By Dr. Vishal Chauhan (Unit-IV)

The Del Operator

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}}$$

There are three ways to operate ∇

1. On a scalar - (The gradient) ∇T
2. On a vector function via dot product (The Divergence) $\nabla \cdot \vec{v}$
3. On a vector function via cross product (The Curl) $\nabla \times \vec{v}$

(i) $r = \sqrt{x^2+y^2+z^2}$

$$\nabla r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)\hat{i} + \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right)\hat{j} + \frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)\hat{k}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{1}{\cancel{2}}\frac{\cancel{2}z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\boxed{\hat{r} = \frac{\vec{r}}{r}}$$

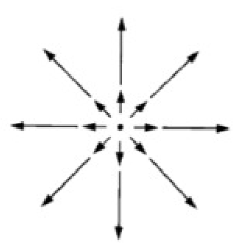
The Divergence

If $\vec{v}$ is a vector function, then

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

$$\nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$\boxed{\nabla \cdot \vec{v} = \frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y + \frac{\partial}{\partial z}V_z}$$



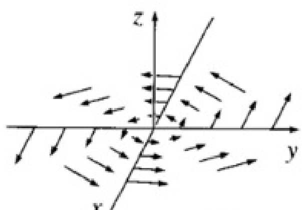
*The divergence is a measure of how much the vector v spreads out (diverges) from the point in question.

*Positive Divergence - spreading out
*Negative Divergence - converging

The Curl

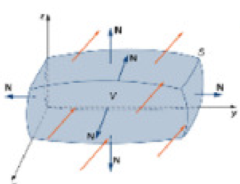
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$



(a)

The Gauss Divergence theorem



$$\int_{\mathcal{V}} (\nabla \cdot \vec{v})\, d\tau = \oint_{S} \vec{v} \cdot d\vec{s}.$$

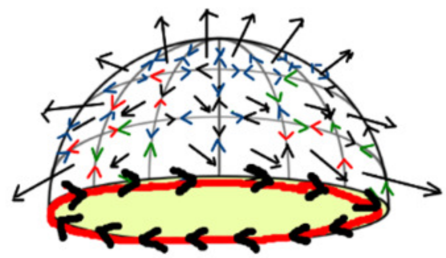
$$\vec{ds} = \hat{n}\, ds$$

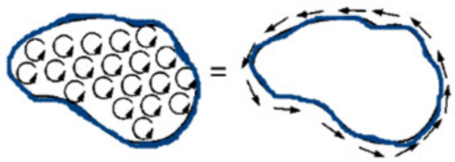
$$\int (\text{faucets within the volume}) = \oint (\text{flow out through the surface}).$$

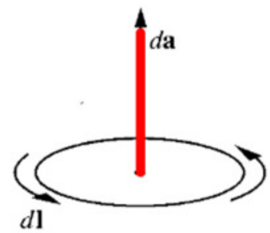
- **Statement:** The Gauss divergence theorem states that the vector's outward flux through a closed surface is equal to the volume integral of the divergence over the area within the surface.
- Gauss divergence theorem is the result that describes the flow of a vector field by a surface to the behaviour of the vector within it.
- It relates the flux of a vector field through the closed surface to the divergence of the field in the volume enclosed.

By Dr. Vishal Chauhan (Unit-IV)

Stokes' theorem







Statement: The line integral of a vector field around a closed path is equal to the surface integral of the normal component of its curl over any closed surface.

$$\boxed{\oint_c \vec{v}\cdot\vec{dl} = \oiint(\vec{\nabla}\times\vec{v})\cdot\vec{ds}}$$

from R.H.S →

$$(\vec{\nabla}\times\vec{v})\cdot\vec{ds} = (\vec{\nabla}\times\vec{v})\cdot\hat{n}\,ds \quad \text{——— (1)}$$

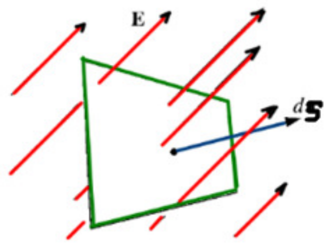
$\because$ we know that $(\vec{\nabla}\times\vec{v}) = \dfrac{\oint_c \vec{v}\cdot\vec{dl}}{ds}\,\hat{n}$

Then from eq^n (1) → $(\vec{\nabla}\times\vec{v})\cdot\hat{n}\,ds = \dfrac{\oint\vec{v}\cdot\vec{dl}}{\cancel{ds}}\,\hat{n}\cdot\hat{n}\,\cancel{ds}$

$\because$ No net curl in complete closed surface due to opposite directions. Only curl due to outer rim.

$$\boxed{\oiint(\vec{\nabla}\times\vec{v})\,\vec{ds} = \oint\vec{v}\cdot\vec{dl} \quad (\text{L.H.S.})}$$

Gauss's Law



Statement: Gauss Law states that the total electric flux out of a closed surface is equal to the charge enclosed divided by the permittivity. The electric flux in an area is defined as the electric field multiplied by the area of the surface projected in a plane and perpendicular to the field

$$\phi = \frac{q}{\epsilon_0} \quad \text{———— (1)}$$

$\because$ Charge Density $(\rho) = \frac{q}{V} = \frac{\text{Charge}}{\text{Volume}}$

For a small area ds

$$\rho = \frac{dq}{dv}, \quad dq = \rho\, dv$$

$$\int dq = \int \rho\, dv$$

$$\boxed{q = \oint_V \rho\, dv} \quad \text{———— (2)}$$

&

$$d\phi = \vec{E}\cdot\vec{ds}$$

$$\int d\phi = \oint_S \vec{E}\cdot\vec{ds}$$

$$\boxed{\phi = \oint_S \vec{E}\cdot\vec{ds}} \quad \text{———— (3)}$$

Put the value of ϕ from eq^n(1)

$$\oint_S \vec{E}\cdot\vec{ds} = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\oint_V \rho\, dv$$

$$\left(\because q = \oint_V \rho\, dv\right)$$

$$\boxed{\oint_S \vec{E}\cdot d\vec{s} = \oint_V \frac{\rho}{\epsilon_0}\,dv} \quad \text{———} (4)$$

From Gauss Divergence Theorem

$$\left(\because \oint_S \vec{E}\cdot d\vec{s} = \oint_V \text{div.}E\,dv\right)$$

$$\oint_V \text{div.}E\,dv = \oint_V \frac{\rho}{\epsilon_0}\,dv$$

$$\boxed{\text{div.}E = \frac{\rho}{\epsilon_0} \quad \text{or} \quad \nabla\cdot E = \frac{\rho}{\epsilon_0}}$$

Maxwell's First Equation.

Modified first equation

Electric displacement vector: $\because \vec{E} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r}$ ——— (1)

If electric field does not depents upon medium then it is known as displacemetn vector.

$$\vec{D} = \frac{q}{4\pi r} \quad \text{———} (2)$$

$$(1)\div(2) \Rightarrow \frac{\vec{E}}{\vec{D}} = \frac{\frac{\cancel{q}}{\cancel{4\pi}\epsilon_0\cancel{r}}}{\frac{\cancel{q}}{\cancel{4\pi r}}} = \frac{1}{\epsilon_0}$$

$$\boxed{\vec{D} = \epsilon_0\vec{E}} \quad \text{———} (3)$$

$$\because \text{div}E = \frac{\rho}{\epsilon_0}$$

$$\rho = \epsilon_0\cdot\text{div}E$$

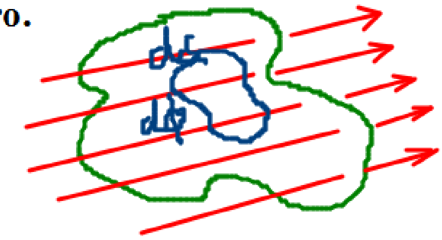
$$\rho = \text{div}(\epsilon_0 E)$$

$$\rho = \text{div.}(\vec{D})$$

$$\boxed{\text{div}D = \rho}$$

Maxwell's Second Equation

Gauss's Law for Magnetism: Magnetic flux passing through a closed surface placed in amagnetic field is equal to zero.



$$\phi_B = 0$$

$$d\phi_B = \vec{B} \cdot \vec{ds}$$

$$\int d\phi_B = \oint_S \vec{B} \cdot \vec{ds}$$

$$\phi_B = \oint_S \vec{B} \cdot \vec{ds}$$

$$\oint_S \vec{B} \cdot \vec{ds} = 0 \quad (\because \phi_B = 0)$$

By Gauss Div. Theorem

$$\oint_S \vec{B} \cdot \vec{ds} = \oint_V \mathrm{div}\, B \, dv$$

$$\therefore \oint_V \mathrm{div}\, B \, dv = 0$$

$$\boxed{\mathrm{div}\, B = 0 \quad \text{or} \quad \vec{\nabla} \cdot \vec{B} = 0}$$

Maxwell's Second Equation

Maxwell's third Equation

Faraday's law of electromagnetic induction:

$$\text{induced emf} = \frac{-d\phi_B}{dt}$$

$$\because \quad \phi_B = \oint_s \vec{B}\cdot\vec{ds}$$

Also e^- field (E) = Potential Gradient

$$E = \frac{V}{d} \text{ or } \frac{de}{dl} \quad (\text{small emf})$$

$$de = E\,dl$$

$$\int de = \oint_l \vec{E}\cdot\vec{dl}$$

$$\underset{(\text{emf})}{e} = \oint_l \vec{E}\cdot\vec{dl}$$

$$\oint_l \vec{E}\cdot\vec{dl} = \frac{-d\phi_B}{dt}$$

$$= -\frac{d}{dt}\oint_s \vec{B}\cdot\vec{ds}$$

$$= \oint_s -\frac{\partial\vec{B}}{\partial t}\,ds$$

By Stoke's theorem $\rightarrow$

$$\oint_l \vec{E}\cdot\vec{dl} = \oint_s \text{Curl}\,\vec{E}\cdot\vec{ds}$$

$$\therefore \oint_s \text{curl}\,\vec{E}\cdot\vec{ds} = \oint_s -\frac{\partial\vec{B}}{\partial t}\,\vec{ds}$$

$$\boxed{\nabla\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}}$$

Maxwell's Fourth Equation

Ampere's Circuital Law

$$\oint_l \vec{B}\cdot dl = \mu_0 I$$

$$\because \; I = \oint_s J\cdot ds \,, \quad J = \text{Current Density}$$

$$\therefore \; \oint_l \vec{B}\cdot dl = \mu_0 \oint_s J\cdot ds$$

$$\oint_l \vec{B}\cdot dl = \oint_s \mu_0 J\, ds$$

By Stokes Theorem

$$\oint_l \vec{B}\, dl = \oint_s \text{Curl}\, \vec{B}\cdot ds$$

$$\oint_s \text{Curl}\, \vec{B}\cdot \vec{ds} = \oint_s \mu_0 \vec{J}\, \vec{ds}$$

$$\therefore \; \text{Curl}\, \vec{B} = \mu_0 \vec{J}$$

or $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

or $\boxed{\vec{\nabla} \times \vec{H} = \vec{J}}$ $\quad \because \vec{B} = \mu_0 \vec{H}$

Fourth Equation

Modified Fourth Equation of Maxwell

$$\because \vec{\nabla}\cdot\vec{D} = \rho$$

$$\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = \frac{\partial \rho}{\partial t}$$

$$\vec{\nabla}\frac{\partial \vec{D}}{\partial t} = \frac{\partial \rho}{\partial t}$$

Add $\vec{\nabla}\cdot\vec{J}$ both sides

$$\vec{\nabla}\cdot\vec{J} + \vec{\nabla}\frac{\partial \vec{D}}{\partial t} = \underbrace{\left[\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t}\right]}_{\text{Equation of continuity}}$$

$$\vec{\nabla}\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0$$

$$\vec{J} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J} = \vec{J} + \vec{J}_D$$