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* Smallest group of order one is $\{e\}$ consisting of the identity element 'e' alone.

* $\{e\}$ is also abelian.

Ex: Let Q^+ be the set of all positive rational numbers and $*$ be a binary operation defined on Q^+ as:

$$a * b = \frac{ab}{3}$$

then

(i) Determine the identity element in Q^+
(ii) Determine the inverse of any element a.
(iii) Does $(Q, *)$ form an abelian group?

Sol^n We define an operation $*$ on Q^+ as follows:

$$a * b = \frac{ab}{3} \quad \forall\ a, b \in Q^+$$

To show that $(Q^+, *)$ forms an abelian group, we need to show that the following conditions hold in $(Q^+, *)$:

(1.) Closure Property: for every $a, b \in Q^+$ $\frac{ab}{3}$ is also in Q^+, therefore Q^+ is closed w.r.t. the operation $*$.

(2.) Associativity: Let $a, b, c \in Q^+$, then

$$a*(b*c) = a*\left(\frac{bc}{3}\right)$$

$$= \frac{abc}{3.3} = \frac{ab}{3}.\frac{c}{3} = \left(\frac{ab}{3}\right)*c$$

$$= (a*b)*c$$

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therefore Q^+ is associative w.r.t. $*$.

(3) Existence of identity element : An element $e \in Q^+$ will be the identity element if

$$a * e = e * a = a \quad \forall \; a \in Q^+$$

Now $a = a * e = \frac{ae}{3}$

$$\Rightarrow a = \frac{ae}{3}$$

$$\Rightarrow 3a - ae = 0$$

$$\Rightarrow a(3-e) = 0$$

$$\Rightarrow 3-e = 0 \quad \text{since } a \neq 0$$

$$\Rightarrow e = 3 \in Q^+$$

So 3 is the identity element in Q^+ under $*$

(4) Existence of inverse : Let $a \in Q^+$, if $b \in Q^+$ is inverse of a then we must have

$$b * a = e = a * b$$

Now $b * a = e$

$$\Rightarrow \frac{ba}{3} = e = 3$$

$$\Rightarrow ba = 9 \Rightarrow b = \frac{9}{a}$$

We know that for every $a \in Q^+$, $\frac{9}{a} \in Q^+$

$\Rightarrow$ every non-zero element a in Q^+ has its inverse in Q^+.

(5.) Commutativity : Q^+ is abelian under $*$ if

$$a * b = b * a \quad \forall \; a, b \in Q^+$$

Now $a * b = \frac{ab}{3} = \frac{ba}{3} = b * a$

So $(Q^+, *)$ is commutative.

Hence $(Q^+, *)$ forms an abelian group.