13.29 INVERSE LAPLACE TRANSFORM BY CONVOLUTION

$$\mathrm{L}\left\{\int_0^t f_1(x) * f_2(t-x)\,dx\right\} = F_1 \cdot (s) \cdot F_2(s) \quad \text{or} \quad \int_0^t f_1(x) \cdot f_2(t-x)\,dx = \mathrm{L}^{-1}\, F_1(s) \cdot F_2(s)$$

Example 45. *Using the convolution theorem, find*

$$\mathrm{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}, \quad a \neq b.$$

Solution. We have

$$\mathrm{L}(\cos at) = \frac{s}{s^2+a^2} \quad \text{and} \quad \mathrm{L}(\cos bt) = \frac{s}{s^2+b^2}$$

Hence by the convolution theorem

$$\mathrm{L}\left\{\int_0^t \cos ax \cos b(t-x)\,dx\right\} = \frac{s^2}{(s^2+a^2)(s^2+b^2)}$$

Therefore,

$$\mathrm{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos ax \cos b(t-x)\,dx$$

$$= \frac{1}{2}\int_0^t \left\{\cos(ax+bt-bx) + \cos(ax-bt+bx)\right\} dx$$

$$= \frac{1}{2}\int_0^t \cos[(a-b)x+bt]\,dx + \frac{1}{2}\int_0^t \cos[(a+b)x-bt]\,dx$$

$$= \left[\frac{\sin[(a-b)x+bt]}{2(a-b)}\right]_0^t + \left[\frac{\sin[(a+b)x-bt]}{2(a+b)}\right]_0^t$$

$$= \frac{\sin at - \sin bt}{2(a-b)} + \frac{\sin at + \sin bt}{2(a+b)}$$

$$= \frac{a\sin at - b\sin bt}{a^2-b^2} \qquad \textbf{Ans.}$$

$$- \frac{}{a^2-b^2} \qquad \textbf{Ans.}$$

Example 46. *Obtain* $L^{-1}\dfrac{1}{s(s^2+a^2)}$.

Solution. $\mathrm{L}^{-1}\dfrac{1}{s} = 1$ and $\mathrm{L}^{-1}\dfrac{1}{s^2+a^2} = \dfrac{\sin at}{a}$.

Hence by the convolution theorem

$$\mathrm{L}\int_0^t \left\{1 \cdot \frac{\sin a(t-x)}{a}\,dx\right\} = \left(\frac{1}{s}\right)\left(\frac{1}{s^2+a^2}\right)$$

$$\mathrm{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{\sin a(t-x)}{a}\,dx = \left[\frac{-\cos(at-ax)}{-a^2}\right]_0^t$$

$$= \frac{1}{a^2}[1-\cos at] \qquad \textbf{Ans.}$$

Obtain the inverse Laplace transform by convolution.

1. $\dfrac{s^2}{(s^2+a^2)^2}$ **Ans.** $\dfrac{1}{2}t\cos at+\dfrac{1}{2a}\sin at$

2. $\dfrac{1}{(s^2+1)^3}$ **Ans.** $\dfrac{1}{8}\left\{(3-t^2)\sin t-3t\cos t\right\}$

3. $\dfrac{s}{(s^2+a^2)^2}$ **Ans.** $\dfrac{t\sin at}{2a}$;

4. $\dfrac{1}{s^2(s^2-a^2)}$ **Ans.** $\dfrac{1}{a^3}\left[-at+\sin h\ at\right]$

5. $\dfrac{1}{(s+1)\ (s^2+1)}$; **Ans.** $\dfrac{1}{2}\ (\cos t-\sin t\ -e^{-t})$