Lecture 10

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Topics covered in this lecture session

1. Differential equations: Introduction

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1. Differential equations: Introduction
2. Terminology

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1. Differential equations: Introduction
2. Terminology
3. Solving variable separable form of ODEs

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1. Differential equations: Introduction
2. Terminology
3. Solving variable separable form of ODEs
4. Applications.

Differential equations: Introduction

What is a Differential Equation?

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e.g. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + xy = 0$ is a differential equation.

Differential equations: Introduction

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- velocity (distance and time)
- acceleration (velocity and time)
- pressure (force and area)
- charge (current and time).

Differential equations: Introduction

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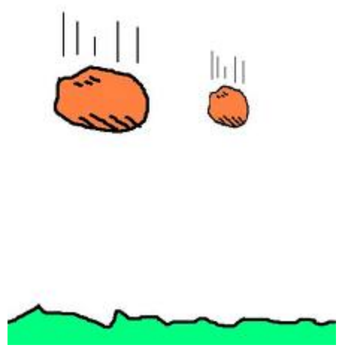
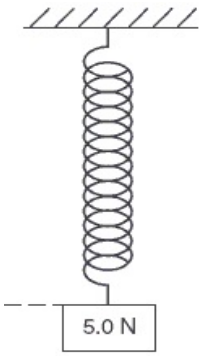
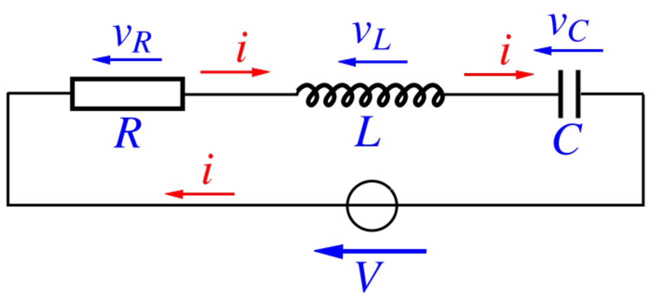
What do these pictures have in common?

Differential equations: Introduction

What do these pictures have in common?

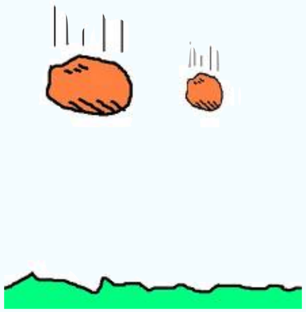


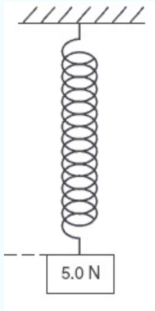


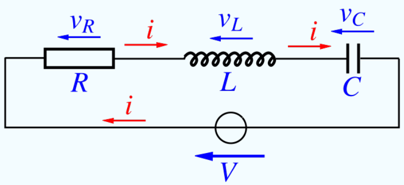


Differential equations: Introduction

Differential equations govern these processes (that changes with time).







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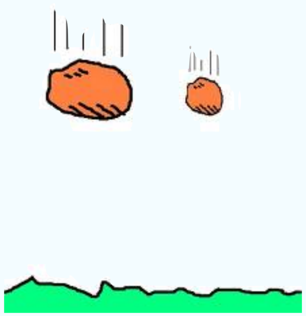
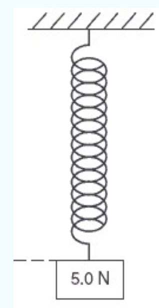
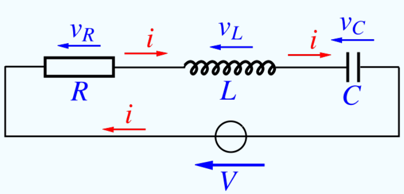


$$\frac{d^2 s}{dt^2} = -g$$

g gravitational acceleration
s distance
t time travelled





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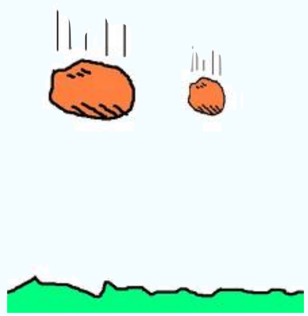

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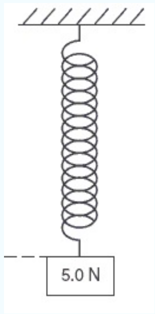
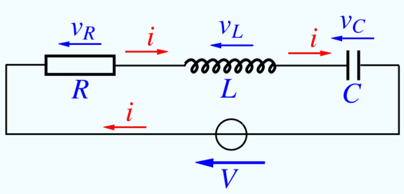


$$m\,\frac{d^2 y}{dt^2} = -\,k\,y$$

y	vertical displacement
m	mass
k	spring constant ($k > 0$)
t	time



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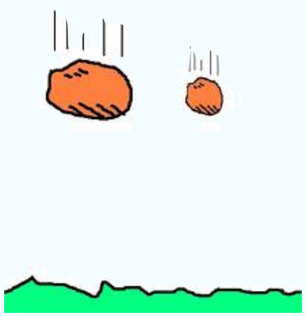

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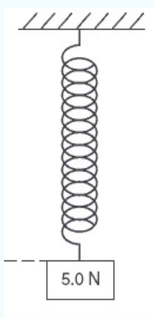

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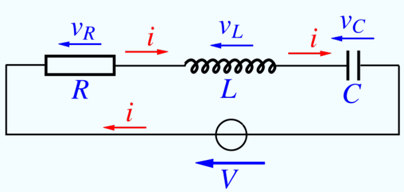

$$L\,\frac{d^2 q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = E$$

q	charge on capacitor
L	inductance
C	capacitance
R	resistance
E	voltage
t	time

Differential equations: Introduction

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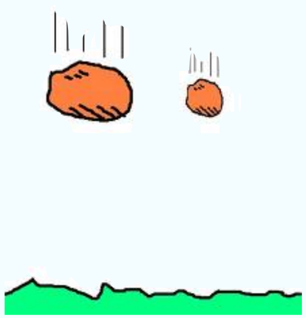

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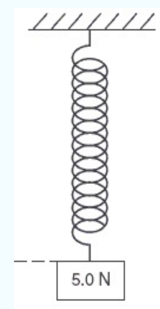

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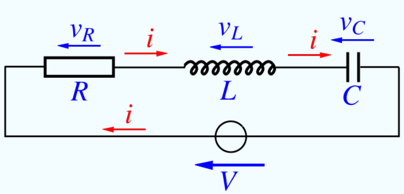

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q	charge on capacitor
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$$\frac{dm}{dt} = k\,m \quad (k < 0)$$

m	mass of radioactive substance
k	constant
t	time

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i.e. studying only the methods of solving ODEs.

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A theoretical perspective:

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An applied perspective:

i.e. using differential equations to model real world problems.

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An equation involving derivatives of more than one dependent variable with respect to one or more independent variable.

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We shall focus only on ODEs in this module.

Terminology: Order and Degree

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The order of the highest derivative involved in a differential equation is called the order of the differential equation.

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Terminology: Solution of an ODE

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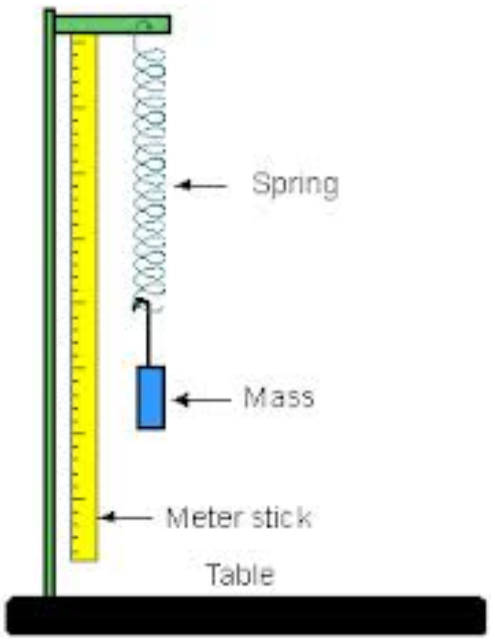
$y = A\sin mx + B\cos mx$ is a solution of the ODE $\dfrac{d^2y}{dx^2} + m^2 y = 0$

Terminology: Solution of an ODE

e.g. The function

$$x = x(t) = c_1 \cos \sqrt{\frac{k}{m}}\, t + c_2 \sin \sqrt{\frac{k}{m}}\, t$$

where c_1 and c_2 are arbitrary constants, m is mass, and k is the spring constant is a solution of the differential equation for the vibrating spring $m \dfrac{d^2x}{dt^2} + k\,x = 0$



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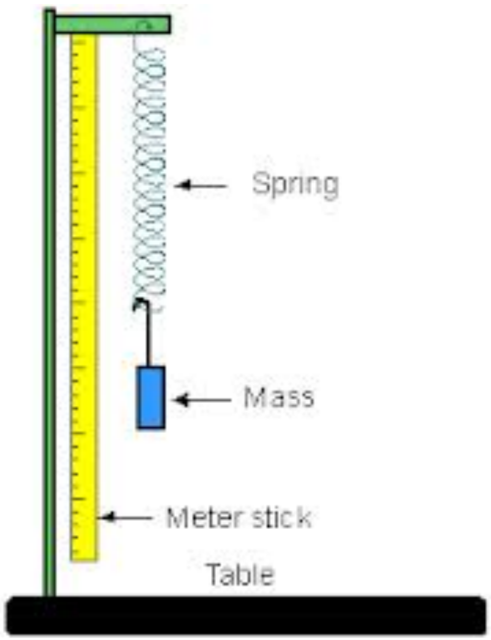

The corresponding motion of the spring is referred to as the **Simple Harmonic Motion**.

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2) Particular solution: having a particular value of C based on some given condition called **initial condition**.

Terminology: Integral curves

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Suppose that the general solution to the differential equation

$$F\left(x, y, \frac{dy}{dx}, \ldots., \frac{d^n y}{dx^n}\right) = 0$$ is $f(x, y, C) = 0.$

Terminology: Integral curves

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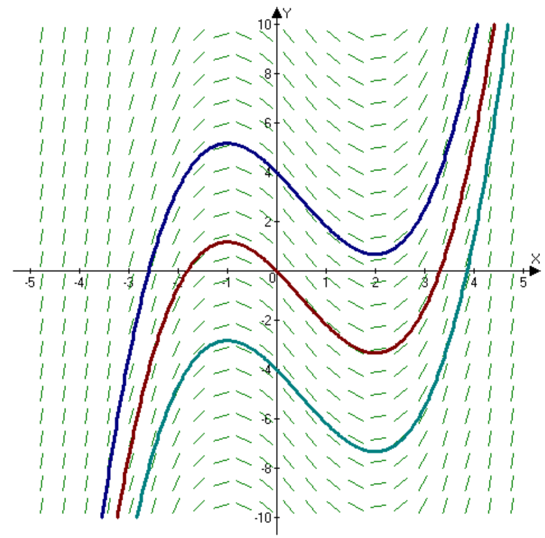

Integral curves of $\dfrac{dy}{dx} = x^2 - x - 1$

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e.g. the general solution to the

ODE $$\frac{dy}{dx} = -\frac{x}{y}$$

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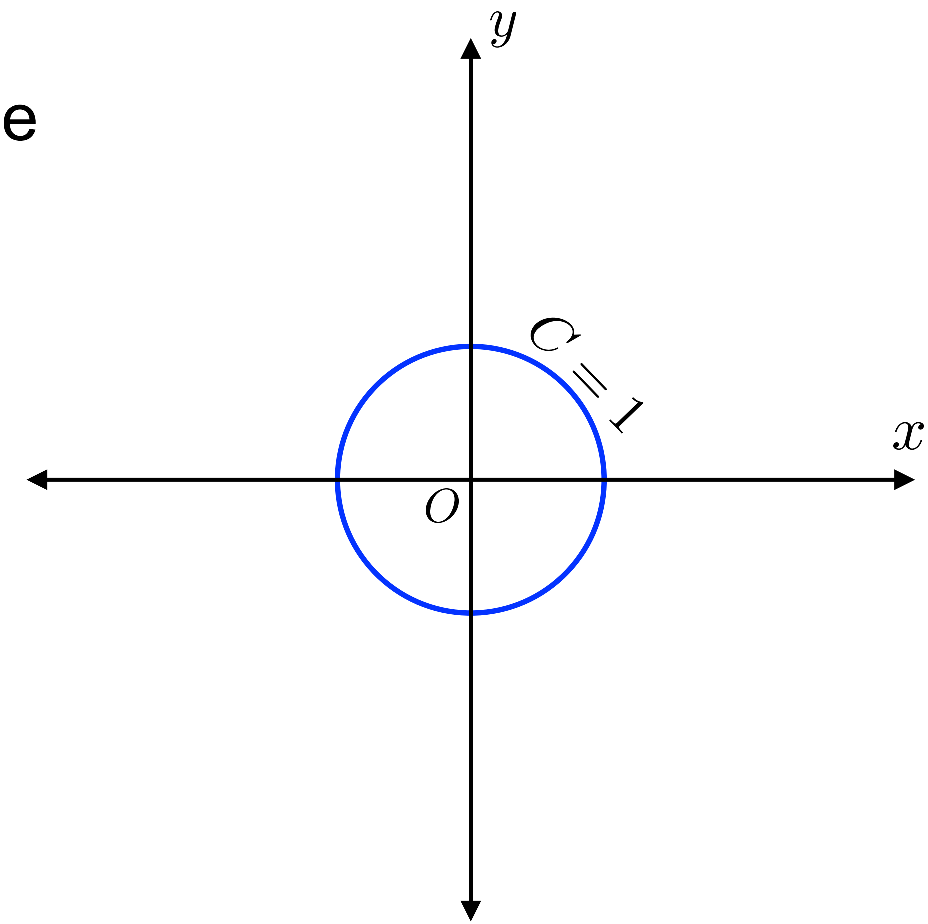

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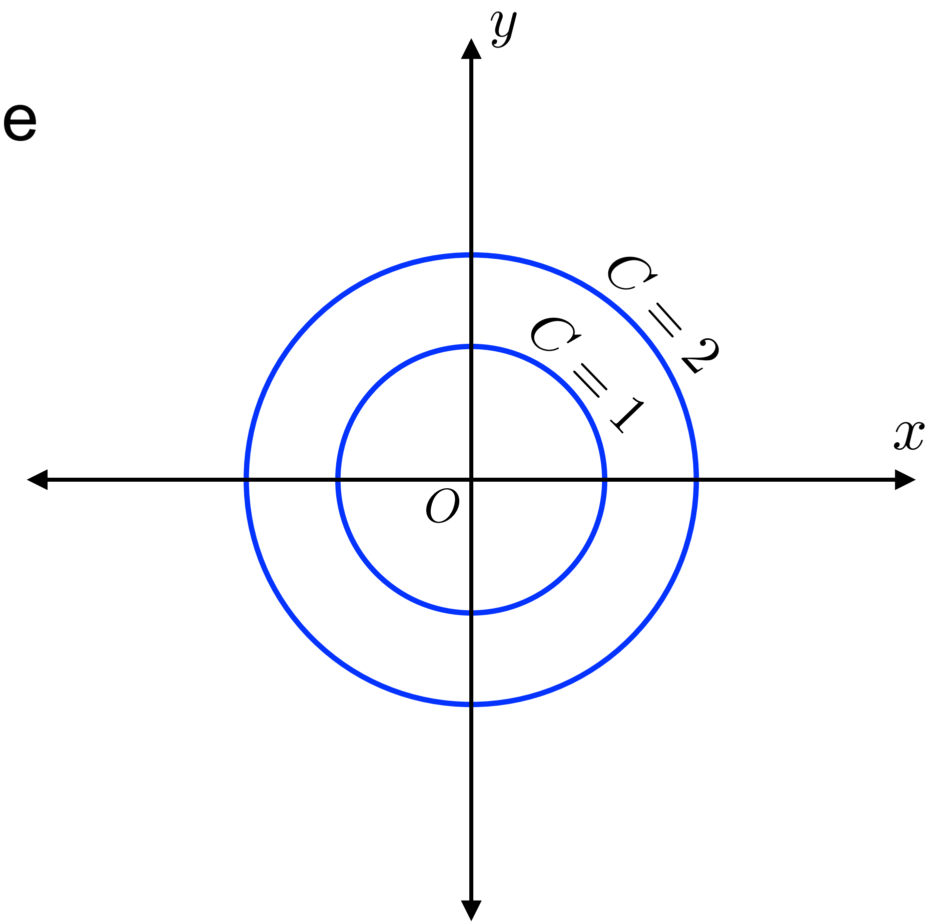

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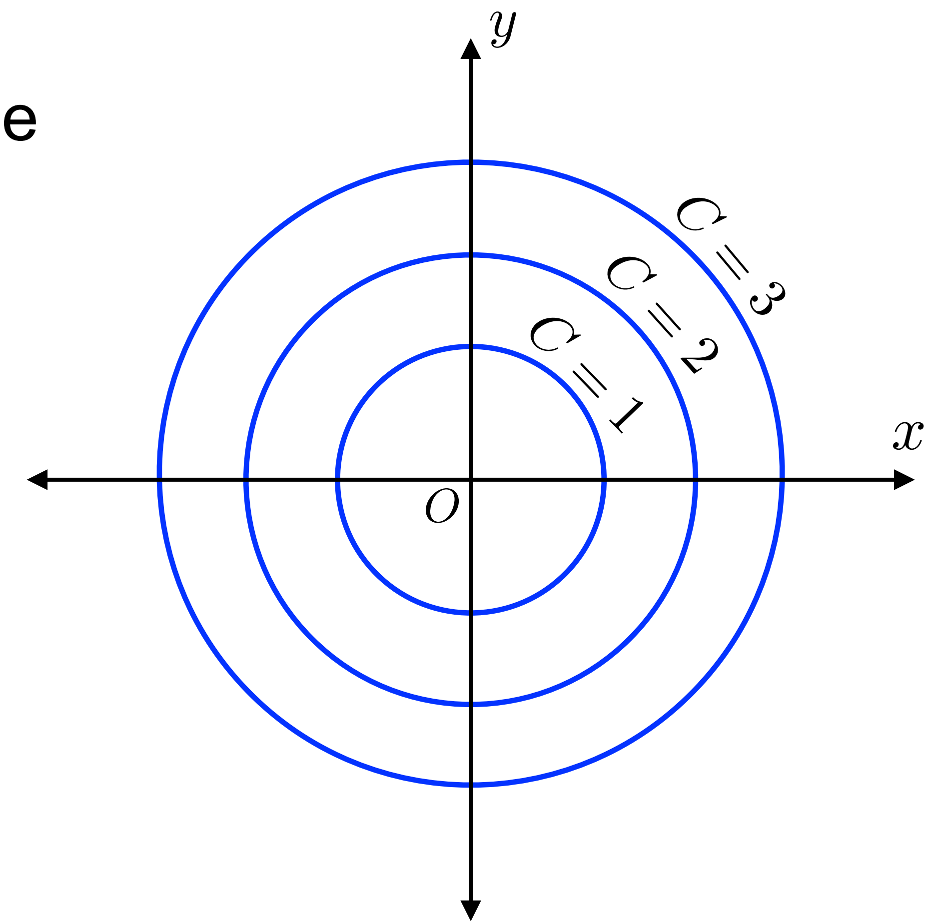

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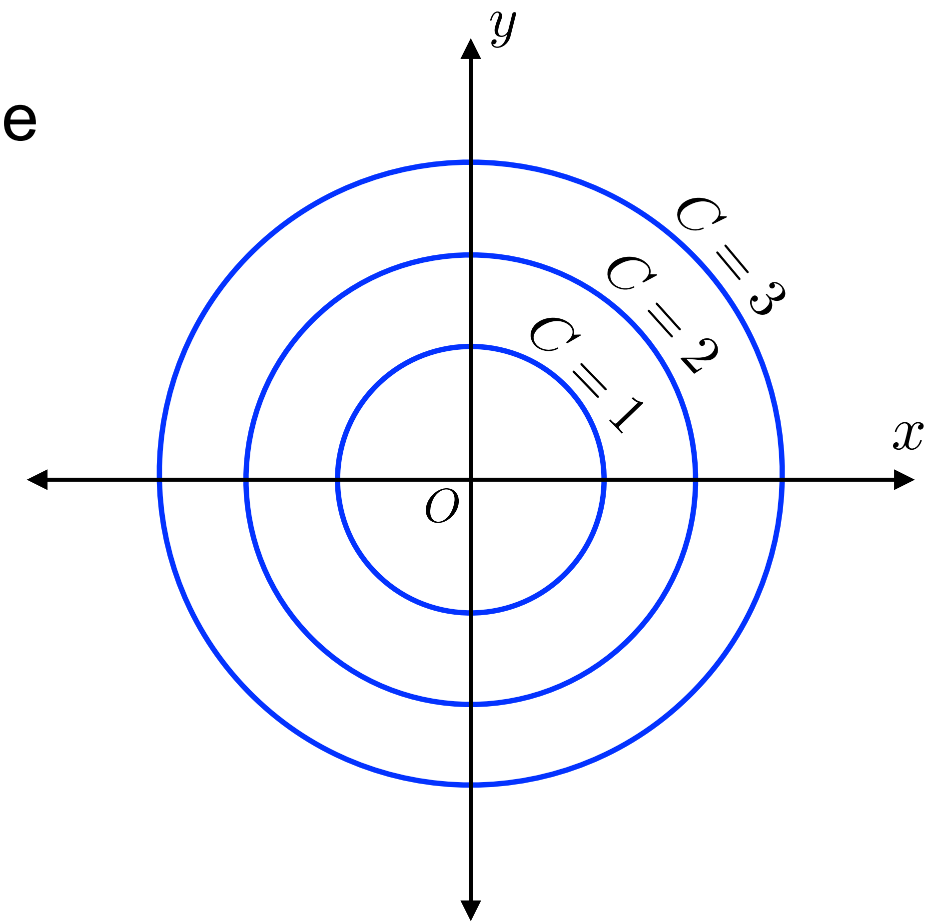
$\therefore$ The integral curves are concentric circles.



Solving ODEs

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In this module, we study only one analytical method to solve the ODE of order 1 and degree 1
(i.e. first order-first degree ODE).

Solving ODEs

There are two fundamental methods to solve an ODE:

1) Analytical methods
2) Numerical methods.

In this module, we study only one analytical method to solve the ODE of order 1 and degree 1
(i.e. first order-first degree ODE).

The ODE is called: Variable-Separable type ODE.

Solving variable-separable type ODEs

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$$\Rightarrow \ G(y) = F(x) + C$$

where $G(y)$ and $F(x)$ are antiderivatives of $g(y)$ and $f(x)$ resp.

Solving variable-separable type ODEs

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$\Rightarrow \ y^2 + \sin y = 2x^3 + C$ is the general solution of the given ODE, where C is an arbitrary constant.

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$$\therefore \quad y^3 = 3x^2 + 9$$

is a particular solution.

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Applications of ODEs

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1. Exponential growth model

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Applications of ODEs

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t			
P			

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t	0		
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t	0	10	
P	P_0	$2P_0$	

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Applications of ODEs

$$\Rightarrow \ln P = k\,t + C$$

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When $t = 0, \; P = P_0$

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$$\Rightarrow \ \ln P = k\,t + \ln P_0$$

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$$\Rightarrow \ln P = k\,t + \ln P_0$$

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = k\,t$$

Applications of ODEs

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P	P_0	$2P_0$	$?$

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Applications of ODEs

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$$\therefore \quad \ln\left(\frac{2P_0}{P_0}\right) = k\,(10)$$

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$$\Rightarrow \ln P = k\,t + C$$

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t	0	10	30
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When $t = 30,$

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Applications of ODEs

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t	0	10	30
P	P_0	$2P_0$	$?$

When $t = 30,$

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$$\Rightarrow P = 8\,P_0$$

Applications of ODEs

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = \frac{t}{10}\ln 2$$

t	0	10	30
P	P_0	$2P_0$	$?$

When $t = 30,$

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = \frac{30}{10}\ln 2 = \ln 8$$

$$\Rightarrow P = 8\,P_0$$

$\therefore$ The population will be 8 times the original after 30 years.

Applications of ODEs

2. Exponential decay model

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t			
m			

Applications of ODEs

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t	0		
m	m_0		

Applications of ODEs

Example 5: Radium decays at a rate proportional to the amount (m) present at time t.

If it takes 1600 years for half the original amount to decay, find the percentage of the original amount that remains after 200 years.

t	0	1600	
m	m_0	$\frac{m_0}{2}$	

Applications of ODEs

Example 5: Radium decays at a rate proportional to the amount (m) present at time t.

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t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

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$$\frac{dm}{dt} \propto m$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

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$$\frac{dm}{dt} \propto m$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

Applications of ODEs

Example 5: Radium decays at a rate proportional to the amount (m) present at time t.

If it takes 1600 years for half the original amount to decay, find the percentage of the original amount that remains after 200 years.

$$\frac{dm}{dt} \propto m$$

$$\Rightarrow \quad \frac{dm}{dt} = k\,m$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

Applications of ODEs

Example 5: Radium decays at a rate proportional to the amount (m) present at time t.

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$$\Rightarrow \int \frac{dm}{m} = k \int dt$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

Applications of ODEs

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$$\frac{dm}{dt} \propto m$$

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$$\Rightarrow \ln m = k\,t + C$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	?

Applications of ODEs

$$\Rightarrow \ln m = k\,t + C$$

t	0	1600	200
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Applications of ODEs

$$\Rightarrow \ln m = k\,t + C$$

When $t = 0,\ m = m_0$

t	0	1600	200
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Applications of ODEs

$$\Rightarrow \quad \ln m = k\,t + C$$

When $t = 0,\ m = m_0$

$$\therefore \quad \ln m_0 = k\,(0) + C$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

Applications of ODEs

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t	0	1600	200
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When $t = 0, \ m = m_0$

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Applications of ODEs

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Applications of ODEs

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$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = k\,t$$

Applications of ODEs

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Applications of ODEs

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When $t = 1600,\ m = \dfrac{m_0}{2}$

$$\therefore \ \ln\left(\frac{m_0}{2\,m_0}\right) = k\,(1600)$$

Applications of ODEs

$$\Rightarrow \ \ln m = k\,t + C$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

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When $t = 1600, \ m = \dfrac{m_0}{2}$

$$\therefore \ \ln\left(\frac{m_0}{2\,m_0}\right) = k\,(1600)$$

$$\therefore \ k = \frac{1}{1600}\ln\left(\frac{1}{2}\right)$$

Applications of ODEs

$$\Rightarrow \ln m = k\,t + C$$

t	0	1600	200
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$$\therefore\ \ln\left(\frac{m_0}{2\,m_0}\right) = k\,(1600)$$

$$\therefore\ k = \frac{1}{1600}\ln\left(\frac{1}{2}\right) < 0$$

Applications of ODEs

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Applications of ODEs

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Applications of ODEs

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = \frac{t}{1600}\ln\left(\frac{1}{2}\right)$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

When $t = 200,$

Applications of ODEs

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = \frac{t}{1600}\ln\left(\frac{1}{2}\right)$$

t	0	1600	200
m	m_0	$\frac{m_0}{2}$	$?$

When $t = 200,$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = \frac{200}{1600}\ln\left(\frac{1}{2}\right)$$

Applications of ODEs

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$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = \frac{200}{1600}\ln\left(\frac{1}{2}\right)$$

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Applications of ODEs

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Applications of ODEs

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$\therefore$ After 200 years, 91.7% of the original amount remains.

Applications of ODEs

3. Newton's law of cooling

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Experiments show that the time rate of change of the temperature T of a body is proportional to the difference between T and the temperature of the surrounding medium.

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This law is **Newton's law of cooling**.

Applications of ODEs

Let
$$\begin{aligned} T &\equiv \text{temperature inside the building (dependent on t)} \\ t &\equiv \text{time (independent variable)} \\ T_0 &\equiv \text{outside temperature (constant)} \end{aligned}$$

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$$\frac{dT}{dt} \propto (T - T_0)$$

Applications of ODEs

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\begin{aligned}
T &\equiv \text{temperature inside the building (dependent on t)} \\
t &\equiv \text{time (independent variable)} \\
T_0 &\equiv \text{outside temperature (constant)}
\end{aligned}
$$

$\therefore$ By Newton's law of cooling,

$$\frac{dT}{dt} \propto (T - T_0) \Rightarrow \frac{dT}{dt} = k\,(T - T_0)$$

where $k < 0$ is the proportionality constant.

Applications of ODEs

Example 6:

According to Newton's law of cooling, the rate at which the temperature of a body falls is proportional to the difference between the temperature of the body and the room temperature.

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A room is at a constant temperature of 20^{o}C. An object with temp. 80^{o}C is brought into the room and 5 minutes later, its temperature falls to 65^{o}C. What will its temperature be after a further interval of 5 minutes?

Applications of ODEs

t			
T			

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Applications of ODEs

t	0		
T	80		

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Applications of ODEs

t	0	5	
T	80	65	

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Applications of ODEs

t	0	5	10
T	80	65	$?$

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Applications of ODEs

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Given $T_0 = 20$

$$\therefore \ln{(T - 20)} = k\,t + C$$

Applications of ODEs

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Given $T_0 = 20$

$$\therefore \ \ln(T - 20) = k\,t + C$$

t	0	5	10
T	80	65	?

$$\therefore \ \ln(80 - 20) = k\,(0) + C$$

Applications of ODEs

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Given $T_0 = 20$

$$\therefore \ \ln(T - 20) = k\,t + C$$

t	0	5	10
T	80	65	?

$$\therefore \ \ln(80 - 20) = k\,(0) + C$$

$$\Rightarrow \ C = \ln 60$$

Applications of ODEs

$$\therefore \ \ln(T-20) = k\,t + \ln 60$$

Applications of ODEs

$$\therefore \ \ln(T-20) = k\,t + \ln 60$$

$$\therefore \ \ln\left(\frac{T-20}{60}\right) = k\,t$$

Applications of ODEs

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$$\therefore \ \ln\left(\frac{T-20}{60}\right) = k\,t$$

When $t = 5,\ T = 65$

Applications of ODEs

$$\therefore \quad \ln(T-20) = k\,t + \ln 60$$

$$\therefore \quad \ln\left(\frac{T-20}{60}\right) = k\,t$$

When $t = 5,\ T = 65$

$$\therefore \quad \ln\left(\frac{65-20}{60}\right) = k\,(5)$$

Applications of ODEs

$$\therefore \ \ln(T - 20) = k\,t + \ln 60$$

$$\therefore \ \ln\left(\frac{T - 20}{60}\right) = k\,t$$

When $t = 5, \ T = 65$

$$\therefore \ \ln\left(\frac{65 - 20}{60}\right) = k\,(5)$$

$$\Rightarrow \ k = \frac{1}{5}\ln\left(\frac{3}{4}\right)$$

Applications of ODEs

$$\therefore \ \ln(T-20) = k\,t + \ln 60$$

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When $t = 5, \ T = 65$

$$\therefore \ \ln\left(\frac{65-20}{60}\right) = k\,(5)$$

$$\Rightarrow \ k = \frac{1}{5}\ln\left(\frac{3}{4}\right) < 0$$

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$$\therefore \ \ln\left(\frac{T-20}{60}\right) = \frac{t}{5}\ln\left(\frac{3}{4}\right)$$

Applications of ODEs

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$$\Rightarrow \ k = \frac{1}{5}\ln\left(\frac{3}{4}\right) < 0$$

$$\therefore \ \ln\left(\frac{T-20}{60}\right) = \frac{t}{5}\ln\left(\frac{3}{4}\right)$$

$\therefore$ When $t = 10,$

Applications of ODEs

$$\therefore \ \ln(T-20) = k\,t + \ln 60$$

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$$\Rightarrow \ k = \frac{1}{5}\ln\left(\frac{3}{4}\right) < 0$$

$$\therefore \ \ln\left(\frac{T-20}{60}\right) = \frac{t}{5}\ln\left(\frac{3}{4}\right)$$

$\therefore$ When $t = 10,$

$$\ln\left(\frac{T-20}{60}\right) = \ln\left(\frac{3}{4}\right)^2$$

Applications of ODEs

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$\therefore$ When $t = 10,$

$$\ln\left(\frac{T-20}{60}\right) = \ln\left(\frac{3}{4}\right)^2$$

$$\Rightarrow \ T - 20 = \frac{60\cdot 9}{16}$$

Applications of ODEs

$$\therefore \ \ln(T-20) = k\,t + \ln 60$$

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i.e. $T = 56^0\,C$ **after 10 min.**

Next week…… Revision session