1 Chapter 1: Tools for Analysis

Definition 1.1 *For a set of real numbers S that is bounded above, there exist a number such that all upper bounds of S are greater than it. This number is the supremum of the set S.*

Theorem 1.1 *(Archimedean Principle)*

1. *Given any number ϵ, there exist a natural number n such that $n > \epsilon$*
2. $(\forall \epsilon \in \mathbb{R})(\exists n \in \mathbb{N}) \frac{1}{n} < \epsilon$

Theorem 1.2 *The rationals are dense in $\mathbb{R}$*

2 Chapter 2: Convergent Sequences

Definition 2.1 *Given a sequence of numbers a_n, we say that a_n converges to the number a if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$*

$$|a_n - a| < \epsilon$$

Equivalently,

$$\lim_{n\to\infty} a_n = a$$

Theorem 2.1 *(Properties of Limits) Suppose a_n and b_n are sequences that converge. Then,*

1. *(Linearity) $\alpha, \beta \in \mathbb{R}$*

$$\lim_{n\to\infty} \alpha a_n + \beta b_n = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n$$

2. *(Product Rule)*

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

3. *(Quotient Rule) Suppose also that $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n \neq 0$. Then,*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

Definition 2.2 *A sequence a_n is bounded if there exists nonnegative number M such that*

$$|a_n| \leq M$$

for all n.

Theorem 2.2 *Every convergent sequence is bounded*

Theorem 2.3 *A set S is dense in $\mathbb{R} \iff$ every number x is the limit of a sequence of S.*

Theorem 2.4 *(Comparison Lemma) Suppose $\lim_{n\to\infty} b_n = b$. A sequence $\lim_{n\to\infty} a_n$ converges to a number a if for some $C \geq 0$ and $N \in \mathbb{N}$,*

$$|a_n - a| \leq C|b_n - b|$$

for $n \geq N$

Definition 2.3 *A subset S of $\mathbb{R}$ is closed if a_n is a sequence in S that converges in S.*

Theorem 2.5 *Every interval $[a, b]$ over $\mathbb{R}$ is closed.*

Definition 2.4 *A sequence is said to be monotone if*

$$a_n \geq a_{n-1}$$

(increasing) or

$$a_n \leq a_{n-1}$$

(decreasing) for all n.

Theorem 2.6 *A monotone sequence converges $\Longleftrightarrow$ it is bounded*

Definition 2.5 *Consider a sequence a_n. A subsequence is a sequence b_k such that for some sequence of strictly increasing natural numbers $n_1 < n_2 < \ldots$,*

$$b_k = a_{n_k}$$

Theorem 2.7 *If a_n converges to a, then every a_{n_k} converges to a*

Theorem 2.8 *(Sequential Compactness) Every sequence in interval $[a, b]$ has a subsequence that converges to a number in $[a, b]$.*

3 Chapter 3: Continuous Functions

Definition 3.1 *A function $f : D \to \mathbb{R}$ is said to be continuous at $x_0 \in D$ if whenever $\{x_n\}$ is a sequence that converges to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.*

A function is continuous if it is continuous for all $x_0 \in D$.

Theorem 3.1 *(Properties of continuity)*

1. *Products, quotients, and linear combinations of functions f and g are continuous.*
2. *Compositions of continuous functions (provided that the domain and images are consistent) are continuous*

Definition 3.2 *(Epsilon-Delta Convergence)*