Analysis of a Length-Structured Model for Fish

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Introduction

White perch are found in Nebraska lakes, but are not native to Nebraska, and are invasive. The population exhibits stunting, which means that populations are dominated by smaller (less desirable) fish. We introduce a length-based model for this population and study the convergence of the total population, and see how this is affected by different survival rates.

We use a length-based model:

- Easier to collect length data than age data
- Managers might be more interested in size distribution than age distribution
- Life history parameters might be length-based. For instance, larger fish hold more eggs.
- More tractable to mathematical analysis.

Length-Structured Model

Population vector

$$\begin{bmatrix} P_0(t) \\ P_1(t) \\ \vdots \\ P_n(t) \end{bmatrix}$$

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Assumptions:

- The time step is constant, determined by the behavior of species, or by data collection.
- Each stage is corresponds to the size of the fish with P_0 being newborn fish, and P_n being the largest fish possible.
- In one time step a fish can either stay in its length class, or grow into the next length class.

Parameters

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- Average length of fish in i^{th} stage is L_i.
- Survival rate of stage i fish each time step is s_i.
- Fecundity of stage i fish each time step is f_i

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Let p_t be the probability that at time step t grows into the next stage (if it survives).

Matrix

We use a model of the form $\vec{P}(t+1) = A_{p_t}\vec{P}(t)$, where $A_{p_t} =$

$$\begin{bmatrix} 0 & f_1 & f_2 & f_3 & \cdots & f_{n-2} & f_{n-1} & f_n \\ s_0 & s_1(1-p_t) & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & s_1 p_t & s_2(1-p_t) & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & s_2 p_t & s_3(1-p_t) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & s_{n-2}p_t & s_{n-1}(1-p_t) & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & s_{n-1}p_t & s_n \end{bmatrix}$$

Density dependence

Biomass: assume the mass of a fish of length L_i is $W_i = \alpha L_i^3$, where α is the mass-length coefficient.

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The Population Biomass at time step t is

$$B(t) = \sum_{i=0}^{n} W_i P_i(t).$$

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Nonlinearity: $g : [0, \infty) \to (0, 1]$ is strictly decreasing and continuous, with

$$g(0) = 1, \qquad \lim_{y \to \infty} g(y) = 0.$$

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We study the dynamical system

$$\vec{P}(t+1) = A_{p_t} \vec{P}(t), \qquad p_t = g(B(t)).$$

This means:

- If there is low biomass, the probability that fish get larger in a time step is close to one.
- If there is high biomass, the probability that a fish gets larger in a time step is close to zero.
- This means the model penalizes crowding.

White Perch - Nonlinearity

Recall that $B(t)$ is the biomass at year t. We model the probability p_t at year t that a fish grows into the next length class in one year by a Beverton-Holt response function

$$p_t = g(B(t)) = \frac{1}{1 + b_{\text{growth}} B(t)}$$

where $b_{\text{growth}} = 9.0 \times 10^{-6}$ g^{-1} is the scaling parameter for density-dependent growth .

Convergence

- The vector $\vec{P}^*$ is globally asymptotically attracting if for every nonzero, nonnegative $\vec{P}(0)$, $\lim_{t\to\infty} \vec{P}(t) = \vec{P}^*$.
- The spectral radius of a matrix A, $\rho(A)$, is the largest eigenvalue of A. It determines the asymptotic behavior of solutions to $\vec{x}(t+1) = A\vec{x}(t)$.

Main Theorem (Callahan et al 2019)

Assume (s_i) and (f_i) are nondecreasing in i. This is often true, but not always (for instance, in the presence of angling).

1. If $\rho(A_1) < 1$, then the zero population $\vec{0}$ is globally asymptotically attracting.
2. If $\rho(A_0) < 1 < \rho(A_1)$, then the system has a unique nonzero equilibrium $\vec{P}^*$.

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- The condition that s_i nondecreasing is often not satisfied. For instance, new anglers usually try to catch larger fish.
- We study whether the theorem is true even if s_i is not non-decreasing.

Results from "Length-Structured Density Dependent Model for Fish", Callahan et al.

White Perch - Survival

- In the paper it is assumed that the survival rate across age classes is a constant $s = 0.68$, which we assume is the survival rate across length classes.
- In our talk we study many other survivals that did not satisfy the non-decreasing condition.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
	Initial Population vectors														
Population Class 0	353.4217	2353.124	1812.296	4904.518	2883.791	1862.671	3334.658	1944.419	4498.567	614.075	3474.026	93.06387	1878.461	2000.399	2699.525
Population Class 1	4613.723	2803.567	3940.567	1433.102	129.2874	2965.923	4668.628	2273.709	2251.968	2036.592	4171.845	3373.882	2732.769	4159.357	476.8635
Population Class 2	4001.86	1345.458	3901.479	4004.101	2232.655	4362.763	4054.75	1233.436	1028.362	1376.435	3048.148	2192.544	2809.601	671.6917	732.5743
Population Class 3	1429.734	3745.092	3342.561	4480.557	3231.51	4667.508	2422.741	3922.115	4498.255	3583.349	2873.686	2189.101	1979.111	302.3339	3155.706
Population Class 4	2718.316	2519.439	667.5193	2987.633	2606.015	3342.321	3783.746	4414.188	3812.928	1416.922	1630.211	585.1841	1990.654	421.2353	4296.602
Population Class 5	4923.881	3234.048	107.7794	4420.084	1861.563	1033.882	2085.237	4568.558	4412.432	4480.994	2282.123	4073.408	2576.836	819.4916	4871.108
Population Class 6	3578.39	1538.728	2799.204	4718.658	4685.673	3269.253	4858.93	2791.425	1424.751	4132.894	3568.978	1624.277	3287.653	1621.1	2854.192
Population Class 7	4194.848	693.6232	1504.095	2745.79	4147.664	360.2578	4939.874	2994.341	3366.13	1950.133	4422.025	1231.141	4754.576	1508.634	4984.251
Population Class 8	2166.303	2377.865	4697.049	3641.934	4245.427	2033.635	4320.738	744.3836	3321.4	2489.515	3604.278	1713.566	3611.743	58.40496	2767.708
	Limiting Population Vectors														
Population Class 0	1.81E-12	1.83E-12	1.71E-12	1.64E-12	2.22E-12	1.67E-12	1.81E-12	1.59E-12	1.58E-12	1.68E-12	1.61E-12	2.08E-12	2.25E-12	2.22E-12	1.67E-12
Population Class 1	5.22E-13	5.27E-13	4.92E-13	4.74E-13	6.41E-13	4.82E-13	5.24E-13	4.58E-13	4.56E-13	4.85E-13	4.66E-13	6.01E-13	6.50E-13	6.42E-13	4.83E-13
Population Class 2	1.51E-13	1.52E-13	1.42E-13	1.37E-13	1.85E-13	1.39E-13	1.51E-13	1.32E-13	1.32E-13	1.40E-13	1.35E-13	1.74E-13	1.88E-13	1.85E-13	1.39E-13
Population Class 3	1.74E-13	1.76E-13	1.64E-13	1.58E-13	2.14E-13	1.61E-13	1.75E-13	1.53E-13	1.52E-13	1.62E-13	1.56E-13	2.01E-13	2.17E-13	2.14E-13	1.61E-13
Population Class 4	5.03E-14	5.08E-14	4.74E-14	4.56E-14	6.17E-14	4.64E-14	5.04E-14	4.41E-14	4.39E-14	4.67E-14	4.49E-14	5.79E-14	6.26E-14	6.18E-14	4.65E-14
Population Class 5	5.81E-14	5.87E-14	5.48E-14	5.27E-14	7.13E-14	5.36E-14	5.82E-14	5.10E-14	5.08E-14	5.40E-14	5.19E-14	6.69E-14	7.23E-14	7.14E-14	5.37E-14
Population Class 6	1.68E-14	1.69E-14	1.58E-14	1.52E-14	2.06E-14	1.55E-14	1.68E-14	1.47E-14	1.47E-14	1.56E-14	1.50E-14	1.93E-14	2.09E-14	2.06E-14	1.55E-14
Population Class 7	1.94E-14	1.96E-14	1.83E-14	1.76E-14	2.38E-14	1.79E-14	1.94E-14	1.70E-14	1.69E-14	1.80E-14	1.73E-14	2.23E-14	2.41E-14	2.38E-14	1.79E-14
Population Class 8	7.87E-15	7.94E-15	7.42E-15	7.14E-15	9.66E-15	7.26E-15	7.89E-15	6.90E-15	6.87E-15	7.31E-15	7.02E-15	9.06E-15	9.79E-15	9.67E-15	7.28E-15

$$s_i = [.2 .8 .2 .8 .2 .8 .2 .8 .2]$$

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Population Class 1	5.22E-13	5.27E-13	4.92E-13	4.74E-13	6.41E-13	4.82E-13	5.24E-13	4.58E-13	4.56E-13	4.85E-13	4.66E-13	6.01E-13	6.50E-13	6.42E-13	4.83E-13
Population Class 2	1.51E-13	1.52E-13	1.42E-13	1.37E-13	1.85E-13	1.39E-13	1.51E-13	1.32E-13	1.32E-13	1.40E-13	1.35E-13	1.74E-13	1.88E-13	1.85E-13	1.39E-13
Population Class 3	1.74E-13	1.76E-13	1.64E-13	1.58E-13	2.14E-13	1.61E-13	1.75E-13	1.53E-13	1.52E-13	1.62E-13	1.56E-13	2.01E-13	2.17E-13	2.14E-13	1.61E-13
Population Class 4	5.03E-14	5.08E-14	4.74E-14	4.56E-14	6.17E-14	4.64E-14	5.04E-14	4.41E-14	4.39E-14	4.67E-14	4.49E-14	5.79E-14	6.26E-14	6.18E-14	4.65E-14
Population Class 5	5.81E-14	5.87E-14	5.48E-14	5.27E-14	7.13E-14	5.36E-14	5.82E-14	5.10E-14	5.08E-14	5.40E-14	5.19E-14	6.69E-14	7.23E-14	7.14E-14	5.37E-14
Population Class 6	1.68E-14	1.69E-14	1.58E-14	1.52E-14	2.06E-14	1.55E-14	1.68E-14	1.47E-14	1.47E-14	1.56E-14	1.50E-14	1.93E-14	2.09E-14	2.06E-14	1.55E-14
Population Class 7	1.94E-14	1.96E-14	1.83E-14	1.76E-14	2.38E-14	1.79E-14	1.94E-14	1.70E-14	1.69E-14	1.80E-14	1.73E-14	2.23E-14	2.41E-14	2.38E-14	1.79E-14
Population Class 8	7.87E-15	7.94E-15	7.42E-15	7.14E-15	9.66E-15	7.26E-15	7.89E-15	6.90E-15	6.87E-15	7.31E-15	7.02E-15	9.06E-15	9.79E-15	9.67E-15	7.28E-15

$s_i = [.2.8.2.8.2.8.2.8.2]$

- Convergence to zero - Sequence
- Convergence no matter starting population vector

Simulations

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
	Initial Population vectors														
Population Class 0	2207.945	3448.189	4629.29	4865.068	3076.441	1230.348	3981.227	524.1111	739.0871	2790.59	2665.816	3179.416	4189.203	555.9249	1532.484
Population Class 1	2231.078	659.1533	2463.193	1825.163	2915.665	2907.456	3089.253	4291.764	98.82332	2138.963	4773.774	3991.851	4603.951	1486.771	527.8057
Population Class 2	2328.312	617.5042	3274.414	1545.748	3491.27	4688.384	351.0676	3490.999	4821.459	1335.97	1338.738	2508.505	2491.14	1982.093	2969.138
Population Class 3	1395.196	954.5143	4450.617	604.5619	146.6617	238.9365	346.3949	3668.711	4851.865	3768.68	1250.423	3254.061	1388.056	2103.778	1413.638
Population Class 4	3376.877	728.6605	2692.628	4578.829	2639.413	269.8883	680.0369	3252.653	619.3025	4491.882	4638.364	3979.775	3262.6	1557.377	776.1081
Population Class 5	4518.323	2925.218	1411.026	677.391	160.3642	103.0902	3944.457	2581.353	2337.05	3642.22	342.9117	1166.869	4586.494	3469.216	3.293338
Population Class 6	4542.629	366.8085	4879.788	1660.589	4135.712	3407.393	461.9923	1631.941	3283.47	2034.151	1497.002	3004.195	2549.197	459.3592	1417.977
Population Class 7	3735.985	4111.631	182.1276	4487.399	1699.931	2993.143	1189.344	3308.881	1450.928	4691.579	2957.918	562.3119	4870.957	2010.443	2754.054
Population Class 8	1302.558	3614.515	1631.223	2498.244	4233.555	570.1502	1218.24	587.8274	3772.683	1277.137	1016.496	2578.828	986.3947	1475.904	4354.511
	Limiting Population Vectors														
Population Class 0	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03
Population Class 1	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04
Population Class 2	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03
Population Class 3	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03
Population Class 4	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02
Population Class 5	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02
Population Class 6	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02
Population Class 7	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01
Population Class 8	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00

$$s_i = [.9 .7 .7 .7 .7 .7 .7 .7 .5]$$

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Population Class 5	4518.323	2925.218	1411.026	677.391	160.3642	103.0902	3944.457	2581.353	2337.05	3642.22	342.9117	1166.869	4586.494	3469.216	3.293338
Population Class 6	4542.629	366.8085	4879.788	1660.589	4135.712	3407.393	461.9923	1631.941	3283.47	2034.151	1497.002	3004.195	2549.197	459.3592	1417.977
Population Class 7	3735.985	4111.631	182.1276	4487.399	1699.931	2993.143	1189.344	3308.881	1450.928	4691.579	2957.918	562.3119	4870.957	2010.443	2754.054
Population Class 8	1302.558	3614.515	1631.223	2498.244	4233.555	570.1502	1218.24	587.8274	3772.683	1277.137	1016.496	2578.828	986.3947	1475.904	4354.511
	Limiting Population Vectors														
Population Class 0	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03	7.74E+03
Population Class 1	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04	1.45E+04
Population Class 2	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03	5.45E+03
Population Class 3	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03	2.05E+03
Population Class 4	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02	7.69E+02
Population Class 5	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02	2.89E+02
Population Class 6	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02	1.09E+02
Population Class 7	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01	3.12E+01
Population Class 8	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00	8.06E+00

$s_i = [.9 .7 .7 .7 .7 .7 .7 .7 .5]$

- Convergence to unique limiting population
- Convergence no matter starting population vector

Simulations

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
	Initial Population vectors														
Population Class 0	1742.875	3143.467	3355.82	585.7258	2550.202	896.8455	2555.041	68.27783	2567.816	4145.404	1098.28	799.3442	1214.381	3159.774	419.931
Population Class 1	209.605	4388.82	3262.104	1202.118	2478.221	546.806	4367.561	1870.301	4962.949	2559.372	2031.4	3334.193	4171.351	1216.348	1251.52
Population Class 2	711.6986	3311.759	2655.244	3424.542	3256.842	4525.792	351.1074	4613.426	2279.219	2759.809	3149.54	89.60185	4067.956	2857.065	4056.73
Population Class 3	382.9642	4377.068	3575.535	4196.268	3718.531	4381.755	4937.694	2732.398	2130.204	1066.424	2776.538	598.3768	3144.866	4908.644	422.115
Population Class 4	3702.633	2337.585	2524.053	4850.723	1509.767	4998.963	4613.567	2369.434	1066.074	2939.103	637.8953	4760.627	11.19099	4248.41	2656.2
Population Class 5	2282.626	706.6801	2439.997	1075.848	448.0597	4321.273	2821.305	2482.687	966.2474	713.8161	845.9917	4879.27	1898.62	1417.221	4003.11
Population Class 6	3341.247	340.6723	2489.192	3801.717	4129.827	184.3876	2157.31	1544.855	4163.771	261.1176	4.997091	154.5727	4522.028	3412.288	3694.11
Population Class 7	3496.23	3571.212	4679.878	2920.514	1947.935	2723.412	1689.223	4754.188	3633.185	3416.54	2090.812	2469.409	3401.758	1790.623	708.314
Population Class 8	2856.786	1539.93	1946.408	2014.758	3876.524	4988.081	3603.617	4909.966	2648.728	3042.797	2442.428	4313.611	1894.116	4934.634	2189.47
	Limiting Population Vectors														
Population Class 0	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+0
Population Class 1	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+0
Population Class 2	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+0
Population Class 3	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+0
Population Class 4	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+0
Population Class 5	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+0
Population Class 6	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+0
Population Class 7	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+0
Population Class 8	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+0

$s_i = [.9.8.7.6.5.5.5.5.4]$

Simulations

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
	Initial Population vectors														
Population Class 0	1742.875	3143.467	3355.82	585.7258	2550.202	896.8455	2555.041	68.27783	2567.816	4145.404	1098.28	799.3442	1214.381	3159.774	419.931
Population Class 1	209.605	4388.82	3262.104	1202.118	2478.221	546.806	4367.561	1870.301	4962.949	2559.372	2031.4	3334.193	4171.351	1216.348	1251.52
Population Class 2	711.6986	3311.759	2655.244	3424.542	3256.842	4525.792	351.1074	4613.426	2279.219	2759.809	3149.54	89.60185	4067.956	2857.065	4056.73
Population Class 3	382.9642	4377.068	3575.535	4196.268	3718.531	4381.755	4937.694	2732.398	2130.204	1066.424	2776.538	598.3768	3144.866	4908.644	422.115
Population Class 4	3702.633	2337.585	2524.053	4850.723	1509.767	4998.963	4613.567	2369.434	1066.074	2939.103	637.8953	4760.627	11.19099	4248.41	2656.2
Population Class 5	2282.626	706.6801	2439.997	1075.848	448.0597	4321.273	2821.305	2482.687	966.2474	713.8161	845.9917	4879.27	1898.62	1417.221	4003.11
Population Class 6	3341.247	340.6723	2489.192	3801.717	4129.827	184.3876	2157.31	1544.855	4163.771	261.1176	4.997091	154.5727	4522.028	3412.288	3694.11
Population Class 7	3496.23	3571.212	4679.878	2920.514	1947.935	2723.412	1689.223	4754.188	3633.185	3416.54	2090.812	2469.409	3401.758	1790.623	708.314
Population Class 8	2856.786	1539.93	1946.408	2014.758	3876.524	4988.081	3603.617	4909.966	2648.728	3042.797	2442.428	4313.611	1894.116	4934.634	2189.47
	Limiting Population Vectors														
Population Class 0	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+03	2.34E+0
Population Class 1	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+03	3.13E+0
Population Class 2	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+03	1.62E+0
Population Class 3	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+02	6.79E+0
Population Class 4	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+02	2.37E+0
Population Class 5	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+01	8.24E+0
Population Class 6	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+01	2.87E+0
Population Class 7	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+00	9.44E+0
Population Class 8	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+00	3.37E+0

$s_i = [.9 .8 .7 .6 .5 .5 .5 .5 .4]$

- Convergence to unique limiting population - Steep Slope
- Convergence no matter starting population vector

Simulations

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
	Initial Population vectors														
Population Class 0	3915.509	3972.947	4088.174	1220.48	3101.391	2821.344	3390.53	1580.356	1132.672	4930.255	2483.997	4488.233	1414.2	4310.937	3691.28
Population Class 1	3428.436	4629.049	3001.724	4422.113	3733.423	936.9144	262.4277	387.4374	1990.026	3590.905	112.0683	1442.827	1025.906	2214.673	190.007
Population Class 2	2331.096	894.1998	424.9853	3563.234	4886.278	2658.448	4005.862	4253.07	3482.843	2065.917	269.1577	1345.234	2195.67	2740.046	4771.22
Population Class 3	1301.59	2587.706	4611.79	1890.742	1919.568	1775.167	3392.843	722.6345	323.2038	493.1512	704.369	2970.971	136.2511	2834.304	3711.86
Population Class 4	2846.341	3135.027	267.9891	1244.598	1301.028	1573.918	4730.045	1852.429	3738.308	3672.796	4467.372	2379.395	4380.922	3401.975	4687.24
Population Class 5	1243.853	4565.912	2635.125	1264.269	4387.348	3633.707	457.7907	3111.957	2102.002	3186.531	2329.1	1841.555	3050.461	1856.893	2566.8
Population Class 6	1596.508	3319.841	594.2664	3836.218	4030.48	2578.864	4542.192	4987.76	4056.587	369.2094	2804.284	3278.055	1017.962	391.1435	1204.52
Population Class 7	4554.011	1945.964	1900.715	249.3094	2305.605	3953.224	2549.765	2586.721	1898.026	602.5408	2472.282	4691.002	2599.584	2281.754	1299.82
Population Class 8	4426.1	3700.038	4064.163	3426.443	454.8084	1022.463	3074.518	4952.556	1595.339	4907.981	338.9274	3102.126	269.1215	239.219	3794.87
	Limiting Population Vectors														
Population Class 0	3.50E-13	3.19E-13	2.75E-13	2.51E-13	5.91E-13	2.52E-13	3.06E-13	3.75E-13	2.40E-13	2.83E-13	5.14E-13	3.36E-13	4.59E-13	3.83E-13	2.75E-1
Population Class 1	1.87E-13	1.70E-13	1.47E-13	1.34E-13	3.15E-13	1.35E-13	1.63E-13	2.00E-13	1.28E-13	1.51E-13	2.75E-13	1.79E-13	2.45E-13	2.04E-13	1.47E-1
Population Class 2	9.96E-14	9.09E-14	7.82E-14	7.16E-14	1.68E-13	7.19E-14	8.71E-14	1.07E-13	6.84E-14	8.06E-14	1.47E-13	9.56E-14	1.31E-13	1.09E-13	7.84E-1
Population Class 3	5.32E-14	4.85E-14	4.18E-14	3.82E-14	8.98E-14	3.84E-14	4.65E-14	5.70E-14	3.65E-14	4.30E-14	7.82E-14	5.10E-14	6.98E-14	5.82E-14	4.18E-1
Population Class 4	1.42E-14	1.29E-14	1.11E-14	1.02E-14	2.40E-14	1.02E-14	1.24E-14	1.52E-14	9.75E-15	1.15E-14	2.09E-14	1.36E-14	1.86E-14	1.55E-14	1.12E-1
Population Class 5	3.79E-15	3.46E-15	2.97E-15	2.72E-15	6.40E-15	2.73E-15	3.31E-15	4.06E-15	2.60E-15	3.06E-15	5.57E-15	3.63E-15	4.97E-15	4.14E-15	2.98E-1
Population Class 6	1.01E-15	9.22E-16	7.94E-16	7.26E-16	1.71E-15	7.29E-16	8.84E-16	1.08E-15	6.94E-16	8.17E-16	1.49E-15	9.70E-16	1.33E-15	1.11E-15	7.95E-1
Population Class 7	2.70E-16	2.46E-16	2.12E-16	1.94E-16	4.55E-16	1.95E-16	2.36E-16	2.89E-16	1.85E-16	2.18E-16	3.97E-16	2.59E-16	3.54E-16	2.95E-16	2.12E-1
Population Class 8	9.82E-17	8.96E-17	7.71E-17	7.05E-17	1.66E-16	7.08E-17	8.59E-17	1.05E-16	6.74E-17	7.94E-17	1.44E-16	9.42E-17	1.29E-16	1.07E-16	7.72E-1

$$s_i = [.2.2.2.2.1.1.1.1.1]$$

Simulations

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7	Test 8	Test 9	Test 10	Test 11	Test 12	Test 13	Test 14	Test 15
	Initial Population vectors														
Population Class 0	3915.509	3972.947	4088.174	1220.48	3101.391	2821.344	3390.53	1580.356	1132.672	4930.255	2483.997	4488.233	1414.2	4310.937	3691.28
Population Class 1	3428.436	4629.049	3001.724	4422.113	3733.423	936.9144	262.4277	387.4374	1990.026	3590.905	112.0683	1442.827	1025.906	2214.673	190.007
Population Class 2	2331.096	894.1998	424.9853	3563.234	4886.278	2658.448	4005.862	4253.07	3482.843	2065.917	269.1577	1345.234	2195.67	2740.046	4771.22
Population Class 3	1301.59	2587.706	4611.79	1890.742	1919.568	1775.167	3392.843	722.6345	323.2038	493.1512	704.369	2970.971	136.2511	2834.304	3711.86
Population Class 4	2846.341	3135.027	267.9891	1244.598	1301.028	1573.918	4730.045	1852.429	3738.308	3672.796	4467.372	2379.395	4380.922	3401.975	4687.24
Population Class 5	1243.853	4565.912	2635.125	1264.269	4387.348	3633.707	457.7907	3111.957	2102.002	3186.531	2329.1	1841.555	3050.461	1856.893	2566.8
Population Class 6	1596.508	3319.841	594.2664	3836.218	4030.48	2578.864	4542.192	4987.76	4056.587	369.2094	2804.284	3278.055	1017.962	391.1435	1204.52
Population Class 7	4554.011	1945.964	1900.715	249.3094	2305.605	3953.224	2549.765	2586.721	1898.026	602.5408	2472.282	4691.002	2599.584	2281.754	1299.82
Population Class 8	4426.1	3700.038	4064.163	3426.443	454.8084	1022.463	3074.518	4952.556	1595.339	4907.981	338.9274	3102.126	269.1215	239.219	3794.87
	Limiting Population Vectors														
Population Class 0	3.50E-13	3.19E-13	2.75E-13	2.51E-13	5.91E-13	2.52E-13	3.06E-13	3.75E-13	2.40E-13	2.83E-13	5.14E-13	3.36E-13	4.59E-13	3.83E-13	2.75E-1
Population Class 1	1.87E-13	1.70E-13	1.47E-13	1.34E-13	3.15E-13	1.35E-13	1.63E-13	2.00E-13	1.28E-13	1.51E-13	2.75E-13	1.79E-13	2.45E-13	2.04E-13	1.47E-1
Population Class 2	9.96E-14	9.09E-14	7.82E-14	7.16E-14	1.68E-13	7.19E-14	8.71E-14	1.07E-13	6.84E-14	8.06E-14	1.47E-13	9.56E-14	1.31E-13	1.09E-13	7.84E-1
Population Class 3	5.32E-14	4.85E-14	4.18E-14	3.82E-14	8.98E-14	3.84E-14	4.65E-14	5.70E-14	3.65E-14	4.30E-14	7.82E-14	5.10E-14	6.98E-14	5.82E-14	4.18E-1
Population Class 4	1.42E-14	1.29E-14	1.11E-14	1.02E-14	2.40E-14	1.02E-14	1.24E-14	1.52E-14	9.75E-15	1.15E-14	2.09E-14	1.36E-14	1.86E-14	1.55E-14	1.12E-1
Population Class 5	3.79E-15	3.46E-15	2.97E-15	2.72E-15	6.40E-15	2.73E-15	3.31E-15	4.06E-15	2.60E-15	3.06E-15	5.57E-15	3.63E-15	4.97E-15	4.14E-15	2.98E-1
Population Class 6	1.01E-15	9.22E-16	7.94E-16	7.26E-16	1.71E-15	7.29E-16	8.84E-16	1.08E-15	6.94E-16	8.17E-16	1.49E-15	9.70E-16	1.33E-15	1.11E-15	7.95E-1
Population Class 7	2.70E-16	2.46E-16	2.12E-16	1.94E-16	4.55E-16	1.95E-16	2.36E-16	2.89E-16	1.85E-16	2.18E-16	3.97E-16	2.59E-16	3.54E-16	2.95E-16	2.12E-1
Population Class 8	9.82E-17	8.96E-17	7.71E-17	7.05E-17	1.66E-16	7.08E-17	8.59E-17	1.05E-16	6.74E-17	7.94E-17	1.44E-16	9.42E-17	1.29E-16	1.07E-16	7.72E-1

$s_i = [.2.2.2.2.1.1.1.1.1]$

- Convergence to zero - Gentle slope
- Convergence no matter starting population vector

The following appears to be true based on numerical simulations:

- If $\rho(A_1) < 1$, then the zero population $\vec{0}$ is globally asymptotically attracting.
- If $\rho(A_0) < 1 < \rho(A_1)$, then for every nonzero non-negative initial state $\vec{P}(0)$,
$$\vec{P}(t) \to \vec{P}^* \text{ as } t \to \infty.$$

Future work

Since smaller fish are less desirable than larger fish, we would like to identify survivals (s_i) that lead to limiting populations which are not dominated by small fish. This might lead to management recommendations in the form of fishing regulations.

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