

Likelihood model

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April 13, 2020

Let D be a sample of trajectories in the dataset, such that $D = \langle X_1^{[0,T]}, X_2^{[0,T]}, X_3^{[0,T]} \rangle$, and the set of parameters to the system $\theta = \langle Q_1, Q_2, \pi, \Phi \rangle$, where Φ is observation model, π is optimal stochastic policy, Q_1 and Q_2 are the transition intensity matrices of X_1 and X_2 , respectively. Then likelihood of the sample trajectory D can be written as:

$$P(D | \theta) = P(X_1^{[0,T]}, X_2^{[0,T]}, X_3^{[0,T]} | Q_1, Q_2, \pi, \Phi) \quad (1)$$

$$= P(X_3^{[0,T]} | X_1^{[0,T]}, X_2^{[0,T]}, Q_1, Q_2, \pi, \Phi) P(X_1^{[0,T]} | Q_1) P(X_2^{[0,T]} | Q_2) \quad (2)$$

$$= P(X_3^{[0,T]} | X_1^{[0,T]}, X_2^{[0,T]}, \pi, \Phi) P(X_1^{[0,T]} | Q_1) P(X_2^{[0,T]} | Q_2) \quad (3)$$

$$= P(X_3^{[0,T]} | Q_3^{[0,T]}) P(X_1^{[0,T]} | Q_1) P(X_2^{[0,T]} | Q_2) \quad (4)$$

where $Q_3^{[0,T]}$ is the trajectory of transition intensity matrix of X_3 and is a deterministic function of $X_1^{[0,T]}, X_2^{[0,T]}, \pi$ and Φ .

Marginalizing the likelihood over Q_1 and Q_2 :

$$P(D | \pi, \Phi) = \int \int P(D | \theta) P(Q_1) P(Q_2) dQ_1 dQ_2 \quad (5)$$

$$= \int \int P(X_3^{[0,T]} | Q_3^{[0,T]}) P(X_1^{[0,T]} | Q_1) P(X_2^{[0,T]} | Q_2) P(Q_1) P(Q_2) dQ_1 dQ_2 \quad (6)$$

$$= P(X_3^{[0,T]} | Q_3^{[0,T]}) \int P(X_1^{[0,T]} | Q_1) P(Q_1) dQ_1 \int P(X_2^{[0,T]} | Q_2) P(Q_2) dQ_2 \quad (7)$$

X_1 and X_2 are independent homogenous Markov processes, with state space $Val(X_{1,2}) = \{0, 1\}$. The transition intensity matrices Q_1 and Q_2 can be written in the following form for convenience,

$$\begin{pmatrix} -q_0 & q_0 \\ q_1 & -q_1 \end{pmatrix}$$

where the transition intensities q_0 and q_1 are gamma-distributed with parameters α_0, β_0 and α_1, β_1 , respectively. The marginal likelihood of a sample trajectory from binary-valued homogenous Markov process X with transition intensity matrix Q can be written as follows:

$$P(X^{[0,T]}) = \int P(X^{[0,T]} | Q) P(Q) dQ \quad (8)$$

$$= \int_0^\infty \left(\prod_x \exp(-q_x T_x) \prod_{x'} q_{xx'}^{M[x,x']} \right) \frac{\beta_{xx'}^{\alpha_{xx'}} q_{xx'}^{\alpha_{xx'}-1} \exp(-\beta_{xx'} q_{xx'})}{\Gamma(\alpha_{xx'})} dq_{xx'} \quad (9)$$

$$= \prod_{x \in 0,1} \int_0^\infty q_x^{M_x} \exp(-q_x T_x) \frac{\beta_x^{\alpha_x} q_x^{\alpha_x-1} \exp(-\beta_x q_x)}{\Gamma(\alpha_x)} dq_x \quad (10)$$

$$= \prod_{x \in 0,1} \frac{\beta_x^{\alpha_x}}{\Gamma(\alpha_x)} \int_0^\infty q_x^{M_x + \alpha_x - 1} \exp(-q_x(T_x + \beta_x)) dq_x \quad (11)$$

$$= \prod_{x \in 0,1} \frac{\beta_x^{\alpha_x}}{\Gamma(\alpha_x)} \left(-(T_x + \beta_x)^{M_x + \alpha_x} \Gamma(M_x + \alpha_x, q_x(T_x + \beta_x)) \right) \Big|_0^\infty \quad (12)$$

$$= \prod_{x \in 0,1} \frac{\beta_x^{\alpha_x}}{\Gamma(\alpha_x)} ((T_x + \beta_x)^{M_x + \alpha_x} \Gamma(M_x + \alpha_x)) \quad (13)$$

where T_x , the amount of time spent in state x , $M[x, x']$ the number of transitions from state x to x' and $M[x] = \sum_{x' \neq x} M[x, x']$.

From Eq.11, the integral is solved using computer algebra system WolframAlpha as follows:

$$\int x^a \exp(-xb) dx = -b^{-a-1} \Gamma(a+1, bx) + C \quad (14)$$

Plugging Eq.13 in Eq.7 for both X_1 and X_2 :

$$P(D | \pi, \Phi) = P(X_3^{[0,T]} | Q_3^{[0,T]}) \prod_{x_1 \in 0,1} \frac{\beta_{x_1}^{\alpha_{x_1}}}{\Gamma(\alpha_{x_1})} (T_{x_1} + \beta_{x_1})^{M_{x_1} + \alpha_{x_1}} \Gamma(M_{x_1} + \alpha_{x_1}) \quad (15)$$

$$\prod_{x_2 \in 0,1} \frac{\beta_{x_2}^{\alpha_{x_2}}}{\Gamma(\alpha_{x_2})} (T_{x_2} + \beta_{x_2})^{M_{x_2} + \alpha_{x_2}} \Gamma(M_{x_2} + \alpha_{x_2})$$