GRAPHIC ERA HILL UNIVERSITY

Department of Mathematics

TMA-316 : Discrete Mathematical Structures and Combinatorics

(Assignment No: 5)

Last Date of Submission : 30-Nov-2023

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1. How many different ways are there to place four different colored tiles in a row? Assume the tiles are red, blue, green and yellow. **Answer : 24**

2. How many different ways are there to place three different colored tiles chosen form a set of five different colored tiles in a row? Assume the five tiles are red, blue, green, yellow and orange. **Answer : 60**

3. In a school soccer league with seven teams, in how many ways can they finish in the positions "winner", "runner-up" and "third place?" **Answer : 210**

4. The school hiking club has 10 members.
   (1) In how many ways can 3 members of the club be chosen for the Rules Committee?
   (2) In how many ways can 3 members of the club be chosen for the offices of President, Vice President and Secretary? **Answers : 120, 720**

5. If we reduce the number of elements by two, the number of permutations reduces thirty times. Find the number of elements. **Answer :** $\mathbf{P(x-2) = \frac{P(x)}{30} \wedge x \geq 2 \Rightarrow x = 6}$

6. The registration number of the vehicle consists of two letters, three numbers and two letters. How many registration numbers can we form if we use 25 letters ? **Answers :** $\mathbf{10^3 \times 25^4}$

7. There are places where in the buses are used tickets with nine squares numbered from 1 to 9. When a passenger boards the bus, he inserts the ticket into the machine which makes little holes through three or four of the squares. How many different ways exists for the perforation of the bus ticket ? **Answer : 210**

8. There are 20 students in the class. In how many ways can we choose a couple for a weekly service ? **Answer : 190**

9. How many players participated in the tournament in table tennis, where 21 matches was played and each player played with each other exactly once ? **Answer : 7**

10. How many possible ways there is for 12 visitors of the cinema to sit in one row, if each of the six married couples wants to sit next to each other ? **Answer :** $\mathbf{2^6 \times 6!}$

Pigeonhole Principle

The Pigeonhole Principle (also known as the Dirichlet box principle, Dirichlet principle or box principle) states that if $n+1$ or more pigeons are placed in n holes, then one hole must contain two or more pigeons.

The extended version of the Pigeonhole Principle states that if k objects are placed in n boxes then at least one box must hold at least $\left\lceil \frac{k}{n} \right\rceil$ objects. Here $\lceil \cdot \rceil$ denotes the ceiling function.

11. If a Martian has an infinite number of red, blue, yellow, and black socks in a drawer, how many socks must the Martian pull out of the drawer to guarantee he has a pair?

    Solution : The Martian must pull 5 socks out of the drawer to guarantee he has a pair. In this case the pigeons are the socks he pulls out and the holes are the colors. Thus, if he pulls out 5 socks, the Pigeonhole Principle states that some two of them have the same color. Also, note that it is possible to pull out 4 socks without obtaining a pair.

12. Suppose S is a set of $n+1$ integers. Prove that there exist distinct $a, b \in S$ such that $a-b$ is a multiple of n.

    Solution : Consider the residues of the elements of S, modulo n. By the Pigeonhole Principle, there exist distinct $a, b \in S$ such that $a \equiv b \pmod{n}$, as desired.

13. Seven line segments, with lengths no greater than 10 inches, and no shorter than 1 inch, are given. Show that one can choose three of them to represent the sides of a triangle.

    Solution : If we have 2 line segments of lengths a, b (so that $a \geq b$ and $a-b<1$); then if we have at least a line segment of length c left to check (so that $b \geq c$), we will get that a, b and c are sides of a triangle. This is true because $c \geq 1$. This means if we check, for example, 5.5 and 5 then any number less than or equal to 5 will satisfy the condition of them being sides of a triangle.

14. There are 51 senators in a senate. The senate needs to be divided into n committees such that each senator is on exactly one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does 'not' necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

    Solution : In the worst case, consider that senator S hates a set of 3 senators, while he himself is hated by a completely different set of 3 other senators. Thus, given one senator, there may be a maximum of 6 other senators whom he cannot work with. If we have a minimum of 7 committees, there should be at least one committee suitable for the senator S after the assignment of the 6 conflicting senators.

15. Given a group of 100 people, at minimum, how many people were born in the same month?

    Solution : We have 12 months, so $100/12 = 8.33$ gives us the content of each month. But we cannot have partial people in each month, so we round up: $\left\lceil \frac{100}{12} \right\rceil = 9$.

16. Among any set of 21 decimal digits there must be 3 that are the same.

    Solution : This follows because when 21 objects are distributed into 10 boxes, one box must have $\left\lceil \frac{21}{10} \right\rceil = 3$ elements.

17. What is the minimum number of students required in a course to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D and F?

    Solution : We have $k = 5$ and $r = 6$. We need to compute N such that $\left\lceil \frac{N}{k} \right\rceil = r$ or more precisely $\left\lceil \frac{N}{5} \right\rceil = 6$. We can compute the smallest integer with this property as $N = k(r-1)+$. Plugging k and r into this equation gives us $N = 5(6-1)+1 = 26$. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

18. What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10 digit telephone numbers? (Assume that telephone numbers are of the form $NXX - NXX - XXXX$, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

    Solution : There are eight million different phone numbers of the form $NXX - XXXX$ (as we have shown previously). Hence, by the generalized pigeonhole principle, among 25 million telephones, at least $\left\lceil \frac{25,000,000}{8,000,000} \right\rceil = 4$ of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 10 digit numbers are different.

    Recurrence relation

    A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i < n$).

    Example :- Fibonacci series - $F_n = F_{n-1} + F_{n-2}$, Tower of Hanoi - $F_n = 2F_{n-1} + 1$

19. Identify each of the given recurrence as linear homogeneous of constant coefficients. If the relation is a linear homogeneous relation, give its degree :-
    a) $a_n = 3a_{n-1}$, b) $a_n = 2^n a_{n-1}$, c) $a_n = 6a_{n-1} + 7$, d) $a_n = 3a_{n-1} - 2a_{n-2}$,
    e) $a_n = a_{n-4}a_{n-2}$, f) $a_n = 3n\, a_{n-1}$, g) $a_n = a_{n-1}^2 + a_{n-2}$.

20. Show that $a_n = 0, a_n = 4^n, a_n = n4^n$ and $a_n = 2.4^n + 3n4^n$ are all solutions of the same recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$.

21. Show that $a_n = C_1 + C_2 2^n - n$ is a solution of the recurrence relation $a_n - 3a_{n-1} + 2a_{n-2} = 1$.

22. Show that $a_n = C_1 2^n + C_2 4^n$ is a solution of the recurrence relation $a_n - 6a_{n-1} + 8a_{n-2} = 0$.

23. Consider the sequence $\{a_n\}, a_n = \frac{n-1}{n+1}$, find first six terms of the sequence. Hence, show that $a_{n+1} - a_n = \frac{2}{(n+1)(n+2)}$.

24. Use the iterative method to find the solution to each of the following recurrence relations and initial condition :-
    a) $a_n = a_{n-1} + 2; a_0 = 1$, b) $a_n = 2a_{n-1} + 1; a_1 = 7$, c) $a_n = a_{n-1} + n; a_1 = 4$,
    d) $a_n = na_{n-1}; a_0 = 5$, e) $a_n = a_{n-1} + 6a_{n-2}; n \geq 2, a_0 = 1, a_1 = 1$.

25. Show that $a_n = -2^{n+1}$ is a solution of non-homogeneous recurrence relation $a_n = 3a_{n-1} + 2^n$.

26. Show that $a_n = (1/6)\,5^n$ is a solution of recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 2^n$.

27. Find the generating function for the sequence $1, a, a^2, a^3, ...$, where a is a fixed constant.

28. Use generating function to solve the recurrence relations :-
    a) $a_n = 3a_{n-1} + 2; a_0 = 1$,
    b) $a_n - 9a_{n-1} + 20a_{n-2} = 0; a_0 = -3, a_1 = -10$,
    c) $a_{n+2} - 2a_{n+1} + a_n = 12; a_0 = 2, a_1 = 1$.

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End of the assignment..