Likelihood model

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Let D be a sample of trajectories in the dataset, such that $D = \left\langle X_1^{[0,T]}, X_2^{[0,T]}, X_3^{[0,T]} \right\rangle$, and the set of parameters to the system $\theta = \langle Q_1, Q_2, \pi, \Phi \rangle$, where Φ is observation model, π is optimal stochastic policy, Q_1 and Q_2 are the transition intensity matrices of X_1 and X_2, respectively. Then likelihood of the sample trajectory D can be written as:

$$
\begin{aligned}
P(D \mid \theta) &= P(X_1^{[0,T]}, X_2^{[0,T]}, X_3^{[0,T]} \mid Q_1, Q_2, \pi, \Phi) && (1)\\
&= P(X_3^{[0,T]} \mid X_1^{[0,T]}, X_2^{[0,T]}, Q_1, Q_2, \pi, \Phi)\ P(X_1^{[0,T]} \mid Q_1)\ P(X_2^{[0,T]} \mid Q_2) && (2)\\
&= P(X_3^{[0,T]} \mid X_1^{[0,T]}, X_2^{[0,T]}, \pi, \Phi)\ P(X_1^{[0,T]} \mid Q_1)\ P(X_2^{[0,T]} \mid Q_2) && (3)\\
&= P(X_3^{[0,T]} \mid Q_3^{[0,T]})\ P(X_1^{[0,T]} \mid Q_1)\ P(X_2^{[0,T]} \mid Q_2) && (4)
\end{aligned}
$$

where $Q_3^{[0,T]}$ is the trajectory of transition intensity matrix of X_3 and is a deterministic function of $X_1^{[0,T]}, X_2^{[0,T]}, \pi$ and Φ.

Marginalizing the likelihood over Q_1 and Q_2:

$$
\begin{aligned}
P(D \mid \pi, \Phi) &= \int\int P(D \mid \theta)\ P(Q_1)\ P(Q_2)\ dQ_1 dQ_2 && (5)\\
&= \int\int P(X_3^{[0,T]} \mid Q_3^{[0,T]})\ P(X_1^{[0,T]} \mid Q_1)\ P(X_2^{[0,T]} \mid Q_2)\ P(Q_1)\ P(Q_2)\ dQ_1 dQ_2 && (6)\\
&= P(X_3^{[0,T]} \mid Q_3^{[0,T]}) \int P(X_1^{[0,T]} \mid Q_1)\ P(Q_1)\ dQ_1 \int P(X_2^{[0,T]} \mid Q_2)\ P(Q_2)\ dQ_2 && (7)
\end{aligned}
$$

X_1 and X_2 are independent homogenous Markov processes, with state space $Val(X_{1,2}) = \{0, 1\}$. The transition intensity matrices Q_1 and Q_2 can be written in the following form for convenience,

$$
\begin{pmatrix} -q_0 & q_0 \\ q_1 & -q_1 \end{pmatrix}
$$

where the transition intensities q_0 and q_1 are gamma-distributed with parameters α_0, β_0 and α_1, β_1, respectively. The marginal likelihood of a sample trajectory from binary-valued homogenous Markov process X with transition intensity matrix Q can be written as follows:

$$
\begin{aligned}
P(X^{[0,T]}) &= \int P(X^{[0,T]} \mid Q)P(Q)dQ & (8)\\
&= \int_0^\infty \left(\prod_x \exp(-q_x T_x)\right)\left(\prod_{x'} q_{xx'}^{M[x,x']}\right) \frac{\beta_{xx'}^{\alpha_{xx'}}\, q_{xx'}^{\alpha_{xx'}-1} \exp(-\beta_{xx'} q_{xx'})}{\Gamma(\alpha_{xx'})}\; dq_{xx'} & (9)\\
&= \prod_{x\in 0,1} \int_0^\infty q_x^{M_x}\; \exp(-q_x T_x)\; \frac{\beta_x^{\alpha_x}\; q_x^{\alpha_x-1}\; \exp(-\beta_x q_x)}{\Gamma(\alpha_x)}\; dq_x & (10)\\
&= \prod_{x\in 0,1} \frac{\beta_x^{\alpha_x}}{\Gamma(\alpha_x)} \int_0^\infty q_x^{M_x+\alpha_x-1}\; \exp(-q_x(T_x+\beta_x))\; dq_x & (11)\\
&= \prod_{x\in 0,1} \frac{\beta_x^{\alpha_x}}{\Gamma(\alpha_x)} \left(-(T_x+\beta_x)^{M_x+\alpha_x}\; \Gamma(M_x+\alpha_x,\; q_x(T_x+\beta_x))\right)\Big|_0^\infty & (12)\\
&= \prod_{x\in 0,1} \frac{\beta_x^{\alpha_x}}{\Gamma(\alpha_x)} \left((T_x+\beta_x)^{M_x+\alpha_x}\; \Gamma(M_x+\alpha_x)\right) & (13)
\end{aligned}
$$

where T_x, the amount of time spent in state x, $M[x, x']$ the number of transitions from state x to x' and $M[x] = \sum_{x \neq x'} M[x, x']$.

From Eq.11, the integral is solved using computer algebra system WolframAlpha as follows:

$$
\int x^a\; \exp(-xb)\; dx = -b^{-a-1}\; \Gamma(a+1,\; bx) + C \tag{14}
$$

Plugging Eq.13 in Eq.7 for both X_1 and X_2:

$$
\begin{aligned}
P(D \mid \pi, \Phi) = P(X_3^{[0,T]} \mid Q_3^{[0,T]}) &\prod_{x_1\in 0,1} \frac{\beta_{x_1}^{\alpha_{x_1}}}{\Gamma(\alpha_{x_1})}\; (T_{x_1}+\beta_{x_1})^{M_{x_1}+\alpha_{x_1}}\; \Gamma(M_{x_1}+\alpha_{x_1})\\
&\prod_{x_2\in 0,1} \frac{\beta_{x_2}^{\alpha_{x_2}}}{\Gamma(\alpha_{x_2})}\; (T_{x_2}+\beta_{x_2})^{M_{x_2}+\alpha_{x_2}}\; \Gamma(M_{x_2}+\alpha_{x_2})
\end{aligned} \tag{15}
$$