Notes on Engineering Mathematics – II Unit V

French Mathematician: **Friedrich Wilhelm Bessel(1784-1846)**

French Mathematician: **Adrien-Marie Legendre (1752 – 1833)**

Special Functions

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Power Series Solution of Differential Equation:

Example 1: Find the power series solution of $\left(1-x^2\right)y''-2xy'+2y=0$ about $x=0$.

Solution: Suppose $y\left(x\right)=\sum_{r=0}^{\infty}a_r x^r$ be its general solution, then

$y'\left(x\right)=\sum_{r=0}^{\infty}a_r r x^{r-1}$ and $y''\left(x\right)=\sum_{r=0}^{\infty}a_r r\left(r-1\right)x^{r-2}$

Substitute in the equation.

$$\left(1-x^2\right)\sum_{r=0}^{\infty}a_r r\left(r-1\right)x^{r-2}-2x\sum_{r=0}^{\infty}a_r r x^{r-1}+2\sum_{r=0}^{\infty}a_r x^r=0$$

$$\sum_{r=0}^{\infty}a_r r\left(r-1\right)x^{r-2}-\sum_{r=0}^{\infty}a_r r\left(r-1\right)x^r-2\sum_{r=0}^{\infty}a_r r x^r+2\sum_{r=0}^{\infty}a_r x^r=0$$

$$\sum_{r=0}^{\infty}a_r r\left(r-1\right)x^{r-2}-\sum_{r=0}^{\infty}a_r\left(r^2+r-2\right)x^r=0 \qquad \text{...... (1)}$$

Replace r by $r+2$ in the first term of (1).

$$\sum_{r=0}^{\infty}a_{r+2}\left(r+2\right)\left(r+1\right)x^r-\sum_{r=0}^{\infty}a_r\left(r^2+r-2\right)x^r=0$$

Equate the coefficient of identical powers of x^r from both sides.

$$a_{r+2}\left(r+2\right)\left(r+1\right)-a_r\left(r^2+r-2\right)=0$$

$$a_{r+2}=\frac{r^2+r-2}{\left(r+2\right)\left(r+1\right)}a_r;\ r\geq 0$$

$$a_2=-\frac{-2a_0}{2}=-a_0;\quad a_3=0$$

$$a_4=\frac{4a_2}{12}=-\frac{a_0}{3};\quad a_5=0$$

$$a_6=\frac{18a_4}{30}=-\frac{a_0}{5}$$

Use these values in $y\left(x\right)=\sum_{r=0}^{\infty}a_r x^r=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+...$

$$y\left(x\right)=a_0+a_1x-a_0x^2-\frac{a_0}{3}x^4-\frac{a_0}{5}x^6...$$

$$y\left(x\right)=a_0\left(1-x^2-\frac{1}{3}x^4-\frac{1}{5}x^6...\right)+a_1x$$

Example 2: Find the power series solution of $\left(1-x^2\right)y''-xy'+4y=0.$

Solution: Here $p(x)=\dfrac{x}{1-x^2}, q(x)=\dfrac{4}{1-x^2}$ are analytic functions at $x=0$.

Hence, $x=0$ is an ordinary point.

Suppose $y(x)=\sum_{r=0}^{\infty}a_r x^r$ be its general solution, then

$y'(x)=\sum_{r=0}^{\infty}a_r r x^{r-1}$ and $y''(x)=\sum_{r=0}^{\infty}a_r r(r-1)x^{r-2}$

Substitute in the equation.

$$\left(1-x^2\right)\sum_{r=0}^{\infty}a_r r(r-1)x^{r-2}-x\sum_{r=0}^{\infty}a_r r x^{r-1}+4\sum_{r=0}^{\infty}a_r x^r=0$$

$$\sum_{r=0}^{\infty}a_r r(r-1)x^{r-2}-\sum_{r=0}^{\infty}a_r r(r-1)x^{r}-\sum_{r=0}^{\infty}a_r r x^{r}+4\sum_{r=0}^{\infty}a_r x^r=0$$

$$\sum_{r=0}^{\infty}a_r r(r-1)x^{r-2}-\sum_{r=0}^{\infty}\left(r^2-4\right)a_r x^r=0 \qquad \ldots\ldots (1)$$

Replace r by $r+2$ in the first term of (1).

$$\sum_{r=0}^{\infty}a_{r+2}(r+2)(r+1)x^r-\sum_{r=0}^{\infty}\left(r^2-4\right)a_r x^r=0$$

$$\sum_{r=0}^{\infty}\left[a_{r+2}(r+2)(r+1)-\left(r^2-4\right)a_r\right]x^r=0$$

Equate the coefficient of identical powers of x from both sides.

$$a_{r+2}(r+2)(r+1)-\left(r^2-4\right)a_r=0$$

$$a_{r+2}=\frac{r^2-4}{(r+1)(r+2)}a_r;\ r\geq 0$$

$$a_2=-2a_0$$

$$a_3=\frac{-3}{6}a_1=-\frac{1}{2}a_1$$

$$a_4=0$$

$$a_5=\frac{1}{4}a_3=-\frac{1}{8}a_1$$

$$a_6=a_8=a_{10}=\ldots..=0$$

Use these values in $y(x)=\sum_{r=0}^{\infty}a_r x^r=a_0+a_1x+a_2x^2-\frac{1}{2}a_1x^3+a_5x^5+\ldots$

$$y(x)=a_0+a_1x-2a_0x^2-\frac{a_1}{2}x^3-\frac{1}{8}a_1x^5+\ldots$$

$$\boxed{y(x)=a_0\left(1-2x^2\ldots\right)+a_1\left(x-\frac{1}{2}x^3-\frac{1}{8}x^5\ldots\right)}$$

Example 3: Find the power series solution of the equation

$$\left(1-x^2\right)y''+2xy'+y=0;\ y(0)=1,\ y'(0)=1$$

Solution: Suppose $y(x)=\sum_{r=0}^{\infty}a_r x^r$ be its general solution, then

$y'(x)=\sum_{r=0}^{\infty}a_r r x^{r-1}$ and $y''(x)=\sum_{r=0}^{\infty}a_r r(r-1)x^{r-2}$

Substitute in the equation.

$$\left(1-x^2\right)\sum_{r=0}^{\infty}a_r r(r-1)x^{r-2}+2x\sum_{r=0}^{\infty}a_r r x^{r-1}+\sum_{r=0}^{\infty}a_r x^r=0$$

$$\sum_{r=0}^{\infty}a_r r(r-1)x^{r-2}-\sum_{r=0}^{\infty}a_r r(r-1)x^r+2\sum_{r=0}^{\infty}a_r r x^r+\sum_{r=0}^{\infty}a_r x^r=0$$

$$\sum_{r=0}^{\infty}a_r r(r-1)x^{r-2}-\sum_{r=0}^{\infty}\left[r^2-3r-1\right]a_r x^r=0 \qquad \ldots\ldots (1)$$

Replace r by $r+2$ in the first term of (1).

$$\sum_{r=0}^{\infty}a_{r+2}(r+2)(r+1)x^r-\sum_{r=0}^{\infty}\left[r^2-3r-1\right]a_r x^r=0$$

Equate the coefficient of identical powers of x^r from both sides.

$$a_{r+2}(r+2)(r+1)-a_r\left(r^2-3r-1\right)=0$$

$$a_{r+2}=\frac{r^2-3r-1}{(r+2)(r+1)}a_r;\ r\geq 0$$

$$\Rightarrow a_2=-\frac{a_0}{2};\quad a_3=-\frac{a_1}{2},\quad a_4=-\frac{a_2}{4}=\frac{a_0}{8};\quad a_5=-\frac{a_3}{20}=\frac{a_1}{60}$$

Use these values in $y(x)=\sum_{r=0}^{\infty}a_r x^r=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\ldots$

$$y(x)=a_0+a_1x-\frac{a_0}{2}x^2-\frac{a_1}{2}x^3+\frac{a_0}{8}x^4+\frac{a_1}{60}x^5+\ldots$$

$$y(x)=a_0\left(1-\frac{x^2}{2}+\frac{x^4}{8}\ldots\right)+a_1\left(x-\frac{x^3}{2}+\frac{x^5}{60}\ldots\right) \qquad \ldots\ldots (2)$$

Differentiate with respect to x.

$$y'(x)=a_0\left(-x+\frac{x^3}{2}\ldots\right)+a_1\left(1-\frac{3x^2}{2}+\frac{x^4}{12}\ldots\right) \qquad \ldots\ldots (3)$$

Substitute $x=0$ in (2)

$y(0)=a_0=1$

Substitute $x=0$ in (3)

$y'(0)=a_1=1$

Therefore,

$$\boxed{y(x)=1+x-\frac{x^2}{2}-\frac{x^3}{2}+\frac{x^4}{8}+\frac{x^5}{60}\ldots}$$

FROBENIUS METHOD : If $x = 0$ is a regular singularity of the equation.

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0 \qquad \ldots (1) \qquad [P\,(0) = 0]$$

Then the series solution is $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots) = \sum_{k=0}^{\infty} a_k x^{m+k}$

The value of m will be determined by substituting the expressions for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1), we get the identity.

On equating the coefficient of lowest power of x in the identity to zero, a quadratic equation in m (**indicial equation**) is obtained.

Thus, we will get two values of m. The series solution of (1) will depend on the nature of the roots of the indicial equation.

(*i*) **Case 1 : When roots m_1, m_2 are distinct and not differing by an integer)** $m_1 - m_2 \neq$ 0 or a positive integer. *e.g.*, $m_1 = \frac{1}{2}, m_2 = 2$.

The complete solution is $y = c_1(y)_{m_1} + c_2(y)_{m_2}$

(*ii*) **Case 2 : When roots m_1, m_2 *are equal i.e.* $m_1 = m_2$**

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

(*iii*) **Case 3 : When roots m_1, m_2 are distinct and differ by an integer** $(m_1 < m_2)$ *e.g.*, $m_1 = \frac{3}{2}, m_2 = \frac{5}{2}$ or $m_1 = 2$, $m_2 = 4$.

If some of the coefficients of y series become infinite when $m = m_1$, to overcome this difficulty, replace a_0 by $b_0\,(m - m_1)$. We get a solution which is only a constant multiple of the first solution.

Complete solution is $y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}$

(*iv*) **Case 4 : Roots are distinct and differing by an integer, making some coefficient indeterminate**

Complete solution is $y = c_1(y)_{m_1} + c_2(y)_{m_2}$

if the coefficients do not become infinite when $m = m_2$.

Study Case 1 and 2 only

Case 1

Example 1. Solve in series $2x(1-x)y'' + (5-7x)y' - 3y = 0$

Solution: $2x(x-1)y'' + (5-7x)y' - 3y = 0 \qquad \ldots(1)$

Divide by $2x(1-x)$

$$y''+\frac{5-7x}{2x(1-x)}y'+\frac{1}{2x(1-x)}y=0$$

Compare with $y''+P(x)y'+Q(x)y=0$

$$P(x)=\frac{5-7x}{2x(1-x)},\ Q(x)=\frac{1}{2x(1-x)}$$

At $x=0$, $P(x)$ and $Q(x)$ are not analytic. Hence $x=0$ is a regular singular point.

Assume $y(x)=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+\ldots.+a_nx^{m+n}+\ldots$ be the solution.

Then

$$y'(x)=a_0mx^{m-1}+a_1(m+1)x^m+a_2(m+2)x^{m+1}+a_3(m+3)x^{m+2}+\ldots.+a_n(m+n)x^{m+n-1}+\ldots$$

$$\begin{aligned}y''(x)&=a_0m(m-1)x^{m-2}+a_1m(m+1)x^{m-1}+a_2(m+2)(m+1)x^m+a_3(m+3)x^{m+2}+\ldots.\\&\quad+a_n(m+n)(m+n-1)x^{m+n-1}+\ldots\end{aligned}$$

Substitute in (1).

$$2x(1-x)\begin{bmatrix}a_0m(m-1)x^{m-2}+a_1m(m+1)x^{m-1}+a_2(m+2)(m+1)x^m+a_3(m+3)x^{m+2}+\ldots.\\ +a_n(m+n)(m+n-1)x^{m+n-1}+\ldots\end{bmatrix}$$

$$+(5-7x)\left[a_0mx^{m-1}+a_1(m+1)x^m+a_2(m+2)x^{m+1}+a_3(m+3)x^{m+2}+\ldots.+a_n(m+n)x^{m+n-1}+\ldots\right]$$

$$-3\left(a_0x^m+a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+\ldots.+a_nx^{m+n}+\ldots\right)=0$$

Now equate to zero the coefficient of lowest degree term i.e coefficient of $x^{m-1}=0$

$$2a_0m(m-1)+5a_0m=0$$

$$2a_0m(m-1)+5a_0m=0\Rightarrow a_0\left(2m(m-1)+5m\right)=0$$

$$2m^2+3m=0\Rightarrow m(2m+3)=0$$

$$\Rightarrow m=0,-3/2$$

Roots are distinct and do not differ by an integer.

Now coefficient of $x^m=0$

$$2(m+1)ma_1-2m(m-1)a_0+5(m+1)a_1-7ma_0-3a_0=0$$

$$(m+1)(2m+5)a_1=\left(2m^2-2m+7m+3\right)a_0$$

$$a_1=\frac{(m+1)(2m+3)}{(m+1)(2m+5)}a_0\Rightarrow a_1=\frac{2m+3}{2m+5}a_0$$

and coefficient of $x^{m+1}=0$

$$2(m+2)(m+1)a_2-2(m+1)ma_1+5(m+2)a_2-7(m+1)a_1-3a_1=0$$

$$2(m+2)(m+1)a_2=\left(2m^2+9m+10\right)a_1=(2m+5)(m+2)a_1$$

$$a_2=\frac{2m+5}{2m+7}a_1$$

$$a_2=\frac{2m+5}{2m+7}\cdot\frac{2m+3}{2m+5}a_0$$

$$a_2 = \frac{2m+3}{2m+7} a_0$$

Similarly, $a_3 = \frac{2m+7}{2m+9} a_2 = \frac{2m+7}{2m+9} \cdot \frac{2m+3}{2m+7} a_0$

$a_3 = \frac{2m+3}{2m+9} a_0$ and so on

Use in $y(x) = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \dots + a_n x^{m+n} + \dots$

$$y(x) = x^m \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots \right)$$

$$y(x) = x^m \left(a_0 + \frac{2m+3}{2m+5} a_0 x + \frac{2m+3}{2m+7} a_0 x^2 + \frac{2m+3}{2m+9} a_0 x^3 + \dots \right)$$

$$y(x) = a_0 x^m \left(1 + \frac{2m+3}{2m+5} x + \frac{2m+3}{2m+7} x^2 + \frac{2m+3}{2m+9} x^3 + \dots \right)$$

Now $y_1 = (y)_{m=0} = a_0 \left(1 + \frac{3}{5} x + \frac{3}{7} x^2 + \frac{3}{9} x^3 + \dots \right)$

and $y_2 = (y)_{m=-3/2} = a_0 x^{-3/2} \left(1 + 0.x + 0.x^2 + 0.x^3 + \dots \right)$

$$y_2 = (y)_{m=-3/2} = a_0 x^{-3/2}$$

Hence complete solution is given by

$$y(x) = c_1 y_1 + c_2 y_2$$

$$y(x) = c_1 a_0 \left(1 + \frac{3}{5} x + \frac{3}{7} x^2 + \frac{3}{9} x^3 + \dots \right) + c_2 a_0 x^{-3/2}$$

$$y(x) = A \left(1 + \frac{3}{5} x + \frac{3}{7} x^2 + \frac{3}{9} x^3 + \dots \right) + B x^{-3/2}$$

Case 2

Example 2. Solve in series $x(x-1) y'' + (3x-1) y' + y = 0$

Solution: $x(x-1) y'' + (3x-1) y' + y = 0 \qquad \dots(1)$

Divide by $x(x-1)$

$$y'' + \frac{3x-1}{x(x-1)} y' + \frac{1}{x(x-1)} y = 0$$

Compare with $y''+P(x)y'+Q(x)y=0$

$$P(x)=\frac{3x-1}{x(x-1)},\ Q(x)=\frac{1}{x(x-1)}$$

At $x=0$, $P(x)$ and $Q(x)$ are not analytic. Hence $x=0$ is a regular singular point.

Assume $y(x)=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+\ ....\ +a_nx^{m+n}+\ ...$ be the solution.

Then

$$y'(x)=a_0mx^{m-1}+a_1(m+1)x^m+a_2(m+2)x^{m+1}+a_3(m+3)x^{m+2}+\ ....\ +a_n(m+n)x^{m+n-1}+\ ...$$

$$y''(x)=a_0m(m-1)x^{m-2}+a_1m(m+1)x^{m-1}+a_2(m+2)(m+1)x^m+a_3(m+3)x^{m+2}+\ ....$$
$$+a_n(m+n)(m+n-1)x^{m+n-1}+\ ...$$

Substitute in (1).

$$x(x-1)\begin{bmatrix}a_0m(m-1)x^{m-2}+a_1m(m+1)x^{m-1}+a_2(m+2)(m+1)x^m+a_3(m+3)x^{m+2}+\ ....\\ +a_n(m+n)(m+n-1)x^{m+n-1}+\ ...\end{bmatrix}$$
$$+(3x-1)\left[a_0mx^{m-1}+a_1(m+1)x^m+a_2(m+2)x^{m+1}+a_3(m+3)x^{m+2}+\ ....\ +a_n(m+n)x^{m+n-1}+\ ...\right]$$
$$+a_0x^m+a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+\ ....\ +a_nx^{m+n}+\ ...=0$$

Now equate to zero the coefficient of lowest degree term i.e coefficient of $x^{m-1}=0$

$$-a_0m(m-1)-a_0m=0$$
$$-a_0m(m-1)-a_0m=0\Rightarrow -a_0\left(m(m-1)+m\right)=0$$
$$m(m-1)+m=0\Rightarrow m^2=0$$
$$\Rightarrow m=0,0$$

Now coefficient of $x^m=0$

$$a_0m(m-1)-a_1m(m+1)+3a_0m-a_1(m+1)+a_0=0$$
$$\left[m(m+1)+(m+1)\right]a_1=\left[m(m-1)+3m+1\right]a_0$$
$$\left[m(m+1)+(m+1)\right]a_1=\left[m(m-1)+3m+1\right]a_0$$
$$a_1=\frac{m^2+2m+1}{(m+1)^2}a_0\Rightarrow a_1=a_0$$

and coefficient of $x^{m+1}=0$

$$a_1m(m+1)-a_2(m+2)(m+1)+3a_1(m+1)-a_2(m+2)+a_1=0$$
$$a_1m(m+1)-a_2(m+2)(m+1)+3a_1(m+1)-a_2(m+2)+a_1=0$$
$$a_2\left[(m+2)(m+1)+(m+2)\right]=a_1m(m+1)+3a_1(m+1)+a_1$$
$$a_2(m+2)^2=a_1\left[m(m+1)+3(m+1)+1\right]$$
$$a_2(m+2)^2=a_1(m+2)^2$$
$$a_1=a_0$$

Similarly, $a_3=a_4=....a_n=a_0$

Therefore, $y(x)=a_0\left(x^m+x^{m+1}+x^{m+2}+x^{m+3}+\ ....\ +x^{m+n}+\ ...\right)$

$$y(x) = a_0 x^m \left(1 + x + x^2 + x^3 + \dots + x^n + \dots\right)$$

$$y_1 = (y)_{m=0} = a_0 \left(1 + x + x^2 + x^3 + \dots + x^n + \dots\infty\right)$$

$$\frac{\partial y}{\partial m} = a_0 x^m \log x.\left(1 + x + x^2 + x^3 + \dots + x^n + \dots\right)$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = \left[a_0 x^m \log x.\left(1 + x + x^2 + x^3 + \dots + x^n + \dots\infty\right)\right]_{m=0}$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = a_0 \log x.\left(1 + x + x^2 + x^3 + \dots + x^n + \dots\infty\right)$$

Thus, the complete solution is given by.

$$y(x) = c_1 y_1 + c_2 y_2$$

$$y(x) = c_1 a_0 \left(1 + x + x^2 + x^3 + \dots + x^n + \dots\infty\right) + c_2 a_0 \log x.\left(1 + x + x^2 + x^3 + \dots + x^n + \dots\infty\right)$$

$$y(x) = (A + B\log x)\left(1 + x + x^2 + x^3 + \dots + x^n + \dots\infty\right)$$

Legendre Equation:

$$\left(1 - x^2\right) y'' - 2xy' + n(n+1) y = 0$$

Differential form:

$$\frac{d}{dx}\left\{\left(1 - x^2\right)\frac{dy}{dx}\right\} + n(n+1) y = 0$$

Series solution of Legendre Equation:

An equation of the form

$$\left(1 - x^2\right) y'' - 2xy' + n(n+1) y = 0 \qquad \dots\dots (1)$$

Where n is real number, is called Legendre's equation of order n.
The solution is given by

$$y(x) = AP_n(x) + BQ_n(x)$$

$$P_n(x) = \frac{1.3.5\dots(2n-1)}{n!}\left[x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)x^{n-4}}{2(2n-1)(2n-3)} - \dots\right]$$

In compact form

$$P_n(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \frac{(2n-2r)!\,x^{n-2r}}{2^n r!(n-r)!(n-2r)!}, \qquad \text{Where } \left[\frac{n}{2}\right] = \begin{cases} n/2 & ; n \text{ is even} \\ (n-1)/2; & n \text{ is odd} \end{cases}$$

$$Q_n(x)=\frac{n!}{1.3.5...(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)x^{-n-3}}{2.(2n+3)}+\frac{(n+1)(n+2)(n+3)x^{-n-5}}{2.4.(2n+3)(2n+5)}+...\right]$$

Legendre Polynomial $P_n(x)$:

A part of solution of Legendre's equation is known as Legendre's polynomial $P_n(x)$.

$$y(x)=AP_n(x)+BQ_n(x)$$

$$P_n(x)=\frac{1.3.5...(2n-1)}{n!}\left[x^n-\frac{n(n-1)x^{n-2}}{2(2n-1)}+\frac{n(n-1)(n-2)(n-3)x^{n-4}}{2(2n-1)(2n-3)}-...\right]$$

In compact form

$$P_n(x)=\sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r\frac{(2n-2r)!x^{n-2r}}{2^n r!(n-r)!(n-2r)!},\quad \text{Where } \left[\frac{n}{2}\right]=\begin{cases} n/2 & ; n \text{ is even} \\ (n-1)/2; & n \text{ is odd}\end{cases}$$

Example 4: Express $f(x)=x^4+2x^3+2x^2-x-3$ in terms of Legendre's polynomials.

Solution: We know that

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\frac{(3x^2-1)}{2},$$

$$P_3(x)=\frac{(5x^3-3x)}{2},\ P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$$

In order to express $f(x)$ in terms of $P_n(x)$ we need only P_0, P_1, P_2, P_3, P_4. Because $f(x)$ is a polynomial of degree 4. Now, from $P_4(x)$;

$$x^4=\frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35}; \qquad \text{...... (1)}$$

From $P_3(x)$; $x^3=\frac{2}{35}P_3(x)+\frac{3}{5}x;$ ...... (2)

From $P_2(x)$; $x^2=\frac{2}{3}P_2(x)+\frac{1}{3};$ ...... (3)

and $x=P_1(x);\quad 1=P_0(x)$

Substitute these values in $f(x)=x^4+2x^3+2x^2-x-3.$

$$f(x)=\left[\frac{8}{35}P_4(x)+\frac{6}{7}x^2-\frac{3}{35}\right]+2\left[\frac{2}{35}P_3(x)+\frac{3}{5}x\right]+2\left[\frac{2}{3}P_2(x)+\frac{1}{3}\right]-\left[P_1(x)\right]-3.$$

Further substitute the values of x, x^2.

$$f(x)=\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{21}P_2(x)+\frac{1}{5}P_1(x)-\frac{224}{105}P_0(x)$$

Example 1: Show that:

(a) $P_n(1)=1$

(b) $P_n(-x)=(-1)^n P_n(x)$ and $P_n(-1)=(-1)^n$

(c) $P_{2n+1}(0)=0$ and $P_{2n}(0)=(-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$

Solution: **(a)** We know that

$$\sum_{n=0}^{\infty} z^n P_n(x)=\left(1-2xz+z^2\right)^{-1/2} \qquad \ldots\ldots (1)$$

Substitute $x=1$.

$$\sum_{n=0}^{\infty} z^n P_n(1)=\left(1-2z+z^2\right)^{-1/2}=(1-z)^{-1}$$

$$\sum_{n=0}^{\infty} z^n P_n(1)=\left(1-2z+z^2\right)^{-1/2}=1+z+z^2+z^3+\ldots$$

$$\sum_{n=0}^{\infty} z^n P_n(1)=\sum_{n=0}^{\infty} z^n \qquad \ldots\ldots (2)$$

Equate the coefficient of z^n.

$P_n(1)=1.$

(b) If $x=-x$ then equation (1) becomes

$$\sum_{n=0}^{\infty} z^n P_n(-x)=\left(1+2z+z^2\right)^{-1/2}=\left\{1-2x(-z)+(-z)^2\right\}^{-1/2}$$

$$=\sum_{n=0}^{\infty}(-z)^n P_n(x) \qquad \text{from (1)}$$

$$\sum_{n=0}^{\infty} z^n P_n(-x)=\sum_{n=0}^{\infty}(-1)^n z^n P_n(x) \qquad \ldots\ldots (3)$$

Equate the coefficient of z^n.

$$P_n(-x)=(-1)^n P_n(x) \qquad \ldots\ldots (4)$$

Substitute $x=1$.

$P_n(-1)=(-1)^n P_n(1)$

$P_n(-1)=(-1)^n \qquad \left[\because\ P_n(1)=1\right]$

(c) Substitute $x=0$ in equation (1).

$$\sum_{n=0}^{\infty} z^n P_n(0)=\left(1+z^2\right)^{-1/2}$$

$$\sum_{n=0}^{\infty} z^n P_n(0)=\left\{1-\left(-z^2\right)\right\}^{-1/2}$$

$$\sum_{n=0}^{\infty} z^n P_n(0)=1+\frac{1}{2}\left(-z^2\right)+\frac{1.3}{2.4}\left(-z^2\right)^2+\frac{1.3.5}{2.4.6}\left(-z^2\right)^3+\ldots+\frac{1.3.5\ldots(2r-1)}{2.4.6\ldots 2r}\left(-z^2\right)^r$$

$$\sum_{n=0}^{\infty} z^n P_n(0) = \sum_{n=0}^{\infty} (-1)^m z^{2m} \frac{1.3.5...(2m-1)}{2.4.6...2m}$$

$\Rightarrow \quad P_{2n+1}(0) = 0$ [As there is no term containing odd powers of z.]

and $P_{2n}(0) = (-1)^n \dfrac{1.3.5...(2n-1)}{2.4.6...2n}$

Multiply and divide by $(2.4.6...2n)$

$$P_{2n}(0) = (-1)^n \frac{1.2.3.4.5.6...(2n-1).2n}{(2.4.6...2n)(2.4.6...2n)}$$

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^n(1.2.3.4....n)2^n(1.2.3.4....n)}$$

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(1.2.3.4....n)^2}$$

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$$

Example 6: Show that $\dfrac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) P_n(x) z^n$.

Solution: We have $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$ …… (1)

Differentiate (1) with respect to z.

$$-\frac{1}{2}(2z-2x)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$$

$$(x-z)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$$

Multiply by $2z$.

$$2(x-z)z(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} 2nz^n P_n(x) \qquad \text{…… (2)}$$

Add (1) and (2).

$$(1-2xz+z^2)^{-1/2} + 2(x-z)z(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n(x) + \sum_{n=0}^{\infty} 2nz^n P_n(x)$$

$$(1-2xz+z^2)^{-3/2}\left[(1-2xz+z^2) + 2(x-z)z\right] = \sum_{n=0}^{\infty} (2n+1) P_n(x) z^n$$

$$\frac{(1-2xz+z^2) + 2(x-z)z}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) P_n(x) z^n$$

$$\boxed{\frac{1-z^2}{\left(1-2xz+z^2\right)^{3/2}}=\sum_{n=0}^{\infty}\left(2n+1\right)P_n\left(x\right)z^n}$$

ORTHOGONAL PROPERTIES OF LEGENDRE POLYNOMIAL

$(a)\quad \int_{-1}^{1}P_m\left(x\right)P_n\left(x\right)dx=0\quad \text{if } m\neq n$

$(b)\quad \int_{-1}^{1}\left|P_n\left(x\right)\right|^2dx=\frac{2}{2n+1}\quad \text{if } m=n.$

Proof: (a) Legendre's equation may be written as

$$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dy}{dx}\right\}+n\left(n+1\right)y=0$$

$$\therefore\quad \frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\}+n\left(n+1\right)P_n=0\qquad \text{...... (1)}$$

and $$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_m}{dx}\right\}+m\left(m+1\right)P_m=0\qquad \text{...... (2)}$$

Multiply equation (1) P_m and equation (2) by P_n and then subtract.

$$P_m\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\}-P_n\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_m}{dx}\right\}+\left\{n\left(n+1\right)-m\left(m+1\right)\right\}P_nP_m=0$$

Integrating between the limits $-1 \text{ to } 1$.

$$\int_{-1}^{1}P_m\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\}dx-\int_{-1}^{1}P_n\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dP_m}{dx}\right\}dx+\left\{n\left(n+1\right)-m\left(m+1\right)\right\}\int_{-1}^{1}P_nP_mdx=0$$

$$\left[P_m\left(1-x^2\right)\frac{dP_n}{dx}\right]_{-1}^{1}-\int_{-1}^{1}\frac{dP_m}{dx}\left\{\left(1-x^2\right)\frac{dP_n}{dx}\right\}dx-\left[P_n\left(1-x^2\right)\frac{dP_m}{dx}\right]_{-1}^{1}+\int_{-1}^{1}\frac{dP_n}{dx}\left\{\left(1-x^2\right)\frac{dP_m}{dx}\right\}dx+$$

$$\left\{n\left(n+1\right)-m\left(m+1\right)\right\}\int_{-1}^{1}P_nP_mdx=0$$

$$\therefore\quad \left\{n\left(n+1\right)-m\left(m+1\right)\right\}\int_{-1}^{1}P_nP_mdx=0$$

Hence $\int_{-1}^{1}P_mP_ndx=0$

$(b)\quad \int_{-1}^{1}\left|P_n\left(x\right)\right|^2dx=\frac{2}{2m+1}\quad \text{if } m=n.$

Proof: We have

$$\sum_{n=0}^{\infty}z^nP_n\left(x\right)=\left(1-2xz+z^2\right)^{-1/2}\qquad \text{......(1)}$$

$$\sum_{m=0}^{\infty}z^mP_m\left(x\right)=\left(1-2xz+z^2\right)^{-1/2}\qquad \text{......(2)}$$

Multiply the corresponding sides of (1) and (2).

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}P_m(x)P_n(x)z^{m+n}=\left(1-2xz+z^2\right)^{-1}$$

Integrating between the limits -1 to 1.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\int_{-1}^{1}P_m(x)P_n(x)\,dx\right]z^{m+n}=\int_{-1}^{1}\left(1-2xz+z^2\right)^{-1}dx$$

When $m=n$

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1}\{P_n(x)\}^2\,dx\right]h^{2n}=\int_{-1}^{1}\frac{1}{\left(1-2xz+z^2\right)}dx=\left[\frac{\log\left(1-2xz+z^2\right)}{-2z}\right]_{-1}^{1}$$

$$=-\frac{1}{2z}\left[\log\left(1-2z+z^2\right)-\log\left(1+2z+z^2\right)\right]$$

$$=-\frac{1}{2z}\left[\log(1-z)^2-\log(1+z)^2\right]$$

$$=-\frac{1}{2z}\left[2\log(1-z)-2\log(1+z)\right]$$

$$=\frac{1}{z}\left[\log(1+z)-\log(1-z)\right]$$

$$=\frac{1}{z}\left[\left(z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\ldots\right)-\left(-z-\frac{z^2}{2}-\frac{z^3}{3}-\frac{z^4}{4}-\ldots\right)\right]$$

$$=\frac{1}{z}\left[\left(z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\ldots\right)+\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}++\ldots\right)\right]$$

$$=\frac{2}{z}\left(z+\frac{z^3}{3}+\frac{z^5}{5}+\ldots\right)$$

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1}\{P_n(x)\}^2\,dx\right]z^{2n}=2\left(1+\frac{z^2}{3}+\frac{z^4}{5}+\ldots\right)$$

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1}\{P_n(x)\}^2\,dx\right]z^{2n}=2\sum_{n=0}^{\infty}\frac{z^{2n}}{2n+1}$$

Equate the coefficient of z^{2n}.

$$\boxed{\int_{-1}^{1}\{P_n(x)\}^2\,dx=\frac{2}{2n+1}}$$

Thus,

$$\boxed{\int_{-1}^{1}P_m(x)P_n(x)\,dx=\begin{cases}0, & m\neq n\\ \dfrac{2}{2n+1}, & m=n\end{cases}}$$

RECURRENCE RELATIONS:

(I) $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

Use $\left(1-2xz+z^2\right)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$ …… (1)

Differentiate both sides with respect to 'z'

$$-\frac{1}{2}\left(1-2xz+z^2\right)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$\left(1-2xz+z^2\right)^{-1/2}(x-z) = \left(1-2xz+z^2\right)\sum_{n=0}^{\infty} nz^{n-1}P_n(x) \qquad \text{…… (2)}$$

Use (1) in (2)

$$\sum_{n=0}^{\infty} z^n P_n(x)(x-z) = \left(1-2xz+z^2\right)\sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$(x-z)\left[P_0(x)+zP_1(x)+z^2P_2(x)+...+z^{n-1}P_{n-1}(x)+z^nP_n(x)...\right]$$

$$=\left(1-2xz+z^2\right)\left[P_1(x)+zP_2(x)+...+(n-1)z^{n-2}P_{n-1}(x)+nz^{n-1}P_n(x)+(n+1)z^nP_{n+1}(x)+...\right]$$

Equate the coefficient of z^n.

$$xP_n(x)-P_{n-1}(x)=(n+1)P_{n+1}(x)-2xnP_n(x)+(n-1)P_{n-1}(x)$$

$$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x)$$

$$\boxed{(2n+1)xP_n=(n+1)P_{n+1}+nP_{n-1}}$$

(II) $nP_n = xP_n' - P_{n-1}'$

Poof:

Use $\left(1-2xz+z^2\right)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$ …… (1)

Differentiate both sides with respect to 'z'

$$-\frac{1}{2}\left(1-2xz+z^2\right)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$\left(1-2xz+z^2\right)^{-3/2}(x-z) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x) \qquad \text{…… (2)}$$

Again differentiate (1) with respect to 'x'.

$$-\frac{1}{2}(-2z)\left(1-2xz+z^2\right)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n'(x)$$

$$z\left(1-2xz+z^2\right)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n'(x) \qquad \ldots\ldots (3)$$

Divide (2) by (3)

$$\frac{\left(1-2xz+z^2\right)^{-3/2}(x-z)}{z\left(1-2xz+z^2\right)^{-3/2}} = \frac{\sum_{n=0}^{\infty} nz^{n-1}P_n(x)}{\sum_{n=0}^{\infty} z^n P_n'(x)}$$

$$z\sum_{n=0}^{\infty} nz^{n-1}P_n(x) = (x-z)\sum_{n=0}^{\infty} z^n P_n'(x)$$

$$z\left[P_1(x)+2zP_2(x)+\ldots+nz^{n-1}P_n(x)+\ldots\right] = (x-z)\left[P_0'(x)+zP_1'(x)+\ldots+z^{n-1}P_{n-1}'(x)+z^nP_n'(x)+\ldots\right]$$

Equate the coefficient of z^n.

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

$$\boxed{nP_n = xP_n' - P_{n-1}'}$$

(III) $(2n+1)P_n = P_{n+1}' - P_{n-1}'$

Poof:

From recurrence relation *I*, we have

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

Differentiate with respect to '*x*'.

$$(2n+1)xP_n' + (2n+1)P_n = (n+1)P_{n+1}' + nP_{n-1}' \qquad \ldots\ldots (1)$$

From recurrence relation (*II*), we have

$$nP_n = xP_n' - P_{n-1}' \qquad \ldots\ldots (2)$$

Put the value of xP_n' from (2) into (1).

$$(2n+1)\left(nP_n + P_{n-1}'\right) + (2n+1)P_n = (n+1)P_{n+1}' + nP_{n-1}'$$

$$(2n+1)(n+1)P_n = (n+1)P_{n+1}' + nP_{n-1}' - (2n+1)P_{n-1}'$$

$$(2n+1)(n+1)P_n = (n+1)P_{n+1}' - (n+1)P_{n-1}'$$

$$\boxed{(2n+1)P_n = P_{n+1}' - P_{n-1}'}$$

(IV) $(n+1)P_n = P_{n+1}' - xP_{n-1}'$

Proof: From recurrence relation *II* and *III*, we have

$$nP_n = xP_n' - P_{n-1}' \qquad \ldots\ldots (1)$$

$$(2n+1)P_n = P_{n+1}' - P_{n-1}' \qquad \ldots\ldots (2)$$

Subtract (1) from (2).

$$(2n+1)P_n - nP_n = P'_{n+1} - P'_{n-1} - xP'_n + P'_{n-1}$$

$$\boxed{(n+1)P_n = P'_{n+1} - xP'_{n-1}}$$

Bessel's Functions

The ordinary differential equation of second order

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0 \qquad \ldots\ldots (1)$$

Where n is a constant (real or complex), is called "Bessel's equation".
The point $x = 0$ is regular point of equation (1).
The series solution of equation (1) in neighbourhood of $x = 0$ is

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \qquad \ldots\ldots (2)$$

Equation (2) shows one of the solutions of Bessel's equation (1) which is called Bessel's function of first kind of order *n*. The other solution $J_{-n}(x)$ is called Bessel's function of second kind. The general solution of equation (1) is

$$y(x) = AJ_n(x) + BJ_{-n}(x)$$

Where *A* and *B* are constants.

Example 1: Show that when *n* is

(a) Positive integer, $J_{-n}(x) = (-1)^n J_n(x)$

(b) any integer, $J_n(-x) = (-1)^n J_n(x)$

(a) We know

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \qquad \ldots\ldots (1)$$

Then

$$J_{-n}(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r} \qquad \ldots\ldots (2)$$

$$J_{-n}(x) = \sum_{r=0}^{n-1}\frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r} + \sum_{r=n}^{\infty}\frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

$$J_{-n}(x) = 0 + \sum_{r=n}^{\infty}\frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r} \quad \because \sum_{r=0}^{n-1}\frac{(-1)^r}{r!\,\Gamma(-(n-1)+r)}\left(\frac{x}{2}\right)^{-20+2r} = 0,\ as\ -19+r<0$$

Put $-n+r = s$ $\qquad\qquad [r = n+s]$

$$J_{-n}(x)=\sum_{s=0}^{\infty}\frac{(-1)^{n+s}}{(n+s)!\,\overline{|(s+1)}}\left(\frac{x}{2}\right)^{n+2s}$$

$$J_{-n}(x)=(-1)^{n}\sum_{s=0}^{\infty}\frac{(-1)^{s}}{(n+s)!\,\overline{|(s+1)}}\left(\frac{x}{2}\right)^{n+2s}$$

$$J_{-n}(x)=(-1)^{n}\sum_{s=0}^{\infty}\frac{(-1)^{s}}{s!\,\overline{|(n+s+1)}}\left(\frac{x}{2}\right)^{n+2s} \qquad \left[\overline{|n+1}=n!\right]$$

Thus,

$$J_{-n}(x)=(-1)^{n}J_{n}(x)$$

(b) We know

$$J_{n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!\,\overline{|(n+r+1)}}\left(\frac{x}{2}\right)^{n+2r} \qquad \ldots\ldots (1)$$

Replace x by $-x$.

$$J_{n}(-x)=\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!\,\overline{|(n+r+1)}}\left(-\frac{x}{2}\right)^{n+2r} \qquad \ldots\ldots (2)$$

$$J_{n}(-x)=(-1)^{n}\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!\,\overline{|(n+r+1)}}\left(\frac{x}{2}\right)^{n+2r}$$

$$\boxed{J_{n}(-x)=(-1)^{n}J_{n}(x)}$$

RECURRENCE RELATIONS

(I) $\quad$ $\boldsymbol{x.J_n'(x)=n.J_n(x)-x.J_{n+1}(x)}$

Proof: We know

$$J_{n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^{r}}{r!\,\overline{|(n+r+1)}}\left(\frac{x}{2}\right)^{n+2r} \qquad \ldots\ldots (1) \text{ where } n \text{ is positive integer.}$$

Differentiate with respect to x.

.

$$J_{n}'(x)=\sum_{r=0}^{\infty}\frac{(-1)^{r}(n+2r)}{r!\,\overline{|(n+r+1)}}\cdot\frac{1}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$xJ_{n}'(x)=\sum_{r=0}^{\infty}\frac{(-1)^{r}(n+2r)}{r!\,\overline{|(n+r+1)}}\left(\frac{x}{2}\right)^{n+2r}$$

$$xJ_n^{'}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r n}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}+\sum_{r=0}^{\infty}\frac{(-1)^r 2r}{r!\,\Gamma(n+r+1)}\cdot\frac{x}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$xJ_n^{'}(x)=nJ_n+\sum_{r=0}^{\infty}\frac{(-1)^r 2r}{r!\,\Gamma(n+r+1)}\cdot\frac{x}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$xJ_n^{'}(x)=nJ_n+x\sum_{r=0}^{\infty}\frac{(-1)^r r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$xJ_n^{'}(x)=nJ_n+x\sum_{r=0}^{\infty}\frac{(-1)^r r}{r(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$xJ_n^{'}(x)=nJ_n+x\sum_{r=0}^{\infty}\frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

Substitute $r-1=s$, s is a symbol for $\sum$ notation

$$xJ_n^{'}(x)=nJ_n+x\sum_{s=0}^{\infty}\frac{(-1)^{s+1}}{s!\,\Gamma(n+s+2)}\left(\frac{x}{2}\right)^{n+2s+1}$$

$$xJ_n^{'}(x)=nJ_n+x\sum_{s=0}^{\infty}\frac{(-1)^{s+1}}{s!\,\Gamma((n+1)+s+1)}\left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$xJ_n^{'}(x)=nJ_n-x\sum_{s=0}^{\infty}\frac{(-1)^{s}}{s!\,\Gamma((n+1)+s+1)}\left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$\boxed{xJ_n^{'}(x)=nJ_n-xJ_{n+1}}\qquad\left[\because\quad J_{n+1}=\sum_{s=0}^{\infty}\frac{(-1)^{s}}{s!\,\Gamma((n+1)+s+1)}\left(\frac{x}{2}\right)^{(n+1)+2s}\right]$$

(II) $\quad$ $\boldsymbol{x.J_n'(x)=-n.J_n(x)+x.J_{n-1}(x)}$

Proof: We know

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}\qquad\ldots\ldots(1)$$

$$J_n^{'}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(n+2r)}{r!\,\Gamma(n+r+1)}\cdot\frac{1}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

Multiply both sides by *x*.

$$xJ_n^{'}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(n+2r)}{r!\,\Gamma(n+r+1)}\cdot\frac{x}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$xJ_n^{'}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(n+2r)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$xJ_n^{'}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(2n+2r-n)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}\qquad \text{[Add and subtract } n\text{]}$$

$$xJ_n^{'}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r\,2(n+r)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}-n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$xJ_n^{'}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r\,2(n+r)}{r!\,\Gamma(n+r+1)}\frac{x}{2}\left(\frac{x}{2}\right)^{n+2r-1}-n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$xJ_n^{'}(x)=x\sum_{r=0}^{\infty}\frac{(-1)^r(n+r)}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}-n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$xJ_n^{'}(x)=x\sum_{r=0}^{\infty}\frac{(-1)^r(n+r)}{r!(n+r)\Gamma(n+r)}\left(\frac{x}{2}\right)^{n-1+2r}-n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$xJ_n^{'}(x)=x\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma((n-1)+r+1)}\left(\frac{x}{2}\right)^{n-1+2r}-n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$\boxed{xJ_n^{'}(x)=xJ_{n-1}(x)-nJ_n(x)}$$

(III) $\quad$ $\boldsymbol{2J_n^{'}(x)=J_{n-1}(x)-J_{n+1}(x)}$

Proof: By recurrence I

$x.J_n^{'}(x)=n.J_n(x)-x.J_{n+1}(x)$ …… (1)

By recurrence *II*

$x.J_n(x)=-n.J_n(x)+x.J_{n-1}(x)$ …… (2)

Add (1) and (2).

$x.J_n^{'}(x)+x.J_n^{'}(x)=n.J_n(x)-x.J_{n+1}(x)-n.J_n(x)+x.J_{n-1}(x)$

$2x.J_n^{'}(x)=x\left(J_{n-1}(x)-J_{n+1}(x)\right)$

$$\boxed{2J_n^{'}(x)=J_{n-1}(x)-J_{n+1}(x)}$$

(IV) $\quad$ $\boldsymbol{2nJ_n=x(J_{n-1}+J_{n+1})}$

Proof: By recurrence *I*

$x.J_n^{'}(x)=n.J_n(x)-x.J_{n+1}(x)$ …… (1)

By recurrence *II*

$x.J_n(x)=-n.J_n(x)+x.J_{n-1}(x)$ …… (2)

Subtract (2) from (1).

$$0 = n.J_n(x) - x.J_{n+1}(x) - \left(-n.J_n(x) + x.J_{n-1}(x)\right)$$
$$0 = n.J_n(x) - x.J_{n+1}(x) + n.J_n(x) - x.J_{n-1}(x)$$
$$0 = 2n.J_n(x) - x.J_{n+1}(x) - x.J_{n-1}(x)$$

$$\boxed{2n.J_n = x\left(J_{n+1} + J_{n-1}\right)}$$

Example 2: Show that:

$$(a)\ J_{-1/2}(x) = \sqrt{\left\{\frac{2}{\pi x}\right\}}.\cos x$$

$$(b)\ J_{1/2}(x) = \sqrt{\left\{\frac{2}{\pi x}\right\}}.\sin x$$

$$(c)\ \left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}$$

$$(d)\ J_{3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\frac{\sin x}{x} - \cos x\right\}$$

$$(e)\ J_{-3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left\{-\frac{\cos x}{x} - \sin x\right\}$$

Solution: (a) We know that

$$J_n(x) = \frac{x^n}{2^n\left|\overline{(n+1)}\right.}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\right]$$

Put $n = -\frac{1}{2}$

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\left|\overline{\frac{1}{2}}\right.}\left[1 - \frac{x^2}{2(-1+2)} + \frac{x^4}{2.4(-1+2)(-1+4)} - \ldots\right]$$

$$J_{-1/2}(x) = \sqrt{\left\{\frac{2}{\pi x}\right\}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right] \qquad \left[\left|\overline{\frac{1}{2}}\right. = \sqrt{\pi}\right]$$

$$\boxed{J_{-1/2}(x) = \sqrt{\left\{\frac{2}{\pi x}\right\}}\cos x} \qquad \ldots\ldots (1)$$

Solution: (b) We know that

$$J_n(x) = \frac{x^n}{2^n\left|\overline{(n+1)}\right.}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \ldots\right]$$

Put $n = \frac{1}{2}$

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\left|\frac{3}{2}\right.}\left[1 - \frac{x^2}{2(1+2)} + \frac{x^4}{2.4(1+2)(1+4)} - \ldots\right]$$

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\left|\frac{3}{2}\right.}\left[1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \ldots\right]$$

$$J_{1/2}(x) = \sqrt{\left\{\frac{x}{2}\right\}}\frac{1}{\frac{1}{2}\sqrt{x}}.\frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right]$$

$$J_{1/2}(x) = \sqrt{\left\{\frac{2}{\pi x}\right\}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right]$$

$$\boxed{J_{1/2}(x) = \sqrt{\left\{\frac{2}{\pi x}\right\}}\sin x} \qquad \ldots\ldots (2)$$

Solution: (c) Square and add (1) and (2).

$$\left[J_{-1/2}(x)\right]^2 + \left[J_{1/2}(x)\right]^2 = \left[\sqrt{\left\{\frac{2}{\pi x}\right\}}\cos x\right]^2 + \left[\sqrt{\left\{\frac{2}{\pi x}\right\}}\sin x\right]^2$$

$$\left[J_{-1/2}(x)\right]^2 + \left[J_{1/2}(x)\right]^2 = \frac{2}{\pi x}\left(\cos^2 x + \sin^2 x\right)$$

$$\left[J_{-1/2}(x)\right]^2 + \left[J_{1/2}(x)\right]^2 = \frac{2}{\pi x}$$

$$\left[J_{1/2}(x)\right]^2 + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}$$

Solution (*d*) : $J_{3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\frac{\sin x}{x} - \cos x\right\}$

From recurrence relation IV

$$2nJ_n = x\left(J_{n-1} + J_{n+1}\right)$$

$$J_{n-1} + J_{n+1} = \frac{2n}{x}J_n$$

Put $n = 1/2$

$$J_{\frac{1}{2}-1} + J_{\frac{1}{2}+1} = \frac{2.\frac{1}{2}}{x}J_{\frac{1}{2}}$$

$$J_{-1/2} + J_{3/2} = \frac{1}{x}J_{1/2}$$

$$J_{3/2} = \frac{1}{x}J_{1/2} - J_{-1/2}$$

Put $J_{1/2}(x)=\sqrt{\left\{\frac{2}{\pi x}\right\}}.\sin x$ and $J_{-1/2}(x)=\sqrt{\left\{\frac{2}{\pi x}\right\}}\cos x$.

$$J_{3/2}=\frac{1}{x}\sqrt{\left\{\frac{2}{\pi x}\right\}}.\sin x-\sqrt{\left\{\frac{2}{\pi x}\right\}}\cos x$$

$$J_{3/2}=\sqrt{\left\{\frac{2}{\pi x}\right\}}\left(\frac{\sin x}{x}-\cos x\right)$$

Solution: (e) $J_{-3/2}(x)=\sqrt{\left(\frac{2}{\pi x}\right)}\left\{-\frac{\cos x}{x}-\sin x\right\}$

From recurrence relation IV

$$2nJ_n = x\left(J_{n-1}+J_{n+1}\right)$$

$$J_{n-1}+J_{n+1}=\frac{2n}{x}J_n$$

Put $n=-1/2$

$$J_{\frac{1}{2}-1}+J_{\frac{1}{2}+1}=-\frac{2.\frac{1}{2}}{x}J_{\frac{1}{2}}$$

$$J_{-1/2-1}+J_{-1/2+1}=-\frac{1}{x}J_{-1/2}$$

$$J_{-3/2}=-\frac{1}{x}J_{-1/2}-J_{1/2}$$

Put $J_{1/2}(x)=\sqrt{\left\{\frac{2}{\pi x}\right\}}.\sin x$ and $J_{-1/2}(x)=\sqrt{\left\{\frac{2}{\pi x}\right\}}\cos x$.

$$J_{-3/2}=-\frac{1}{x}J_{-1/2}-J_{1/2}$$

$$J_{-3/2}=-\frac{1}{x}\sqrt{\left\{\frac{2}{\pi x}\right\}}\cos x-\sqrt{\left\{\frac{2}{\pi x}\right\}}.\sin x$$

$$J_{-3/2}=\sqrt{\left\{\frac{2}{\pi x}\right\}}\left(-\frac{\cos x}{x}-\sin x\right)$$