Solution: Exercise 1:
Biophysical properties of cell membranes

Problem 1: Electroneutrality in neurons

(i) From the relationship $Q = CV$ we find for the required number N of monovalent ions:

$$N = \frac{Q}{e} = \frac{CV}{e} = \frac{C_m A_{\rm cell} V}{e} \tag{1}$$

where $A_{\rm cell} = 4\pi r_{\rm cell}^2$ is the surface area of the cell and $e = 1.6 \cdot 10^{-19}$ C is the elementary charge.

With numbers inserted we get

$$\begin{aligned} N &= \frac{1\mu\mathrm{F} \cdot \mathrm{cm}^{-2} \cdot 4\pi (25\mu\mathrm{m})^2 \cdot 100 \text{ mV}}{1.6 \cdot 10^{-19} \text{ C}} \\ &= \frac{1 \cdot 10^{-6} \cdot 10^4 \cdot 4\pi (25)^2 \cdot (10^{-6})^2 \cdot 100 \cdot 10^{-3}}{1.6 \cdot 10^{-19}} \frac{FV}{C} = 4.9 \cdot 10^7 \end{aligned} \tag{2}$$

In the last step we have used that the ratio $1\mathrm{F} \cdot 1\mathrm{V}/1\mathrm{C}$ is 1. If you are surprised by this, remember (i) that F is the SI unit for capacitance (farad), V is the SI unit for voltage (volt), and C is the SI unit for charge (coulomb), and (ii) that the fundamental relationship relating these quantities is charge equals capacitance multiplied by voltage.

(ii) N is the number of required 'uncompensated' Cl^- ions to get a membrane potential of 100 mV. This must be compared with the total number of Cl^- ions inside the neuron. This total number is given by the concentration c of Cl^- (100 mM) multiplied by the volume $v_{\rm cell}$ of the cell (here: $v_{\rm cell} = 4\pi r_{\rm cell}^3/3$):

$$\begin{aligned} f_{\rm uncomp} &= \frac{N}{c\, v_{\rm cell}} = \frac{3N}{4\pi c\, r_{\rm cell}^3} = \frac{3N}{4\pi \cdot 100 \cdot 10^{-3} \cdot N_A \cdot 10^{-3} \cdot r_{\rm cell}^3} \\ &= \frac{3 \cdot 4.9 \cdot 10^7}{4\pi \cdot 100 \cdot 10^{-3} \cdot 6.0 \cdot 10^{23} \cdot 10^3 \cdot (2.5 \cdot 10^{-5})^3} = 1.1 \cdot 10^{-5} \simeq 0.001\% \end{aligned} \tag{3}$$

The fraction of uncompensated Cl^- ions is thus very small, about 0.001%

(iii) As the fraction of uncompensated ions is only about 10^{-5} even at a membrane potential of 100 mV (which in magnitude is about as big as it gets), we can safely conclude that the membrane currents have a negligible effect on the ion concentrations in the *cytoplasm*, i.e., inside the cell.

Problem 2: Calcium ions in dendritic spines

(i) We find for the required number $N_{\rm Ca}$ of calcium ions:

$$N_{\rm Ca} = \frac{Q}{2e} = \frac{C_m A_{\rm sp} V}{2e} \tag{4}$$

where $A_{\mathrm{sp}} = \pi d_{\mathrm{sp}} h_{\mathrm{sp}} + 2\pi d_{\mathrm{sp}}^2/4 = \pi d_{\mathrm{sp}}(h_{\mathrm{sp}} + d_{\mathrm{sp}}/2)$ correspond to the sum of the side surface and top and bottom surfaces of the spine-head cylinder.

With numbers inserted we get

$$\begin{aligned} N_{\mathrm{Ca}} &= \frac{1\mu\mathrm{F} \cdot \mathrm{cm}^{-2} \cdot \pi \cdot 0.4\ \mu\mathrm{m}\ (0.2\ \mu\mathrm{m} + 0.4/2\ \mu\mathrm{m}) \cdot 10\ \mathrm{mV}}{2 \cdot 1.6 \cdot 10^{-19}\ \mathrm{C}} \\ &= \frac{1 \cdot 10^{-6} \cdot 10^{4} \cdot \pi \cdot 0.4 \cdot 0.4 \cdot (10^{-6})^2 \cdot 10 \cdot 10^{-3}}{2 \cdot 1.6 \cdot 10^{-19}} = 157 \end{aligned} \tag{5}$$

Thus 157 Ca^{2+} ions are needed to increase spine membrane potential with 10 mV.

(ii) The (average) number of Ca ions inside the spine head when the concentration is 70 nM, is given by the concentration c multiplied by the volume v_{sp} of the spine head (here: $v_{\mathrm{sp}} = \pi h_{\mathrm{sp}} d_{\mathrm{sp}}^2/4$):

$$\begin{aligned} N_{\mathrm{Ca,bkg}} &= \pi\, c\, h_{\mathrm{sp}}\, d_{\mathrm{sp}}^2/4 \\ &= \pi \cdot 70 \cdot 10^{-9} \cdot 6.0 \cdot 10^{23} \cdot 10^{3} \cdot 0.2 \cdot 10^{-6} \cdot (0.4 \cdot 10^{-6})^2/4 = 1.06 \simeq 1 \end{aligned} \tag{6}$$

Thus in the background state there is on average only a single Ca^{2+} ion in the spine head .

(iii) Clearly, the influx of 157 Ca^{2+} ions into the spine head will have a major effect on the concentration since there is only about 1 Ca^{2+} ion there to begin with.

For more info, see Box 2.3 in Sterratt et al.: ”Principles ...”.