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Bioinspired Communication Systems

Bayesian Inference of Information Transfer in Graph-Based Continuous-Time Multi-Agent Systems

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Eingereicht von

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Abstract

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Contents

List of Symbols

χ	state space of stochastic process X
$X(t)$	value of stochastic process X at time t
$X^{[0,T]}$	discrete valued trajectory of stochastic process X in time interval $[0, T]$

1 Introduction

1.1 Motivation

1.2 Related Work

1.3 Contributions

1.4 Structure of the Thesis

2 Foundations

This chapter presents the theory and models applied in this thesis. First, the details of the communication problem is described, then the mathematical theory of the frameworks used to model this problem is introduced.

2.1 Problem Formulation

The communication model considered in this thesis is given in Figure 2.1. The parent nodes, X_1 and X_2, emit messages which carry information about their states. These messages are translated by an observation model, ψ, and agent node, X_3 makes a decision based on this translated message, y. The main objective is to infer the observation model given the trajectories of nodes.

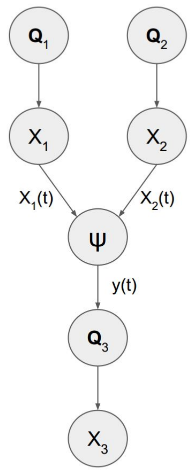

Figure 2.1: Graphical model.

The messages that are emitted by the parent nodes X_1 and X_2 are modelled as independent homogeneous continuous-time Markov processes $X_i(t)$, with state space $\mathcal{X}_i = \{x_1, x_2, ..., x_n\}$ for $i \in \{1, 2\}$. These processes are defined by transition intensity matrices Q_i, where intensities do not depend on time. These matrices are assumed to be gamma distributed.

$$\mathbf{Q}_i \sim Gam(\alpha_i, \beta_i) \quad for \; i \in \{1, 2\}$$

The agent node does not have a direct access to the messages, but observes a translation of them. The observation model is defined as the likelihood of a translation given the parent messages.

$$\psi := p(y(t) \mid X_1(t), X_2(t)) \tag{2.1}$$

The agent X_3 is modelled as inhomogeneous continuous-time Markov process with state space $\mathcal{X}_3 = \{x_1, x_2, ..., x_n\}$, set of actions $a \in \{a_0, a_1, ..., a_k\}$ and set of transition intensity matrices $\boldsymbol{Q}_3 = \left\{\mathbf{Q}_{a_0}, \mathbf{Q}_{a_1}, ..., \mathbf{Q}_{a_k}\right\}$.

$$\mathbf{Q}_a \sim Gam(\alpha_a, \beta_a) \tag{2.2}$$

Given the observation, the agent forms a belief over the parent states, $b(x_1, x_2; t)$, that summarizes the past observations.[1] The policy of the agent, $\pi(a \mid b)$, is assumed to be shaped by evolution (close) to optimality. Based on the belief state, the agent takes an action, which in the setting described above means to change its internal dynamics through choice of intensity matrix.

2.2 Continuous Time Markov Processes

A continuous-time Markov process (CTMP) is a continuous-time stochastic process which satisfies Markov property, namely, the probability distribution over the states at a future time is conditionally independent of the past states given the current state.[2] Let X be a CTMP with state space $\mathcal{X} = \{x_1, x_2, ..., x_n\}$. Then the Markov property can be written as follows:

$$\Pr\left(X^{(t_k)} = x_{t_k} | X^{(t_{k-1})} = x_{t_{k-1}}, \dots, X^{(t_0)} = x_{t_0}\right) = \Pr\left(X^{(t_k)} = x_{t_k} | X^{(t_{k-1})} = x_{t_{k-1}}\right) \quad (2.3)$$

A CTMP is represented by its transition intensity matrix, $\mathbf{Q}$. In this matrix, the intensity q_i represents the instantaneous probability of leaving state x_i and $q_{i,j}$ represents the instantaneous probability of switching from state x_i to x_j.

$$\mathbf{Q} = \begin{bmatrix} -q_1 & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & -q_2 & \cdots & q_{2,n} \\ \vdots & \vdots & \ddots & \dots \\ q_{n,1} & q_{n,2} & \cdots & -q_n \end{bmatrix} \quad (2.4)$$

where $q_i = \sum_{i \neq j} q_{i,j}$.[3]

2.2.1 Homogenous Continuous Time Markov Processes

A continuous-time Markov process is time-homogenous when the transition intensities do not depend on time. Let X be a homogenous CTMP, with transition intensity matrix $\mathbf{Q}$. Infinitesimal transition probability from state x_i to x_j in terms of the transition intensities $q_{i,j}$ can be written as [2]:

$$p_{i,j}(h) = \delta_{ij} + q_{i,j}h + o(h) \quad (2.5)$$

where $p_{i,j}(h) \equiv Pr(X^{(t+h)} = j \mid X^{(t)} = i)$ are Markov transition functions and o(.) is a function decaying to zero faster than its argument.

The *master equation* is then derived as follows:

$$\begin{aligned} p_j(t) &= \Pr(X(t) = x_j) \\ &= \sum_{\forall i} p_{i,j}(h) p_i(t-h) \\ \lim_{h \to 0} p_j(t) &= \lim_{h \to 0} \sum_{\forall i} \left[\delta_{ij} + q_{i,j}h + o(h)\right] p_i(t-h) \\ &= \lim_{h \to 0} p_j(t-h) + \lim_{h \to 0} h \sum_{\forall i} q_{i,j} p_i(t-h) \\ \lim_{h \to 0} \frac{p_j(t) - p_j(t-h)}{h} &= \lim_{h \to 0} \sum_{\forall i} q_{i,j} p_i(t-h) \\ \frac{d}{dt} p_j(t) &= \sum_{\forall i} q_{i,j} p_i(t) \end{aligned} \quad (2.6)$$

Eq.2.6 can be written in matrix form:

$$\frac{d}{dt}p(t) = p(t)\mathbf{Q} \tag{2.7}$$

where the time-dependent probability distribution $p(t)$ is a row vector with entries $p_i(t)_{x_i \in \mathcal{X}}$. The solution of this ODE is,

$$p(t) = p(0)\exp(t\mathbf{Q}) \tag{2.8}$$

with initial distribution $p(0)$.

The amount of time staying in a state x_i is exponentially distributed with parameter q_i. The probability density function f and cumulative distribution function F for staying in the state x_i [3]:

$$f(t) = q_i \exp\left(-q_i t\right), t \geq 0 \tag{2.9}$$
$$F(t) = 1 - \exp\left(-q_i t\right), t \geq 0 \tag{2.10}$$

Given the transitioning from state x_i, the probability of landing on state x_j is $q_{i,j}/q_i$.

2.2.1.1 Likelihood Function

Consider a single transition denoted as $d =< x_i, x_j, t >$, where transition occurs from state x_i to x_j after spending t amount of time at state x_i. The likelihood of this transition is the product of the probability of having remained at state x_i for that long, and the probability of transitioning to x_j.

$$\Pr(d \mid \mathbf{Q}) = (q_i exp(-q_i t)) \left(\frac{q_{i,j}}{q_i}\right) \tag{2.11}$$

The likelihood of a trajectory sampled from a homogenous CTMC, $X^{[0,T]}$, can be decomposed as the product of the likelihood of single transitions. The sufficient statistics summarizing this trajectory can be written as $T[x_i]$, total amount of time spent in state x_i, $M[x_i, x_j]$ total number of transitions from state x_i to x_j, we can write down the likelihood of a trajectory $X^{[0,T]}$,

$$\begin{aligned}
\Pr(X^{[0,T]} \mid \mathbf{Q}) &= \prod_{d \in X^{[0,T]}} \Pr(d \mid \mathbf{Q}) \\
&= \left(\prod_i q_i^{M[x_i]} \exp\left(-q_i T[x_i]\right)\right) \left(\prod_i \prod_{j \neq i} \left(\frac{q_{i,j}}{q_i}\right)^{M[x_i,x_j]}\right) \\
&= \prod_i exp(-q_i T[x_i]) \prod_{j \neq i} q_{i,j}^{M[x_i,x_j]} \qquad (2.12)
\end{aligned}$$

where $M[x_i] = \sum_{j \neq i} M[x_i, x_j]$ is the total number transitions leaving state x_i.

2.2.1.2 Marginalized Likelihood Function

Let X be a homogenous CTMP. For convenience, it is assumed to be binary-valued, $\mathcal{X} = \{x_0, x_1\}$. The transition intensity matrix can be written in the following form:

$$\mathbf{Q} = \begin{bmatrix} -q_0 & q_0 \\ q_1 & -q_1 \end{bmatrix} \tag{2.13}$$

where the transition intensities q_0 and q_1 are gamma-distributed with parameters α_0, β_0 and α_1, β_1, respectively. The marginal likelihood of a sample trajectory $X^{[0,T]}$ can be written as follows:

$$\begin{aligned}
P(X^{[0,T]}) &= \int P(X^{[0,T]} \mid Q)P(Q)dQ \\
&= \int_0^\infty \left(\prod_x \exp(-q_x T_x) \prod_{x'} q_{xx'}^{M[x,x']}\right) \frac{\beta_{xx'}^{\alpha_{xx'}} q_{xx'}^{\alpha_{xx'}-1} \exp(-\beta_{xx'} q_{xx'})}{\Gamma(\alpha_{xx'})}\, dq_{xx'} \\
&= \prod_{i \in 0,1} \int_0^\infty q_i^{M[x_i]} \, \exp(-q_i T[x_i]) \, \frac{\beta_i^{\alpha_i} \, q_i^{\alpha_i - 1} \, \exp(-\beta_i q_i)}{\Gamma(\alpha_i)} \, dq_i \\
&= \prod_{i \in 0,1} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \int_0^\infty q_i^{M[x_i]+\alpha_i - 1} \, \exp(-q_i(T[x_i] + \beta_i)) \, dq_i && (2.14) \\
&= \prod_{i \in 0,1} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \Big(-(T_i + \beta_i)^{M[x_i]+\alpha_i} \, \Gamma(M[x_i] + \alpha_i, \; q_i(T[x_i] + \beta_i)) \Big) \Big|_0^\infty && (2.15) \\
&= \prod_{i \in 0,1} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \Big((T[x_i] + \beta_i)^{M[x_i]+\alpha_i} \, \Gamma(M[x_i] + \alpha_i) \Big) && (2.16)
\end{aligned}$$

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2.2.2 Inhomogeneous Continuous Time Markov Processes

A continuous-time Markov process is time-inhomogenous when the transition intensities changes over time. For CTMP, Eq.2.9 becomes:

$$f(t) = q_i(t) \exp\left(-\int_0^t q_i(u)du\right) \tag{2.17}$$

2.2.2.1 Likelihood Function

Let X be an inhomogeneous Markov process. $X^{[0,T]}$ is a trajectory sampled from this process with m number of transitions, $0 = t_0 < t_1 < ... < t_m$ are the times where transition occurred, and $x_{t_0}, x_{t_1}, ..., x_{t_m}$ are the observed states. The likelihood of trajectory $X^{[0,T]}$ can be written as follows:

$$L(\mathbf{Q}_X : X^{[0,T]}) = \prod_{k=1}^{m} \left[q_{x_{k-1}}(t_k) \exp\left(-\int_{t_{k-1}}^{t_k} q_{x_{k-1}}(u)du\right) \frac{q_{x_{k-1},x_k}(t_k)}{q_{x_{k-1}}(t_k)} \right] \tag{2.18}$$

2.3 Belief State in Partially Observable Markov Decision Processes

Partially observable Markov decision process (POMDP) framework provides a model of an agent which interacts with its environment, but unable to obtain certain information about its state. Instead, the agent gets an observation which is a stochastic function of the true state. The main goal, as similar to Markov decision processes (MDPs), is to learn a policy solving a task by optimizing a reward function. The problem of decision making under uncertainty can be decomposed into two parts for the agent. The first is to keep a belief state which is a sufficient statistic of its past experiences, and the second is to generate an optimal policy which will give an action based on the belief state. [4, 1]

In the problem considered in this thesis, the agent node X_3 cannot observe the incoming messages directly, rather a summary of them. This presents a POMDP problem. However, since the optimal policy of the agent is assumed to be given, the theory for policy optimization is skipped. In this section, update methods for belief state is introduced.

In the following, belief state refers to the posterior probability distribution over the environment states.

2.3.1 Exact/Bayes(?) Belief State Update

Consider a POMDP problem, with state space S, action space A, observation space Ω. In a scneario where a compact representation of the *transition model*, $T(s, a, s')$, and *observation model*, $O(s', a, o)$, is available, the belief state update can be obtain via Bayes' theorem [1]:

$$\begin{aligned} b'\left(s'\right) &= \Pr\left(s'|o, a, b\right) \\ &= \frac{\Pr\left(o|s', a, b\right)\Pr\left(s'|a, b\right)}{\Pr(o|a, b)} \\ &= \frac{\Pr\left(o|s', a\right)\sum_{s\in\mathcal{S}}\Pr\left(s'|a, b, s\right)\Pr(s|a, b)}{\Pr(o|a, b)} \\ &= \frac{O\left(s', a, o\right)\sum_{s\in\mathcal{S}}T\left(s, a, s'\right)b(s)}{\Pr(o|a, b)} \end{aligned} \tag{2.19}$$

2.3.2 Filtering for CTMP

Eq.2.19 is discrete-time solution of belief state. However, since in the model described in Section 2.1, the parent nodes are modelled as CTMPs, and the environment state for the agent is the state of an CTMP, the belief state should be solved in continuous-time. This is achieved by the inference of posterior probability of CTMP.

Let X be a CTMC with transition intensity matrix $\mathbf{Q}$. Assume discrete-time observations denoted by $y_1 = y(t_1), ..., y_N = y(t_N)$. The belief state can be written as:

$$b(x_i; t_N) = \Pr(X(t_N) = x_i \mid y_1, ..., y_N) \tag{2.20}$$

From the master equation given in Eq.2.6, it follows that:

$$\frac{d}{dt} b(x_j; t) = \sum_{\forall i} q_{i,j} b(x_i; t) \tag{2.21}$$

The time-dependent belief state $b(t)$ is a row vector with $\{b(x_i; t)_{x_i \in \mathcal{X}}\}$. This posterior probability can be described by a system of ODEs:

$$\frac{db(t)}{dt} = b(t)\mathbf{Q} \tag{2.22}$$

where the initial condition $b(0)$ is row vector with $\{b(x_i; t)_{x_i \in \mathcal{X}}\}$ [5].

The belief state update at discrete times of observation y_t is derived as

$$\begin{aligned}
b(x_i; t_N) &= \Pr(X(t_N) = x_i, \mid y_1, ..., y_N) \\
&= \frac{\Pr(y_1, ..., y_N, X(t_N) = x_i)}{\Pr(y_1, ..., y_N)} \\
&= \frac{\Pr(y_N \mid y_1, ..., y_{N-1}, X(t_N) = x_i)}{\Pr(y_N \mid y_1, ..., y_{N-1})} \frac{\Pr(y_1, ..., y_{N-1}, X(t_N) = x_i)}{\Pr(y_1, ..., y_{N-1})} \\
&= Z_N^{-1} \; \Pr(y_N \mid X(t_N) = x_i) \; \Pr(X(t_N) = x_i \mid y_1, ..., y_{N-1}) \\
&= Z_N^{-1} \Pr(y_N \mid X(t_N) = x_i) \; b(x_i; t_N^-)
\end{aligned} \tag{2.23}$$

where $Z_N = \sum_{x_i \in \mathcal{X}} \Pr(y_N \mid X(t_N) = x_i) \; b(x_i; t_N^-)$ is the normalization factor [5].

2.3.3 Belief State Update using Particle Filter

In a more realistic scenario, the exact update of belief state may not be feasible for several reasons.

2.3.3.1 Marginalized Continuous Time Bayesian Networks

The marginal process description of a CTMC/CTBN is given as

$$\begin{aligned}
&p(X_n(t + dt) = x' \mid X_n(t) = x, U_n(t) = u, X_{[0,t)}) \\
&\qquad = \int p(X_n(t + dt) = x' \mid X_n(t) = x, U_n(t) = u, Q_n^u, X_{[0,t)}) p(Q_n^u) dQ_n^u \\
&\qquad = \delta(x, x') + \underbrace{\mathbb{E}[Q_n^u(x, x') \mid X_{[0,t]} = x_{[0,t]}]}_{\equiv f_n^u(x' \mid x_{[0,t]})} dt + o(dt),
\end{aligned} \tag{2.24}$$

where we introduced f for brevity. This expression also holds for K trajectories that have been independently drawn from the CTBN; in line with the paper, we denote the joint history of all K trajectories until time point t as $\xi_t = \{x^1_{[0,t]}, ..., x^K_{[0,t]}\}$.

By integrating out the rate matrices Q^u_n, we introduced dependencies between all trajectories; this is readily seen by evaluation of the above expectation:

$$\mathbb{E}[Q^u_n(x,x') \mid \xi_t] = \frac{\alpha^u_n(x,x') + r^u_n(x,x')}{\beta^u_n(x,x') + T^u_n(x)} \tag{2.25}$$

$$= \frac{\alpha^u_n(x,x') + \sum_k r^u_{n,k}(x,x')}{\beta^u_n(x,x') + \sum_k T^u_{n,k}(x)}. \tag{2.26}$$

The expectation of the rate matrix depends on the summary statistics $T^u_n = \sum_k T^u_{n,k}$, $r^u_n = \sum_k r^u_{n,k}$ from all observed trajectories up to the current point in time. Hence, to draw K trajectories from a marginalised CTBN exactly, they have to be simulated jointly, inducing a joint path measure $p(\{X^k_{[0,T]} : k = 1, ..., K\})$. Because this is computationally infeasible, we would like to approximate this resulting joint distribution by a set of K factorising variational distributions: $\prod_k q(X^k_{[0,T]})$ such that

$$\min_q \mathrm{KL}\left(\prod_k q(X^k_{[0,T]}) \mid\mid p(\{X^k_{[0,T]}\}_k)\right). \tag{2.27}$$

2.3.3.2 Particle Filter

Algorithm 1: Marginal particle filter

Input: Measurement data y_k at time t_k, set of particles $\mathbf{p}^{k-1}$, estimated $\hat{Q}$
Result: New set of particles $\mathbf{p}^k$, representing $b(t_k)$

for $p_m \in \mathbf{p}^{k-1}$ **do**
  $p_m = \left\{x_m, \hat{Q}\right\} \leftarrow$ *Propagate particle through marginal process model from* t_{k-1} *to* t_k
  $w_m \leftarrow p(y_k \mid X(t_k) = x_m)$ `// observation likelihood`
  $\hat{Q} \leftarrow$ *sufficient statistics added from* $p_m[t_{k-1}, t_k]$
end for
$w_m \leftarrow \frac{w_m}{\sum_m w_m}$ `// normalize weights`
for $p_m \in \mathbf{p}_k$ **do**
  $p_m \leftarrow$ *Sample from* p_k *with probabilities* w_m *with replacement*
end for

2.4 Likelihood Model of Communication System (?)

3 Experimental Setup

Environment with exact belief update and belief update using particle filter $\mathbf{T}$ is the joint transition intensity matrix of X_1 and X_2 and given by amalgamation operation between $\mathbf{Q}_1$ and $\mathbf{Q}_2$ [3].

$$\mathbf{T} = \mathbf{Q}_1 * \mathbf{Q}_2 \tag{3.1}$$

$$b(x_1, x_2; t) = P(X_1(t) = x_1, X_2(t) = x_2 \mid y_1, ..., y_t) \tag{3.2}$$

3.1 Data Generation

3.1.1 Sampling Trajectories

3.1.1.1 Gillespie Algorithm

3.1.1.2 Thinning Algorithm

3.2 Inference of Deterministic Observation Model

Our dataset contains a number of trajectories from all the nodes involved in the communication. $\mathbf{D} = \{D_1, ..., D_N\}$. Every trajectory comprises of state transitions in time interval $[0, T]$, and the times of these transitions.

4 Experimental Results and Evaluation

4.1 Results

4.2 Evaluation

5 Conclusion

5.1 Discussion

5.2 Future Work

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