AE1MCS: Mathematics for Computer Scientists

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 9, Section 9.1 Relations and Their Properties
- Chapter 9, Section 9.2 Database and Relations
- Chapter 9, Section 9.5 Equivalence Relations
- Chapter 9, Section 9.6 Partial Orderings

Relations

- Relationships between elements of sets occur in many contexts.
- Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets.

Binary Relations

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. Hence, sets of ordered pairs are called binary relations.

Definition

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

We use $a\ R\ b$ or $R(a, b)$ to denote that $(a, b) \in R$.

Exercise

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \textit{ divides } b\}$?

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Answer:
$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

Relations on a Set

Relations from a set A to itself are of special interest.

Definition

A *relation on a set* A is a relation from A to A.

How many relations are there on a set with n elements?

Reflexive Relations

There are several properties that are used to classify relations on a set.

Definition

A relation R on a set A is called *reflexive*, if $(a, a) \in R$ for every element $a \in A$.

How to use quantifiers to express it?

Symmetric Relations

Definition

A relation R on a set A is called *symmetric*, if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

How to use quantifiers to express it?

Antisymmetric Relations

Definition

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

- The terms symmetric and antisymmetric are not opposites.
- A relation can have both of these properties or may lack both of them. Examples?

Transitive Relations

Definition

A relation R on a set A is called *transitive*, if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

How to use quantifiers to express it?

Examples

Consider these relations on the set of integers:

$$
\begin{aligned}
R_1 &= \{(a, b) \mid a \leq b\}, \\
R_2 &= \{(a, b) \mid a > b\}, \\
R_3 &= \{(a, b) \mid a = b \text{ or } a = -b\}, \\
R_4 &= \{(a, b) \mid a = b\}, \\
R_5 &= \{(a, b) \mid a = b + 1\}, \\
R_6 &= \{(a, b) \mid a + b \leq 3\}.
\end{aligned}
$$

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

Exercise

Let R_1 be the 'less than' relation on the set of real numbers and let R_2 be the 'greater than' relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

Combining Relations: Composite

There is another way that relations are combined that is analogous to the composition of functions.

Definition

Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Exercise

What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Composing a Relation with Itself

Definition

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Exercise

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find the powers R^n, $n = 2,3,4,....$

A Theorem

Theorem

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

See Rosen's textbook, p.581

More Examples

- $A = \mathbb{Z}, \quad xRy \quad \text{if } x \equiv y \pmod 5$
- $A = \mathbb{Z}^+, \quad xRy \quad \text{if } x|y$
- $A = \mathbb{N}, \quad xRy \quad \text{if } x \leq y$

Equivalence Relations

Equivalence relation: a reflexive, symmetric, and transitive relation.

Two elements a and b that are related by an equivalence relation are called **equivalent.** The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Equivalence Classes

$[a]_R$ (equivalence class of a with respect to R): the set of all elements of A that are equivalent to a

$$[a]_R = \{s | (a, s) \in R\}.$$

Example

What are the equivalence classes of 0 and 1 for congruence modulo 4?

Partition

A partition of a set S is a collection of disjoint nonempty subsets of S t hat have S as their union.

- $A_i = \phi$ for $i \in I$
- $A_i \cap A_j = \phi$ when$i \neq q$
- $\cup_{i \in I} A_i = S$

What are the sets in the partition of the integers arising from congruence modulo 4?

Partial Orderings

Partial ordering: a relation that is reflexive, antisymmetric, and transitive

A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.

Partial Orderings

In different posets different symbols such as $\leq$, $\subseteq$, and $|$, are used for a partial ordering.

- $\leq$
- $|$
- $\subseteq$

Comparable V.S. Incomparable

Comparable: the elements a and b in the poset $(A, \preceq)$ are comparable if $a \preceq b$ or $b \preceq a$

Incomparable: When a and b are elements of S such that neither $a \preceq b$ nor $b \preceq a$, a and b are called incomparable.

- $(P(\mathbb{Z}), \subseteq)$
- $(\mathbb{Z}^+, |)$

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