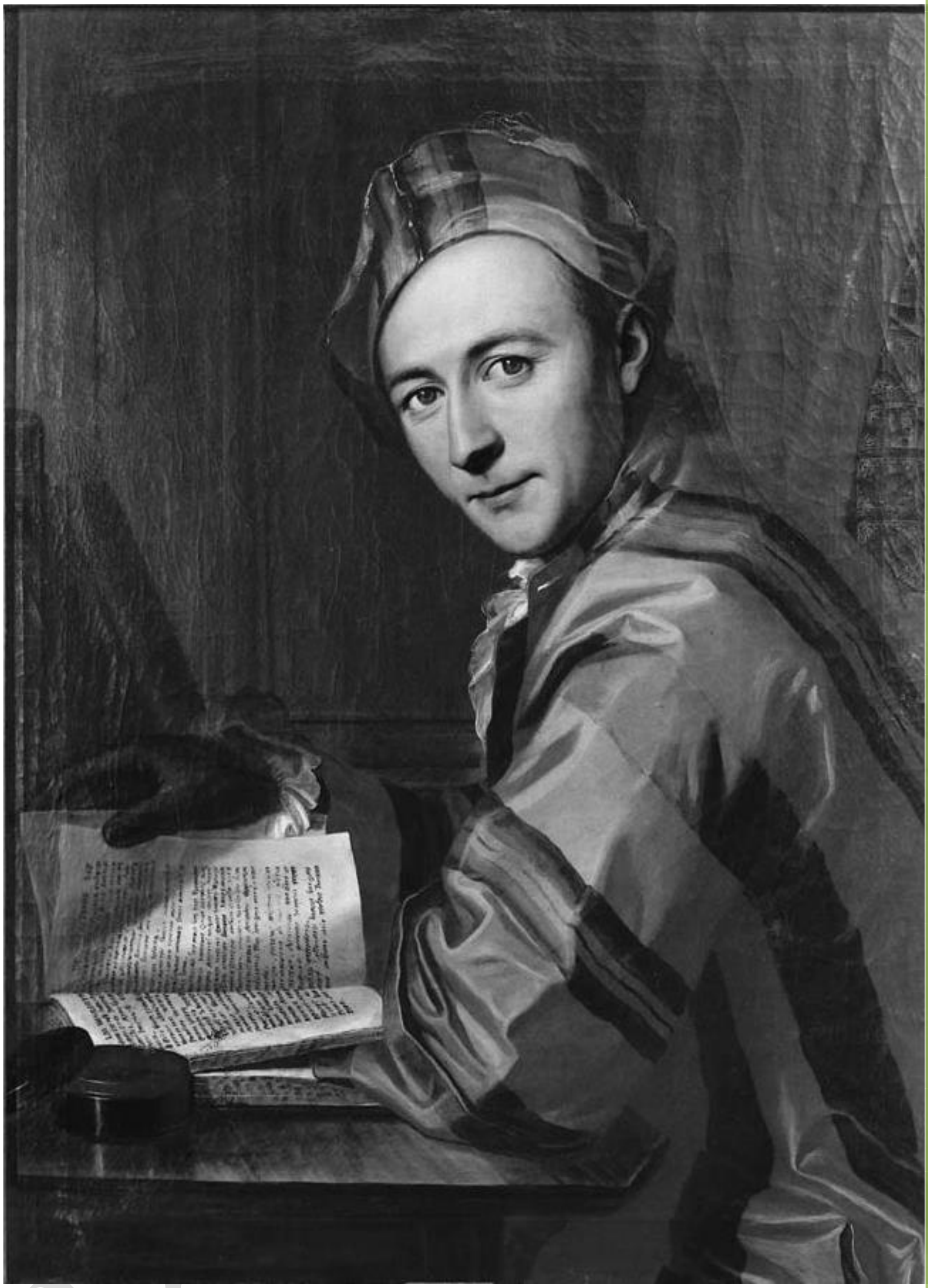
Partial Differential Equations and Applications



Johann Albrecht Euler Swiss-Russian Astronomer and Mathematician (1768 - 1830). He played an important role in developing partial differential equations

Unit IV Partial Differential Equations

LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS OF nTH ORDER WITH CONSTANT COEFFICIENTS

An equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots\ldots + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \ldots (1)$$

is called a homogeneous linear partial differential equation of nth order with constant coefficients. It is called homogeneous because all the terms contain derivatives of the same order.

Putting $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$, (1) becomes

$$(a_0 + D^n + a_1 D^{n-1} \; D' + \ldots\ldots + a_n \; D'^{\,n}) z = F(x,y)$$

$$f(D, D')\, z = F(x, y)$$

RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0 \qquad \text{or} \qquad (a_0 D^2 + a_1 DD' + a_2 D'^2) z = 0$$

1st step : Put $D = m$ and $D' = 1$

$$a_0 m^2 + a_1 m + a_2 = 0$$

This is the auxiliary equation.

2nd step : Solve the auxiliary equation

Case 1. If the roots of the auxiliary equation are real and different; say m_1, m_2

Then C.F. $= f_1(y + m_1 x) + f_2(y + m_2 x)$.

Case 2. If the roots are equal; say m

Then C.F. $= f_1(y + mx) + x f_2(y + mx)$

Example 1.

Solve $(D^3 - 4D^2 \; D' + 3D \; D'^2) z = 0$.

Solution. $(D^3 - 4D^2 \; D' + 3D \; D'^2) z = 0$

Its auxiliary equation is

$$m^3 - 4m^2 + 3m = 0 \quad \Rightarrow \quad m(m^2 - 4m + 3) = 0$$

$$m(m-1)(m-3) = 0 \quad \Rightarrow \quad m = 0, 1, 3$$

The required solution is $z = f_1(y) + f_2(y + x) + f_3(y + 3x)$

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x^2} + \frac{4\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = f_1(y + x) + f_2(y - 5x)$
2. $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = f_1(2y - x) + f_2(y - 2x)$
3. $(D^3 - 6D^2 D' + 11 D D'^2 - 6D'^3) z =$ **Ans.** $z = f_1(y + x) + f_2(y + 2x) + f_3(y + 3x)$
4. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = f_1(y + x) + x f_2(y + x)$

RULES FOR FINDING THE PARTICULAR INTEGRAL

Given partial differential equation is

$$f(D,\ D')\,z = F(x,\ y)$$

$$P.I. = \frac{1}{f(D,D')}\ F(x,\ y)$$

(*i*) When $F(x,y) = e^{ax+by}$

$$P.I. = \frac{1}{f(D,D')}e^{ax+by} = \frac{e^{ax+by}}{f(a,b)} \qquad [\text{Put } D = a,\ D' = b]$$

(*ii*) When $F(x,y) = \sin(ax + by)$ or $\cos(ax + by)$

$$P.I. = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by) \text{ or } \cos(ax + by)$$

$$= \frac{\sin(ax+by)\, or \cos(ax+by)}{f(-a^2, -ab, -b^2)} \qquad \begin{bmatrix} \text{Put } D^2 = -a^2 \\ DD' = -ab,\ D'^2 = -b^2 \end{bmatrix}$$

(*iii*) When $F(x,y) = x^m y^n$

$$P.I. = \frac{1}{f(D,D')}x^m y^n = [f(D,D')]^{-1} x^m y^n$$

Expand $[f(D,\ D')]^{-1}$ in ascending power of D or D' and operate on $x^m y^n$ term by term.

(*iv*) When = Any function $F(x,\ y)$

$$P.I. = \frac{1}{f(D,D')}F(x,y)$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions

Considering $f(D,\ D')$ as a function of D alone

$$P.I. = \frac{1}{D-mD'}F(x,y) = \int F(x, c-mx)\,dx$$

where c is replaced by $y + mx$ after integration.

Case 1. *When R.H. S.* $= e^{ax+by}$

Example 1.

Solve : $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

Solution. $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

Given equation in symbolic form is

$$(D^3 - 3D^2 D' + 4D'^3)z = e^{x+2y}$$

Its A.E. is $m^3 - 3m^2 + 4 = 0$ whence, $m = -1, 2, 2$.

$$\text{C.F.} = f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$$

$$P.I. = \frac{1}{D^3 - 3D^2D' + 4D'^3}e^{x+2y}$$

Put $D = 1, D' = 2$ $$= \frac{1}{1-6+32} e^{x+2y} = \frac{e^{x+2y}}{27}$$

Hence complete solution is

$$z = f_1(y-x) + f_2(y+2x) + xf_3(y+2x) + \frac{e^{x+2y}}{27}$$

Solve the following equations:

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x+2y}$ **Ans.** $z = f_1(y+x) + f_2(y-x) - \frac{e^{x+2y}}{3}$

2. $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = e^{x+y}$ **Ans.** $z = f_1(y+2x) + f_2(y+3x) + \frac{1}{2} e^{x+y}$

Case II. When R.H.S. = $\sin(ax+by)$ or $\cos(ax+by)$

Example 1.

Solve $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x+2y)$

Solution. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x+2y)$

Putting $$\frac{\partial}{\partial x} = D, \quad \frac{\partial}{\partial y} = D'$$

$$D^3 z - 4D^2 D' z + 4D D'^2 z = 2\sin(3x+2y)$$

A.E. is $D^3 - 4D^2 D' + 4D D'^2 = 0 \Rightarrow D(D^2 - 4DD' + 4D'^2) = 0$

Put $D = m, D' = 1$

$$m(m^2 - 4m + 4) = 0 \Rightarrow m(m-2)^2 = 0 \Rightarrow m = 0, 2, 2$$

C.F. is $f_1(y) + f_2(y+2x) + xf_3(y+2x)$

$$P.I. = \frac{1}{D^3 - 4D^2 D' + 4DD'^2} 2\sin(3x+2y) = 2.\frac{1}{D(D^2 - 4DD' + 4D'^2)}\sin(3x+2y)$$

$$= 2.\frac{1}{D[-9-4(-6)+4(-4)]}\sin(3x+2y) = -\frac{2}{D}\sin(3x+2y)$$

$$= -\frac{2}{3}[-\cos(3x+2y)] = \frac{2}{3}\cos(3x+2y)$$

General solution is

$$z = f_1(y) + f_2(y+2x) + xf_3(y+2x) + \frac{2}{3}\cos(3x+2y)$$

Case III. When R.H.S. = $x^m y^n$

Example 1.

Find the general integral of the equation

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$$

Solution. $\dfrac{\partial^2 z}{\partial x^2}+3\dfrac{\partial^2 z}{\partial x\partial y}+2\dfrac{\partial^2 z}{\partial y^2}=x+y$

with $D=\dfrac{\partial}{\partial x}, D'=\dfrac{\partial}{\partial y}$, the given equation can be written in the form

$(D^2+3\,DD'+2\,D'^2)\,z=x+y$

Writing $D=m$ and $D'=1$, the auxiliary equation is

$$m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2$$

$\therefore$ C.F. $=f_1(y-x)+f_2(y-2x)$

$$\text{P.I.}=\frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$=\frac{1}{D^2}\left(1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)^{-1}(x+y)=\frac{1}{D^2}\left(1-\frac{3D'}{D}....\right)(x+y)$$

$$=\frac{1}{D^2}\left[x+y-3\frac{1}{D}(1)\right]=\frac{1}{D^2}[x+y-3x]$$

$$=\frac{1}{D^2}[y-2x]=\frac{x^2}{2}y-\frac{x^3}{3}$$

Hence the complete solution is

$$z=f_1(y-x)+f_2(y-2x)+\frac{x^2y}{2}-\frac{x^3}{3}$$

Example 2. Solve $\dfrac{\partial^2 z}{\partial x^2}-3\dfrac{\partial^2 z}{\partial x\partial y}+2\dfrac{\partial^2 z}{\partial y^2}=e^{x+y}+\cos(x+2y)+e^{2x-y}$

Solution : $\dfrac{\partial^2 z}{\partial x^2}-3\dfrac{\partial^2 z}{\partial x\partial y}+2\dfrac{\partial^2 z}{\partial y^2}=e^{x+y}+\cos(x+2y)+e^{2x-y}$

Write the equation in operator form

$\left(D^2-3DD'+2D'^2\right)z=e^{x+y}+\cos(x+2y)+e^{2x-y}$

Auxiliary equation: $m^2-3m+2=0 \Rightarrow (m-1)(m-2)=0 \Rightarrow m=1,\ 2$

$$CF=f_1(y+x)+f_2(y+2x)$$

$$PI=\frac{1}{D^2-3DD'+2D'^2}e^{x+y}+\frac{1}{D^2-3DD'+2D'^2}\cos(x+2y)+\frac{1}{D^2-3DD'+2D'^2}e^{2x-y}$$

$$PI=\frac{x}{2D-3D'}e^{x+y}+\frac{1}{D^2-3DD'+2D'^2}\cos(x+2y)+\frac{1}{D^2-3DD'+2D'^2}e^{2x-y}$$

$$PI=-xe^{x+y}-\frac{1}{3}\cos(x+2y)+\frac{1}{12}e^{2x-y}$$

Complete solution is: $y=CF+PI$

$$\boxed{z=f_1(y+x)+f_2(y+2x)-xe^{x+y}-\frac{1}{3}\cos(x+2y)+\frac{1}{12}e^{2x-y}}$$

Q.4 Solve $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial x\partial y}-6\dfrac{\partial^2 z}{\partial y^2}=y\cos x$

Solution: $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial x\partial y}-6\dfrac{\partial^2 z}{\partial y^2}=y\cos x$

Write the equation is operator form

$\left(D^2+DD'-6D'^2\right)z=y\cos x$

The auxiliary equation is

$m^2+m-6=0\Rightarrow(m+3)(m-2)=0$

$m=2,-3$

$CF=f_1\left(y+2x\right)+f_2\left(y-3x\right)$

$$\text{P.I.}=\frac{1}{D^2+DD'-6D'^2}y\cos x=\frac{1}{\left(D-2D'\right)\left(D+3D'\right)}y\cos x$$

$=\dfrac{1}{D-2D'}\int\left(c+3x\right)\cos x\,dx$ Put $y=c+3x$

$=\dfrac{1}{D-2D'}[(c+3x)\sin x+3\cos x]=\dfrac{1}{D-2D'}[y\sin x+3\cos x]$ Put $c+3x=y$

$=\int[(c-2x)\sin x+3\cos x]\,dx$ Put $y=c-2x$

$=(c-2x)(-\cos x)-2\sin x+3\sin x=-y\cos x+\sin x$ Put $c-2x=y$

Hence the complete solution is

$z=f_1(y+2x)+f_2(y-3x)+\sin x-y\cos x$

METHOD OF SEPARATION OF VARIABLES

In this method, we assume that the dependent variable is the product of two functions, each of which involves only one of the independent variables. So two ordinary differential equations are formed.

Q. 1 Solve $\dfrac{\partial u}{\partial x}=2\dfrac{\partial u}{\partial t}+u$; $u(x,0)=6e^{-3x}$

Solution: Let $u=X\left(x\right).T\left(t\right)$ be the solution.

$\dfrac{\partial u}{\partial x}=T.\dfrac{\partial X}{\partial x};\ \dfrac{\partial u}{\partial t}=X.\dfrac{\partial T}{\partial t}$ Put in the given equation.

$T\dfrac{dX}{dx}=2X\dfrac{dT}{dt}+X.T$

Separate the variables.

$\dfrac{1}{X}\dfrac{dX}{dx}=\dfrac{2}{T}\dfrac{dT}{dt}+1=k(say)$

$\dfrac{1}{X}\dfrac{dX}{dx}=k \Rightarrow \dfrac{dX}{dx}=kX$ Separate the variables	$\dfrac{2}{T}\dfrac{dT}{dt}+1=k$ $\dfrac{2}{T}\dfrac{dT}{dt}=k-1$

$$\frac{dX}{X} = kdx$$

Integrate

$$\int \frac{dX}{X} = k\int dx + C$$

$$\log X = kx + C$$

$$X = e^{kx+C}$$

$$X = e^{kx}.e^{C}$$

$$\Rightarrow \quad X = C_1 e^{kx}$$

$$\frac{dT}{T} = \frac{1}{2}(k-1)dt$$

Integrate

$$\int \frac{dT}{T} = \frac{1}{2}(k-1)\int dt + C$$

$$\log T = \frac{1}{2}(k-1)t + C$$

$$T = e^{\frac{1}{2}(k-1)t}.e^{C}$$

$$T = C_2 e^{\frac{1}{2}(k-1)t}$$

Therefore the solution is:

$u(x,t) = \quad C_1 C_2 e^{kx} e^{\frac{1}{2}(k-1)t}$

Put $t = 0, \quad u(x,0) = 6e^{-3x}$

$6e^{-3x} \; = \; C_1 C_2 e^{kx} e^{\frac{1}{2}(k-1)t}$

$C_1 C_2 = 6, \quad k = -3$

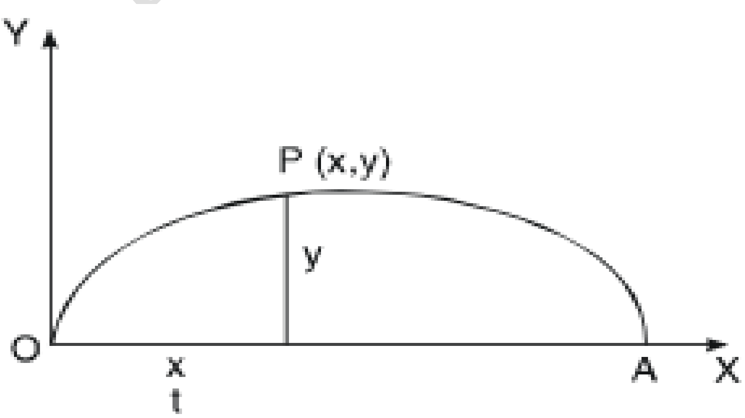
Therefore,

$$\boxed{u(x,t) = 6e^{-3x-2t}}$$

EQUATION OF VIBRATING STRING

Consider an elastic string tightly stretched between two points O and A. Let O be the origin and OA as x-axis. On giving a small displacement to the string, perpendicular to its length (parallel to the y-axis). Let y be the displacement at the point $P(x, y)$ at any time. The wave equation.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$



Q.2 Solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Solution: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ...(1)

Let $u = X(x).T(t)$ be the solution.

$\frac{\partial u}{\partial x} = T.\frac{dX}{dx}$; $\frac{\partial^2 u}{\partial x^2} = T.\frac{d^2X}{dx^2}$ and $\frac{\partial u}{\partial t} = X.\frac{dT}{dt}$, $\frac{\partial^2 u}{\partial t^2} = X.\frac{d^2T}{dt^2}$

Put in the given equation.

$X.\frac{d^2T}{dt^2} = c^2T.\frac{d^2X}{dx^2}$

Divide by c^2XT

$\frac{1}{c^2T}.\frac{d^2T}{dt^2} = \frac{1}{X}.\frac{d^2X}{dx^2} = -p^2(say)$

$\frac{1}{X}.\frac{d^2X}{dx^2} = -p^2$	$\frac{1}{c^2T}.\frac{d^2T}{dt^2} = -p^2$
$\frac{d^2X}{dx^2} = -p^2X$	$\frac{d^2T}{dt^2} = -p^2c^2T$
$\frac{d^2X}{dx^2} + p^2X = 0$	$\frac{d^2T}{dt^2} + p^2c^2T = 0$
$(D^2 + p^2)X = 0$	$(D^2 + p^2c^2)T = 0$
A.E. $m^2 + p^2 = 0$	A.E. $m^2 + p^2c^2 = 0$
$m = \pm pi$	$m = \pm pci$
$X = c_1 \cos px + c_2 \sin px$	$T = c_3 \cos pct + c_4 \sin pct$

The solution is given by

$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct)$ ...(2)

Boundary conditions are:

$u(0,t) = 0$ ....(3)

$u(l,t) = 0$ ...(4)

$u(x,0) = 0$ ....(5)

$\left(\frac{\partial u}{\partial t}\right)_{t=0} = b\sin^3 \frac{\pi x}{l}$ ...(6)

Put (3) in (2), $0 = c_1(c_3 \cos pct + c_4 \sin pct) \Rightarrow c_1 = 0$

Put $c_1 = 0$ in (2), $u(x,t) = c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt)$ ...(7)

Put (4) in (7), $0 = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt)$ $\Rightarrow$ $\sin pl = 0 = \sin n\pi \Rightarrow p = \frac{n\pi}{l}$

Put the value of p in (7)

$$u(x,t)=c_2\sin\frac{n\pi x}{l}\left(c_3\cos\frac{n\pi ct}{l}+c_4\sin\frac{n\pi ct}{l}\right) \quad ...(8)$$

Put $t=0,\ u=0$ in (8), $\quad 0=c_2\sin\frac{n\pi x}{l}(c_3)\ \Rightarrow c_3=0$

Put $c_3=0$ in (8) $\ u(x,t)=c_2c_4\sin\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$

$$\Rightarrow\ u(x,t)=b_n\sin\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

The general solution is:

$$u(x,t)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \qquad ...(9)$$

Differentiate (9) w.r.t.'t'

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}b_n.\frac{n\pi c}{l}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \qquad ...(10)$$

Put (6) in (10), $\quad b\sin^3\frac{\pi x}{l}=\sum_{n=1}^{\infty}b_n.\frac{n\pi c}{l}.\sin\frac{n\pi x}{l}$

$$\Rightarrow\ \frac{b}{4}\left[3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right]=b_1\frac{\pi c}{l}\sin\frac{\pi x}{l}+\frac{2b_2\pi c}{l}\sin\frac{2\pi x}{l}+3b_3\frac{\pi c}{l}\sin\frac{3\pi c}{l}+...$$

Equate coefficient of sin terms.

$$b_1=\frac{3bl}{4\pi c},\quad b_0=0,\quad b_3=-\frac{bl}{12\pi c},$$

Put all b values in (9)

$$\boxed{u(x,t)=\frac{bl}{12\pi c}\left[9\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l}-\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right]}$$

Q.3 Solve $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2}$

Solution: $\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$

Let $u=X(x).T(t)$ be the solution.

$$\frac{\partial u}{\partial x}=T.\frac{\partial X}{\partial x};\ \frac{\partial^2 u}{\partial x^2}=T.\frac{\partial^2 X}{\partial x^2}\ \text{ and }\ \frac{\partial u}{\partial t}=X.\frac{\partial T}{\partial t}\ ,\frac{\partial^2 u}{\partial t^2}=X.\frac{\partial^2 T}{\partial t^2}$$

Put in the given equation.

$$X.\frac{\partial^2 T}{\partial t^2}=c^2T.\frac{\partial^2 X}{\partial x^2}=-p^2(say)$$

Divide by c^2XT

$$\frac{1}{c^2T}.\frac{d^2T}{dt^2}=\frac{1}{X}.\frac{d^2X}{dx^2}=-p^2(say)$$

$$\begin{array}{c|c}
\dfrac{1}{X}.\dfrac{d^2X}{dx^2} = -p^2 & \dfrac{1}{c^2T}.\dfrac{d^2T}{dt^2} = -p^2 \\
\dfrac{d^2X}{dx^2} = -p^2X & \dfrac{d^2T}{dt^2} = -p^2c^2T \\
\dfrac{d^2X}{dx^2} + p^2X = 0 & \dfrac{d^2T}{dt^2} + p^2c^2T = 0 \\
(D^2 + p^2)X = 0 & (D^2 + p^2c^2)T = 0 \\
\text{A.E. } m^2 + p^2 = 0 & \text{A.E. } m^2 + p^2c^2 = 0 \\
m = \pm pi & m = \pm pci \\
X = c_1 \cos px + c_2 \sin px & T = c_3 \cos pct + c_4 \sin pct
\end{array}$$

The solution is given by

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pct + c_4 \sin pct) \qquad ...(2)$$

Boundary conditions are:

$$u(0,t) = 0 \qquad ....(3)$$

$$u(l,t) = 0 \qquad ...(4)$$

$$u(x,0) = f(x) \qquad ....(5)$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \qquad ...(6)$$

Put (3) in (2), $\quad 0 = (c_3 \cos cpt + c_4 \sin cpt)c_1 \quad \Rightarrow \quad c_1 = 0$

Put $c_1 = 0$ in (2), $u(x,t) = (c_3 \cos cpt + c_2 \sin cpt)c_2 \sin px \quad ...(7)$

Put (4) in (7), $\quad 0 = (c_3 \cos cpt + c_4 \sin cpt)c_2 \sin pl \quad \Rightarrow \quad \sin pl = 0 \Rightarrow p = \dfrac{n\pi}{l}$

Put the value of p in (7)

$$u(x,t) = \left(c_3 \cos\frac{n\pi ct}{l} + c_4 \sin\frac{n\pi ct}{l}\right)c_2 \sin\frac{n\pi x}{l} \qquad ...(8)$$

Differentiate (8) w.r.t.'t'

$$\frac{\partial u}{\partial t} = \left(-\frac{n\pi c}{l}c_3 \sin\frac{n\pi ct}{l} + \frac{n\pi c}{l}c_4 \cos\frac{n\pi ct}{l}\right)c_2 \sin\frac{n\pi x}{l} \qquad ...(9)$$

Put (6) in (9), $\quad 0 = c_2c_4 \sin\dfrac{n\pi x}{l} \quad \Rightarrow c_4 = 0$

Therefore, $u(x,t) = c_2c_3 \sin\dfrac{n\pi x}{l}\cos\dfrac{n\pi ct}{l}$

$$u(x,t) = b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad ...(10)$$

Put (5) in (10), $\quad f(x) = b_n \sin\dfrac{n\pi x}{l}$ where

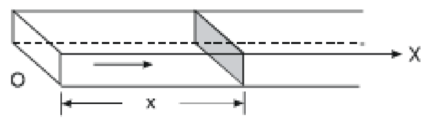
$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

Thus the solution is:

$$\boxed{u(x,t) = b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}}$$

ONE DIMENSIONAL HEAT FLOW

Let heat flow along a bar of uniform cross-section, in the direction perpendicular to the cross-section. Take one end of the bar as origin and the direction of heat flow is along x-axis.



Let the temperature of the bar at any time t at a point x distance from the origin be $u(x,t)$. Then the equation of one dimensional heat flow is $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Q.4

A rod of length l with insulated sides is initially at a uniform temperature u. Its ends are suddenly cooled to O°C and are kept at that temperature. Prove that the temperature function u (x, t) is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.e^{-\frac{c^2\pi^2 n^2 t}{l^2}}$$ *where b_n is determined from the equation.*

Solution. Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Let us assume that $u = XT$, where X is a function of x alone and T that of t alone.

$$\therefore \qquad \frac{\partial u}{\partial t} = X\frac{dT}{dt} \qquad and \qquad \frac{\partial^2 u}{\partial x^2} = T\frac{d^2 X}{dx^2}$$

Substituting these values in (1), we get $X\frac{dT}{dt} = c^2 T \frac{d^2X}{dx^2}$

i.e. $$\frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \qquad \ldots(2)$$

Let each side be equal to a constant $(-p^2)$.

$$\frac{1}{c^2T}\frac{dT}{dt} = -p^2 \quad \Rightarrow \quad \frac{dT}{dt} + p^2c^2T = 0 \qquad \ldots(3)$$

and $$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} + p^2X = 0 \qquad \ldots(4)$$

Solving (3) and (4) we have

$$T = c_1 e^{-p^2c^2t} \quad \text{and} \quad X = c_2 \cos px + c_3 \sin px$$

$$\therefore \quad u = c_1 e^{-p^2c^2t}(c_2 \cos px + c_3 \sin px) \qquad \ldots(5)$$

Putting $x = 0$, $u = 0$ in (5), we get

$$0 = c_1 e^{-p^2c^2t}(c_2) \quad \Rightarrow \quad c_2 = 0 \text{ since } c_1 \neq 0$$

(5) becomes $$u = c_1 e^{-p^2c^2t} c_3 \sin px \qquad \ldots(6)$$

Again putting $x = l$, $u = 0$ in (6), we get

$$0 = c_1 e^{-p^2c^2t}.c_3 \sin pl \Rightarrow \quad \sin pl = 0 = \sin n\pi$$

$$\Rightarrow \quad pl = n\pi \quad \Rightarrow \quad p = \frac{n\pi}{l}, n \text{ is any integer}$$

Hence (6) becomes $u = c_1c_3 e^{\frac{-n^2c^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l} = b_n e^{-\frac{n^2c^2\pi^2t}{l^2}} \sin\frac{n\pi}{l}x, \qquad b_n = c_1c_3$

This equation satisfies the given conditions for all integral values of n. Hence taking $n = 1, 2, 3, \ldots\ldots$, *the most general solution is*

$$u = \sum_{n=1}^{\infty} b_n e^{\frac{-n^2c^2\pi^2t}{l^2}} \sin\frac{n\pi}{l}x$$

By initial conditions $u = U_0$ when $t = 0$

$$U_0 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Q.5

An insulated rod of length l has its ends A and B maintained at 0^0C and 100^0C respectively until steady state conditions prevail. If B is suddenly reduced to 0^0C and maintained at 0^0C, find the temperature at a distance x from A at time t.

Solution: One dimensional heat flow equation is given by

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \ldots (1)$$

Its solution can be expressed as

$u(x,t) = X(x)T(t)$ or

$$u = XT \qquad \ldots (2)$$

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx}$$

$$\frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$$

Use in (1)

$$X\frac{dT}{dt} = c^2T\frac{d^2X}{dx^2}$$

Divide by c^2XT

$$\frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{c^2X}\frac{d^2X}{dx^2} = -p^2 \text{ (say)}$$

$$\frac{1}{c^2X}\cdot\frac{d^2X}{dx^2} = -p^2$$

$$\frac{d^2X}{dx^2} = -p^2c^2X$$

$$\frac{d^2X}{dx^2} + p^2c^2X = 0$$

$$(D^2 + p^2)X = 0$$

A.E. $m^2 + p^2 = 0$

$$m = \pm pi$$

$$X = c_1\cos px + c_2\sin px$$

$$\frac{1}{c^2T}\cdot\frac{dT}{dt} = -p^2$$

$$\frac{dT}{dt} = -p^2c^2T$$

$$\frac{dT}{dt} + p^2c^2T = 0$$

$$(D + p^2c^2)T = 0$$

A.E. $m + p^2c^2 = 0$

$$m = -p^2c^2$$

$$T = c_3e^{-p^2c^2t}$$

Use in (2)

$$u(x,t) = (c_1\cos px + c_2\sin px)c_3e^{-p^2c^2t} \qquad \ldots. (3)$$

When steady state conditions prevails

$\frac{\partial u}{\partial t} = 0$ then by (1)

$$0 = c^2\frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

Integrate

$$\frac{\partial u}{\partial x} = A_1$$

Again integrate.

$$u(x) = A_1 x + B_1 \qquad \dots (4)$$

When $x = 0, \quad u(0) = 0$

$u(0) = A_1.0 + B_1 \Rightarrow 0 = B_1$

$u(x) = A_1 x$

When $x = l, \quad u(l) = 100$

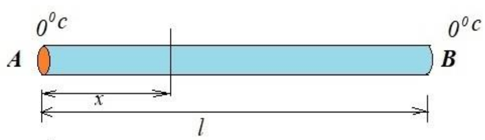
$u(l) = A_1 l \Rightarrow 100 = A_1 l$

$$A_1 = \frac{100}{l}$$

Therefore,

$$u(x,0) = \frac{100}{l}x \qquad \dots (5)$$

When B is suddenly reduced to $0^0 C$ and maintained at $0^0 C$,then use these conditions in (3).



At A $x = 0,\ u(0,t) = 0$

$u(0,t) = (c_1 \cos 0 + c_2 \sin 0) c_3 e^{-p^2 c^2 t}$

$0 = c_1 c_3 e^{-p^2 c^2 t} \Rightarrow c_1 = 0$ use in (3).

$$u(x,t) = c_2 \sin px.c_3 e^{-p^2 c^2 t} \qquad \dots (6)$$

At B $x = l,\ u(l,t) = 0$ use in (6).

$0 = c_2 \sin pl.c_3 e^{-p^2 c^2 t} \Rightarrow \sin pl = 0 = \sin n\pi$

$pl = n\pi \Rightarrow p = \frac{n\pi}{l}$ use in (6).

$$u(x,t) = c_2 c_3 \sin \frac{n\pi}{l} x.e^{-\frac{n^2\pi^2 c^2}{l^2} t}$$

The general solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x.e^{-\frac{n^2\pi^2 c^2}{l^2} t} \qquad \dots (7)$$

At $t = 0$ then by (4)

$u(x,0) = \frac{100}{l}x$ use in (7).

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x.e^0$$

$$\frac{100}{l}x=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

By Fourier series

$$b_n=\frac{2}{l}\int_0^l u(x)\sin\frac{n\pi x}{l}dx$$

$$b_n=\frac{2}{l}\int_0^l \frac{100}{l}x.\sin\frac{n\pi x}{l}\,dx$$

$$b_n=\frac{200}{l^2}\int_0^l x.\sin\frac{n\pi x}{l}\,dx$$

$$b_n=-\frac{200}{n\pi}(-1)^n$$

$$b_n=\frac{200}{n\pi}(-1)^{n+1}\text{ then}$$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{200}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{l}x.e^{-\frac{n^2\pi^2c^2}{l^2}t}$$

$$u(x,t)=\frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^{n+1}\sin\frac{n\pi}{l}x.e^{-\frac{n^2\pi^2c^2}{l^2}t}$$

Q.6

The ends A and B of a rod 20 cm long have the temperatures at 30° C and at 80°C until steady state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.

Solution:

The initial temperature distribution in the rod is

$$u=30+\frac{50}{20}x$$

$$u=30+\frac{5}{2}x$$

and the final distribution (*i.e.* in steady state)is

$$u=40+\frac{20}{20}x=40+x$$

To get u in the intermediate period, reckoning time from the instant when the end temperature were changed, we assumed

$$u=u_1(x,t)+u_2(x)$$

where $u_2(x)$ is the steady state temperature distribution in the rod (*i.e.* temperature after sufficiently long time) and $u_1(x, t)$ is the transient temperature distribution which tends to zero a t increases.

Thus $$u_2(x) = 40 + x$$

Now $u_1(x, t)$ satisfies the one-dimensional heat-flow equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Hence u is of the form

$$u = 40 + x + \sum (a_k \cos kx + b_k \sin kx) e^{-c^2 k^2 t}$$

Since $u = 40°$, when $x = 0$ and $u = 60°$, when $x = 20$, we get

$$a_k = 0, k = \frac{n\pi}{20}$$

Hence $$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-c^2 \left(\frac{n\pi}{20}\right)^2 t} \qquad ...(1)$$

Using the initial condition *i.e.*,

$$u = 30 + \frac{5}{2} x \text{ when } t = 0, \text{ we get}$$

$$30 + \frac{5}{2} x = 40 + x + \sum b_n \sin \frac{n\pi}{20} x \quad \Rightarrow \quad \frac{3}{2} x - 10 = \sum b_n \sin \frac{n\pi x}{20}$$

Hence $$b_n = \frac{2}{20} \int_0^{20} \left(\frac{3}{2} x - 10 \right) \sin \frac{n\pi x}{20} dx$$

$$= \frac{1}{10} \left[\left(\frac{3x}{2} - 10 \right) \left(-\frac{20}{n\pi} \cos \frac{n\pi x}{20} \right) - \frac{3}{2} \left(\frac{-400}{n^2 \pi^2} \sin \frac{n\pi x}{20} \right) \right]_0^{20}$$

$$= \frac{1}{10} \left[-20 \left(\frac{20}{n\pi} \right) (-1)^n - (-10) \left(\frac{20}{n\pi} \right) \right] = -\frac{20}{n\pi} \left[2(-1)^n + 1 \right]$$

Putting this value of b_n in (1), we get

$$u = 40 + x - \frac{20}{\pi} \sum \left[\left(\frac{2(-1)^n}{n} \right) \sin \frac{n\pi x}{20} . e^{-\left(\frac{cn\pi}{20}\right)^2 t} \right]$$