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Submission Deadline:
Wednesday 2. November, 2022, 12:00 (noon)

NTNU

Project #2

10. October 2022

Notes

Groups. This project is a group project and can be solved in groups of *up to three* students. You could for example continue in the groups of three you build in the exercises. You will have to register these groups in Inspera. One person (per group) can create a group and will get a PIN code which can be used by the other group members for registering.

Requirements for submission. The submission is in **Inspera**. Each group must submit their own report. It is not allowed to copy from other groups.
All code – also the tests – should be in individual cells that can just be run (as soon as the necessary functions are defined). Functions should only be used in cells *after* their definition, such that an evaluation in order of the notebook does not yield errors.
It is not possible to have an extension for this project.

Supervision. For questions, the usual exercise time, Monday, 10.15–12.00 can be used. Questions can also be asked in the Mattelab forum.

The project is obligatory and counts 20% on the final grade.

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Introduction

In this project we want to consider numerical methods for signal and image processing that are based on Fourier series. We will first consider the Fourier Transform of *periodic functions* and *periodic signals.* A (complex-valued) function $f\colon \mathbb{R} \to \mathbb{C}$ is periodic with periodic T if

$$f(x) = f(x+T) \qquad \text{holds for all } x \in \mathbb{R}.$$

For such a function we just need to know one *period*, for example $[-\frac{T}{2}, \frac{T}{2})$ or $[0, T)$. Note that these intervals are *half-open*, since e. g. $f(0) = f(T)$ and we only need one of these values to determine f. We will consider periodic functions with a period of $T = 1$. We refer to this period as write $\mathbb{T} \equiv [0, 1)$. We further collect all *square integrable* functions in the set

$$L^2(\mathbb{T}) := \left\{ f\colon \mathbb{T} \to \mathbb{R} \,\middle|\, \int_0^1 \lvert f(x)\rvert^2 \,\mathrm{d}x < \infty \right\}.$$

Based on these functions we can create a vector space consisting of functions. We first define an *inner product* for two functions $f, g \in L^2(\mathbb{T})$ by

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)} \,\mathrm{d}x,$$

where $\overline{a}$ denotes the complex conjugate of a complex number $a \in \mathbb{C}$. With a few more details (concerning integration and the Lebesgue measure) we get a norm $\lVert f \rVert = \sqrt{\langle f, f \rangle} < \infty$ from this inner product as well if we build suitable congruence classes.
Note that in both cases we could —similar to the note above— also integrate from $-\frac{1}{2}$ to $\frac{1}{2}$ instead, which we can also refer to with $\mathbb{T}$ (because the integral is still the same due to periodicity).

Finally, again similar to vector spaces, given a set of functions $f_1, \ldots, f_N$ we write

$$\operatorname{span}(f_1, \ldots f_N) := \left\{ g \,\middle|\, \text{there exist coefficients } a_1, \ldots, a_N \text{ such that } g(x) = \sum_{i=1}^{N} a_i f_i(x) \right\},$$

which we even could build for infinitely many functions h_k, $k \in \mathbb{N}$. We just need a sequence a_k of coefficients then.

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Task 1: The (Discrete) Fourier Transform

a) We consider the functions $\mathrm{e}^{2\pi ikx}$, $k \in \mathbb{Z}$, $x \in \mathbb{T}$. Prove that for any $k, h \in \mathbb{Z}$ we have

$$\langle \mathrm{e}^{2\pi ik\cdot}, \mathrm{e}^{2\pi ih\cdot} \rangle = \begin{cases} 1 & \text{if } k = h \\ 0 & \text{else.} \end{cases}$$

Notation. Since similar to the inner product definition above, we would like to avoid the variable x in the inner product, we write a $\cdot$ instead. This emphasizes that we plug in the complete function (and not the function evaluated at some x). So the best way to "read" $\cdot$ is a variable, that we did not assign a name to, because it is not so important in that place. We do the same with functions when we write just $f = f(\cdot)$ and $g = g(\cdot)$, just that we even leave out the variable completely.

b) We consider the functions of the form $\sqrt{2}\sin(2\pi mx)$, $m = 1, 2, \ldots$, $\cos(2\pi 0x)$ and $\sqrt{2}\cos(2\pi nx)$, $n = 1, 2, \ldots$, $x \in \mathbb{T}$. Prove that for these functions form an orthonormal system, i. e. we have

- $\langle \sqrt{2}\sin(2\pi n\cdot), \sqrt{2}\cos(2\pi m\cdot) \rangle = 0$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$
- $\langle \sqrt{2}\sin(2\pi n\cdot), \sqrt{2}\sin(2\pi m\cdot) \rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n, \end{cases} \qquad m, n \in \{1, 2, \ldots\}$
- $\langle \sqrt{2}\cos(2\pi n\cdot), \sqrt{2}\cos(2\pi m\cdot) \rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n \neq 0, \\ 2 & \text{if } m = n = 0, \end{cases} \qquad m, n \in \{0, 1, \ldots, \}$

c) We introduce the two spaces. The first one is

$$\mathcal{T}_n := \operatorname{span}(\mathrm{e}^{-2\pi in\cdot}, \ldots, \mathrm{e}^{2\pi in\cdot}) = \left\{ f \,\middle|\, f(x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{2\pi ikx}, \quad \text{where } c_{-n}, c_{-n+1}, \ldots, c_n \in \mathbb{C}, \right\},$$

where we further restrict the coefficients to $c_k = \overline{c_{-k}}$, $k = 0, \ldots, n$, where $\overline{z}$ denotes the complex conjugate of a complex number $z \in \mathbb{C}$. andThe second one is

$$\begin{aligned} \mathcal{S}_n &:= \operatorname{span}(\cos(0\cdot), \cos(2\pi\cdot), \ldots, \cos(2\pi n\cdot), \sin(2\pi\cdot), \sin(2\pi 2\cdot), \ldots, \sin(2\pi n\cdot)) \\ &= \left\{ f \,\middle|\, f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(2\pi kx) + b_k \sin(2\pi kx), \text{ where } a_0, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R} \right\} \end{aligned}$$

Use the results from Item a) and Item b) to find orthonormal bases for these spaces. User Euler's identity to argue that both spaces are the same, i.e. $\mathcal{T}_n = \mathcal{S}_n$. What is the dimension of $\mathcal{T}_n$?

d) Use the representation of $\mathcal{S}_n$ from Item c) to prove that the *Fourier coefficients* $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ of a function $f \in \mathcal{S}_n$ can be computed as

$$a_k = 2\langle f, \cos(2\pi k\cdot)\rangle = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\cos(2\pi kx)\,\mathrm{d}x, \qquad k = 0, 1, \ldots, n$$

and

$$b_k = 2\langle f, \sin(2\pi k\cdot)\rangle = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\sin(2\pi kx)\,\mathrm{d}x, \qquad k = 1, \ldots, n,$$

where we might write $a_k(f)$ and $b_k(f)$ to emphasize that these are the coefficients belonging to f.

Remarks.

- The same way works for the space $\mathcal{T}_n$ where

$$c_k = c_k(f) = \langle f, \mathrm{e}^{2\pi \mathrm{i}k\cdot}\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\mathrm{e}^{-2\pi \mathrm{i}kx}\,\mathrm{d}x, \qquad k = -n, \ldots, n. \tag{1}$$

  which is more often used, since we just have to keep track of one set of coefficients. These are also referred to as *Fourier coefficients.*

- We can actually compute these coefficients for arbitrary $k \in \mathbb{Z}$ and for an arbitrary periodic function f. Computing $c_{-n}(f), \ldots, c_n(f)$ for an arbitrary function, we obtain the *best approximation* $f_n \in \mathcal{T}_n$, i. e. $f_n = \underset{g\in\mathcal{T}_n}{\arg\min}\|f - g\|$.

- For $f, g \in L_2(\mathbb{T})$ the *Parseval identity* holds, that is

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} c_k(f)\overline{c_k(g)}. \tag{2}$$

e) We want to use equidistant points $x_0, \ldots, x_{N-1}$, $x_j = \frac{j}{N}$, $j = 0, \ldots, N$, for some $N \in \mathbb{N}$ to approximate the integral required for the Fourier coefficients $c_k(f)$ of a function f from (1). We introduce for ease of notation $f_j = f(x_j)$ and $\boldsymbol{f} = (f_0, \ldots, f_{N-1})$.

Show that using the composite trapezoidal rule, we obtain

$$c_k(f) \approx \hat{f}_k := \frac{1}{N}\sum_{j=0}^{N-1} f_j \mathrm{e}^{-2\pi \mathrm{i} jk/N}.$$

Show further, that the $\hat{f}_k$ are N periodic, that is $\hat{f}_k = \hat{f}_{k+N}$ for all $k \in \mathbb{Z}$.
What does that mean for the approximation in the equation above?

Remark. The coefficients $\hat{\boldsymbol{f}} = (\hat{f}_0, \ldots, \hat{f}_{N-1})$ are called the *Discrete Fourier Transform (DFT)* of f. There exist fast transforms, we will see in the lecture, called the *Fast Fourier Transform (FFT)* to compute these

f) Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ be given. Prove that

$$\frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{-2\pi \mathrm{i} jk/N} = \begin{cases} 1 & \text{if } k \bmod N \equiv 0, \\ 0 & \text{else.} \end{cases}$$

g) We collect the discrete Fourier transform into a matrix, that is we want to write $\hat{\boldsymbol{f}} = \mathcal{F}_N \boldsymbol{f}$ with $\mathcal{F} \in \mathbb{C}^{N\times N}$ given by

$$\mathcal{F}_N = \frac{1}{N}\left(\mathrm{e}^{-2\pi ikl/N}\right)_{k,l=0}^{N-1}$$

We further introduce for a vector $\boldsymbol{a} = (a_0, \ldots, a_{N-1})^\mathsf{T}$ the *circulant matrix*

$$\operatorname{circ} \boldsymbol{a} = \left(a_{k-l \bmod N}\right)_{k,l=0}^{N-1} = \begin{pmatrix} a_0 & a_{N-1} & \cdots & a_2 & a_1 \\ a_1 & a_0 & \cdots & a_3 & a_2 \\ \vdots & & \ddots & & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_1 & a_0 \end{pmatrix}$$

Prove that the Fourier matrix $\mathcal{F}_N$ diagonalizes the circulant matrix, i. e. using $\hat{\boldsymbol{a}} = \mathcal{F}_N \boldsymbol{a}$ we get

$$\operatorname{circ} \boldsymbol{a} = N^2 \overline{\mathcal{F}_N} \operatorname{diag}(\hat{\boldsymbol{a}}) \mathcal{F}_N,$$

where $\operatorname{diag}(\cdot)$ denotes the diagonal matrix.

Hint. Consider a single entry of the result from the matrix product on the right and use the result from Item f)

Derive further a formula for the inverse $\mathcal{F}_N^{-1}$ using either the just proven property with the help of 1f).

h) Write a function `transform(f,N,start=0.0)` that takes a function f as its first parameter and the number of samples N as its second and returns the vector $\boldsymbol{f} = (f_0, \ldots, f_{N-1})^{\mathrm{T}}$ of function values. The optional third parameter `start` would by default be zero, and state where to start the sampling. This way the function can perform sampling on both intervals we discussed until now. In this task we want to use this to compute the DFT $\hat{\boldsymbol{f}}$ of $\boldsymbol{f}$, which is available as scipy.fft.fft|.

Consider the following functions defined on $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2})$ as periodic functions (by periodic continuation),

- $f_1(x) = \sin(8\pi x), \qquad x \in \mathbb{T},$
- $f_2(x) = \sin(32\pi x) + \cos(128\pi x), \qquad x \in \mathbb{T},$
- $f_3(x) = x, \qquad x \in \mathbb{T},$
- $f_4(x) = 1 - |x|, \quad x \in \mathbb{T},$

with $N = 5, 17, 257$ and plot $\boldsymbol{f}$ and $\hat{\boldsymbol{f}}$ for these functions side by side. Take into account that these signals might be complex.

Note that the $\hat{f}_k$ are complex values, so you have to plot the real and imaginary parts in the same plot in different colors.

For which cases does $\boldsymbol{f}$ approximate f well?

i) Take f_2 from Item h) plot its discrete Fourier coefficients $\hat{\boldsymbol{f}}$ after applying `fftshift` and plot the result again for $N = 17, 65, 257$.

- State $a_k(f_2)$ and $b_k(f_2)$ without solving an integral.
- Compute $c_k(f_2)$ (again without computing the integral)
- What does the `fftshift`-function do?
- Using Euler's identity – Can you use the coefficients to "remove" the second summand of f_2 by just modifying $\hat{\boldsymbol{f}} \in \mathbb{C}^{257}$
- Can you do the same for the case $N = 17$, i. e. $\hat{\boldsymbol{f}} \in \mathbb{C}^{17}$? Try to find a reason for your answer.

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Task 2: Interpolation and Translation invariant Spaces

In this task we want to combine the discrete Fourier coefficients with interpolation. We assume for this task that $f\colon \mathbb{T} \to \mathbb{C}$ can be written by its Fourier coefficients $c_k(f)$ from Equation (1) as

$$f(x) = \sum_{k\in\mathbb{Z}} c_k(f)\mathrm{e}^{2\pi\mathrm{i}kx}$$

where we assume that the sum converges absolutely and uniformly. This especially means that we are allowed to arbitrarily reorder the summands. It also means that the series of Fourier coefficients $c_k(f)$ is absolutely summable.

We further consider shifts of a function, that is $\tau_N f = f(\cdot - \frac{1}{N})$, that is, we shift the function by $\frac{1}{N}$ to the right. We can write $\tau_N^j f = f(\cdot - \frac{j}{N})$. Note that $\tau_n^N f = f$ since a shift by 1 is a shift by a whole period.

We define the *space of translates* as

$$V_{N,f} = \operatorname{span}\Bigl(f, \tau_N f, \tau_N^2 f, \ldots, \tau_N^{N-1} f\Bigr),$$

Or in other words, $V_{N,f}$ consists of all functions that can be represented by shifted versions of f, that is

$$g \in V_{N,f} \Leftrightarrow \text{there exist } a_0,\ldots,a_{N-1} \in \mathbb{R} \text{ sucht that } g(x) = \sum_{j=0}^{N-1} a_j f\Bigl(x - \frac{j}{N}\Bigr) \tag{3}$$

We will first need two small helper tasks, one of them answers more precisely the observations from Task 1, Item i)s last point

a) Use the result from Task 1, Item f) to prove that the *Aliasing formula* holds, that is for any $N \in \mathbb{N}$ we have for our continuous function with uniformly convergent Fourier series that

$$\hat{f}_k = \sum_{\ell\in\mathbb{Z}} c_{k+\ell N}(f), \qquad k \in \mathbb{Z}.$$

This is also called the *aliasing formula*.

b) Let f and its Fourier coefficients $c_k(f)$, $k \in \mathbb{Z}$ be given. Prove that for any k it holds

$$c_k(\tau_N f) = \mathrm{e}^{-2\pi\mathrm{i}k/N} c_k(f). \tag{4}$$

State a formula for $c_k(\tau_N^j f)$.

c) Let $g \in V_{N,f}$ as in Equation (3) show that $\tau_N g \in V_{N,f}$ as well and state the coefficients of $\tau_N g$.

d) Let f be a function with an absolutely convergent Fourier series We consider the translates that span $V_{N,f}$. Prove that

$$G = \left(\langle \tau_N^k f\,, \tau_N^j f \rangle\right)_{k,j=0}^{N-1} \in \mathbb{C}^{N\times N}$$

is a circulant matrix.
What does the diagonal matrix in Task 1, Item g) look like here?

Derive a statement that characterises linear independence of the translates in terms of Fourier coefficients. When the translates are orthonormal?

e) Let $g \in V_{N,f}$ be given with coefficients $\boldsymbol{a} = (a_0, \ldots, a_{N-1}) \in \mathbb{R}^N$. We denote by $\hat{\boldsymbol{a}} = (\hat{a}_0, \ldots, \hat{a}_{N-1})$ the discrete Fourier transform of $\boldsymbol{a}$. We introduce for $k \in \mathbb{Z}$ the decomposition into $k = h + Nz$, $h \in \{0, \ldots, N-1\}, z \in \mathbb{Z}$ (in other words: modulo) Prove that $g \in V_{N,f}$ is equivalent to the statement that

$$c_k(g) = \left(\sum_{j=0}^{N-1} a_j \mathrm{e}^{-2\pi \mathrm{i} kj/N}\right) c_k(f) = N\hat{a}_h c_k(f).$$

holds for all $k \in \mathbb{Z}$.

Remark. Phrased differently, we can check whether $g \in V_{N,f}$ holds by checking N pairs of Fourier coefficients $c_k(g)$ and $c_k(f)$, respectively.

f) We now perform the task of *Interpolation*: Given the sampling values $s_0, \ldots, s_{N-1}$ (for example $s_j = h(x_j)$ as in Task 1 for some arbitrary periodic function h you want to interpolate/approximate, at the points $x_j = \frac{j}{N}$, $j = 0, \ldots, N-1$), find a function $g \in V_{N,f}$ such that $g(x_j) = s_j$ for all $j = 0, \ldots, N-1$. Prove first that the so-called *Fundamental interpolant* $I_N : \mathbb{T} \to \mathbb{C}$ with

$$I_N(x_i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$$

exists in $V_{N,f}$ is and if only if

$$\sum_{z\in\mathbb{Z}} c_{h+Nz}(f) \neq 0, \qquad \text{holds for all } h = 0, \ldots, N-1.$$

Hint. Use Item e) together with the Aliasing formula Item a).

g) Derive an algorithm for *periodic interpolation* based on $I_N(x)$, that is:
Given samples $s_0, \ldots, s_{N-1}$ and a function f given by its Fourier coefficients $c_k(f)$, $k \in \mathbb{Z}$, whose translates are linearly independent (you may assume that without checking), derive the coefficients $a_0, \ldots, a_{N-1}$ of the function $g \in V_{N,f}$, such that

$$g\Bigl(\frac{i}{N}\Bigr) = s_i, \qquad i = 0, \ldots, N-1. \tag{5}$$

Implement this algorithm as a function `interpolate(s,ckf)`, where you assume that the Fourier coefficients of f are only finitely many and *centred*, that is only the coefficients $c_{-L}(f), c_{-L+1}(f), \ldots, c_{L-1}(f), c_L(f)$, $\frac{N}{2} < L \in \mathbb{N}$, are passed to the function and all other Fourier coefficients of f are zero anyway.

Hint. Write g first as a linear combination of shifts of I_N and compute the coefficients $a_0, \ldots, a_{N-1}$ from there.

Task 3: Interpolation and Signal Decomposition

We want to combine the interpolation and what we learned about functions in a translation invariant space to decompose a signal into different parts – to be precise in to a high frequency and a low frequency part.

a) We consider the so-called *Dirichlet kernel* for $n \in \mathbb{N}_0$ given by

$$D_n(x) = \frac{1}{2n+1} \sum_{k=-n}^{n} \mathrm{e}^{2\pi i k x}.$$

Derive the Fourier coefficients $c_k(D_n)$ and prove that the fundamental interpolant in V_{2n+1,D_n} exists.

Plot both D_n and its fundamental Interpolant I_N for $n = 5, 10, 17$. Illustrate where the function is zero and where it is 1 by putting points there as well

b) Show that the method from Item a) also work for D_n and V_{2n,D_n}. Compute the interpolant and implement it in closed form. That is usually called the *modified Dirichlet Kernel*. What does change in this case?

Plot again the modified Dirichlet Kernel, its zeros and the one point where it is 1. Do you notice a difference compared to the previous plot?
Use interpolation with the modified dirichlet kernel to provide a higher resolution signal of `s5` from `signals.csv` in the supplementary data, that is: produce a new `s6` such that between any two points of the given signal you obtain at least 3 new values

c) Similar to the analysis of Fourier coefficients, we would like to use V_{2n,D_n} to split the space into different parts.

Prove that for even n we have that $g = D_{\frac{n}{2}} \in V_{2n,D_n}$ and derive the vector $\boldsymbol{a} = (a_0, \ldots, a_{2n-1})$ that fulfils

$$D_{\frac{n}{2}}(x) = \sum_{k=0}^{2n-1} a_k \tau_{2n}^k D_n(x) = \sum_{k=0}^{2n-1} a_k D_n\big(x - \frac{k}{2n}\big).$$

Conclude that hence $V_{n,D_{\frac{n}{2}}} \subset V_{2n,D_n}$.

d) For decomposition, we are looking for a second space $W_{n,E_{\frac{n}{2}}} \subset V_{2n,D_n}$ such that

$$\langle f, g\rangle = 0 \text{ for all } f \in V_n, D_{\frac{n}{2}}, g \in W_{n,E_{\frac{n}{2}}}.$$

Let $\mathbf{a}$ denote the vector from the previous task and as usual $\hat{\mathbf{a}}$ denote its DFT.

Prove that for $h = 0, \ldots, N-1$, $z \in \mathbb{Z}$, $N = 2n$, can take $E_{\frac{n}{2}}$ defined by

$$c_{h+Nz}(E_{\frac{n}{2}}) = N\hat{b}_h c_{h+Nz}(D_n), \qquad \text{with } \hat{b}_h = \hat{a}_{h+n \bmod N}\mathrm{e}^{-h\pi\mathrm{i}/n}$$

to generate the space $W_{n,E_{\frac{n}{2}}}$ that is orthogonal to $V_{n,D_{\frac{n}{2}}}$. While $W_{n,E_{\frac{n}{2}}}$ is also (just) a space of translates, the W is used to emphasize that it is orthogonal to $V_{n,D_{\frac{n}{2}}}$

Prove further that the translates $\tau_n^j E_{\frac{n}{2}}$, $j = 0, \ldots, n-1$ are linearly independent.

e) Let $n = 512$ generate D_n $D_{\frac{n}{2^j}}$ and $E_{\frac{n}{2^j}}$, $j = 1, 2, 3$ and use these to decompose the signals `s1`, `s2`, `s3` from the signals file. into its components belonging to the corresponding subspaces.

The spaces V_{2n,D_n} are called scale spaces, the W_{2n,E_n} wavelet spaces. For each j in the decomposition above, we obtain one *wavelet level.*
Which spaces to you need to reconstruct the signal?

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Task 4: Periodic Wavelets and Image Decomposition

For this last part we want to consider the multivariate FFT, which can for example be interpreted as a Fourier transform first on each column of an image followed by one on each row. To be precise, for 2D data – e. g. a gray scale image $F \in \mathbb{C}^{N_1 \times N_2}$ we can write the 2D discrete Fourier Transform as

$$\begin{aligned}\hat{F}_{k_1,k_2} &:= \frac{1}{N_1 N_2} \sum_{j_1=0}^{N_1-1} \sum_{j_2=0}^{N_2-1} F_{j_1,j_2} \mathrm{e}^{-2\pi \mathrm{i}\left(j_1 k_1/N_1 + j_2 k_2/N_2\right)} \\ &= \hat{F}_{\boldsymbol{k}} = \frac{1}{\det(\operatorname{diag}(\boldsymbol{N}))} \sum_{\boldsymbol{j}=(0,0)}^{(N_1,N_2)} F_{\boldsymbol{j}} \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{j}^\top ((\operatorname{diag}(\boldsymbol{N}))^{-1} \boldsymbol{k})}, \\ &\qquad k_1 = 0, \ldots, N_1 - 1, k_2 = 0, \ldots, N_2 - 1, \quad \boldsymbol{k} = (k_1, k_2)^\top, \boldsymbol{j} = (j_1, j_2)^\top, \boldsymbol{N} = (N_1, N_2)^\top\end{aligned}$$

This can also be written as $\hat{F} = \mathcal{F}_{N_1} F \mathcal{F}_{N_2}$, i.e. a Fourier transform first for all columns (first multiplication) seen as signals followed by a Fourier transform on all rows (second multiplication).

Splitting every entry $\hat{F}_{\boldsymbol{k}} = |\hat{F}_{\boldsymbol{k}}|\mathrm{e}^{\mathrm{i}\phi_{\boldsymbol{k}}}$ into an Amplitude matrix $A = (|\hat{F}_{\boldsymbol{k}}|)_{\boldsymbol{k}}$ and a phase $P = (\phi_{\boldsymbol{k}})_{\boldsymbol{k}}$,

a) Load the images `Barbara.gif` and `klaus.gif`. Perform a 2D discrete Fourier transform, exchange the phases (but keep amplitudes) and compute the inverse Fourier transforms. What do you notice?

b) Consider the image `Yarimton.png`. It is a usual image from a newspaper that is printed with so-called "half-toning". Compute and plot its amplitude. Find a way to remove the dot artefacts using the Fourier transform. Do the same with the lighthouse and the munkholmen image.

   Remark. The inverse – that is to generate good half-toning (or dithering-based) images is also not-trivial, see for example `DitherPunk.jl` at https://github.com/JuliaImages/DitherPunk.jl.

c) Wavelet approaches are used in several image processing tasks. So we want to generalize the approach from Task 2 to images as well.

State an idea how to write the spaces $V_{\mathbf{N},F}$ for the 2D case of considering a function $F\colon \mathbb{T}^2 = [0,1)^2 \to \mathbb{C}$ and $\boldsymbol{N} = (N_1, N_2)^\mathsf{T} \in \mathbb{N}^2$. We assume that both N_1 and N_2 are powers of 2.
Hint. Take a tensor product approach, that is define $F(\boldsymbol{x})$ as a product $f_1(x_1)f_2(x_2)$, i. e. functions of the form $D_{\mathbf{N}}(\boldsymbol{x}) = D_{N_1}(x_1)D_{N_2}(x_2)$

d) We finally treat D_{N_1} and D_{N_2} the same way as in Task 3 Item d) and Item e). We consider one wavelet scale. What does the scale space look like?

Implement this wavelet decomposition and use it to automatically remove the dithering from the images provided in the supplementary data.

Can you also do something funny / not so useful with this method?