Notes on Engineering Mathematics – II for First year / semester - II

Laplace Transforms

French mathematician Pierre-Simon Laplace (1749–1827)

Laplace transform, in mathematics, a particular integral transform invented by the French mathematician Pierre-Simon Laplace (1749–1827), and systematically developed by the British physicist Oliver Heaviside (1850–1925), to simplify the solution of many differential equations that describe physical processes. Today it is used most frequently by electrical engineers in the solution of various electronic circuit problems.

<u>Unit – II Laplace Transform:</u>

Laplace Transform: Let $f(t)$ be a function of *t* defined for all positive values, then

$$L\left[f(t)\right]=\int_0^\infty e^{-st}.f(t)\,dt$$

Some important formulae:

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1. $L[1]=\dfrac{1}{s}$
2. $L\left[e^{at}\right]=\dfrac{1}{s-a},\quad s>a$
3. $L\left[t^n\right]=\dfrac{\lceil n+1}{s^{n+1}}=\dfrac{n\lceil n}{s^{n+1}}=\dfrac{\lfloor n}{s^{n+1}},\quad n=0,1,2,3,.....$
4. $L[\sin at]=\dfrac{a}{s^2+a^2}$
5. $L[\cos at]=\dfrac{s}{s^2+a^2}$
6. $L[\sinh at]=\dfrac{a}{s^2-a^2},\ s^2>a^2$
7. $L[\cosh at]=\dfrac{s}{s^2-a^2},\ s^2>a^2$

Q. 1(*a*) Find the Laplace transform of $f(t)=\begin{cases}1, & 0\le t<1\\ t, & 1\le t<2\\ t^2, & 2\le t<\infty\end{cases}$

Solution 1(*a*): Here, $f(t)=\begin{cases}1, & 0\le t<1\\ t, & 1\le t<2\\ t^2, & 2\le t<\infty\end{cases}$

Use $L\left[f(t)\right]=\int_0^\infty e^{-st}.f(t)\,dt$

$$L\left[f(t)\right]=\int_0^1 e^{-st}f(t)\,dt+\int_1^2 e^{-st}f(t)\,dt+\int_2^\infty e^{-st}f(t)\,dt$$

$$L\left[f(t)\right]=\int_0^1 e^{-st}.1\,dt+\int_1^2 e^{-st}.t\,dt+\int_2^\infty e^{-st}.t^2\,dt$$

$$L\left[f(t)\right]=\left[\frac{e^{-st}}{-s}\right]_0^1+\left[t.\frac{e^{-st}}{-s}-\int 1.\frac{e^{-st}}{-s}dt\right]_1^2+\left[t^2.\frac{e^{-st}}{-s}-\int 2t.\frac{e^{-st}}{-s}dt\right]_2^\infty$$

$$L\left[f(t)\right]=\left[-\frac{e^{-st}}{s}\right]_0^1+\left[-t.\frac{e^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_1^2+\left[-t^2.\frac{e^{-st}}{s}+2\int t.\frac{e^{-st}}{s}dt\right]_2^\infty$$

$$L\left[f(t)\right]=\left[-\frac{e^{-st}}{s}\right]_0^1+\left[-t.\frac{e^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_1^2+\left[-t^2.\frac{e^{-st}}{s}+2\left\{-t.\frac{e^{-st}}{s^2}-\int 1.\frac{e^{-st}}{-s^2}dt\right\}\right]_2^\infty$$

$$L\left[f(t)\right]=\left[-\frac{e^{-st}}{s}\right]_0^1+\left[-t.\frac{e^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_1^2+\left[-t^2.\frac{e^{-st}}{s}+2\left\{-t.\frac{e^{-st}}{s^2}-\frac{e^{-st}}{s^3}\right\}\right]_2^\infty$$

$$L\left[f(t)\right]=\left[-\left(\frac{e^{-s}-e^0}{s}\right)\right]+\left[\left(-2.\frac{e^{-2s}}{s}-\frac{e^{-2s}}{s^2}\right)-\left(-2.\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}\right)\right]+\left[\begin{array}{l}-0-\left(-4\dfrac{e^{-2s}}{s}\right)\\+2\left\{0-\left(-2.\dfrac{e^{-2s}}{s^2}-\dfrac{e^{-2s}}{s^3}\right)\right\}\end{array}\right]$$

Simplify

$$L\left[f(t)\right]=\frac{1}{s}+\frac{2}{s}e^{-2s}+\frac{1}{s^2}e^{-s}+\frac{3}{s^2}e^{-2s}+\frac{2}{s^3}e^{-2s}$$

First shifting Theorem:

If $L\left[f(t)\right]=F(s)$ then $L\left[e^{at}.f(t)\right]=F(s-a)$.

Q.1 Evaluate $L\left[e^{2t}\sin t\right]$.

Solution: Here $f(t)=\sin t$

$L\left[f(t)\right]=L[\sin t]$

$L\left[f(t)\right]=\dfrac{1}{s^2+1}$ then

$L\left[e^{2t}\sin t\right]=\dfrac{1}{(s-2)^2+1}$ $\qquad\because\quad L\left[e^{at}.f(t)\right]=F(s-a)$

Laplace Transform of $t^n.f(t)$ (Multiplication by t^n):

If $L\left[f(t)\right]=F(s)$ then

$L\left[t^n.f(t)\right]=(-1)^n\dfrac{d^n}{ds^n}\left[F(s)\right],\quad n=1,2,3,......$

Q. 2 Evaluate $L\left[t^2\cos at\right]$.

Solution: Here $f(t)=\cos at$

$L\left[f(t)\right]=L[\cos at]$

$$L\left[f(t)\right]=\frac{s}{s^2+a^2}=F(s)$$

Therefore,

$$L\left[t^2\cos at\right]=(-1)^2\frac{d^2}{ds^2}\left[\frac{s}{s^2+a^2}\right]$$

$$L\left[t^2\cos at\right]=\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)\right]$$

$$L\left[t^2\cos at\right]=\frac{d}{ds}\left[\frac{(s^2+a^2).1-2s.s}{(s^2+a^2)^2}\right]$$

$$L\left[t^2\cos at\right]=\frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$L\left[t^2\cos at\right]=\left[\frac{(s^2+a^2)^2(-2s)-2(a^2-s^2)(s^2+a^2).2s}{(s^2+a^2)^4}\right]$$

$$L\left[t^2\cos at\right]=-2s(s^2+a^2)\left[\frac{(s^2+a^2)+2(a^2-s^2)}{(s^2+a^2)^4}\right]$$

$$L\left[t^2\cos at\right]=-2s\left[\frac{3a^2-s^2}{(s^2+a^2)^3}\right]$$

Q. 3 Find the Laplace transform of the function $f(t)=te^{-t}\sin 2t$

Solution: $f(t)=te^{-t}\sin 2t$

$$L[\sin 2t]=\frac{2}{s^2+4}$$

$$L\left[e^{-t}\sin 2t\right]=\frac{2}{(s+1)^2+4}=F(s)\quad (say)$$

$$L\left[te^{-t}\sin 2t\right]=-\frac{d}{ds}\left[\frac{2}{(s+1)^2+4}\right]=\frac{2.2(s+1)}{\left\{(s+1)^2+4\right\}^2}$$

$$L\left[te^{-t}\sin 2t\right]=\frac{4(s+1)}{\left\{(s+1)^2+4\right\}^2}$$

Q. 3 Evaluate $L\left[t^2e^{2t}\sin t\right]$. (Solved example)

Laplace Transform of $\frac{1}{t}f(t)$ (Division by t):

If $L\left[f(t)\right]=F(s)$ then

$$L\left[\frac{1}{t}f(t)\right]=\int_s^{\infty}F(s)\,ds$$

Q.4 Find the Laplace Transform of $\dfrac{\cos at-\cos bt}{t}$.

Solution: $f(t)=\cos at-\cos bt$

$$L\left[f(t)\right]=L\left[\cos at-\cos bt\right]$$

$$L\left[f(t)\right]=\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}=F(s)$$

Therefore,

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\int_s^{\infty}\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]ds$$

Integrate without substitution

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\left[\frac{1}{2}\log\left(s^2+a^2\right)-\frac{1}{2}\log\left(s^2+b^2\right)\right]_s^{\infty}$$

Simplify the RHS.

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^{\infty}$$

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\left[\log\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^{\infty}$$

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\left[\log\frac{\left(1+\frac{a^2}{s^2}\right)}{\left(1+\frac{b^2}{s^2}\right)}\right]_s^{\infty}$$

Substitute the limits.

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\left[\log\frac{\left(1+\frac{a^2}{\infty}\right)}{\left(1+\frac{b^2}{\infty}\right)}-\log\frac{\left(1+\frac{a^2}{s^2}\right)}{\left(1+\frac{b^2}{s^2}\right)}\right]$$

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\left[\log\frac{(1+0)}{(1+0)}-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\left[\log 1-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

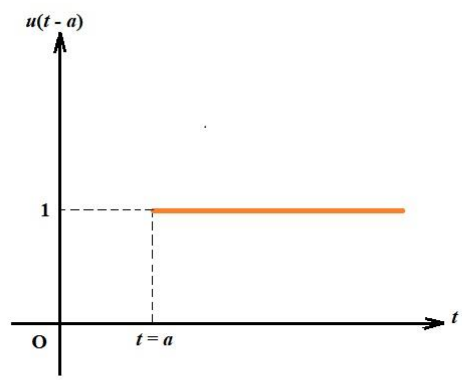
$$L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\left[0-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$L\left[\frac{\cos at-\cos bt}{t}\right]=\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)=\log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

Q.5 Find the Laplace Transform of $\frac{1-\cos t}{t^2}$. (Solved example)

Unit Step Function: A unit step function can be defined as

$$u(t-a)=\begin{cases}0, & \text{when } t<a\\ 1, & \text{when } t\ge a\end{cases}$$



Second Shifting Theorem:
If $L\left[f(t)\right]=F(s)$ then

$$\boxed{L\left[f(t-a).u(t-a)\right]=e^{-as}.F(s)}$$

Q.6 Express the following function in terms of unit step function

$$f(t)=\begin{cases}t-1, & 1<t<2\\ 3-t, & 2<t<3\end{cases}$$

Solution: $f(t)=\begin{cases}t-1, & 1<t<2\\ 3-t, & 2<t<3\end{cases}$

The function $f(t)$ can be written as

$$f(t)=(t-1)\left[u(t-1)-u(t-2)\right]+(3-t)\left[u(t-2)-u(t-3)\right]$$

$$f(t)=(t-1)\left[u(t-1)-u(t-2)\right]-(t-3)\left[u(t-2)-u(t-3)\right]$$

$$f(t)=\left[(t-1)u(t-1)-(t-1)u(t-2)\right]-\left[(t-3)u(t-2)-(t-3)u(t-3)\right]$$

$$f(t)=\left[(t-1)u(t-1)-(t-2+1)u(t-2)\right]-\left[(t-2-1)u(t-2)-(t-3)u(t-3)\right]$$

$$f(t)=(t-1)u(t-1)-(t-2)u(t-2)-u(t-2)-\left[(t-2)u(t-2)-u(t-2)-(t-3)u(t-3)\right]$$

$$f(t)=(t-1)u(t-1)-(t-2)u(t-2)-u(t-2)-(t-2)u(t-2)+u(t-2)+(t-3)u(t-3)$$

Simplify.

$$f(t)=(t-1)u(t-1)-2(t-2)u(t-2)+(t-3)u(t-3)$$

Take *L. T.* of both sides.

$$L\left[f(t)\right]=L\left[(t-1)u(t-1)-2(t-2)u(t-2)+(t-3)u(t-3)\right]$$

$$L\left[f(t)\right]=L(t-1)u(t-1)-2L(t-2)u(t-2)+L(t-3)u(t-3)$$

Use second shifting theorem

$$L\left[f(t-a).u(t-a)\right]=e^{-as}.F(s)$$

$$L\left[f(t)\right]=e^{-s}L(t)-2e^{-2s}L(t)+e^{-3s}L(t)$$

$$L\left[f(t)\right]=e^{-s}\frac{1}{s^2}-2e^{-2s}\frac{1}{s^2}+e^{-3s}\frac{1}{s^2}$$

$$L\left[f(t)\right]=e^{-s}\frac{1}{s^2}\left(1-2e^{-s}+e^{-2s}\right)$$

$$L\left[f(t)\right]=e^{-s}\frac{1}{s^2}\left(1-e^{-s}\right)^2$$

Laplace Transform of a Periodic Function:

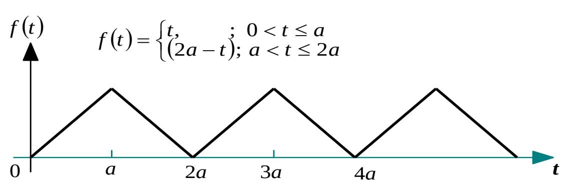
Let $f(t)$ be a periodic function of period T

$$\boxed{L\left[f(t)\right]=\frac{\int_0^T e^{-st}.f(t)\,dt}{1-e^{-sT}}}$$

Q.7 Draw the graph of the following periodic function and find its Laplace transform,

where $f(t)=\begin{cases} t, & 0<t\le a \\ (2a-t); & a<t\le 2a \end{cases}$

Solution $f(t)=\begin{cases} t, & 0<t\le a \\ (2a-t); & a<t\le 2a \end{cases}$



Laplace Transform of a periodic function is:

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T f(t) e^{-st}\, dt$$

Here, $T = 2a$ therefore,

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} f(t) e^{-st}\, dt + \int_a^{2a} f(t) e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^{a} t e^{-st}\, dt + \int_a^{2a} (2a-t) e^{-st}\, dt \right]$$

$$= \frac{1}{s^2\left(1-e^{-2as}\right)} \left(1 + e^{-2as} - 2e^{-as}\right)$$

$$= \frac{1}{s^2} \left(\frac{1-e^{-as}}{1+e^{-as}} \right)$$

$$= \frac{1}{s^2} \left(\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{e^{\frac{as}{2}} + e^{-\frac{as}{2}}} \right) = \left(\frac{\frac{e^{\frac{as}{2}} - e^{-\frac{as}{2}}}{2}}{\frac{e^{\frac{as}{2}} + e^{-\frac{as}{2}}}{2}} \right)$$

$$\boxed{L[f(t)] = \frac{1}{s^2} \tanh \frac{as}{2}}$$

Q. 8 A periodic function of period $\frac{2\pi}{\omega}$ is defined as:

$$f(t) = \begin{cases} E \sin \omega t, & ; \ 0 \le t \le \frac{\pi}{\omega} \\ 0 & ; \ \frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega} \end{cases}$$

Where E and ω are constants, find the Laplace transform**.**

Solution: Here, $f(t) = \begin{cases} E \sin \omega t, & ; \ 0 \le t \le \frac{\pi}{\omega} \\ 0 & ; \ \frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega} \end{cases}$

Laplace Transform of a periodic function is:

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T f(t) e^{-st}\, dt$$

Here, $T = \frac{2\pi}{\omega}$ therefore,

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} f(t)e^{-st}dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} f(t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st} E\sin\omega t dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}.0\, dt\right]$$

$$= \frac{E}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st} \sin\omega t dt\right]$$

$$\boxed{L[f(t)] = \frac{E\omega}{(s^2+\omega^2)\left(1-e^{-\frac{\pi s}{\omega}}\right)}}$$

Q.9 Draw the graph of the following periodic function and find its Laplace transform,

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$$ (Solved Example)

Evaluation of integrals with the help of Laplace Transform:

Q.10 Evaluate $\int_0^\infty te^{-3t}\sin t\,dt$.

Solution $\int_0^\infty te^{-3t}\sin t\,dt$

Find the L.T. of integrand neglecting the exponential term.

$$L[t\sin t] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) \quad \because \quad L[\sin t] = \frac{1}{s^2+1}$$

$$L[t\sin t] = \frac{2s}{(s^2+1)^2}$$

Compare the integral with the definition of $L.T.$ and find the value of s.

Here, $s = 3$

$$\int_0^\infty te^{-3t}\sin t\,dt = \frac{2\times 3}{(3^2+1)^2} = \frac{3}{50}$$

Q.11 Evaluate $\int_0^\infty te^{-4t}\sin t\,dt$.

Q.12 Evaluate $\int_0^\infty \frac{e^{-t}-e^{-4t}}{t}dt$.

Inverse Laplace Transform:

If $L\left[f(t)\right]=F(s)$ then $L^{-1}\left[F(s)\right]=f(t)$

S.No.	$L.T$	Inverse $L.T$
1	$L[1]=\dfrac{1}{s}$	$L^{-1}\left[\dfrac{1}{s}\right]=1$
2	$L\left[e^{at}\right]=\dfrac{1}{s-a},\quad s>a$	$L^{-1}\left[\dfrac{1}{s-a}\right]=e^{at}$
3	$L\left[t^n\right]=\dfrac{\overline{\lvert n+1}}{s^{n+1}}=\dfrac{n\overline{\lvert n}}{s^{n+1}}=\dfrac{\lfloor n}{s^{n+1}},\quad n=0,1,2,3,.....$	$L^{-1}\left[\dfrac{1}{s^n}\right]=\dfrac{t^{n-1}}{\lfloor n-1}$
4	$L[\sin at]=\dfrac{a}{s^2+a^2}$	$L^{-1}\left[\dfrac{1}{s^2+a^2}\right]=\dfrac{1}{a}\sin at$
5	$L[\cos at]=\dfrac{s}{s^2+a^2}$	$L^{-1}\left[\dfrac{s}{s^2+a^2}\right]=\cos at$
6	$L[\sinh at]=\dfrac{a}{s^2-a^2},\ s^2>a^2$	$L^{-1}\left[\dfrac{1}{s^2-a^2}\right]=\dfrac{1}{a}\sinh at$
7	$L[\cosh at]=\dfrac{s}{s^2-a^2},\ s^2>a^2$	$L^{-1}\left[\dfrac{s}{s^2-a^2}\right]=\cosh at$

Multiplication by *s*:

If $L^{-1}\left[F(s)\right]=f(t)$ then

$$\boxed{L^{-1}\left[s.F(s)\right]=\frac{d}{dt}f(t)+f(0)\delta(t)}$$

Q. 13 Evaluate $L^{-1}\left[\dfrac{s^2}{s^2+a^2}\right]$.

Solution: $L^{-1}\left[\dfrac{s^2}{s^2+a^2}\right]=L^{-1}\left[s.\dfrac{s}{s^2+a^2}\right]$

Here, $F(s)=\dfrac{s}{s^2+a^2}$

$$L^{-1}\left[F(s)\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]=\cos at=f(t)$$

Therefore,

$$L^{-1}\left[s.\frac{s}{s^2+a^2}\right]=\frac{d}{dt}\cos at+\cos(0)\delta(t)$$

$$\boxed{L^{-1}\left[\frac{s^2}{s^2+a^2}\right]=-a\sin at+1}$$

Division by *s*:

If $L^{-1}\left[F(s)\right]=f(t)$ then

$$L^{-1}\left[\frac{1}{s}.F(s)\right]=\int_0^t f(t)\,dt$$

Q. 14 Evaluate $L^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]$

Solution: Here, $F(s)=\frac{1}{s^2+1}$

$$L^{-1}\left[F(s)\right]=L^{-1}\left[\frac{1}{s^2+1}\right]=\sin t=f(t)$$

Therefore,

$$L^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]=\int_0^t \sin t\,dt$$

$$L^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]=-\left[\cos t\right]_0^t$$

$$L^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]=-\left[\cos t-\cos 0\right]$$

$$L^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]=1-\cos t$$

First Shifting Property:

If $L^{-1}\left[F(s)\right]=f(t)$ then

$$\boxed{L^{-1}\left[F(s-a)\right]=e^{at}.f(t)}$$

Q15 Find the inverse Laplace transform of $\frac{s+1}{s^2-6s+25}$.

Solution: $L^{-1}\left[\frac{s+1}{s^2-6s+25}\right]$

$$=L^{-1}\left[\frac{s+1}{s^2-6s+9+16}\right]$$

$$=L^{-1}\left[\frac{s-3+4}{(s-3)^2+4^2}\right]$$

$$= L^{-1}\left[\frac{s-3}{(s-3)^2+4^2}\right] + L^{-1}\left[\frac{4}{(s-3)^2+4^2}\right]$$

$$= e^{3t}\cos 4t + 4e^{3t}\frac{1}{4}\sin 4t$$

$$= e^{3t}\left(\cos 4t + \sin 4t\right)$$

Therefore,

$$\boxed{L^{-1}\left[\frac{s+1}{s^2-6s+25}\right] = e^{3t}\left(\cos 4t + \sin 4t\right)}$$

Second Shifting Property:

If $L^{-1}\left[F(s)\right] = f(t)$ then

$$\boxed{L^{-1}\left[e^{-as}F(s)\right] = f(t-a).u(t-a)}$$

Q.16 Find the inverse Laplace transform of $\dfrac{e^{-\pi s}}{(s+3)}$.

Solution: $L^{-1}\left[\dfrac{e^{-\pi s}}{(s+3)}\right]$

Here, $F(s) = \dfrac{1}{s+3}$

$$L^{-1}\left[F(s)\right] = L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t} = f(t)$$

Therefore,

$$\boxed{L^{-1}\left[\frac{e^{-\pi s}}{(s+3)}\right] = e^{-3(t-\pi)}.u(t-\pi)}$$

Inverse *L.T.* of derivative:

$$L\left[t.f(t)\right] = -\frac{d}{ds}\left[F(s)\right]$$

$$t.f(t) = -L^{-1}\left[\frac{d}{ds}\left[F(s)\right]\right]$$

$$t.L^{-1}F(s) = -L^{-1}\left[\frac{d}{ds}\left[F(s)\right]\right] \Rightarrow \quad \boxed{L^{-1}F(s) = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left[F(s)\right]\right]}$$

Q.17 Find the inverse Laplace transform of $\cot^{-1}\left(\frac{s+3}{2}\right)$

Solution: $L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right]$

$$L^{-1}[F(s)] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$$

$$L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\cot^{-1}\left(\frac{s+3}{2}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{-\frac{1}{2}}{1+\left(\frac{s+3}{2}\right)^2}\right]$$

$$= -\frac{e^{-3t}}{t}\sin 2t$$

Therefore,

$$\boxed{L^{-1}\left[\cot^{-1}\left(\frac{s+3}{2}\right)\right] = -\frac{e^{-3t}}{t}\sin 2t}$$

Partial Fraction Method:

Q.18 Find the inverse Laplace transform of $\frac{2s^2+5s-4}{s^3+s^2-2s}$

Solution: $L^{-1}\left[\frac{2s^2+5s-4}{s^3+s^2-2s}\right]$

$$\frac{2s^2+5s-4}{s^3+s^2-2s} = \frac{2s^2+5s-4}{s\left(s^2+s-2\right)}$$

$$= \frac{2s^2+5s-4}{s(s+2)(s-1)}$$

Resolve in to partial fractions.

$$\frac{2s^2+5s-4}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

Calculate A, B and C.

$$A = \lim_{s\to 0}\left[s.\frac{2s^2+5s-4}{s(s+2)(s-1)}\right]$$

$$A = \lim_{s\to 0}\left[\frac{2s^2+5s-4}{(s+2)(s-1)}\right] = \frac{-4}{-2} = 2$$

$$B = \lim_{s\to -2}\left[(s+2).\frac{2s^2+5s-4}{s(s+2)(s-1)}\right]$$

$$B=\lim_{s\to -2}\left[\frac{2s^2+5s-4}{s(s-1)}\right]=\frac{2(-2)^2+5(-2)-4}{-2(-2-1)}$$

$$B=\frac{-6}{6}=-1$$

$$C=\lim_{s\to 1}\left[(s-1).\frac{2s^2+5s-4}{s(s+2)(s-1)}\right]$$

$$C=\lim_{s\to 1}\left[\frac{2s^2+5s-4}{s(s+2)}\right]=\frac{2+5-4}{1(1+2)}=1$$

Therefore,

$$\frac{2s^2+5s-4}{s(s+2)(s-1)}=\frac{2}{s}-\frac{1}{s+2}+\frac{1}{s-1}$$

Take inverse L.T.

$$L^{-1}\left[\frac{2s^2+5s-4}{s(s+2)(s-1)}\right]=2L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+2}\right]+L^{-1}\left[\frac{1}{s-1}\right]$$

$$\boxed{L^{-1}\left[\frac{2s^2+5s-4}{s(s+2)(s-1)}\right]=2-e^{-2t}+e^{t}}$$

Convolution Theorem:

If $L^{-1}\left[F_1(s)\right]=f_1(t)$ and $L^{-1}\left[F_2(s)\right]=f_2(t)$ exist then

$$L^{-1}\left[F_1(s).F_2(s)\right]=\int_0^t f_1(x).f_2(t-x)\,dx$$

Q.19 Using Convolution Theorem find $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$.

Solution: $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=L^{-1}\left\{\frac{s}{(s^2+a^2)}.\frac{s}{(s^2+b^2)}\right\}$

Here, $F_1(s)=\frac{s}{s^2+a^2}$ and $F_2(s)=\frac{s}{s^2+b^2}$

$L^{-1}\left[F_1(s)\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]$ and $L^{-1}\left[F_2(s)\right]=L^{-1}\left[\frac{s}{s^2+b^2}\right]$

$f_1(t)=\cos at$ and $f_2(t)=\cos bt$

Use convolution theorem

$$L^{-1}\left[F_1(s).F_2(s)\right]=\int_0^t f_1(x).f_2(t-x)\,dx$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\int_0^t \cos ax.\cos b(t-x)\,dx$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\int_0^t 2\cos ax.\cos b(t-x)\,dx$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\int_0^t\left[\cos\{ax+b(t-x)\}+\cos\{ax-b(t-x)\}\right]dx$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\int_0^t\left[\cos\{(a-b)x+bt\}+\cos\{(a+b)x-bt\}\right]dx$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\left[\frac{\sin\{(a-b)x+bt\}}{a-b}+\frac{\sin\{(a+b)x-bt\}}{a+b}\right]_0^t$$

Substitute the limits.

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\left[\left\{\frac{\sin\{(a-b)t+bt\}}{a-b}+\frac{\sin\{(a+b)t-bt\}}{a+b}\right\}-\left\{\frac{\sin bt}{a-b}+\frac{\sin(-bt)}{a+b}\right\}\right]$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\left[\left\{\frac{\sin at}{a-b}+\frac{\sin at}{a+b}\right\}-\left\{\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right\}\right]$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\left[\frac{\sin at}{a-b}+\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}+\frac{\sin bt}{a+b}\right]$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\left[\left(\frac{1}{a-b}+\frac{1}{a+b}\right)\sin at-\left(\frac{1}{a-b}-\frac{1}{a+b}\right)\sin bt\right]$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\left[\left(\frac{a+b+a-b}{a^2-b^2}\right)\sin at-\left(\frac{a+b-a+b}{a^2-b^2}\right)\sin bt\right]$$

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{2}\left[\left(\frac{2a}{a^2-b^2}\right)\sin at-\left(\frac{2b}{a^2-b^2}\right)\sin bt\right]$$

$$\boxed{L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{(a^2-b^2)}(a\sin at-b\sin bt)}$$

Q. 20 Use convolution theorem to find, $L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$

Solution 2(*e*): $L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]$

Suppose $F_1(s)=\frac{s}{s^2+1}$, $F_2(s)=\frac{1}{s^2+4}$

$L^{-1}[F_1(s)]=\cos t, \quad L^{-1}[F_2(s)]=\frac{1}{2}\sin 2t$

Use Convolution theorem: If $L^{-1}\left[F_1(s)\right]=f_1(t)$ *and* $L^{-1}\left[F_2(s)\right]=f_2(t)$ exist, then

$$L^{-1}[F_1(s).F_2(s)]=\int_0^t f_1(x).f_2(t-x)dx$$

Therefore, $L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]=\frac{1}{2}\int_0^t \cos x.\sin 2(t-x)dx$

$$=\frac{1}{4}\int_0^t 2\cos x.\sin 2(t-x)dx$$

$$=\frac{1}{4}\int_0^t [\sin(x+2t-2x)-\sin(x-2t+2x)]dx$$

$$=\frac{1}{4}\int_0^t [\sin(-x+2t)-\sin(3x-2t)]dx$$

$$=\frac{1}{4}\left[\cos(-x+2t)+\frac{1}{3}\cos(3x-2t)\right]_0^t$$

$$\boxed{L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)}\right]=\frac{1}{3}[\cos t-\cos 2t]}$$

Solution of Linear differential equations with constant coefficients:

Procedure:

1. **Take L.T. of both sides.**
2. **Substitute the boundary conditions.**
3. **Simplify for** $\bar{y}$ where $\bar{y}=L[y]$.
4. **Take inverse L.T of both sides.**
5. **Find an expression of *y* in terms of *t*. If the equation is given in** $\frac{dy}{dt}$.

Q. 21 Solve the differential equation using Laplace transform method:

$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=e^x.\sin x$, where, $y(0)=0, y'(0)=1$

Solution : $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=e^x.\sin x$ where, $y(0)=0, y'(0)=1$

Take *L.T.* of both sides

$$L\left[\frac{d^2y}{dx^2}\right]+2\left[\frac{dy}{dx}\right]+5L[y]=L\left[e^x.\sin x\right]$$

$$s^2\bar{y}-sy(0)-y'(0)+2\left[s\bar{y}-y(0)\right]+5\bar{y}=\frac{1}{(s+1)^2+1}$$

Put the given values

$$\left(s^2+2s+5\right)\bar{y}=1+\frac{1}{s^2+2s+2}=\frac{s^2+2s+3}{s^2+2s+2}$$

$$\bar{y}=\frac{s^2+2s+3}{\left(s^2+2s+5\right)\left(s^2+2s+2\right)}$$

$$\bar{y}=\frac{2}{3}\frac{1}{s^2+2s+5}+\frac{1}{3}\frac{1}{s^2+2s+2}$$

Take inverse $L.T$

$$y=\frac{2}{3}L^{-1}\left[\frac{1}{s^2+2s+5}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

$$y=\frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+4}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$\boxed{y=\frac{1}{3}e^{-x}\left(\sin 2x+2\sin x\right)}$$

Q. 22 Solve the differential equation using Laplace transform method

$\frac{d^2x}{dt^2}+9x=\cos 2t,\quad x(0)=1,\ x\left(\frac{\pi}{2}\right)=-1$ (Solved example in book)