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Bioinspired Communication Systems

Bayesian Inference of Information Transfer in Graph-Based Continuous-Time Multi-Agent Systems

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Elektro- und Informationstechnik

Eingereicht von

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Abstract

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Contents

List of Symbols

$X(t)$	value of stochastic process X at time t
$X^{[0,T]}$	discrete valued trajectory of stochastic process X in time interval $[0, T]$
x	position
v	velocity
a	acceleration
t	time
F	force

1 Introduction

1.1 Motivation

1.2 Related Work

1.3 Contributions

1.4 Structure of the Thesis

2 Foundations

This chapter presents the theory and models applied in this thesis. First, the details of the communication problem is described, then the mathematical theory of the frameworks used to model this problem is introduced.

2.1 Problem Formulation

The communication model considered in this thesis is given in Figure 2.1. The parent nodes, X_1 and X_2, emit messages which carry information about their states. These messages are translated by an observation model, ψ, and agent node, X_3 makes a decision based on this translated message, y. The main objective is to infer the observation model given the trajectories of nodes.

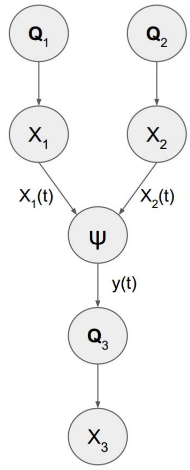

Figure 2.1: Graphical model.

The messages that are emitted by the parent nodes X_1 and X_2 are modelled as independent homogeneous continuous-time Markov processes $X_i(t)$, with state space $\mathcal{X}_i = \{x_1, x_2, ..., x_n\}$ for $i \in \{1, 2\}$. These processes are defined by transition intensity matrices Q_i, where intensities do not depend on time. These matrices are assumed to be gamma distributed.

$$\mathbf{Q}_i \sim Gam(\alpha_i, \beta_i) \quad for\ i \in \{1, 2\}$$

The agent node does not have a direct access to the messages, but observes a translation of them. The observation model is defined as the likelihood of a translation given the parent messages.

$$\psi := p(y(t) \mid X_1(t), X_2(t)) \tag{2.1}$$

The agent X_3 is modelled as inhomogeneous continuous-time Markov process with state space $\mathcal{X}_3 = \{x_1, x_2, ..., x_n\}$, set of actions $a \in \{a_0, a_1, ..., a_k\}$ and set of transition intensity matrices $\boldsymbol{Q}_3 = \left\{\mathbf{Q}_{a_0}, \mathbf{Q}_{a_1}, ..., \mathbf{Q}_{a_k}\right\}$.

$$\mathbf{Q}_a \sim Gam(\alpha_a, \beta_a) \tag{2.2}$$

Given the observation, the agent forms a belief over the parent states, $b(x_1, x_2; t)$, that summarizes the past observations.[1] The policy of the agent, $\pi(a \mid b)$, is assumed to be shaped by evolution (close) to optimality. Based on the belief state, the agent takes an action, which in the setting described above means to change its internal dynamics through choice of intensity matrix.

2.2 Continuous Time Markov Processes

A continuous-time Markov process (CTMP) is a continuous-time stochastic process which satisfies Markov property, namely, the probability distribution over the states at a future time is conditionally independent of the past states given the current state.[2] Let X be a CTMP with state space $\mathcal{X} = \{x_1, x_2, ..., x_n\}$. Then the Markov property can be written as follows:

$$\Pr\left(X^{(t_k)} = x_{t_k} | X^{(t_{k-1})} = x_{t_{k-1}}, \ldots, X^{(t_0)} = x_{t_0}\right) = \Pr\left(X^{(t_k)} = x_{t_k} | X^{(t_{k-1})} = x_{t_{k-1}}\right) \quad (2.3)$$

A CTMP is represented by its transition intensity matrix, $\mathbf{Q}$. In this matrix, the intensity q_i represents the instantaneous probability of leaving state x_i and $q_{i,j}$ represents the instantaneous probability of switching from state x_i to x_j.

$$\mathbf{Q} = \begin{bmatrix} -q_1 & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & -q_2 & \cdots & q_{2,n} \\ \vdots & \vdots & \ddots & \ldots \\ q_{n,1} & q_{n,2} & \cdots & -q_n \end{bmatrix} \quad (2.4)$$

where $q_i = \sum_{i \neq j} q_{i,j}$.[3]

2.2.1 Homogenous Continuous Time Markov Processes

A continuous-time Markov process is time-homogenous when the transition intensities do not depend on time. Let X be a homogenous CTMP, with transition intensity matrix $\mathbf{Q}$. Infinitesimal transition probability from state x_i to x_j in terms of the transition intensities $q_{i,j}$ can be written as [2]:

$$p_{i,j}(h) = \delta_{ij} + q_{i,j} h + o(h) \quad (2.5)$$

where $p_{i,j}(t) \equiv Pr(X^{(t+s)} = j \mid X^{(s)} = i)$ are Markov transition functions and o(.) is a function decaying to zero faster than its argument.

The *master equation* is then derived as follows:

$$\begin{aligned} p_j(t) &= \sum_{\forall i} p_{i,j}(h) p_i(t-h) \\ \lim_{h\to 0} p_j(t) &= \lim_{h\to 0} \sum_{\forall i} \left[\delta_{ij} + q_{i,j} h + o(h)\right] p_i(t-h) \\ &= \lim_{h\to 0} p_j(t-h) + \lim_{h\to 0} h \sum_{\forall i} q_{i,j} p_i(t-h) \\ \lim_{h\to 0} \frac{p_j(t) - p_j(t-h)}{h} &= \lim_{h\to 0} \sum_{\forall i} q_{i,j} p_i(t-h) \\ \frac{d}{dt} p_j(t) &= \sum_{\forall i} q_{i,j} p_i(t) \\ &= \sum_{\forall i \neq j} \left[q_{i,j} p_i(t) - q_{j,i} p_j(t)\right] \end{aligned} \quad (2.6)$$

Eq.2.6 can be written in matrix form:

$$\frac{d}{dt}p = p\mathbf{Q} \tag{2.7}$$

The solution to ODE, the time-dependent probability distribution $p(t)$ is,

$$p(t) = p(0)\exp(t\mathbf{Q}) \tag{2.8}$$

with initial distribution $p(0)$.

The amount of time staying in a state x_i is exponentially distributed with parameter q_i. The probability density function f and cumulative distribution function F for staying in the state x_i [3]:

$$f(t) = q_i \exp(-q_i t), t \geq 0 \tag{2.9}$$
$$F(t) = 1 - \exp(-q_i t), t \geq 0 \tag{2.10}$$

Given the transitioning from state x_i, the probability of landing on state x_j is $q_{i,j}/q_i$.

2.2.1.1 Likelihood Function

Consider a single transition denoted as $d =< x_i, x_j, t >$, where transition occurs from state x_i to x_j after spending t amount of time at state x_i. The likelihood of this transition is the product of the probability of having remained at state x_i for that long, and the probability of transitioning to x_j.

$$\Pr(d \mid \mathbf{Q}) = (q_i exp(-q_i t))\left(\frac{q_{i,j}}{q_i}\right) \tag{2.11}$$

The likelihood of a trajectory sampled from a homogenous CTMC, $X^{[0,T]}$, can be decomposed as the product of the likelihood of single transitions. The sufficient statistics summarizing this trajectory can be written as $T[x_i]$, total amount of time spent in state x_i, $M[x_i, x_j]$ total number of transitions from state x_i to x_j, we can write down the likelihood of a trajectory $X^{[0,T]}$,

$$\begin{aligned}
\Pr(X^{[0,T]} \mid \mathbf{Q}) &= \prod_{d \in X^{[0,T]}} \Pr(d \mid \mathbf{Q}) \\
&= \left(\prod_i q_i^{M[x_i]} \exp(-q_i T[x_i])\right)\left(\prod_i \prod_{j \neq i} \left(\frac{q_{i,j}}{q_i}\right)^{M[x_i,x_j]}\right) \\
&= \prod_i exp(-q_i T[x_i]) \prod_{j \neq i} q_{i,j}^{M[x_i,x_j]}
\end{aligned} \tag{2.12}$$

where $M[x_i] = \sum_{j \neq i} M[x_i, x_j]$ is the total number transitions leaving state x_i.

2.2.1.2 Marginalized Likelihood Function

Let X be a homogenous CTMP. For convenience, it is assumed to be binary-valued, $\mathcal{X} = \{x_0, x_1\}$. The transition intensity matrix can be written in the following form:

$$\mathbf{Q} = \begin{bmatrix} -q_0 & q_0 \\ q_1 & -q_1 \end{bmatrix} \tag{2.13}$$

where the transition intensities q_0 and q_1 are gamma-distributed with parameters α_0, β_0 and α_1, β_1, respectively. The marginal likelihood of a sample trajectory $X^{[0,T]}$ can be written as follows:

$$\begin{aligned}
P(X^{[0,T]}) &= \int P(X^{[0,T]} \mid Q)P(Q)dQ \\
&= \int_0^\infty \left(\prod_x \exp(-q_x T_x) \prod_{x'} q_{xx'}^{M[x,x']}\right) \frac{\beta_{xx'}^{\alpha_{xx'}} q_{xx'}^{\alpha_{xx'}-1} \exp(-\beta_{xx'} q_{xx'})}{\Gamma(\alpha_{xx'})} \, dq_{xx'} \\
&= \prod_{i\in 0,1} \int_0^\infty q_i^{M[x_i]} \, \exp(-q_i T[x_i]) \, \frac{\beta_i^{\alpha_i} \, q_i^{\alpha_i - 1} \, \exp(-\beta_i q_i)}{\Gamma(\alpha_i)} \, dq_i \\
&= \prod_{i\in 0,1} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \int_0^\infty q_i^{M[x_i]+\alpha_i-1} \, \exp(-q_i(T[x_i]+\beta_i)) \, dq_i && (2.14) \\
&= \prod_{i\in 0,1} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \left(-(T_i+\beta_i)^{M[x_i]+\alpha_i} \, \Gamma(M[x_i]+\alpha_i, \; q_i(T[x_i]+\beta_i))\right)\Big|_0^\infty && (2.15) \\
&= \prod_{i\in 0,1} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \left((T[x_i]+\beta_i)^{M[x_i]+\alpha_i} \, \Gamma(M[x_i]+\alpha_i)\right) && (2.16)
\end{aligned}$$

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2.2.2 Inhomogeneous Continuous Time Markov Processes

A continuous-time Markov process is time-inhomogenous when the transition intensities changes over time. For CTMP, Eq.2.9 becomes:

$$f(t) = q_i(t) \exp\left(-\int_0^t q_i(u)du\right) \tag{2.17}$$

2.2.2.1 Likelihood Function

Let X be an inhomogeneous Markov process. $X^{[0,T]}$ is a trajectory sampled from this process with m number of transitions, $0 = t_0 < t_1 < ... < t_m$ are the times where transition occurred, and $x_{t_0}, x_{t_1}, ..., x_{t_m}$ are the observed states. The likelihood of trajectory $X^{[0,T]}$ can be written as follows:

$$L(\mathbf{Q}_X : X^{[0,T]}) = \prod_{k=1}^{m} \left[q_{x_{k-1}}(t_k) \exp\left(-\int_{t_{k-1}}^{t_k} q_{x_{k-1}}(u)du\right) \frac{q_{x_{k-1},x_k}(t_k)}{q_{x_{k-1}}(t_k)} \right] \tag{2.18}$$

2.3 Belief State in Partially Observable Markov Decision Processes

As the agent node X_3 cannot observe the incoming messages directly, it needs to infer In our problem, cell Z does not have a direct access to its parents' states, rather it observes a summary of them. This is different than observing the states through noisy binary channels, and presents a POMDP problem. Even though it is assumed that the policy of the agent is optimal given the true states of parents, the problem of inferring the observational model remains unsolved and to be addressed.

2.3.1 Exact Belief State Update

In a conventional POMDP, given the *transition function*, $T(s, a, s')$ and *observation function*, $O(s', a, o)$, the belief state update is computed as follows [1] [1]:

$$\begin{aligned} b'(s') &= \Pr(s'|o, a, b) = \Pr(s'|o, b) \\ &= \frac{\Pr(o|s', b)\Pr(s'|b)}{\Pr(o|b)} \\ &= \frac{\Pr(o|s')\sum_{s\in\mathcal{S}}\Pr(s'|b, s)\Pr(s|b)}{\Pr(o|b)} \\ &= \frac{O(s', o)\sum_{s\in\mathcal{S}} T(s, s')\, b(s)}{\Pr(o|b)} \end{aligned} \tag{2.19}$$

In Eq.2.19, the transition function is time-independent. However, in our setting, the parent nodes X and Y are modelled as CTMCs with time-dependent transition matrices. Now we can derive the belief state as follows:

$$b(x_1, x_2; t) = P(X_1(t) = x_1, X_2(t) = x_2 \mid y_1, ..., y_t) \tag{2.20}$$

Denote $b(t)$, $t \geq 0$, as row vector with $\{b(x_1, x_2; t)_{x_i \in \mathcal{X}_i}\}$. This posterior probability can be described by a system of ODEs

$$\frac{db(t)}{dt} = b(t)\ \mathbf{T} \tag{2.21}$$

where the initial condition $b(0)$ is row vector with $\{P(X_1(0) = x_1, X_2(0) = x_2)_{x_i \in \mathcal{X}_i}\}$ [5]. $\mathbf{T}$ is the joint transition intensity matrix of X_1 and X_2 and given by amalgamation operation between $\mathbf{Q}_1$ and $\mathbf{Q}_2$ [3].

$$\mathbf{T} = \mathbf{Q}_1 * \mathbf{Q}_2 \tag{2.22}$$

[1] Since it is assumed that there is no affect of agent X_3's action on the observation or transition function, a is omitted from the equation.

The belief update at discrete times of observation y_t

$$
\begin{aligned}
b(x_1, x_2; t) &= P(X_1(t) = x_1, X_2(t) = x_2 \mid y_1, ..., y_t) \\
&= \frac{P(y_1, ..., y_t, X_1(t) = x_1, X_2(t) = x_2)}{P(y_1, ..., y_t)} \\
&= \frac{P(y_t \mid y_1, ..., y_{t-1}, X_1(t) = x_1, X_2(t) = x_2)}{P(y_t \mid y_1, ..., y_{t-1})} \frac{P(y_1, ..., y_{t-1}, X_1(t) = x_1, X_2(t) = x_2)}{P(y_1, ..., y_{t-1})} \\
&= Z_t^{-1} \; P(y_t \mid x_1, x_2) \; P(X_1(t) = x_1, X_2(t) = x_2 \mid y_1, ..., y_{t-1}) \\
&= Z_t^{-1} \; P(y_t \mid x_1, x_2) \; b(x_1, x_2; t^-)
\end{aligned} \tag{2.23}
$$

where $Z_t = \sum_{x_1, x_2 \in X} P(y_t \mid x_1, x_2) \; b(x_1, x_2; t^-)$ is the normalization factor [5].

2.3.2 Belief State Update using Particle Filter

2.3.2.1 Marginalized Continuous Time Bayesian Networks

2.3.2.2 Particle Filter

2.4 Likelihood Model of Communication System (?)

3 Simulation and Experiments

Environment with exact belief update and belief update using particle filter

3.1 Data Generation

3.1.1 Sampling Trajectories

3.1.1.1 Gillespie Algorithm

3.1.1.2 Thinning Algorithm

3.2 Inference of Deterministic Observation Model

Our dataset contains a number of trajectories from all the nodes involved in the communication. $\mathbf{D} = \{D_1, ..., D_N\}$. Every trajectory comprises of state transitions in time interval $[0, T]$, and the times of these transitions.

4 Experimental Results and Evaluation

4.1 Results

4.2 Evaluation

5 Conclusion

5.1 Discussion

5.2 Future Work

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