Example 52 Use convolution theorem to find

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$$

lution We know that $L^{-1}\left(\frac{1}{s+a}\right)=e^{-at}$ and $L^{-1}\left(\frac{1}{s+b}\right)=e^{-bt}$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = e^{-at} * e^{-bt} = \int_0^t e^{-au} \cdot e^{-b(t-u)}\, du$$

$$= e^{-bt} \int_0^t e^{-(a-b)u}\, du = e^{-bt} \left[\frac{e^{-(a-b)u}}{b-a} \right]_0^t$$

$$= \frac{e^{-at} - e^{-bt}}{b-a}$$

Example 53 Find $L^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\}$.

Solution

$$L^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\} = L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} * L^{-1}\left\{\frac{1}{s-1}\right\}$$

[By convolution theorem]

$$= \frac{1}{\sqrt{\pi t}} * e^t$$

$$= \int_0^t (\pi u)^{-1/2} \cdot e^{t-u}\, du$$

[put $u = x^2 \Rightarrow du = 2x\, dx$]

$$= e^t \cdot \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\, dx$$

$$= e^t \cdot \operatorname{erf}(\sqrt{t})$$

Example 54 Find $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$.

Solution

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} \cdot \frac{1}{s^2+1} = f(s) \cdot g(s)$$

so that $F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t$

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$\therefore$ by convolution theorem,

$$L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = F * G = \int_0^t (t-u)\sin u\, du$$

$$= -t\cos t + t + t\cos t - \sin t$$

$$= (t - \sin t)$$

Example 55 Find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$.

Solution

$$\frac{1}{(s^2+a^2)^2} = \left(\frac{1}{(s^2+a^2)}\right)\cdot\left(\frac{1}{(s^2+a^2)}\right) = f(s)\cdot g(s)$$

$\therefore$
$$F(t) = L^{-1}\{f(s)\} = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{2}\sin at$$

and
$$G(t) = L^{-1}\{g(s)\} = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{2}\sin at$$

By convolution theorem,

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = F*G = \int_0^t \frac{\sin au}{a}\cdot\frac{\sin a(t-u)}{a}\,du$$

$$= \frac{1}{2a^2}\int_0^t [\cos a(2u-t) - \cos at]\,du$$

$$= \frac{1}{2a^2}\left[\frac{\sin a(2u-t)}{2a} - \cos at\cdot u\right]_{u=0}^{t}$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at - \frac{\sin(-at)}{2a}\right]$$

$$= \frac{1}{2a^3}[\sin at - at\cos at]$$