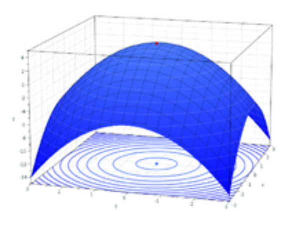
Lecture 8 - Linear Mappings Part 1

COMP1046 - Maths for Computer Scientists

Dr. Ferrante Neri / Dr. Tony Bellotti

The University of Nottingham



Learning outcomes

By the end of this lecture we will have learned:

- Mappings
- Linear Mappings
- Linear Mappings and Vector Spaces

Based on Sections 10.1 and 10.2 of textbook (Neri 2018).

Mappings

Mapping and Domain

Definition

Let $(E, +, \cdot)$ and $(F, +, \cdot)$ be two vector spaces defined over the scalar field $\mathbb{K}$. Let $f : E \rightarrow F$ be a relation.
Let U be a set such that $U \subseteq E$.
The relation f is said *mapping* when

$$\forall \mathbf{u} \in U : \exists! \mathbf{w} \in F \text{ such that } f(\mathbf{u}) = \mathbf{w}.$$

The set U is said *domain* and is indicated with $\mathrm{dom}(f)$.

A vector $\mathbf{w}$ such that

$$\mathbf{w} = f(\mathbf{u})$$

is said to be the *mapped* (or *transformed*) of $\mathbf{u}$ through f.

Image

Definition

Let f be a mapping $E \longrightarrow F$, where E and F are sets associated with the vector spaces $(E, +, \cdot)$ and $(F, +, \cdot)$. The *image* of f, indicated with $\mathrm{Im}\left(f\right)$, is a set defined as

$$\mathrm{Im}\left(f\right) = \left\{\mathbf{w} \in F \quad | \quad \exists \mathbf{u} \in E \text{ such that } f\left(\mathbf{u}\right) = \mathbf{w}\right\}.$$

Mapping and Image

Example

Let $(\mathbb{R}, +, \cdot)$ be a vector space.
An example of mapping $\mathbb{R} \to \mathbb{R}$ is $f(x) = x^2 + 2x + 2$.

- The domain of the mapping $\text{dom}(f) = \mathbb{R}$.
- The image $\text{Im}(f) = [1, \infty[$.

Mapping and Image

Example

Let $(\mathbb{R}^2, +, \cdot)$ and $(\mathbb{R}, +, \cdot)$ be two vector spaces.
An example of mapping $\mathbb{R}^2 \longrightarrow \mathbb{R}$ is $f(x, y) = x + 2y + 2$.

- The domain of the mapping $\operatorname{dom}(f) = \mathbb{R}^2$.
- The image $\operatorname{Im}(f) = \mathbb{R}$.

Example

From the vector spaces $(\mathbb{R}^2, +, \cdot)$ and $(\mathbb{R}^3, +, \cdot)$ an example of mapping $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is
$f(x, y, z) = (x + 2y + -z + 2, 6y - 4z + 2)$.

- The domain of the mapping $\operatorname{dom}(f) = \mathbb{R}^3$.
- The image $\operatorname{Im}(f) = \mathbb{R}^2$.

Surjective, Injective, Bijective

Definition

Let f be a mapping $E \longrightarrow F$, where E and F are sets associated with the vector spaces $(E, +, \cdot)$ and $(F, +, \cdot)$.

- The mapping f is said *surjective* if the image of f coincides with F: $\mathrm{Im}\left(f\right) = F$.
- The mapping f is said *injective* if

$$\forall \mathbf{u}, \mathbf{v} \in E \text{ with } \mathbf{u} \neq \mathbf{v} \Rightarrow f\left(\mathbf{u}\right) \neq f\left(\mathbf{v}\right).$$

- The mapping f is said *bijective* if f is injective and surjective.

Surjective, Injective, Bijective

Example

Consider these mappings:

- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$:
  - Not surjective since $\mathrm{Im}(f) = \mathbb{R}^+$.
  - Not injective because $\exists x_1, x_2$ with $x_1 \neq x_2$ such that $x_1^2 = x_2^2$. For example if $x_1 = 3$ and $x_2 = -3$, thus $x_1 \neq x_2$, it occurs that $x_1^2 = x_2^2 = 9$.
  - Hence not bijective.
- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = e^x$ is injective but not surjective. Hence, this mapping is not bijective.
- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 2x + 2$ is injective and surjective. Hence, this mapping is bijective.

Linear Mappings

Linear Mapping

Definition

Let f be a mapping $E \longrightarrow F$, where E and F are sets associated with the vector spaces $(E, +, \cdot)$ and $(F, +, \cdot)$.
The mapping f is said *linear mapping* if the following properties are valid:

- additivity: $\forall \mathbf{u}, \mathbf{v} \in E : f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$
- homogeneity: $\forall \lambda \in \mathbb{K}$ and $\forall \mathbf{v} \in E : f(\lambda \mathbf{v}) = \lambda f(\mathbf{v})$

Linear Mapping

Example

Let us check the linearity of the mapping $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\forall x : f(x) = e^x.$$

Consider two vectors (numbers in this case) x_1 and x_2.
Calculate $f(x_1 + x_2) = e^{x_1 + x_2}$.
From basic calculus we know that

$$e^{x_1 + x_2} \neq e^{x_1} + e^{x_2}.$$

Therefore, additivity is not verified. Hence the mapping is not linear.

Linear Mapping

Example

Let us consider the linearity of the mapping $f : \mathbb{R} \to \mathbb{R}$

$$\forall x : f(x) = 2x.$$

Consider two vectors (numbers in this case) x_1 and x_2. Then,

$$f(x_1 + x_2) = 2(x_1 + x_2)$$
$$f(x_1) + f(x_2) = 2x_1 + 2x_2.$$

It follows that $f(x_1 + x_2) = f(x_1) + f(x_2)$.
Hence, this mapping is additive.

Continued...

Example

Let us check the homogeneity by considering a generic scalar λ. We have that

$$f(\lambda x) = 2\lambda x$$
$$\lambda f(x) = \lambda 2x.$$

It follows that $f(\lambda x) = \lambda f(x)$.

Hence, since also homogeneity is verified this mapping is linear.

Affine Mapping

Definition

Let f be a mapping $E \longrightarrow F$, where E and F are sets associated with the vector spaces $(E, +, \cdot)$ and $(F, +, \cdot)$.

The mapping f is said *affine mapping* if the mapping

$$g(\mathbf{v}) = f(\mathbf{v}) - f(\mathbf{o})$$

is linear.

Affine Mapping

Example

Consider the mapping $f : \mathbb{R} \to \mathbb{R}, \forall x : f(x) = x + 2$.

Consider two vectors (numbers in this case) x_1 and x_2. Then

$$\begin{aligned} f(x_1 + x_2) &= x_1 + x_2 + 2 \\ f(x_1) + f(x_2) &= x_1 + 2 + x_2 + 2 = x_1 + x_2 + 4. \end{aligned}$$

It follows that $f(x_1 + x_2) \neq f(x_1) + f(x_2)$. Hence, this mapping is not linear. Still, $f(0) = 2$ and

$$g(x) = f(x) - f(0) = x$$

which is a linear mapping. This means that $f(x)$ is an affine mapping.

Linear Mapping

Proposition

Let f be a linear mapping $E \longrightarrow F$.
Let us indicate with $\mathbf{o_E}$ and $\mathbf{o_F}$ the null vectors of the vector spaces $(E, +, \cdot)$ and $(F, +, \cdot)$, respectively.
It follows that

$$f(\mathbf{o_E}) = \mathbf{o_F}.$$

Proof.

$$f(\mathbf{o_E}) = f(0\mathbf{o_E}) = 0f(\mathbf{o_E}) = \mathbf{o_F}.$$

$\square$

Exercise 1: Linear and Affine Mappings

Which of these mappings are linear and which are affine (or both or neither)?:

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^2 + 2$.
- $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 2x - 3y$.
- $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (2x - y + 1, 2y - x - 2)$.

Exercise 1: *Solution*

- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 2x^2 + 2$:
  - Take $x_1 = 1$ and $x_2 = 2$, then
    $f(x_1 + x_2) = 2(x_1 + x_2)^2 + 2 = 11$.
  - However, $f(x_1) + f(x_2) = 4 + 10 = 14$, hence not additive, and so it is not a linear mapping.
  - Also, $f(0) = 2 \Rightarrow g(x) = 2x^2$.
    Again, $g(x_1 + x_2) = 2(x_1 + x_2)^2 = 18$ and
    $g(x_1) + g(x_2) = 2 + 8 = 10$,
    hence not an affine mapping either.

Exercise 1: *Solution*

- $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f\left(x, y\right) = 2x - 3y$:
  - For any $(x_1, y_1) \in \mathbb{R}^2$ and $(x_2, y_2) \in \mathbb{R}^2$, $f(x_1, y_1) + f(x_2, y_2) = (2x_1 - 3y_1) + (2x_2 - 3y_2) = 2(x_1 + x_2) - 3(y_1 + y_2) = f((x_1, y_1) + (x_2, y_2))$ which shows additivity.
  - For any $(x, y) \in \mathbb{R}^2$, $\lambda f(x, y) = \lambda(2x - 3y) = 2(\lambda x) - 3(\lambda y) = f(\lambda(x, y))$ which shows homogeneity.
  - Hence this is a linear mapping.
  - Since $f(x, y) = 0$, $g = f$ and hence this is an affine mapping too.

Exercise 1: *Solution*

- $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f\left(x, y\right) = (2x - y + 1, 2y - x - 2)$.
  - $f(0,1) + f(1,0) = (0,0) + (3,-3) = (3,-3)$. However, $f((0,1) + (1,0)) = f(1,1) = (2,-1)$, so it is not additive and hence not a linear mapping.
  - $f(0,0) = (1,-2) \Rightarrow g(x,y) = (2x - y, 2y - x)$.
  - For any $(x_1, y_1) \in \mathbb{R}^2$ and $(x_2, y_2) \in \mathbb{R}^2$, $g(x_1, y_1) + g(x_2, y_2) = (2x_1 - y_1) + (2x_2 - y_2) = 2(x_1 + x_2) - (y_1 + y_2) = g((x_1, y_1) + (x_2, y_2))$ which shows g is additive.
  - For any $(x, y) \in \mathbb{R}^2$, $\lambda g(x,y) = \lambda(2x - y) = 2(\lambda x) - \lambda y = g(\lambda(x,y))$ which shows g is homogeneous.
  - Hence g is a linear mapping $\Rightarrow f$ is an affine mapping.

Linear Mappings and Vector Spaces

Definition

Let f be a mapping $U \subset E \longrightarrow F$.

The *image of* U *through* f, indicated with $f(U)$, is the set

$$f(U) = \left\{ \mathbf{w} \in F \quad | \quad \exists \mathbf{u} \in U \text{ such that } f(\mathbf{u}) = \mathbf{w} \right\}.$$

Theorem

Let $f : E \longrightarrow F$ *be a linear mapping and* $(U, +, \cdot)$ *be a vector subspace of* $(E, +, \cdot)$.

It follows that the triple $\left(f(U), +, \cdot\right)$ *is a vector subspace of* $(F, +, \cdot)$.

Proof.

In order to prove that $\left(f\left(U\right),+,\cdot\right)$ is a vector subspace of $(F,+,\cdot)$ we have to show that the set $f\left(U\right)$ is closed with respect to the two composition laws.

By definition, the fact that a vector $\mathbf{w} \in f\left(U\right)$ means that $\exists \mathbf{v} \in U$ such that $f\left(\mathbf{v}\right) = \mathbf{w}$.

Thus, if we consider two vectors $\mathbf{w}, \mathbf{w}' \in f\left(U\right)$ then

$$\mathbf{w} + \mathbf{w}' = f\left(\mathbf{v}\right) + f\left(\mathbf{v}'\right) = f\left(\mathbf{v} + \mathbf{v}'\right).$$

Since for hypothesis $(U,+,\cdot)$ is a vector space, then $\mathbf{v} + \mathbf{v}' \in U$. Hence, $f\left(\mathbf{v} + \mathbf{v}'\right) \in f\left(U\right)$. The set $f\left(U\right)$ is therefore closed with respect to the internal composition law. *continued...*

Proof.

Let us now consider a generic scalar $\lambda \in \mathbb{K}$ and calculate

$$\lambda \mathbf{w} = \lambda f(\mathbf{v}) = f(\lambda \mathbf{v}).$$

Since for hypothesis $(U, +, \cdot)$ is a vector space, then $\lambda \mathbf{v} \in U$. Hence, $f(\lambda \mathbf{v}) \in f(U)$. The set $f(U)$ is closed with respect to the external composition law.

Since the set $f(U)$ is closed with respect to both the composition laws the triple $\left(f(U), +, \cdot\right)$ is a vector subspace of $(F, +, \cdot)$. $\square$

Inverse image

Definition

Let f be a mapping $E \longrightarrow W \subset F$.

The *inverse image of* W *through* f, indicated with $f^{-1}(W)$, is a set defined as

$$f^{-1}(W) = \left\{ \mathbf{u} \in E \quad | \quad f(\mathbf{u}) \in W \right\}.$$

Theorem

Let $f : E \longrightarrow F$ *be a linear mapping.*

If $(W, +, \cdot)$ *is a vector subspace of* $(F, +, \cdot)$*,*

then $\left(f^{-1}(W), +, \cdot\right)$ *is a vector subspace of* $(E, +, \cdot)$*.*

Proof.

In order to prove that $\left(f^{-1}(W), +, \cdot\right)$ is a vector subspace of $(E, +, \cdot)$ we have to prove the closure of $f^{-1}(W)$ with respect to the two composition laws.
If a vector $\mathbf{v} \in f^{-1}(W)$ then $f(\mathbf{v}) \in W$.

We can write for the linearity of f

$$f(\mathbf{v} + \mathbf{v}') = f(\mathbf{v}) + f(\mathbf{v}').$$

Since $(W, +, \cdot)$ is a vector space, $f(\mathbf{v}) + f(\mathbf{v}') \in W$.
Hence, $f(\mathbf{v} + \mathbf{v}') \in W$ and $\mathbf{v} + \mathbf{v}' \in f^{-1}(W)$.
Thus, the set $f^{-1}(W)$ is closed with respect to the first composition law.

continued...

Inverse image

Proof.

Let us consider a generic scalar $\lambda \in \mathbb{K}$ and calculate

$$f(\lambda \mathbf{v}) = \lambda f(\mathbf{v}).$$

Since $(W, +, \cdot)$ is a vector space, $\lambda f(\mathbf{v}) \in W$. Since $f(\lambda \mathbf{v}) \in W$ the $\lambda \mathbf{v} \in f^{-1}(W)$. Thus, the set $f^{-1}(W)$ is closed with respect to the second composition law.

Hence, $\left(f^{-1}(W), +, \cdot\right)$ is a vector subspace of $(E, +, \cdot)$. $\square$

Example

Let us consider the linear mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f\left(x, y\right) = \left(x + y, x - y\right)$$

and the set $U = \{\left(x, y\right) \in \mathbb{R}^2 \mid 2x - y = 0\}$.

It can be easily checked that $(U, +, \cdot)$ is a vector space. This set can be represented by means of the vector

$$U = \alpha\left(1, 2\right)$$

with $\alpha \in \mathbb{R}$.

Continued...

Note: Here we use the shorthand $\alpha\mathbf{v}$ for $\{\alpha\mathbf{v} : \alpha \in \mathbb{R}\}$.

Example

Let us now calculate $f(U)$ by replacing (x, y) with $(\alpha, 2\alpha)$:

$$f(U) = (\alpha + 2\alpha, \alpha - 2\alpha) = (3\alpha, -\alpha) = \alpha(3, -1)$$

that is a line passing through the origin. Hence also

$$(f(U), +, \cdot)$$

is a vector space.

Summary and next lecture

Summary

- Mappings
- Linear Mappings
- Linear Mappings and Vector Spaces

The next lecture

We will learn about Endomorphisms, Kernels and Rank-Nullity Theorem.