Notes and Commentary on Peskin & Schroeder's QFT

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1 Special Relativity

1.1 Galileo's Relativity and Einstein's Relativity

1.2 Lorentz Transformation

1.3 Four-Vectors and Minkowski Space

1.4 Electromagnetic Quantities

1.5 Covariant Electrodynamics

2 Invitation: Pair Production in e^+e^- Annihilation

2.1 Feynman Rules for QED

For any given set of spin orientations, it is conventional to write the differential cross section for our process, with the μ^-produced into a solid angle $d\Omega$, as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\mathrm{cm}}^2} \cdot |\mathcal{M}|^2 \tag{2.1}$$

The good news is that Feynman has invented a beautiful way to organize and visualize the perturbation series: the method of Feynman diagrams. Roughly speaking, the diagrams display the flow of electrons and photons during the scattering process.

For our particular calculation, the lowest-order term in the perturbation series can be represented by a single diagram, shown in Fig. 1.2.

We introduce the Feynman diagram rules for $e^+e^- \to \mu^+\mu^-$ scattering in Fig. 1.2.

The diagram is made up of three types of components:

1. external lines (representing the four incoming and outgoing particles),
2. internal lines (representing "virtual" particles, in this case one virtual photon), and
3. vertices.

It is conventional to use straight lines for fermions and wavy lines for photons. The arrows on the straight lines denote the direction of negative charge flow, not momentum. We assign a 4 -momentum vector to each external line, as shown.

In this diagram, the momentum q of the one internal line is determined by momentum conservation at either of the vertices: $q = p + p' = k + k'$. We must also associate a spin state (either "up" or "down") with each external fermion.

The rules assign a short algebraic factor to each element of a diagram, and the product of these factors gives the value of the corresponding term in the perturbation series.

2.2 The Calculation of Cross Section $\sigma(e^+e^- \to \mu^+\mu^-)$

We then will use some heuristic arguments instead of the actual Feynman rules. In our case the initial state is $|e^+e^-\rangle$and the final state is $\langle\mu^+\mu^-|$. But our interaction Hamiltonian couples electrons to muons only through the electromagnetic field (that is, photons), not directly. So the first-order result (1.2) vanishes, and we must go to the second-order expression.

$$\mathcal{M} \sim \left\langle \mu^+\mu^- \right| H_I |\gamma\rangle^\mu \langle\gamma| H_I \left| e^+e^- \right\rangle_\mu \tag{2.2}$$

The external electron lines correspond to the factor $|e^+e^-\rangle$; the external muon lines correspond to $\langle\mu^+\mu^-|$. The vertices correspond to H_I, and the internal photon line corresponds to the operator $|\gamma\rangle\langle\gamma|$. We have added vector indices (μ) because the photon is a vector particle with four components. There are four possible intermediate states, one for each component, and according to the rules of perturbation theory we must sum over intermediate states. Note that since the sum in (2.2) takes the form of a 4 -vector dot product, the amplitude $\mathcal{M}$ will be a Lorentz-invariant scalar as long as each half of (2.2) is a 4-vector.

Because the matrix element should be proportional to e,considering the possibility of spin orientations, we have four results for $\mathcal{M}$:

$$\begin{aligned}
\mathcal{M}(RL \rightarrow RL) &= -e^2(1+\cos\theta) \\
\mathcal{M}(RL \rightarrow LR) &= -e^2(1-\cos\theta) \\
\mathcal{M}(LR \rightarrow RL) &= -e^2(1-\cos\theta) \\
\mathcal{M}(LR \rightarrow LR) &= -e^2(1+\cos\theta)
\end{aligned} \tag{2.3}$$

and

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\mathrm{cm}}^2}\left(1+\cos^2\theta\right) \tag{2.4}$$

where $\alpha = e^2/4\pi \simeq 1/137$. Integrating over the angular variables θ and ϕ gives the total cross section,

$$\sigma_{\text{total}} = \frac{4\pi\alpha^2}{3E_{\mathrm{cm}}^2} \tag{2.5}$$

Some Commentary on Dirac Spinors and Gamma Matrices!!!!

2.3 Embellishments and Questions

For the internal photon line we write $-ig_{\mu\nu}/q^2$, where $g_{\mu\nu}$ is the usual Minkowski metric tensor and q is the 4 -momentum of the virtual photon. This factor corresponds to the operator $|\gamma\rangle\langle\gamma|$ in our heuristic expression (2.2).

For each vertex we write $-ie\gamma^\mu$, corresponding to H_I in (2.2). The objects γ^μ are a set of four 4×4 constant matrices. They do the "addition of angular momentum" for us, coupling a state of two spin $-1/2$ particles to a vector particle.

The external lines carry expressions for four-component column-spinors u, v, or row-spinors $\bar{u}, \bar{v}$. These are essentially the momentum-space wavefunctions of the initial and final particles, and correspond to $|e^+e^-\rangle$and $\langle\mu^+\mu^-|$ in (2.2). The indices s, s', r, and r' denote the spin state, either up or down. We can now write down an expression for $\mathcal{M}$, reading everything straight off the diagram:

$$\begin{aligned}
\mathcal{M} &= \bar{v}^{s'}(p')\left(-ie\gamma^\mu\right)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)\left(-ie\gamma^\nu\right)v^{r'}(k') \\
&= \frac{ie^2}{q^2}\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right)
\end{aligned} \tag{2.6}$$

It is instructive to compare this in detail with Eq. (2.2).

3 The Klein-Gordon Field

3.1 The Necessity of Field Viewpoint

In nonrelativistic quantum mechanics we have $E = \mathbf{p}^2/2m$, so

$$\begin{aligned}
U(t) &= \langle\mathbf{x}|e^{-i\left(\mathbf{p}^2/2m\right)t}|\mathbf{x}_0\rangle \\
&= \int\frac{d^3p}{(2\pi)^3}\langle\mathbf{x}|e^{-i\left(\mathbf{p}^2/2m\right)t}|\mathbf{p}\rangle\langle\mathbf{p}\mid\mathbf{x}_0\rangle \\
&= \frac{1}{(2\pi)^3}\int d^3p e^{-i\left(\mathbf{p}^2/2m\right)t}\cdot e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \\
&= \left(\frac{m}{2\pi it}\right)^{3/2}e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t}
\end{aligned} \tag{3.1}$$

This expression is nonzero for all x and t, indicating that a particle can propagate between any two points in an arbitrarily short time.

Quantum field theory provides a natural way to handle not only multiparticle states, but also transitions between states of different particle number. It solves the causality problem by introducing antiparticles, then goes on to explain the relation between spin and statistics. But most important, it provides the tools necessary to calculate innumerable scattering cross sections, particle lifetimes, and other observable quantities.

3.2 The Classical Field Theory

3.2.1 Lagrangian and Hamiltonian Field Theory

We have some essential equations. As a simple example, consider the theory of a single field $\phi(x)$, governed by the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2.\end{aligned}$$

For now we take ϕ to be a real-valued field. The quantity m will be interpreted as a mass in Section 2.3, but for now just think of it as a parameter. From this Lagrangian the usual procedure gives the equation of motion

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0 \quad \text{or} \quad \left(\partial^\mu\partial_\mu + m^2\right)\phi = 0,$$

which is the well-known Klein-Gordon equation. (In this context it is a classical field equation, like Maxwell's equations-not a quantum-mechanical wave equation.) Noting that the canonical momentum density conjugate to $\phi(x)$ is $\pi(x) = \dot{\phi}(x)$, we can also construct the Hamiltonian:

$$H = \int d^3x\mathcal{H} = \int d^3x\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$$

3.2.2 Noether's Theorem

3 examples of Noether's theorem!!!!

We have some results about the energy-momentum tensor and conserved quantities.

The Lagrangian is also a scalar, so it must transform in the same way:

$$\mathcal{L} \to \mathcal{L} + a^\mu\partial_\mu\mathcal{L} = \mathcal{L} + a^\nu\partial_\mu\left(\delta^\mu{}_\nu\mathcal{L}\right).$$

Comparing this equation to (2.10), we see that we now have a nonzero $\mathcal{J}^\mu$. Taking this into account, we can apply the theorem to obtain four separately conserved currents:

$$T^\mu{}_\nu \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\nu\phi - \mathcal{L}\delta^\mu{}_\nu$$

This is precisely the stress-energy tensor, also called the energy-momentum tensor, of the field ϕ. The conserved charge associated with time translations is the Hamiltonian:

$$H = \int T^{00}d^3x = \int\mathcal{H}d^3x$$

By computing this quantity for the Klein-Gordon field, one can recover the result (2.8). The conserved charges associated with spatial translations are

$$P^i = \int T^{0i}d^3x = -\int\pi\partial_i\phi d^3x$$

and we naturally interpret this as the (physical) momentum carried by the field (not to be confused with the canonical momentum).

3.3 The Klein-Gordon Field an Harmonic Oscillators

3.3.1 Quantization of the Real Klein-Gordon Field

We firsr introduce the canonical commutation relations for the field $\phi(\mathbf{x},t)$ and its conjugate momentum $\pi(\mathbf{x},t)$ at equal times:

$$\begin{aligned}[\phi(\mathbf{x}),\pi(\mathbf{y})] &= i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\ [\phi(\mathbf{x}),\phi(\mathbf{y})] &= [\pi(\mathbf{x}),\pi(\mathbf{y})] = 0\end{aligned} \tag{3.2}$$

Let us seek guidance by writing the Klein-Gordon equation in Fourier space. If we expand the classical Klein-Gordon field as

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p},t)$$

(with $\phi^*(\mathbf{p}) = \phi(-\mathbf{p})$ so that $\phi(\mathbf{x})$ is real), the Klein-Gordon equation (2.7) becomes

$$\left[\frac{\partial^2}{\partial t^2} + \left(|\mathbf{p}|^2 + m^2\right)\right]\phi(\mathbf{p},t) = 0.$$

This is the same as the equation of motion for a simple harmonic oscillator with frequency

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}.$$

We can find the spectrum of the Klein-Gordon Hamiltonian using the same trick, but now each Fourier mode of the field is treated as an independent oscillator with its own a and $a^\dagger$. In analogy with (2.23) we write

$$\begin{aligned}\phi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)e^{i\mathbf{p}\cdot\mathbf{x}}; \\ \pi(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right)e^{i\mathbf{p}\cdot\mathbf{x}}.\end{aligned} \tag{3.3}$$

Take attention for the meaning of $\pi(\mathbf{x})$!!!

We have new formulas for the commutation realation:

$$\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}\right] = (2\pi)^3\delta^{(3)}\left(\mathbf{p}-\mathbf{p}'\right), \tag{3.4}$$

We are now ready to express the Hamiltonian in terms of ladder operators. Starting from its expression (2.8) in terms of ϕ and π, we have

$$\begin{aligned}H = &\int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}\{-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4}\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p}'} - a^\dagger_{-\mathbf{p}'}\right) \\ &+ \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}\left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right)\left(a_{\mathbf{p}'} + a^\dagger_{-\mathbf{p}'}\right)\Big\} \\ = &\int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2}\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}\right]\right)\end{aligned}$$

Take attention to the second term in the last line, which arises from normal ordering the operators. It is an infinite constant!!!!

3.3.2 Interpretation of the Spectrum of $\mathcal{H}$

Having found the spectrum of the Hamiltonian, let us try to interpret its eigenstates. From

$$P^i = \int T^{0i}d^3x = -\int \pi\partial_i\phi d^3x \tag{3.5}$$

and a calculation similar to (2.31) we can write down the total momentum operator,

$$\mathbf{P}=-\int d^3x\pi(\mathbf{x})\nabla\phi(\mathbf{x})=\int\frac{d^3p}{(2\pi)^3}\mathbf{p}a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}$$

So the operator $a_{\mathbf{p}}^{\dagger}$ creates momentum $\mathbf{p}$ and energy $\omega_{\mathbf{p}}=\sqrt{|\mathbf{p}|^2+m^2}$. Similarly, the state $a_{\mathbf{p}}^{\dagger}a_{\mathbf{q}}^{\dagger}\cdots|0\rangle$ has momentum $\mathbf{p}+\mathbf{q}+\cdots$. It is quite natural to call these excitations particles, since they are discrete entities that have the proper relativistic energy-momentum relation. (By a particle we do not mean something that must be localized in space; $a_{\mathbf{p}}^{\dagger}$ creates particles in momentum eigenstates.) From now on we will refer to $\omega_{\mathbf{p}}$ as $E_{\mathbf{p}}$ (or simply E), since it really is the energy of a particle. Note, by the way, that the energy is always positive: $E_{\mathbf{p}}=+\sqrt{|\mathbf{p}|^2+m^2}$.

3.3.3 The normalization of $|\mathbf{p}\rangle$

We naturally choose to normalize the vacuum state so that $\langle 0\mid 0\rangle=1$. The one-particle states $|\mathbf{p}\rangle\propto a_{\mathbf{p}}^{\dagger}|0\rangle$ will also appear quite often, and it is worthwhile to adopt a convention for their normalization. The simplest normalization $\langle\mathbf{p}\mid\mathbf{q}\rangle=(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$ (which many books use) is not Lorentz invariant, as we can demonstrate by considering the effect of a boost in the 3 -direction. Under such a boost we have $p_3'=\gamma\left(p_3+\beta E\right), E'=\gamma\left(E+\beta p_3\right)$. Using the delta function identity

$$\delta\left(f(x)-f\left(x_0\right)\right)=\frac{1}{\left|f'\left(x_0\right)\right|}\delta\left(x-x_0\right)$$

We can compute that

$$\delta^{(3)}(\mathbf{p}-\mathbf{q})=\delta^{(3)}\left(\mathbf{p}'-\mathbf{q}'\right)\frac{E'}{E}. \tag{3.6}$$

We therefore define

$$|\mathbf{p}\rangle=\sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^{\dagger}|0\rangle, \tag{3.7}$$

so that

$$\langle\mathbf{p}\mid\mathbf{q}\rangle=2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{3.8}$$

By considering a Lorentz Transformation Λ which will be implemented as some unitary operator $U(\Lambda)$, we have a representation for 1-partial state:

$$(\mathbf{1})_{1-\text{ particle }}=\int\frac{d^3p}{(2\pi)^3}|\mathbf{p}\rangle\frac{1}{2E_{\mathbf{p}}}\langle\mathbf{p}|, \tag{3.9}$$

3.3.4 Interpretation of the state of $\phi(\mathbf{x})\,|0\rangle$

Finally let us consider the interpretation of the state $\phi(\mathbf{x})|0\rangle$. From the expansion (2.25) we see that

$$\phi(\mathbf{x})|0\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle$$

is a linear superposition of single-particle states that have well-defined momentum. Except for the factor $1/2E_{\mathbf{p}}$, this is the same as the familiar nonrelativistic expression for the eigenstate of position $|\mathbf{x}\rangle$; in fact the extra factor is nearly constant for small (nonrelativistic) $\mathbf{p}$. We will therefore put forward the same interpretation, and claim that the operator $\phi(\mathbf{x})$, acting on the vacuum, creates a particle at position $\mathbf{x}$. This interpretation is further confirmed when we compute

$$\begin{aligned}\langle 0|\phi(\mathbf{x})|\mathbf{p}\rangle&=\langle 0|\int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}'}}}\left(a_{\mathbf{p}'}e^{i\mathbf{p}'\cdot\mathbf{x}}+a_{\mathbf{p}'}^{\dagger}e^{-i\mathbf{p}'\cdot\mathbf{x}}\right)\sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^{\dagger}|0\rangle\\&=e^{i\mathbf{p}\cdot\mathbf{x}}\end{aligned}$$

We can interpret this as the position-space representation of the single-particle wavefunction of the state $|\mathbf{p}\rangle$, just as in nonrelativistic quantum mechanics $\langle\mathbf{x}\mid\mathbf{p}\rangle\propto e^{i\mathbf{p}\cdot\mathbf{x}}$ is the wavefunction of the state $|\mathbf{p}\rangle$.

3.4 The Klein-Gordon Field in Space-Time

3.4.1 The Dual Particle and Wave Interpretations of the Quantum Field $\phi(\mathbf{x})$

We can compute the Heisenberg-picture field operator $\phi(\mathbf{x}, t)$ by solving the Heisenberg equation of motion,

$$\begin{aligned} \phi(\mathbf{x}, t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right) \Big|_{p^0 = E_{\mathbf{p}}} ; \\ \pi(\mathbf{x}, t) &= \frac{\partial}{\partial t} \phi(\mathbf{x}, t). \end{aligned} \tag{3.10}$$

Equation (2.47) makes explicit the dual particle and wave interpretations of the quantum field $\phi(x)$.

On the one hand, $\phi(x)$ is written as a Hilbert space operator, which creates and destroys the particles that are the quanta of field excitation.

On the other hand, $\phi(x)$ is written as a linear combination of solutions ($e^{ip\cdot x}$ and $e^{-ip\cdot x}$) of the Klein-Gordon equation.

Both signs of the time dependence in the exponential appear: We find both $e^{-ip^0 t}$ and $e^{+ip^0 t}$, although p^0 is always positive. If these were single-particle wavefunctions, they would correspond to states of positive and negative energy; let us refer to them more generally as positive- and negative-frequency modes.

The connection between the particle creation operators and the waveforms displayed here is always valid for free quantum fields:

A positive-frequency solution of the field equation has as its coefficient the operator that destroys a particle in that single-particle wavefunction.

A negative-frequency solution of the field equation, being the Hermitian conjugate of a positive-frequency solution, has as its coefficient the operator that creates a particle in that positive-energy single-particle wavefunction.

In this way, the fact that relativistic wave equations have both positive- and negative-frequency solutions is reconciled with the requirement that a sensible quantum theory contain only positive excitation energies.

3.4.2 Calculation of Contour Integrals and Causality

The main point of this section is to calculate the propagator amplitude by contour integrals and to show that the commutator of two field operators vanishes for spacelike separations, thus preserving causality. First consider the case where the difference $x - y$ is purely in the timedirection: $x^0 - y^0 = t, \mathbf{x} - \mathbf{y} = 0$. (If the interval from y to x is timelike, there is always a frame in which this is the case.) Then we have

$$\begin{aligned} D(x-y) &= \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t} \\ &= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2 - m^2} e^{-iEt} \\ &\underset{t\to\infty}{\sim} e^{-imt} \end{aligned}$$

Next consider the case where $x - y$ is purely spatial: $x^0 - y^0 = 0, \mathbf{x} - \mathbf{y} = \mathbf{r}$. The amplitude is then

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{r}} \\ &= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_{\mathbf{p}}} \frac{e^{ipr} - e^{-ipr}}{ipr} \\ &= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}} \underset{r\to\infty}{\sim} e^{-mr} \end{aligned}$$

Specific details of contour integration need to be added here, along with Jordan's Lemma!!!!

Because of the calculation:

$$\begin{aligned}
[\phi(x), \phi(y)] =& \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \\
& \times \left[\left(a_{\mathbf{p}} e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger} e^{ip\cdot x} \right), \left(a_{\mathbf{q}} e^{-iq\cdot y} + a_{\mathbf{q}}^{\dagger} e^{iq\cdot y} \right) \right] \\
=& \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) \\
=& D(x-y) - D(y-x)
\end{aligned} \tag{3.11}$$

When $(x-y)^2 < 0$, we can perform a Lorentz transformation on the second term (since each term is separately Lorentz invariant), taking $(x-y) \to -(x-y)$, as shown in Fig. 2.4. The two terms are therefore equal and cancel to give zero; causality is preserved. Note that if $(x-y)^2 > 0$ there is no continuous Lorentz transformation that takes $(x-y) \longrightarrow -(x-y)$. In this case, by Eq. (2.51), the amplitude is (fortunately) nonzero, roughly ($e^{-imt} - e^{imt}$) for the special case $\mathbf{x} - \mathbf{y} = 0$. Thus we conclude that no measurement in the Klein-Gordon theory can affect another measurement outside the light-cone.

3.4.3 The Klein-Gordon Propagator

For $x^0 > y^0$ we can close the contour below, picking up both poles to obtain the previous line of (2.54). For $x^0 < y^0$ we may close the contour above, giving zero. Thus the last line of (2.54), together with the prescription for going around the poles, is an expression for what we will call

$$D_R(x-y) \equiv \theta\left(x^0 - y^0\right) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle.$$

To understand this quantity better, let's do another computation:

$$\begin{aligned}
\left(\partial^2 + m^2\right) D_R(x-y) =& \left(\partial^2 \theta\left(x^0 - y^0\right)\right) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \\
& + 2\left(\partial_\mu \theta\left(x^0 - y^0\right)\right) \left(\partial^\mu \langle 0 | [\phi(x), \phi(y)] | 0 \rangle\right) \\
& + \theta\left(x^0 - y^0\right) \left(\partial^2 + m^2\right) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \\
=& -\delta\left(x^0 - y^0\right) \langle 0 | [\pi(x), \phi(y)] | 0 \rangle \\
& + 2\delta\left(x^0 - y^0\right) \langle 0 | [\pi(x), \phi(y)] | 0 \rangle + 0 \\
=& -i\delta^{(4)}(x-y)
\end{aligned}$$

This says that $D_R(x-y)$ is a Green's function of the Klein-Gordon operator. Since it vanishes for $x^0 < y^0$, it is the retarded Green's function.

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3.4.4 Particle Creation by a Classical Source

3.4.5 Problems and Solutions