Unit I

Differential Equations:
An equation containing dependent variable, independent variable and differential coefficient of independent variable with respect to dependent variable is known as differential equation.

Example: (i) $\frac{d^2y}{dx^2}+y=\sec x$ (ii) $\frac{dy}{dx}+y=x$

Order and Degree of a differential equation:
The order of a differential equation is the order of the highest differential coefficient used in the equation.

Examples: (i) $\frac{d^2y}{dx^2}+y=\sec x$ Order is 2

(ii) $\frac{dy}{dx}+y=x$ Order is 1

(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=\frac{d^2y}{dx^2}$ Order is 2

The degree of a differential equation is the degree of the highest differential coefficient removing the radical sign present in the equation.

Examples: (i) $\frac{d^2y}{dx^2}+y=\sec x$ Degree is 2

(ii) $\frac{dy}{dx}+y=x$ Degree is 1

(iii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}=\frac{d^2y}{dx^2}$ Degree is 2

How to form a differential equation with the help of its solution?

Eliminate the arbitrary constants used in the solution through differentiation.

How much freedom you have to do so?

Differentiate the solution number of times as the number of arbitrary constants.

Example: Form the differential equation whose solution is $y=c_1\cos x+c_2\sin x$

Solution: $y=c_1\cos x+c_2\sin x$

Differentiate wrt '*x*'

$$\frac{dy}{dx}=-c_1\sin x+c_2\cos x$$

Again differentiate wrt '*x*'

$$\frac{d^2y}{dx^2}=-c_1\cos x-c_2\sin x$$

$$\frac{d^2y}{dx^2}=-\left(c_1\cos x+c_2\sin x\right)$$

$$\frac{d^2y}{dx^2} = -y$$

$$\boxed{\frac{d^2y}{dx^2} + y = 0}$$

Equations solvable by variable separable method:

Example 1: Solve $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Solution: $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

$$\sec^2 y \tan x\, dy = -\sec^2 x \tan y\, dx$$

$$\frac{\sec^2 y}{\tan y} dy = -\frac{\sec^2 x}{\tan x} dx$$

Integrate both sides.

$$\int \frac{\sec^2 y}{\tan y} dy = -\int \frac{\sec^2 x}{\tan x} dx$$

$$\log(\tan y) = -\log(\tan x) + \log C$$

$$\log(\tan y) + \log(\tan x) = \log C$$

$$\log(\tan x \tan y) = \log C$$

Thus, the solution is

$$\boxed{\tan x \tan y = C}$$

Example 2: Solve $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Solution: $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Substitute $x + y = z$

$$1 + \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$$

$$\frac{dz}{dx} - 1 = \sin z + \cos z$$

$$\frac{dz}{dx} = 1 + \sin z + \cos z$$

Separate the variables.

$$\frac{dz}{1 + \sin z + \cos z} = dx$$

Integrate both sides.

$$\int \frac{dz}{1 + \sin z + \cos z} = \int dx + C$$

$$\int \frac{dz}{1+\frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}}+\frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}} = \int dx + C \qquad \left[\because\ \sin z = \frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}},\quad \cos z = \frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}\right]$$

$$\int \frac{dz}{\frac{1+\tan^2\frac{z}{2}+2\tan\frac{z}{2}+1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}} = \int dx + C$$

$$\int \frac{dz}{\frac{2+2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}}} = \int dx + C$$

$$\frac{1}{2}\int \frac{dz}{\frac{\left(1+\tan\frac{z}{2}\right)}{1+\tan^2\frac{z}{2}}} = \int dx + C$$

$$\frac{1}{2}\int \frac{\sec^2\frac{z}{2}}{1+\tan\frac{z}{2}}\,dz = \int dx + C$$

Substitute $1+\tan\frac{z}{2} = t,\ \frac{1}{2}\sec^2\frac{z}{2}\,dz = dt$

$$\int \frac{1}{t}\,dt = \int dx + C$$

$$\log t = x + C$$

$$\log\left(1+\tan\frac{z}{2}\right) = x + C$$

Thus, the solution is

$$\boxed{\log\left(1+\tan\frac{x+y}{2}\right) = x + C}$$

Example 3: Solve $\frac{d^2y}{dx^2}=\left[1-\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$

Solution: $\frac{d^2y}{dx^2}=\left[1-\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$

Put $\frac{dy}{dx}=u$, $\frac{d^2y}{dx^2}=\frac{du}{dx}$

$$\frac{du}{dx}=\left[1-u^2\right]^{\frac{1}{2}} \Rightarrow \frac{du}{dx}=\sqrt{1-u^2} \Rightarrow \frac{du}{\sqrt{1-u^2}}=dx$$

Integrate

$$\int\frac{du}{\sqrt{1-u^2}}=\int dx$$

$$\sin^{-1}u=x\Rightarrow u=\sin x$$

$$\frac{dy}{dx}=\sin x\Rightarrow dy=\sin x dx$$

Integrate

$$\int dy=\int \sin x dx+C\Rightarrow y=-\cos x+C$$

$$\boxed{y=-\cos x+C}$$

Homogeneous Differential Equations:

A differential equation of the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ is known as a homogeneous differential equation if each term of $f(x,y)$ and $g(x,y)$ is of the same degree.

Example: Solve $\left(y^2-xy\right)dx+x^2dy=0$

Procedure:

1. Express in the form of $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$

2. Substitute $y=vx$, $\frac{dy}{dx}=v+x\frac{dv}{dx}$. In this way the equation is reduced in variable separable form.

Example 1: Solve: $\frac{dy}{dx}=\frac{xy-y^2}{x^2}$

Solution: $\frac{dy}{dx}=\frac{xy-y^2}{x^2}$

$$v+x\frac{dv}{dx}=\frac{x.vx-v^2x^2}{x^2}$$

$$v+x\frac{dv}{dx}=\frac{x^2\left(v-v^2\right)}{x^2}$$

$$v + x\frac{dv}{dx} = v - v^2$$

$$x\frac{dv}{dx} = -v^2$$

Separate the variables.

$$v^{-2}dv = \frac{dx}{x}$$

Integrate both sides.

$$\int v^{-2}dv = \int \frac{dx}{x} + C$$

$$\left[\frac{v^{-2+1}}{-2+1}\right] = \log x + C$$

$$-\frac{1}{v} = \log x + C$$

$$-\frac{1}{y/x} = \log x + C$$

Thus, the solution is

$$\boxed{-\frac{x}{y} = \log x + C}$$

Practice problems:

1. Solve: $\left(x^2 + y^2\right)dy = xy.dx$
2. Solve: $\left(y^2 - xy\right)dx + x^2dy = 0.$
3. Solve: $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$.
4. Solve: $\left(x^3 - 3xy^2\right)dx + \left(y^3 - 3x^2y\right)dy = 0;\ y(0) = 1.$

Linear Differential Equations (LDE):

A differential equation of the form $\frac{dy}{dx} + Py = Q$ is known as a linear differential equation, where P and Q, are functions of x or constants.

The solution of linear differential equation is given by

$y.IF = \int Q.IF\, dx + C$ where $IF = e^{\int Pdx}$

Example 1: Solve: $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$

Solution: $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$

This is Be1rnoulli's form of LDE.

Divide the equation by $z(\log z)^2$.

$$\frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{1}{x\log z}=\frac{1}{x^2}$$

Substitute $\frac{1}{\log z}=u$

$$-\frac{1}{z(\log z)^2}\frac{dz}{dx}=\frac{du}{dx}$$

$$-\frac{du}{dx}+\frac{1}{x}u=\frac{1}{x^2}$$

$$\frac{du}{dx}+\left(-\frac{1}{x}\right)u=\frac{1}{x^2}$$ [Linear differential equation]

$$IF=e^{\int Pdx}$$

$$IF=e^{\int\left(-\frac{1}{x}\right)dx}$$

$$IF=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$$

The solution is

$$u.IF=\int Q.IF\,dx+C$$

$$u.\frac{1}{x}=\int\frac{1}{x^2}.\frac{1}{x}dx+C$$

$$u.\frac{1}{x}=\int x^{-3}dx+C$$

$$u.\frac{1}{x}=\left[\frac{x^{-3+1}}{-3+1}\right]+C$$

$$\frac{1}{x\log z}=-\frac{1}{2x^2}+C$$

Practice problems:

1. Solve: $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$
2. Solve: $x(x-1)\frac{dy}{dx}-(x-2)y=x^2(2x-1)$
3. Solve: $\frac{dy}{dx}+y\cot x=\cos x$.
4. Solve: $(1-x^2)\frac{dy}{dx}+2xy=x(1-x^2)^{1/2}$
5. Solve: $(1+y^2)dx=(\tan^{-1}y-x)dy$
6. Solve: $y\log y\frac{dx}{dy}+x-\log y=0$

Exact Differential Equation

If the solution of a differential equation is differentiated an exact differential equation is formed.
Standard form of an exact differential equation:

$$Mdx + Ndy = 0$$

This equation is said to be an exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution of an exact differential equation

$$\int M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy = C$$

Example 1: Find the value of λ for which the differential equation:
$\left(xy^2 + \lambda x^3 y^2\right)dx + \left(x^3 y + yx\right)xdy = 0$ is exact.

Solution: $\left(xy^2 + \lambda x^3 y^2\right)dx + \left(x^3 y + yx\right)xdy = 0$

Here, $M = xy^2 + \lambda x^3 y^2$, $N = x^4 y + yx^2$

$$\frac{\partial M}{\partial y} = 2xy + 2\lambda x^3 y \quad \frac{\partial N}{\partial x} = 4x^3 y + 2xy$$

The equation is exact. (given)
Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$2xy + 2\lambda x^3 y = 4x^3 y + 2xy$$

This implies

$$2\lambda = 4$$

$$\boxed{\lambda = 2}$$

Example: 2 Solve $\left(1 + 3e^{x/y}\right)dx + 3e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Solution: $\left(1 + 3e^{x/y}\right)dx + 3e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

Here, $M = 1 + 3e^{x/y}$, $N = 3e^{x/y}\left(1 - \frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = 3e^{x/y}\left(-\frac{x}{y^2}\right), \quad \frac{\partial N}{\partial x} = 3\left\{e^{x/y}\left(-\frac{1}{y}\right) + \frac{1}{y}e^{x/y}\left(1 - \frac{x}{y}\right)\right\}$$

$$\frac{\partial M}{\partial y} = -\frac{3x}{y^2}e^{x/y}, \quad \frac{\partial N}{\partial x} = 3\left\{-\frac{1}{y}e^{x/y} + \frac{1}{y}e^{x/y} - \frac{x}{y^2}e^{x/y}\right\}$$

$$\frac{\partial M}{\partial y} = -\frac{3x}{y^2}e^{x/y}, \quad \frac{\partial N}{\partial x} = \frac{3x}{y^2}e^{x/y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This shows the equation is exact.

The solution is given by

$$\int M\,dx+\int(\text{Terms of } N \text{ not containing } x)\,dy=C$$

$$\int\left(1+3e^{x/y}\right)dx+\int(0)\,dx=C$$

$$x+\frac{3}{1/y}e^{x/y}=C$$

$$\boxed{x+3ye^{x/y}=C}$$

Example: 3 Solve $xdx+ydy=\dfrac{a^2(xdy-ydx)}{x^2+y^2}$.

Solution: $xdx+ydy=\dfrac{a^2(xdy-ydx)}{x^2+y^2}$

$$\Rightarrow\left(x+\frac{a^2y}{x^2+y^2}\right)dx+\left(y-\frac{a^2x}{x^2+y^2}\right)dy=0$$

Here, $M=x+\dfrac{a^2y}{x^2+y^2}$, $N=y-\dfrac{a^2x}{x^2+y^2}$

$$\frac{\partial M}{\partial y}=a^2\left\{\frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}\right\},\ \frac{\partial N}{\partial x}=-a^2\left\{\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}\right\}$$

$$\frac{\partial M}{\partial y}=a^2\left\{\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right\},\ \frac{\partial N}{\partial x}=-a^2\left\{\frac{(y^2-x^2)}{(x^2+y^2)^2}\right\}$$

$$\frac{\partial M}{\partial y}=a^2\left\{\frac{(x^2-y^2)}{(x^2+y^2)^2}\right\},\ \frac{\partial N}{\partial x}=a^2\left\{\frac{(x^2-y^2)}{(x^2+y^2)^2}\right\}$$

$$\frac{\partial M}{\partial y}=a^2\frac{(x^2-y^2)}{(x^2+y^2)^2},\quad \frac{\partial N}{\partial x}=a^2\frac{(x^2-y^2)}{(x^2+y^2)^2}$$

This show

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

This implies the equation is exact.

The solution is given by

$$\int M\,dx+\int(\text{Terms of } N \text{ not containing } x)\,dy=C$$

$$\int\left(x+\frac{a^2y}{x^2+y^2}\right)dx+\int ydy=C$$

$$\frac{x^2}{2}+\frac{a^2y}{y}\tan^{-1}\frac{x}{y}+\frac{y^2}{2}=C$$

$$\boxed{\frac{x^2+y^2}{2}+a^2\tan^{-1}\frac{x}{y}=C}$$

Problem: 1 Solve:

$\left(y^2 e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$ [Example 30. HK Dass Vol. II, Page No. 20]

Problem: 2 Solve $(1+xy)\,ydx+(1-xy)\,xdy=0$ S [Ans: $x=cye^{1/(xy)}$]

[Rule III HK Dass Vol. II page No. 24]

Equations Reducible to Exact Equations

If $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ then the equation is not exact. To reduce in exact form multiply the equation by integrating factor.

Case I: $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$ then $IF=e^{\int f(x)dx}$

Example: Solve $\left(x\sec^2 y-x^2\cos y\right)dy=\left(\tan y-3x^4\right)dx.$

Solution: $\left(x\sec^2 y-x^2\cos y\right)dy=\left(\tan y-3x^4\right)dx$

$\left(\tan y-3x^4\right)dx-\left(x\sec^2 y-x^2\cos y\right)dy=0$

Here, $M=\left(\tan y-3x^4\right),\ N=-\left(x\sec^2 y-x^2\cos y\right)$

$\frac{\partial M}{\partial y}=\sec^2 y,\ \frac{\partial N}{\partial x}=-\sec^2 y+2x\cos y$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{\sec^2 y+\sec^2 y-2x\cos y}{-\left(x\sec^2 y-x^2\cos y\right)}$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2\left(\sec^2 y-x\cos y\right)}{-x\left(\sec^2 y-x\cos y\right)}=-\frac{2}{x}=f(x)$$

$IF=e^{\int f(x)dx}=e^{\int\left(-\frac{2}{x}\right)dx}$

$IF=e^{-2\int\frac{1}{x}dx}=e^{-2\log x}=e^{\log x^{-2}}=x^{-2}$

Multiply the equation by x^{-2}.

$\left(x^{-2}\tan y-3x^2\right)dx-\left(x^{-1}\sec^2 y-\cos y\right)dy=0$

The solution is given by

$\int M\,dx+\int(\text{Terms of } N \text{ not containing } x)\,dy=C$

$\int\left(x^{-2}\tan y-3x^2\right)dx+\int\cos y\,dy=C$

$\left(\frac{x^{-2+1}}{-2+1}\right)\tan y-3\frac{x^{2+1}}{2+1}+\sin y=C$

$$\boxed{-\frac{1}{x}\tan y - x^3 + \sin y = C}$$

Case II: $\dfrac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M} = f(y)$ then $IF = e^{\int f(y)dy}$

Example: Solve $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0.$

Solution: $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$

Here, $M = y^4+2y,\ N = xy^3+2y^4-4x$

$$\frac{\partial M}{\partial y} = 4y^3+2,\ \frac{\partial N}{\partial x} = y^3-4$$

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y^4+2y}$$

$$\frac{\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}}{M} = \frac{-3y^3-6}{y^4+2y} = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y} = f(y)$$

$$IF = e^{\int f(y)dy} = e^{\int\left(-\frac{3}{y}\right)dy}$$

$$IF = e^{-3\log y} = e^{\log y^{-3}} = y^{-3}$$

Multiply the equation by y^{-3}.

$$y^{-3}(y^4+2y)dx + y^{-3}(xy^3+2y^4-4x)dy = 0.$$

$$(y+2y^{-2})dx + (x+2y-4xy^{-3})dy = 0.$$

The solution is given by

$$\int M\,dx + \int(\text{Terms of } N \text{ not containing } x)\,dy = C$$

$$\int(y+2y^{-2})dx + \int 2y\,dy = C \qquad \boxed{xy + \frac{2x}{y^2} + y^2 = C}$$

Case II: If the form of differential equation is $x^m y^n(aydx+bxdy) + x^{m'}y^{n'}(a'ydx+b'xdy) = 0$

Then $IF = x^h y^k$ where $\dfrac{m+h+1}{a} = \dfrac{n+k+1}{b}$ and $\dfrac{m'+h+1}{a'} = \dfrac{n'+k+1}{b'}$

Example: Solve $(3y-2xy^3)dx + (4x-3x^2y^2)dy = 0$

Solution: $(3y-2xy^3)dx + (4x-3x^2y^2)dy = 0$

$$\Rightarrow (3ydx+4xdy) + xy^2(-2ydx-3xdy) = 0$$

Compare with $x^m y^n(aydx+bxdy) + x^{m'}y^{n'}(a'ydx+b'xdy) = 0$

$m=0, n=0, a=3, b=4$

$m'=1, n'=2, a'=-2, b'=-3$

$IF = x^h y^k$ Where,

$$\frac{m+h+1}{a}=\frac{n+k+1}{b} \text{ and } \frac{m'+h+1}{a'}=\frac{n'+k+1}{b'}$$

$$\frac{h+1}{3}=\frac{k+1}{4} \text{ and } \frac{1+h+1}{-2}=\frac{2+k+1}{-3}$$

$$\boxed{4h-3k=-1 \text{ and } 3h-2k=0 \Rightarrow h=2 \text{ and } k=3}$$

$IF = x^2 y^3$

Multiply by $x^2 y^3$ to the given equation.

$$x^2y^3\left(3y-2xy^3\right)dx + x^2y^3\left(4x-3x^2y^2\right)dy=0$$

$$\left(3x^2y^4-2x^3y^6\right)dx+\left(4x^3y^4-3x^4y^5\right)dy=0 \text{ (Exact diff. equation)}$$

It' solution is: $\int\left(3x^2y^4-2x^3y^6\right)dx=C$

$$\boxed{x^3y^4-\frac{1}{2}x^4y^6=C}$$

Linear Differential Equation with constant coefficient:

The general form of linear differential equation is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

The complete solution of linear differential equation is

$$\boxed{y=CF+PI}$$

If $R=0,$ then the solution is $y=CF$ only.

Rule for finding CF:

For the equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$

(i) Convert the equation in differential operator form
$D^2y+PDy+Qy=0 \Rightarrow \left(D^2+PD+Q\right)y=0$

(ii) Replace D by m for auxiliary equation (AE).
$m^2+Pm+Q=0$

(iii) Solve the AE for m.
Suppose $m=m_1, m_2$ (Distinct and Real)
Then $CF=c_1e^{m_1x}+c_2e^{m_2x}$

(iv) Suppose $m=m_1=m_2$ (Repeated and Real)
Then $CF=\left(c_1+xc_2\right)e^{m_1x}$

(v) Suppose the roots are $m=\alpha\pm i\beta$ (Complex)
Then $CF=e^{\alpha x}\left(c_1\cos\beta x+c_2\sin\beta x\right)$

(vi) Suppose the roots are complex an repeated $m = \alpha \pm i\beta, \alpha \pm i\beta$

Then $CF = e^{\alpha x}(c_1 + xc_2)(A\cos\beta x + B\sin\beta x)$

Rule for finding Particular Integration (*PI*):

For the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

(i) If $RHS = e^{ax}$

$$PI = \frac{1}{f(D)}e^{ax}$$

If $f(a) = 0$ then $PI = \frac{x}{f^{'}(a)}e^{ax}$ and so on

(ii) If $RHS = x^n$ (Algebraic function)

$$PI = \frac{1}{f(D)}x^n$$

$PI = [f(D)]^{-1}x^n$, Expand $[f(D)]^{-1}$ binomially.

(iii) If $RHS = \sin ax$ or $\cos ax$

$$PI = \frac{1}{f(D^2)}\sin ax$$

$$PI = \frac{1}{f(-a^2)}\sin ax$$

If $f(-a^2) = 0$ then $PI = \frac{x}{f^{'}(-a^2)}\sin ax$

(iv) If $RHS = e^{ax}.\phi(x)$

$$PI = \frac{1}{f(D)}e^{ax}.\phi(x)$$

$$PI = e^{ax}.\frac{1}{f(D+a)}\phi(x)$$

(v) If $RHS = x^n \sin ax$

$$PI = \frac{1}{f(D)}x^n \sin ax$$

$PI =$ Imaginary part of $e^{iax}\frac{1}{f(D+ia)}x^n$

If $RHS = x^n \cos ax$

$$PI = \frac{1}{f(D)}x^n \cos ax$$

$PI =$ Real part of $e^{iax}\frac{1}{f(D+ia)}x^n$

Example 1. Solve $\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=e^{ax}+e^{bx}$.

Solution: $\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=e^{ax}+e^{bx}$

$\left[D^2-(a+b)D+ab\right]y=e^{ax}+e^{bx}$

Auxiliary equation

$m^2-(a+b)m+ab=0$

$m^2-am-bm+ab=0$

$m(m-a)-b(m-a)=0$

$(m-a)(m-b)=0$

$m=a,b$

$CF=c_1e^{ax}+c_2e^{bx}$

$$PI=\frac{1}{f(D)}\left(e^{ax}+e^{bx}\right)$$

$$PI=\frac{1}{\left(D^2-(a+b)D+ab\right)}\left(e^{ax}+e^{bx}\right)$$

$$PI=\frac{1}{\left(D^2-(a+b)D+ab\right)}e^{ax}+\frac{1}{\left(D^2-(a+b)D+ab\right)}e^{bx}$$

Here $f(a)=0$ and $f(b)=0$. The Rule is failed.

$$PI=\frac{x}{2D-(a+b)}e^{ax}+\frac{x}{2D-(a+b)}e^{bx}$$

$$PI=\frac{x}{2a-(a+b)}e^{ax}+\frac{x}{2b-(a+b)}e^{bx}$$

$$PI=\frac{x}{a-b}e^{ax}+\frac{x}{b-a}e^{bx}$$

$$PI=\frac{x}{a-b}\left(e^{ax}-e^{bx}\right)$$

The complete solution is

$y(x)=CF+PI$

$$\boxed{y(x)=c_1e^{ax}+c_2e^{bx}+\frac{x}{a-b}\left(e^{ax}-e^{bx}\right)}$$

Example 2. Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x$

Solution: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x$

$(D^2+2D+1)y = x$

Auxiliary equation

$m^2+2m+1=0$

$(m+1)^2=0$

$\Rightarrow m=-1,-1$

$CF = (c_1 + xc_2)e^{-x}$

$PI = \frac{1}{f(D)}x$

$PI = \frac{1}{D^2+2D+1}x$

$PI = (1+D^2+2D)^{-1}x$

$PI = \left[1-(D^2+2D)+...\right]x$

$PI = \left[x-(D^2x+2Dx)+...\right]$

$PI = \left[x-(0+2)+0\right]$

$PI = x-2$

The complete solution is

$y(x) = CF + PI$

$$\boxed{y(x) = (c_1 + xc_2)e^{-x} + x - 2}$$

Example 3. Solve $(D^2-4D+3)y = x^3$

Solution: $(D^2-4D+3)y = x^3$

Auxiliary equation

$m^2-4m+3=0$

$m^2-3m-m+3=0$

$m(m-3)-1(m-3)=0$

$(m-3)(m-1)=0$

$m=1,3$

$CF = c_1e^x + c_2e^{3x}$

$PI = \frac{1}{f(D)}x^3$

$PI = \frac{1}{D^2-4D+3}x^3$

$PI = \frac{1}{3\left(1+\frac{D^2-4D}{3}\right)}x^3$

$$PI=\frac{1}{3}\left(1+\frac{D^2-4D}{3}\right)^{-1}x^3$$

$$PI=\frac{1}{3}\left(1-\frac{D^2-4D}{3}+\left(\frac{D^2-4D}{3}\right)^2-\left(\frac{D^2-4D}{3}\right)^3+...\right)x^3$$

$$PI=\frac{1}{3}\left(1-\frac{1}{3}\left(D^2-4D\right)+\frac{1}{9}\left(D^4-8D^3+16D^2\right)-\frac{1}{27}\left(D^3-12D^5+48D^4-64D^3\right)+...\right)x^3$$

$$PI=\frac{1}{3}\left(x^3-\frac{1}{3}\left(D^2x^3-4Dx^3\right)+\frac{1}{9}\left(D^4x^3-8D^3x^3+16D^2x^3\right)-\frac{1}{27}\left(D^3x^3-12D^5x^3+48D^4x^3-64D^3x^3\right)+...\right)$$

$$PI=\frac{1}{3}\left(x^3-\frac{1}{3}\left(6x-12x^2\right)+\frac{1}{9}\left(0-48+96x\right)-\frac{1}{27}\left(6-0+0-384\right)+0\right)$$

$$PI=\frac{1}{3}\left(x^3-2x+4x^2+\frac{1}{9}\left(0-48+96x\right)-\frac{1}{27}\left(6-0+0-384\right)+0\right)$$

$$PI=\frac{1}{27}\left(9x^3+36x^2+78x+80\right)$$

The complete solution is

$y(x)=CF+PI$

$$\boxed{y(x)=c_1e^x+c_2e^{3x}+\frac{1}{27}\left(9x^3+36x^2+78x+80\right)}$$

Example 4. Solve $\left(2D^2+5D+2\right)y=\left(5+2x\right)$

Solution: $\left(2D^2+5D+2\right)y=\left(5+2x\right)$

Auxiliary equation

$2m^2+5m+2=0$

$2m^2+4m+m+2=0$

$2m(m+2)+1(m+2)=0$

$(m+2)(2m+1)=0$

$m=-\frac{1}{2},-2$

$CF=c_1e^{-\frac{1}{2}x}+c_2e^{-2x}$

$$PI=\frac{1}{f(D)}\left(5+2x\right)$$

$$PI=\frac{1}{2D^2+5D+2}\left(5+2x\right)$$

$$PI=\frac{1}{2\left(1+\frac{2D^2+5D}{2}\right)}\left(5+2x\right)$$

$$PI=\frac{1}{2}\left(1+\frac{2D^2+5D}{2}\right)^{-1}\left(5+2x\right)$$

Expand binomially.

$$PI=\frac{1}{2}\left(1-\frac{2D^2+5D}{2}+\left(\frac{2D^2+5D}{2}\right)^2-\ldots\right)(5+2x)$$

$$PI=\frac{1}{2}\left(5+2x-\frac{2D^2+5D}{2}(5+2x)+\left(\frac{2D^2+5D}{2}\right)^2(5+2x)-\ldots\right)$$

$$PI=\frac{1}{2}\left(5+2x-\frac{2D^2+5D}{2}(5+2x)+\left(\frac{4D^4(5+2x)+10D^3(5+2x)+25D^2(5+2x)}{2}\right)-\ldots\right)$$

$$PI=\frac{1}{2}\left(5+2x-\frac{5\times 2}{2}+0-\ldots\right)$$

$$PI=x$$

Complete solution is

$$y(x)=CF+PI$$

$$\boxed{y(x)=c_1e^{-\frac{1}{2}x}+c_2e^{-2x}+x}$$

Example 5. Solve $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x$

Solution: $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x$

$$\left(D^3-3D^2+4D-2\right)y=e^x+\cos x$$

Auxiliary equation

$$m^3-3m^2+4m-2=0$$

$$\Rightarrow (m-1)\left(m^2-2m+2\right)=0$$

$$m=1,1\pm i$$

$$CF=c_1e^x+e^x\left(c_2\cos x+c_3\sin x\right)$$

$$PI=\frac{1}{f(D)}\left(e^x+\cos x\right)$$

$$PI=\frac{1}{\left(D^3-3D^2+4D-2\right)}e^x+\frac{1}{\left(D^3-3D^2+4D-2\right)}\cos x$$

$$PI=\frac{x}{\left(3D^2-6D+4\right)}e^x+\frac{1}{\left(D.D^2-3D^2+4D-2\right)}\cos x$$

$$PI=\frac{x}{(3-6+4)}e^x+\frac{1}{\left(D.\left(-1^2\right)-3\left(-1^2\right)+4D-2\right)}\cos x$$

$$PI=xe^x+\frac{1}{(-D+3+4D-2)}\cos x$$

$$PI = xe^x + \frac{1}{(3D+1)}\cos x$$

$$PI = xe^x + \frac{(3D-1)}{(3D+1)(3D-1)}\cos x$$

$$PI = xe^x + \frac{(3D-1)}{(9D^2-1)}\cos x$$

$$PI = xe^x + \frac{(3D-1)}{\left(9\left(-1^2\right)-1\right)}\cos x$$

$$PI = xe^x - \frac{(3D\cos x - \cos x)}{10}$$

$$PI = xe^x - \frac{(-3\sin x - \cos x)}{10}$$

$$PI = xe^x + \frac{1}{10}(3\sin x + \cos x)$$

Complete solution is

$$y(x) = CF + PI$$

$$\boxed{y(x) = c_1e^x + e^x(c_2\cos x + c_3\sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x)}$$

Example 6. Solve $\left(D^4-1\right)y = e^x\cos x$.

Solution: $\left(D^4-1\right)y = e^x\cos x$

Auxiliary equation

$$m^4 - 1 = 0$$

$$\left(m^2-1\right)\left(m^2+1\right) = 0$$

$$(m-1)(m+1)\left(m^2+1\right) = 0$$

$$m = -1, 1, -i, i$$

$$CF = c_1e^{-x} + c_2e^x + (c_3\cos x + c_4\sin x)$$

$$PI = \frac{1}{\left(D^4-1\right)}e^x\cos x$$

$$PI = e^x.\frac{1}{(D+1)^4-1}\cos x$$

$$PI = e^x.\frac{1}{D^4+6D^3+4D^2+6D}\cos x$$

$$PI = e^x.\frac{1}{\left(-1^2\right)\left(-1^2\right)+6\left(-1^2\right)D+4\left(-1^2\right)+6D}\cos x$$

$$PI = e^x.\frac{1}{1-6D-4+6D}\cos x$$

$PI = e^x . \frac{1}{-3} \cos x$

$PI = -\frac{1}{3} e^x \cos x$

Complete solution is

$y(x) = CF + PI$

$$\boxed{y(x) = c_1 e^{-x} + c_2 e^x + (c_3 \cos x + c_4 \sin x) - \frac{1}{3} e^x \cos x}$$

Example 7. Solve $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

Solution: $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

Auxiliary equation: $m^2 - 4m + 4 = 0 \ \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2,\ 2$

$CF = (C_1 + xC_2) e^{2x}$

$PI = \frac{1}{D^2 - 4D + 4} 8x^2 e^{2x} \sin 2x$

$= \frac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x$

$= \frac{8e^{2x}}{(D+2-2)^2} x^2 \sin 2x$

$= 8e^{2x} \frac{1}{D^2} x^2 \sin 2x$

$PI = -e^{2x} \left[4x \cos 2x + (2x^2 - 3) \sin 2x \right]$

Complete solution is: $y = CF + PI$

$$\boxed{y = (C_1 + xC_2) e^{2x} - e^{2x} \left[4x \cos 2x + (2x^2 - 3) \sin 2x \right]}$$

Example 8. Solve $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = x \sin x$

Solution: $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = x \sin x$

$(D^2 - 2D + 1) y = x \sin x$

Auxiliary equation

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1,1$

$CF = (c_1 + c_2 x) e^x$

$PI = \frac{1}{(D^2 - 2D + 1)} x \sin x$

$PI = \text{Imaginary part of } \frac{1}{\left(D^2-2D+1\right)} x\left(\cos x+i\sin x\right)$

$PI = \text{Imaginary part of } \frac{1}{\left(D^2-2D+1\right)} x.e^{ix}$ $\quad\left[\because e^{ix}=\cos x+i\sin x\right]$

$PI = \text{Imaginary part of } e^{ix}\frac{1}{\left(\left(D+i\right)^2-2\left(D+i\right)+1\right)} x$

$PI = \text{Imaginary part of } e^{ix}\frac{1}{\left(D^2-2\left(D+i\right)-2i\right)} x$

$PI = \text{Imaginary part of } e^{ix}\frac{1}{-2i\left[1-\left(1+i\right)D-\frac{1}{2i}D^2\right]} x$

$PI = \text{Imaginary part of } e^{ix}\frac{1}{-2i}\left[1-\left\{\left(1+i\right)D+\frac{1}{2i}D^2\right\}\right]^{-1} x$

Expand binomially.

$PI = \text{Imaginary part of } e^{ix}\frac{1}{-2i}\left[1+\left(1+i\right)D+\frac{1}{2i}D^2+\ldots\right] x$

$PI = \text{Imaginary part of } e^{ix}\frac{1}{-2i}\left[x+\left(1+i\right)Dx+\frac{1}{2i}D^2x+\ldots\right]$ $\quad\left[\because \left(1-x\right)^{-1}=1+x+x^2+\ldots\right]$

$PI = \text{Imaginary part of } e^{ix}\frac{1}{-2i}\left[x+1+i\right]$

Multiply by i to the numerator and denominator.

$PI = \text{Imaginary part of } e^{ix}\frac{1}{2}\left[xi+i-1\right]$

$PI = \text{Imaginary part of } \frac{1}{2}\left(\cos x+i\sin x\right)\left[\left(x+1\right)i-1\right]$

$PI = \text{Imaginary part of } \frac{1}{2}\left[\left\{-\cos x-\left(x+1\right)\sin x\right\}+i\left\{\left(x+1\right)\cos x-\sin x\right\}\right]$

$PI = \frac{1}{2}\left\{\left(x+1\right)\cos x-\sin x\right\}$

Complete solution is

$y\left(x\right)=CF+PI$

$$\boxed{y\left(x\right)=\left(c_1+c_2x\right)e^x+\frac{1}{2}\left\{\left(x+1\right)\cos x-\sin x\right\}}$$

Example 9. Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x\cos x$ $\quad$ [Example 36 HK. Dass Vol. II page No. 64]

Cauchy Euler Homogeneous Linear Differential Equation:

The general Cauchy Euler Homogeneous Linear Differential Equation is

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_0 y = \phi(x)$$

Where $a_0, a_1, a_2, \ldots$ are constants

Put $x = e^z$, $z = \log x$, $\frac{d}{dz} = D$

$$x\frac{dy}{dx} = Dy, \quad x^2 \frac{d^2 y}{dx^2} = D(D-1)y, \quad x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

Example: Solve $x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + y = \log x.$

Solution: $x^2 \frac{d^2 y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Put $x = e^z$, $z = \log x$, $\frac{d}{dz} = D$

$x\frac{dy}{dx} = Dy$ and $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$

$$D(D-1)y - Dy + y = z$$

$$(D^2 - 2D + 1)y = z$$

Auxiliary equation: $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1,1$

$$CF = (c_1 + zc_2)e^z$$

$$PI = \frac{1}{(D^2 - 2D + 1)} z$$

$$PI = \frac{1}{(1-D)^2} z$$

$$PI = (1-D)^{-2} z$$

Expand binomially.

$$PI = (1 + 2D + \ldots) z$$

$$PI = (z + 2Dz + \ldots)$$

$$PI = z + 2$$

Complete solution is: $y(z) = CF + PI$

$$y(z) = (c_1 + zc_2)e^z + z + 2$$

$$y(x) = (c_1 + c_2 \log x)x + \log x + 2$$

$$\boxed{y(x) = (c_1 + c_2 \log x)x + \log x + 2}$$

Example: Solve $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=(\log x)\sin(\log x)$.

Solution: $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=(\log x).\sin(\log x)$

Put $x=e^z \Rightarrow \log x=z$

$x\dfrac{dy}{dx}=Dy,\quad x^2\dfrac{d^2y}{dx^2}=D(D-1)y,\ D\equiv\dfrac{d}{dx}$

The transformed equation is:

$D(D-1)y+Dy+y=z\sin z \quad\Rightarrow\ (D^2+1)y=z\sin z$

Auxiliary equation: $m^2+1=0\ \Rightarrow\ m=\pm i$

$CF=C_1\cos z+C_2\sin z$

$$PI=\frac{1}{D^2+1}z\sin z$$

$$=\text{Imaginary part of }\frac{1}{D^2+1}z(\cos z+i\sin z)=\text{Imaginary part of }\frac{1}{D^2+1}ze^{iz}$$

$$=\text{Imaginary part of }e^{iz}\frac{1}{(D+i)^2+1}z$$

$$=\text{Imaginary part of }e^{iz}\frac{1}{2i}\left(\frac{z^2}{2}-\frac{z}{2i}\right)$$

$$\boxed{PI=-\frac{z^2}{4}\cos z+\frac{z}{4}\sin z}$$

Complete solution is: $y=CF+PI$

$$y=C_1\cos z+C_2\sin z-\frac{z^2}{4}\cos z+\frac{z}{4}\sin z$$

$$\boxed{y=C_1\cos(\log x)+C_2\sin(\log x)-\frac{(\log x)^2}{4}\cos(\log x)+\frac{\log x}{4}\sin(\log x)}$$

Question : Solve $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=x^3\log x.$

Solution. We have, $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=x^3\log x$

Let $x=e^z$ so that $z=\log x$, $D\equiv\frac{d}{dz}$

The equation becomes after substitution

$$[D(D-1)(D-2)+3D(D-1)+D]\,y=z\,e^{3z} \quad\Rightarrow\quad D^3y=ze^{3z}$$

Auxiliary equation is $m^3=0 \Rightarrow \quad m=0,0,0.$

C.F. $=C_1+C_2z+C_3z^2=C_1+C_2\log x+C_3(\log x)^2$

$$\text{P.I.}=\frac{1}{D^3}\cdot ze^{3z}=e^{3z}\cdot\frac{1}{(D+3)^3}\cdot z$$

$$=e^{3z}\,\frac{1}{27}\left(1+\frac{D}{3}\right)^{-3}z=\frac{e^{3z}}{27}(1-D)\,z=\frac{e^{3z}}{27}(z-1)=\frac{x^3}{27}(\log x-1)$$

Complete solution is $\quad y=C_1+C_2\log x+C_3(\log x)^2+\frac{x^3}{27}(\log x-1)$

Variation of Parameters Method:

Procedure:

(i) Find $C.F$

(ii) Compare the $C.F$ to $c_1y_1+c_2y_2$ and find y_1,y_2.

(iii)Find $P.I.=uy_1+vy_2$.

(iv)Find u and v by the formula

$$u=\int\frac{-y_2X}{y_1y_2^{'}-y_1^{'}y_2}dx,\quad v=\int\frac{y_1X}{y_1y_2^{'}-y_1^{'}y_2}dx$$

(v) Find the general solution, $y(x)=C.F.+P.I.$

Example: Solve $\frac{d^2y}{dx^2}+y=\sec x$ by using variation of parameters method.

Solution: $\frac{d^2y}{dx^2}+y=\sec x$

$$(D^2+1)y=\sec x$$

Auxiliary equation: $m^2+1=0 \quad\Rightarrow\quad m=\pm i$

$$CF=C_1\cos x+C_2\sin x$$

Here $\quad y_1=\cos x,\quad y_2=\sin x$

$$PI=uy_1+vy_2$$

Where, $\quad u=\int\frac{-y_2X}{y_1y_2^{'}-y_1^{'}y_2}dx,\quad v=\int\frac{y_1X}{y_1y_2^{'}-y_1^{'}y_2}dx$

$$u=\int\frac{-\sin x.\sec x}{\cos x.\cos x+\sin x.\sin x}dx\Rightarrow\ u=\int\frac{-\tan x}{1}dx=\log\cos x$$

$$v=\int\frac{y_1\sec x}{y_1y_2^{'}-y_1^{'}y_2}dx$$

$$v=\int\frac{\cos x.\sec x}{\cos x.\cos x+\sin x.\sin x}dx\Rightarrow\ v=\int dx=x$$

$PI = \cos x.\log\cos x + x\sin x$

Complete solution is: $y = CF + PI$

$$\boxed{y = C_1\cos x + C_2\sin x + \cos x.\log x + x.\sin x}$$

Example:

Apply the mehtod of variation of parameters to solve $\dfrac{d^2y}{dx^2} + y = \tan x$

Solution. We have, $\dfrac{d^2y}{dx^2} + y = \tan x$

$$(D^2+1)y = \tan x$$

A.E. is $m^2 = -1$ or $m = \pm i$

C. F. $y = A\cos x + B\sin x$

Here, $y_1 = \cos x,\quad y_2 = \sin x$

$$y_1 \cdot y'_2 - y'_1 \cdot y_2 = \cos x\,(\cos x) - (-\sin x)\sin x = \cos^2 x + \sin^2 x = 1$$

P. I. $= u.\,y_1 + v.\,y_2$ where

$$u = \int \frac{-y_2\tan x}{y_1 \cdot y'_2 - y'_1 \cdot y_2}\,dx = -\int \frac{\sin x\tan x}{1}\,dx = -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= \int (\cos x - \sec x)\,dx = \sin x - \log(\sec x + \tan x)$$

$$v = \int \frac{y_1\tan x}{y_1 \cdot y'_2 - y'_1 \cdot y_2}\,dx = \int \frac{\cos x \cdot \tan x}{1}\,dx = \int \sin x\,dx = -\cos x$$

P. I. $= u.\,y_1 + v.\,y_2$

$$= [\sin x - \log(\sec x + \tan x)]\cos x - \cos x\sin x = -\cos x\log(\sec x + \tan x)$$

Complete solution is

$$y = A\cos x + B\sin x - \cos x\log(\sec x + \tan x)$$

Example

Solve by method of variation of parameters: $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Solution. $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

A. E. is $(m^2 - 1) = 0$

$m^2 = 1, \quad m = \pm 1$

$C.F. = C_1 e^x + C_2 e^{-x}$

$\therefore \quad P.I. = uy_1 + vy_2$

Here, $y_1 = e^x, \quad y_2 = e^{-x}$

and $y_1 \cdot y'_2 - y'_1 \cdot y_2 = -e^x \cdot e^{-x} - e^x . e^{-x} = -2$

$$u = \int \frac{-y_2 X}{y_1 . y'_2 - y'_1 . y_2} dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x} dx$$

$$= \int \frac{e^{-x}}{1+e^x} dx = \int \frac{dx}{e^x (1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) dx$$

$$= \int e^{-x} dx - \int \frac{e^{-x}}{e^{-x}+1} dx = -e^{-x} + \log (e^{-x}+1)$$

$$v = \int \frac{y_1 X}{y_1 . y'_2 - y'_1 . y_2} dx = \int \frac{e^x}{-2} \frac{2}{1+e^x} dx = -\int \frac{e^x}{1+e^x} dx = -\log (1+e^x)$$

$$\text{P.I.} = u. y_1 + v. y_2 = [-e^{-x} + \log (e^{-x}+1)] e^x - e^{-x} \log (1+e^x)$$

$$= -1 + e^x \log (e^{-x}+1) - e^{-x} \log (e^x + 1)$$

Complete solution = $y = C_1 e^x + C_2 e^{-x} - 1 + e^x \log (e^{-x}+1) - e^{-x} \log (e^x+1)$ **Ans.**