Lecture 10

Topics covered in this lecture session

1. Method of Partial Fractions
   - Non reported linear factors
   - Non repeated quadratic factors
   - Repeated linear factors
2. Sequences
   - Arithmetic sequence (A.P.)
   - Geometric sequence (G.P.)
   - Harmonic sequence
   - Fibonacci sequence

Partial Fractions

Process: Simplifying algebraic fractions

$$\frac{1}{(x+1)} - \frac{1}{(x+2)} = \frac{(x+2)-(x+1)}{(x+1)\,(x+2)} = \frac{1}{(x^2+3x+2)}$$

Process: Finding partial fractions for a given expression

Partial Fractions

Thus, in the method of partial fraction, we decompose a rational fraction

$$f(x) = \frac{p(x)}{q(x)} \quad ; \quad q(x) \neq 0,$$

where $p(x)$ and $q(x)$ are polynomials, as a sum of several fractions with a simpler denominator.

$$\frac{1}{(x^2+3x+2)}$$

$$\Downarrow$$

$$\frac{1}{(x+1)} - \frac{1}{(x+2)}$$

Partial Fractions

The method is applicable when the following conditions are satisfied.

a) $\deg[\,p(x)\,] < \deg[\,q(x)\,]$,

b) The expression in the denominator is factorable.

e.g. $\dfrac{2x+3}{x^2+3x+2}$ $\quad \therefore \deg[p(x)] = 1 < 2 = \deg[q(x)]$

$\dfrac{3x+1}{(x-1)^2\,(x+2)}$ $\quad \therefore \deg[p(x)] = 1 < 3 = \deg[q(x)]$

Partial Fractions

1. Non-repeated Linear Factors

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

e.g. $$\frac{3x}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

Partial Fractions

2. Non-repeated Quadratic Factors

$$\frac{1}{(x+a)(x^2+b)} = \frac{A}{(x+a)} + \frac{Bx+C}{(x^2+b)}$$

$$\frac{1}{(ax^2+bx+c)(x+d)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$$

e.g. $$\frac{13}{(x^2+1)\,(2x+3)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(2x+3)}$$

Partial Fractions

Non repeated linear factors

$$\frac{1}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

$$\Rightarrow\ A\,(x+b) + B\,(x+a) = 1$$

Put $x = -a$ to find the value of A

and then

put $x = -b$ to find the value of B.

Non repeated quadratic factor

$$\frac{1}{(x^2+a)(x+b)} = \frac{Ax+B}{(x^2+a)} + \frac{C}{(x+b)}$$

$$\Rightarrow\ (Ax+B)\,(x+b) + C\,(x^2+a) = 1$$

Put $x = -b$ to find the value of C

and then

equate the terms in x^2 or x

or constants, to find A and B.

Partial Fractions

Step 1:

Express the given rational function of the form $\frac{p(x)}{q(x)}$ as a sum of partial fractions with constants A and B (and C).

Step 2:

Find the constants A and B (and C) as explained earlier.

Step 3:

Finally, write the given expression as a sum of partial fractions with obtained values of constants A and B (and C)..

Examples

Express the following expressions as a sum of partial fractions.

1. $\dfrac{1}{(x+1)(x+2)}$

2. $\dfrac{3}{(x+1)(x^2+2)}$

Ans: $\dfrac{1}{x+1} - \dfrac{1}{x+2}$

$\dfrac{1}{x+1} + \dfrac{1}{x^2+2} - \dfrac{x}{x^2+2}$

Partial Fractions

3. Repeated Linear Factors

$$\frac{1}{(x+a)^2(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+b)}$$

$$\frac{1}{(x+a)^3(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3} + \frac{D}{(x+b)}$$

e.g. $\dfrac{x}{(x-3)^2(2x+1)} = \dfrac{A}{(x-3)} + \dfrac{B}{(x-3)^2} + \dfrac{C}{(2x+1)}$

Example

Express the following expressions as a sum of partial fractions.

3. $\dfrac{9}{(x-1)^2(x+2)}$

Ans:

$$\frac{1}{x+2} - \frac{1}{x-1} + \frac{3}{(x-1)^2}$$

Sequences - Introduction

A sequence is an ordered list of numbers (objects).

Mathematically,
A sequence is a function defined on a set of natural numbers.

that is,

A sequence is a function $f : \mathbb{N} \to A$, where A is any non-empty set of numbers (or objects).

Sequences - Introduction

Some examples of sequences are:

2, 4, 6, 8, 10, …. is a sequence of even numbers.

1, 2, 4, 8, 16, …. is a sequence of numbers of the form

$$2^{n-1} \quad ; \quad n = 1, 2, 3, 4, \dots$$

4, 9, 16, 25, 36, …. is a sequence of numbers of the form

$$n^2 \quad ; \quad n = 2, 3, 4, \dots$$

Sequences - Introduction

Each member of the set is called the term of the sequence, and is denoted by

$$a_1,\ a_2,\ a_3,\ \dots. \quad \text{or} \quad T_1,\ T_2,\ T_3,\ \dots. \quad \text{or} \quad f(1),\ f(2),\ f(3),\ \dots..$$

Thus, a sequence may be denoted by: $\{a_n\}_{n=1}^{n=k}$

If k is a finite number, the sequence is called finite sequence; otherwise infinite.

Sequences - Introduction

Some sequences have a general formula, some not.

e.g.

- The sequence of numbers

  2, 5, 8, 11, ….. has a general formula

  $$f(n) = 3n - 1 \quad ; \quad n \in \mathbb{N}.$$

- The sequence of primes

  2, 3, 5, 7, 11, ….. has no general formula.

Arithmetic Sequence/Progression (A.P.)

An Arithmetic Progression (A.P.) is a sequence in which difference between any two consecutive terms is constant.

e.g. 1, 5, 9, 13, 17, 21, ….. is an A.P.

- The constant difference, called the common difference is denoted by d.
- The first term of the sequence is denoted by a.

Arithmetic Sequence/Progression (A.P.)

Some examples of A.P. are:

2, 4, 6, 8, 10, …. where $a = 2, \quad d = 2.$

5, 8, 11, 14, 17, …. where $a = 5, \quad d = 3.$

8, 5, 2, -1, - 4, …. where $a = 8, \quad d = -3.$

Thus, an A.P. takes the form:

$$a, \quad a+d, \quad a+2d, \quad ,\ldots.., \quad a+(n-1)\,d$$

Arithmetic Sequence/Progression (A.P.)

$a,$	$a+d,$	$a+2d,$	$a+3d,$	……… ,	$a+(n-1)d$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$
1^{st} Term	2^{nd} Term	3^{rd} Term	4^{th} Term		n^{th} Term

$\therefore$ the n^{th} term of an A.P. is $\boxed{a_n = a + (n-1)d}$

a is the first term of an A.P.

d is the common difference.

Arithmetic Sequence/Progression (A.P.)

Example:

For an A.P., the seventh term is 19 and the eighteenth term is 41. Find a and d. Hence, write first seven terms of the A.P.

The seventh (7^{th}) term is 19 $\Rightarrow \quad a + (7-1)d = 19$

$i.e. \quad a + 6\,d = 19 \longrightarrow (1)$

and eighteenth (18^{th}) term is 41 $\Rightarrow \quad a + 17d = 41 \longrightarrow (2)$

$(2) - (1)$ gives: $(a + 17d) - (a + 6\,d) = 41 - 19 \Rightarrow 11d = 22 \Rightarrow \boldsymbol{d = 2}$

$\therefore \mathrm{a} = 19 - 6\mathrm{d} = 19 - 12 \Rightarrow \mathbf{a = 7}$

Hence, the A.P. is: $7,\ 9,\ 11,\ 13,\ 15,\ 17,\ 19.$

Geometric Sequence/Progression (G.P.)

A Geometric Progression (G.P.) is a sequence in which ratio of any two consecutive terms is constant.

e.g. 4, 12, 36, 108, 324, ….. is a G.P.

- The constant ratio, called common ratio is denoted by r.
- The first term of the sequence is denoted by a.

Geometric Sequence/Progression (G.P.)

Some examples of G.P. are:

$2, \quad 4, \quad 8, \quad 16, \quad 32, \; .....$ where $a = 2, \quad r = 2$

$1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \; .....$ where $a = 1, \quad r = \frac{1}{2}$

Thus, a G.P. takes the form:

$$a, \quad a\,r, \quad a\,r^2, \quad a\,r^3, \quad , ..., \quad a\,r^{n-1}$$

Geometric Sequence/Progression (G.P.)

$$\underset{\text{Term}}{\overset{a,}{\downarrow}}\;\; 1^{st} \qquad \underset{\text{Term}}{\overset{a\,r,}{\downarrow}}\;\; 2^{nd} \qquad \underset{\text{Term}}{\overset{a\,r^2,}{\downarrow}}\;\; 3^{rd} \qquad \underset{\text{Term}}{\overset{a\,r^3,}{\downarrow}}\;\; 4^{th} \qquad ... \; ... \; ... \; , \qquad \underset{\text{Term}}{\overset{a\,r^{n-1}}{\downarrow}}\;\; n^{th}$$

$\therefore$ the n^{th} term of the G.P. is $\boxed{a_n = a\,r^{n-1}}$

a is the first term of the G.P.

r is the common ratio.

Geometric Sequence/Progression (G.P.)

Example:

For a G.P., the third term is 18 and the seventh term is 1458. Find a and r. Hence, write first seven terms of the G.P.

The third (3^{rd}) term is 18 $\Rightarrow \quad a\,r^2 = 18 \longrightarrow$ (1)

and seventh (7^{th}) term is 1458 $\Rightarrow \quad a\,r^6 = 1458 \longrightarrow$ (2)

(2) $\div$ (1) gives: $\frac{ar^6}{ar^2} = \frac{1458}{18} \Rightarrow r^4 = 81 \Rightarrow \mathbf{r = \pm 3} \quad \therefore \mathrm{a} = \frac{18}{\mathrm{r}^2} = \frac{18}{9} \Rightarrow \mathbf{a = 2}$

Hence, the G.P. is: $2, \; 6, \; 18, \; 54, \; 162, \; 486, \; 1458$

OR $2, \; -6, \; 18, \; -54, \; 162, \; -486, \; 1458.$

Harmonic Sequence

A general harmonic progression (or harmonic sequence) is a progression formed by taking the reciprocals of an arithmetic progression.

i.e. it is a sequence of the form: $\frac{1}{a}, \; \frac{1}{a+d}, \; \frac{1}{a+2d}, \; \frac{1}{a+3d}, \; .....$

A particular case of general harmonic sequence is given by:

$$\frac{1}{1}, \; \frac{1}{2}, \; \frac{1}{3}, \; \frac{1}{4}, \; ....., \; \frac{1}{n}, \; ......$$

$\therefore n^{th}$ term of a Harmonic sequence is: $\boxed{f(n) = \frac{1}{n}}$

Fibonacci Sequence

- Fibonacci sequence is named after Leonardo Fibonacci.
- It consists of (Fibonacci) numbers in the following integer sequence: $0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots\ldots$

Mathematically, Fibonacci numbers is defined by the recurrence relation:

$$\boxed{f(n) = f(n-1) + f(n-2)} \quad ; \quad n \in \mathbb{N},\ n > 1$$

with $f(0) = 0$ and $f(1) = 1$.

Fibonacci Sequence and the Golden ratio

- The ratio of neighbouring Fibonacci numbers tends to the Golden ratio.

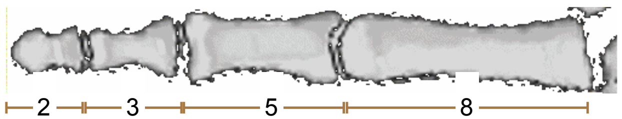
$$\phi = \frac{1+\sqrt{5}}{2}$$

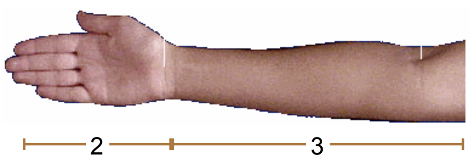
$$= \boxed{1.6180339887}$$

Golden Ratio

2 / 1 = 2.0
3 / 2 = 1.5
5 / 3 = 1.67
8 / 5 = 1.6
13 / 8 = 1.625
21 / 13 = 1.615
34 / 21 = 1.619
55 / 34 = 1.618
89 / 55 = 1.618

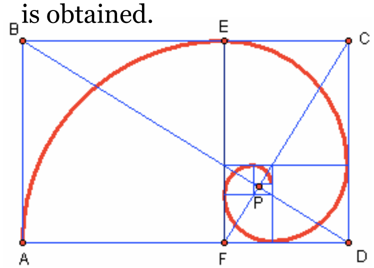
Fibonacci Sequence and the Golden ratio





Fibonacci Spiral

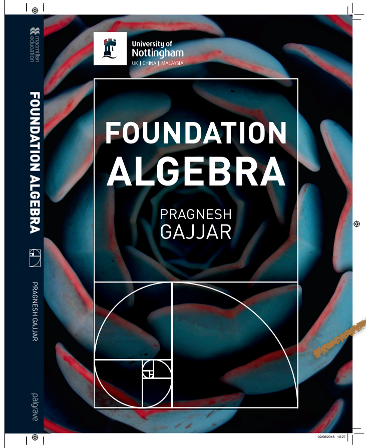
When a Golden Rectangle is progressively subdivided into smaller and smaller Golden Rectangles the pattern below is obtained.

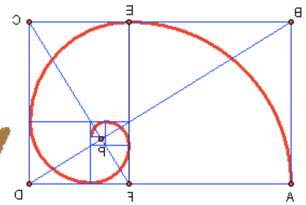


From this, a spiral can be drawn which grows logarithmically, where the radius of the spiral, at any given point, is the length of the corresponding square to a Golden Rectangle.

This is called the **Golden Spiral**.

Fibonacci Spiral





Fibonacci Sequence

Applications

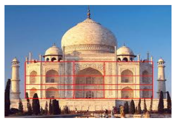
- In Computer Algorithms, e.g. Fibonacci search technique.
- In Biological settings, such as, branching in trees, arrangement of leaves on a stem, the fruit sprouts of a pineapple, etc.

Fibonacci Sequence

- Many artists and architects have been fascinated by the presumption that the golden rectangle (with length of sides as neighbouring Fibonacci numbers) is considered aesthetically pleasing.

1.618

1



Fibonacci Sequence

- Universe has a 'golden ratio' that keeps everything in order, researchers claim.

More info: