The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN/SPRING SEMESTER 2022-2023

Mathematics for Computer Scientists

Time allowed: 2 Hours and 0 Minutes

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced

Answer all FOUR questions. All questions are worth 25 marks each, hence the total mark is 100.

No calculators are permitted in this examination.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

ADDITIONAL MATERIAL: None.
INFORMATION FOR INVIGILATORS: None.

Question 1: This question is about *Logic, Proof, and Functions*.

[**overall 25 marks**]

a. Use a truth table to prove or disprove the following statements:

(i) $\neg(p \vee (q \wedge r)) \equiv (\neg p) \wedge (\neg q \vee \neg r)$

(ii) $\neg(p \wedge (q \vee r)) \equiv (\neg p) \vee (\neg q \vee \neg r)$

(6 marks)

b. Is the following implication always true? If yes, prove it using the rules of predicate logic; otherwise, provide a counter-example.

$$(\exists x P(x) \wedge \exists Q(x)) \rightarrow \forall x(P(x) \vee Q(x))$$

(4 marks)

c. Use induction to prove that for all integers $n \geq 1$,

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

(9 marks)

d. If $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are both one-to-one and onto, the function $(f+g) : \mathbb{R} \rightarrow \mathbb{R}$ is defined by the formula $(f+g)(x) = f(x) + g(x)$ for all real numbers. Is the function $f+g$ one-to-one? Is the function $f+g$ onto? Justify your answer. (6 marks)

Question 2: This question is about *Set, Relations, Counting, and Probability*.

[**overall 25 marks**]

a. For all sets A and B, prove or disprove $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$, where $\mathcal{P}(X)$ is the power set of X. (5 marks)

b. Let A be the set of all propositional statements. A relation **R** is defined on A as follows: For all $p, q \in A$,

$$\{(p, q) \in \mathbf{R} \mid p \to q \text{ is true.}\}$$

Determine whether the relation is reflexive, symmetric, anti-symmetric, transitive or none of these. Justify your answer.

(8 Marks)

c. A four-sided die is engraved with the number 1 through 4 on its four different sides. Suppose that when rolled, each side (and hence each number) has an equal probability of being the bottom face when it lands. We roll two such dice. Let X be the sum of the numbers on the bottom faces of the two dice.

(i) What is the probability that X is at least five? (2 marks)

(ii) How does your answer to (i) change if you are told that the bottom face of the first die has the number "2" on it? (5 Marks)

(iii) How does your answer to (i) change if you are told that the bottom face of one of the dice has the number "3" on it? (5 Marks)

Question 3: This question is about *Vector Spaces*.

[**overall 25 marks**]

For this question, the scalar set $\mathbb{R}$, and "+" and "." are the usual internal and external composition laws for real numbers; i.e. the real number vector sum and scalar product respectively.

a. Consider these two sets:

$$U = \{(x, y) \in \mathbb{R}^2 \mid 2x + y = 0\}$$
$$V = \{(x, y) \in \mathbb{R}^2 \mid 2x + y \neq 0\}$$

Show whether or not each set has closure with respect to the internal and external composition laws. Then determine whether or not it is a vector space.

(6 marks)

b. Let $E = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - 2z = 0\}$.

(i) For what value of γ do the vectors $(0, 1, \gamma), (1, 0, 1)$ form a basis of E?

(3 marks)

(ii) What is dim(E)? (1 mark)

c. For what value of k, are the vectors $(0, 1, 1), (1, 0, 1), (1, 1, k)$ linearly dependent?

(2 marks)

d. Give the definition of linear span. (2 marks)

e. Show that the linear span

$$L((0, 1, 0, 1, 0), (0, 2, 1, 0, 0)) = \{(v, w, x, y, z) \in \mathbb{R}^5 \mid v = 0, w - 2x - y = 0, z = 0\}.$$

(4 marks)

f. Let $(E, +, \cdot)$ be a vector space, with some vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in E$.
Prove that the span $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ with the composition laws is a vector subspace of $(E, +, \cdot)$. (6 marks)

g. Let $(E, +, \cdot)$ be a vector space with vector subspace $(F, +, \cdot)$.
Which one of these statements is false? (1 mark)

(i) dim$(F) \leq$ dim(E);

(ii) dim$(F) =$ dim(E) if and only if $F = E$;

(iii) dim$(F) >$ dim(E).

Question 4: This question is about *Matrices and Geometric Mappings*.

[**overall 25 marks**]

a. Write the identity matrix with 3 rows. (1 mark)

b. Invert the following three matrices using any method, or if it is not invertible, explain why. (6 marks)

$$\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -2 & 1 \\ 6 & -3 \end{pmatrix}$$

c. Let **A**, **B** and **C** be three matrices, and **A** is a $n \times n$ matrix (i.e. it has n rows and n columns). Consider the expression

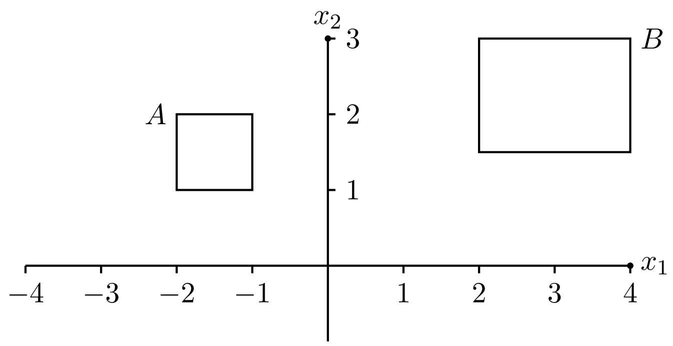
$$\mathbf{E} = \left[\mathbf{C}^T \left(\mathbf{A}^T \mathbf{B} + \mathbf{C}^{-1}\right)^T \mathbf{A}^{-1}\right]^T$$

(i) Show that if **E** can be computed, then **B** and **C** are $n \times n$ matrices. (2 marks)

(ii) Show that $\mathbf{E} = \mathbf{BC} + \left(\mathbf{A}^{-1}\right)^T$. (6 marks)

(iii) How many rows and columns does **E** have? (2 marks)

d. Consider the geometric mapping from square A to rectangle B.



Each point in this space is represented by a vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(i) Express the geometric mapping from A to B as two transformations in sequence: a reflection followed by a scaling. Use matrix representation to express both transformations.

(4 marks)

(ii) Express the geometric mapping from A to B as a single matrix representation.

(2 marks)

(iii) If the two transformations are reversed, i.e. the scaling followed by the rotation, will this give the same geometric mapping from A to B? (2 marks)