Companion Notes to Peskin & Schroeder's QFT

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January 5, 2026

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1 Introduction

These notes serve as a commentary on Peskin & Schroeder's textbook [1], supplementing the original text by providing the missing intermediate calculation steps. They are designed to be accessible to any graduate student with a foundation in classical mechanics and quantum mechanics.

The notes also include the authors personal perspectives. Given the author's background in mathematics, more rigorous mathematical proofs will be integrated into future versions. The exercise solutions provided in each section reference A Complete Solution to Problems in 'An Introduction to Quantum Field Theory' by Zhong-Zhi Xianyu (Harvard University).

The primary references for these notes are:

1. An Introduction to Quantum Field Theory by Peskin and Schroeder [1].
2. Modern Quantum Mechanics by Sakurai [2].
3. A Complete Solution to Problems in "An Introduction to Quantum Field Theory" by Zhong-Zhi Xianyu [3].
4. Modern Electrodynamics by Zangwill [4].

2 General Electromagnetic Fields

2.1 Maxwell's Equations

We begin by formulating Maxwell's equations in a vacuum. This fundamental set comprises four partial differential equations: Gauss's law for electricity, Gauss's law for magnetism, Faraday's law of induction, and the AmpèreMaxwell law. We first establish the fundamental quantities that act as sources for the electromagnetic field. For detailed derivations, please refer to Chapter 2 of Zangwill's Modern Electrodynamics [4, Chapter 2].

In the macroscopic limit, we treat charge not as discrete points but as a continuous volume charge density, defined as:

$$\rho(\mathbf{r}, t) = \frac{dq}{dV},$$

which represents the average electric charge per unit volume.

To describe the dynamic transport of this charge, we introduce the current density vector $\mathbf{j}$. For a charge distribution moving with a velocity field $\mathbf{v}$, this is defined as:

$$\mathbf{j} = \rho \mathbf{v},$$

representing the charge flux through a unit area per unit time.

Finally, the physical principle of charge conservation dictates that these quantities are intrinsically coupled: any temporal change in the amount of charge within a volume must be accounted for by the flow of current across its boundary. This conservation law is mathematically codified in the continuity equation:

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$$

By the expirmental results, we define equation:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \rho\left(\mathbf{r}'\right) \frac{\mathbf{r} - \mathbf{r}'}{\left|\mathbf{r} - \mathbf{r}'\right|^3}. \tag{2.1}$$

to be the electric field for any choice of charge density $\rho(\mathbf{r})$. Given (2.1), we can compute the divergence of $\mathbf{E}$:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \mathbf{E} = 0. \tag{2.2}$$

The first equation in (2.2) is **Gauss's law** in differential form. The second equation states is only valid for electrostatics, where charges are stationary.

The magnetic field $\mathbf{B}$ is defined by the Biot-Savart in 1820, and we define the magnetic field produced by any time-independent current density $\mathbf{j}(\mathbf{r})$ as:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{2.3}$$

In 1851, William Thomson (later Lord Kelvin) used an equivalence between current loops and permanent magnets due to Ampère to show that the magnetic field produced by both types of sources satisfies

$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j}. \tag{2.4}$$

We confirm that (2.3) is consistent with both equations in (2.4) as long as the current density satisfies the steady-current condition: $\nabla \cdot \mathbf{j} = 0$. The first equation in (2.4) states is called **Gauss's law for magnetism**, and the second equation is known as **Ampère's law**.

By Faraday's observation and Stokes' theorem, we yields the differential form of **Faraday's law**:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.5}$$

The displacement current $\mathbf{j}_D = \epsilon_0 \partial \mathbf{E}/\partial t$ is Maxwells transcendent contribution to the theory of electromagnetism. We insert $\mathbf{j}_D$ into the second equation of (2.4) to obtain the modified **Ampère-Maxwell law**:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{2.6}$$

where $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light in vacuum.

Taken (2.2),(2.4),(2.5) and (2.6) together, we have the complete set of **Maxwell's equations in vacuum**:

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = \rho/\epsilon_0 \tag{2.7}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} \tag{2.8}$$

2.2 Scalar Potential and Vector Potential

To simplify the description of the electric and magnetic fields, we introduce the scalar potential and the vector potential, respectively.

By using the Helmholtz theorem, we can provide the explicit formula of $\mathbf{E}(\mathbf{r})$ [4, Chapter 3]:

$$\mathbf{E}(\mathbf{r}) = -\nabla \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.9}$$

The electric field integral(2.1) is difficult to evaluate for all but the simplest choices of $\rho(\mathbf{r})$. However, a glance back at (2.9) shows that we can define a function called the electrostatic scalar potential,

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|},$$

and write the electric field in the form

$$\mathbf{E}(\mathbf{r}) = -\nabla\varphi(\mathbf{r}). \tag{2.10}$$

Similarly, we can get an explicit formula of $\mathbf{B}(\mathbf{r})$ by Helmholtz theorem. [4, Chapter 10]:

$$\mathbf{B}(\mathbf{r}) = \nabla \times \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{j}\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|} \tag{2.11}$$

The magnetic scalar potential formula $\mathbf{B} = -\nabla\psi$ is not valid at points in space where $\mathbf{j}(\mathbf{r}) \neq 0$. A more general approach to $\mathbf{B}(\mathbf{r})$ exploits the zero-divergence condition $\nabla \cdot \mathbf{B} = 0$ to infer that a vector potential $\mathbf{A}(\mathbf{r})$ exists such that

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}). \tag{2.12}$$

We can easily appreciated this fact already from our application of the Helmholtz theorem to get(2.11), where

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{j}\left(\mathbf{r}'\right)}{\left|\mathbf{r} - \mathbf{r}'\right|}.$$

Thus, in general, we can express the electrostatics and magnetostatics in terms of the scalar(2.10) and vector potentials(2.12).

2.3 Electromagnetic Potentials

The scalar potential $\varphi(\mathbf{r})$ and vector potential $\mathbf{A}(\mathbf{r})$ played prominent simplifying roles in electrostatics and magnetostatics. Their time-dependent counterparts do the same for time-varying fields. The starting point, as always, is the Maxwell equations in vacuum (2.7) and (2.8).

We consider the time-varying generalization of (2.12):

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t). \tag{2.13}$$

Further progress comes from inserting (2.13) into Faraday's law on the left side of (2.7). This gives

$$0 = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right)$$

Recall now that $\nabla \times \nabla f = 0$ is an identity for any $f(\mathbf{r}, t)$. We can set $\varphi\left(\mathbf{r}, t\right)$ in such a way that:

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\varphi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}. \tag{2.14}$$

Thus,the potentials reduce the number of functions to be determined from six (the scalar components of $\mathbf{E}$ and $\mathbf{B}$) to four (the scalar components of φ and the vector components of $\mathbf{A}$).

2.3.1 Gauge Invariance

Equations of motion for the potentials follow by substituting (2.14) and (2.13) into the inhomogeneous Maxwell equations, the equations on the right sides of (2.7) and (2.8) where the charge density and current density appear. Making use of $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, we find

$$\nabla^2\varphi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\rho/\epsilon_0 \tag{2.15}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) = -\mu_0\mathbf{j} \tag{2.16}$$

Like their static counterparts, $\varphi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ are not uniquely defined. This has no observable consequences because the non-uniqueness does not affect the electric and magnetic fields that enter the Coulomb-Lorentz law. We therefore let $\Lambda(\mathbf{r},t)$ be an arbitrary gauge function of space and time and define a new vector potential $\mathbf{A}'$ and a new scalar potential φ' using

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \tag{2.17}$$

and

$$\varphi' = \varphi - \frac{\partial\Lambda}{\partial t}. \tag{2.18}$$

We can verify that the fields $\mathbf{E}$ and $\mathbf{B}$, calculated using (2.13) and (2.14), remain invariant under (2.18) and (2.17).

Thus, we need to choose a gauge function $\Lambda(\mathbf{r},t)$ so that (2.15) and (2.16) become simple and easy to solve when written in the primed variables. Two common choices are the Coulomb gauge and the Lorenz gauge, which are defined by the condition

$$\nabla\cdot\mathbf{A} = 0 \qquad \text{(Coulomb gauge)} \tag{2.19}$$

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0 \qquad \text{(Lorenz gauge)} \tag{2.20}$$

2.3.2 The Coulomb Gauge and the Lorenz Gauge

The **Coulomb gauge** choice (2.19) reduces (2.15) and (2.16) to

$$\nabla^2\varphi_C = -\rho/\epsilon_0$$

and

$$\nabla^2\mathbf{A}_C - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_C}{\partial t^2} = -\mu_0\mathbf{j} + \frac{1}{c^2}\nabla\frac{\partial\varphi_C}{\partial t}.$$

The scalar potential obeys Poisson's equation. Therefore, if we specify that $\varphi_C(\mathbf{r},t)\to 0$ as $|\mathbf{r}|\to\infty$, we know from electrostatics that

$$\varphi_C(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho\left(\mathbf{r}',t\right)}{|\mathbf{r}-\mathbf{r}'|} \tag{2.21}$$

This shows that at least a part of the electric field (2.14) is the familiar, instantaneous Coulomb electric field.

The **Lorenz gauge** choice (2.20) uncouples (2.15) and (2.16) to give

$$\nabla^2\varphi_L - \frac{1}{c^2}\frac{\partial^2\varphi_L}{\partial t^2} = -\rho/\epsilon_0$$

and

$$\nabla^2\mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} = -\mu_0\mathbf{j}.$$

We obeserve that the charge density ρ determines φ_L in exactly the same way that the Cartesian components of the current density $\mathbf{j}$ determine the Cartesian components of $\mathbf{A}_L$. This characteristic makes the Lorenz gauge very popular for problems where the Coulomb potential (2.21) does not simplify the physics.

It is crucial to distinguish the Lorenz gauge named after the Danish physicist**Ludvig Lorenz** who formulated this gauge condition from the works of **H.A. Lorentz** (the Dutch physicist famous for the Lorentz force and transformations in special relativity), as well as the meteorologist **Edward Lorenz**, who pioneered chaos theory and the butterfly effect.

2.4 Conversation of Energy

We now turn to consider the work done by electromagnetic [4, Chapter 15]. The magnetic Lorentz force does no work , so all the work is done by the electric Coulomb force. Specifically, the rate at which $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ do mechanical work on a collection of particles with charge density $\rho(\mathbf{r},t)$ and current density $\mathbf{j}(\mathbf{r},t)=\rho(\mathbf{r},t)\boldsymbol{v}(\mathbf{r},t)$ confined to a volume V is

$$\frac{dW_{\text{mech}}}{dt}=\int_V d^3r(\rho\mathbf{E}+\mathbf{j}\times\mathbf{B})\cdot\boldsymbol{v}=\int_V d^3r\mathbf{j}\cdot\mathbf{E}. \tag{2.22}$$

We try to rewrite this in the form of a conservation law. The first step eliminates the current density $\mathbf{j}$ on the far right side of (2.22) using the Ampère-Maxwell equation. This gives

$$\frac{dW_{\text{mech}}}{dt}=\int_V d^3r\left[\frac{1}{\mu_0}\nabla\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right]\cdot\mathbf{E}$$

Next, using Faraday's law and specific identity, we have

$$\nabla\cdot(\mathbf{E}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{B})=-\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}-\mathbf{E}\cdot(\nabla\times\mathbf{B}).$$

Combining the result of this substitution with the original equation (2.22) gives the desired conversation of energy law in a form known as Poynting's theorem:

$$\int_V d^3r\frac{\partial}{\partial t}\frac{1}{2}\epsilon_0\left[\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B}\right]=-\int_V d^3r\mathbf{j}\cdot\mathbf{E}-\int_V d^3r\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}).$$

We therefore define the total electromagnetic energy as:

$$U_{\text{EM}}=\frac{1}{2}\epsilon_0\int_V d^3r\left[\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B}\right]. \tag{2.23}$$

This definition is physically plausible because it reduces to the sum of the electrostatic total energy U_E [4, Section 3.6] and the magnetostatic total energy U_B [4, Section 12.6] in the static limit.

A spatially local statement of energy conservation follows from (2.22) if we use (2.23) to define an electromagnetic energy density,

$$u_{\text{EM}}=\frac{1}{2}\epsilon_0\left(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B}\right).$$

We will rewrite this in terms of the electromagnetic tensor $F^{\mu\nu}F_{\mu\nu}$ later.

3 Special Relativity

Special relativity is the theory of how different observers, moving at constant velocity with respect to one another, report their experience of the same physical event.

Our discussion begins with the physical postulates of relativity, the Lorentz transformation, and some of the simpler consequences of the Lorentz transformation. We treat the kinematics and dynamics of point particles quite briefly and do not discuss spin at all. A central topic is the transformation laws for electromagnetic quantities like charge density, current density, the electromagnetic potentials, and the electromagnetic fields. Using these, we revisit the physics of moving point charges and plane electromagnetic waves. We then introduce the concept of the Lorentz tensor and derive manifestly covariant representations for the Maxwell equations and for the conservation laws of electrodynamics.

3.1 Galileo's and Einstein's Relativity

Before special relativity was formulated, the fundamental laws of physics were understood to obey Galileos principle of relativity.Galileos relativity principle states that the laws of bodily motion are the same in all **inertial frames**, where we define an inertial frame as one in which a body not subject to any forces moves with constant velocity.

Consider, for example, the inertial frames K and K' shown in Figure 1.

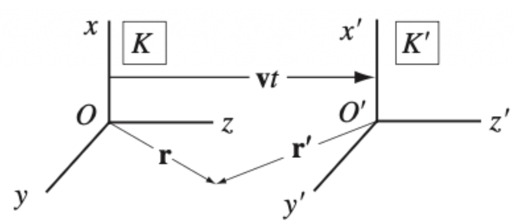

Figure 1: Two inertial frames K and K' moving at a constant relative velocity $\mathbf{v}$ along their common z-axis. At time $t = t' = 0$, the origins of the two frames coincide.

Special relativity appeared at a time when scientists were struggling to understand the absence of Galilean relative motion effects in light-propagation experiments. Einstein resolved the conceptual issues associated with the electrodynamics of moving bodies by rejecting the universal validity of Newtons laws and embracing the universal validity of Maxwells laws. His famous and highly readable 1905 paper on the subject frames the solution using two postulates:

1. **The Principle of Relativity**: The laws of physics are the same in all inertial frames.

2. **The Constancy of the Speed of Light**: The speed of light in vacuum has the same value c in all inertial frames, independent of the motion of the source or observer.

Postulate 1 is a generalization of Galileos relativity principle to include Maxwells laws of electrodynamics.Postulate 2 explained the failure to detect relative motion between light and the aether by the simple expedient of making the aether superfluous. We will show below that Einsteins two postulates (and the tacit assumptions that empty space is isotropic and spatially homogeneous) are sufficient to construct the entire edifice of special relativity. Part of the program is to discover the transformation laws that preserve the forms of the wave equation and the Maxwell equations, and to discover the dynamical law of motion that replaces Newtons laws. An equally important part of the program is to discover the physical consequences of the postulates. We begin with the most important of the physical consequences.

3.2 Lorentz Transformation

The different perception of time by different inertial observers leads us to treat space and time on an equal footing and to locate events in a venue called space-time. The most general transformation law between two inertial frames K and K' in space-time is

$$x' = x'(x, y, z, t) \quad y' = y'(x, y, z, t) \quad z' = z'(x, y, z, t) \quad t' = t'(x, y, z, t) \, .$$

Because **space are homogeneou**, the infinitesimal displacement

$$dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy + \frac{\partial x'}{\partial z} dz + \frac{\partial x'}{\partial t} dt = 0. \tag{3.1}$$

cannot be an explicit function of (x, y, z, t), which means the partial derivatives in (3.1) are constants. We therefore let r_μ (with $\mu = 1, 2, 3, 4$) stand for x, y, z, ct and write the transformation law to this point in the form

$$r'_\mu = L_{\mu\nu} r_\nu + a_\mu .$$

3.2.1 Boosting the Standard Configuration

3.2.2 The Invariant Interval

3.3 Four-Vectors

3.4 Electromagnetic Quantities

3.5 Covariant Electrodynamics

3.6 Minkowski Metric

3.7 Lagrangian of Free Electromagnetism Field

4 The Klein-Gordon Field

4.1 The Necessity of Field Viewpoint

In nonrelativistic quantum mechanics we have $E = \mathbf{p}^2/2m$, so

$$
\begin{aligned}
U(t) &= \langle \mathbf{x} | e^{-i\left(\mathbf{p}^2/2m\right)t} | \mathbf{x}_0 \rangle \\
&= \int \frac{d^3p}{(2\pi)^3} \langle \mathbf{x} | e^{-i\left(\mathbf{p}^2/2m\right)t} | \mathbf{p} \rangle \langle \mathbf{p} \mid \mathbf{x}_0 \rangle \\
&= \frac{1}{(2\pi)^3} \int d^3p e^{-i\left(\mathbf{p}^2/2m\right)t} \cdot e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \\
&= \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t}
\end{aligned}
$$

This expression is nonzero for all x and t, indicating that a particle can propagate between any two points in an arbitrarily short time.

Quantum field theory provides a natural way to handle not only multiparticle states, but also transitions between states of different particle number. It solves the causality problem by introducing antiparticles, then goes on to explain the relation between spin and statistics. But most important, it provides the tools necessary to calculate innumerable scattering cross sections, particle lifetimes, and other observable quantities.

4.2 The Classical Field Theory

4.2.1 Lagrangian and Hamiltonian Field Theory

We have some essential equations. As a simple example, consider the theory of a single field $\phi(x)$, governed by the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \\
&= \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2.
\end{aligned}
$$

For now we take ϕ to be a real-valued field. The quantity m will be interpreted as a mass in Section 2.3, but for now just think of it as a parameter. From this Lagrangian the usual procedure gives the equation of motion

$$
\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0 \quad \text{or} \quad \left(\partial^\mu\partial_\mu + m^2\right)\phi = 0,
$$

which is the well-known Klein-Gordon equation. (In this context it is a classical field equation, like Maxwell's equations-not a quantum-mechanical wave equation.) Noting that the canonical momentum density conjugate to $\phi(x)$ is $\pi(x) = \dot{\phi}(x)$, we can also construct the Hamiltonian:

$$
H = \int d^3x \mathcal{H} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]
$$

4.2.2 Noether's Theorem

3 examples of Noether's theorem!!!!

We have some results about the energy-momentum tensor and conserved quantities.

The Lagrangian is also a scalar, so it must transform in the same way:

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu \left(\delta^\mu{}_\nu \mathcal{L}\right).$$

Comparing this equation to (2.10), we see that we now have a nonzero $\mathcal{J}^\mu$. Taking this into account, we can apply the theorem to obtain four separately conserved currents:

$$T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \partial_\nu \phi - \mathcal{L} \delta^\mu{}_\nu$$

This is precisely the stress-energy tensor, also called the energy-momentum tensor, of the field ϕ. The conserved charge associated with time translations is the Hamiltonian:

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x$$

By computing this quantity for the Klein-Gordon field, one can recover the result (2.8). The conserved charges associated with spatial translations are

$$P^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi d^3x$$

and we naturally interpret this as the (physical) momentum carried by the field (not to be confused with the canonical momentum).

4.3 The Klein-Gordon Field an Harmonic Oscillators

4.3.1 Quantization of the Real Klein-Gordon Field

We firsr introduce the canonical commutation relations for the field $\phi(\mathbf{x}, t)$ and its conjugate momentum $\pi(\mathbf{x}, t)$ at equal times:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$
$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0$$

Let us seek guidance by writing the Klein-Gordon equation in Fourier space. If we expand the classical Klein-Gordon field as

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t)$$

(with $\phi^*(\mathbf{p}) = \phi(-\mathbf{p})$ so that $\phi(\mathbf{x})$ is real), the Klein-Gordon equation (2.7) becomes

$$\left[\frac{\partial^2}{\partial t^2} + \left(|\mathbf{p}|^2 + m^2\right)\right] \phi(\mathbf{p}, t) = 0.$$

This is the same as the equation of motion for a simple harmonic oscillator with frequency

$$\omega_\mathbf{p} = \sqrt{|\mathbf{p}|^2 + m^2}.$$

We can find the spectrum of the Klein-Gordon Hamiltonian using the same trick, but now each Fourier mode of the field is treated as an independent oscillator with its own a and $a^\dagger$. In analogy with (2.23) we write

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger\right) e^{i\mathbf{p}\cdot\mathbf{x}};$$
$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger\right) e^{i\mathbf{p}\cdot\mathbf{x}}.$$

Take attention for the meaning of $\pi(\mathbf{x})$!!!
We have new formulas for the commutation realation:

$$\left[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger\right] = (2\pi)^3 \delta^{(3)} \left(\mathbf{p} - \mathbf{p}'\right),$$

We are now ready to express the Hamiltonian in terms of ladder operators. Starting from its expression (2.8) in terms of ϕ and π, we have

$$\begin{aligned} H = & \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \{ -\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger\right)\left(a_{\mathbf{p}'} - a_{-\mathbf{p}'}^\dagger\right) \\ & + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger\right)\left(a_{\mathbf{p}'} + a_{-\mathbf{p}'}^\dagger\right) \Big\} \\ = & \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}\left[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger\right]\right) \end{aligned}$$

Take attention to the second term in the last line, which arises from normal ordering the operators. It is an infinite constant!!!!

4.3.2 Interpretation of the Spectrum of $\mathcal{H}$

Having found the spectrum of the Hamiltonian, let us try to interpret its eigenstates. From

$$P^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi d^3x$$

and a calculation similar to (2.31) we can write down the total momentum operator,

$$\mathbf{P} = -\int d^3x \pi(\mathbf{x}) \nabla\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$$

So the operator $a_{\mathbf{p}}^\dagger$ creates momentum $\mathbf{p}$ and energy $\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$. Similarly, the state $a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger \cdots |0\rangle$ has momentum $\mathbf{p} + \mathbf{q} + \cdots$. It is quite natural to call these excitations particles, since they are discrete entities that have the proper relativistic energy-momentum relation. (By a particle we do not mean something that must be localized in space; $a_{\mathbf{p}}^\dagger$ creates particles in momentum eigenstates.) From now on we will refer to $\omega_{\mathbf{p}}$ as $E_{\mathbf{p}}$ (or simply E), since it really is the energy of a particle. Note, by the way, that the energy is always positive: $E_{\mathbf{p}} = +\sqrt{|\mathbf{p}|^2 + m^2}$.

4.3.3 The normalization of $|\mathbf{p}\rangle$

We naturally choose to normalize the vacuum state so that $\langle 0 \mid 0\rangle = 1$. The one-particle states $|\mathbf{p}\rangle \propto a_{\mathbf{p}}^\dagger|0\rangle$ will also appear quite often, and it is worthwhile to adopt a convention for their normalization. The simplest normalization $\langle \mathbf{p} \mid \mathbf{q}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q})$ (which many books use) is not Lorentz invariant, as we can demonstrate by considering the effect of a boost in the 3 -direction. Under such a boost we have $p_3' = \gamma\left(p_3 + \beta E\right), E' = \gamma\left(E + \beta p_3\right)$. Using the delta function identity

$$\delta\left(f(x)-f\left(x_0\right)\right)=\frac{1}{\left|f^{\prime}\left(x_0\right)\right|} \delta\left(x-x_0\right)$$

We can compute that

$$\delta^{(3)}(\mathbf{p}-\mathbf{q})=\delta^{(3)}\left(\mathbf{p}^{\prime}-\mathbf{q}^{\prime}\right) \frac{E^{\prime}}{E}.$$

We therefore define

$$|\mathbf{p}\rangle=\sqrt{2 E_{\mathbf{p}}} a_{\mathbf{p}}^{\dagger}|0\rangle,$$

so that

$$\langle\mathbf{p} \mid \mathbf{q}\rangle=2 E_{\mathbf{p}}(2 \pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})$$

By considering a Lorentz Transformation Λ which will be implemented as some unitary operator $U(\Lambda)$, we have a representation for 1-partial state:

$$(\mathbf{1})_{1-\text { particle }}=\int \frac{d^3 p}{(2 \pi)^3}|\mathbf{p}\rangle \frac{1}{2 E_{\mathbf{p}}}\langle\mathbf{p}|,$$

4.3.4 Interpretation of the state of $\phi(\mathbf{x})\,|0\rangle$

Finally let us consider the interpretation of the state $\phi(\mathbf{x})|0\rangle$. From the expansion (2.25) we see that

$$\phi(\mathbf{x})|0\rangle=\int \frac{d^3 p}{(2 \pi)^3} \frac{1}{2 E_{\mathbf{p}}} e^{-i \mathbf{p} \cdot \mathbf{x}}|\mathbf{p}\rangle$$

is a linear superposition of single-particle states that have well-defined momentum. Except for the factor $1/2E_{\mathbf{p}}$, this is the same as the familiar nonrelativistic expression for the eigenstate of position $|\mathbf{x}\rangle$; in fact the extra factor is nearly constant for small (nonrelativistic) $\mathbf{p}$. We will therefore put forward the same interpretation, and claim that the operator $\phi(\mathbf{x})$, acting on the vacuum, creates a particle at position $\mathbf{x}$. This interpretation is further confirmed when we compute

$$\begin{aligned}
\langle 0|\phi(\mathbf{x})| \mathbf{p}\rangle & =\langle 0| \int \frac{d^3 p^{\prime}}{(2 \pi)^3} \frac{1}{\sqrt{2 E_{\mathbf{p}^{\prime}}}}\left(a_{\mathbf{p}^{\prime}} e^{i \mathbf{p}^{\prime} \cdot \mathbf{x}}+a_{\mathbf{p}^{\prime}}^{\dagger} e^{-i \mathbf{p}^{\prime} \cdot \mathbf{x}}\right) \sqrt{2 E_{\mathbf{p}}} a_{\mathbf{p}}^{\dagger}|0\rangle \\
& =e^{i \mathbf{p} \cdot \mathbf{x}}
\end{aligned}$$

We can interpret this as the position-space representation of the single-particle wavefunction of the state $|\mathbf{p}\rangle$, just as in nonrelativistic quantum mechanics $\langle\mathbf{x} \mid \mathbf{p}\rangle \propto e^{i \mathbf{p} \cdot \mathbf{x}}$ is the wavefunction of the state $|\mathbf{p}\rangle$.

4.4 The Klein-Gordon Field in Space-Time

4.4.1 The Dual Particle and Wave Interpretations of the Quantum Field $\phi(\mathbf{x})$

We can compute the Heisenberg-picture field operator $\phi(\mathbf{x}, t)$ by solving the Heisenberg equation of motion,

$$\begin{aligned}
\phi(\mathbf{x}, t) & =\left.\int \frac{d^3 p}{(2 \pi)^3} \frac{1}{\sqrt{2 E_{\mathbf{p}}}}\left(a_{\mathbf{p}} e^{-i p \cdot x}+a_{\mathbf{p}}^{\dagger} e^{i p \cdot x}\right)\right|_{p^0=E_{\mathbf{p}}} ; \\
\pi(\mathbf{x}, t) & =\frac{\partial}{\partial t} \phi(\mathbf{x}, t).
\end{aligned}$$

Equation (2.47) makes explicit the dual particle and wave interpretations of the quantum field $\phi(x)$.

On the one hand, $\phi(x)$ is written as a Hilbert space operator, which creates and destroys the particles that are the quanta of field excitation.

On the other hand, $\phi(x)$ is written as a linear combination of solutions $\left(e^{ip\cdot x} \text{ and } e^{-ip\cdot x}\right)$ of the Klein-Gordon equation.

Both signs of the time dependence in the exponential appear: We find both e^{-ip^0t} and e^{+ip^0t}, although p^0 is always positive. If these were single-particle wavefunctions, they would correspond to states of positive and negative energy; let us refer to them more generally as positive- and negative-frequency modes.

The connection between the particle creation operators and the waveforms displayed here is always valid for free quantum fields:

A positive-frequency solution of the field equation has as its coefficient the operator that destroys a particle in that single-particle wavefunction.

A negative-frequency solution of the field equation, being the Hermitian conjugate of a positive-frequency solution, has as its coefficient the operator that creates a particle in that positive-energy single-particle wavefunction.

In this way, the fact that relativistic wave equations have both positive- and negative-frequency solutions is reconciled with the requirement that a sensible quantum theory contain only positive excitation energies.

4.4.2 Calculation of Contour Integrals and Causality

The main point of this section is to calculate the propagator amplitude by contour integrals and to show that the commutator of two field operators vanishes for spacelike separations, thus preserving causality. First consider the case where the difference $x-y$ is purely in the timedirection: $x^0-y^0=t, \mathbf{x}-\mathbf{y}=0$. (If the interval from y to x is timelike, there is always a frame in which this is the case.) Then we have

$$\begin{aligned}
D(x-y) &= \frac{4\pi}{(2\pi)^3}\int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t} \\
&= \frac{1}{4\pi^2}\int_m^\infty dE \sqrt{E^2-m^2} e^{-iEt} \\
&\underset{t\to\infty}{\sim} e^{-imt}
\end{aligned}$$

Next consider the case where $x-y$ is purely spatial: $x^0-y^0=0, \mathbf{x}-\mathbf{y}=\mathbf{r}$. The amplitude is then

$$\begin{aligned}
D(x-y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{r}} \\
&= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp \frac{p^2}{2E_{\mathbf{p}}} \frac{e^{ipr}-e^{-ipr}}{ipr} \\
&= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}} \underset{r\to\infty}{\sim} e^{-mr}
\end{aligned}$$

Specific details of contour integration need to be added here, along with Jordan's Lemma!!!!

Because of the calculation:

$$\begin{aligned}
[\phi(x),\phi(y)] = &\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}} \\
&\times \left[\left(a_{\mathbf{p}}e^{-ip\cdot x}+a_{\mathbf{p}}^\dagger e^{ip\cdot x}\right), \left(a_{\mathbf{q}}e^{-iq\cdot y}+a_{\mathbf{q}}^\dagger e^{iq\cdot y}\right)\right] \\
=&\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\right) \\
=&D(x-y)-D(y-x)
\end{aligned}$$

When $(x-y)^2 < 0$, we can perform a Lorentz transformation on the second term (since each term is separately Lorentz invariant), taking $(x-y) \to -(x-y)$, as shown in Fig. 2.4. The two terms are therefore equal and cancel to give zero; causality is preserved. Note that if $(x-y)^2 > 0$ there is no continuous Lorentz transformation that takes $(x-y) \longrightarrow -(x-y)$. In this case, by Eq. (2.51), the amplitude is (fortunately) nonzero, roughly ($e^{-imt} - e^{imt}$) for the special case $\mathbf{x} - \mathbf{y} = 0$. Thus we conclude that no measurement in the Klein-Gordon theory can affect another measurement outside the light-cone.

4.4.3 The Klein-Gordon Propagator

For $x^0 > y^0$ we can close the contour below, picking up both poles to obtain the previous line of (2.54). For $x^0 < y^0$ we may close the contour above, giving zero. Thus the last line of (2.54), together with the prescription for going around the poles, is an expression for what we will call

$$D_R(x-y) \equiv \theta\left(x^0 - y^0\right) \langle 0|[\phi(x), \phi(y)]|0\rangle.$$

To understand this quantity better, let's do another computation:

$$\begin{aligned}
\left(\partial^2 + m^2\right) D_R(x-y) =& \left(\partial^2 \theta\left(x^0 - y^0\right)\right) \langle 0|[\phi(x), \phi(y)]|0\rangle \\
&+ 2\left(\partial_\mu \theta\left(x^0 - y^0\right)\right)\left(\partial^\mu \langle 0|[\phi(x), \phi(y)]|0\rangle\right) \\
&+ \theta\left(x^0 - y^0\right)\left(\partial^2 + m^2\right) \langle 0|[\phi(x), \phi(y)]|0\rangle \\
=& -\delta\left(x^0 - y^0\right) \langle 0|[\pi(x), \phi(y)]|0\rangle \\
&+ 2\delta\left(x^0 - y^0\right) \langle 0|[\pi(x), \phi(y)]|0\rangle + 0 \\
=& -i\delta^{(4)}(x-y)
\end{aligned}$$

This says that $D_R(x-y)$ is a Green's function of the Klein-Gordon operator. Since it vanishes for $x^0 < y^0$, it is the retarded Green's function.

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4.4.4 Particle Creation by a Classical Source

4.4.5 Problems and Solutions

References

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