Exercise 2: Leaky integrate-and-fire (LIF) neuron

The subthreshold membrane-potential dynamics of the LIF neuron is determined by

$$\tau_m \frac{dV}{dt} = -V + R\,I(t) \tag{1}$$

where R is the neuron membrane resistance, and $\tau_m = RC$ is the membrane time constant (see 8.2 in Sterratt). The resting potential is here chosen to define the zero of the electrical potential. Spikes are emitted whenever the voltage reaches a threshold θ, i.e., if $V(t^*) = \theta$. After each spike emission (spike time denoted t^*), the potential V is reset to zero.

Pen-and-paper problems:

Do not use a computer to solve i) and ii).

(i) For a constant input current $I(t) = I = constant$, and an initial potential $V(t = 0) = 0$, the solution of (1) is given by

$$V(t) = R\,I\,(1 - e^{-t/\tau_m}), \tag{2}$$

provided $V(t) < \theta$. Find an analytical formula for, and sketch (by hand), the firing rate f of the neuron as a function of the input current I (what is typically known as the '$f - I$ curve').
Hint: The firing rate f is by the number of spikes per time unit, and in the present noise-free case with a fixed current input, it is given by the inverse of the inter-spike interval T, that is, $1/T$

(ii) Guess how the shape of the $f - I$ curve would qualitatively change in the presence of (a small amount of) additive noise, i.e., when $I(t) = I$ in equation (1) is replaced by $I(t) = I + noise(t)$.

Python exercises:

(iii) We shall now implement and investigate the LIF model by means of simulations. One possible discretized version of the differential equation (1) reads:

$$\tau_m \frac{V(t_{n+1}) - V(t_n)}{t_{n+1} - t_n} = -V(t_n) + R\,I(t_n) \tag{3}$$

This suggests the following simple numerical scheme, the so called *forward Euler method*, for a numerical solution:

$$V_{n+1} = V_n + \frac{h}{\tau_m}(-V_n + R_m I_n). \tag{4}$$

Here $V_{n+1} \equiv V(t_{n+1})$, $V_n \equiv V(t_n)$, $I_n \equiv I(t_n)$, and $h \equiv t_{n+1} - t_n$.
(a) Make a Python script that implements the LIF dynamics in discrete time $t = 0,\ h,\ 2h,\ \ldots,\ T$ using the numerical forward Euler scheme.
(b) Simulate a LIF neuron with time constant $\tau_m = 10$ ms, membrane resistance $R = 0.04$ GΩ, and threshold voltage $\theta = 15$ mV for a constant input current $I = 400$ pA (both the resting and the reset potential are to zero). Set the initial voltage $V(t = 0)$ to zero. Record and plot the voltage $V(t)$ and the spike times (threshold crossings) for a time resolution of $h = 0.1$ ms and total simulation time $T_{\text{simtime}} = 1000$ ms

(iv) Measure and plot the neuron's $f - I$ curve by repeating the simulation for a range of constant input currents I (for example, $I = 0,\ 10,\ 20,\ \ldots,\ 1000$ pA) and measuring the corresponding firing rates

$$f = \frac{\text{total number of emitted spikes}}{\text{simulation time}} \tag{5}$$

(v) Add a small amount of noise (cf., problem (ii) above) to the input current and investigate how the $f - I$ curves change. Was your inital guess in problem (ii) in correct?