Propositional Logic
and
Logical Operators
(Negation, Disjunction and Conjunction)

Check whether the following sentences are true/ false?

The sum of five and two is seven. TRUE

The sum of four and three is eight. FALSE

The sum of x and three is five. CAN NOT SAY

Go to the classroom. "

The sun will rise tomorrow. "

A proposition is a sentence which is either true or false but not both.

We use letters like P, Q, R, etc. to denote a proposition.

For example, P : 2+3 = 5

Q : Delhi is the capital of India.

R : The sum of two odd integers is an odd integer.

p, q, r

Logical Operators

Logical operators are used to join two more propositions to form a new proposition.

Negation (~):

The negation of a proposition P is denoted by $\sim P$ (not P).

Example:

P : I go to market.
$\sim P$: I do not go to market.

Truth Table (Negation)

P	$\sim P$
T	F
F	T

Disjunction (OR / $\vee$):

The disjunction of two propositions P and Q is denoted by $P \vee Q$.

Example:

P : I go to market.
Q : I play football.

$P \vee Q$: I go to market or I play football.

Truth Table (Disjunction)

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction (AND / $\wedge$):

The disjunction of two propositions P and Q is denoted by $P \wedge Q$.

Example:

P : I go to market.
Q : I play football.

$P \wedge Q$: I go to market and I play football.

Truth Table (Conjunction)

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: Let P : 2+3 = 5

Q : Delhi is the capital of India.

R : Five is an even number.

Find the truth value of the following propositions:

(i) $P \wedge Q$

(ii) $P \vee Q$

(iii) $P \vee R$

(iv) $P \wedge \sim R$

(v) $\sim Q \wedge \sim R$

(vi) $(P \vee \sim Q) \wedge \sim R$

Example: Let $P : 2+3 = 5$ T

Q : Delhi is the capital of India. T

R : Five is an even number. F

Find the truth value of the following propositions:

(i) $P \wedge Q$ T

(ii) $P \vee Q$ T

(iii) $P \vee R$ T

(iv) $P \wedge \sim R$ $T \wedge T$ T

(v) $\sim Q \wedge \sim R$ $F \wedge T$ F

(vi) $(P \vee \sim Q) \wedge \sim R$ $(T \vee F) \wedge T \equiv T \wedge T \equiv T$

Conditional ($\rightarrow$):

Let P and Q be two propositions. The statement $P \rightarrow Q$ is called the conditional statement. Here P is condition (sufficient) and Q is conclusion. It is defined as follows:

- If P, then Q
- P implies Q
- P is sufficient for Q.
- Q is necessary for P.
- Q whenever P.

Example: P : I go to market. Q : I play football.

$P \to Q$: If I go to market, then I play football.

: To play football, it is sufficient for me to go market.

: Playing football is necessary for me to go market.

Truth Table (Conditional)

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Bi-Conditional ($\leftrightarrow$):

The bi-conditional of two propositions P and Q is denoted by $P \leftrightarrow Q$. It is the proposition 'P if and only if Q' or 'P is necessary and sufficient for Q.

If P, then Q and if Q, then P

$(P \to Q) \wedge (Q \to P)$

Example:

P : I go to market.

Q : I play football.

$P \leftrightarrow Q$: I go to market if and only if I play football.

$P \leftrightarrow Q$: To go market is necessary and sufficient for me to play football.

Truth Table (Bi-Conditional)

$P \rightarrow$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Well Formed Formula (WFF)

A statement that cannot be broken down into smaller statements is called an atomic statement. For example, ' P : Today is Monday' is an atomic statement and P is the variable of the statement. A statement formula is said to be a well-formed formula (WFF) if it has following properties:

1. Every atomic statement is a WFF.
2. If P is WFF, then $\sim P$ is also WFF.
3. If P and Q are WFF, then $(P \wedge Q)$, $(P \vee Q)$, and $(P \rightarrow Q)$ are WFF.
4. Nothing else is WFF.

For example, $((P \wedge Q) \longrightarrow (\sim P))$ is a WFF, while $P \wedge Q \longrightarrow \sim P$ is not a WFF.

Rules of Precedence

A statement which is not a WFF can be converted into WFF using the rules of precedence of logical operators. The precedence of logical operators is as follows:

1. $\sim$
2. $\wedge$
3. $\vee$
4. $\longrightarrow$
5. $\longleftrightarrow$

Example: $P \wedge Q \longrightarrow \sim P$

Exclusive OR ($\oplus$):

For two propositions P and Q, exclusive OR is the proposition which is true when exactly one of the two propositions is true, otherwise false.

Truth Table (Exclusive OR)

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

NAND (↑):

NAND is equivalent to NOT AND. For two propositions P and Q, it is $\sim(P \wedge Q)$.

Truth Table (NAND)

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

NOR (↓):

NOR is equivalent to NOT OR. For two propositions P and Q, it is $\sim(P \vee Q)$.

Truth Table (NOR)

P	Q	$P \vee Q$	$\sim(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Converse, Contrapositive and Inverse

Let P and Q are two propositions. For the statement $P \longrightarrow Q$

Converse : $Q \longrightarrow P$

Contrapositive: $\sim Q \longrightarrow \sim P$

Inverse: $\sim P \longrightarrow \sim Q$

P : I go to market, Q : I buy a pen.

Converse: If I buy a pen, then I go to market.

Contrapositive: If I do not buy a pen, then I do not go to market.

Inverse: If I do not go to market, then I do not buy a pen.

Propositional Logic

Tautology, Contradiction, Logical Equivalence and Logical Implication

Tautology

A statement is called a tautology if it is true for all combinations of truth values of the variables included in it. We denote a tautology by T.

Ex $\quad P \vee \sim P$

Truth Table of $P \vee \sim P$

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

Contradiction

A statement is called a Contradiction if it is false for all combinations of truth values of the variables included in it. We denote a contradiction by $\underline{F}$.

Ex. $P \wedge \sim P$

Truth Table of $P \wedge \sim P$

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

Contingency

A statement is called a Contingency if it is neither a tautology nor a contradiction.

$P \to Q$, $P \vee Q$, P

Logical Equivalence

Two propositions are said to be logically equivalent or simply equivalent if both have the same truth values for all combinations of truth values of the variables included in them. $P \equiv Q$

Ex. $\underline{P \to Q} \equiv \sim\underline{P \vee Q}$

Truth Table of $P \to Q$ and $\sim P \vee Q$

P	Q	$P \to Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Some Logical Equivalences

Idempotent Laws	$P \vee P \equiv P,\ P \wedge P \equiv P$
Commutative Laws	$P \wedge Q \equiv Q \wedge P,\ P \vee Q \equiv Q \vee P$
Associative Laws	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R,\ P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
Distributive Laws	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R),\ P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
Domination Laws	$P \vee T \equiv T,\ P \wedge F \equiv F$
Identity Laws	$P \wedge T \equiv P,\ P \vee F \equiv P$
Negation Laws	$P \vee \sim P \equiv T,\ P \wedge \sim P \equiv F$
Absorption Laws	$P \wedge (P \vee Q) \equiv P,\ P \vee (P \wedge Q) \equiv P$
De Morgan's Laws	$\sim(P \vee Q) \equiv \sim P \wedge \sim Q,\ \sim(P \wedge Q) \equiv \sim P \vee \sim Q$
Double negation law	$\sim(\sim P) \equiv P$

Some Other Logical Equivalences

$$P \to Q \equiv \sim P \vee Q$$

$$P \to Q \equiv \sim Q \to \sim P$$

$$(P \to Q) \wedge (P \to R) \equiv P \to (Q \wedge R)$$

$$(P \to R) \wedge (Q \to R) \equiv (P \vee Q) \to R$$

$$(P \to Q) \vee (P \to R) \equiv P \to (Q \vee R)$$

$$(P \to R) \vee (Q \to R) \equiv (P \wedge Q) \to R$$

$$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$$

$$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$$

Some Logical Implications

$$P \wedge Q \Rightarrow P$$
$$P \wedge Q \Rightarrow Q$$
$$P \Rightarrow P \vee Q$$
$$Q \Rightarrow P \vee Q$$
$$\sim P \Rightarrow P \to Q$$
$$Q \Rightarrow P \to Q$$

$P \wedge (P \to Q) \Rightarrow Q$ (modus ponens)

$\sim Q \wedge (P \to Q) \Rightarrow \sim P$ (modus tollens)

$(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$ (hypothetical syllogism)

$(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$ (dilemma)

Example 1: Show that $\sim(P \vee (\sim P \wedge Q)) \equiv \sim P \wedge \sim Q$ without constructing truth table.

$\sim(P \vee (\sim P \wedge Q)) \equiv \sim P \wedge \sim(\sim P \wedge Q)$ [Using DeMorgan's law)

$\equiv \sim P \wedge (P \vee \sim Q)$ ["]

$\equiv (\sim P \wedge P) \vee (\sim P \wedge \sim Q)$ [" Distributive law]

$\equiv F \vee (\sim P \wedge \sim Q)$ [Since $\sim P \wedge P \equiv F$]

$\equiv \sim P \wedge \sim Q$ [Since $F \vee P \equiv P$]

$A \equiv B \qquad A \Rightarrow B \qquad \sim Q$

Example 2: Show that $\sim(P \longrightarrow Q) \Longrightarrow Q \longrightarrow P$ without constructing truth table.

$\sim(P \to Q) \Rightarrow \sim(\sim P \vee Q)$ (Since $P \to Q \equiv \sim P \vee Q$)

$\Rightarrow P \wedge \sim Q$ (Since $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$ DeMorgan's law)

$\Rightarrow \sim Q$ (Since $P \wedge Q \Rightarrow Q$)

$\Rightarrow Q \to P$ (Since $\sim P \Rightarrow P \to Q$)

Propositional Logic:
Argument and its validity

Argument

Let us consider a sequence of propositions, called premises $H_1, H_2, \dots, H_n$ and another proposition C called conclusion. This sequence is called an argument.

The conclusion C follows logically from the set of premises $H_1, H_2, \dots, H_n$ if and only if

$$H_1 \wedge H_2 \wedge \cdots \wedge H_n \Rightarrow C,$$

that is, $H_1 \wedge H_2 \wedge \cdots \wedge H_n \rightarrow C$ is a tautology. In this case the argument is called a **valid argument.**

Example1 : Check whether the following argument is a valid argument?
"If I go to Delhi, then I visit parliament. I go to Delhi. Therefore, I visit parliament."

$H_1 \wedge H_2 \Rightarrow C$, $H_1 \wedge H_2 \to C$

Solution: Let P: I go to Delhi .

Q: I visit parliament.

H_1 : $P \to Q$
H_2 : P
C : Q

H_2: P	C: Q	H_1: $P \to Q$	$H_1 \wedge H_2$	$(H_1 \wedge H_2) \to C$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Example 2: Check whether the following argument is a valid argument?
"If I go to Delhi, then I visit parliament. I do go to Delhi. Therefore, I do not visit parliament."

Solution: Let P: I go to Delhi

Q: I visit parliament.

$H_1 \wedge H_2 \rightarrow C$

$H_1: P \rightarrow Q$

$H_2: \sim P$

$C: \sim Q$

P	Q	H_2: $\sim P$	C: $\sim Q$	H_1: $P \rightarrow Q$	$H_1 \wedge H_2$: $(P \rightarrow Q) \wedge (\sim P)$	$H_1 \wedge H_2 \rightarrow C$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Example 3: Check whether the following argument is a valid argument?
"If I go to market and buy a book, then I play football and dance. I go to market and I buy a book. I sing a song and write a poem. Therefore, I paly football and sing a song."

Solution: Let P: I go to market, Q: I buy a book, R: I play football, S: I dance, T: I sing a song, U: I write a poem.

$(P \to Q) \wedge P \Rightarrow Q$ (Modus Ponens)

$H_1: (P \wedge Q) \to (R \wedge S)$

$H_2: (P \wedge Q)$

$H_3: T \wedge U$

$C: R \wedge T$

From $H_1 \wedge H_2$, we get

$H_1 \wedge H_2 \Rightarrow R \wedge S$ — ① (Since $(P \to Q) \wedge P \Rightarrow Q$, using modus ponens)

$H_1 \wedge H_2 \wedge H_3 \Rightarrow (R \wedge S) \wedge (T \wedge U)$

$\Rightarrow (R \wedge T) \wedge (S \wedge U)$ (Using associative law)

$\Rightarrow R \wedge T$ (Since $P \wedge Q \Rightarrow P$)

Predicates

$P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_{50}$

Let us consider the following sentences:

1. Shekhar has I division in 12th standard. : P_1
2. Nikhil has I division in 12th standard. P_2
3. Suman has I division in 12th standard. P_3

$P(x)$: x has I division in 12th standard $x \in$ set of students of 12th class

Universe of discourse

$Q(x)$: x is an even no $x \in Z$

$P(x, y)$: $x > y$ $x, y \in Z$

Quantifiers:

Universal Quantifier

When a statement is true for all values in its universe of discourse, we use universal quantifier.

$\forall$ for all

$X = \{2, 4, 6, 8, 10\}$

$x \in \{2, 4, 6, 8, 10\}$

All no. in X are even

$P(x)$: x is an even no.

$\forall x\, P(x)$

$\forall x\, P(x)$

$\exists x\, P(x)$

Quantifiers:

Existential Quantifier

When a statement is true for some of the values in its universe of discourse, we use existential quantifier. $\exists$ there exist

$x = \{1, 2, 3, 4, 5, 6\} = X$

$P(x)$: x is an even no.

Some no. in X are even no.

$\exists x\, P(x)$

Example 1: Symbolize the following sentences:

(1) All students are happy

with using universe of discourse

Let $x \in$ set of students

$P(x)$: x is happy.

$\forall x \, P(x)$

without universe of discourse

$P(x)$: x is a student

$Q(x)$: x is happy

$\forall x \, (P(x) \rightarrow Q(x))$

(2) Some farmers are rich.

Let $x \in$ set of farmers

$P(x)$: x is rich

$\exists x \, P(x)$

$P(x)$: x is a farmer

$Q(x)$: x is rich

$\exists x \, (P(x) \wedge Q(x))$

$\exists x \, (P(x) \rightarrow Q(x))$ ✗

y $\quad P(y) \rightarrow Q(y)$

F $\quad$ T

Free and bounded variables

A variable is bounded if it is bounded by a quantifier. A variable is free if it is not bounded.

$\underline{\forall x} P(x, y)$

Removing quantifiers from predicates

Let $x \in \{x_1, x_2, \dots, x_n\}$. Remove quantifier from the followings:

1. $\forall x\ P(x)$ $\qquad P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$
2. $\exists x\ P(x)$ $\qquad P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$
3. $\forall x \sim P(x)$ $\qquad \sim P(x_1) \wedge \sim P(x_2) \wedge \cdots \wedge \sim P(x_n)$

$\sim \forall x P(x) \equiv \exists x \sim P(x)$

$\sim \exists x P(x) \equiv \forall x \sim P(x)$

Negations of quantifiers

Find the negation of the following sentences:

1. All players are healthy.

$x \in$ set of players

$P(x)$: x is healthy

$\forall x P(x)$

$\sim \forall x P(x) \equiv \exists x \sim P(x)$

Some players are not healthy.

2. Some students are well-mannered .

$x \in$ set of students

$P(x)$: x is well mannered

$\exists x P(x)$

$\sim \exists x P(x) \equiv \forall x \sim P(x)$

All students are not well mann