AMSC 466 Notes

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Contents

1 Nonlinear Equations

Our goal is to find real number(s) x_* such that

$$f(x_*) = 0$$

where $f(x)$ is a continuous function.

Theorem 1.1 (Intermediate Value Theorem). *If $f(x)$ is continuous on $[a, b]$ and c is a constant such that*

$$f(a) \leq c \leq f(b) \quad \textit{or} \quad f(b) \leq c \leq f(c)$$

then there exists an $x^* \in [a, b]$ such that $f(x^*) = c$

Lemma 1.2. *A solution to $f(x) = 0$ exists if $a < x < b$ and $f(a)f(b) < 0$.*

Proof. Either $f(a)$ or $f(b)$ is negative while the other must be positive by the assumed inequality. Then employ Intermediate Value Theorem. □

By algorithm:

1. Choose midpoint c between a and b
2. If $f(c)f(a) < 0$ then c is the new b, otherwise c is the new a
3. Assess $|f(c) - 0| < \epsilon$ for accuracy

Example 1.1. *Consider $f(x) = \sin(x) - x + \frac{1}{3}$, $0 \leq x \leq \pi$. Show that there exists $x_* \in (0, \pi)$ s.t. $f(x_*) = 0$. $f(0) = \frac{1}{3} > 0$ $f(\pi) = -\pi + \frac{1}{3} > 0$ By IVT, $\exists x_* \in (0, \pi)$ s.t.$f(x_*) = 0$.*

Example 1.2. *Show that there exists unique $x_* \in (0, 1)$ s.t. $f(x_*) = 0$ where $f(x) = e^{-x} - x$.*

By IVT, x_* exists. To show uniqueness. there are several ways:

1. *Observe that $f'(x) < 0$ over $(0, 1)$, so it is monotincally decreasing and thus 1-1.*
2. *$f(x) = 0 \leftrightarrow g(x) = e^{-x} = x$. This is known as a fixed point.*

Claim Some systematic numeraical methods for solving $f(x) = 0$ are based on solving, instead, the equation $g(x) = x$.

1.1 Introduction to fixed point methods

Suppose $a \leq x \leq b$ and $(x) \in [a,b] \forall x \in [a,b]$. There exists $x_* \in [a,b]$ such that $g(x_*) = x_*$.

Theorem 1.3 (Brower's Fixed Point Theorem)**.** *Suppose $g(x)$ is $C[a,b]$. Also assme that $[a,b]$ is mapped onto itself via g i.e.e $g(x)]in[a,b]\forall x \in [a,b]$. Then, there exists $c \in [a,b]$ s.t. $g(c) = c$.*

Definition 1.1. *The point c of theorem 1 is called "fixed-point" of function $g(x)$*

Proof. Define $f(x) = g(x) - x$. Assume without loss of generality, $a \leq g(a)$ and $b \geq g(b)$.

At $x = a$, $f(a) - a \leq 0$

At $x = b$, $f(b) - b \geq 0$

$\Rightarrow f(a)f(b) \leq 0$. By IVT, $\exists c \in [a,b]$s.t.$f(c) = 0 \leftrightarrow g(c) = c$ $\square$

Example 1.3. *Show that the eq. $f(x) = e^x = x^2 - 3 = 0$ has a solution in $(1,2)$. Taking the log,*

$$x = \ln(x^2 + 3)$$

Note that $g'(x) > 0 \quad \forall x \in [a,b]$ Thus, g is monotonically increasing. Show that $g(1) \leq g(x) \leq g(2)$

Goal Solve $f(x) = 0$ by using (generating) sequence $\{x_n\}_{n=0}^{\infty}$ where $x_n \to x_*$ and $f(x_*) = 0$.

1.2 Review of Basic Methods

1. Bisection Method $x_0 = \frac{a+b}{2}$
2. Newton's Method. If $f(x)$ is continuously differentiable with $f'(x_n) = 0 \forall n$ the scheme is defined by $x_{n+1} = x_n - \frac{f(x_n))}{f'(x_n)}$. Give $x_0 \in \mathbb{R}$
3. Secant Method
   It follows from the scheme of Newtonn's method by replacing $f'(x_n)$ with a finite difference $c = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$
   Scheme: $x_{n+1} = x_n - f(x_n)c$; $n = 0, 1$. Need two points x_0, x_1 as initial guess.

1.3 Rigorous Study of Iterative Methods

Idea start with an initial guess x_0. Then, generate $\{x_n\}$, $x_n \to x_*$ as $n \to \infty$. For a fixed point, we can guess $x_{n+1} = g(x_n)$.

Remark If $x_n rightarrow c$ and g is continuous then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} g(x_n) = g(\lim_{n\to\infty} x_n)$

Definition 1.2. *Suppose $g(x)$ is $C[a,b]$. Then $g(x)$ is called a contraction on $[a,b]$ if there is a real constant L s.t.*

$$0 < L < 1 \quad \text{and} \quad |g(x) - g(y)| < L|x-y| \forall x, y \in [a,b]$$

Remark 1.1. *By the contraction property, interval mapped by g shrinks $\epsilon_1 \le L\epsilon < \epsilon$.*

Definition 1.3 (Secant Method)**.** *The scheme:*

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{1}$$

Proof. Suppose $f'(x) = \beta > 0$ (without loss of generality). Since $f'(x)$ is continuous in I_h for any $\epsilon > 0$ we can choose $\delta > 0$ s.t. for every x in $I = [x_* - \delta, x_* + \delta]$ where $0 \le \delta \le h$ we have

$$|f'(x) - \beta| < \epsilon$$

$\Leftrightarrow \beta - \epsilon < f'(x) < \epsilon + \beta$

Pick $\epsilon = \beta/4$: $3\beta/4 < f'(x) < 5\beta/4 \forall x \in I_\delta$.

Secant Method:

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

By the MVT: $f(x_n) - f(x_{n-1}) = f'(\xi)(x_n - x_{n-1})$ for some ξ_n between x_n and x_0.

$$x_{n+1} - x_* = x_n - x_* - f'(\zeta)(x_n - x_*)\frac{1}{f'(\zeta)} \tag{2}$$

$$x_{n+1} - x_* = x_n - x_* \left(1 - \frac{f'(\zeta_n)}{f'(\xi_n)}\right) \tag{3}$$

$$x_{n+1} - x_* \le |x_n - x_*| \left(1 - \frac{f'(\zeta_n)}{f'(\xi_n)}\right) = \frac{2}{3}|x_n - x_*| \tag{4}$$

$\square$

1.3.1 Heuristic Derivation of $q = \frac{1+\sqrt{5}}{2}$

Define the n-th interation error $e_n = x_n - x_*$. Assume $x_n \to x_*$. SM scheme:

$$e_{n+1} = \frac{e_{n-1}f(x_n) - e_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$e_{n+1} = e_n e_{n-1} \frac{\frac{f(x_1)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}}}{x_n - x_{n-1}} \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{5}$$

We will show $e_{n+1} \approx c e_n e_{n-1}$ for $n >> 1$.

Taylor expansion:

$$f(x_n) = f(x_*) + (x_n - x_*)f'(x_*) + \frac{1}{2}(x_n - x_*)^2 f''(x_*) + \frac{1}{6}(x_n - x_*)^3 f'''(\xi_n) \tag{6}$$

$$\frac{f(x_n)}{e_n} = f'(x_*) + \frac{1}{2}(x_n - x_*)f''(x_*) + \frac{1}{6}(x_n - x_*)^2 f'''(\xi_n) \tag{7}$$

$$\frac{f(x_{n-1})}{e_{n-1}} = f'(x_*) + \frac{1}{2}(x_{n-1} - x_*)f''(x_*) + \frac{1}{6}(x_{n-1} - x_*)^2 f'''(\xi_{n-1}) \tag{8}$$

$$\frac{f(x_n)}{e_n} - \frac{f(x_{n-1})}{e_{n-1}} = \frac{1}{2}(e_n - e_{n-1})f''(x_*) + \frac{1}{6}\left[e_n^2 f'''(\xi_n) - e_{n-1}^2 f'''(\xi_{n-1})\right] \tag{9}$$

$$e_{n+1} = e_n e_{n-1}\left\{\frac{1}{2}f''(x_*) + \frac{1}{6}\frac{e_n^2 f'''(\xi_n) - e_{n-1}^2 f'''(\xi_{n-1})}{e_n - e_{n-1}}\right\}\frac{1}{f'(\sigma_n)} \tag{10}$$

Suppose

$$\frac{1}{6}\frac{e_n^2 f'''(\xi_n) - e_{n-1}^2 f'''(\xi_{n-1})}{e_n - e_{n-1}} \to 0$$

Then for $n > 1$ we can write $e_{n+1} \approx e_n e_{n-1} c$ where $c = \frac{1}{2}f''(x_*)$. Suppose that the limit $A \approx \frac{|e_{n+1}|}{|e_n|^q}$

$$A|e_n|^q \approx |e_{n+1}| \tag{11}$$

$$|e_{n-1}| = A^{-1/q}|e_n|^{-1/q} \tag{12}$$

Combining with the bound,

$$A|e_{n+1}|^q \approx |e_n| A^{-1/q}|e_n|^{-1/q} \tag{13}$$

$$|e_n|^{q-\frac{1}{q}-1} \approx |c| A^{-1-\frac{1}{q}} \tag{14}$$

In order for the approximation to be satisfied (arguing that e_n cannot go to infinity or zero if $q - \frac{1}{q} - 1$ is greater or less than 0, called a scaling argument). So,

$$q - \frac{1}{q} - 1 = 0 \tag{15}$$

$$q^2 - q - 1 = 0 \tag{16}$$

$$q = \frac{1 + \sqrt{5}}{2} \tag{17}$$

Must deal with:

$$\frac{1}{6} \frac{e_n^2 f'''(\xi_n) - e_{n-1}^2 f'''(\xi_{n-1})}{e_n - e_{n-1}} \to 0$$

Write numerator as

$$e_n^2 f'''(\xi_n) - e_{n-1}^2 f'''(\xi_{n-1}) = (e_n^2 - e_{n-1}^2) f'''(\xi_n) + e_n^2 \left[f'''(\xi_n) - f'''(\xi_{n-1}) \right] \tag{18}$$

$$\tag{19}$$

So,

$$\frac{1}{6} \frac{e_n^2 f'''(\xi_n) - e_{n-1}^2 f'''(\xi_{n-1})}{e_n - e_{n-1}} = \frac{1}{6} (e_n + e_{n-1}) f'''(\xi_n) + e_n^2 f^{(4)}(\bar{\xi}_n) \frac{\xi_n - \xi_{n-1}}{x_n - x_{n-1}} \tag{20}$$

A sufficient condition for the last term to approach 0 as $n \to \infty$ is

$$|\xi_n - \xi_{n-1}| \leq C|x_n - x_{n-1}|$$

Then

$$e_{n-1}^2 \leq |e_{n-1}|^2 \frac{C|x_n - x_{n-1}|}{e_n - e_{n-1}}$$

2 Introduction to Polynomial Interpolation

Consider $n+1$ pairs of points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ where x_i are distinct.

Goal is to construct a function $Q(x)$ from a known family of functions, e.g. polynomials, such that

$$Q(x_j) = y_j \quad \forall j \; 0 \leq j \leq n$$

Definition 2.1. *The points $x_0, x_1, \ldots, x_n$ are "interpolation points". The requirement that $Q(x_j) = y_j$ where y_j are "data", is also referred to as "interpolating the data".*

The function $Q(x)$ is the "interpolant" or the "interpolating function".

2.0.1 Notation

In $Q_n(x)$, n indicates the number of interpolation points (minus 1).

Problem: Given distinct points $x_0, \ldots, x_n$. **Find** a polynomial $Q_n(x)$ of the *lowest* possible degree such that

$$Q(x_j) = y_j$$

Theorem 2.1. *If $x_0, x_1, \ldots, x_n \in \mathbb{R}$, then for any data points $y_0, \ldots, y_n \in \mathbb{R}$ there exists a unique polynomial $Q_n(x)$, $\deg(Q_n) \leq n$ such that $Q_n(x_j) = y_j$ for all j.*

Uniqueness of Q_n. If Q_n exists by construction, it has to be unique. Suppose Q_n is not unique. Then there exists another polynomial $P_n(x)$, $\deg(P_n) \leq n$ such that it meets every data point. Consider

$$H_n(x) = P_n(x) - Q_n(x)$$

Then $\deg(H_n) \leq n$; Also, $H_m(x_j) = 0$ for $j = 0, \ldots, n$ so it has $n + 1$ zeros. However, the only polynomial whose degree is less than its number of zeros is the zero polynomial (corollary of the Fundamental Theorem of Algebra). So, $H_n(x) \equiv 0$. □

Existence of Q_n. By induction, start with $n = 0$. Take $Q_0(x) = y_0 = Q(x)$.

Given $Q_{n-1}(x)$, find a $Q_n(x)$ which satsifies $Q_n(x_n) = y_n$. Take

$$Q_n(x) = Q_{n-1}(x) + c(x_n - x_0) \ldots (x_n - x_{n-1})$$

The idea being that other points $j = 0, \ldots, n-1$ do not affect the new function. However, for point x_n,

$$c = \frac{y_n - Q_{n-1}(x_n)}{(x_n - x_0)(x_n - x_1) \ldots (x_n - x_{n-1})}$$

One can also write the entire process By

$$Q_n(x) = a_0 + \sum_{i=1}^{n} a_i \left\{ \prod_{k=0}^{i-1} (x_j - x_0) \right\} \tag{21}$$

□

Vandermonde Determinant

Express $Q_n(x) = \sum_{k=0}^{n} b_k x^k$. Each $Q_n(x_j) = y_j$ is a constraint equation. In matrix form,

$$\begin{pmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix} \tag{22}$$

This system has a unique solution if

$$\det\left(\begin{pmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{pmatrix}\right) \neq 0$$

Lemma 2.2.

$$\mathcal{D}_n = \prod_{i,j=0|i>j}^{n} (x_i - x_j)$$