## 13.30. SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace Transform method, without finding the general solution and the arbitrary constants.

The method will be clear from the following examples:

*Using Laplace transforms, find the solution of the initial value problem*

$$y'' - 4y' + 4y = 64 \sin 2t$$

$$y(0) = 0,\ y'(0) = 1.$$

**Solution.**

$$y'' - 4y' + 4y = 64 \sin 2t \qquad \text{... (1)}$$

$$y(0) = 0,\ y'(0) = 1$$

Taking Laplace transform of both sides of (1), we have

$$[s^2 \bar{y} - sy(0) - y'(0)] - 4[s\bar{y} - y(0)] + 4\bar{y} = \frac{64 \times 2}{s^2+4} \qquad \text{... (2)}$$

On putting the values of $y(0)$ and $y'(0)$ in (2), we get

$$s^2 \bar{y} - 1 - 4s\bar{y} + 4\bar{y} = \frac{128}{s^2+4}$$

$$(s^2 - 4s + 4)\bar{y} = 1 + \frac{128}{s^2+4}, \quad \text{or} \quad (s-2)^2 \bar{y} = 1 + \frac{128}{s^2+4}$$

$$\bar{y} = \frac{1}{(s-2)^2} + \frac{128}{(s-2)^2(s^2+4)} = \frac{1}{(s-2)^2} - \frac{8}{s-2} + \frac{16}{(s-2)^2} + \frac{8s}{s^2+4}$$

$$y = L^{-1}\left[-\frac{8}{s-2} + \frac{17}{(s-2)^2} + \frac{8s}{s^2+4}\right]$$

$$y = -8e^{2t} + 17t\,e^{2t} + 8\cos 2t \qquad \textbf{Ans.}$$

*Using the Laplace transfors, find the solution of the initial value problem*

$$y'' + 25y = 10 \cos 5t$$

$$y(0) = 2, \quad y'(0) = 0.$$

**Solution.** Taking Laplace transform of the given differential equation, we get

$$[s^2 \bar{y} - sy(0) - y'(0)] + 25\bar{y} = 10\frac{s}{s^2+25}$$

$$s^2 \bar{y} - 2s + 25\bar{y} = \frac{10s}{s^2+25}$$

$$(s^2+25)\bar{y} = 2s + \frac{10s}{s^2+25}$$

$$\bar{y} = \frac{2s}{s^2+25} + \frac{10s}{(s^2+25)^2}$$

$$y = L^{-1}\left[\frac{2s}{s^2+25} + \frac{10s}{(s^2+25)^2}\right] = 2\cos 5t + L^{-1}\left[\frac{10s}{(s^2+25)^2}\right]$$

$$= 2\cos 5t + L^{-1}\frac{d}{ds}\left[\frac{-5}{(s^2+25)}\right]$$

$$= 2\cos 5t + t \sin 5t \qquad \textbf{Ans.}$$

*Solve* $[t\,D^2 + (1-2t)\,D - 2]\,y = 0,$ *where* $y(0) = 1,\; y'(0) = 2.$

*(R.G.P.V. June, 2002)*

**Solution.** Here, $t\,D^2 y + (1-2t)\,Dy - 2y = 0 \quad \Rightarrow \quad t\,y'' + y' - 2\,t\,y' - 2\,y = 0$

Taking Laplace transform of given differential equation, we get

$$L(t\,y'') + L(y') - 2\,L(t\,y') - 2\,L(y) = 0 \;\Rightarrow\; -\frac{d}{ds}L\{y''\} + L\{y'\} + 2\frac{d}{ds}L(y') - 2\,L(y) = 0$$

$$-\frac{d}{ds}\,[\,s^2\,\bar{y} - s\,y(0) - y'(0)\,] + [\,s\,\bar{y} - y(0)\,] + 2\frac{d}{ds}\,[\,s\,\bar{y} - y(0)] - 2\,\bar{y} = 0$$

Putting the values of $y(0)$ and $y'(0)$, we get

$$-\frac{d}{ds}(s^2\,\bar{y} - s - 2) + (s\,\bar{y} - 1) + 2\,\frac{d}{ds}\,(s\,\bar{y} - 1) - 2\,\bar{y} = 0 \qquad [\because y(0) = 1,\; y'(0) = 2]$$

$$\Rightarrow -\frac{s^2\,d\bar{y}}{d\,s} - 2\,s\,\bar{y} + 1 + s\,\bar{y} - 1 + 2\left(s\,\frac{d\bar{y}}{d\,s} + \bar{y}\right) - 2\,\bar{y} = 0 \;\Rightarrow\; -(s^2 - 2\,s)\,\frac{d\,\bar{y}}{d\,s} - s\,\bar{y} = 0$$

$$\Rightarrow\; -\frac{d\bar{y}}{\bar{y}} + \frac{1}{s-2}\,ds = 0 \qquad \text{(Separating the variables)}$$

$$\Rightarrow \int\frac{d\bar{y}}{\bar{y}} + \int\frac{d\,s}{s-2} = 0 \quad \Rightarrow\; \log\bar{y} + \log(s-2) = \log C$$

$$\Rightarrow\; \bar{y}\,(s-2) = C \quad \Rightarrow\; \bar{y} = \frac{C}{s-2} \quad \Rightarrow\; y = C\,L^{-1}\left\{\frac{1}{s-2}\right\} \;\Rightarrow\; y = C\,e^{2t} \quad \text{...(1)}$$

Putting $y(0) = 1$ in (1), we get $\quad 1 = C\,e^0 \quad \Rightarrow \quad C = 1$

Putting $C = 1$ in (1), we get $y = e^{2t}$

This is the required solution. **Ans.**

*Solve the initial value problem*

$$2\,y'' + 5\,y' + 2\,y = e^{-2t}, \qquad y(0) = 1, \quad y'(0) = 1,$$

*ısing the Laplace transforms.* *(A.M.I.E.T.E., Summer 1995)*

**Solution.** $\quad 2\,y'' + 5\,y' + 2\,y = e^{-2t}, \quad y(0) = 1, y'(0) = 1$

Taking the Laplace Transform of both sides, we get

$$2\,[s^2\,\bar{y} - sy(0) - y'(0)] + 5\,[s\,\bar{y} - y(0)] + 2\,\bar{y} = \frac{1}{s+2} \qquad \text{...(1)}$$

On substituting the values of $y(0)$ and $y'(0)$ in (1), we get

$$2\,[s^2\,\bar{y} - s - 1] + 5\,[s\,\bar{y} - 1] + 2\,\bar{y} = \frac{1}{s+2}$$

$$[2\,s^2 + 5\,s + 2]\,\bar{y} - 2\,s - 2 - 5 = \frac{1}{s+2}$$

$$\bar{y} = \frac{1}{(s+2)\,(2\,s^2 + 5\,s + 2)} + \frac{2\,s + 7}{2\,s^2 + 5\,s + 2} = \frac{1 + 2\,s^2 + 7\,s + 4\,s + 14}{(2\,s^2 + 5\,s + 2)\,(s+2)} = \frac{2\,s^2 + 11\,s + 15}{(2\,s+1)\,(s+2)^2}$$

$$= \frac{4/9}{2\,s+1} - \frac{11/9}{s+2} - \frac{1/3}{(s+2)^2} = \frac{4}{9}\,\frac{1}{2}\,\frac{1}{s + \frac{1}{2}} - \frac{11}{9}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{1}{(s+2)^2}$$

$$y = \frac{2}{9}\,e^{-\frac{1}{2}t} - \frac{11}{9}\,e^{-2t} - \frac{1}{3}\,t\,e^{-2t}$$ **Ans.**

. *Using Laplace transforms, find the solution of the initial value problem*

$$y'' + 9y = 9u(t-3), \quad y(0) = y'(0) = 0$$

*ere $u(t-3)$ is the unit step function.* *(A.M.I.E.T.E., Winter 1998)*

**Solution.** $$y'' + 9y = 9u(t-3)\,. \qquad \ldots (1)$$

Taking Laplace transform of (1), we have

$$s^2\,\bar{y} - sy(0) - y'(0) + 9\,\bar{y} = 9\,\frac{e^{-3s}}{s} \qquad \ldots (2)$$

Putting the values of $y(0) = 0$ and $y'(0) = 0$ in (2), we get

$$s^2\,\bar{y} + 9\,\bar{y} = \frac{9\,e^{-3s}}{s}$$

$$(s^2+9)\,\bar{y} = 9\,\frac{e^{-3s}}{s}$$

$$\bar{y} = \frac{9\,e^{-3s}}{s\,(s^2+9)} \quad \Rightarrow \quad y = L^{-1}\,\frac{9\,e^{-3s}}{s\,(s^2+9)}$$

$$L^{-1}\,\frac{3}{s^2+9} = \sin 3t$$

$$3\,L^{-3}\,\frac{3}{s\,(s^2+9)} = 3\int_0^t \sin 3t\,dt = -\left[\cos 3t\right]_0^t = 1 - \cos 3t$$

$$y = L^{-1}\,\frac{9\,e^{-3s}}{s\,(s^2+9)}$$

$$y = [1 - \cos 3\,(t-3)]\,u\,(t-3) \qquad \textbf{Ans.}$$