Engineering Physics
Course Code : TPH 201

By Dr. Vishal Chauhan

L'Hospital's or (L'Hôpital)Rule:

Suppose that we have one of the following cases,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0} \qquad \text{OR} \qquad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where (a) can be a real number, infinity or negative infinity. Inthese cases we have,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

So, L'Hospital's Rule tells us that if we have an indeterminate form 0/0 or ∞ / ∞ all we need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Example:

(a) $\lim_{x\to 0}\frac{\sin x}{x}$

(b) $\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}$

Solutions:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=1$$

$$\lim_{t\to 1}\frac{5t^4-4t^2-1}{10-t-9t^3}=\lim_{t\to 1}\frac{20t^3-8t}{-1-27t^2}=\frac{20-8}{-1-27}=-\frac{3}{7}$$

Distribution of Intensity (N-slit diffraction case)

(i) We know the expression for resultant intensity (I) in case of N-slit diffraction

$$\boxed{I=\frac{A_0^2\sin^2\alpha}{\alpha^2}\left(\frac{\sin^2 N\beta}{\sin^2\beta}\right)} \quad \text{---(1)}$$

(ii) In this expression the value of (β) is equal to $\frac{\delta}{2}$(Phase angle)

(iii) we also know the value of phase angle in case of N-slit diffraction is

$$\boxed{\delta=\frac{2\pi}{\lambda}(e+d)\sin\theta} \quad \text{---(2)}$$

The first factor of equation (1), i.e. $\frac{A_0^2\sin^2\alpha}{\alpha^2}$ represents intensity distribution due to diffraction at a single slit; whereas the second factor, that is $(\sin^2 N\beta/\sin^2\beta)$ gives the distribution of intensity in the diffraction pattern due to the interference in the waves from all the N slits.

1. Principal Maxima (in case of N-slit diff.): $(\because I\propto 1/\sin^2\beta)$

As it is evident from equation (1) that for maximum value of (I) the value of $(\sin\beta)$ should be minimum, i.e. 0.

In this case $\frac{\sin^2 N\beta}{\sin\beta}$ will take the form of 0/0 which is an indeterminate form.

Also the value of $\sin\beta$ is zero when $\beta=\pm n\pi$, $n=0,1,2,3\ldots$

Now Apply the L'Hôpital Rule using limit of $\beta\to\pm n\pi$

By Dr. Vishal Chauhan

$$= \lim_{\beta \to \pm n\pi} \frac{\frac{d}{d\beta}(\sin N\beta)}{\frac{d}{d\beta}(\sin \beta)}$$

$$= \lim_{\beta \to \pm n\pi} \frac{N \cos N\beta}{\cos \beta}$$

$$= \frac{N \cos(\pm n\pi)}{\cos(\pm n\pi)}$$

$$= \boxed{\pm N} \xrightarrow{\text{use this for}} \left(\frac{\sin N\beta}{\sin \beta}\right)$$

$$I = \frac{A_0^2 \sin^2 \alpha}{\alpha^2} (\pm N)^2$$

or

$$\boxed{I_{P.max.} = \frac{A_0^2 \sin^2 \alpha \, N^2}{\alpha^2}} \quad \text{——— (3)}$$

(where N = Number of slits)

Thus, in order to find the resultant intensity of any principal maxima, we have to multiply N^2 by a factor of $\left(A_0^2 \sin^2\alpha / \alpha^2\right)$.

For $I_{P.max}$, $\beta = \pm n\pi$ ——— (4)

& $\beta = \frac{\delta}{2}$, and $\boxed{\delta = \frac{2\pi}{\lambda}(e+d)\sin\theta}$

Then, $\beta = \frac{\pi(e+d)\sin\theta}{\lambda}$ ——— (5)

from (4) & (5) $\Rightarrow$

$$\pm n\cancel{\pi} = \frac{\cancel{\pi}(e+d)\sin\theta}{\lambda}$$

$$\boxed{(e+d)\sin\theta = \pm n\lambda}$$

This is also known as grating equation.

For $n = 0$, zero order max.

" $n = \pm 1$, first " "

$n = \pm 2$, Second " "

and so on....

2. Principal Minima:

$$\because \boxed{I = \frac{A_0^2 \sin^2\alpha}{\alpha^2}\left(\frac{\sin N\beta}{\sin\beta}\right)^2}$$

As $I \propto \sin N\beta$

For minimum value of (I), $\sin N\beta$ should be minimum.

but $\sin\beta \neq 0$.

$$\therefore \boxed{I_{min} = 0}$$

$$\& \quad N\beta = \pm m\pi \qquad \because \beta = \frac{\pi(e+d)\sin\theta}{\lambda}$$

$$\therefore N\left[\frac{\cancel{\pi}(e+d)\sin\theta}{\lambda}\right] = \pm m\cancel{\pi}$$

$$\boxed{N(e+d)\sin\theta = \pm m\lambda}$$

Here, (m) can have all integral values 0, N, 2N, 3N.... because for these values of m, sin β=0 which gives the positions of principal maxima. Positive and negative signs show that the minima lie symmetrically on both sides of the central principal maxima.

For m =0, we get zero order principal maximum.
For m = N, we get principal maximum of first order.
For m = 1, 2, 3, ...(N-1) give minima. It means there are (N-1) minima between two successive principal maxima.

3.Secondary Maxima:

As we know that there are (N-1) minima between two successive principal maxima. Hence, there are (N-2) other maxima coming alternatively with the minima between tow successive principal maxima. These maxima are called secondary maxima. To find the positions of the secondary maxima, we first differentiate (I) with respect to β and equating it to zero.

$$\frac{dI}{d\beta} = 0, \quad \because I = \underbrace{\frac{A_0^2 \sin^2\alpha}{\alpha^2}}_{\text{Constant term}}\left(\frac{\sin N\beta}{\sin\beta}\right)^2$$

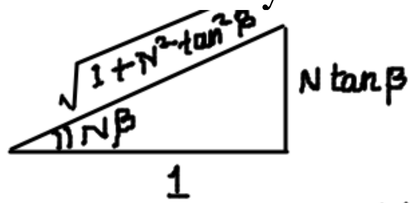
$$\frac{dI}{d\beta} = \frac{A_0^2 \sin^2\alpha}{\alpha^2} \cdot \frac{d}{d\beta}\left(\frac{\sin N\beta}{\sin\beta}\right)^2$$

$$\Rightarrow \frac{A_0^2 \sin^2\alpha}{\alpha^2}\left[2\left(\frac{\sin N\beta}{\sin\beta}\right)\left\{\frac{\sin\beta(N\cos N\beta) - \sin N\beta(\cos\beta)}{\sin^2\beta}\right\}\right] = 0$$

$$\sin\beta(N\cos N\beta) - \sin N\beta(\cos\beta) = 0$$

$$N\sin\beta\cos N\beta = \sin N\beta\cos\beta$$

$$\frac{N\sin\beta}{\cos\beta} = \frac{\sin N\beta}{\cos N\beta} \Rightarrow \boxed{\tan N\beta = \frac{N\tan\beta}{1}}$$



From this $\triangle$ $\sin N\beta = \dfrac{N\tan\beta}{\sqrt{1+N^2\tan^2\beta}}$

$$I_{Sec.Max} = \frac{A_0^2\sin^2\alpha}{\alpha^2}\cdot\frac{(N\tan\beta)^2}{\left(\sqrt{1+N^2\tan^2\beta}\right)^2\cdot\sin^2\beta}$$

$$= \frac{A_0^2\sin^2\alpha}{\alpha^2}\cdot\frac{N^2\tan^2\beta}{(1+N^2\tan^2\beta)\cdot\sin^2\beta}$$

$$= \frac{A_0^2\sin^2\alpha}{\alpha^2}\cdot\frac{N^2\cdot\cancel{\sin^2\beta}}{\cos^2\beta\left(1+N^2\cdot\frac{\sin^2\beta}{\cos^2\beta}\right)\cdot\cancel{\sin^2\beta}}$$

$$= \frac{A_0^2\sin^2\alpha}{\alpha^2}\cdot\frac{N^2}{\cancel{\cos^2\beta}\left(\frac{\cos^2\beta+N^2\sin^2\beta}{\cancel{\cos^2\beta}}\right)}$$

$$= \frac{A_0^2\sin^2\alpha}{\alpha^2}\cdot\frac{N^2}{\cos^2\beta+N^2\sin^2\beta}$$

$$= \frac{A_0^2\sin^2\alpha}{\alpha^2}\left[\frac{N^2}{(1-\sin^2\beta)+N^2\sin^2\beta}\right]$$

$$\boxed{I_{S.Max} = \frac{A_0^2\sin^2\alpha}{\alpha^2}\cdot\left[\frac{N^2}{1+\sin^2\beta(N^2-1)}\right]}$$

$$\frac{I_{S.Max}}{I_{P.Max}} = \frac{\cancel{\frac{A_0^2\sin^2\alpha}{\alpha^2}}\cdot\left[\frac{\cancel{N^2}}{1+\sin^2\beta(N^2-1)}\right]}{\cancel{\frac{A_0^2\sin^2\alpha}{\alpha^2}}\,[\cancel{N^2}]}$$

$$\boxed{\frac{I_{S.Max}}{I_{P.Max}} = \frac{1}{1+(N^2-1)\sin^2\beta}}$$

Greater the value of N, lower the intensity of secondary maxima.

Formation of Multiple Spectra with Grating

Since we know the grating equation, i.e. $\boxed{(e+d)\sin\theta = n\lambda}$

where $n = 0, 1, 2, 3 \ldots$

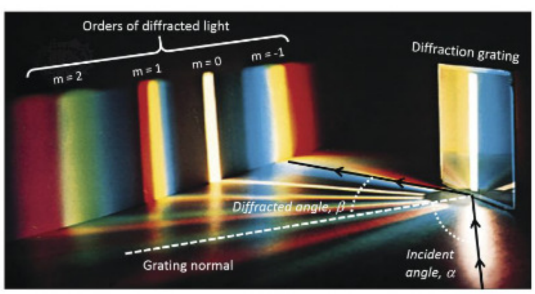
here $\theta \rightarrow$ Angle of diffraction

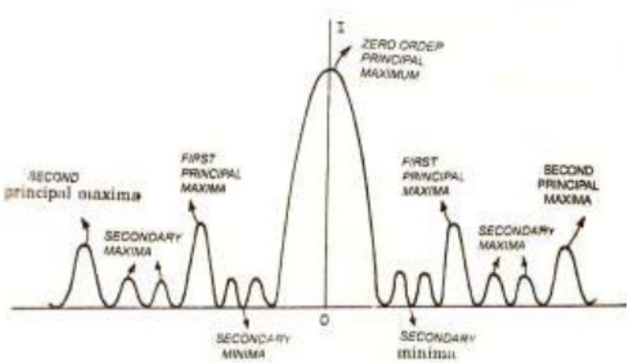
$n \rightarrow$ Number of orders of spectra

$\lambda \rightarrow$ Wavelength of incident light

The above grating equation provides very useful information about grating spercta.

1. For a particular wavelength (λ), the direction of principal maxima of different orders are different.
2. Longer the wavelength (λ), greater the angle of diffraction (θ).
3. If white light incident normally on a diffraction grating, each order of spectrum will contain principal maxima of different wavelengths in different directions.
4. For n=0, θ = 0 for all values of (λ). Thus, zero order principal maxima for all wavelengths lie in same direction. Hence zero order (n=0) principal maxima will be white.
5. The first order (n=1) principal maxima of all visible wavelengths form the first order and the second order principal maxima and second order spectrum and so on.
6. As (λ_r) > (λ_v), the angle of diffraction for red colour is greater than that for violet colour. Hence, in every spectrum the violet colour being in the innermost position and red colour in the outermost position except the central maximum.





Condition for Absent Spectra with a Diffraction Grating

$$\because \quad I = \frac{A_0^2 \sin^2\alpha}{\alpha^2} \cdot \frac{\sin^2 N\beta}{\sin^2\beta}$$

Also, $(e+d)\sin\theta = n\lambda$ ——— (i)

& $e\sin\theta = m\lambda$ ——— (ii)

$m = 1, 2, 3 \ldots$

$$\frac{(i)}{(ii)} \Rightarrow \frac{(e+d)\cancel{\sin}\theta}{e\,\cancel{\sin}\theta} = \frac{n\cancel{\lambda}}{m\cancel{\lambda}}$$

$$\frac{e+d}{e} = \frac{n}{m},$$

e = slit thickness
d = separation b/w two sources
$(e+d)$ = Grating Element

① if $e = d$,

$$\frac{e+e}{e} = \frac{n}{m}$$

$$\frac{2\cancel{e}}{\cancel{e}} = \frac{n}{m} \Rightarrow \boxed{n = 2m},$$ for $m = 1, 2, 3 \ldots$
$n = 2, 4, 6 \ldots$

It means if (e = d), 2,4,6,....order spectra are missing or absent.

② if $d = 2e$, $\frac{e+2e}{e} = \frac{n}{m}$

$$\frac{3\cancel{e}}{\cancel{e}} = \frac{n}{m}$$

$$\boxed{n = 3m}$$

Hence for m=1,2,3.... n=3,6,9,.... It means that for 1st,2nd, 3rd...orders of m, 3rd, 6th, 9th order of n will be absent.

Maximum number of orders with a diffraction grating

The maximum number of spectra available with a diffraction grating in the visible region can be evaluated by using the grating equation for normal incidence as →

$$\because \ (e+d)\sin\theta = n\lambda \qquad \text{where } e+d = \text{Grating Element}$$

$$n = \frac{(e+d)\sin\theta}{\lambda} \qquad \theta = \text{Angle of diffraction}, \quad n = \text{the order of maximum}$$

The maximum possible value of angle of diffraction is 90^0. Therefore, the maximum possible order available with grating is given by

$$n_{max.} = \frac{(e+d)\sin(90^\circ)}{\lambda}$$

$$\boxed{n_{max.} = \frac{e+d}{\lambda}}$$

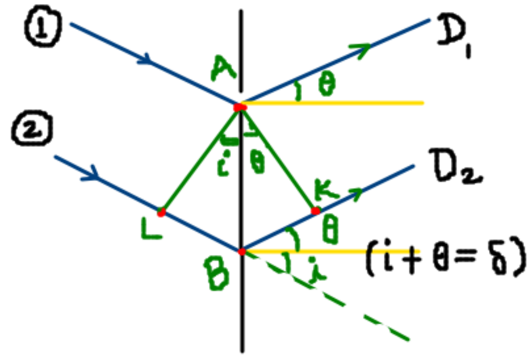
If the grating element lies within (λ) and (2λ) or grating element is $< 2\lambda$

$$\boxed{n_{max} < \frac{2\lambda}{\lambda} < 2}$$

This means if the width of a grating element is less than twice the wavelength of light, then only first order is visible.

By Dr. Vishal Chauhan

Grating under oblique incidence



$$\delta = (i+\theta) \text{ Angle of deviation}$$

$$BL = AB \sin i$$
$$= (e+d)\sin i$$

$$BK = AB \sin\theta$$
$$= (e+d)\sin\theta$$

BL+BK = Total Path Difference

$$\Delta P = BL + BK$$

$$\Delta P = BL + BK$$
$$= (e+d)\sin i + (e+d)\sin\theta$$
$$= (e+d)\left[\sin i + \sin\theta\right]$$

$$\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$$

$$= (e+d)\left[2\sin\left(\frac{i+\theta}{2}\right)\cdot\cos\left(\frac{i-\theta}{2}\right)\right]$$
$$= 2(e+d)\cdot\sin\left(\frac{i+\theta}{2}\right)\cdot\cos\left(\frac{i-\theta}{2}\right)$$

$$\because (i+\theta) = \delta$$

$$\Delta P = 2(e+d)\sin\left(\frac{\delta}{2}\right)\cdot\cos\left(\frac{i-\theta}{2}\right)$$

$$\Delta P = n\lambda$$

$$2(e+d)\sin\left(\frac{\delta}{2}\right)\cdot\cos\left(\frac{i-\theta}{2}\right) = n\lambda$$

δ would be min when $\cos\left(\frac{i-\theta}{2}\right)$ is max.

When δ_{min}, $\cos\left(\frac{i-\theta}{2}\right) = 0$, $\boxed{i = \theta}$

Now the grating equation

$$\boxed{2(e+d)\sin\left(\frac{\delta_{min}}{2}\right) = n\lambda}$$

Resultant Grating Equation

Dispersive Power of Grating

The dispersive power of a diffraction grating is defined as the rate of change of the angle of diffraction with the change in the wavelength of light used. Therefore, dispersive power of grating = $d\theta/d\lambda$

We know that the for plane diffraction grating, the grating equation for normal incidence as $(e+d)\sin\theta = n\lambda$ ——(1)

Differentiate equation(1) w.r.to λ

$$(e+d)\cos\theta \frac{d\theta}{d\lambda} = n$$

$$\boxed{\frac{d\theta}{d\lambda} = \frac{n}{(e+d)\cos\theta}}$$

Dispersive Power of grating $\propto n$

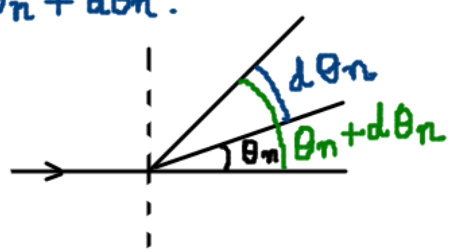
$\propto \frac{1}{e+d}$

$\propto \cos\theta$

Angular half width for grating

$$\because \boxed{(e+d)\sin\theta_n = n\lambda} \text{ ——(1)}$$

Let nth order principal maximum is obtained in the direction (θ_n). and let the first minimum adjacent to the nth maximum is obtained in the direction of $\theta_n + d\theta_n$.



$d\theta_n$ = Angular half with of the nth maximum.

$$\because \boxed{N(e+d)\sin\theta = m\lambda} \text{ ——(2)}$$

Put these → in eq^n (2)

N = Total no. of slits of grating.

m = nN+1

$\theta_n = \theta_n + d\theta_n$

$$N(e+d)\sin\theta_n = n\lambda$$

$$N(e+d)\underbrace{\sin(\theta_n + d\theta_n)} = (nN+1)\lambda$$

$$N(e+d)\left[\sin\theta_n \cdot \cos d\theta_n + \cos\theta_n \cdot \sin d\theta_n\right] = n\lambda N + \lambda$$

$$N\underbrace{(e+d)\sin\theta_n} \cdot \cos d\theta_n + N(e+d)\cos\theta_n \cdot \sin d\theta_n = n\lambda N + \lambda$$

$$\because (e+d)\sin\theta_n = n\lambda \quad [\text{from eq}^n (1)]$$

$$N n\lambda \cos d\theta_n + N(e+d)\cos\theta_n \cdot \sin d\theta_n = n\lambda N + \lambda$$

for small $d\theta_n$, $\cos d\theta_n \cong 1$, $\sin d\theta_n \cong d\theta_n$

$$\cancel{Nn\lambda} + N(e+d)\cos\theta_n \cdot d\theta_n = \cancel{Nn\lambda} + \lambda$$

$$N(e+d)\cos\theta_n \cdot d\theta_n = \lambda$$

$$\boxed{d\theta_n = \frac{\lambda}{N(e+d)\cos\theta_n}} \quad \text{——(3)}$$

$$\left[\begin{aligned} d\theta_n &\propto \lambda \\ &\propto \frac{1}{N} \\ &\propto \frac{1}{e+d} \\ &\propto \frac{1}{\cos\theta_n} \end{aligned}\right.$$

Also,

$$(e+d)\sin\theta_n = n\lambda$$

$$\boxed{(e+d) = \frac{n\lambda}{\sin\theta_n}} \quad \text{——(4)}$$

Put this value in eq^n (3) $\rightarrow$

$$d\theta_n = \frac{\cancel{\lambda}}{N\left(\frac{n\cancel{\lambda}}{\sin\theta_n}\right) \cdot \cos\theta_n}$$

$$\boxed{d\theta_n = \frac{1}{Nn\cot\theta_n}}$$

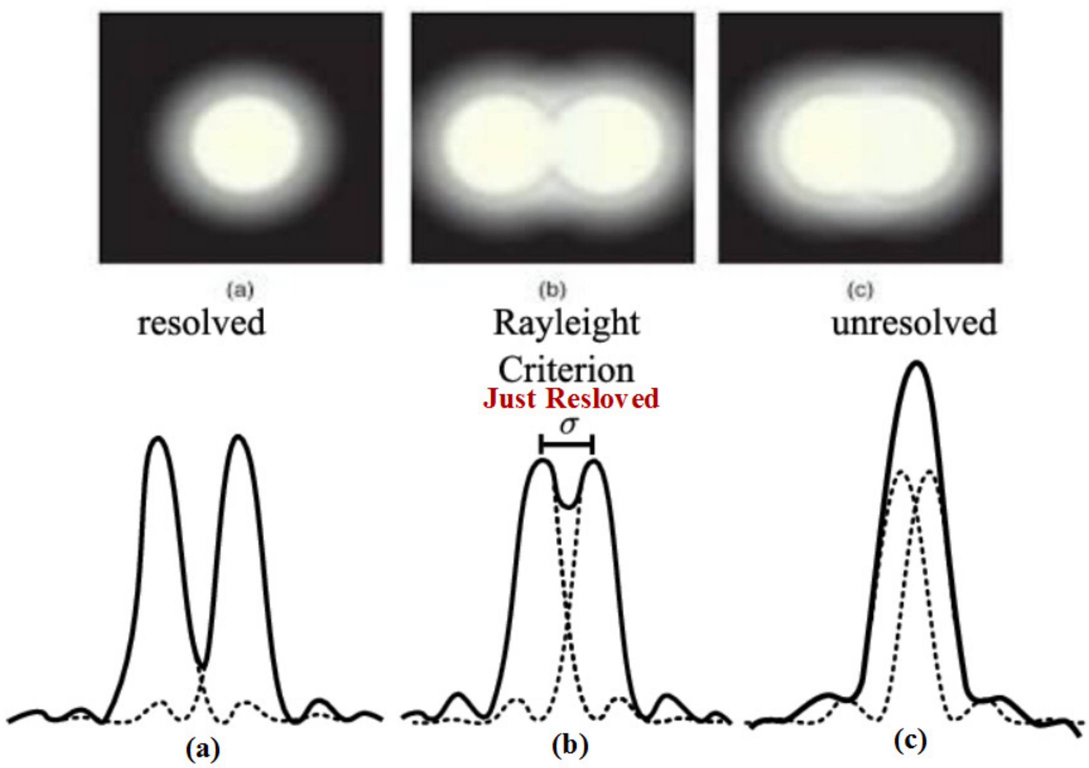
Resolving power of an instrument

When the two objects are very close, they may appear as one object and it may be difficult for the naked eye to see them as separate. Similarly, if there are two point sources very close together, the two diffraction patterns produced by each of them may overlap and hence it may be difficult to distinguish them as separate. To see the the close objects or spectral lines, optical instruments are used.

The capacity of an optical instrument to show two clsoe objects separately is called resolution and the ability of an optical instrument to just reslove the images or two close point objects is called its resolving power.

Rayleigh Criteron of Resolution

According to Rayleigh, the two point sources or two equally intense spectral lines are just resolved by an optical instrument when the central maximum of the diffraction pattern due to one source falls exactly on the first minimum of the diffraction pattern of the other and vice-versa.



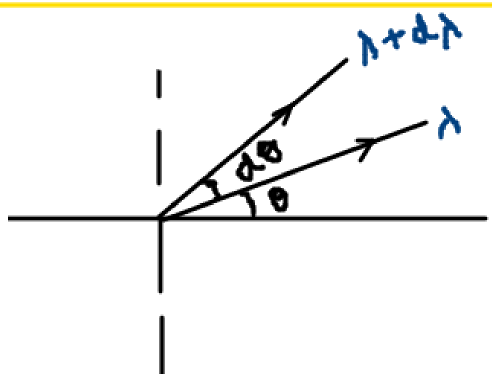
Case-1: (Figure a) Let us consider the intensity distribution curves of two nearby wavelengths (λ) and ($\lambda + d\lambda$). The separation between their central maximum will depend upon the wavelength difference ($d\lambda$). In this case ($d\lambda$) is sufficiently large so that the central maximum due to both wavelength are quite separated and the two spectal lines are distinctly resolved.

Case-2: (Figure b) If, the difference in wavelengths is made smaller and have a limiting value for which the angular separation between their principal maxima is such that the principal central maximum due to one source coincides with the first minimum of the other and vice-versa, the curve shows a distinct dip in the middle, indicating the presence of two spectral lines corresponding to wavelengths (λ) and ($\lambda+d\lambda$). The two spectral lines, under this condition are said to be just resloved.

Case-3: (Figure c) If the wavelength difference ($d\lambda$) is smaller than the limiting value, then the two principal maxima show considerable overlapping. Under this condition, the two spectral lines are not resolved.

Resolving power of a Diffraction Grating

The resolving power of a diffraction grating is defined as its ability to show the two neighbouring spectral lines in a spectrum as separate. It can also be defined as the ratio of the wavelength of any spectral line to the smallest wavelength difference between neighboring lines for which the spectral lines can be just resloved at the wavelength (λ). It can be expressed mathematically as ($\lambda/d\lambda$).



Let a plane diffraction grating be illuminated normally by parallel beam of light of two nearby wavelengths (λ) and ($\lambda+d\lambda$). The light of each wavelength would form a separate diffraction pattern of the slit.

Let n^{th} principal maximum due to wavelength (λ) be formed in the direction (θ); then we have -

$$(e+d)\sin\theta = n\lambda \quad \text{or} \quad N(e+d)\sin\theta = m\lambda$$

here, (e+d) = Grating element

N = total no. of lines on the grating surface.

Let the first minimum adjacent to the n^{th} order maximum be formed in the direction ($\theta + d\theta$). Then from the grating equation for the directions of minima, we have

$$N(e+d)\sin\theta = m\lambda$$

$$\theta = \theta + d\theta$$

$$m = nN+1 \quad \text{, then}$$

$$N(e+d)\sin(\theta+d\theta) = (nN+1)\lambda$$

$$(e+d)\sin(\theta+d\theta) = \frac{(nN+1)\lambda}{N} \quad \text{———} (1)$$

As per Rayleigh criterion, for just resolution of spectral lines n^{th} maximum of ($\lambda + d\lambda$) and first minimum of wavelenth (λ) adjacent to n^{th} maximum should be formed in the direction ($\theta + d\theta$). Hence for n maximum of wavelength ($\lambda + d\lambda$) we have --

$$(e+d)\sin(\theta+d\theta) = n(\lambda+d\lambda) \text{———} (2)$$

$$\because \text{L.H.S of (1)} = (2)$$

$$\therefore \frac{(nN+1)\lambda}{N} = n(\lambda+d\lambda)$$

$$\cancel{nN\lambda} + \lambda = \cancel{nN\lambda} + nN\,d\lambda$$

$$\lambda = nN\,d\lambda$$

$$\boxed{\frac{\lambda}{d\lambda} = nN}$$

This is resolving power of grating.

$$\because (e+d)\sin\theta = n\lambda$$

$$n = \frac{(e+d)\sin\theta}{\lambda}$$

$$\boxed{\frac{\lambda}{d\lambda} = \frac{N(e+d)\sin\theta}{\lambda}}$$