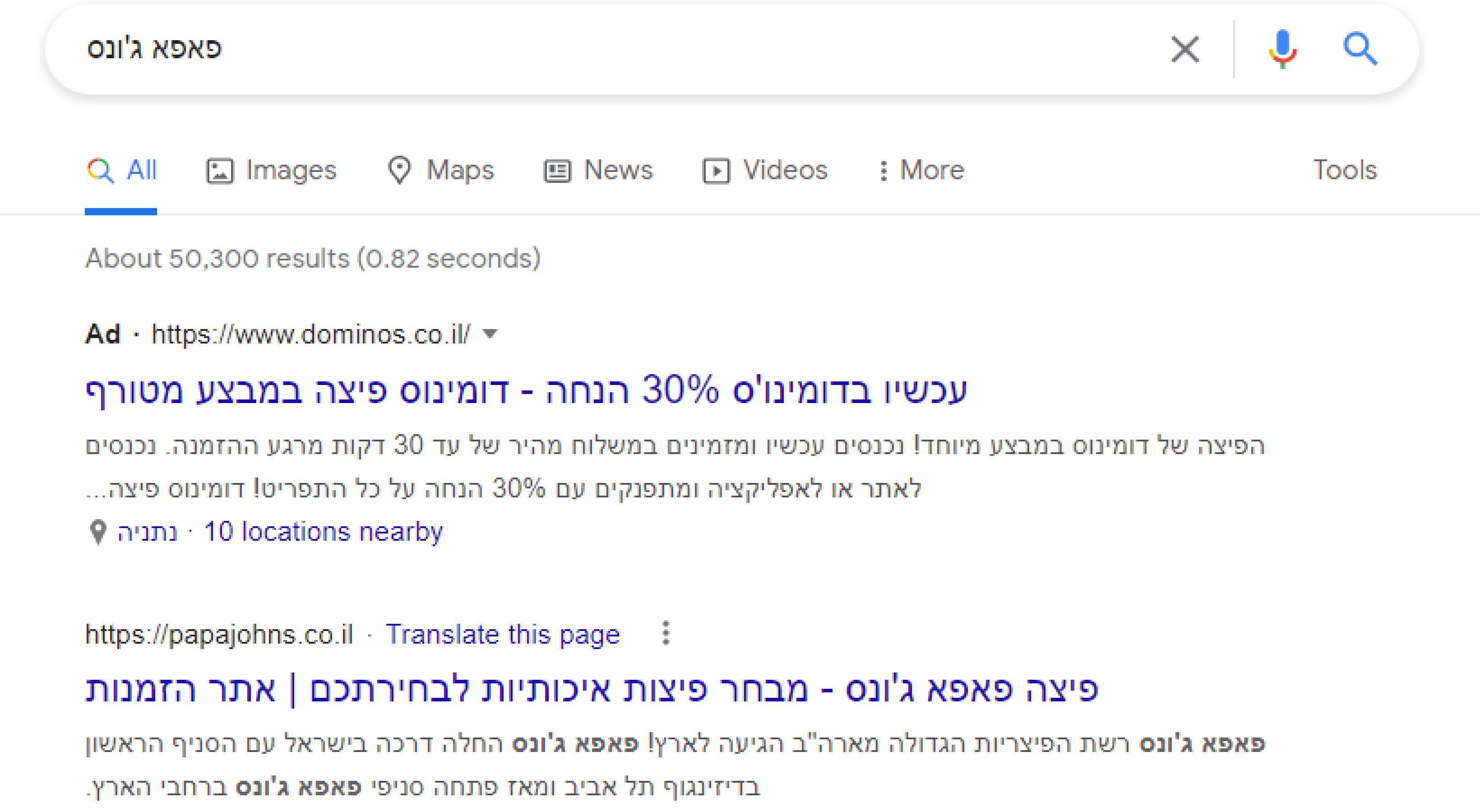
	Max Bid	Quality Score	AdRank	Position	CPC Calculation	Actual CPC
Advertiser 1	4	8	32	1	=27/8+.01	3.39
Advertiser 2	3	9	27	2	=24/9+.01	2.68
Advertiser 3	6	4	24	3	=16/4+.01	4.01
Advertiser 4	8	2	16	4		

Fig. 6 – A typical AdWords auction

A STEP TOWARDS THE REAL WORLD

Bidding on complex expressions

- Supp … nts ski equip
  - I ha
  - Sor
  - Ho … where to go for
  - Ho
- Simil



The big picture

- Many advertisers with complex preferences over queries
  - May gain some value from displaying ad and some from clicks
  - Value may change quickly for some, slowly for others
- Companies have a “total budget” for advertising
  - Need to spend wisely across all terms
- Many interesting research problems
- Challenge for industry:
  - Need an appropriate bidding interface
  - Smart algorithms to determine relevance, quality, etc.

Moving to first price auctions

- Integration of many bidders from several platforms:
  - Google's advertisers would lose
- You need to trust the mechanism when it tells you what the second bid is, “buy-side fees”
- Simpler for the buyer - recent years many optimizations on the second price auctions
- Privacy: can use second bid to learn about competitors
- "In their 2021 blog post, Google says moving AdSense to a first-price auction should have a neutral effect on publishers."

References

Varian, Hal R. "Position auctions." *international Journal of industrial Organization* 25.6 (2007): 1163-1178.

Edelman, Benjamin, Michael Ostrovsky, and Michael Schwarz. *Internet advertising and the generalized second price auction: Selling billions of dollars worth of keywords*. No. w11765. National Bureau of Economic Research, 2005.

Electronic Commerce 096211

Information and Markets

Omer Ben Porat (some of the slides are adopted from Prof. Reshef Meir)

Course Structure

Online Societies
- Web ranking
- Diffusion
- Structure of the web
- Power laws
- Networks

Advertising
- Auctions
- Matching markets
- Sponsored search

Decision Making
- Recommendation systems
- Multi-armed bandits

Advance
- Information in markets
- Externalities
- Sharing economy
- Blockchain

Tools and Techniques
- Game theory
- Algorithms
- Graph theory
- Optimization

Admin

- Last week!
- Information in markets
  - Recitation: Today + recording for those who can’t attend
- Exam

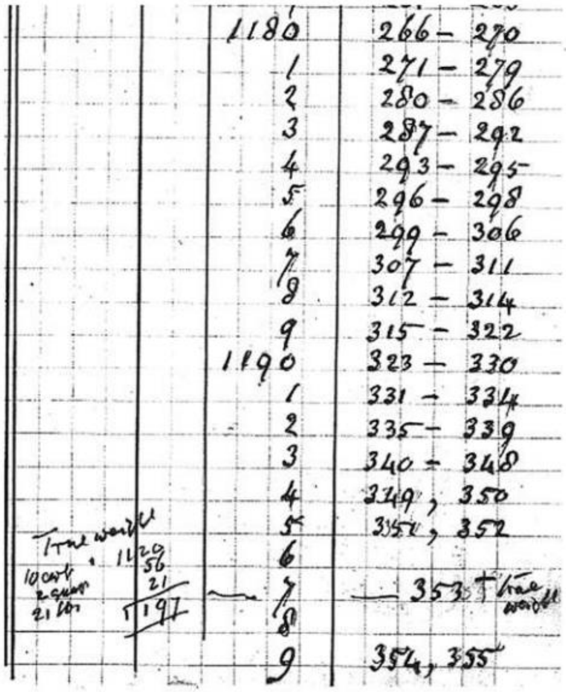
Outline

- Wisdom of the crowds
- Scoring rules
  - Extract an estimate from one expert
- Prediction markets
  - Extract an estimate from many experts
- The Market for Lemons

I discovered a way to get my students interested in microeconomics.

Wisdom of the crowd

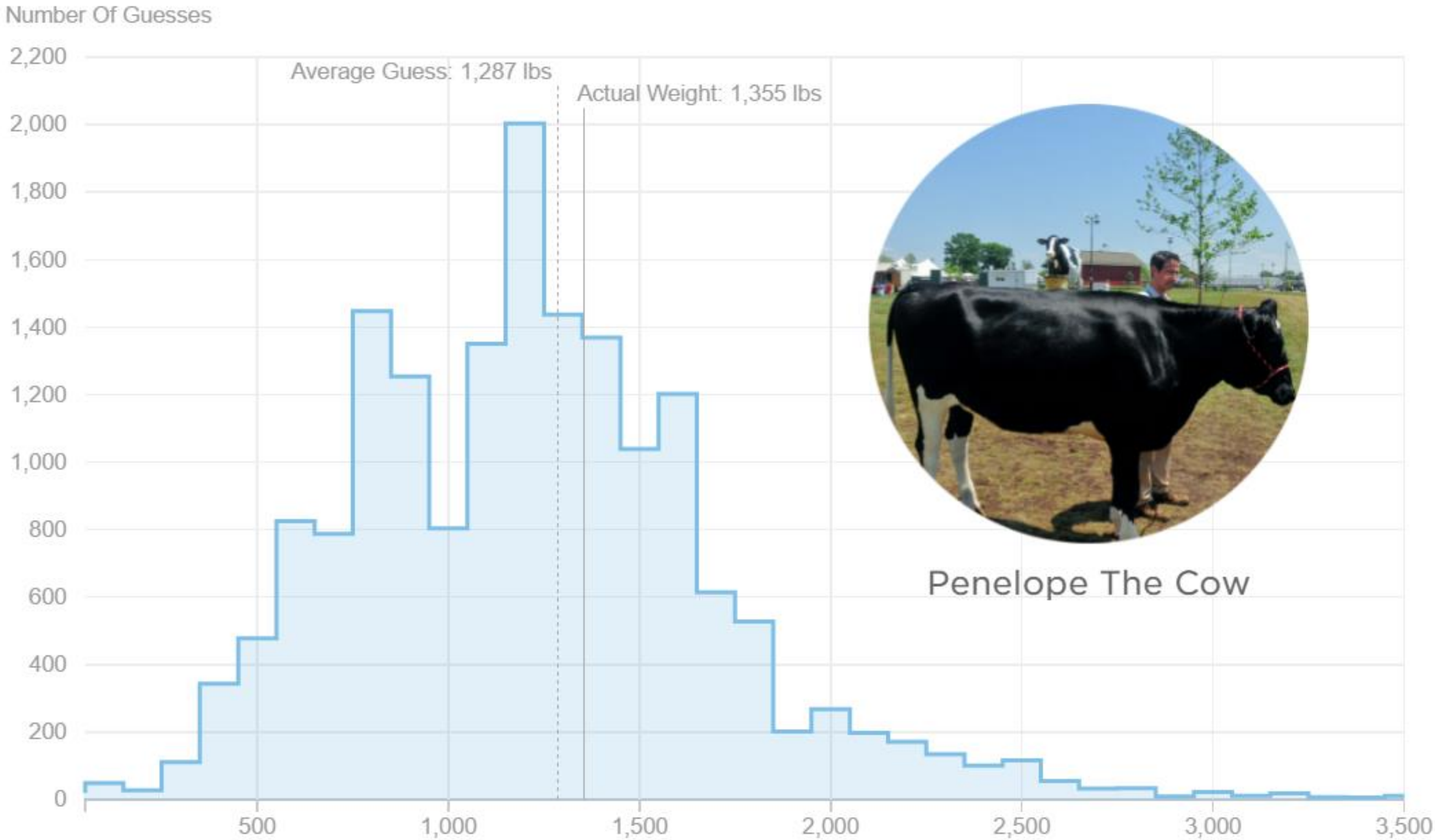
- A famous story by Galton [Wallis, '14]:



1180	266 – 270
1	271 – 279
2	280 – 286
3	287 – 292
4	293 – 295
5	296 – 298
6	299 – 306
7	307 – 311
8	312 – 314
9	315 – 322
1190	323 – 330
1	331 – 334
2	335 – 339
3	340 – 348
4	349, 350
5	351, 352
6	
7	353 True weight
8	
9	354, 355

  - 787 people
  - Median: 1208 lbs.
  - Real: 1197 lbs. !

How did 17,000 people do?



http://www.npr.org/sections/money/2015/08/07/429720443/17-205-people-guessed-the-weight-of-a-cow-heres-how-they-did

Why crowdsourcing succeeds?

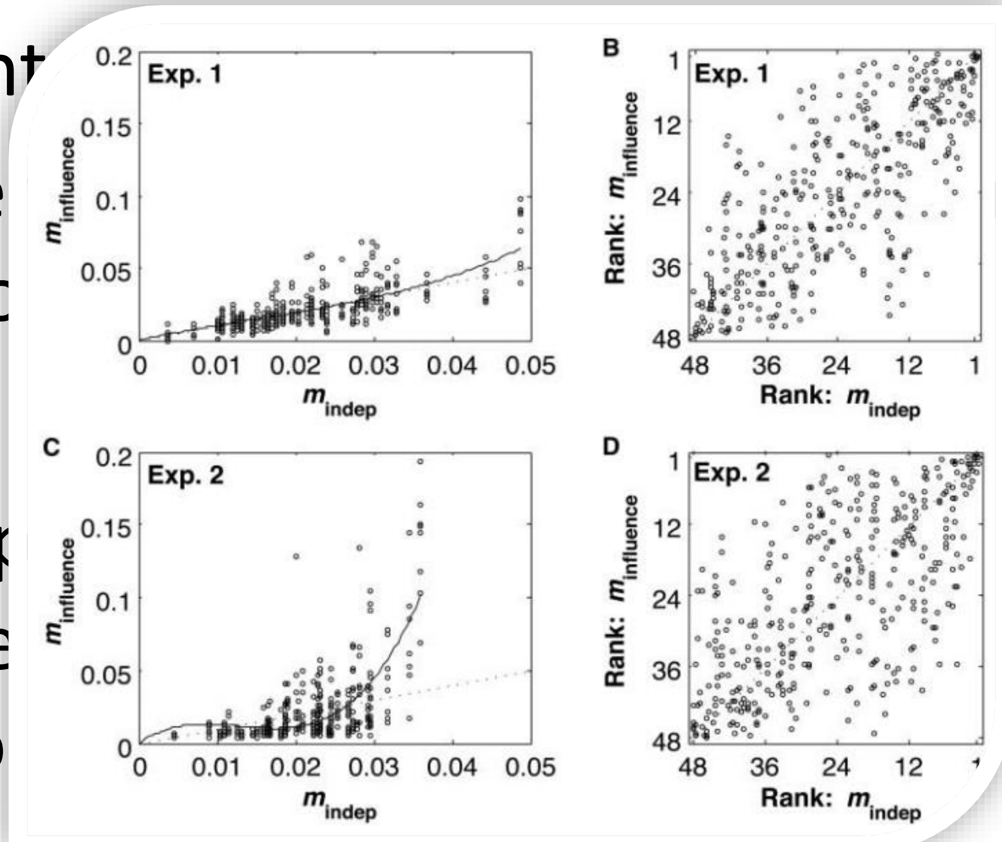
- The Central limit theorem
- The average of many *independent* samples tends to the mean value

- If each person has some (even weak) signal of the true value, the aggregated value will be close to the mean

More on this in RESHEF's advanced course: Social Choice and Preference Aggregation (096578)

Why crowdsourcing succeeds?

- The Cent
- The ave mples tends to



- If each signal of the true will be close to

More on this in RESHEF's advanced course: Social Choice and Preference Aggregation (096578)

Why crowdsourcing fails?

- People have varying levels of information
  - Can sometimes fix the mechanism to use that
- Dependency across people
  - Extreme case: everyone has the same source
  - Mutual influences, imitation
- Incentives
  - People don’t care to tell their true opinion
  - Might want to bias the outcome

Outline

- Wisdom of the crowds
- **Scoring rules**
  - Extract an estimate from one expert
- Prediction markets
  - Extract an estimate from many experts
- The Market for Lemons

Scoring rules

- We want a prediction from a single expert (say, a weatherperson)
  - “Is there going to be rain tomorrow?”
- Prediction can be probabilistic
  - “30% chance of rain”
- Suppose they predict “30%” and it does rain
  - Are they doing a good job?
  - Why would they bother at all?

Scoring rules

- More examples:
  - Political forecasts
  - Success of medical operations
  - Time to complete coding tasks
  - Risk due to environmental actions
  - …
- We want to **pay** in a way that incentivizes honest prediction:
  - Reward accurate forecast
  - Punish poor forecast

Scoring rules

- Basic model:
  - Binary event $A \in \{1,2\}$
  - The expert **believes** that $\Pr(A = 1) = p$
    - "subjective probability"
  - The expert **reports** probability r
    - Expert is "truthful" if $r = p$
  - A **scoring rule** is composed of two functions:
    - Pay $S_1(r)$ if $A = 1$
    - Pay $S_2(r)$ if $A = 2$

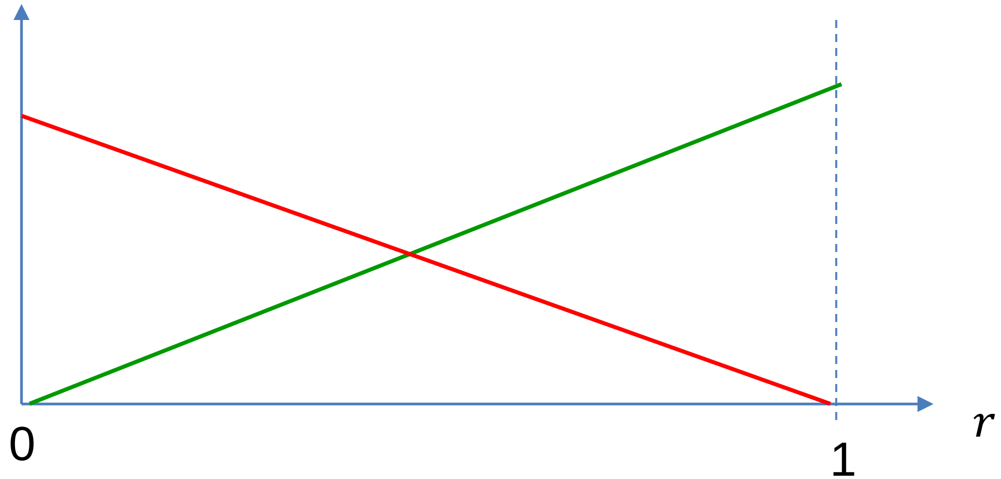
Scoring rules: Protocol

- A scoring rule is declared, $S_1, S_2: [0,1] \to \mathbb{R}$
- The expert **obtains** $p = \Pr(A = 1)$
- The expert **reports** r
- The binary outcome is realized (rain)
  1. If $A = 1$, the expert gets $S_1(r)$
  2. If $A = 2$, the expert gets $S_2(r)$

Scoring rules

r estimates $\Pr(A = 1)$

- Examples of scoring rules:
  - $S_1(r) = r$, $S_2(r) = 1 - S_1(r)$

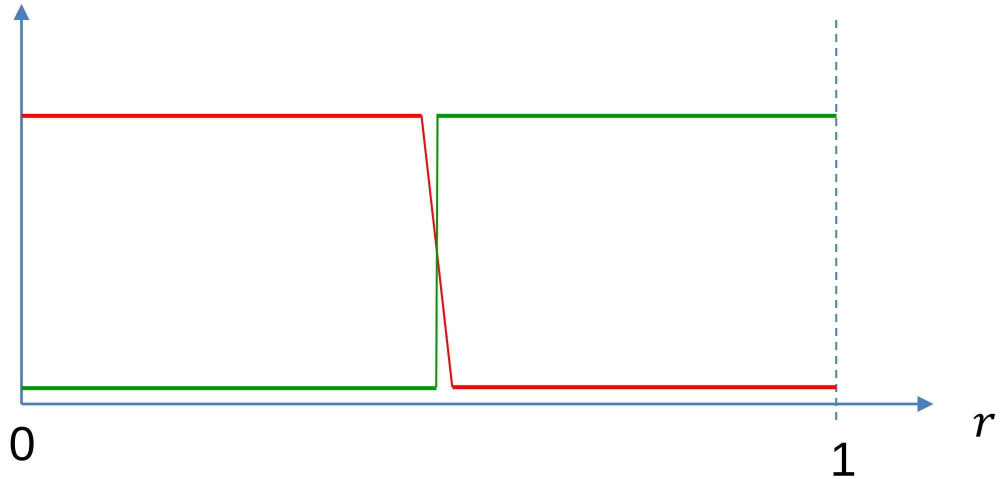


linear

Scoring rules

- Examples of scoring rules:
  - $S_1(r) = r \quad , \quad S_2(r) = 1 - S_1(r)$
  - $S_1(r) = 1 \text{ iff } r > 0.5, \; S_2(r) = 1 - S_1(r)$

r estimates $\Pr(A = 1)$

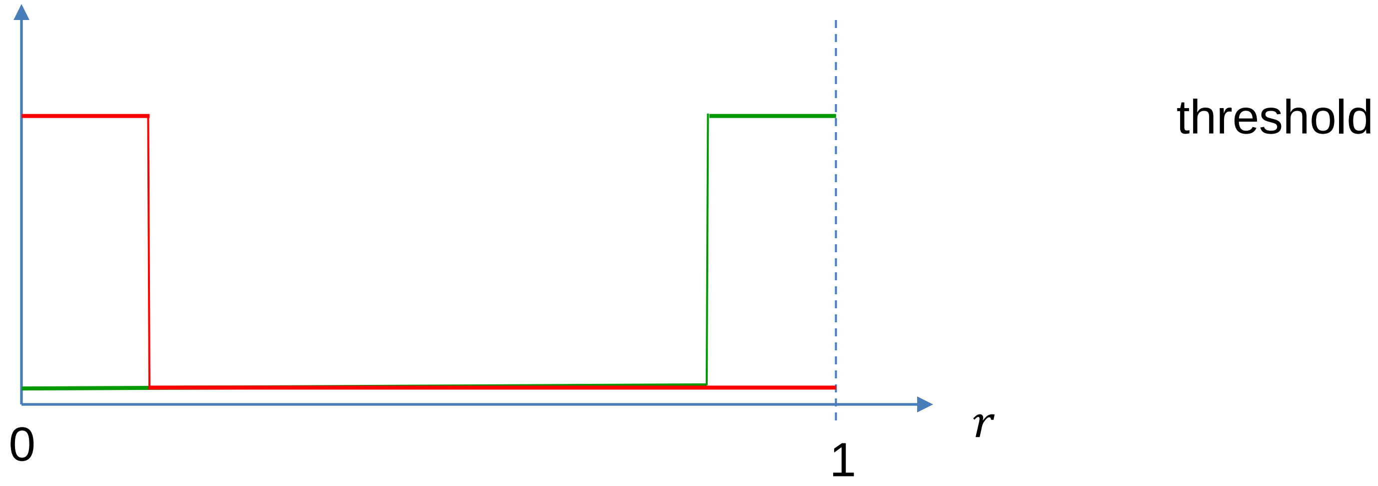


threshold

Scoring rules

r estimates $\Pr(A = 1)$

- Examples of scoring rules:
  - $S_1(r) = r \quad , \quad S_2(r) = 1 - S_1(r)$
  - $S_1(r) = 1$ iff $r > 0.5$, $S_2(r) = 1 - S_1(r)$
  - $S_1(r) = 1$ iff $r > 0.8$, $S_2(r) = 1$ iff $r < 0.2$

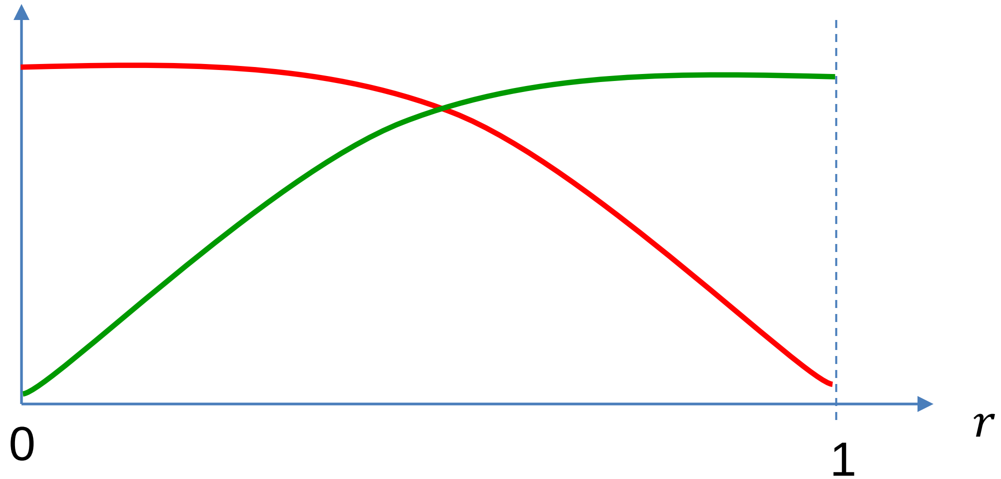


Scoring rules

- Examples of scoring rules:

r estimates $\Pr(A = 1)$

  - $S_1(r) = r$, $S_2(r) = 1 - S_1(r)$
  - $S_1(r) = 1$ iff $r > 0.5$, $S_2(r) = 1 - S_1(r)$
  - $S_1(r) = 1$ iff $r > 0.8$, $S_2(r) = 1$ iff $r < 0.2$
  - $S_1(r) = C - (1-r)^2$, $S_2(r) = C - r^2$

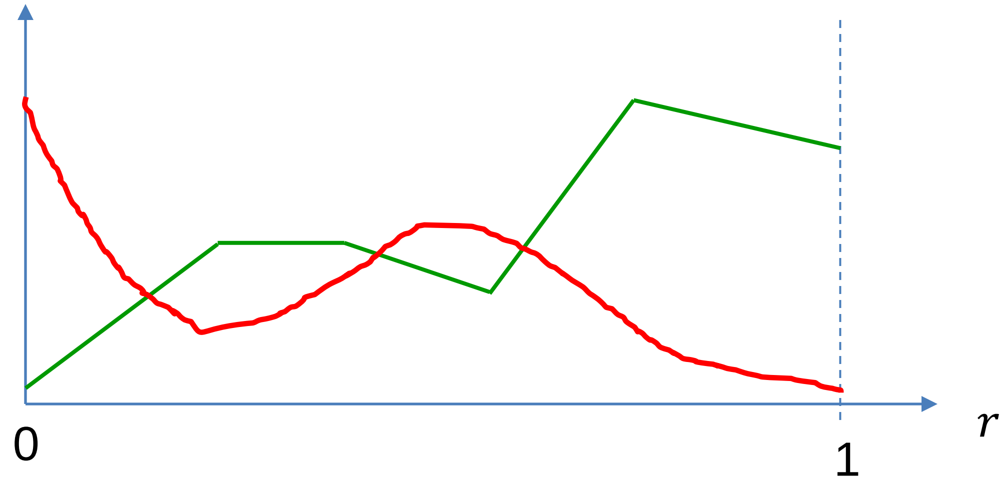


Quadratic
(a.k.a. Brier score)

Scoring rules

- Examples of scoring rules:
  - $S_1(r) = ?$, $S_2(r) = ?$
  - Desired property?

r estimates $\Pr(A = 1)$



Proper scoring rules

- What is the expected score (payment)?

$$E_p(S, r) = p \cdot S_1(r) + (1 - p) S_2(r)$$

Determined by the designer w/o knowing p

Strategy of the expert that knows p

- A scoring rule S is **strictly proper** if

$$E_p(S, p) > E_p(S, r) \quad \text{for all } r \neq p$$

- Incentive for effort?

Proper scoring rules

Claim: The quadratic scoring rule is strictly proper

$$S_1(r) = -(1-r)^2, \;\; S_2(r) = -r^2$$

Proof: $E_p(S,r) = p(-(1-r)^2) + (1-p)(-r^2)$

$$= (-1 - r^2 + 2r)p + (r^2 p - r^2)$$

$$= 2rp - p - r^2$$

Proper scoring rules

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Proof: $$E_p(S,r) = p(-(1-r)^2) + (1-p)(-r^2)$$
$$= (-1 - r^2 + 2r)p + (r^2 p - r^2)$$
$$= 2rp - p - r^2$$

$$0 = \frac{\partial E_p(S,r)}{\partial r} = 2p - 2r \Rightarrow r = p$$

$\square$

Proper scoring rules

- What about non-binary events $A = \{1,2,\dots,k\}$?
- Note that prediction is a vector $r = (r_1, \dots, r_k)$
- The quadratic score easily extends:

$$S_j(r) = 2r_j - \sum_{i \le k} r_i^2$$

- In the binary case $r_1 = 1 - r_2$, so

$$S_1(r_1) = 2r_1 - r_1^2 - r_2^2 = 2r_1 - r_1^2 - 1 + 2r_1 - r_1^2$$
$$= -2r_1^2 + 4r_1 - 1$$
$$\Rightarrow \frac{S_1(r_1) - 1}{2} = -\frac{2r_1^2 - 4r_1 + 2}{2} = -(1 - r_1)^2$$

Proper scoring rules

- Is the quadratic scoring rule the only strictly proper scoring rule?
- We can create a proper scoring rule from *any* strictly convex function $s: R^k \to R$

Set $\forall j,\ S_j(r) := s(r) + \frac{\partial s(r)}{\partial r_j} - \sum_{i \leq k} \frac{\partial s(r)}{\partial r_i} r_i$

Theorem: A scoring rule S is strictly proper if and only if it has the above form.

Proper scoring rules

Sanity check: consider $s(r) = \sum_i r_i^2$

Set $\forall j,\ S_j(r) := s(r) + \frac{\partial s(r)}{\partial r_j} - \sum_{i \le k} \frac{\partial s(r)}{\partial r_i} r_i$

$$= \sum_i r_i^2 + 2r_j - \sum_i 2r_i \cdot r_i = 2r_j - \sum_i r_i^2$$

We obtained exactly the quadratic scoring rule.

<u>Exercise</u>: what do we get from the (weakly convex) function $s(r) = \sum_i r_i$?

More strictly proper scoring rules

- Logarithmic: $S_j(r) = \log(r_j)$
  - Note it is **local** - depends only on r_j
- Spherical: $S_j(r) = \frac{r_j}{\sqrt{\sum_{i \le k} r_i^2}}$
- Can add / multiply by any positive constant
- How to choose?

What about cows?

- We can extend scoring rules to continuous predictions (like weight of a cow)
- The expert reports a distribution (PDF) $r(x)$
- For example, the continuous quadratic rule:

If the real weight is z then the payment is

$$S_z(r) = 2r(z) - \int_{x=-\infty}^{+\infty} \big(r(x)\big)^2 dx$$

HW?!

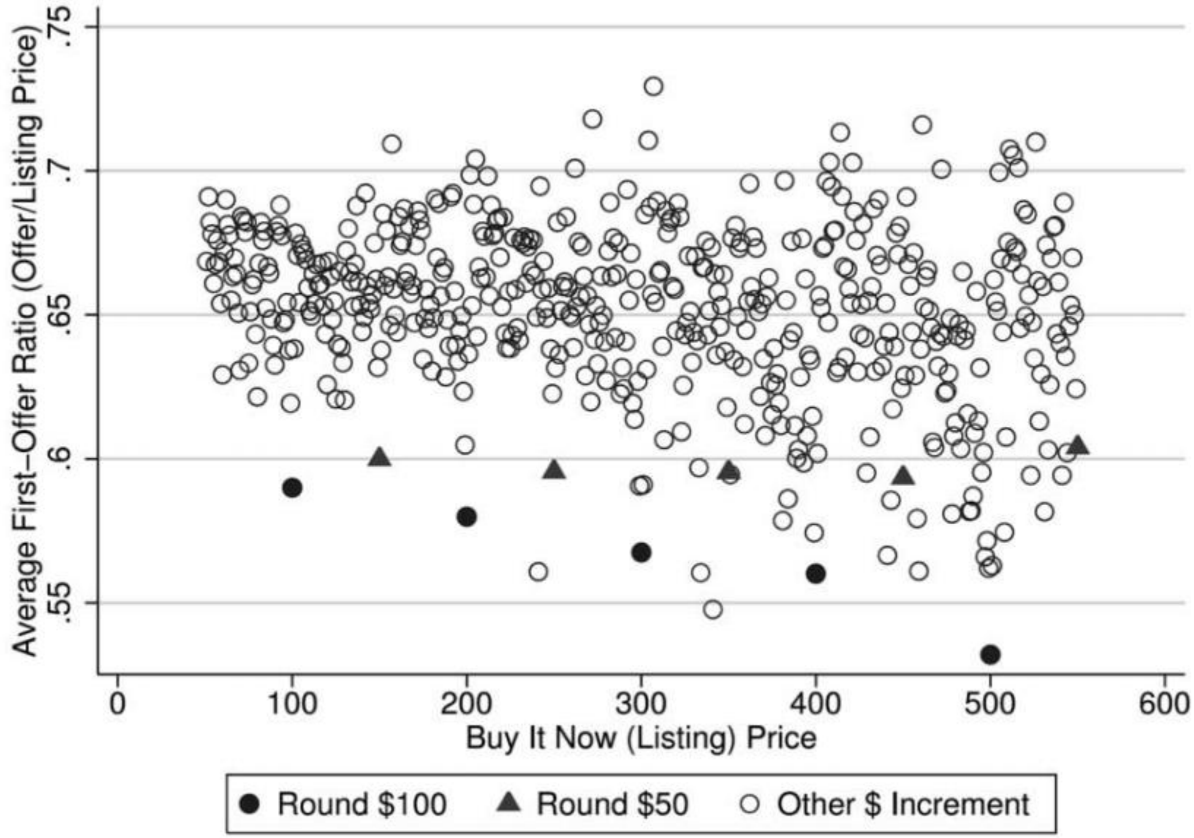


FIG. 2.—Average first offers by BIN price. This scatter plot presents average first offers, normalized by the BIN price to be between zero and one, grouped by unit intervals of the BIN price, defined by $(z-1,\ z]$. When the BIN price is on an interval rounded to a number ending in 00, it is represented by a circle; 50 numbers are represented by a triangle.

Source: https://faculty.haas.berkeley.edu/stadelis/round.pdf

Outline

- Wisdom of the crowds
- Scoring rules
  - Extract an estimate from one expert
- **Prediction markets**
  - Extract an estimate from many experts
- The Market for Lemons

Betting markets

- Situations where people have different information and conflicting interests?
  - Markets! E.g., matching markets
- Somehow, the market was able to aggregate the demands of all people to an efficient outcome
  - The key part: prices aggregate information

Prediction markets

- We want to predict an event $A \in \{1,2,..k\}$
- We can sell k different contracts, where c_i is the contract "pay \$1 if $A = i$"
- What should be the price of c_i ?

Who will win the 2016 Democratic presidential nomination?

Market Type: Linked

End Date: 09/15/2016 12:00 AM (ET)

Status: Open

Contracts | Info | Rules

Trade shares from this page by clicking any price in bold. For more information on an individual prediction, click on the name or image.

	DNOM16	Latest	Buy Yes	Sell Yes	Buy No	Sell No
	**Hillary Clinton** CLINTON.DNOM16	92¢ ↓ 1¢	93¢	92¢	8¢	7¢
	**Bernie Sanders** SANDERS.DNOM16	7¢ ↓ 1¢	8¢	7¢	93¢	92¢
	**Joe Biden** BIDEN.DNOM16	4¢ NC	4¢	3¢	97¢	96¢
	**Elizabeth Warren** WARREN.DNOM16	2¢ NC	2¢	1¢	99¢	98¢
	**Martin O'Malley** O'MALLEY.DNOM16	1¢ NC	1¢	None	None	99¢

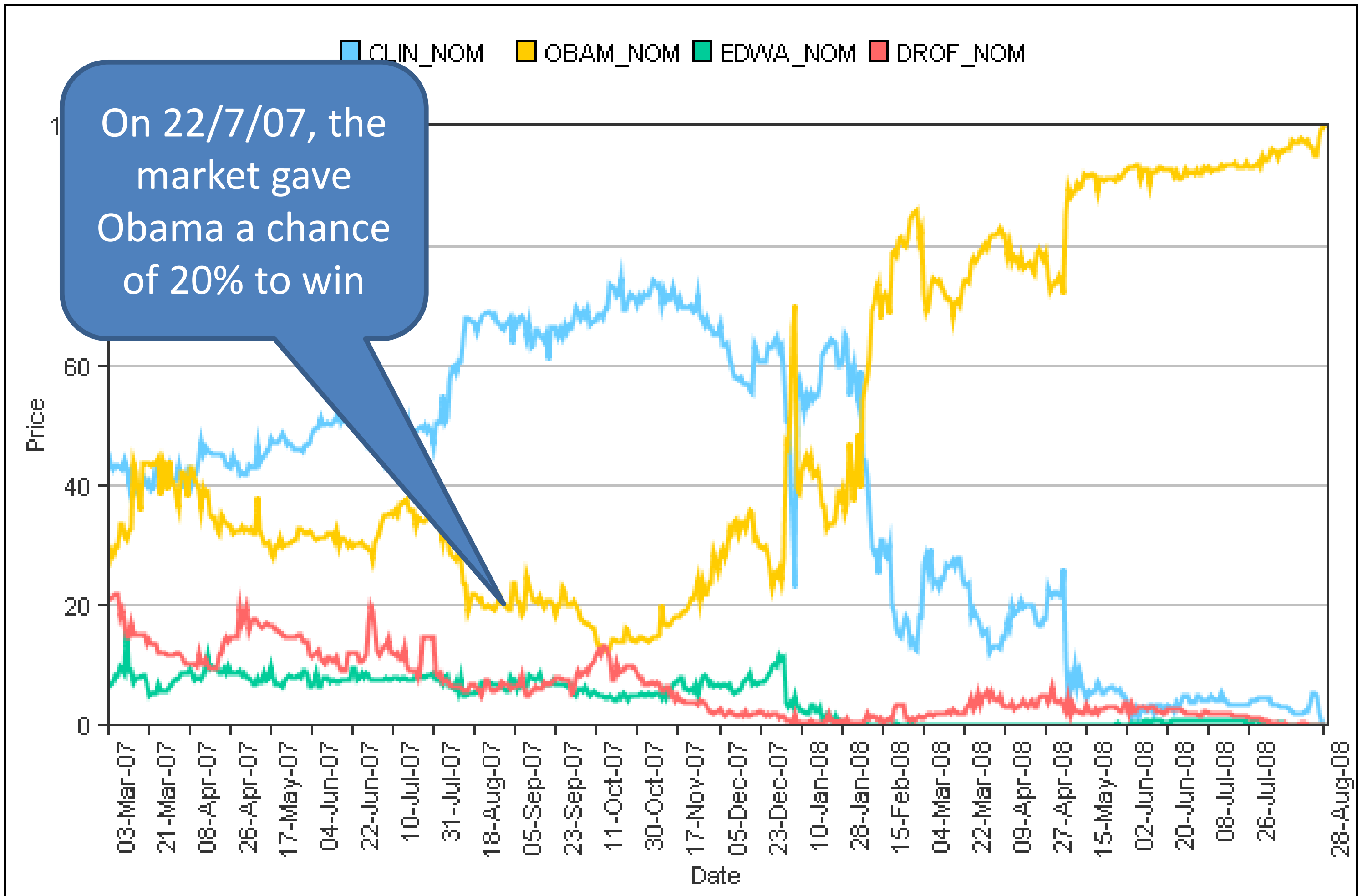
Exercise: find a way to surely gain money in this market (assume no commissions)

Updating prices

- Suppose a participant believes that $A = i$ with some probability q_i.
- Will buy contract c_i if $price(c_i) < q_i$
  - Price will rise
- Will sell contract (short) c_i if price is higher
  - Price will decrease
- When the price is stable, it reflects the beliefs of the market

Prices change

- (2008 Democrats primary elections)



Why do prediction markets work\fail?

- variable information quality
  - Laymen, experts, “inside info”, etc.
  - wishful thinking
- Suppose you “know” that the current market belief is wrong
  - profit
  - -> Improve market prediction
- Overconfident or underconfident
  - Ok if only some are
- Diverse and extreme risk attitudes
  - E.g. Clinton believers very risk-averse, and Obama’s more risk-neutral
- People are extremely correlated
  - E.g. everybody Googles “Iowa prediction market” to decide how to bet

Self-fulfilling expectations

- In some markets the belief affects the outcome
  - If people think a new social network will fail, no one will join (and it will fail)
  - If people think used cars are bad, price will drop. Then owners of good used cars will not sell.
  - If people think some candidates have no chance, they will not get votes
    - True for ≥ 3 candidate races

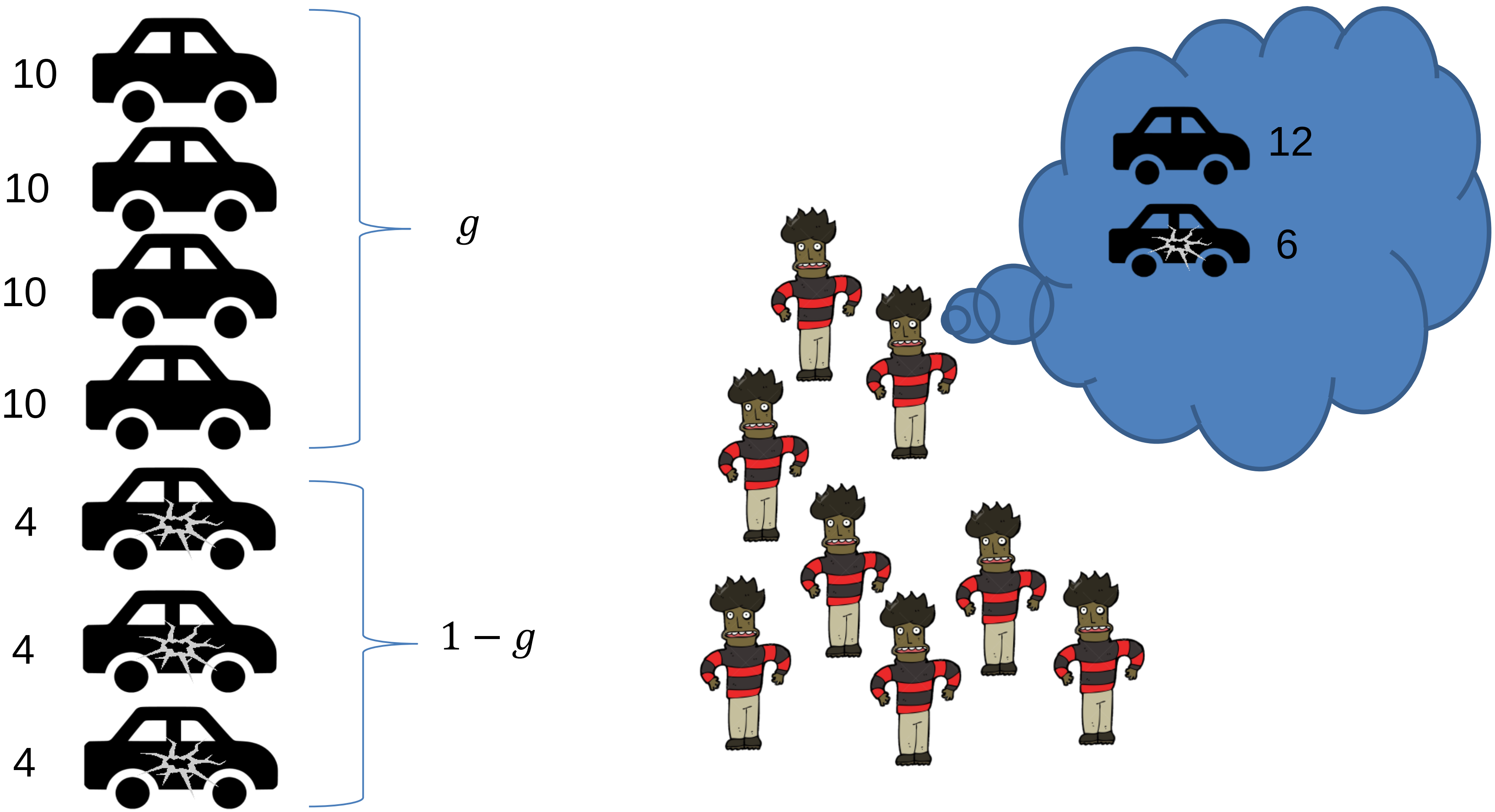
Outline

- Wisdom of the crowds
- Scoring rules
  - Extract an estimate from one expert
- Prediction markets
  - Extract an estimate from many experts
- **The Market for Lemons**

Asymmetric information

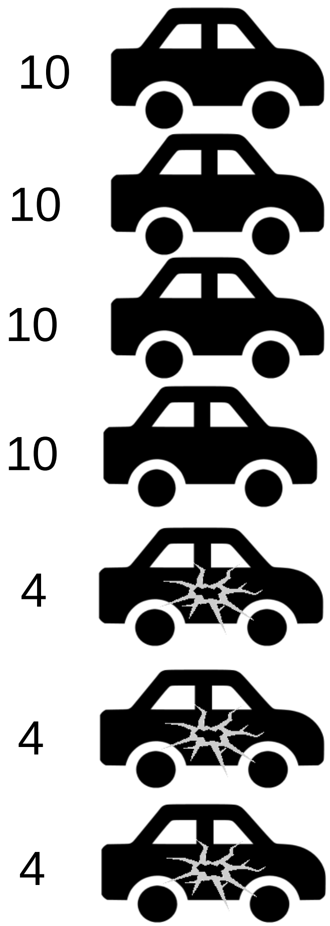
- Do recommender systems lie?
- In the used car market, there is asymmetric information
  - Some sellers know their car is good, but this information does not get to the market
- In other markets such as health insurance, the clients have better information

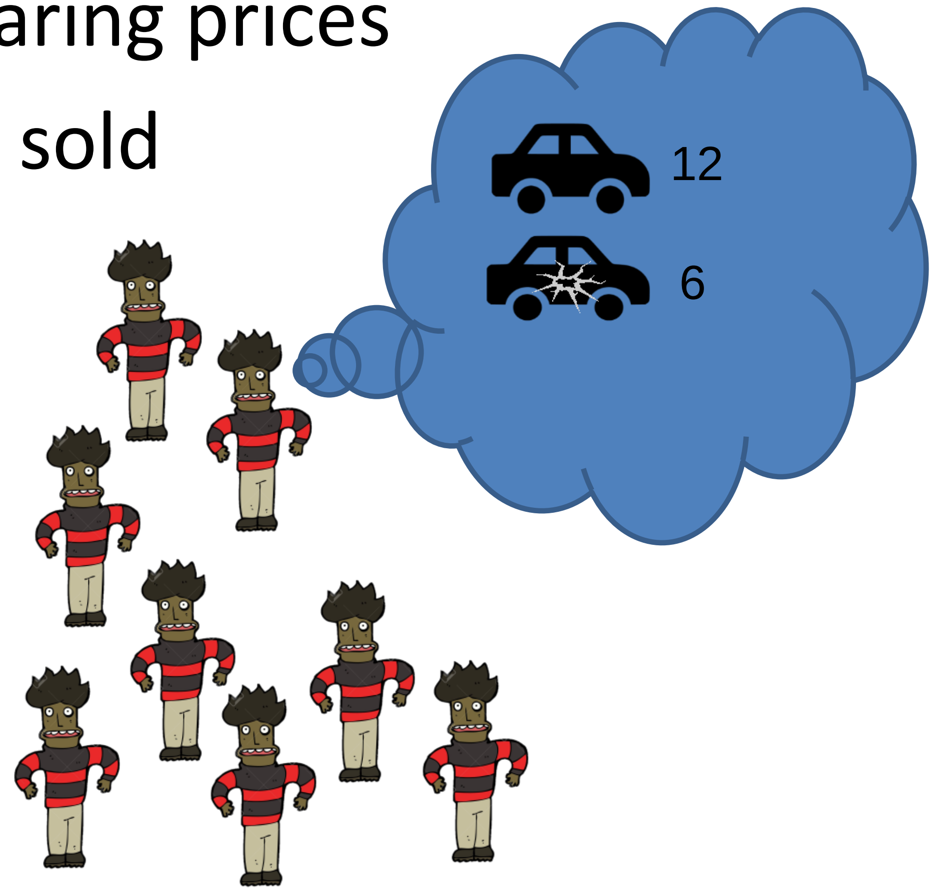
Example [Ackerlof '70]



Example [Ackerlof '70]

- Suppose car types are observable
- Market clearing prices
- All cars are sold

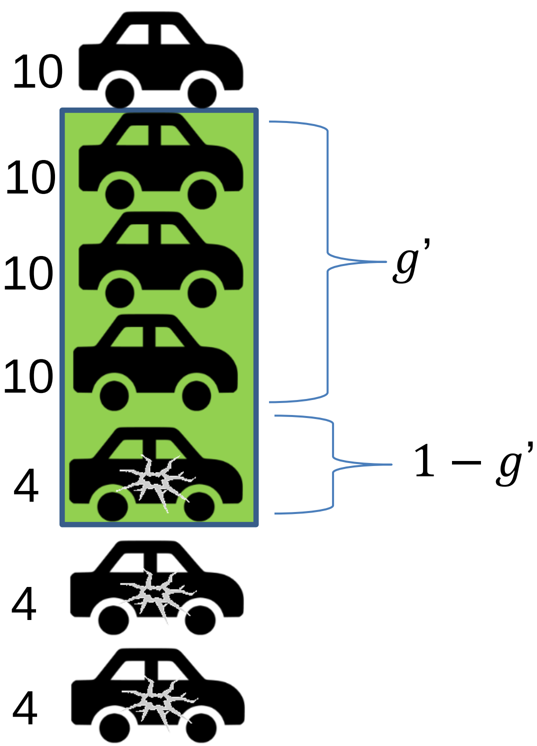


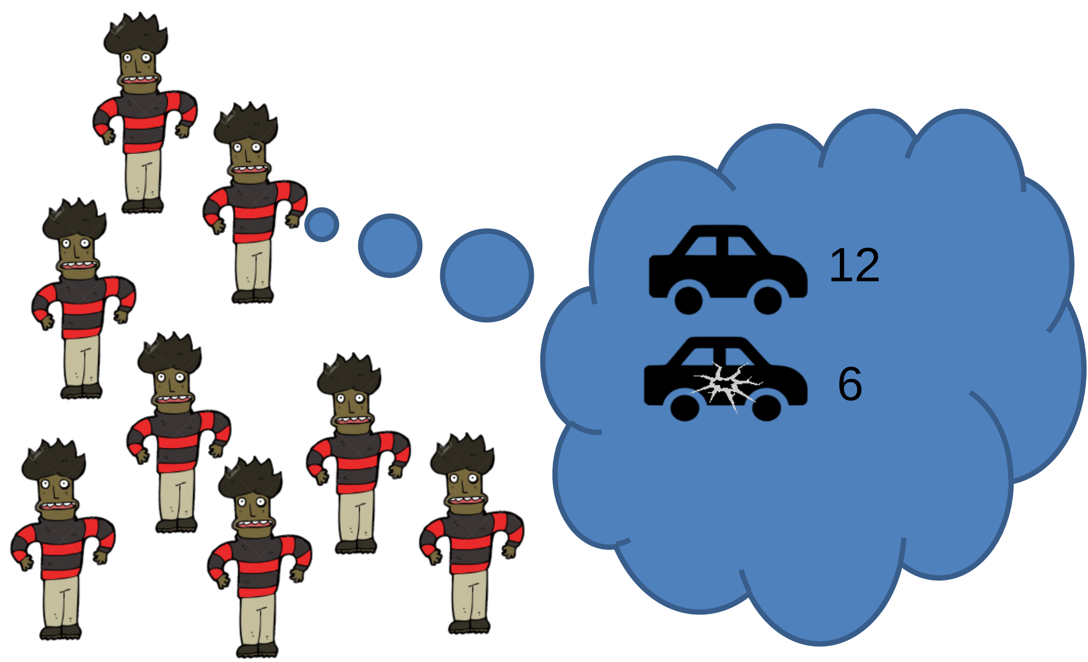


Example [Ackerlof '70]

- Suppose car types are **un**observable
- One price p for all cars

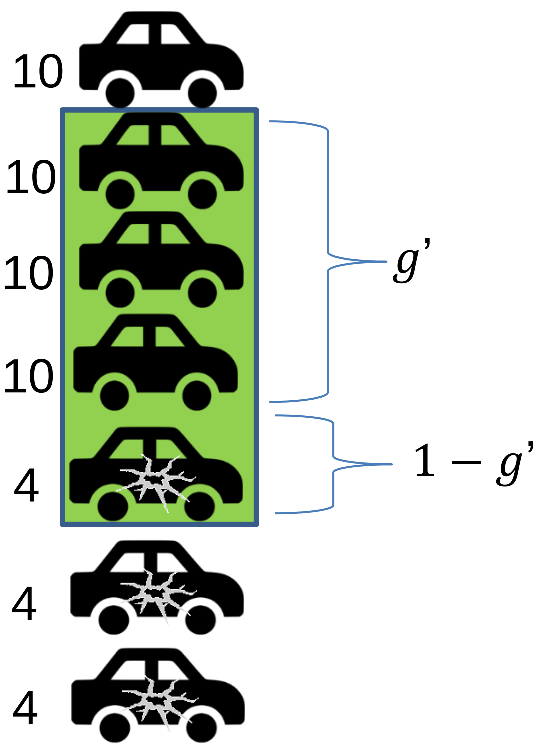
$$v(buy) = 12g' + 6(1 - g') = 6 + 6g'$$

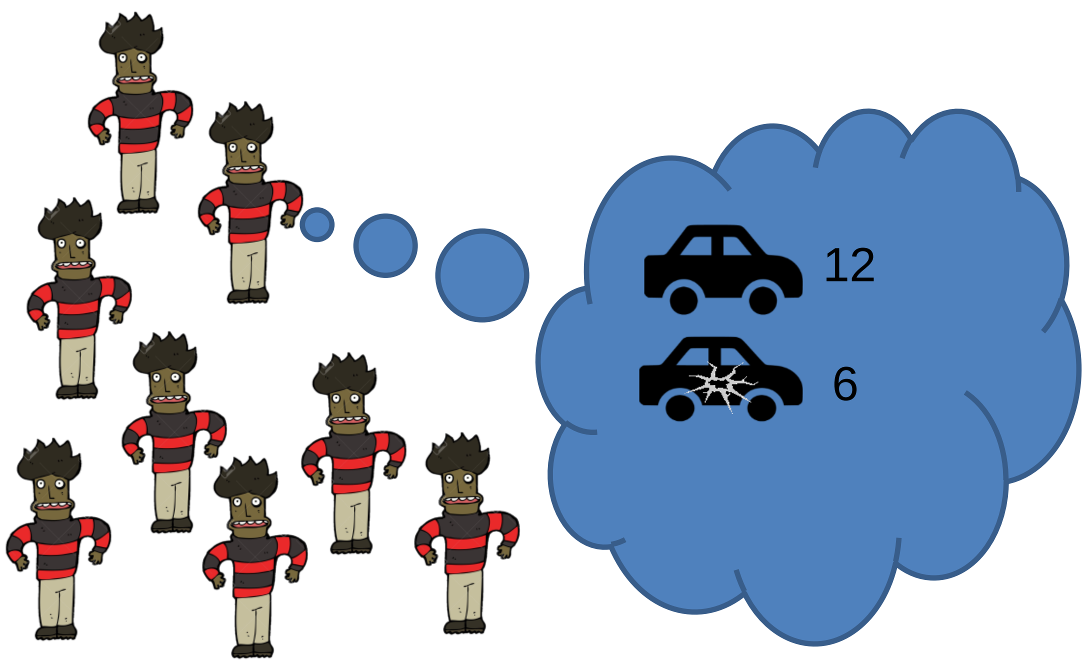




Example [Ackerlof '70]

- $v(buy) = 6 + 6g'$
- Equilibrium: g' and p s.t. buyers & sellers are satisfied





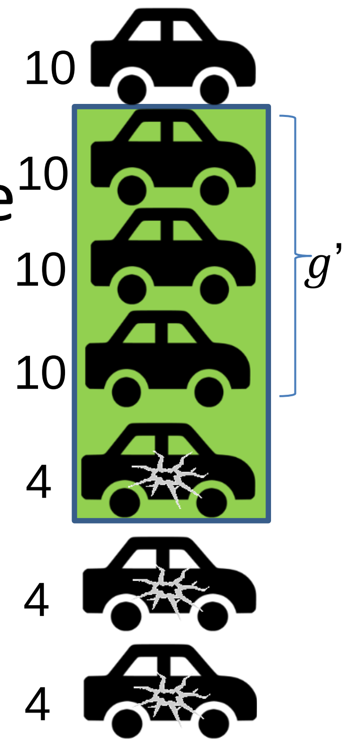
Example [Ackerlof ‘70]

- $v(buy) = 6 + 6g'$
- Equilibrium: $g'$and p s.t. buyers & sellers are satisfied
- Try 1: can $g' = g$ be an equilibrium (all good cars in the market)?
  - Buyers willing to pay $6 + 6g$
  - “Good” sellers require 10
  - In equilibrium $6 + 6g \geq 10$
  - Thus $g \geq \frac{2}{3}$. Otherwise not an equilibrium

Example [Ackerlof '70]

- $v(buy) = 6 + 6g'$
- Equilibrium: g' and p s.t. buyers & sellers are satisfied

- Try 2: Suppose $g = \frac{4}{7} < \frac{2}{3}$. What is g' ?
  - If $g' = \frac{3}{4} > g = \frac{4}{7}$, more bad sellers enter the market



Example [Ackerlof '70]

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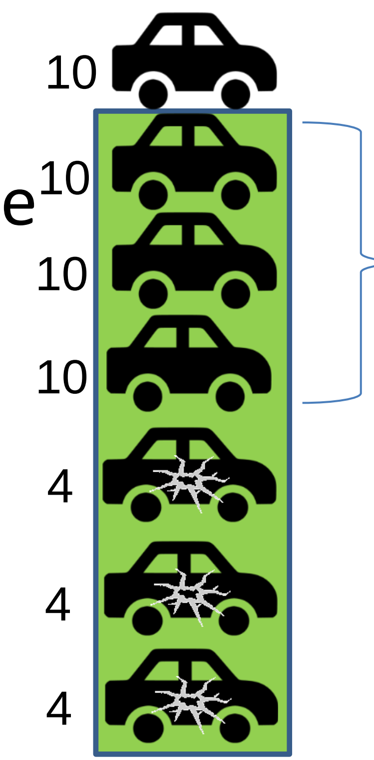


Example [Ackerlof '70]

- $v(buy) = 6 + 6g'$
- Equilibrium: g' and p s.t. buyers & sellers are satisfied
- Try 2: Suppose $g = \frac{4}{7} < \frac{2}{3}$. What is g' ?
  - If $g' = \frac{3}{4} > g = \frac{4}{7}$, more bad sellers enter the market
  - Thus $g' \leq g$ (recall we have $g < \frac{2}{3}$)
  - $v(buy) < 6 + 6 \cdot 2/3 = 10$
  - No good seller stays in the market
  - $g' = 0$ is the unique equilibrium!

The market for lemons

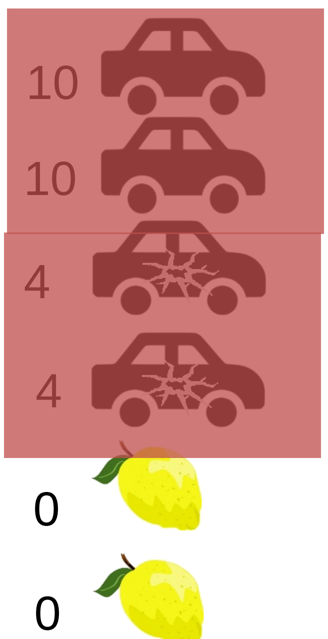
- The used cars example shows a suboptimal market, but one that still works to some extent
  - Both buyers and sellers get some value
- Ackerlof shows how to reach a complete failure by modifying the example
  - In addition to good and bad cars, there are "lemons"
  - A lemon is worthless to everyone

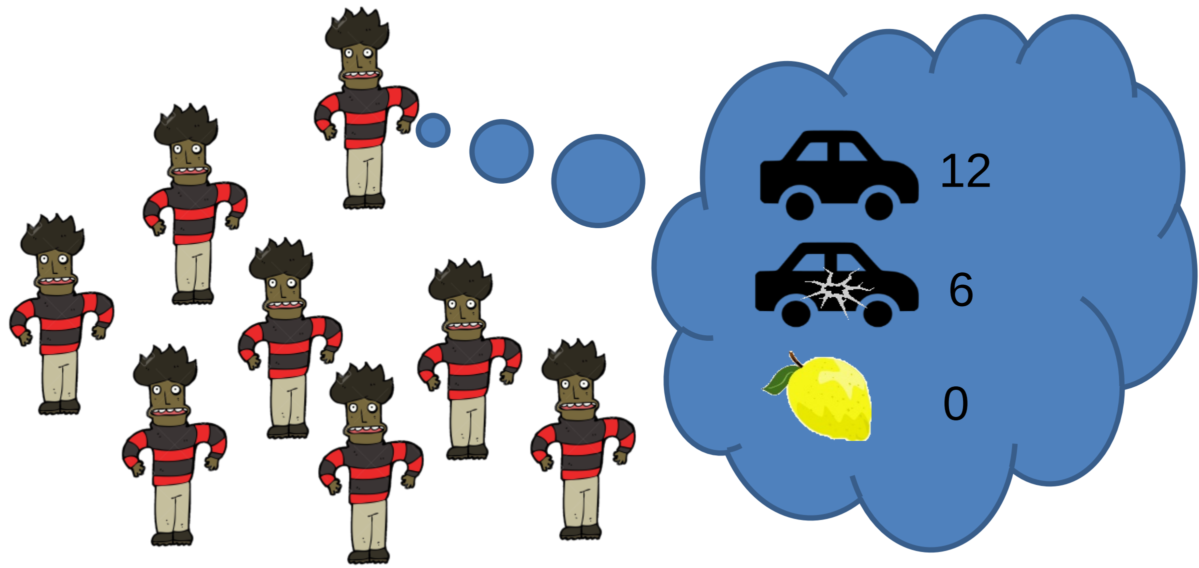
The market for lemons

- The used cars example shows a suboptimal market, but one that still works to some extent
  - Both buyers and sellers get some value
- Ackerlof shows how to reach a complete failure by modifying the example
  - In addition to good and bad cars, there are “lemons”
  - A lemon is worthless to everyone
  - Next example: Each type is $\frac{1}{3}$ of the market

Example 2 [Ackerlof '70]

- <u>Claim</u>: in equilibrium, only lemons are sold.
- $v(buy) = 6b' + 12g' \leq 2 + 4 = 6$
- No good sellers enter. Thus $g' = 0$, $v(buy) \leq 2$
- No bad seller enters either. Only lemons remain.





What happened?

- Due to **information asymmetry**, the market cannot differentiate products
- One price for all
- Price too low for some goods, so market gradually collapsed
- Happens in other markets:
  - Labor market
  - Dating “market”
  - Stock market

Signaling

- Who “suffers” mostly from information asymmetry?
  - The owners of good cars, high skills, good health, etc.
- Need to convince the other side (buyers, employers) that they are “better” than others
- Try to provide signals for high quality
  - Used cars: certificates from mechanics, test drive, and so on.
  - Dating sites: list esoteric interests
  - Labor markets: acquire academic degree ☺

Wait. What?

- Is academic degree just a “signal” for employers?
- What about all the important stuff you learn?
- Regardless of their “actual” worth, academic degrees can help the market reach a better allocation
  - This comes at a cost (“waste” 3-10 years in school)
- Getting a degree is easier for talented people
  - Talented people more likely to invest in getting a degree
  - Hence people with a degree are (on average) more talented
  - The “price” for people with a degree will be higher

Signaling mechanisms on the web

- certifications

- Online reputation systems

Detailed Seller Ratings (last 12 months) ?

Criteria	Average rating	Number of ratings
Item as described	★★★★★	63438
Communication	★★★★★	66595
Dispatch time	★★★★★	63305
Postage and packaging charges	★★★★★	66465

- “real” ad quality

Bangkok Hotel
460 Luxury Suites room in Bangkok Special rates online Booking Now!
www.WindsorSuitesHotel.com

Thaifood.uk

Liked Thailand? You will also like out Thai food!

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- Kahneman, Daniel, and Amos Tversky. "Prospect theory: An analysis of decision under risk." *Econometrica: Journal of the Econometric Society* (1979): 263-291.

CMSC 858G: Bandits, Experts and Games 09/12/16

Lecture 2: Multi-arm Bandits with i.i.d Rewards

Instructor: Alex Slivkins Scribed by: Amr Sharaf, Liqian Zhang

1 Administrivia

In the literature of bandits, experts, and games, there are hundreds of papers, and it will be impossible to cover everything in details. Thus, this course will cover some of the most interesting aspects of the field. We will cover one line of work every lecture, trying to cover the essential concepts, giving higher priority for more intiuitive results rather than complicated techniques.

2 Introduction

Last lecture we had a probability theory recap, including concentration of measure inequalities. Today, we're going to cover the basic model of multi-arm bandits with i.i.d rewards. We will introduce several techniques for solving the problem, and analyze there performance in terms of regret guarantees.

3 Multi-arm Bandits: Mathematical Model

In the multi-arm bandit problem with iid rewards, the learner selects an arm $a \in \mathcal{A}$ at every time step t. Learning proceeds in rounds, and we assume that the number of rounds is fixed, and indexed by $t = 1 \cdots T$. At each round, the algorithm chooses one action a_t (we'll use the arms and actions interchangeably to mean the same thing). After taking the action, a reward for this action is realized and observed by the algorithm. The process is repeated until the end of time horizon T is reached. The goal of the algorithm is to gather as much commulative reward as possible.

It's important to emphasize that the algorithm observes only the reward for the selected action, not all the other actions that could have been selected, that is why it is called a bandit feedback setting.

There is also the iid assumption, where the reward for each action is assumed be i.i.d (independent and identically distributed). More precisely, for each action a, there is a distribution $\mathcal{D}_a$ over real numbers (called "reward distribution"); for simplicity, all rewards will be in the interval $[0, 1]$. Every time this action is chosen, the reward is sampled independently from this distribution. It is crucial that this distribution is unknown to the algorithm, and does not change over time.

Notation: the mean (expected) reward of action a is denoted $\mu(a)$; the time horizon is T; the number of arms is K. Actions are denoted a. Let's use these conventions throughout the course.

Perhaps the simplest reward distribution is the Bernoulli distribution, when the reward of each arm a can be either 0 or 1. ("Success or failure", "heads or tails".) This distribution is parameterized by the probability of success $\mu(a)$, which also defines the mean of the distribution. Note that the problem instance is then completely described by the time horizon T and the vector $(\mu(a) : a \in \mathcal{A})$.

3.1 Some (stylized) examples for MAB with IID rewards

1. **News**: in a very stylized news application, a user visits a news site, the site presents it with a header for an article, and a user either clicks on this header or not. The goal of the website is to maximize #clicks. So each possible header is an arm in a bandit problem, and clicks are the rewards. Note that the rewards are 0-1.

   A typical modeling assumption is that each user is drawn independently from a fixed distribution over users, so that in each round the click happens independently with a probability that depends only on the chosen header.

2. **Selecting Ads**: In website advertising, a user visits a webpage, and a learning algorithm has to select one of many possible ads to display. In this setting, each ad can be considered an arm. The ad is displayed, it is observed whether or not the user clicks on the ad. If the ad a is clicked, the advertiser pays some amount $v_a \in [0, 1]$ which is fixed in advance and known to the algorithm. (This amount depends only on the chosen ad, but does not change over time.) So the paid amount is considered as the observed reward.

   Note that here rewards can be arbitrary numbers, but for each arm a, the reward can take only two values: 0 or v_a.

3. **Medical Trials:** a patient visits a doctor and the doctor can proscribe one of several possible treatments, and observes the treatment effectiveness. Then the next patient arrives, and so forth. For simplicity of this example, the effectiveness of a treatment is quantified as a number in $[0, 1]$. So here each treatment can be considered as an arm, and the reward is defined as the treatment effectiveness.

   Note that for each arm, the reward can, in principle, take arbitrarily many different values.

3.2 Regret

While algorithm's goal is to maximize reward, we have to be more precise about defining what it means. There are several notions for defining reward maximization. One standard notion is **regret**; we will use this notion (or versions thereof) throughout most of this course.

To define regret, we will look at two quantities: the reward accumulated by the best arm, and the average reward accumulated by the algorithm, we define the difference between the two quantities to be the regret:

$$R(t) = \mu^* \times t - \sum_{s=1}^{t} \mu(a_s), \tag{1}$$

where $R(t)$ is the regret after t-time steps, and $\mu^* = \max_{a \in \mathcal{A}} \mu(a)$ is the expected reward for the best arm. Note that the arm a_t chosen by the algorithm is a random quantity, as it may depend on the (random) rewards and also possibly on the internal randomness of the algorithm. Hence, we will typically talk about "expected regret" $\mathbb{E}[R(T)]$.

So why is it called regret? Because it is how much the algorithm "regrets" not knowing what is the best arm!

Remark 3.1. One can view the $\mu^* \times t$ term as a 'benchmark' to which the performance of the algorithm is compared. Hence, it is sometimes called the "best arm" benchmark. While it is the standard benchmark in most work on bandits, in some settings other benchmarks make sense (and sometimes make *more* sense).

Remark 3.2 (Terminology). Since our definition of regret sums over all rounds, we will sometimes call it *cumulative* regret. When/if we need to highlight the distinction between $R(T)$ and $\mathbb{E}[R(T)]$, we will say *realized regret* and *expected regret*; but most of the time, we will just say "regret" and the meaning will be clear from the context. The quantity $\mathbb{E}[R(T)]$ is sometimes called *pseudo-regret* in the literature.

Regret can depend on many different parameters, we care mainly about the dependence on the time horizon, number of arms, and the average reward for each arm (μ). We will usually use big-O notation to focus on the growth rate dependence on the different parameters instead of keeping track of all the constants.

4 Multi-arm bandit Algorithms

In this section, we cover several algorithms for solving the multi-arm bandit problem. For the simplicity of presentation, we start with the simple case where we have only two arms with zero/one rewards. Later, we extend the algorithms for more than two arms with bounded rewards.

4.1 Explore-First

```
1 Exploration phase: try each arm N times;
2 Select the arm a* with the highest average reward (break ties arbitrarily);
3 Exploitation phase: play arm a* in all remaining rounds.
```

Algorithm 1: Explore-First

Algorithm 1 describes the explore-first algorithm: we explore for N time steps, and then select the arm with the highest average reward. The parameter N is fixed in advance; it will be chosen later in the analysis as function of the time horizon and #arms. In the remainder of this subsection, we analyze the algorithm in terms of regret guarantees.

Let the average reward for each action a after exploration phase be denoted $\bar{\mu}(a)$. We want the average reward to be a good estimate of the true expected rewards, i.e. the following quantity should be small: $|\bar{\mu}(a) - \mu(a)|$. We can use the Hoeffding inequality from last lecture to quantify the deviation of the average from the true expectation. By defining the confidence radius $r(a) = \sqrt{\frac{2\log T}{N}}$, and using Hoeffding inequality, we get:

$$\Pr\{|\bar{\mu}(a) - \mu(a)| \le r(a)\} \ge 1 - \frac{1}{T^4} \tag{2}$$

So, the probability that the average will deviate from the true expectation is very small.

We define the *clean event* to be the event that (2) holds for both arms simultaneously. We will argue separately the clean event, and the "bad event" – the complement of the clean event.

Remark 4.1. With this approach, one does not need to worry about probability in the rest of the proof. Indeed, the probability has been taken care of by defining the clean event and observing that (2) holds! And we do not need to worry about the bad event either — essentially, because its probability is so tiny.

We will use this "clean event" approach in many other proofs, to help simplify the technical details. The downside is that it usually leads to worse constants that can be obtained with a more technical proof that argues about probabilities more carefully.

Let us start with the clean event. We will show that if we chose the worse arm, it is not so bad because the expected rewards for the two arms would be close.

Let the best arm be a^*, and suppose we choose the other arm $a \neq a^*$. But why did we choose arm a? This must have been because its average reward was better than that of a^*; in other words, $\bar{\mu}(a) > \bar{\mu}(a^*)$. Since this is a clean event, we have:

$$\mu(a) + r(a) \geq \bar{\mu}(a) > \bar{\mu}(a^*) \geq \mu(a^*) - r(a^*)$$

Re-arranging the terms, it follows that

$$\mu(a^*) - \mu(a) \leq r(a) + r(a^*) = O\left(\sqrt{\frac{\log T}{N}}\right).$$

Thus, each round in the exploitation phase contributes at most $O\left(\sqrt{\frac{\log T}{N}}\right)$ to regret. And each round in exploration trivially contributes at most 1. So we can derive an upper bound on the regret. This regret bound consists of two parts: for the first N rounds of exploration, and then for the remaining $T - 2N$ rounds of exploitation.

$$\begin{aligned} R(T) &\leq N + O(\sqrt{\frac{\log T}{N}} \times (T - 2N)) \\ &\leq N + O(\sqrt{\frac{\log T}{N}} \times T). \end{aligned}$$

Since we can select any value for N (as long as it is known to the algorithm before the first round), we can optimize the right-hand side to get the tightest upper bound. Noting that the two summands are, resp., monotonically increasing and monotonically decreasing in N, we set N so that they are (approximately) equal. For $N = T^{2/3}$, we get the following:

$$\begin{aligned} R(T) &\leq T^{2/3} + O(\sqrt{\frac{\log T}{T^{2/3}}} \times T) \\ &\leq O(\sqrt{\log T} \times T^{2/3}). \end{aligned}$$

To complete the proof, we have to analyze the bad event case. Since regret can be at most T (because each round contributes at most 1), and the bad event happens with a very small probability ($1/T^4$), the (expected) regret from this case can be neglected. Formally,

$$\begin{aligned} \mathbb{E}[R(T)] &= \mathbb{E}[R(T)|\text{clean event}] \times \Pr[\text{clean event}] \; + \; \mathbb{E}[R(T)|\text{bad event}] \times \Pr[\text{bad event}] && (3) \\ &\leq \mathbb{E}[R(T)|\text{clean event}] + T \times O(T^{-4}) && (4) \\ &\leq O(\sqrt{\log T} \times T^{2/3}). && (5) \end{aligned}$$

This completes the proof for $K = 2$ arms.

For $K > 2$ arms, we have to apply the union bound for (2) over the K arms, and then follow the same argument as above. Note that the value of T is greater than K, since we need to explore each arm at least once. For the final regret computation, we will need to take into account the dependence on K: specifically, the confidence radius is now $r(a) = \sqrt{\frac{2 \log T}{N/K}}$. Working through the proof, we obtain $R(T) \leq N + O(\sqrt{\frac{\log T}{N/K}} \times T)$. Plugging in $N = T^{2/3} \times O(K \log T)^{1/3}$, and completing the proof same way as in (3), we obtain:

Theorem 4.2. *Explore-first algorithm achieves regret* $\mathbb{E}[R(T)] \leq T^{2/3} \times O(K \log T)^{1/3}$, *where* K *is the number of arms.*

4.2 Epsilon Greedy

One problem with Explore-first is that the "losses" are concentrated in the initial exploration phase. It may be better to have a more uniform exploration over time. This is done in the epsilon-greedy algorithm.

input : Exploration probability ϵ

Toss a coin with probability of success $= \epsilon$;

if *success* **then**

  explore: choose an arm uniformly at random

else

  exploit: choose the arm with the highest average reward so far

Algorithm 2: Epsilon-Greedy

Note that the exploration is uniform, which is similar to the "round-robin" exploration in explore-first. Choosing the best option in the short term is often called the "greedy" choice in the computer science literature, hence the name "epsilon-greedy".

The analysis for this algorithm may appear on the homework.

Both exploration-first and epsilon-greedy have a big flaw that the exploration schedule does not depend on the history of the observed rewards. Whereas it is usually better to *adapt* exploration to the observed rewards. Informally, we refer to this distinction as *adaptive* vs *non-adaptive* exploration. In the remainder of this class, we will talk about two algorithms that implement adaptive exploration and achieve better regret.

CS224W: Social and Information Network Analysis — Fall 2014

Handout: Influence Maximization

The study of social processes by which ideas and innovations diffuse through social networks has been ongoing for more than half a century and as a result a fair understanding of such processes has been achieved. Modern models of social influence have been augmented with various features allowing for arbitrary network structure, non-uniform interactions, probabilistic events and other aspects. This handout will expose you to the basic stochastic model of social influence, *i.e.*, the *Independent Cascade Model (ICM)*, and show how it can be used to find an influential set of nodes to target in order to maximize the final adoption, *i.e.*, the *Influence Maximization* problem.

1 Independent Cascade Model (ICM)

The ICM was introduced by Goldenberg et.al in 2001 to model the dynamics of viral marketing and is inspired from the field of interacting particle systems. In this model, we start with an initial set S of active individuals. Each active individual u has a single chance to activate each non-active neighbour v of his/her. However, the process of activation is deemed stochastic and succeeds with probability $p_{u,v}$ independently for each attempt. Therefore, from an initial population of active individuals the activation process spreads in a cascading manner as newly activated individuals may activate new nodes that either previous attempts failed to activate or were not before accessible.

To make things more precise and to enable mathematical treatment of the model, we are going to adopt an alternative view of the model utilizing the notion of reachability.

Definition 1 (Reachability) *Given a graph $G = (V, E)$ and a node u, define X_u^E the set of reachable nodes of V from u through the edges in E (including u).*

There is an elegant interpretation of the ICM, in terms of the reachability of nodes via paths from the initial active set S. We can picture the process of a node u activating one of his neighbours v with probability $p_{u,v}$, as flipping a biased coin and if it succeeds declare the edge *live*, otherwise declare it *blocked*. Moreover, we can without loss of generality use the *principle of deferred decision* and consider that all the coins are tossed before the process begins. Therefore, from the initial graph $G(V, E)$, we get a graph $G(V, E_{live})$ where we keep only live edges. Now, in this setting all nodes that are reachable via a live path from the initial set S would become active when the cascade process quiesced. This view is very helpful and will be used to prove a crucial property about our model.

Definition 2 (ICM) *Given a graph $G = (V, E)$ and edge probabilities $\{p_e\}_{e\in E}$, consider $\{U_e\}_{e\in E}$ independent uniform $[0, 1]$ random variables. Define the random set of active edges as $I = \{e \in E : p_e \leq U_e\}$. The Independent Cascade Model for the graph G and probabilities p defines for every initial set of active nodes S, the final set A of active nodes as $A_I(S) = \cup_{u\in S} X_u^I$.*

We can think of X_u^I as the influence set of node u under random realization of edge activations I (where I is a random variable). From here on we will assume implicitly that the graph G and probabilities $\{p_e\}_{e\in E}$ are given.

2 Influence Maximization

Our end goal is to use the knowledge of the interactions to find a set of influential nodes. In order to quantify the goodness of the initial set, the stochastic nature of the ICM necessitates the use of expectations.

Definition 3 (Total Influence) *The total influence function for the ICM is* $\sigma(S) = \mathbb{E}[|A_I(S)|]$

The problem, therefore, is given a social network, *i.e.*, a set of nodes (individuals) and the edges(interactions) between them, to select the optimal "seed" of individuals to influence so that after the activation process terminates the expected number of active nodes is maximal for a seed of size k.

Definition 4 (Influence Maximization) *Given a graph* G *with probabilities* $\{p_e\}_{e\in E}$ *and an integer* k*, the Influence Maximization problem asks for the set* S *of cardinality* k *such that* $\sigma(S)$ *is maximized.*

Theorem 1 *The Influence Maximization Problem is NP-Complete.*

Proof:(Sketch) We prove the statement through a reduction of *Set Cover*. In the Set Cover we are given a "universe" of n elements U, a collection of sets $X_1, \ldots, X_m \subset U$ and an integer k. The decision problem is whether we can select k sets out of the collection such that their union equals U (that is, "covers" U). Given such an instance of Set Cover, we show that we can construct an instance of Influence Maximization such that its solution will imply a solution to the original problem. That means we need to provide a directed graph $G = (V, E)$ and probabilities $\{p_e\}_{e\in E}$. The vertex set V consists of U along with a separate vertex v_i for each set X_i. The edge set includes only the directed edges pointing from v_i to the elements of V corresponding to the elements in X_i. We set all the probabilities of the edges equal to 1. Since, vertices corresponding to elements of U do not influence other vertices, and any vertex v_i would immediately activate the vertices corresponding to X_i, solving the Influence Maximization problem with cardinality k would also tell us whether the universe U can be covered by k sets out of $X_1, \ldots, X_m$. On the other hand the decision version of Influence Maximization obviously belongs to NP as it is possible (but non-trivial) to compute the total influence function for the optimal solution. ■

3 Submodularity and ICM

A crucial property satisfied by the ICM, that will sidestep the hardness result and enable the algorithmic treatment of Influence Maximization, is that of submodularity.

Definition 5 (Submodularity) *A set function* $f : 2^V \to \mathbb{R}$ *is called* submodular *if for all subsets* $S \subseteq T \subseteq V$ *and* $u \in V$ *the following inequality holds:*

$$f(S \cap \{u\}) - f(S) \geq f(T \cup \{u\}) - f(T) \tag{1}$$

Intuitively, submodularity is the set-function analog of concavity. Specifically, a function is called submodular if it satisfies the "diminishing returns" property: the marginal gain by adding an element to a set S it is at least as the marginal gain by adding an element to the superset T. In other words, the higher the ground value is, the smaller is the marginal gain of adding one element. The following property of submodular function is useful in proving that the total influence function is submodular.

Lemma 1 (Conic combinations) *Let $c_1, \dots, c_n \geq 0$ be non-negative numbers and $f_1, \dots, f_n : 2^V \to \mathbb{R}$ be submodular functions, then $\tilde{f} = \sum_{i=1}^n c_i f_i$ is a submodular function.*

Proof: Let $S \subseteq T$ be subsets of V, then for every $u \in V$ we have:

$$\begin{aligned}
\tilde{f}(S \cup \{u\}) - \tilde{f}(S) &= \sum_{i=1}^n c_i \left[f_i(S \cup \{u\}) - f_i(S)\right] \\
&\geq \sum_{i=1}^n c_i \left[f_i(T \cup \{u\}) - f_i(T)\right] \\
&= \tilde{f}(T \cup \{u\}) - \tilde{f}(T)
\end{aligned}$$

where in the middle inequality we used submodularity of the functions f_i and positivity of the coefficients c_i. ■

Theorem 2 *The total influence function $\sigma(S)$ is monotone and submodular.*

Proof: We start by writing out the expression for the total influence. We have:

$$\sigma(S) = \mathbb{E}[|A_I(S)|] = \sum_{i \subseteq E} \mathbb{P}(I = i) \cdot |A_i(S)| \tag{2}$$

where $\mathbb{P}(I = i)$ is the probability according to the ICM that the set of active edges I is $i \subseteq E$. Since probabilities are non-negative, if we could show that $f_i(S) = |A_i(S)|$ is a submodular function, invoking Lemma 1 would complete the proof. Let $S \subseteq T \subseteq V$ and $u \in V$, then:

$$\begin{aligned}
f_i(S \cup \{u\}) - f_i(S) &= \left|A_i(S) \cup X_u^i\right| - |A_i(S)| \\
&= |X_u^i| - \left|A_i(S) \cap X_u^i\right| && (3) \\
&\geq |X_u^i| - \left|A_i(T) \cap X_u^i\right| && (4) \\
&= \left|A_i(T) \cup X_u^i\right| - |A_i(T)| && (5) \\
&= f_i(T \cup \{u\}) - f_i(T)
\end{aligned}$$

where in (4) we used monotonicity of $A_i(S)$ and in (3) and (5) the fundamental property $|A \cup B| = |A| + |B| - |A \cap B|$. Thus we proved the defining inequality of submodularity for $f_i(S)$. ■

4 Hill Climbing Algorithm

Submodularity of the total influence function is a property that can be exploited algorithmically to obtain a good approximation to the Influence Maximization Problem. In particular, there is a hope

Algorithm 1 Hill Climbing Algorithm

Input: a graph $G = (V, E)$, probabilities $\{p_e\}_{e \in E}$ and an integer k.
Output: Initialize $S_0 = \emptyset$

for $i = 1$ to k **do**
  $s_i = \arg\max_{u \in V \setminus S_{i-1}} [\sigma(S_{i-1} \cup \{u\}) - \sigma(S_{i-1})]$
  $S_i = S_{i-1} \cup s_i$
end for

Output: the set S_k.

that locally optimal choices would result in good final spread. The following natural algorithm is particularly tailored for problems where submodularity and monotonicity of the objective function coincide.

Lemma 2 (Telescoping) *For a submodular f, any set A and $B = \{b_1, \ldots, b_k\}$ it holds that*

$$f(A \cup B) - f(A) \leq \sum_{i=1}^{k} [f(A \cup \{b_i\}) - f(A)] \tag{6}$$

Proof: Let $B_i = \{b_1, \ldots, b_i\}$ with $B_0 = \emptyset$ and $B_k = B$. We start by expressing the left hand side as a telescopic sum using the above sequence of sets:

$$\begin{aligned}
f(A \cup B_k) - f(A \cup B_0) &= f(A \cup B_k) - f(A \cup B_{k-1}) + \ldots + f(A \cup B_1) - f(A \cup B_0) \\
&= \sum_{i=1}^{k} [f(A \cup B_i) - f(A \cup B_{i-1})] \\
&= \sum_{i=1}^{k} [(f(A \cup B_{i-1}) \cup \{b_i\}) - f(A \cup B_{i-1})] \qquad (7) \\
&\leq \sum_{i=1}^{k} [f(A \cup \{b_i\}) - f(A)] \qquad (8)
\end{aligned}$$

where we used the fact that $B_i = B_{i-1} \cup \{b_i\}$ in (7) and submodularity of f in (8). ■

Definition 6 (Marginal Increments) *Given the set S_{i-1}, the marginal increment at step i is defined as $\delta_i = f(S_i) - f(S_{i-1}) = \max f(S_{i-1} \cup \{u\}) - f(S_{i-1})$.*

Lemma 3 (Accretion) *Let S_i be the set after i-steps of the HC algorithm and T be any other set of size k. Then:*

$$f(S_{i+1}) \geq \left(1 - \frac{1}{k}\right) f(S_i) + \frac{1}{k} f(T)$$

Proof: Since, the two sets S_i and T can in principle be arbitrarily different we are going to use our telescoping lemma to

$$\begin{aligned}
f(T) - f(S_i) &\leq f(S_i \cup T) - f(S_i) && (9)\\
&\leq \sum_{j=1}^{k} [f(S_i \cup \{t_j\}) - f(S_i)] && (10)\\
&\leq \sum_{j=1}^{k} \delta_{i+1} && (11)\\
&= k \cdot \delta_{i+1} && (12)
\end{aligned}$$

where in (9) we used monotonicity and in (10) we the greedy property of the HC algorithm. Now recalling that $\delta_{i+1} = f(S_{i+1}) - f(S_i)$ and substituting the last inequality for δ_{i+1} gives the required statement. ∎

Theorem 3 *The hill-climbing algorithm finds a set $\tilde{S}$ such that $\sigma(\tilde{S}) \geq (1 - \frac{1}{e})\sigma(S^*)$.*

Proof: To prove our theorem we are going to prove a stronger result, namely

$$f(S_i) \geq \left[1 - \left(1 - \frac{1}{k}\right)^i\right] f(T) \tag{13}$$

To that end we are going to employ induction on i. For $i = 0$, (13) trivially holds as $f(\emptyset) \geq 0$. Next, we carry out the inductive step:

$$\begin{aligned}
f(S_{i+1}) &\geq \left(1 - \frac{1}{k}\right) f(S_i) + \frac{1}{k} f(T) && (14)\\
&\geq \left(1 - \frac{1}{k}\right) \left[1 - \left(1 - \frac{1}{k}\right)^i\right] f(T) + \frac{1}{k} f(T) && (15)\\
&= \left[1 - \left(1 - \frac{1}{k}\right)^{i+1}\right] f(T) && (16)
\end{aligned}$$

where in (14) we used Lemma 3 and in (15) the inductive hypothesis. Using (13) for $i = k$ and $T = S^*$ (the optimal set of cardinality k) we get: $f(S_k) \geq \left[1 - \left(1 - \frac{1}{k}\right)^k\right] f(S^*)$. Since, the right hand side is decreasing in k, we have that always $f(S_k) \geq \lim_{k\to\infty} \left[1 - \left(1 - \frac{1}{k}\right)^k\right] \cdot f(S^*) = \left(1 - \frac{1}{e}\right) f(S^*)$ as $\lim_{x\to\infty}(1 - 1/x)^x = 1/e$. ∎