MATH 405 Notes

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1 Fields

Definition 1.1. *A field is a set F such that $+, *$ where $x, y \in F$ are defined such that $x + y \in F$ and $x * y \in F$.*

The operations satisfy

1. $x + y = y + x$ *(commutative)*
2. $(x + y) + z = x + (y + z)$
3. *there is a* 0*-element where* $0 + x = x$
4. $x \in$ *there exist* $(-x)]inF$ *such that* $x + (-x) = 0$
5. $x * y = y * x$
6. $(xy)z = x(yz)$
7. *there exists* $1 \in F$ *such that* $\forall x \in F, 1 * x = x$
8. $x \in F$ $x \neq 0$ *there exists* $x^{-1} \in F$ *such that* $x * x^{-1} = 1$
9. $x * (y + z) = x * y + x * z$

Lemma 1.1 (Uniqueness of One)**.** *Let F be a field. $\forall x, 1x = x$. If $yx = x$, then $1 = y$.*

Theorem 1.2. $\mathbb{Q}$ *is a field.*

Proof. $\mathbb{Q} \subseteq \mathbb{R}$ so all properties hold, provided that $+, *, 1, 0, -x, x^{-1}$ are defined in $\mathbb{Q}$. Continue with proof of each operation acting on $x, y \in \mathbb{Q}$ □

Example Proof. $\mathbb{C}$ is a field.

1. closed under operations $+, *$ (obvious)
2. show $z, w \in \mathbb{C} \to z + w = w + z$
3. multiplicative inverse

□

1.1 Solving Simulatenous Equations over the Reals

Given $a_{11}x_1 + \ldots + a_{1nx_n} = b_1$ through $a_{m1}x_1 + \ldots + a_{mn}x_n = b_1$. Solve x_i. Homogenous if $b_1 = b_2 = \ldots = 0$. Nonhomogeneous otherwise.

1.2 Gaussian Elmination

For any matrix A

1. multiply row by $C \neq 0$
2. add multiple $C(\text{row } j)$ to row i
3. switch 2 rows

A is in reduced row echelon form if

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1}$$

The first non-zero entry in each row is 1, called "pivots". All entries above/below pivots are 0. Pivots go down left to right. All rows which are 0's at bottom.

Theorem 1.3. *A is any $m \times n$ real matrix. Then, row operations $\rightarrow E$ in row echelon form. Then, E is unique.*

Proof. By Gaussian Elimination:

1. Beginning from top-left, scale for pivot
2. Add a multiple of the previous row to next row to null entry below pivot. Repeat for the remainder of the rows.
3. Repeat steps 1 and 2 for the remainder of rows.

$\square$

Example 1.1 (Solving Homoegeneous Equations)**.** *Given $A\mathbf{x} = \mathbf{0}$*

1. *Use Gaussian Elimination to convert $A \rightarrow E$ (RREF)*
2. $\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ *Denote coumns with pivots.*
3. *Number of free variables in solution is the number of non-pivot columns.*

Solution:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{0}$$

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + 1x_4 + 0x_5 &= 0 \\ x_3 + 2x_4 + 0x_5 &= 0 \\ x_5 &= 0 \end{aligned}$$

Move pivots to right hand side...

$$\begin{aligned} x_1 + 0x_3 + 0x_5 &= -2x_2 + 1x_4 \\ x_3 + 0x_5 &= -2x_4 \end{aligned}$$

Choose x_2, x_4 arrbitrarliy. x_1, x_2, x_3 determined.

$$x_1 = -2x_2 - x_4 \tag{2}$$
$$x_3 = -2x_4 \tag{3}$$
$$x_5 = 0 \tag{4}$$

The general form of the solution (obtaining the basis vectors of the codomain) is obtained from choosing 1 for a single non-pivot while 0 for other non-pivot for each basis vector.

Note that the operations in Gaussian Elimination are well-defined over and field. The same process can be done in other spaces.

Example:

$$\begin{pmatrix} 1 & i & 0 & 1+i \\ 0 & 0 & 1 & 3-i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In order to find the general solution, for each non-pivot $x_i, \ldots, x_{n-r}$, set the variable to 1 while the other non-pivots to 0. Every solution is a linear combination of the resulting vectors. The solution is an $n - r$ subspace of $\mathbb{R}^n$.

Theorem 1.4.

$$A\mathbf{x} = \mathbf{y}$$
$$\begin{pmatrix} A| & y_1 \\ | & y_2 \end{pmatrix} \tag{5}$$

A augmented with RHS is the augmented matrix. Reduce to E using row operations. This has at least one solution if and only if the 0-rows have y-term also 0.

If this holds, find one solution using same algorithm from before. The set of solutions is:

$$\{x_p + x_n | x_n \text{ runs over all solutions of the associated homogeneous equation}\}$$

Definition 1.2. *Product of matrices. Given $A \times B$*

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$$

such that $c_{i,j}$ are the entries of C, the product of A and B.

Theorem 1.5.

$$(AB)C = A(BC) \tag{6}$$
$$(A+B)C = AC + BC \tag{7}$$
$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \tag{8}$$
$$I_m A = A \tag{9}$$
$$A I_n = A \tag{10}$$

Corollary 1.1. *A square matrix with either a left or right inverse is invertible.*

Corollary 1.2. *Let $A = A_1 A_2 \dots A_k$ where $A = A_1 A_2 \dots A_k$ are $n \times n$ (square) matrices. Then A is invertible $\leftrightarrow$ each A_j is invertible.*

Theorem 1.6. *For an $n \times n$ matrix A, the following are equivalent:*

1. *A is invertible*

2. *A is a row-equivalent to the $n \times n$ identity matrix.*

3. *A is a product of elementary matrices*

4. *The homogeneous system $AX = 0$ has only the trivial solution $X = 0$*

5. *The system of equation $AX = Y$ has a solution X for each $n \times 1$ matrix Y*

Corollary 1.3. *If A is invertible $n \times n$ matrix and if a sequence of elementary row operations reduces A to the identity, then that same sequence of operations when applied to I yeilds A^{-1}.*

V, $S = \{W\} = \{W_1, W_2, \ldots, W_n\}$ $W_n \subset V$ set of subspaces. e.g. $\mathbb{R}^3 = \{x - axis, y - axis, z - axis, \{(x, y, z)|x + y + z = 0\}\} \leftarrow \cap W = (0, 0, 0)$

Definition 1.3.

$$\mathrm{Span}(S) = \left\{\sum_{i=1}^{n} \alpha_i v_i | v_i \in S, a_i \in F\right\}$$

Proposition 1.1. 1. $\mathrm{Span}(S)$ *is a subspace*

2. $\mathrm{Span}(S) = \cap W$ *where W all subspaces*

Definition 1.4.

$$\{v_1, \ldots, v_n\}$$

is linearly dependent if

$$v_j = \{\sum_{i=1,\ldots,n|i\neq j} a_i v_i\}$$

(ones is a linear combination of the others) ex: $\{(1, -1, 0), (1, 0, -1), (0, 1, -1)\}$ is linearly dependent : $v_2 = v_1 + v_3$

Lemma 1.7. *$\{v_1, v_2, \ldots, v_n\}$ is linearly dependent if and only if there is a relation $\sum_{i=1}^{n} a_i v_i = 0$ where some $a_i \neq 0$*

Definition 1.5. *$\{v_1, \ldots, v_n\}$ is linearly independent if not linearly dependent*

Lemma 1.8. 1. *$\{v_1, \ldots, v_n\}$ is linearly dependent if $\exists a_1, \ldots, a_n$ not all 0, $a_i v_i + \ldots + a_n v_n = 0$*

2. *$\{v_1, \ldots, v_n\}$ is linearly independent if $a_1 v_1 + \ldots + a_1 v_n = 0 \to a_i = 0\, \forall i$*

Definition 1.6. *Given vector space V and $\{v_1, \ldots, v_n\} = S$. S is a basis of V if:*

1. $\mathrm{Span}(S) = V$ *i.e. every vector $v \in V$ can be written as a linear combination of vectors in S. (Enough vectors)*

2. S is linearly independent (Not too many vectors)

Lemma 1.9. *V, S. S is a basis $\Leftrightarrow v \in V \Rightarrow v = \sum_{i=1}^{n} a_i v_i$ uniquely*

Theorem 1.10. *V vector space. B_1, B_2 are bases of V.*

$$|B_1| = |B_2| \tag{11}$$

(Also applies to $|\cdot| = \infty$)

Definition 1.7. *V, B=basis. Then dimension of V is $|B|$. This is well-defined by the previous theorem.*

Theorem 1.11. *V spanned by $v_1, \ldots, v_n$. Suppose $\{w_1, \ldots, w_n\}$. If $w_1, \ldots, w_n$ linearly independent, then $n \leq m$. Equivalently, if $n > m$ then $\{w_1, \ldots, w_n\}$ is linearly dependent.*

Theorem 1.12. *V, B_1, B_2 bases, finite $|B_1| = |B_2|$.*

Proof. B_1 spans, B_2 linearly independent $\implies |B_2| \leq |B_1$
B_2 spans, B_1 linearly independent $\implies |B_1| \leq |B_2$ □

$$\dim(\mathbb{C}^n, \mathbb{C}) = n$$
$$\dim(\mathbb{C}^n, \mathbb{R}) = 2n$$

Theorem 1.13.

1. *$\{v_1, \ldots, v_n\}$ finite, linearly dependent. Then $\exists v_{m+1} \in V$ s.t. $\{v_1, \ldots, v_m, v_{m+1}\}$ is linearly independent.*

2. *$\{v_1, \ldots, v_m\}$ linearly dependent $\exists \text{subset} S \subset v_1, \ldots, v_m$ s.t. S is linearly independent and $\text{Span}(\{v_1, \ldots, v_m\}) = S$*

Corollary 1.4. *$\{v_1, v_2, \ldots, v_m\}$ finite, lin. ind, V finite dim. then $\exists w_1, \ldots, w_r$ s.t. $\{v_1, v_2, \ldots, v_m\} \cup \{w_1, w_2, \ldots, w_r\}$is a basis.*

Lemma 1.14. *V/F, $B = \{v_1, v_2, \ldots, v_n\}$ is bijective with F^n*

This bijection requires a choice of basis.

Definition 1.8. *(V, B), V vector space over F, $B = \{v_1, v_2, \ldots, v_n\}$. $v \in V$ can be written as a coordinate of V, $(a_1, \ldots, a_n)$ where*

$$\sum a_i v_i = v$$

2 Linear Transformation

Definition 2.1. *V, W vector spaces over field F. A linear transformation T from V to W is a function (mapping) $T : V \to W$.*

Such a linear transformation preserves:

1. $v, w \in V \implies T(v + w) = Tv + Tw$
2. $c \in F, v \in V \implies T(cv) = cT(v)$

Definition 2.2. *A homomorphism from V to W is a linear transformation from V to W.*

Definition 2.3. *An isomorphism is a homomorphism that is both 1-1 and onto.*

Lemma 2.1. *Vector space V, basis $B = \{v_1, v_2, \ldots, v_n\}$, another vector space W and map $T : V \to W$. Then, T is a unique transformation such that $T(v_1), T(v_2), \ldots, T(v_n)$.*