1 Chapter 1: Tools for Analysis

Definition 1.1. *For a set of real numbers S that is bounded above, there exist a number such that all upper bounds of S are greater than it. This number is the supremum of the set S.*

Theorem 1.1. *(Archimedean Principle)*

1. *Given any number ϵ, there exist a natural number n such that $n > \epsilon$*
2. $(\forall \epsilon \in \mathbb{R})(\exists n \in \mathbb{N}) \frac{1}{n} < \epsilon$

Theorem 1.2. *The rationals are dense in $\mathbb{R}$*

2 Chapter 2: Convergent Sequences

Definition 2.1. *Given a sequence of numbers a_n, we say that a_n converges to the number a if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$*

$$|a_n - a| < \epsilon$$

Equivalently,

$$\lim_{n \to \infty} a_n = a$$

Theorem 2.1. *(Properties of Limits) Suppose a_n and b_n are sequences that converge. Then,*

1. *(Linearity) $\alpha, \beta \in \mathbb{R}$*

$$\lim_{n \to \infty} \alpha a_n + \beta b_n = \alpha \lim_{n \to \infty} a_n + \beta \lim_{n \to \infty} b_n$$

2. *(Product Rule)*

$$\lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n$$

3. *(Quotient Rule) Suppose also that $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} b_n \neq 0$. Then,*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n}$$

Definition 2.2. *A sequence a_n is bounded if there exists nonnegative number M such that*

$$|a_n| \leq M$$

for all n.

Theorem 2.2. *Every convergent sequence is bounded*

Theorem 2.3. *A set S is dense in $\mathbb{R} \iff$ every number x is the limit of a sequence of S.*

Theorem 2.4. *(Comparison Lemma) Suppose* $\lim_{n\to\infty} b_n = b$. *A sequence* $\lim_{n\to\infty} a_n$ *converges to a number* a *if for some* $C \geq 0$ *and* $N \in \mathbb{N}$,

$$|a_n - a| \leq C|b_n - b|$$

for $n \geq N$

Definition 2.3. *A subset* S *of* $\mathbb{R}$ *is closed if* a_n *is a sequence in* S *that converges in* S.

Theorem 2.5. *Every interval* $[a, b]$ *over* $\mathbb{R}$ *is closed.*

Definition 2.4. *A sequence is said to be monotone if*

$$a_n \geq a_{n-1}$$

(increasing) or

$$a_n \leq a_{n-1}$$

(decreasing) for all n.

Theorem 2.6. *A monotone sequence converges* $\iff$ *it is bounded*

Definition 2.5. *Consider a sequence* a_n. *A subsequence is a sequence* b_k *such that for some sequence of strictly increasing natural numbers* $n_1 < n_2 < \ldots$,

$$b_k = a_{n_k}$$

Theorem 2.7. *If* a_n *converges to* a, *then every* a_{n_k} *converges to* a

Theorem 2.8. *(Sequential Compactness) Every sequence in interval* $[a, b]$ *has a subsequence that converges to a number in* $[a, b]$.

3 Chapter 3: Continuous Functions

Definition 3.1. *A function* $f : D \to \mathbb{R}$ *is said to be continuous at* $x_0 \in D$ *if whenever* $\{x_n\}$ *is a sequence that converges to* x_0, *the sequence* $\{f(x_n)\}$ *converges to* $f(x_0)$.

A function is continuous if it is continuous for all $x_0 \in D$.

Theorem 3.1. *(Properties of continuity)*

1. *Products, quotients, and linear combinations of functions* f *and* g *are continuous.*
   (a) $f + g : D \to \mathbb{R}$ *is continuous*
   (b) $f \cdot g : D \to \mathbb{R}$ *is continuous*
   (c) $f/g : D \to \mathbb{R}$ *is continuous given that* g *is not 0 anywhere in* D

2. *Compositions of continuous functions (provided that the domain and images are consistent) are continuous*

$$f \circ g(x)$$

is continuous

Proof. Properties of convergent sequences. □

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. There exist a maximum and a minimum of the function.*

Proof.

1. f is bounded above – Suppose not bounded. Consider a sequence such that $f(x_n) > n$. x_{n_k} converges but f continuous.

2. $\sup f(D)$ is a functional value – Consider a sequence such that $f(x_n) > c - \frac{1}{n}$. x_{n_k} converges to x_0 and f continuous so $f(x_0) = c$.

□

Theorem 3.3. *Let $f[a, b] \to \mathbb{R}$ be a continuous function. For any number $c \in (a, b)$, there exist a point $x_0 \in [a, b]$ such that*

$$f(x_0) = c$$

Proof. Bisection method: define $m_n = \frac{a_n+b_n}{2}$. Consider first $f(a) < f(b)$. If $c \geq f(m_n)$, set $a_{n+1} = m_n$ (alternatively, same with b_{n+1}). Now, $a_n \leq a_{n+1} < b_{n+1} \leq b_n$. Also, $b_n - a_n = \frac{b-a}{2^{n-1}}$ so it converges to 0. By Nested Interval, $\{a_n\}$ and $\{b_n\}$ converge to a point x_0. $f(x_0) \leq c$ since $f(a_n) \leq c$ and $f(x_0) \geq c$ since $f(b_n) \geq c$, so $f(x_0) = c$. □

Definition 3.2. *An interval I over $\mathbb{R}$ is a set of numbers such that any $x < y$ where $x, y \in I$ defines an interval $[x, y]$ in I.*

Theorem 3.4. *Let $f : I \to \mathbb{R}$ function where I is an interval. Then, the image of f is also an interval.*

Proof. Definition of interval and Intermediate Value Theorem. □

Theorem 3.5. *Let $f : I \to \mathbb{R}$ be a function where I is an interval. Then, f is continuos if its image $f(I)$ is an interval.*

Definition 3.3. *$f : D \to R$ is said to converge uniformly in D if whenever $\{u_n\}$ and $\{v_n\}$ are two sequences in D such that*

$$\lim_{n\to\infty} |u_n - v_n| = 0$$

then

$$\lim_{n\to\infty} |f(u_n) - f(v_n)| = 0$$

Definition 3.4. *(Epsilon-Delta Convergence)*