From “Along the Path to Interstellar Flight” paper

Relativistic Corrections – There are several relativistic corrections that modify the non-relativistic calculations that become important as we proceed to relativistic speeds. The full solution is given in Kulkarni and Lubin 2017, but the physical differences help us understand the corrections:

a) From the viewpoint of the laser the spacecraft reflection/ absorption of the photons is redshifted and hence the power and thus the force is reduced by the reduction in photon energy and momentum. The energy and momentum is conserved by considering the photons emitted and returned (if reflected) redshifted photons. Additonally the moving spacecraft has a perceived increased mass.

   If v_0 is the frequency of the photons emitted by the laser and v is the frequency received by the receding spacecraft moving at speed β then:

$$v = v_0\gamma(1-\beta) = v_0\left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

   For non relativistic speeds where β is small the corrections are of order β $v = v_0\gamma(1-\beta) \sim v_0(1-\beta)$. The increased relativistic mass of the spacecraft system is $m = m_0\gamma$ where m_0 is the spacecraft system rest mass and these corrections are modified by γ. Of the two effects the redshift is the most important at lower speeds.

b) From the viewpoint of the spacecraft the photons “hitting it” are redshifted as the laser is “perceived” to be receding away and are thus the photons are redshifted. In addition, the rate at which the photons “hit” the spacecraft are reduced due to the time dilation with the rate of the photons emitted by the perceived receding being reduced. Here the two effects are the same with the photon redshift being as above and the time dilation being modified by γ.

These two points of view give the same physical solution but are extremely instructive to understanding the physics of the problem. For high precision calculations or as we approach the speed of light the fully relativistic solution must be used as detailed in Kulkarni and Lubin 2017.

Relativistic solution - The solution is for the case of the beam fully on the sail during the time t below.

It is given by t vs β (v/c) and $\gamma=(1-\beta^2)^{-1/2}$ as below (assuming $\varepsilon_r=1$) with $m=m_{sail}+m_0$ with m_0=bare spacecraft mass. This assumes the reflector is large enough so that $L<L_0$:

$$t = \frac{mc^2}{6P_0}\left[\frac{(1+\beta)(2-\beta)\gamma}{(1-\beta)} - 2\right]$$

Define $t_E \equiv \dfrac{mc^2}{P_0}$ ($\equiv$ time for emitted photon energy = spacecraft rest mass energy)

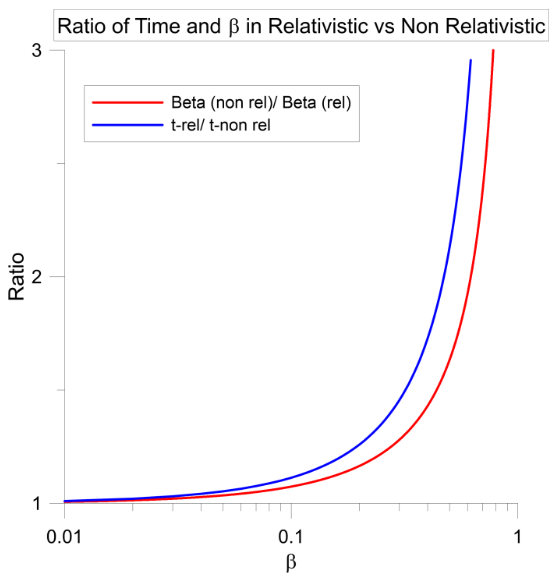
$$\rightarrow t = \frac{t_E}{6}\left[\frac{(1+\beta)(2-\beta)\gamma}{(1-\beta)} - 2\right]$$

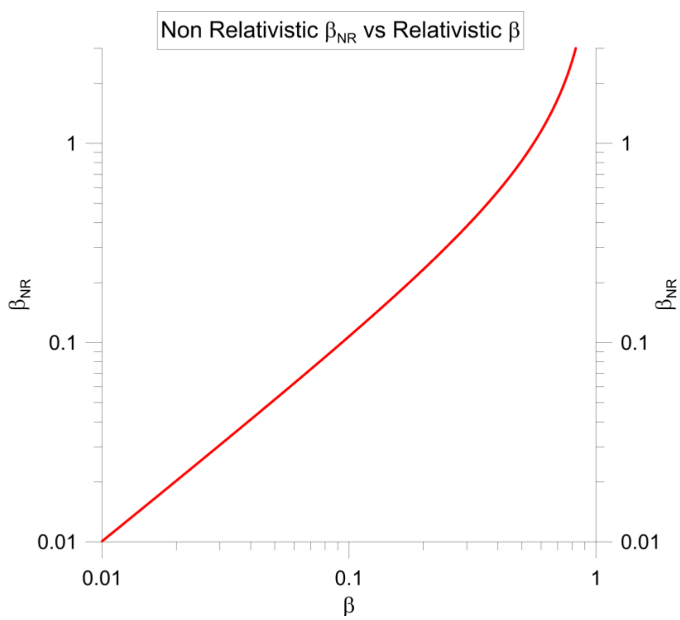
In the non relativistic limit this becomes:

$$t_{NR} = \frac{t_E}{2}\beta = \frac{mcv}{2P_0}\,(\beta << 1)$$

$$t / t_{NR} = \frac{1}{3\beta}\left[\frac{(1+\beta)(2-\beta)\gamma}{(1-\beta)} - 2\right]$$

The time to a given speed and distance is longer and the speed at a given distance is less in the relativistic solution. For low β the difference is small.





Using conservation of momentum for the entire photons + spacecraft system we obtain the equations of motion for the relativistic case as:

$$\dot{\beta} = \begin{cases} \dfrac{2P_0}{mc^2\gamma^3}\left(\dfrac{1-\beta}{1+\beta}\right) & x < L_0 \\ \dfrac{2P_0}{mc^2\gamma^3}\left(\dfrac{1-\beta}{1+\beta}\right)\left(\dfrac{L_0}{x}\right)^2 & x > L_0 \end{cases}$$

We can integrate this directly noting that dβ/dt= dβ/dx* dx/dt=cβ dβ/dx. We then get:

Noting $\int \beta(1-\beta)^{-5/2}(1+\beta)^{-1/2}d\beta = \dfrac{2\beta-1}{3\gamma(1-\beta)^2}+const$

$$\frac{2\beta-1}{3\gamma(1-\beta)^2}+\frac{1}{3}=\frac{2P_0}{mc^3}x \quad for\ x<L_0$$

The speed β_0 at L_0 is thus given by:

$$\rightarrow \frac{2\beta_0-1}{3\gamma_0(1-\beta_0)^2}+\frac{1}{3}=\frac{2P_0}{mc^3}L_0=\frac{2P_0}{mc^2}\frac{L_0}{c}=2\frac{t_L}{t_E}$$

where $t_L \equiv \dfrac{L_0}{c} \equiv$ light travel time over distance L_0

Expanding this to order $\beta_0{}^2$ we have (at small β_0) :

$$\frac{2\beta_0-1}{3\gamma_0(1-\beta_0)^2}+\frac{1}{3}=\frac{2\beta_0-1+\gamma_0(1-\beta_0)^2}{3\gamma_0(1-\beta_0)^2}\sim\frac{2\beta_0-1+(1+\frac{\beta_0{}^2}{2})(1-\beta_0)^2}{3\gamma_0(1-\beta_0)^2}=\frac{\beta_0{}^2(\frac{3}{2}-\beta_0)}{3\gamma_0(1-\beta_0)^2}\sim\frac{\beta_0{}^2}{2}$$

$$\rightarrow \beta_0{}^2=4\frac{t_L}{t_E}\text{(for small }\beta_0\text{ or non-relativistic limit)}$$

This is identical to the non-relativistic limit derived earlier of:

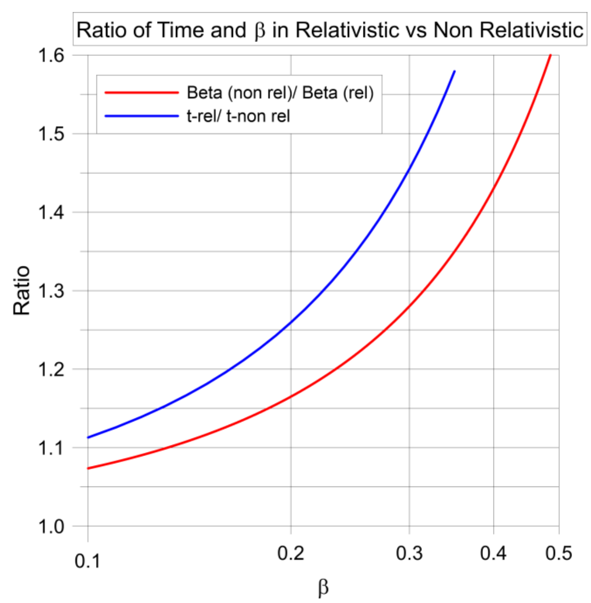
$$\beta_{0-NR}=\sqrt{\frac{4P_0}{mc^3}L_0}=\sqrt{4\frac{t_L}{t_E}}$$

We can rewite the relationship between the correct relativistic speed β_0 at L_0 compared to the non-relativistic speed β_{0-NR} solution as below. The same relationship holds at any point up to L_0. Note that the non-relativistic β_{0-NR} always overestimates the correct speed β_0 up to and including at L_0.

$$\frac{2\beta_0-1}{3\gamma_0(1-\beta_0)^2}+\frac{1}{3}=\frac{2P_0}{mc^3}L_0=2\frac{t_L}{t_E}=\frac{\beta_{0-NR}{}^2}{2}$$

$$\rightarrow \beta_{0-NR}=\sqrt{2}\left[\frac{2\beta_0-1}{3\gamma_0(1-\beta_0)^2}+\frac{1}{3}\right]^{1/2}$$

$$\rightarrow \frac{\beta_{0-NR}}{\beta_0}=\frac{\sqrt{2}}{\beta_0}\left[\frac{2\beta_0-1}{3\gamma_0(1-\beta_0)^2}+\frac{1}{3}\right]^{1/2}$$



The relationship between the non-relativistic and relativistic solution is particularly useful in that the computations and insight from the non-relativistic solutions are much easier and given the above ratios of t/t_{NR} and $\beta_0/\beta_{0\text{-}NR}$ allow us to compute $\beta_{0\text{-}NR}$ and $t_{0\text{-}NR}$ from the system parameters and then translate to the relativistic solution for β_0 and t_0. For example at $\beta_0=0.10$ $\beta_{0\text{-}NR}$ is computed at 8% higher than it should be, at $\beta_0=0.20$ $\beta_{0\text{-}NR}$ is 16% higher than it should be and at $\beta_0=0.30$ $\beta_{0\text{-}NR}$ is 28% higher than it should be. The full relativistic solution can and is be used but it is less intuitive and often the non-relativistic solution for mildly relativistic systems gives much more insight into a system design. For highly relativistic solutions it is easier to use the fully relativistic solution.

Final Speed – The final speed is the speed if the laser is left on in the limit of infinite time. In the non rel case the final speed is $2^{1/2}$ times the speed at Lo. This is obvious not true for the relativistic case.
From Kulkarni and Lubin 2016 we have:

$$\dot{\beta}=\frac{d\beta}{dx}\frac{dx}{dt}=\frac{d\beta}{dx}v=\frac{d\beta}{dx}c\beta=\frac{2P_0}{mc^2\gamma^3}\left(\frac{1-\beta}{1+\beta}\right)\left(\frac{L_0}{x}\right)^2 \quad \text{for } x>L_0$$

$$\frac{mc^3}{2P_0}\left(\frac{1+\beta}{1-\beta}\right)\gamma^3\beta d\beta=\left(\frac{L_0}{x}\right)^2 dx$$

Integrating yields:

$$\frac{1-2\beta}{3\gamma(1-\beta)^2}=\frac{2P_0}{mc^3}\frac{L_0{}^2}{x}+const$$

At $x=\infty$:

$$\frac{1-2\beta_\infty}{3\gamma_\infty(1-\beta_\infty)^2}=const$$

We solve for the constant noting that at x=$L_0, \beta=\beta_0$

Hence const= $\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\frac{2P_0L_0}{mc^3}=\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\frac{\beta_{0-NR}{}^2}{2}=\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\left[\frac{2\beta_0-1}{3\gamma_0\left(1-\beta_0\right)^2}+\frac{1}{3}\right]$

$$=\frac{2(1-2\beta_0)}{3\gamma_0\left(1-\beta_0\right)^2}-\frac{1}{3}$$

Since $\beta_{0-NR}=\sqrt{\frac{4P_0}{mc^3}L_0}\ -->\beta_{0-NR}{}^2/2=\frac{2P_0L_0}{mc^3}=\frac{2\beta_0-1}{3\gamma_0\left(1-\beta_0\right)^2}+\frac{1}{3}$

Note from above that $\frac{\beta_{0-NR}}{\beta_0}=\frac{\sqrt{2}}{\beta_0}\left[\frac{2\beta_0-1}{3\gamma_0\left(1-\beta_0\right)^2}+\frac{1}{3}\right]^{1/2}$

Thus:

$$\frac{1-2\beta_\infty}{3\gamma_\infty(1-\beta_\infty)^2}=const=\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\frac{2P_0L_0}{mc^3}=\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\frac{\beta_{0-NR}{}^2}{2}=\frac{2(1-2\beta_0)}{3\gamma_0\left(1-\beta_0\right)^2}-\frac{1}{3}$$

To solve for β_∞ for a given β_0 we need to numerically solve for the value of β_∞ such that:

$$\frac{1-2\beta_\infty}{3\gamma_\infty(1-\beta_\infty)^2}=\frac{2(1-2\beta_0)}{3\gamma_0\left(1-\beta_0\right)^2}-\frac{1}{3}$$

In the NR limit with γ=1 and $(1-\beta)^{-2}\sim 1+2\beta$

$$(1-2\beta_\infty)(1+2\beta_\infty)=2(1-2\beta_0)(1+2\beta_0)-1$$

$-->4\beta_\infty{}^2=8\beta_0{}^2-->\beta_\infty=\sqrt{2}\beta_0$ as expected

In general at $x>L_0$

$$\frac{1-2\beta}{3\gamma(1-\beta)^2}=\frac{2P_0}{mc^3}\frac{L_0{}^2}{x}+\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\frac{2P_0L_0}{mc^3}=\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\frac{2P_0L_0}{mc^3}(1-\frac{L_0}{x})$$

$$=\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\frac{\beta_{0-NR}{}^2}{2}(1-\frac{L_0}{x})=\frac{1-2\beta_0}{3\gamma_0(1-\beta_0)^2}-\left[\frac{2\beta_0-1}{3\gamma_0\left(1-\beta_0\right)^2}+\frac{1}{3}\right](1-\frac{L_0}{x})$$

Optimization of reflector and spacecraft mass in the relativistic limit – In the non-relativistic limit we showed that the maximum speed is obtained when the reflector mass is equal to the spaceraft mass (Lubin 2015). In the full relativistic case the same condition holds, namely the maximum speed is when the reflector and spacecraft mass are equal [m_{sail} = m_0] (Kulkarni and Lubin 2016).

Kinetic Energy – The kinetic energy is:

$$KE = m_0(\gamma - 1)c^2$$