# AE1MCS: Mathematics for Computer Scientists

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# Aim and Learning Objectives

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
- To gain a good understanding of other definitions: tautology, contradiction, contingency, logical equivalence, converse, contrapositive and inverse.
- To be able to draw truth tables and use them as a tool to solve logical problems;
- To be able to apply important logical equivalences to solve logical problems.

# Proposition

**Definition**

A *proposition* is a statement that is either true or false.

The area of logic that deals with propositions is called the *propositional logic* or *propositional calculus*.

# Is it a proposition?

1. Beijing is the capital of China.
2. $1 + 1 = 2$.
3. $2 + 2 = 3$.
4. What time is it?
5. Read this sentence carefully.
6. $x + 1 = 2$.
7. $x + y = z$.
8. If $x > 0$, then $x > 1$.

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4. Every map can be colored with 4 colors so that adjacent regions have different colors [Four Color Theorem].
5. Every even integer greater than 2 is the sum of two primes [Goldbach's conjecture, 1742].

# Propositional Variable

- a variable that represents a proposition
- denoted using a letter $p, q, r, s, \ldots$
- truth value: T (true); F (false)

# Logical Operators

- **Compound Proposition:** formed from existing propositions using logical operators
- Logical Operators
  - Negation
  - Conjunction
  - Disjunction
  - Implication
  - ...

# Negation

## Definition (Negation)

Let $p$ be a proposition. The *negation* of $p$, denoted by $\neg p$, is the statement

'It is not the case that p'.

The proposition $\neg p$ is read 'not $p$'. The truth value of $\neg p$ is the opposite of the truth value of $p$.

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

# Conjunction

## Definition (Conjunction)

Let $p$ and $q$ be propositions. The *conjunction* of $p$ and $q$, denoted by $p \wedge q$, is the proposition

'$p$ and $q$'.

The proposition $p \wedge q$ is true when both $p$ and $q$ are true and is false otherwise.

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

# Disjunction

## Definition (Disjunction)

Let $p$ and $q$ be propositions. The *disjunction* of $p$ and $q$, denoted by $p \vee q$, is the proposition

$$\text{`}p \text{ or } q\text{'}.$$

The proposition $p \vee q$ is false when both $p$ and $q$ are false and is true otherwise.

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Exclusive Or

### Definition (Exclusive Or)

Let $p$ and $q$ be propositions. The *exclusive or* of $p$ and $q$, denoted by $p \oplus q$, is the proposition that is true when *exactly one* of $p$ and $q$ is true and is false otherwise.

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

# Implication

## Definition (Implication)

Let $p$ and $q$ be propositions. The *implication* $p \rightarrow q$ is the proposition

'if $p$, then $q$'.

The proposition $p \rightarrow q$ is false when $p$ is true and $q$ is false, and true otherwise. $p$ is called the hypothesis or premise and $q$ is called the conclusion or consequence.

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The proposition $p \rightarrow q$ is true, if $p$ is false or $q$ is true.

# Examples

- If Goldbach's Conjecture is true,
  then $x^2 \geq 0$ for every real number $x$.
- If pigs fly, then your account will not get hacked.

# Bi-Implication

## Definition (Bi-Implication)

Let $p$ and $q$ be propositions. The *bi-implication* $p \leftrightarrow q$ is the proposition

'$p$ if and only if $q$'.

The bi-implication $p \leftrightarrow q$ is true when $p$ and $q$ have the same truth values, and is false otherwise.

The words 'if and only if' are sometimes abbreviated 'iff'.

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

$p \leftrightarrow q$ is true when both $p \to q$ and $q \to p$ are true, and is false otherwise.

# More Definitions

- Tautology, Contradiction and Contingency
- Logical Equivalence
- Converse, Contrapositive and Inverse

# Tautology, Contradiction and Contingency

**Definition (Tautology, Contradiction and Contingency)**

A compound proposition that is *always true*, no matter what the truth values of the propositions that occur in it, is called a **tautology**. A compound proposition that is *always false* is called a **contradiction**. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

# Exercise

How to construct a tautology, a contradiction and a contingency using just one propositional variable?

# Logical Equivalence

**Definition (Equivalence)**

The compound propositions $p$ and $q$ are logically equivalent, if they always have the same truth value (i.e. $p \leftrightarrow q$ is a tautology). The notation $p \equiv q$ denotes that $p$ and $q$ are logically equivalent.

# Exercise

- Are $\neg(p \vee q)$ and $\neg p \wedge \neg q$ logically equivalent? Why?
- Are $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent? Why?

[Hint: construct truth tables]

# Converse, Contrapositive and Inverse

- The **converse** of $p \to q$ is the proposition $q \to p$.
- The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$.
- The **inverse** of $p \to q$ is the proposition $\neg p \to \neg q$.

Which pairs of the following propositions are equivalent? Why?

- a conditional statement and its converse
- a conditional statement and its contrapositive
- a conditional statement and its inverse

# Some Important Logical Equivalences

| | Equivalence | Name |
|---|---|---|
| 1 | $p \wedge T \equiv p$ | Identity laws |
| 2 | $p \vee F \equiv p$ | |
| 3 | $p \vee T \equiv T$ | Domination laws |
| 4 | $p \wedge F \equiv F$ | |
| 5 | $p \vee p \equiv p$ | Idempotent laws |
| 6 | $p \wedge p \equiv p$ | |
| 7 | $\neg(\neg p) \equiv p$ | Double negation law |
| 8 | $p \vee q \equiv q \vee p$ | Commutative laws |
| 9 | $p \wedge q \equiv q \wedge p$ | |

# Some Important Logical Equivalences

| | Equivalence | Name |
|---|---|---|
| 10 | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ | Associative laws |
| 11 | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ | |
| 12 | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | Distributive laws |
| 13 | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ | |
| 14 | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ | De Morgan's laws |
| 15 | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ | |
| 16 | $p \vee (p \wedge q) \equiv p$ | Absorption laws |
| 17 | $p \wedge (p \vee q) \equiv p$ | |
| 18 | $p \vee \neg p \equiv T$ | Negation laws |
| 19 | $p \wedge \neg p \equiv F$ | |

# Logical Equivalences involving Implications

| | |
|---|---|
| 20 | $p \rightarrow q \equiv \neg p \vee q$ |
| 21 | $p \rightarrow q \equiv \neg q \rightarrow \neg p$ |
| 22 | $p \vee q \equiv \neg p \rightarrow q$ |
| 23 | $p \wedge q \equiv \neg(p \rightarrow \neg q)$ |
| 24 | $\neg(p \rightarrow q) \equiv p \wedge \neg q$ |
| 25 | $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$ |
| 26 | $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$ |
| 27 | $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$ |
| 28 | $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ |

# Logical Equivalences involving Bi-Implications

| | |
|---|---|
| 29 | $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ |
| 30 | $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ |
| 31 | $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ |
| 32 | $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$ |

# Using De Morgan's Laws

Use De Morgan's laws to express the negations of the following sentences.

- Tony has a cellphone and he has a laptop computer.
- Heather will go to the concert or Steve will go to the concert.

# Constructing New Logical Equivalences

- A proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.
- Prove two propositions are logically equivalent by developing a series of logical equivalences.

# Exercise

- Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.
- Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.
- Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

# Expected Learning Outcomes

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
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# Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Sections 1.1-1.3.