An Iterative Ensemble-Smoother Solution of the HM Problem Formulated with Consistent Error Statistics

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Presentation is available from
`https://github.com/geirev/Presentations`

Background

- Conditioning reservoir models on rate data . . . (Evensen and Eikrem, 2018).
  - We must take correlated measurment errors into account. Not done in practice!
  - We neglect rate errors when forcing the reservoir simulators. Not done in practice!
- Accounting for model errors in iterative ensemble smoothers (Evensen, 2019).
  - Explained how to include model errors (like rate errors) in iterative smoothers.
  - Augment errors to state vector and estimate them.
- Ensemble subspace EnRML method (Evensen et al., 2019, Raanes et al., 2019).
  - Iterative ensemble smoother that allow for correlated measurement errors
- Formulating the history matching problem with consistent error statistics (Evensen, 2021).
  - The current presentation!

Standard formulation of the history-matching problem

Nonlinear model and measurements

$$\mathbf{y} = \mathbf{g}(\mathbf{x}) \qquad \mathbf{d} \leftarrow \mathbf{y} + \mathbf{e}$$

Bayesian formulation

$$f(\mathbf{x}, \mathbf{y}|\mathbf{d}) \propto f(\mathbf{d}|\mathbf{y}) f(\mathbf{y}|\mathbf{x}) f(\mathbf{x})$$

Model pdf

$$f(\mathbf{y}|\mathbf{x}) = \delta\big(\mathbf{y} - \mathbf{g}(\mathbf{x})\big)$$

Marginal pdf

$$f(\mathbf{x}|\mathbf{d}) \propto \int f(\mathbf{d}|\mathbf{y}) f(\mathbf{y}|\mathbf{x}) f(\mathbf{x}) d\mathbf{y} = f\big(\mathbf{d}|\mathbf{g}(\mathbf{x})\big) f(\mathbf{x})$$

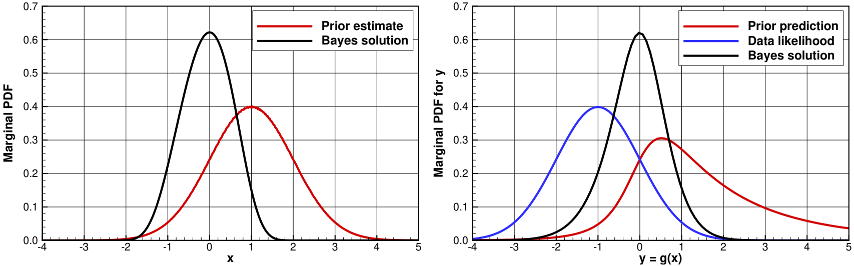
Standard Bayesian inverse problem

$$f(\mathbf{x}|\mathbf{d}) = f\big(\mathbf{d}|\mathbf{g}(\mathbf{x})\big) f(\mathbf{x})$$

Illustration of the parameter estimation problem

Bayes theorem gives posterior probability function for parameters $\boldsymbol{x}$

$$f(\boldsymbol{x}|\boldsymbol{d}) \propto f(\boldsymbol{d}\,|\,\boldsymbol{g}(\boldsymbol{x}))\,f(\boldsymbol{x})$$



- $\boldsymbol{x}$ represents model input parameters like, porosity, permeability, fault multipliers
- $\boldsymbol{y}$ could represent predicted production of oil, gas, and water
- Prior pdf represents uncertainty of $\boldsymbol{x}$.
- Prior prediction pdf represents uncertainty of $\boldsymbol{y} = g(\boldsymbol{x})$.
- Data likelihood represents uncertainty of measurement $\boldsymbol{d}$.

Approximate sampling of the posterior pdf

Gaussian priors and randomized-maximum-likelihood sampling of

$$f(\boldsymbol{x}|\boldsymbol{d}) \propto f\left(\boldsymbol{d} \,|\, \boldsymbol{g}(\boldsymbol{x})\right) f(\boldsymbol{x})$$

by minimizing an ensemble of cost functions

$$\mathcal{J}(\boldsymbol{x}_j) = \underbrace{\left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right)^{\mathrm{T}} \boldsymbol{C}_{xx}^{-1} \left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right)}_{\text{Prior misfit}} + \underbrace{\left(\boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{d}_j\right)^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \left(\boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{d}_j\right)}_{\text{data misfit}}.$$

Solutions methods:

1. Ensemble Smoother (ES) (van Leeuwen and Evensen, 1996) and (Evensen, 2009, Chap. 10).
2. Ensemble Randomized Likelihood (EnRML) (Chen and Oliver, 2013).
3. Ensemble Smoother with Multiple Data Assimilation (ESMDA) (Emerick and Reynolds, 2013).

Ensemble Smoother

Approximately solves $\nabla \mathcal{J}_j = 0$

$$\boldsymbol{C}_{xx}^{-1}(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}) + \boldsymbol{G}_j^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \big(\boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{d}_j\big) = 0.$$

- Apply the linearization $\boldsymbol{g}(\boldsymbol{x}_j) = \boldsymbol{g}(\boldsymbol{x}_j^{\mathrm{f}}) + \boldsymbol{G}_j(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}})$.
- Replace model sensitivities by least-squares fit $\boldsymbol{C}_{yx} = \boldsymbol{G}\boldsymbol{C}_{xx}$.
- ES uses ensemble covariances $\overline{\boldsymbol{C}}_{xy}$, $\overline{\boldsymbol{C}}_{xx}$, and $\overline{\boldsymbol{C}}_{dd}$.

$$\begin{aligned}
\boldsymbol{x}_j^{\mathrm{f}} &\leftarrow \mathcal{N}(\boldsymbol{x}^{\mathrm{f}}, \boldsymbol{C}_{xx}^{\mathrm{f}}), \qquad \boldsymbol{d}_j \leftarrow \mathcal{N}(\boldsymbol{d}, \boldsymbol{C}_{dd}),\\
\boldsymbol{y}_j^{\mathrm{f}} &= \boldsymbol{g}\big(\boldsymbol{x}_j^{\mathrm{f}}\big),\\
\boldsymbol{x}_j^{\mathrm{a}} &= \boldsymbol{x}_j^{\mathrm{f}} + \overline{\boldsymbol{C}}_{xy}\Big(\overline{\boldsymbol{C}}_{yy} + \overline{\boldsymbol{C}}_{dd}\Big)^{-1}\Big(\boldsymbol{d}_j - \boldsymbol{y}_j\Big),\\
\boldsymbol{y}_j^{\mathrm{a}} &= \boldsymbol{g}\big(\boldsymbol{x}_j^{\mathrm{a}}\big).
\end{aligned}$$

EnRML

State-space formulation with Gauss-Newton iterations

$$\mathcal{J}(\boldsymbol{x}_j) = \left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right)^{\mathrm{T}} \boldsymbol{C}_{xx}^{-1} \left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right) + \left(\boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{d}_j\right)^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \left(\boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{d}_j\right)$$

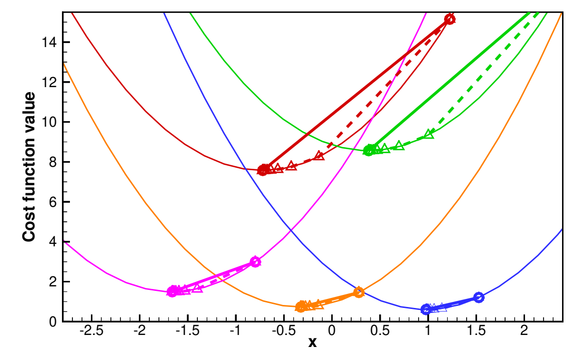
with gradient and Hessian

$$\begin{aligned} \nabla_x \mathcal{J} &= \boldsymbol{C}_{xx}^{-1}\left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right) + \boldsymbol{G}_j^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \left(\boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{d}_j\right), \\ \nabla_x \nabla_x \mathcal{J} &\approx \boldsymbol{C}_{xx}^{-1} + \boldsymbol{G}_j^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \boldsymbol{G}_j \end{aligned}$$

Iterate

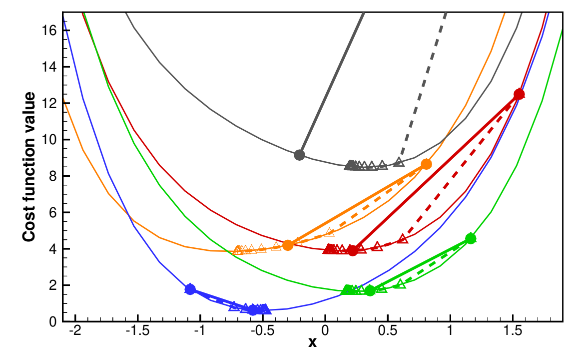
$$\begin{aligned} \boldsymbol{x}_j^{i+1} &= \boldsymbol{x}_j^i - \gamma \left(\nabla\nabla \mathcal{J}_j^i\right)^{-1} \nabla \mathcal{J}_j^i \\ \boldsymbol{y}_j^{i+1} &= \boldsymbol{g}\left(\boldsymbol{x}_j^{i+1}\right) \end{aligned}$$

ES and EnRML illustration: Linear model



- EnRML and ES both finds global minimum.
- Samples exactly posterior pdf.

ES and EnRML illustration: Non-linear model



- EnRML gets closer to minimum than ES
- Approximate sampling of posterior pdf.

ESMDA uses tapering of likelihood

Approximate sampling of $f(\boldsymbol{x}|\boldsymbol{d})$ by gradually introducing the measurements (Neal, 1996)

$$\begin{aligned}
f(\boldsymbol{x}|\boldsymbol{d}) &= f(\boldsymbol{d}|\boldsymbol{y})f(\boldsymbol{x}) \\
&= f(\boldsymbol{d}|\boldsymbol{y})^{\left(\sum_{i=1}^{N}\frac{1}{\alpha_i}\right)}f(\boldsymbol{x}) \quad \text{with} \quad \sum_{i=1}^{N}\frac{1}{\alpha_i} = 1 \\
&= f(\boldsymbol{d}|\boldsymbol{y})^{\frac{1}{\alpha_N}} \cdots f(\boldsymbol{d}|\boldsymbol{y})^{\frac{1}{\alpha_2}} f(\boldsymbol{d}|\boldsymbol{y})^{\frac{1}{\alpha_1}} f(\boldsymbol{x})
\end{aligned}$$

We compute N ES steps with “inflated” observation errors.

- Small updates reduce impact of the linear approximation.
- ESMDA is identical to ES in the linear case.

Subspace EnRML: (Evensen et al., 2019, Raanes et al., 2019)

Original cost functions

$$\mathcal{J}(\boldsymbol{x}_j) = \left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right)^{\mathrm{T}} \boldsymbol{C}_{xx}^{-1} \left(\boldsymbol{x}_j - \boldsymbol{x}_j^{\mathrm{f}}\right) + \left(\boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{d}_j\right)^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \left(\boldsymbol{g}(\boldsymbol{x}_j) - \boldsymbol{d}_j\right).$$

Solution is contained in the ensemble subspace, thus

$$\boldsymbol{x}_j^{\mathrm{a}} = \boldsymbol{x}_j^{\mathrm{f}} + \boldsymbol{A}\boldsymbol{w}_j,$$

and,

$$\mathcal{J}(\boldsymbol{w}_j) = \boldsymbol{w}_j^{\mathrm{T}} \boldsymbol{w}_j + \left(\boldsymbol{g}\left(\boldsymbol{x}_j^{\mathrm{f}} + \boldsymbol{A}\boldsymbol{w}_j\right) - \boldsymbol{d}_j\right)^{\mathrm{T}} \boldsymbol{C}_{dd}^{-1} \left(\boldsymbol{g}\left(\boldsymbol{x}_j^{\mathrm{f}} + \boldsymbol{A}\boldsymbol{w}_j\right) - \boldsymbol{d}_j\right)$$

Reduces dimension of problem from state size to ensemble size.

$$\boldsymbol{w}_j^{i+1} = \boldsymbol{w}_j^{i} - \gamma\left(\nabla\nabla\mathcal{J}_i\right)^{-1} \nabla\mathcal{J}_j^{i}$$

Practical implementation

Historical production rates have time-correlated errors acting as model errors.

1. Create an ensemble of historical OPR_j, GPR_j and WPR_j with correct error statistics.
2. Force the simulation of each realization by its corresponding RESV_j.
3. Use the realizations OPR_j, GPR_j and WPR_j to define $\boldsymbol{C}_{dd} = \boldsymbol{E}\boldsymbol{E}^{\mathrm{T}}$.
4. Update OPR_j, GPR_j and WPR_j as we do for the state ensemble.

We can include and estimate stochastic rates as part of the history-matching process.
We can account for correlated measurent errors in the inversion.

Subspace EnRML algorithm: (Evensen, 2021)

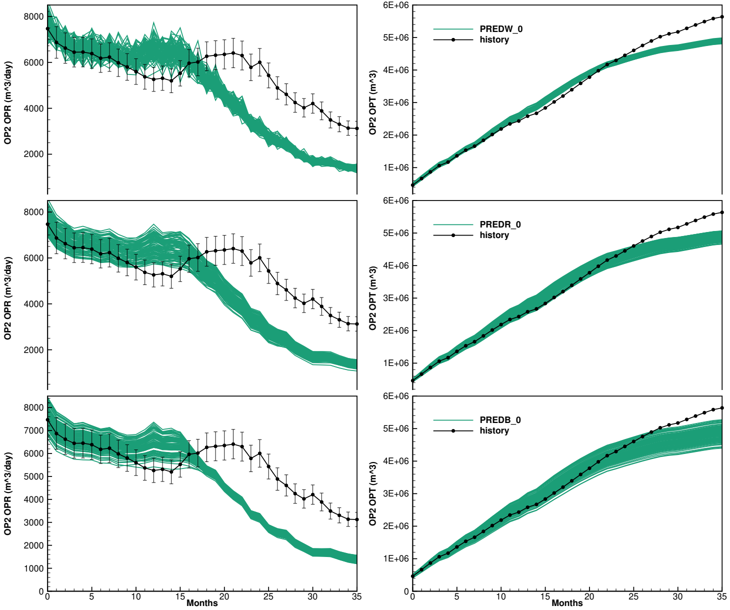
```
Input: $\mathbf{X}_0 \in \Re^{n\times N}$                ▹ prior model ensemble
Input: $\mathbf{D} \in \Re^{m\times N}$                  ▹ perturbed measurements
Input: $\mathbf{E}_0 \in \Re^{m_u\times N}$              ▹ initial rate perturbations
$\mathbf{W}_0 = 0$                                       ▹ $\mathbf{W} \in \Re^{N\times N}$
$\boldsymbol{\Pi} = \left(\mathbf{I} - \frac{1}{N}\mathbf{1}\mathbf{1}^\mathrm{T}\right) / \sqrt{N-1}$   ▹ projection subtracting ensemble mean; $\boldsymbol{\Pi} \in \Re^{N\times N}$
$\mathbf{E} = \mathbf{D}\boldsymbol{\Pi}$                ▹ scaled measurement perturbations; $\mathbf{E} \in \Re^{m\times N}$
i=0
repeat
    $\mathbf{Y}_i = \mathbf{g}(\mathbf{X}_i, \mathbf{E}_i)\boldsymbol{\Pi}$        ▹ $\mathbf{Y} \in \Re^{m\times N}$
    $\boldsymbol{\Omega}_i = \mathbf{I} + \mathbf{W}_i\boldsymbol{\Pi}$            ▹ $\boldsymbol{\Omega} \in \Re^{N\times N}$
    $\mathbf{S}_i = \mathbf{Y}_i\boldsymbol{\Omega}_i^{-1}$                      ▹ $\mathbf{S} \in \Re^{m\times N}$
    $\mathbf{H}_i = \mathbf{S}_i\mathbf{W}_i + \mathbf{D} - \mathbf{g}(\mathbf{X}_i, \mathbf{E}_i)$   ▹ $\mathbf{H} \in \Re^{m\times N}$
    $\mathbf{W}_{i+1} = \mathbf{W}_i - \gamma\left(\mathbf{W}_i - \mathbf{S}_i^\mathrm{T}\left(\mathbf{S}_i\mathbf{S}_i^\mathrm{T} + \mathbf{E}\mathbf{E}^\mathrm{T}\right)^{-1}\mathbf{H}_i\right)$
    $\mathbf{T}_i = \left(\mathbf{I} + \mathbf{W}_{i+1} \middle/ \sqrt{N-1}\right)$   ▹ $\mathbf{T} \in \Re^{N\times N}$
    $\mathbf{X}_{i+1} = \mathbf{X}\mathbf{T}_i$
    $\mathbf{E}_{i+1} = \mathbf{E}_0\mathbf{T}_i$
    i=i+1
until convergence
```

Reservoir examples

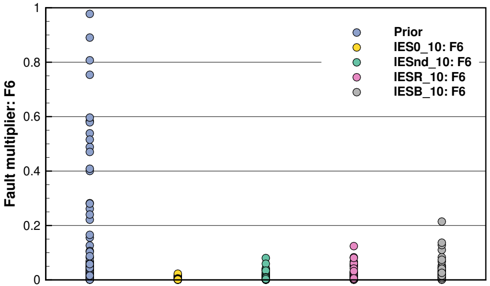
Case	C_{dd}	Noise Model	Use $\boldsymbol{E}_i$
PREDW		White	
PREDR		Red	
PREDB		Bias	
IES0	$\boldsymbol{I}$	White	no
IESnd	$\boldsymbol{E}\boldsymbol{E}^{\mathrm{T}}$	Red	no
IESR	$\boldsymbol{E}\boldsymbol{E}^{\mathrm{T}}$	Red	yes
IESB	$\boldsymbol{E}\boldsymbol{E}^{\mathrm{T}}$	Bias	yes

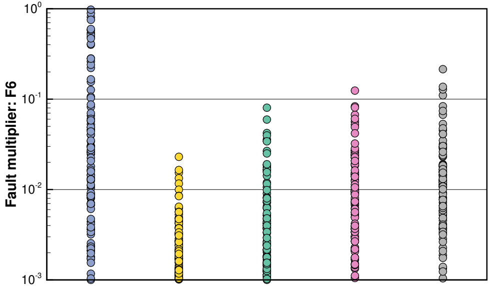
1. PRED cases are forcing-ensemble predictions (without parameter perturbations).
2. IES0 is the standard case neglecting error correlations and errors in controls.
3. IESnd includes error correlations in inversion.
4. IESR adds time-correlated perturbations to rates and updates them.
5. IESB adds perturbation biases to rates and updates them.

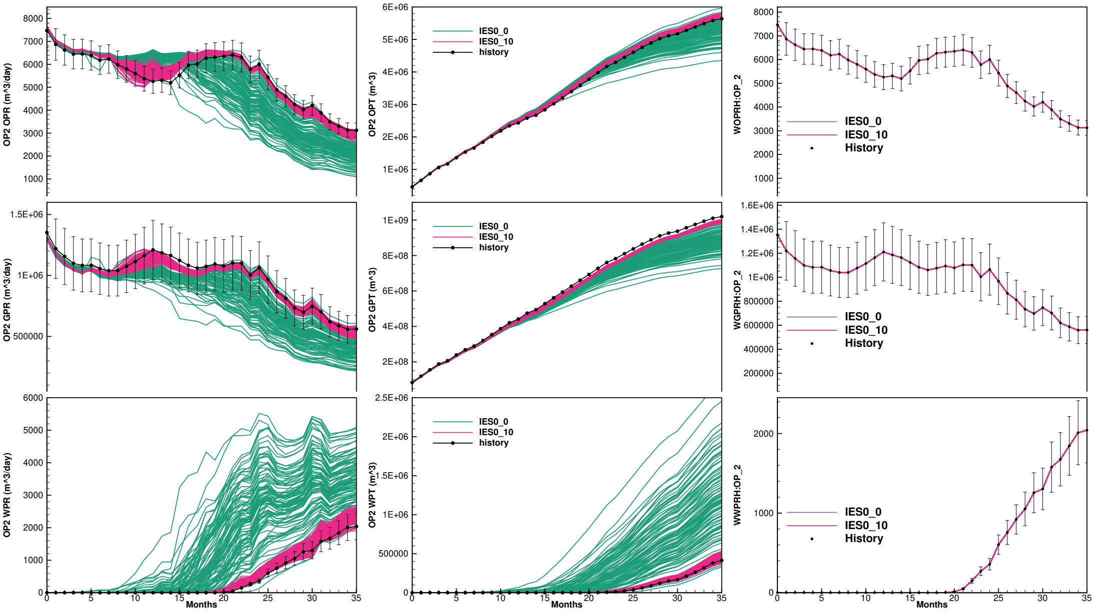
Predictions

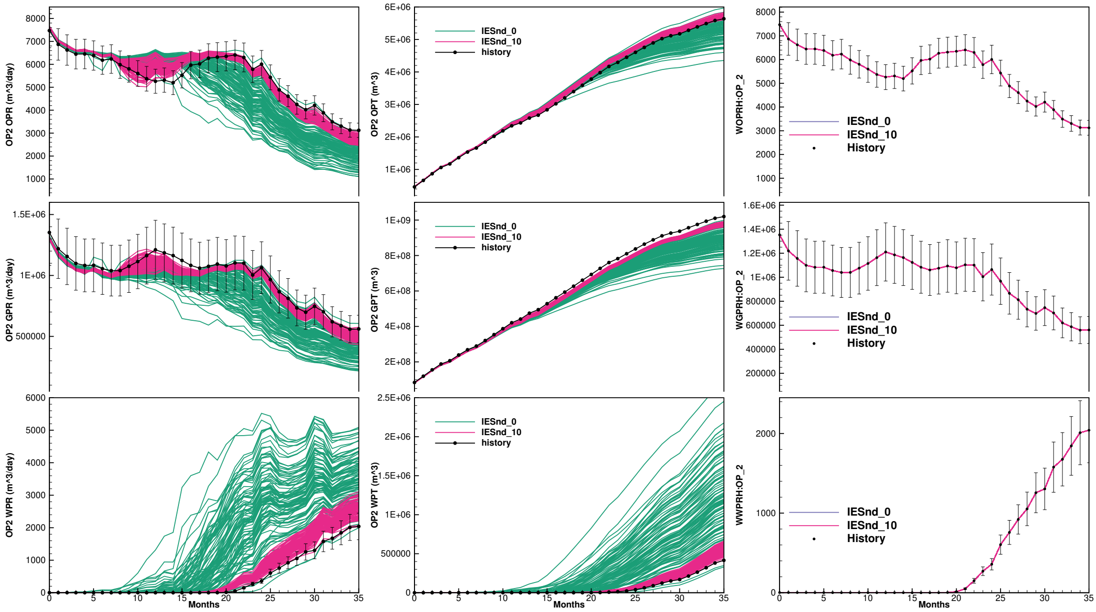


Fault multiplier F6






OP2 OPR (m^3/day)
OP2 OPT (m^3)
WOPRH:OP_2
OP2 GPR (m^3/day)
OP2 GPT (m^3)
WGPRH:OP_2
OP2 WPR (m^3/day)
OP2 WPT (m^3)
WWPRH:OP_2
IES0_0
IES0_10
history
History
Months



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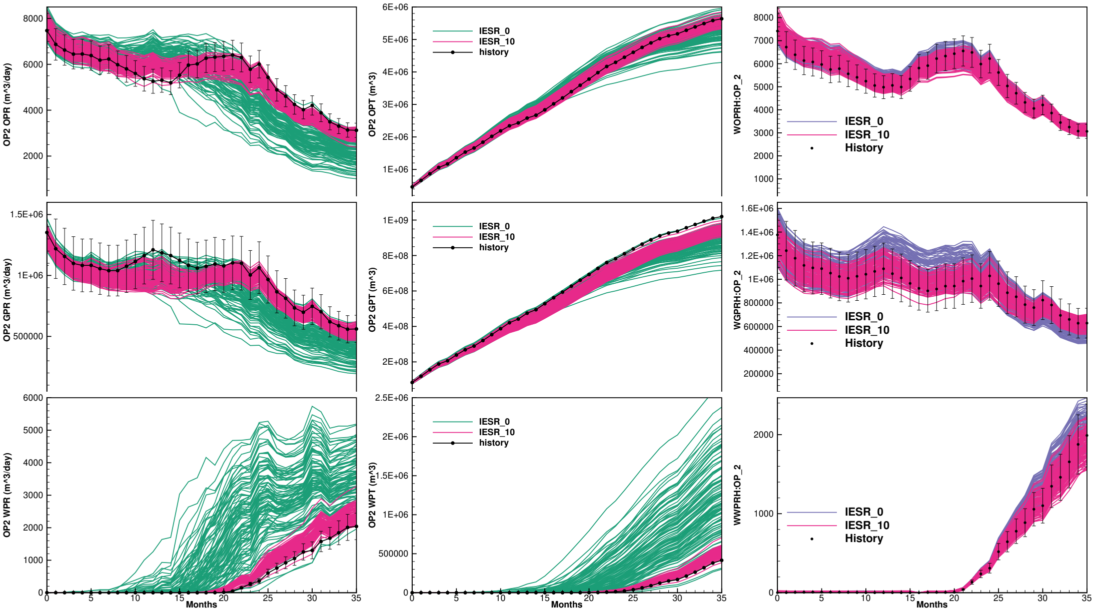
WWPRH:OP_2
IESnd_0
IESnd_10
history
History
Months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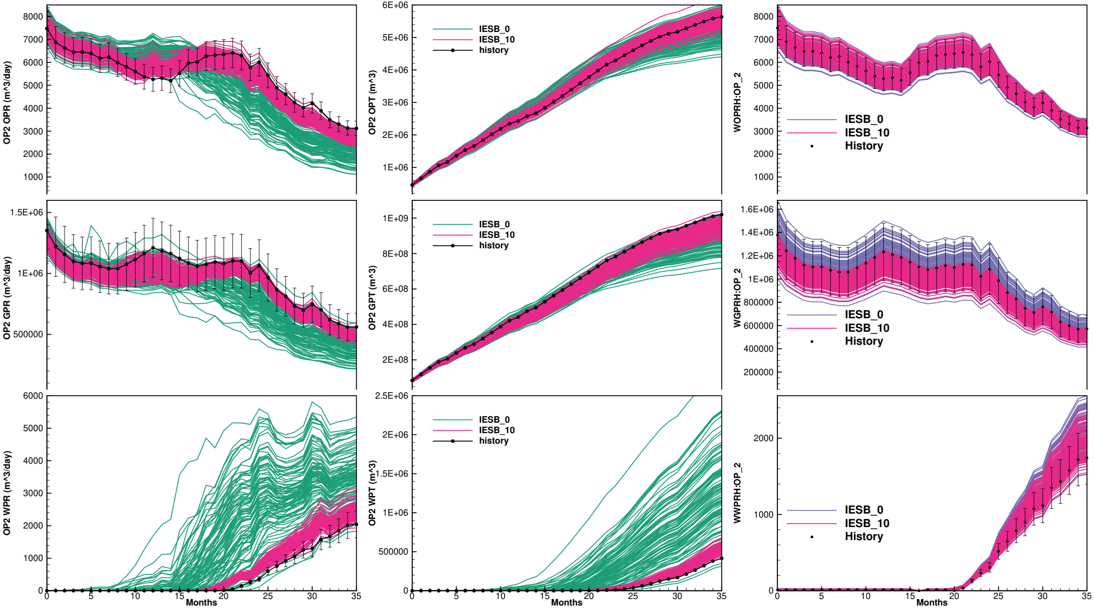
WWPRH:OP_2
IESR_0
IESR_10
history
History
Months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WOPRH:OP_2
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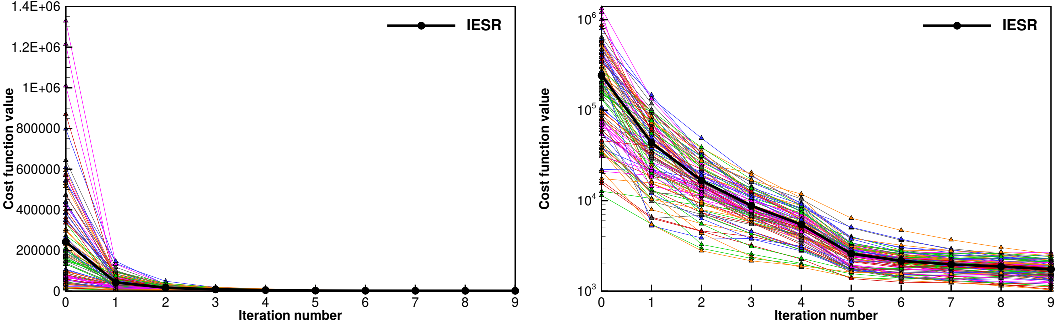
OP2 GPT (m^3)
WGPRH:OP_2
OP2 WPR (m^3/day)
OP2 WPT (m^3)
WWPRH:OP_2
IESB_0
IESB_10
history
History
Months

Ensemble of cost functions



Summary

Rederive consistent HM formulation from Bayes'.

- Condition on rate data while allowing for time-correlated errors.
- Force the model using historical rates with stochastic errors.
- Update stochastic rates as part of the state vector.

Leads to:

- Realistic posterior error statistics.
- Avoids underestimated posterior errors.

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