Subgroup

A non empty subset H of a group $(G,*)$ is called a subgroup of G if H itself forms a group under the same binary operation $*$.

Proper and Improper Subgroups

Every group G of order greater than 1 has atleast two subgroups which are G and $\{e\}$. These two subgroups are known as improper or trivial subgroups.

A subgroup other than these two is known as a proper subgroup.

Ex: (i) $(I,+)$ is a subgroup of $(Q,+)$ and $(R,+)$.

(ii) $(2I,+)$ where $2I = \{\ldots -6,-4,-2,0,2,4,6\ldots\}$ is a subgp. of $(I,+)$.

(iii) $\{1,-1\}$ is a subgp. of $\{1,-1,i,-i\}$ under multiplication.

* If H is a subgroup of group G under $*$, then
(i) the identity element of H = identity element of G i.e. identity element is same.
(ii) If $a \in H$ then a^{-1} is same in H and G.
(iii) If G is finite then H is also finite.

* A non empty subset H of $(G,*)$ is a subgroup iff $a * b^{-1} \in H \quad \forall\, a,b \in H$.

* A non empty finite subset H of group G is a subgroup iff:
$a,b \in H \Rightarrow ab \in H$.

* The intersection of two subgroups of G is again a subgp. of G.

or If H_1 and H_2 are two subgroups of a group G, then show that $H_1 \cap H_2$ is also a subgp. of G.

Proof:- Let H_1 and H_2 be any two subgroups of G, then $H_1 \cap H_2 \neq \{\}$ or ϕ since at least the identity element e is common to both H_1 and H_2.

In order to prove that $H_1 \cap H_2$ is a subgp it is sufficient to prove that

$$a \in H_1 \cap H_2, \; b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$$

Now, $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

But H_1 and H_2 are subgroups, therefore

$a \in H_1, \; b \in H_1 \Rightarrow ab^{-1} \in H_1$

$a \in H_2, \; b \in H_2 \Rightarrow ab^{-1} \in H_2$

Finally, $ab^{-1} \in H_1, \; ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Hence $H_1 \cap H_2$ is a subgroup of G.

* Arbitrary intersection of subgroups i.e. the intersection of any family of subgroups of a group is a subgroup.

* The union of two subgroups is not necessarily a subgroup.

Coset

If $(G, *)$ is a group, $(H, *)$ be a subgroup of $(G, *)$ and $a \in G$ then The 'left coset' $a * H$ is the set of elements s.t. $a * H = \{a * h : h \in H\}$

Similarly, right coset $H * a = \{h * a : h \in H\}$

* H is itself a left & right coset since $e * H = H * e = H$