Fourier Series

French Mathematician Jean Baptiste Joseph Fourier (1768 - 1830)

Unit – III Fourier Series:

Q.1 Given that $f(x)=x+x^2$ for $-\pi<x<\pi$, find the Fourier expression of $f(x)$. Hence deduce that $\dfrac{\pi^2}{6}=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+...$

Solution: Here, $f(x)=x+x^2,\ -\pi<x<\pi$

Fourier series for a function $f(x)$ can be written as:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\text{, where} \qquad \ldots\ldots(1)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

Now

$a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}f(x)dx$, substitute $f(x)=x+x^2$.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)dx$$

Integrate

$$a_0=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{(-\pi)^2}{2}+\frac{(-\pi)^3}{3}\right)\right]$$

$$a_0=\frac{1}{\pi}\left[\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)\right]=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]\text{ (Simplify)}$$

$$a_0=\frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$a_0=\frac{2}{3}\pi^2$$

Find a_n.

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

Integrate by parts.

$$a_n=\frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n}-\int\left(1+2x\right)\frac{\sin nx}{n}dx\right]_{-\pi}^{\pi}$$

$$a_n = \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\frac{\cos nx}{n^2} + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

Substitute the limits.

$$a_n = \frac{1}{\pi}\left[(2\pi+1)\frac{\cos nx}{n^2} - (-2\pi+1)\frac{\cos(-nx)}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[4\pi\frac{\cos nx}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

Find b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Substitute $f(x) = x + x^2$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx\, dx$$

Integrate by parts.

$$b_n = \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - \int (1+2x)\left(-\frac{\cos nx}{n}\right)dx\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-(x+x^2)\left(\frac{\cos nx}{n}\right) + \frac{1}{n}\int (1+2x)\cos nx\, dx\right]_{-\pi}^{\pi}$$

Use again integrate by parts.

$$b_n = \frac{1}{\pi}\left[-(x+x^2)\left(\frac{\cos nx}{n}\right) + \frac{1}{n}\left\{(1+2x)\frac{\sin nx}{n} - \int 2\frac{\sin nx}{n}dx\right\}\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-(x+x^2)\left(\frac{\cos nx}{n}\right) + \frac{1}{n}\left\{(1+2x)\frac{\sin nx}{n} + 2\frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[-(x+x^2)\left(\frac{\cos nx}{n}\right) + (1+2x)\frac{\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

Substitute the limits.

$$b_n = \frac{1}{\pi}\left[\begin{array}{l} \left\{-(\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right) + (1+2\pi)\frac{\sin n\pi}{n^2} + 2\frac{\cos n\pi}{n^3}\right\} - \\ \left\{-(-\pi+(-\pi)^2)\left(\frac{\cos n(-\pi)}{n}\right) + (1+2(-\pi))\frac{\sin n(-\pi)}{n^2} + 2\frac{\cos n(-\pi)}{n^3}\right\}\end{array}\right]$$

$$b_n = \frac{1}{\pi}\left[\begin{array}{l} \left\{-(\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right) + (1+2\pi)\frac{\sin n\pi}{n^2} + 2\frac{\cos n\pi}{n^3}\right\} + \\ (-\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right) + (1-2\pi)\frac{\sin n\pi}{n^2} - 2\frac{\cos n\pi}{n^3}\end{array}\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n}\right] \qquad [\because\ \sin n\pi = 0]$$

$$b_n = -\frac{2\pi}{n}(-1)^n \qquad \left[\because\ \cos n\pi = (-1)^n\right]$$

Substitute the values of a_0, a_n and b_n in (1).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty}\left(-\frac{2\pi}{n}(-1)^n\right)\sin nx$$

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\right] - 2\left[-\sin x + \frac{1}{2^2}\sin 2x - \frac{1}{3}\sin 3x + \ldots\right] \qquad \ldots\ldots(2)$$

Substitute $x = \pi$.

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \ldots\right] - 2\left[-\sin\pi + \frac{1}{2^2}\sin 2\pi - \frac{1}{3}\sin 3\pi + \ldots\right] \qquad \ldots\ldots(3)$$

Now substitute $x = -\pi$ in (2).

$$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[-\cos(-\pi) + \frac{1}{2^2}\cos 2(-\pi) - \frac{1}{3^2}\cos 3(-\pi) + \ldots\right] - 2\left[-\sin(-\pi) + \frac{1}{2^2}\sin 2(-\pi) - \frac{1}{3}\sin 3(-\pi) + \ldots\right] \qquad \ldots\ldots(4)$$

Add (3) and (4).

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\ldots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}\ldots$$

Q.2 Find the Fourier series of $f(x)$, if $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Solution: Here, $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Fourier series for a function $f(x)$ can be written as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{, where} \qquad \ldots\ldots(1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Now

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \text{ substitute } f(x) = \begin{bmatrix} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{bmatrix}.$$
$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi} x\,dx\right]$$

Integrate

$$a_0 = \frac{1}{\pi}\left[-\pi\left\{x\right\}_{-\pi}^{0} + \left\{\frac{x^2}{2}\right\}_{0}^{\pi}\right]$$

Substitute the limits.

$$a_0 = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$
$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\, dx + \int_{0}^{\pi} x\cos nx\, dx\right]$$

Integrate

$$a_n = \frac{1}{\pi}\left[-\pi\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0} + \left\{\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right\}_{0}^{\pi}\right]$$

$$a_n = \frac{1}{n^2\pi}\left(\cos n\pi - 1\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$
$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\, dx + \int_{0}^{\pi} x\sin nx\, dx\right]$$

Integrate

$$b_n = \frac{1}{\pi}\left[\left\{\frac{x\cos nx}{n}\right\}_{-\pi}^{0} + \left\{-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right\}_{0}^{\pi}\right]$$

$$b_n = \frac{1}{n}(1-\cos n\pi)$$

Substitute the values of a_0, a_n and b_n in (1).

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} \quad \dots (2)$$

At the point 0f discontinuity

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = \frac{1}{2}(0-0) = 0$$

Substitute $x = 0$ in (2).

$$0 = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

This shows

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

Q.3 Obtain the Fourier series expansion of $f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0 \end{cases}$

Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$

Solution: Here

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0 \end{cases}$$

Fourier series for a function $f(x)$ can be written as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{, where} \qquad \text{......(1)}$$

Here $f(x)$ is an even function therefore, $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}(-x)dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = -\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(-x)\cos nx\,dx = -\frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = -\frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[1-(-1)^n\right]$$

$$a_n = \begin{cases} 0, & n \text{ is odd} \\ \frac{4}{n^2\pi}, & n \text{ is even} \end{cases}$$

Substitute the values of a_0, a_n and b_n in (1).

$$f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right] \qquad ......(2)$$

At the point 0f discontinuity

$$f(0) = \frac{1}{2}\left[f(0-0) + f(0+0)\right] = \frac{1}{2}(0-0) = 0$$

Substitute $x = 0$ in (2).

$$0 = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right)$$

This shows

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$$

Q.4 Fin the Fourier series expansion of the periodic function of period 2π.

$$f(x) = e^x, \quad 0 < x < 2\pi$$

Solution: Here, $f(x) = e^x$

Fourier series for a function $f(x)$ can be written as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{, where} \qquad ......(1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

Now

$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$, substitute $f(x) = e^x$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^x dx = \frac{1}{\pi}\left[e^x\right]_0^{2\pi} = \frac{1}{\pi}\left[e^{2\pi} - e^0\right]$$

$$a_0 = \frac{1}{\pi}\left(e^{2\pi} - 1\right)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\, dx$$

$$a_n = \frac{1}{\pi}\left[\frac{e^x}{1^2 + n^2}\{1.\cos nx + n\sin nx\}\right]_0^{2\pi} \qquad \because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}\{a\cos bx + b\sin bx\}$$

$$a_n = \frac{1}{\pi}\left[\left\{\frac{e^{2\pi}}{1^2+n^2}\left(\cos 2n\pi + n\sin 2n\pi\right)\right\} - \left\{\frac{e^0}{1^2+n^2}\left(\cos 0 + n\sin 0\right)\right\}\right]$$

$$a_n = \frac{1}{\pi}\left[\left\{\frac{e^{2\pi}}{1^2+n^2}\left(\cos 2n\pi\right)\right\} - \left\{\frac{1}{1^2+n^2}\right\}\right]$$

$$a_n = \frac{1}{\pi}\left[\left\{\frac{e^{2\pi}}{1^2+n^2}\right\} - \left\{\frac{1}{1^2+n^2}\right\}\right]$$

$$a_n = \frac{1}{\pi}\frac{1}{\left(1^2+n^2\right)}\left(e^{2\pi}-1\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\, dx$$

$$b_n = \frac{1}{\pi}\left[\frac{e^x}{1^2+n^2}\left\{1.\sin nx - n\cos nx\right\}\right]_0^{2\pi} \qquad \because \quad \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left\{a\sin bx - b\cos bx\right\}$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{2\pi}}{1^2+n^2}\left\{1.\sin 2n\pi - n\cos 2n\pi\right\} - \frac{e^0}{1^2+n^2}\left\{1.\sin 0 - n\cos 0\right\}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{2\pi}}{1^2+n^2}\left\{-n\right\} - \frac{1}{1^2+n^2}\left\{-n\right\}\right]$$

$$b_n = -\frac{n}{\pi}\frac{1}{\left(1^2+n^2\right)}\left(e^{2\pi}+1\right)$$

Substitute the values of a_0, a_n and b_n in (1).

$$f(x) = \frac{1}{2\pi}\left(e^{2\pi}-1\right) + \frac{1}{\pi}\left(e^{2\pi}-1\right)\sum_{n=1}^{\infty}\frac{1}{\left(1^2+n^2\right)}\cos nx - \frac{1}{\pi}\left(e^{2\pi}+1\right)\sum_{n=1}^{\infty}\frac{n}{\left(1^2+n^2\right)}\sin nx$$

Q.5 Find Fourier series expansion of $f(x) = x^2,\ -\pi < x < \pi$

Solution: Here, $f(x) = x^2,\ -\pi < x < \pi$

Fourier series for a function $f(x)$ is written as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$f(x) = x^2$ is an even function. Therefore, $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \qquad \left[\text{As } f(x) \text{ is even function}\right]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \qquad \left[\text{As } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad f(x) \text{ and } \cos nx \text{ are even functions}\right]$$

Therefore,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$a_0 = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{2}{3}\pi^2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx$$

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \int 2x\frac{\sin nx}{n}dx\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{\pi^2 \sin n\pi}{n} + \frac{2x\cos n\pi}{n^2} - \frac{2\sin n\pi}{n^3}\right]$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

$b_n = 0$ [As x^2 is an even function]

Therefore, Fourier series is:

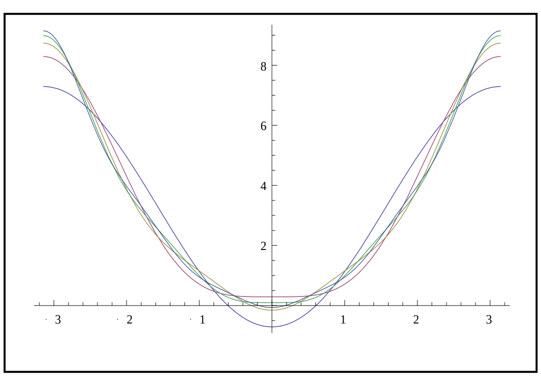
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$ https://youtu.be/QhCjXDGzW4c

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$$

$$f(x) = \frac{\pi^2}{3} + \frac{4(-1)}{1^2}\cos x + \frac{4(-1)^2}{2^2}\cos 2x + \frac{4(-1)^3}{3^2}\cos 3x + \frac{4(-1)^4}{4^2}\cos 4x + \frac{4(-1)^5}{5^2}\cos 5x + \ldots$$

Put $x = 0$

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots\ldots = \frac{\pi^2}{12}}$$



Q.6 Expand the function $f(x)=x\sin x,$ as a Fourier series in the interval $-\pi\le x\le\pi$. Hence deduce that $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+...=\frac{\pi-2}{4}$

Solution: Here, $f(x)=x\sin x,\ -\pi\le x\le\pi$

Fourier series for a function $f(x)$ is written as:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$f(x)=x\sin x$ is an even function. Therefore, $b_n=0$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx \qquad [\text{As } f(x) \text{ is even function}]$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx \qquad \left[\text{As } a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx,\quad f(x)\text{ and }\cos nx\text{ are even functions}\right]$$

Therefore,

$$a_0=\frac{2}{\pi}\int_0^{\pi}x\sin x\,dx$$

$$=\frac{2}{\pi}\left[x(-\cos x)-\int 1(-\cos x)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[x(-\cos x)-(-\sin x)\right]_0^{\pi}=\frac{2}{\pi}[\pi]=2$$

$$\boxed{a_0=2}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos nxdx$$

$$a_n=\frac{1}{\pi}\int_0^{\pi}x\left(2\cos nx\sin x\right)dx$$

$$a_n=\frac{1}{\pi}\int_0^{\pi}x\left\{\sin(n+1)x-\sin(n-1)\right\}dx \qquad [\because\ \ 2\cos A\sin B=\sin(A+B)-\sin(A-B)]$$

$$a_n=\underbrace{\frac{1}{\pi}\int_0^{\pi}x\sin(n+1)x\,dx}_{I_1}-\underbrace{\frac{1}{\pi}\int_0^{\pi}x\sin(n+1)x\,dx}_{I_2}$$

$$I_1=\frac{1}{\pi}\int_0^{\pi}x\sin(n+1)x\,dx$$

$$I_1=\frac{1}{\pi}\left[-x\frac{\cos(n+1)x}{n+1}-\int\left(-1.\frac{\cos(n+1)x}{n+1}\right)dx\right]_0^{\pi} \quad \text{[Integration by parts]}$$

$$I_1=\frac{1}{\pi}\left[-x\frac{\cos(n+1)x}{n+1}+\frac{\sin(n+1)x}{(n+1)^2}\right]_0^{\pi}$$

$$I_1 = \frac{1}{\pi}\left[\left\{-\pi\frac{\cos(n+1)\pi}{n+1} + \frac{\sin(n+1)\pi}{(n+1)^2}\right\} - \{0\}\right] \qquad \text{[Substitute the upper and lower limits]}$$

$$I_1 = \frac{1}{\pi}\left[\left\{-\pi\frac{\cos(n+1)\pi}{n+1}\right\}\right] \qquad [\because \sin(n+1)\pi = 0]$$

$$I_1 = -\frac{(-1)^{n+1}}{n+1}$$

Similarly

$$I_2 = -\frac{(-1)^{n-1}}{n-1}$$

Therefore,

$$a_n = I_1 - I_2$$

$$a_n = -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^{n+1}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$a_n = \frac{2(-1)^{n+1}}{n^2-1} \qquad \text{[This value of } a_n \text{ will fail for } n = 1]$$

Now

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x dx = \frac{1}{\pi}\int_0^\pi x\left(2\sin x\cos x\right)dx$$

$$a_1 = \frac{1}{\pi}\int_0^\pi x\sin 2x\ dx$$

$$a_1 = \frac{1}{\pi}\left[-x\frac{\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^\pi \qquad \text{[Integration by parts]}$$

$$a_1 = \frac{1}{\pi}\left[-\frac{\pi}{2}\right] = -\frac{1}{2}$$

Hence,

$$f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

$$x\sin x = 1 + 2\left[-\frac{1}{4}\cos x - \frac{1}{1.3}\cos 2x + \frac{1}{2.4}\cos 3x - \frac{1}{3.5}\cos 4x + \ldots\right]$$

Put $x = \frac{\pi}{2}$ for hence part

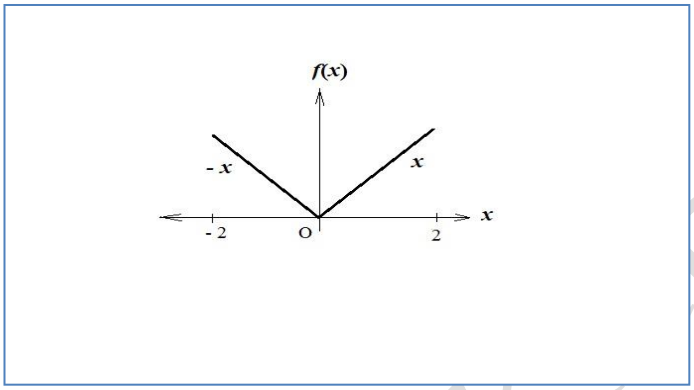
$$\frac{\pi}{2} = 1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\right\}$$

$$\boxed{\frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots}$$

Q.7 A periodic function of period 4 is defined as $f(x) = |x|, -2 < x < 2.$ Find its Fourier series expansion.

Solution: $f(x) = |x|, -2 < x < 2$

$$f(x) = \begin{cases} x, & 0 < x < 2 \\ -x, & -2 < x < 0 \end{cases}$$



Fourier series for a function $f(x)$ can be written as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$f(x) = |x|$ is an even function. Therefore, $b_n = 0$

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)dx = \frac{1}{2}\int_{-2}^{2} f(x)dx$$

$$a_0 = \frac{1}{2}\int_{-2}^{0}(-x)dx + \frac{1}{2}\int_{0}^{2} x\,dx$$

$$a_0 = \frac{1}{2}\left[-\frac{x^2}{2}\right]_{-2}^{0} + \frac{1}{2}\left[\frac{x^2}{2}\right]_{0}^{2} = \frac{1}{4}[0+4] + \frac{1}{4}[4-0]$$

$$\boxed{a_0 = 2}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx$$

$$a_n = \frac{1}{2}\int_{-2}^{0}(-x)\cos\left(\frac{n\pi x}{c}\right)dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\pi x}{c}\right)dx$$

$$a_n = \frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right) - (-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0}$$

$$+\frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right) - 1\left(-\frac{4}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{2}\right)\right]_{0}^{2}$$

$$a_n = \frac{1}{2}\left[0 - \frac{4}{n^2\pi^2} + \frac{4}{n^2\pi^2}(-)^n\right] + \frac{1}{2}\left[0 + \frac{4}{n^2\pi^2}(-)^n - \frac{4}{n^2\pi^2}\right]$$

$$a_n = \frac{1}{2}\frac{4}{n^2\pi^2}\left[-1+(-1)^n+(-1)^n-1\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$a_n = \begin{cases} -\dfrac{4}{n^2\pi^2}, & \text{If } n \text{ is odd.} \\ 0, & \text{If } n \text{ is even.} \end{cases}$$

Fourier series is

$$f(x) = \frac{a_0}{2} + a_1\cos\left(\frac{\pi x}{c}\right) + a_2\cos\left(\frac{2\pi x}{c}\right) + \ldots$$

$$\boxed{|x| = 1 - \frac{8}{\pi^2}\left[\frac{\cos\left(\frac{\pi x}{2}\right)}{1^2} + \frac{\cos\left(\frac{3\pi x}{2}\right)}{3^2} + \frac{\cos\left(\frac{5\pi x}{2}\right)}{5^2} + \ldots\right]}$$

Half period series:

Cosine Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{c}\int_0^c f(x)dx$, $a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx$

Sine Series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}$$

Where, $b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx$

Q.8 Expand $f(x) = e^x$ in a cosine series over $(0,1)$.

Solution: $f(x) = e^x$

Fourier cosine series can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where, $a_0 = \frac{2}{c}\int_0^c f(x)dx$, $a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx$

$$a_0 = \frac{2}{1}\int_0^1 e^x dx = 2\left[e^x\right]_0^1 = 2\left[e^1 - e^0\right]$$

$$a_0 = 2(e-1)$$

$$a_n = \frac{2}{1}\int_0^1 e^x \cos\left(\frac{n\pi x}{1}\right)dx$$

$$a_n = 2\int_0^1 e^x \cos n\pi x\,dx$$

$$a_n = 2\left[\frac{e^x}{1^2+n^2\pi^2}\left(1\cos n\pi x + n\pi \sin n\pi x\right)\right]_0^1 \qquad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)\right]$$

$$a_n = 2\left[\left\{\frac{e^1}{1^2+n^2\pi^2}\left(1\cos n\pi(1) + n\pi \sin n\pi(1)\right)\right\} - \left\{\frac{e^0}{1^2+n^2\pi^2}\left(1\cos n\pi(0) + n\pi \sin n\pi(0)\right)\right\}\right]$$

$$a_n = 2\left[\left\{\frac{e}{1+n^2\pi^2}\left(\cos n\pi + 0\right)\right\} - \left\{\frac{1}{1+n^2\pi^2}\left(\cos(0) + n\pi \sin(0)\right)\right\}\right]$$

$$a_n = 2\left[\left\{\frac{e}{1+n^2\pi^2}(-1)^n\right\} - \left\{\frac{1}{1+n^2\pi^2}(1+0)\right\}\right]$$

$$a_n = 2\left[\frac{e}{1+n^2\pi^2}(-1)^n - \frac{1}{1+n^2\pi^2}\right]$$

$$a_n = \frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right]$$

Therefore, the series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$e^x = \frac{2(e-1)}{2} + \sum_{n=1}^{\infty}\frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right]\cos nx$$

$$e^x = e - 1 + \frac{2}{1+1^2\pi^2}[-e-1]\cos x + \frac{2}{1+2^2\pi^2}[e-1]\cos x \frac{2}{1+3^2\pi^2}[-e-1]\cos x + \ldots$$

$$e^x = e - 1 + \frac{2}{1+\pi^2}(-e-1)\cos x + \frac{2}{1+4\pi^2}(e-1)\cos x \frac{2}{1+9\pi^2}(-e-1)\cos x + \ldots$$

Q.9 Obtain the half rage sine series for the function $f(x) = x^2$ in the interval $0 < x < 3$.

Solution: Do yourself