# Contents

# 1 Volatility Modeling

## 1.1 The different types of volatility

**Motivation.** The B&S model does a decent job for pricing derivatives yet one of its most important flaws concerns the volatility that we will now focus on. They are four types of volatility:

**(i) Actual (aka Local).** This is the volatility that goes into the B&S equation.

**(ii) Historical.** This measures the amount of randomness over some period in the past. This can be looked at a Backward-looking measure.

**(iii) Implied.** This is the volatility that when input into the B&S option pricing formula gives the market price of the option.

**(iv) Forward.** It refers to volatility over some period in the future.

## 1.2 Volatility estimation by statistical means

**(i) Constant volatility Model.** The volatility of an asset is defined as the standard deviation of the prices returns over a given period. In maths, let $S_i$ be the price of an asset at time $i$, then its return (in percentage) is

$$R_i = \left(\frac{S_i - S_{i-1}}{S_{i-1}}\right) \times 100; \quad i = 1, .., n$$

Consequently its variance is:

$$\widehat{Var}(R) = \frac{1}{N}\sum_{i=1}^{N}(R_i - \bar{R})^2; \quad \bar{R} = \frac{1}{N}\sum_{i=1}^{N} R_i$$

**- Example.** We look at the tick NFLX (Netflix) between 01-01-2009 and 01-03-2018. More precisely we'll look at its monthly returns data. We calculate its statistics based on the formula above

$$\widehat{Var}(R)_{month} = 17.62\%; \quad \bar{R}_{month} = 5.20\%$$

As a consequence the typical monthy return of Netflix is comprised between:

$$\text{Return}_{NFLX}^{month} \in [-12.42\%, 22.82\%]$$

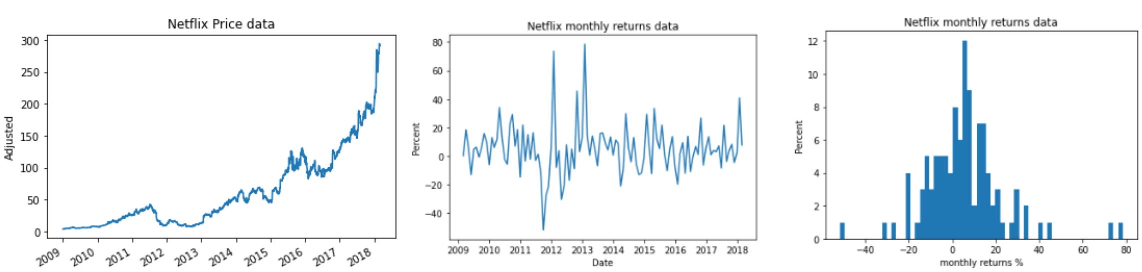



Figure 1: Price of NFLX and its monthly returns in percentage

**(ii) Autoregressive Conditional Heteroscedasticity [ARCH(p)] model (1982 by Engle).** This model extends $AR(p)$ but focus on variance. The variance of the white noise process was constant which is a strong assumption that we will alleviate.

We first explain the terms:

- **Heteroscedasticity.** means volatility or standard-deviation or variance
- **Conditional.** means that the volatility is not fixed over time eg it can be low in summer and high at the end of the year
- **Autoregressive**. The volatility depends on the previous timestamp

**- Definition.** We first build the returns $r_t = \frac{S_t - S_{t-1}}{S_{t-1}} \times 100$ and we obtain here an $ARCH(1)$ model:

$$\begin{cases} r_t = \sigma_t z_t; \quad z_t \sim N(0, \sigma^2) \\ \sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 \end{cases}$$

**- ARCH(1) as an AR(1).**

$$\begin{array}{r} r_t^2 = \sigma_t^2 z_t^2; \quad z_t^2 \sim \chi^2(1) \\ -\alpha_0 + \alpha_1 r_{t-1}^2 = \sigma_t^2 \\ \hline \end{array}$$

$$r_t^2 - \alpha_1 r_{t-1}^2 - \alpha_0 = \sigma_t^2 (z_t^2 - 1)$$
$$\iff r_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + v_t; \quad v_t \sim \textit{centered } \chi^2(1)$$

We have showcased that the returns are defined as an $AR(1)$ process with mean and non-gaussian noise.

**- Example.** Building on our sequences of prices $[S_t]$ we construct the returns vector $r_t = \frac{S_t - S_{t-1}}{S_{t-1}} \times 100$. Subsequently we perform a conditional MLE to obtain the parameters $\alpha_0, \alpha_1$

$$L(\alpha_0, \alpha_1 \mid y_1) = \prod_{t=2}^{n} f_{\alpha_0, \alpha_1}(y_t \mid y_{t-1})$$

Equivalently its log-MLE

$$l(\alpha_0, \alpha_1) = \frac{1}{2} \sum_{t=2}^{n} \ln\left(\alpha_0 + \alpha_1 y_{t-1}^2\right) + \frac{1}{2} \sum_{t=2}^{n} \left( \frac{y_t^2}{\alpha_0 + \alpha_1 y_{t-1}^2} \right).$$

What is left now is the algorithm:

$$\begin{aligned} &\text{for i=1:T} \\ &\quad \sigma_0 = \frac{\alpha_0}{1 - \alpha_1} \\ &\quad \sigma_i = \alpha_0 + \alpha_1 r_{t-1}^2 \end{aligned}$$

**(iii) Generalized Autoregressive Conditional Heteroscedasticity [GARCH(p)] model (1986 by Bollerslev which was Engle's student).** We first explain the terms:

- **Generalized.** We also consider the volatility of previous periods
- **Heteroscedasticity.** means volatility or standard-deviation or variance
- **Conditional.** means that the volatility is not fixed over time eg it can be low in summer and high at the end of the year
- **Autoregressive**. The volatility depends on the previous timestamp

**- Definition.** We first build the returns $r_t = \frac{S_t - S_{t-1}}{S_{t-1}} \times 100$ and we obtain here an $ARCH(1)$ model:

$$\begin{cases} r_t = \sigma_t z_t; \quad z_t \sim N(0, \sigma^2) \\ \sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \end{cases}$$

We can compare both models through their causal diagram. We observe that GARCH(1,1) has an extra causal parameter $\sigma_{t-1}$. It means that if the volatility was high yesterday it will likely impact today. Additionally adding the $\beta_1$ term is similar to an $MA(q)$ model here it enables to makes jumps and remains on a high plateau rather than bursting.

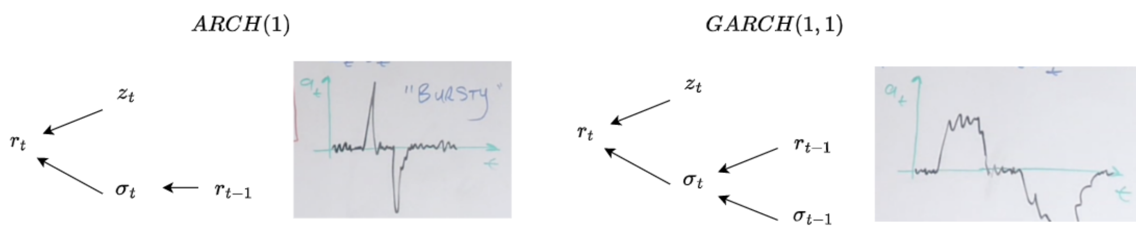
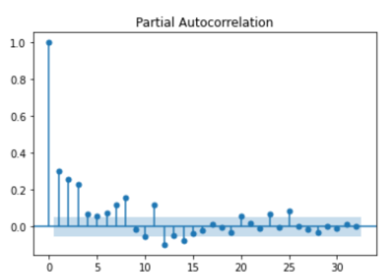



Figure 2: Price of NFLX and its monthly returns in percentage

**Example.** We look at the Disney stock over the span of 15 years and we plot its returns $r_t = \frac{S_t - S_{t-1}}{S_{t-1}} \times 100$. Additionally to define the orders of the $p$ and $q$ we look at their PACF and ACF. We obtain that the GARCH(3,0) aka the ARCH(3,0) is the most suitable:




Volatility Model

| | coef | std err | t | P>\|t\| | 95.0% Conf. Int. |
|---|---|---|---|---|---|
| omega | 0.8619 | 0.138 | 6.230 | 4.678e-10 | [ 0.591, 1.133] |
| alpha[1] | 0.0886 | 4.530e-02 | 1.955 | 5.052e-02 | [-2.023e-04, 0.177] |
| alpha[2] | 0.2621 | 9.123e-02 | 2.873 | 4.060e-03 | [8.334e-02, 0.441] |
| alpha[3] | 0.3558 | 0.169 | 2.102 | 3.559e-02 | [2.397e-02, 0.688] |

Figure 3: Price of NFLX and its monthly returns in percentage

Subsequently we perform a conditional MLE to obtain the parameters $\alpha_0, \alpha_1$

$$L(\alpha_0, \alpha_1, \beta_1 \mid y_1) = \prod_{t=2}^{n} f_{\alpha_0, \alpha_1, \beta_1}(y_t \mid y_{t-1})$$

What is left now is the algorithm:

for i=1:T

$$\sigma_0 = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

$$\sigma_i = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

As a consequence the input of the model is $[r_t]_{3814}$ and the outcome of our algorithm is a vector $[\sigma_t]_{3814}$ that evolves through time. Visually we observe in blue the returns $r_t$ and in orange the volatility which is overall steady except when the return's std increase like in 2020 during the coronavirus outbreak where it becomes very volatile.

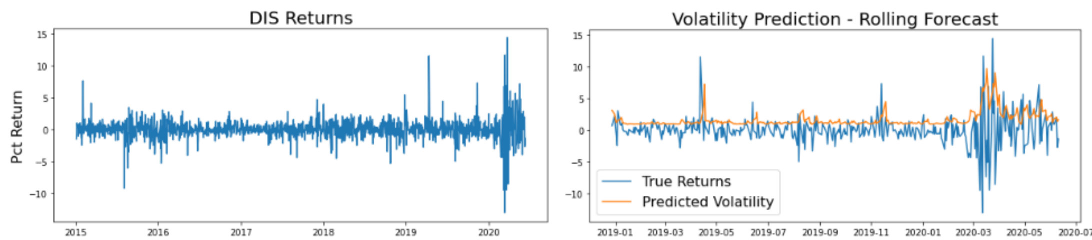



Figure 4: Price of NFLX and its monthly returns in percentage

**- Expected future volatility.** To do

## 1.3 Stochastic volatility model

**(i) Hull and White (1987).** They derived a semi-closed solution for volatility. Recollect that black scholes model was in 1973

**(ii) (1987)** it was the Black monday on the market since ..

### 1.3.1 Stein and Stein (1991)

**(iii) 1991 - Stein and Stein.** Using the Ohlstein Ulhenbeck process; We will look at how to fit the Ornstein Uhlenbeck to a volatility index. 1

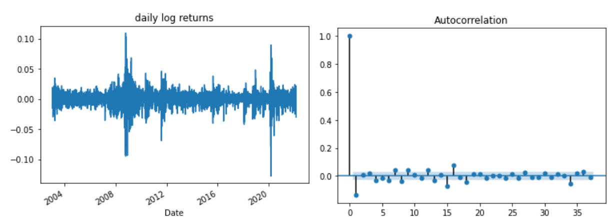
From the paper Rama Cont here in 2005 described three properties of financial time-series:

- Excess volatility
- Heavy Tails
- Volatility Clustering: it means that the magnitude of volatility are clustered together.

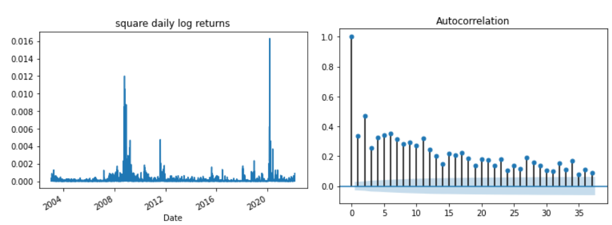
Denote by $S_t$ the price of a financial asset and $X_t = \ln S_t$ its logarithm. Given a tim e scale $\Delta$, the logreturn at scale $\Delta$ is defined as:

$$r_t = X_{t+\Delta} - X_t = \ln\left(S_t + \Delta S_t\right)$$

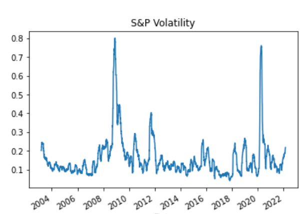
We plot on the LHS the log returns and on the right the$ACF(\rho)$. What we observe is there is not correlations




Instead here we plot the $|r_t|$ and there the ACF $\rho_{rf^2}(h)$ we observe strong correlations up to several weeks. It means that during the period of high volatility/low volatility it is likely to continue. This is also known as volatility clustering.




On typical real data we plot the rolling volatility over 40 days it yields a realized volatility. We would like to model this process with the Ornstein Uhlenbeck process



$$dX_t = \kappa\left(\theta - X_t\right)dt + \sigma dW_t$$

Step 1. We can then rearrange the terms:

$$\begin{aligned} dX_t &= \kappa\left(\theta - X_t\right)dt + \sigma dW_{t^-} \\ dX_t &= \kappa\theta dt - \kappa X_t dt + \sigma dW_{t^-} \\ dX_t + \kappa X_t dt &= \kappa\theta dt + \sigma dW_t \end{aligned}$$

We recognize a terme on the LHS

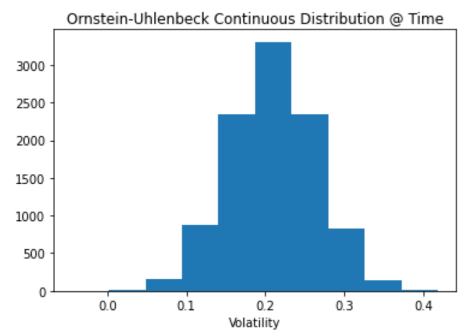
$$d\left(e^{\kappa t}X_t\right) = e^{\kappa t}dX_t + \kappa e^{\kappa t}X_t dt = \kappa\theta dt + \sigma dW_t$$

**Step 2.** so multiply equation through by $e^{\kappa t}$ term. Take integral over time horizon on $t \in [0, T]$

$$\int_0^T d\left(e^{\kappa t}X_t\right) = \int_0^T \kappa\theta e^{\kappa t}dt + \int_0^T \sigma e^{\kappa t}dW_t - e^{\kappa T}X_{T^-}X_0 = \kappa\theta\frac{e^{\kappa T}-1}{\kappa} + \sigma\int_0^T e^{\kappa t}dW_{t^-}$$

$$X_T = X_0 e^{-\kappa T} + \theta\left(1 - e^{-\kappa T}\right) + \sigma\int_0^T e^{-\kappa(T-t)}dW_t$$

**Step 3. Moments.**




### 1.3.2 VIX Index (1993)

**(iv) 1993 - VIX index.** introduction to this index. it tells you how frightened equity markets are. The VIX (Volatility Index) aka fear index (indice de la peur) is an indicator managed by the C.B.O.E. chicago board options exhcnage. It is calculated based on the price of options on the S&P 500.

Broadly speaking it is calculated as the average of the implicit volatility of the call options and put options on the S&P500. This is how you interpret it:

- If it lies between 10 and 15 the volatility is low the market is going up
- Between 20 and 30, the volatility is somewhat more important the market is edgy but can be still going up
- Above 30, the volatility is significant there is a importante chute des cours. Eg: VIX=[40,50] 1998, 2001, 2002, 2008, 2011

### 1.3.3 Heston Model (1993)

**(v) 1993.** a stochastic volatility model was defined by Heston. It is a popular choice cause it has an easy closed-form solution it has no negatives variances and the levarage effect first defined by black.

### 1.3.4 Dupire (1994)

**(vi) local volatility by Dupire (1994).**

**(vii) 1996 - Jump diffusion + stochastic volatility model (SVJ) by Bates.** He uses heston's 1993 and incorporate the Merton's 1994 model. THis led to a new area of research.

**(iix) 1998 rought volatility by Comte and Renault.** They use fractional BM to measure long terms persistent memory in markets.

**(ix) realized variance by Barndorff Nielson and Shepard (2002).** a new estimation of real-world variances.

**(x) 2003** an update to the previous VIX

**(xi) 2004.** VIX futures by CBOE trade a liquid volatility.

**(xii) 2006.** COBE introduces VIX options and it increase the risk management methods;

**(xiii 2008.** VIX reaches its intraday high of 89.52

**2009 .** Double Heston Model by Christoffersen, Heston and Jacobs, two variance processes with correlations between returns and volatility.

## 1.4 Implied volatility

**Data provider reuters.** screenshot of call and put option. The spot value is 3800; anad here we have some information to quote: we see the maturities T like 31-Mar-2021. Here Options are spaced quarterly basis. If you want to buy it is always higher than selling. Also two importantant element $\sigma_{IV}$ and $\Delta_{IV}$, it is calculated based on a mid price. Also note that for different strike we have different IV. If we plot this we get (see). If we want to calibrate different options with different strike we coldn't use this model, causae here we change strike we get a different IV. Also $\Delta_{mid}$

**Motivation.**
**- Premium Call Option $\sigma_{hist}$.** Suppose we have given a standard call option $V_C$ on 100 shares of company $Z$. With $K = \$75, \tau = 55 \; days, r = 5\%, S_0 = \$85$ and from historical data (realized volatility But market is not looking at the past it is forward looking) we have obtained $\sigma_{hist} = 0.25$. So the call price is given by the BS model:

$$BS\left(\sigma_{\text{hist}}, r, T, K, S_0\right) = BS\left(25\%, 5\%, \frac{55}{365}, 75, 85\right) = 10.8667$$

But in the market the price of such a call option \$12.25. What does it mean? An arbitage ?

**- Premium Call Option $\sigma_{IV}$.** Based on the standard BS pricing model, we find the volatility implied by the market price $C_m$ to be 43.89% i.e.

$$\sigma_{\text{IV}} = g\left(C_m\right) = 43.89\%$$

In order to check the calculation we substitute the $\sigma_{\text{market}}$ to the pricing model, i.e.,

$$\begin{aligned} V_c(t, S) &= BS\left(\sigma_{\text{IV}}, r, T, K, S_0\right) \\ &= BS\left(43.89\%, 5\%, \frac{55}{365}, \$75, \$85\right) = \$12.25 \end{aligned}$$

where $BS$ is monotonically increasing in $\sigma$ (higher volatility corresponds to higher prices).
**- Obtaining $\sigma_{IV}$.** Now, assume the existence of some inverse function

$$g_\sigma(\cdot) = BS^{-1}(\cdot)$$

so that

$$\sigma_{IV} = g_\sigma\left(C_m, r, T, K, S_0\right)$$

By computing the implied volatility for traded options with different strikes and maturities, we can test the Black-Scholes model.

**(i) Recall.** The slope coefficient of a straight line (coefficient directeur) is $f'(a)$ we can derive the tangent equation:

$$
\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{y_B - y_A}{x_B - x_A} = f'(a) \\
&= \frac{y - f(a)}{x - a} = f'(a) \\
y &= f'(a)(x - a) + f(a)
\end{aligned}
$$

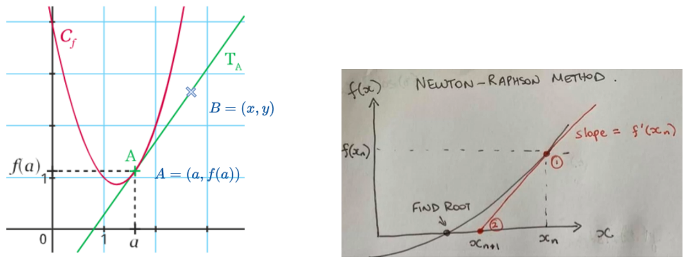



Figure 5: On the left Tangent and on the right the NR algorithm

**(ii) Newton-Raphson algorithm.** We can use the exact same logic as above. Let a point $A = (x_n, f(x_n))$ which lies at the intersection of the tangent and the courbe $f(x)$. We consider an additional point on the x-axis $B = (x_{n+1}, 0)$

$$
\begin{aligned}
&\frac{y_B - y_A}{x_B - x_A} = f'(x_n) \\
&\frac{0 - f(x_n)}{x_{n+1} - x_n} = f'(x_n) \\
&\iff x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\end{aligned}
$$

**(iii) European Call Option Implied Volatility.**

**- Setup.** Given the parameters $r, T, K, S_0$ along with the market price $C_m$ (Gotten from the internet) we want to derive its $\sigma_{IV}$. To do so we cannot directly solve the inverse problem:

$$\sigma_{IV} = g_\sigma\left(C_m, r, T, K, S_0\right); \quad g_\sigma(\cdot) = BS^{-1}(\cdot)$$

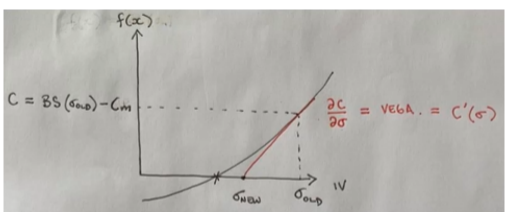
Alternatively we can find the roots of the difference:

$$BS\left(\sigma_{impl}, r, T, K, S_0\right) - C_m = 0$$

**- Newton-Raphson for IV.** Here we will consider two axes: The x-axis is the Implied Volatility (IV) it maps to the Call price which is the difference between the price obtained by the B&S formula and $C_m$ the market price (which can be obtained online). Additionally we know that the vega is the tangent to the curve as $\frac{\partial C}{\partial \sigma} = \nu$
We are searching for $\sigma_{IV}$ such that the above equation is zero.

Using the exact same formula as for Newton-Raphson

$$\sigma_{new} = \sigma_{old} - \frac{C(\sigma_{pld})}{C'(\sigma_{old})}$$
$$\sigma_{new} = \sigma_{old} - \frac{BS(\sigma_{old}) - C_m}{C'(\sigma_{old})}$$



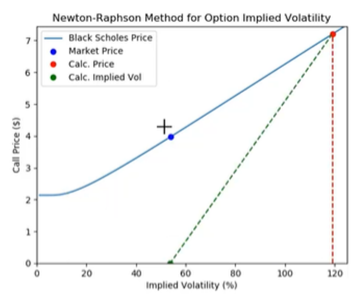




Figure 6: 2 Figures side by side

**procedure** IMPLIED_VOLATILITY($S_0, K, T, r, market\_price$)
  $max\_iter = 200$ #max no. iterations
  $vol\_old = 0.3$ # initial guess
  **for** $k = 0 : max\_iter$ **do**
    $bs\_price = bs(\text{"}call\text{"}, S_0, K, T, r, market\_price)$
    $Cprime = vega(\text{"}call\text{"}, S_0, K, T, r, market\_price)$
    $C = bs\_price - market\_price$

    $vol\_new = vol\_old - \frac{C}{Cprime}$

    **if** $|vol\_old - vol\_new| < tol$ **then**
      break
    **end if**
    $vol\_old = vol\_new$
  **end for**
  implied_vol = vol_new
  **return** implied_vol
**end procedure**

**Example.** If we let $S_0 = 30\$, K = 28\$, T = 0.5, r = 0.025$ and a market price value that we extracted from the web $market\_price = 6.3$. If we plug in the function we obtain a volatility of

$$VI = 28.857\%$$

**(iv) Volatility smile.** First we have to collect a list of $[market\_prices]_n$ along with their corresponding $[strike\_prices]_n$ from historical data.

$S_0 = 730, r = 0.01, T = \frac{3}{252}$
**for** $i = 0 : len(market\_prices)$ **do**
  $market\_price = market\_prices[i]$
  $strike\_price = strike\_prices[i]$
  $implied\_vol[i] = implied\_volatility(market\_price, S_0, strike\_price, r, T)$
**end for**
$plot(strike\_prices, implied\_vol)$

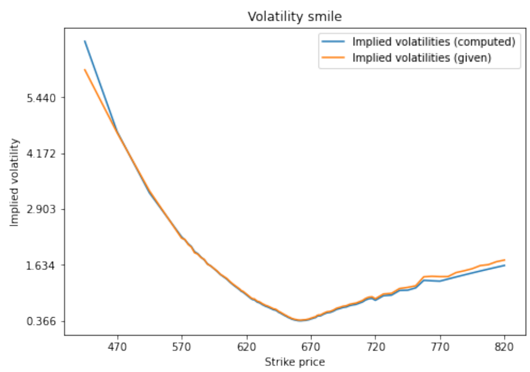



Figure 7: Volatility smile of Tesla's stock (in orange we have the IV from the web)

WHat assumption IV smile destroys ? In B&S model we should have constant volatility in theory.

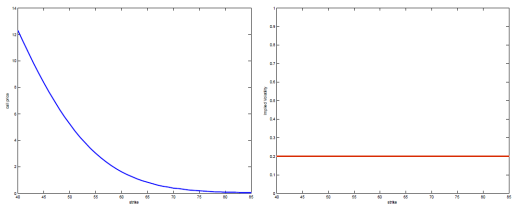


Figure 8: Implivied Volatility in theory

As such it proves that the model must not be working correctly.

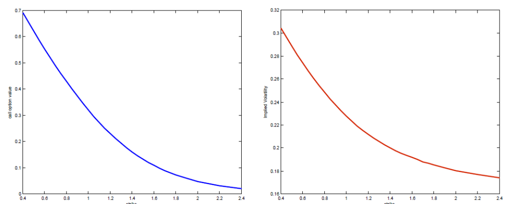


Figure 9: Implied Volatility using the market data

Also BS assumes that the asset price follows a log normal distribution. But if the volatility smile is not flat like a skew it means fatter tail on the left.

Application: crash of phobia, beefore 1987 volatility smile was flatter, and the crashophobia by rubinstein said that OTM put after the crash traders became more concerned about the tail risk (left) .

When there is no skew the calls should be traded for the same values as the put.

But since the recent crashes we have the put traded

**Implied Volatility shapes.** Overall we observe three kind of shapes that were derived with solving the inverse problem:

$$\sigma_{IV} = g_\sigma\left(C_m, r, T, K, S_0\right); \quad g_\sigma(\cdot) = BS^{-1}(\cdot)$$

Recollect that we have gotten a list of $[C_m]$ and $[K]$ on the internet with the same maturity, risk-free rate and asset price.

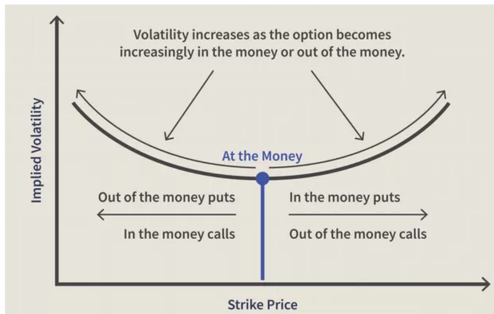



Figure 10: Volatility smile)

**- Three shapes.** We observe three shapes that are divided into ITM, ATM and OTM:

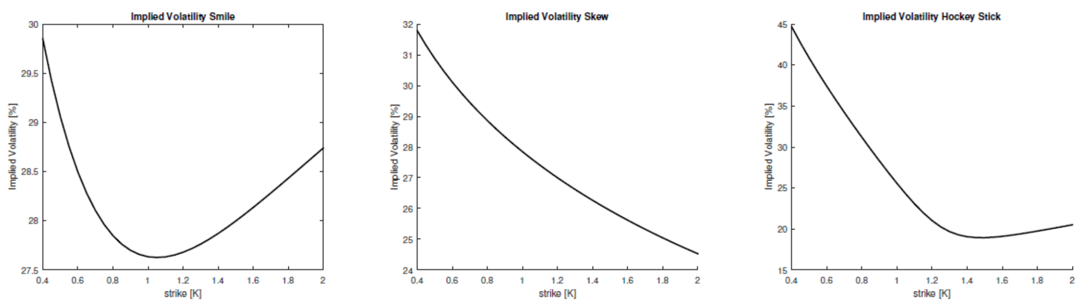



Figure 11: Typical IV shapes: a smile, a skew and the so-called hockey stick. The hockey stick can be seen as a combination of the IV and the skew.

**Implied Volatility surfaces.**