

1 Chapter 1: Tools for Analysis

Definition 1.1. For a set of real numbers S that is bounded above, there exist a number such that all upper bounds of S are greater than it. This number is the supremum of the set S .

Theorem 1.1. (Archimedean Principle)

1. Given any number ϵ , there exist a natural number n such that $n > \epsilon$
2. $(\forall \epsilon \in \mathbb{R})(\exists n \in \mathbb{N}) \frac{1}{n} < \epsilon$

Theorem 1.2. The rationals are dense in \mathbb{R}

2 Chapter 2: Convergent Sequences

Definition 2.1. Given a sequence of numbers a_n , we say that a_n converges to the number a if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$

$$|a_n - a| < \epsilon$$

Equivalently,

$$\lim_{n \rightarrow \infty} a_n = a$$

Theorem 2.1. (Properties of Limits) Suppose a_n and b_n are sequences that converge. Then,

1. (Linearity) $\alpha, \beta \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \alpha a_n + \beta b_n = \alpha \lim_{n \rightarrow \infty} a_n + \beta \lim_{n \rightarrow \infty} b_n$$

2. (Product Rule)

$$\lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

3. (Quotient Rule) Suppose also that $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} b_n \neq 0$. Then,

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$$

Definition 2.2. A sequence a_n is bounded if there exists nonnegative number M such that

$$|a_n| \leq M$$

for all n .

Theorem 2.2. Every convergent sequence is bounded

Theorem 2.3. A set S is dense in $\mathbb{R} \iff$ every number x is the limit of a sequence of S .

Theorem 2.4. (*Comparison Lemma*) Suppose $\lim_{n \rightarrow \infty} b_n = b$. A sequence $\lim_{n \rightarrow \infty} a_n$ converges to a number a if for some $C \geq 0$ and $N \in \mathbb{N}$,

$$|a_n - a| \leq C|b_n - b|$$

for $n \geq N$

Definition 2.3. A subset S of \mathbb{R} is closed if a_n is a sequence in S that converges in S .

Theorem 2.5. Every interval $[a, b]$ over \mathbb{R} is closed.

Definition 2.4. A sequence is said to be monotone if

$$a_n \geq a_{n-1}$$

(increasing) or

$$a_n \leq a_{n-1}$$

(decreasing) for all n .

Theorem 2.6. A monotone sequence converges \iff it is bounded

Definition 2.5. Consider a sequence a_n . A subsequence is a sequence b_k such that for some sequence of strictly increasing natural numbers $n_1 < n_2 < \dots$,

$$b_k = a_{n_k}$$

Theorem 2.7. If a_n converges to a , then every a_{n_k} converges to a

Theorem 2.8. (*Sequential Compactness*) Every sequence in interval $[a, b]$ has a subsequence that converges to a number in $[a, b]$.

3 Chapter 3: Continuous Functions

Definition 3.1. A function $f : D \rightarrow \mathbb{R}$ is said to be continuous at $x_0 \in D$ if whenever $\{x_n\}$ is a sequence that converges to x_0 , the sequence $\{f(x_n)\}$ converges to $f(x_0)$.

A function is continuous if it is continuous for all $x_0 \in D$.

Theorem 3.1. (*Properties of continuity*)

1. Products, quotients, and linear combinations of functions f and g are continuous.

(a) $f + g : D \rightarrow \mathbb{R}$ is continuous

(b) $f \cdot g : D \rightarrow \mathbb{R}$ is continuous

(c) $f/g : D \rightarrow \mathbb{R}$ is continuous given that g is not 0 anywhere in D

2. Compositions of continuous functions (provided that the domain and images are consistent) are continuous

$$f \circ g(x)$$

is continuous

Proof. Properties of convergent sequences. \square

Theorem 3.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. There exist a maximum and a minimum of the function.

Proof.

1. f is bounded above – Suppose not bounded. Consider a sequence such that $f(x_n) > n$. x_{n_k} converges but f continuous.
2. $\sup f(D)$ is a functional value – Consider a sequence such that $f(x_n) > c - \frac{1}{n}$. x_{n_k} converges to x_0 and f continuous so $f(x_0) = c$.

\square

Theorem 3.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. For any number $c \in (a, b)$, there exist a point $x_0 \in [a, b]$ such that

$$f(x_0) = c$$

Proof. Bisection method: define $m_n = \frac{a_n + b_n}{2}$. Consider first $f(a) < f(b)$. If $c \geq f(m_n)$, set $a_{n+1} = m_n$ (alternatively, same with b_{n+1}). Now, $a_n \leq a_{n+1} < b_{n+1} \leq b_n$. Also, $b_n - a_n = \frac{b-a}{2^{n-1}}$ so it converges to 0. By Nested Interval, $\{a_n\}$ and $\{b_n\}$ converge to a point x_0 . $f(x_0) \leq c$ since $f(a_n) \leq c$ and $f(x_0) \geq c$ since $f(b_n) \geq c$, so $f(x_0) = c$. \square

Definition 3.2. An interval I over \mathbb{R} is a set of numbers such that any $x < y$ where $x, y \in I$ defines an interval $[x, y]$ in I .

Theorem 3.4. Let $f : I \rightarrow \mathbb{R}$ function where I is an interval. Then, the image of f is also an interval.

Proof. Definition of interval and Intermediate Value Theorem. \square

Theorem 3.5. Let $f : I \rightarrow \mathbb{R}$ be a function where I is an interval. Then, f is continuous if its image $f(I)$ is an interval.

Definition 3.3. $f : D \rightarrow \mathbb{R}$ is said to converge uniformly in D if whenever $\{u_n\}$ and $\{v_n\}$ are two sequences in D such that

$$\lim_{n \rightarrow \infty} |u_n - v_n| = 0$$

then

$$\lim_{n \rightarrow \infty} |f(u_n) - f(v_n)| = 0$$

Definition 3.4. (*Epsilon-Delta Criterion*) A function is said to satisfy the $\epsilon - \delta$ criterion at a point x_0 , if $\forall \epsilon > 0$ there exists $\delta > 0$ such that,

$$|f(x) - f(x_0)| < \epsilon \quad \text{if} \quad |x - x_0| < \delta$$

($\epsilon - \delta$ Over a Domain) A function is said to satisfy the $\epsilon - \delta$ over a domain if $\forall \epsilon > 0$ there exists $\delta > 0$ such that for all $x \in D$,

$$|f(u) - f(v)| < \epsilon \quad \text{if} \quad |u - v| < \delta$$

Theorem 3.6.

1. A function satisfies the $\epsilon - \delta$ criterion at a point $x_0 \iff$ it is continuous at x_0 .
2. A function satisfies the $\epsilon - \delta$ criterion over a domain \iff it is uniformly continuous over the domain.

4 Chapter 9: Sequences and Series of Functions

Definition 4.1. A sequence is said to be Cauchy if for all $\epsilon > 0 \exists N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon$$

for all $n, m \geq N$

Theorem 4.1. A sequence converges \iff it is Cauchy.

Proof.

1. (\Rightarrow) –

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a|$$
2. (\Leftarrow) – A Cauchy sequence is bounded since $|a_n| \leq |a_n| + |a_n - a_N| = |a_n| + 1$ where N is chosen with Cauchy assumption. By sequential compactness, a_{n_k} converges to a number a . Then there exist $N' = \max\{N_{Cauchy}, N_{sub}\}$ such that for $n \geq N'$,

$$|a_n - a| \leq |a_n - a_{n_k}| + |a - a_{n_k}|$$

□

Theorem 4.2. (*Necessary condition for Series Convergence*) Suppose $\sum_{i=1}^{\infty} a_i$ converges. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Theorem 4.3. Suppose $\{a_n\}$ is nonnegative. $\sum_{i=1}^{\infty} a_i$ converges $\iff S_n$ is bounded

Theorem 4.4. (*Comparison Test*) Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences. And $a_n \leq b_n$. Then,

1. $\sum_{i=1}^{\infty} a_i$ converges if $\sum_{i=1}^{\infty} b_i$ converges. (Example: $\frac{1}{k2^k} \leq \frac{1}{2^k}$)
2. $\sum_{i=1}^{\infty} a_i$ diverges only if $\sum_{i=1}^{\infty} b_i$ diverges. (Example: $\frac{1}{\sqrt{k}} \geq \frac{1}{k}$)

Theorem 4.5. (Integral Test) Suppose a sequence $\{a_k\}$ and a function f that has the property

$$f(k) = a_k$$

and is continuous and monotonically decreasing. for all k . Then, $\{a_k\}$ converges \iff the sequence $\{\int_1^n f(x) dx\}$ is bounded.

Theorem 4.6. (p -Test) For a positive number p , the series

$$\sum_{i=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$

Theorem 4.7. (Alternating Series) Suppose $\{a_k\}$ is a nonnegative sequence, monotonically decreasing, and converges to 0. The series

$$\sum_{i=1}^{\infty} (-1)^{k+1} a_k$$

converges.

Theorem 4.8. If $\sum_{i=1}^{\infty} |a_i|$ converges, then $\sum_{i=1}^{\infty} a_i$ converges.

Theorem 4.9. Given the series $\sum_{i=1}^{\infty} a_i$. Suppose there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_{n+1}| \leq r|a_n| \quad \text{for } 0 \leq r < 1$$

Then, $\sum_{i=1}^{\infty} a_i$ converges absolutely.

Proof. For $n \geq N$,

$$|a_1| + |a_2| + \dots + |a_{N+k}| \leq |a_1| + |a_2| + \dots + |a_N| \frac{1 - r^{k+1}}{1 - r}$$

so the partial sum is bounded for every index n . □

Theorem 4.10. (Ratio Test) Suppose for the sequence $\sum_{i=1}^{\infty} a_i$,

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = l$$

Then, $\sum_{i=1}^{\infty} a_i$ converges absolutely if $l < 1$ and diverges if $l > 1$.

4.1 Sequence of Functions

Theorem 4.11. (*Differentiability of Limit Function*)

Suppose $\{f_n : I \rightarrow \mathbb{R}\}$ is a sequence of continuously differentiable functions and $\{f'_n\}$ the derived sequence. If

1. $\{f_n : I \rightarrow \mathbb{R}\}$ converges pointwise to f
2. $\{f'_n\}$ converges uniformly to a function g

Then,

$$f'(x) = g(x) \quad \forall x \in I$$