1 Chapter 1: Tools for Analysis

Definition 1.1. *For a set of real numbers S that is bounded above, there exist a number such that all upper bounds of S are greater than it. This number is the supremum of the set S.*

Theorem 1.1. *(Archimedean Principle)*

1. *Given any number ϵ, there exist a natural number n such that $n > \epsilon$*
2. $(\forall \epsilon \in \mathbb{R})(\exists n \in \mathbb{N}) \frac{1}{n} < \epsilon$

Theorem 1.2. *The rationals are dense in $\mathbb{R}$*

2 Chapter 2: Convergent Sequences

Definition 2.1. *Given a sequence of numbers a_n, we say that a_n converges to the number a if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$*

$$|a_n - a| < \epsilon$$

Equivalently,

$$\lim_{n \to \infty} a_n = a$$

Theorem 2.1. *(Properties of Limits) Suppose a_n and b_n are sequences that converge. Then,*

1. *(Linearity) $\alpha, \beta \in \mathbb{R}$*

$$\lim_{n \to \infty} \alpha a_n + \beta b_n = \alpha \lim_{n \to \infty} a_n + \beta \lim_{n \to \infty} b_n$$

2. *(Product Rule)*

$$\lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$$

3. *(Quotient Rule) Suppose also that $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} b_n \neq 0$. Then,*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}$$

Definition 2.2. *A sequence a_n is bounded if there exists nonnegative number M such that*

$$|a_n| \leq M$$

for all n.

Theorem 2.2. *Every convergent sequence is bounded*

Theorem 2.3. *A set S is dense in $\mathbb{R}$ $\iff$ every number x is the limit of a sequence of S.*

Theorem 2.4. *(Comparison Lemma) Suppose* $\lim_{n\to\infty} b_n = b$*. A sequence* $\lim_{n\to\infty} a_n$ *converges to a number* a *if for some* $C \geq 0$ *and* $N \in \mathbb{N}$*,*

$$|a_n - a| \leq C|b_n - b|$$

for $n \geq N$

Definition 2.3. *A subset* S *of* $\mathbb{R}$ *is closed if* a_n *is a sequence in* S *that converges in* S*.*

Theorem 2.5. *Every interval* $[a, b]$ *over* $\mathbb{R}$ *is closed.*

Definition 2.4. *A sequence is said to be monotone if*

$$a_n \geq a_{n-1}$$

(increasing) or

$$a_n \leq a_{n-1}$$

(decreasing) for all n*.*

Theorem 2.6. *A monotone sequence converges* $\iff$ *it is bounded*

Definition 2.5. *Consider a sequence* a_n*. A subsequence is a sequence* b_k *such that for some sequence of strictly increasing natural numbers* $n_1 < n_2 < \ldots$*,*

$$b_k = a_{n_k}$$

Theorem 2.7. *If* a_n *converges to* a*, then every* a_{n_k} *converges to* a

Theorem 2.8. *(Sequential Compactness) Every sequence in interval* $[a, b]$ *has a subsequence that converges to a number in* $[a, b]$*.*

3 Chapter 3: Continuous Functions

Definition 3.1. *A function* $f : D \to \mathbb{R}$ *is said to be continuous at* $x_0 \in D$ *if whenever* $\{x_n\}$ *is a sequence that converges to* x_0*, the sequence* $\{f(x_n)\}$ *converges to* $f(x_0)$*.*

A function is continuous if it is continuous for all $x_0 \in D$*.*

Theorem 3.1. *(Properties of continuity)*

1. *Products, quotients, and linear combinations of functions* f *and* g *are continuous.*
   - (a) $f + g : D \to \mathbb{R}$ *is continuous*
   - (b) $f \cdot g : D \to \mathbb{R}$ *is continuous*
   - (c) $f/g : D \to \mathbb{R}$ *is continuous given that* g *is not 0 anywhere in* D

2. *Compositions of continuous functions (provided that the domain and images are consistent) are continuous*

$$f \circ g(x)$$

is continuous

Proof. Properties of convergent sequences. □

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. There exist a maximum and a minimum of the function.*

Proof.

1. f is bounded above – Suppose not bounded. Consider a sequence such that $f(x_n) > n$. x_{n_k} converges but f continuous.

2. $\sup f(D)$ is a functional value – Consider a sequence such that $f(x_n) > c - \frac{1}{n}$. x_{n_k} converges to x_0 and f continuous so $f(x_0) = c$.

□

Theorem 3.3. *Let $f[a, b] \to \mathbb{R}$ be a continuous function. For any number $c \in (a, b)$, there exist a point $x_0 \in [a, b]$ such that*

$$f(x_0) = c$$

Proof. Bisection method: define $m_n = \frac{a_n + b_n}{2}$. Consider first $f(a) < f(b)$. If $c \geq f(m_n)$, set $a_{n+1} = m_n$ (alternatively, same with b_{n+1}). Now, $a_n \leq a_{n+1} < b_{n+1} \leq b_n$. Also, $b_n - a_n = \frac{b-a}{2^{n-1}}$ so it converges to 0. By Nested Interval, $\{a_n\}$ and $\{b_n\}$ converge to a point x_0. $f(x_0) \leq c$ since $f(a_n) \leq c$ and $f(x_0) \geq c$ since $f(b_n) \geq c$, so $f(x_0) = c$. □

Definition 3.2. *An interval I over $\mathbb{R}$ is a set of numbers such that any $x < y$ where $x, y \in I$ defines an interval $[x, y]$ in I.*

Theorem 3.4. *Let $f : I \to \mathbb{R}$ function where I is an interval. Then, the image of f is also an interval.*

Proof. Definition of interval and Intermediate Value Theorem. □

Theorem 3.5. *Let $f : I \to \mathbb{R}$ be a function where I is an interval. Then, f is continuos if its image $f(I)$ is an interval.*

Definition 3.3. *$f : D \to R$ is said to converge uniformly in D if whenever $\{u_n\}$ and $\{v_n\}$ are two sequences in D such that*

$$\lim_{n\to\infty} |u_n - v_n| = 0$$

then

$$\lim_{n\to\infty} |f(u_n) - f(v_n)| = 0$$

Definition 3.4. *(Epsilon-Delta Criterion) A function is said to satisfy the* $\epsilon-\delta$ *criterion at a point* x_0*, if* $\forall\epsilon > 0$ *there exists* $\delta > 0$ *such that,*

$$|f(x) - f(x_0)| < \epsilon \quad \text{if} \quad |x - x_0| < \delta$$

($\epsilon - \delta$ *Over a Domain) A function is said to satisfy the* $\epsilon - \delta$ *over a domain if* $\forall\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x \in D$*,*

$$|f(u) - f(v)| < \epsilon \quad \text{if} \quad |u - v| < \delta$$

Theorem 3.6.

1. *A function satisfies the* $\epsilon - \delta$ *criterion at a point* $x_0 \iff$ *it is continuous at* x_0*.*

2. *A function satisfies the* $\epsilon - \delta$ *criterion over a domain* $\iff$ *it is uniformly continuous over the domain.*

4 Chapter 8: Taylor Polynomials

Definition 4.1. *Suppose* $f : I \to \mathbb{R}$ *has* n *derivatives. The* n*th Taylor Polynomial around the point* $x_0 \in I$ *is defined by*

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Remark 4.1. *A Taylor Series Expansion of a function* $f : I \to \mathbb{R}$ *holds at* x *if and only if*

$$\lim_{n\to\infty}[f(x) - p_n(x)] = 0$$

5 Chapter 9: Sequences and Series of Functions

Definition 5.1. *A sequence is said to be Cauchy if for all* $\epsilon > 0$ $\exists N \in \mathbb{N}$ *such that*

$$|a_n - a_m| < \epsilon$$

for all $n, m \geq N$

Theorem 5.1. *A sequence converges* $\iff$ *it is Cauchy.*

Proof.

1. $(\Rightarrow)$ –

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a|$$

2. $(\Leftarrow)$ – A Cauchy sequence is bounded since $|a_n| \leq |a_n| + |a_n - a_N| = |a_n| + 1$ where N is chosen with Cauchy assumption. By sequential compactness, a_{n_k} converges to a number a. Then there exist $N' = \max\{N_{Cauchy}, N_{sub}\}$ such that for $n \geq N'$,

$$|a_n - a| \leq |a_n - a_{n_k}| + |a - a_{n_k}|$$

□

Theorem 5.2. *(Necessary condition for Series Convergence) Suppose* $\sum_{i=1}^{\infty} a_i$ *converges. Then,* $\lim_{n\to\infty} a_n = 0$.

Theorem 5.3. *Suppose* $\{a_n\}$ *is nonnegative.* $\sum_{i=1}^{\infty} a_i$ *converges* $\iff S_n$ *is bounded*

Theorem 5.4. *(Comparison Test) Suppose* $\{a_n\}$ *and* $\{b_n\}$ *are two sequences. And* $a_n \le b_n$*. Then,*

1. $\sum_{i=1}^{\infty} a_i$ *converges if* $\sum_{i=1}^{\infty} b_i$ *converges. (Example:* $\frac{1}{k2^k} \le \frac{1}{2^k}$*)*
2. $\sum_{i=1}^{\infty} a_i$ *diverges only if* $\sum_{i=1}^{\infty} b_i$ *diverges. (Example:* $\frac{1}{\sqrt{k}} \ge \frac{1}{k}$*)*

Theorem 5.5. *(Integral Test) Suppose a sequence* $\{a_k\}$ *and a function* f *that has the property*

$$f(k) = a_k$$

and is continuous and montonically decreasing. for all k*. Then,* $\{a_k\}$ *converges* $\iff$ *the sequence* $\{\int_1^n f(x)\,dx\}$ *is bounded.*

Theorem 5.6. *(p-Test) For a positive number* p*, the series*

$$\sum_{i=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$

Theorem 5.7. *(Alternating Series) Suppose* $\{a_k\}$ *is a nonnegative sequence, monotonically decreasing, and converges to 0. The series*

$$\sum_{i=1}^{\infty} (-1)^{k+1} a_k$$

converges.

Theorem 5.8. *If* $\sum_{i=1}^{\infty} |a_i|$ *converges, then* $\sum_{i=1}^{\infty} a_i$ *converges.*

Theorem 5.9. *Given the series* $\sum_{i=1}^{\infty} a_i$*. Suppose there exists* $N \in \mathbb{N}$ *such that for all* $n \ge N$*,*

$$|a_{n+1}| \le r|a_n| \quad \textit{for} \quad 0 \le r < 1$$

Then, $\sum_{i=1}^{\infty} a_i$ *converges absolutely.*

Proof. For $n \ge N$,

$$|a_1| + |a_2| + \ldots + |a_{N+k}| \le |a_1| + |a_2| + \ldots + |a_N| \frac{1 - r^{k+1}}{1 - r}$$

so the partial sum is bounded for every index n. □

Theorem 5.10. *(Ratio Test) Suppose for the sequence* $\sum_{i=1}^{\infty} a_i$*,*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l$$

Then, $\sum_{i=1}^{\infty} a_i$ *converges absolutely if* $l < 1$ *and diverges if* $l > 1$.

5.1 Sequence of Functions

Theorem 5.11. *(Integrability of Limit Function)*
Suppose $\{f_n : [a,b] \to \mathbb{R}\}$ *is a sequence of integrable functions. If* $\{f_n\}$ *converges uniformly to* f*,* f *is also integrable and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Theorem 5.12. *(Differentiability of Limit Function)*
Suppose $\{f_n : I \to \mathbb{R}\}$ *is a sequence of continuously differentiable functions and* $\{f_n'\}$ *the derived sequence. If*

1. $\{f_n : I \to \mathbb{R}\}$ *converges pointwise to* f
2. $\{f_n'\}$ *converges uniformly to a function* g

Then,

$$f'(x) = g(x) \quad \forall x \in I$$