

AE1MCS: Mathematics for Computer Scientists

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Aim and Learning Objectives

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
- To gain a good understanding of other definitions: tautology, contradiction, contingency, logical equivalence, converse, contrapositive and inverse.
- To be able to draw truth tables and use them as a tool to solve logical problems;
- To be able to apply important logical equivalences to solve logical problems.

Proposition

Definition

A *proposition* is a statement that is either true or false.

The area of logic that deals with propositions is called the *propositional logic* or propositional calculus.

Is it a proposition?

- { 1 Beijing is the capital of China. T
- 2 $1 + 1 = 2$. T
- 3 $2 + 2 = 3$. F
- 4 What time is it? X
- 5 Read this sentence carefully. X
- 6 $x + 1 = 2$. X
- 7 $x + y = z$. X
- 8 If $x > 0$, then $x > 1$. X

Is a claimed proposition true or false?

Unfortunately, it is not always easy to decide if a claimed proposition is true or false.

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- 3 $313(x^3 + y^3) = z^3$ has no positive integer solutions.

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- 4 Every map can be colored with 4 colors so that adjacent regions have different colors [Four Color Theorem]. 

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- 3 $313(x^3 + y^3) = z^3$ has no positive integer solutions.
- 4 Every map can be colored with 4 colors so that adjacent regions have different colors [Four Color Theorem].
- 5 Every even integer greater than 2 is the sum of two primes [Goldbach's conjecture, 1742].

Propositional Variable

- a variable that represents a proposition
- denoted using a letter p, q, r, s, \dots
- truth value: T (true); F (false)

P.
Truth value T F

Logical Operators

- **Compound Proposition:** formed from existing propositions using logical operators
- Logical Operators
 - {■ Negation
 - Conjunction
 - Disjunction
 - Implication
 - ...

Negation

Definition (Negation)

Let p be a proposition. The *negation* of p , denoted by $\neg p$, is the statement

'It is not the case that p '.

The proposition $\neg p$ is read 'not p '. The truth value of $\neg p$ is the opposite of the truth value of p .

Truth table ✓

p	$\neg p$
T	F
F	T

Conjunction

Definition (Conjunction)

Let p and q be propositions. The *conjunction* of p and q , denoted by $\underline{p \Delta q}$, is the proposition

' p and q '.

The proposition $p \wedge q$ is true when both p and q are true and is false otherwise.

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

Definition (Disjunction)

Let p and q be propositions. The *disjunction* of p and q , denoted by $\underline{p \vee q}$, is the proposition

' p or q '.

The proposition $\underline{p \vee q}$ is false when both \underline{p} and \underline{q} are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or

Definition (Exclusive Or)

Let p and q be propositions. The *exclusive or* of p and q , denoted by $p \oplus q$, is the proposition that is true when *exactly one* of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

P implies q

Definition (Implication)

Let p and q be propositions. The implication $p \rightarrow q$ is the proposition
‘if p , then q ’.

The proposition $p \rightarrow q$ is false when p is true and q is false, and true otherwise. p is called the hypothesis or premise and q is called the conclusion or consequence.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The proposition $p \rightarrow q$ is true, if p is false or q is true.

Examples

- $T \circ P$
- If Goldbach's Conjecture is true,
then $x^2 \geq 0$ for every real number x . *q is true.*
 - If pigs fly, then your account will not get hacked.
 $\overbrace{P=F}^q$

Bi-Implication

Definition (Bi-Implication)

Let p and q be propositions. The *bi-implication* $p \leftrightarrow q$ is the proposition

' p if and only if q '.

$$\begin{cases} p \rightarrow q \\ q \rightarrow p \end{cases}$$

The bi-implication $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

The words 'if and only if' are sometimes abbreviated 'iff'

truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- p is sufficient and necessary for q .
- if p then q , conversely

$p \leftrightarrow q$ is true when both $p \rightarrow q$ and $q \rightarrow p$ are true, and is false otherwise.

More Definitions

- Tautology, Contradiction and Contingency
- Logical Equivalence
- Converse, Contrapositive and Inverse

Tautology, Contradiction and Contingency

Definition (Tautology, Contradiction and Contingency)

A compound proposition that is always true, no matter what the truth values of the propositions that occur in it, is called a tautology. A compound proposition that is always false is called a contradiction. A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Exercise

How to construct a tautology, a contradiction and a contingency using just one propositional variable?

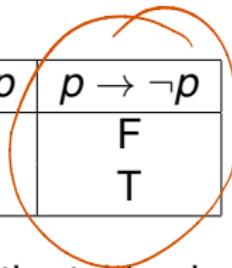
P	$P \vee \neg P$	$P \wedge \neg P$	$P \wedge P$
T	T	F	T
F	T	F	F

Tautology Contradiction Contingency

Exercise Answer

How to construct a tautology, a contradiction and a contingency using just one propositional variable?

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$	$p \rightarrow \neg p$
T	F	T	F	F
F	T	T	F	T



[There are some other ways not shown in the table above...]

Logical Equivalence

Definition (Equivalence)

The compound propositions p and q are logically equivalent, if they always have the same truth value (i.e. $p \leftrightarrow q$ is a tautology). The notation $p \equiv q$ denotes that p and q are logically equivalent.

$$\underline{p \equiv q}$$

$$\underline{p \leftrightarrow q}$$

Exercise

- Are $\neg(p \vee q)$ and $\neg p \wedge \neg q$ logically equivalent? Why?
- Are $p \rightarrow q$ and $\neg p \vee q$ logically equivalent? Why?

[Hint: construct truth tables]

Exercise Answer

Are $\neg(p \vee q)$ and $\neg p \wedge \neg q$ logically equivalent? Why?

Answer: Yes. As shown in the truth table below, $\neg(p \vee q)$ and $\neg p \wedge \neg q$ always have the same truth value. Thus, they are logical equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Exercise Answer

Are $p \rightarrow q$ and $\neg p \vee q$ logically equivalent? Why?

Answer: Yes. As shown in the truth table below, $p \rightarrow q$ and $\neg p \vee q$ always have the same truth value. Thus, they are logical equivalent.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Converse, Contrapositive and Inverse

$$P \rightarrow q$$

- The **converse** of $p \rightarrow q$ is the proposition $q \rightarrow p$.
- The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
- The **inverse** of $p \rightarrow q$ is the proposition $\neg p \rightarrow \neg q$.

Which pairs of the following propositions are equivalent? Why?

- a conditional statement and its converse
- a conditional statement and its contrapositive
- a conditional statement and its inverse

P	$\neg P$	q	$P \rightarrow q$	$\neg q \rightarrow \neg P$
T	F	T	T	T
F	T	F	F	T
T	F	F	F	T
F	T	T	F	F

Converse

P	$\neg P$	q	$\neg q$	$P \rightarrow q$	$\neg q \rightarrow \neg P$
T	F	T	F	T	T
F	T	F	T	F	F
T	F	F	T	F	T
F	T	T	F	T	F

Inverse

P	$\neg P$	q	$\neg q$	$P \rightarrow q$	$\neg q \rightarrow \neg P$
T	F	T	F	T	T
F	T	F	T	F	F
T	F	F	T	F	T
F	T	T	F	T	F

P	$\neg P$	q	$\neg q$	$P \rightarrow q$	$\neg q \rightarrow \neg P$
T	F	T	F	T	T
F	T	F	T	F	F
T	F	F	T	F	T
F	T	T	F	T	F

Some Important Logical Equivalences

	Equivalence	Name
1	$p \wedge T \equiv p$	Identity laws
2	$p \vee F \equiv p$	
3	$p \vee T \equiv T$	Domination laws
4	$p \wedge F \equiv F$	
5	$p \vee p \equiv p$	Idempotent laws
6	$p \wedge p \equiv p$	
7	$\neg(\neg p) \equiv p$	Double negation law
8	$p \vee q \equiv q \vee p$	Commutative laws
9	$p \wedge q \equiv q \wedge p$	

Some Important Logical Equivalences

	Equivalence	Name
10	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
11	$\underline{(p \wedge q) \wedge r} \equiv p \wedge \underline{(q \wedge r)}$	
12	$p \vee (q \wedge r) \equiv \underline{(p \vee q) \wedge (p \vee r)}$	Distributive laws
13	$p \wedge (q \vee r) \equiv \underline{(p \wedge q) \vee (p \wedge r)}$	
14	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
15	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	
16	$p \vee (p \wedge q) \equiv p$	Absorption laws
17	$p \wedge (p \vee q) \equiv p$	
18	$\{ p \vee \neg p \equiv T$	Negation laws
19	$\} p \wedge \neg p \equiv F$	

Logical Equivalences involving Implications

20	$p \rightarrow q \equiv \neg p \vee q$
21	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
22	$\neg p \vee q \equiv \neg p \rightarrow q$
23	$\neg(p \wedge q) \equiv \neg(p \rightarrow \neg q)$
24	$\neg(p \rightarrow q) \equiv p \wedge \neg q$
25	$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
26	$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
27	$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
28	$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences involving Bi-Implications



29	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
30	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
31	$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
32	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Using De Morgan's Laws

Use De Morgan's laws to express the negations of the following sentences.

P

q

- Tony has a cellphone and he has a laptop computer.
- Heather will go to the concert or Steve will go to the concert.

$$\neg(P \wedge q) \equiv \neg P \vee \neg q$$

$$\neg(r \vee s) \equiv \neg r \wedge \neg s$$

Constructing New Logical Equivalences

- A proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.
- Prove two propositions are logically equivalent by developing a series of logical equivalences.

Exercise

- Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.
- Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.
- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$$\begin{aligned}\underline{\neg(p \rightarrow q)} &\equiv \underline{\neg(\neg p \vee q)} && \text{by law } \text{D}\text{o} \\ &\equiv \underline{\neg(\neg p)} \wedge \underline{\neg q} && \text{by De Morgan's law} \\ &\equiv p \wedge \neg q && \text{by double negation law}\end{aligned}$$

Exercise Answer

Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Answer:

$$\begin{aligned}\neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) \\ &\equiv \neg(\neg p) \wedge \neg q \\ &\equiv p \wedge \neg q.\end{aligned}$$

Exercise Answer

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Answer:

$$\begin{aligned} & \neg(p \vee (\neg p \wedge q)) \\ \equiv & \neg p \wedge \neg(\neg p \wedge q) \quad \text{by De Morgan's Law.} \\ \equiv & \neg p \wedge (\neg(\neg p) \vee \neg q) \\ \equiv & \neg p \wedge (p \vee \neg q) \quad \text{by double negation law.} \\ \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) \quad \text{by distributive law.} \\ \equiv & F \vee (\neg p \wedge \neg q) \quad \text{by negation law} \\ \equiv & \neg p \wedge \neg q. \quad \text{by Identity law} \end{aligned}$$

Exercise Answer

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Answer:

$$\begin{aligned} & \underline{(p \wedge q) \rightarrow (p \vee q)} \\ & \equiv \neg(p \wedge q) \vee (p \vee q) \\ & \equiv (\neg p \vee \neg q) \vee (p \vee q) \\ & \equiv (p \vee \neg p) \vee (q \vee \neg q) \\ & \equiv T \vee T \equiv T \end{aligned}$$

List of Symbols

SYMBOL	MEANING	PAGE
$\neg p$	negation of p	3
$p \wedge q$	conjunction of p and q	4
$p \vee q$	disjunction of p and q	4
$p \oplus q$	exclusive or of p and q	6
$p \rightarrow q$	the implication p implies q	6
$p \leftrightarrow q$	biconditional of p and q	9
$p \equiv q$	equivalence of p and q	23
T	tautology	23
F	contradiction	23
$P(x_1, \dots, x_n)$	propositional function	36
$\forall x P(x)$	universal quantification of $P(x)$	38
$\exists x P(x)$	existential quantification of $P(x)$	40
$\exists! x P(x)$	uniqueness quantification of $P(x)$	41
\therefore	therefore	64
$p\{S\}q$	partial correctness of S	364

Expected Learning Outcomes

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Sections 1.1-1.3.