MCS: Tutorial 1
Proposition and Predicate Logic

Proposition

- A ***proposition*** is simply a statement. ***Propositional logic*** studies the ways statements can interact with each other. It is important to remember that propositional logic does not really care about the content of the statements. For example, in terms of propositional logic, the claims, “if the moon is made of cheese then basketballs are round,” and “if spiders have eight legs then Sam walks with a limp” are exactly the same. They are both implications: statements of the form, $P \to Q$.

Suppose P and Q are the statements: P: Jack passed math. Q: Jill passed math.

a. Translate “Jack and Jill both passed math” into symbols.
b. Translate “If Jack passed math, then Jill did not” into symbols.
c. Translate “$P \vee Q$” into English.
d. Translate “$\neg(P \wedge Q) \rightarrow Q$” into English.
e. Suppose you know that if Jack passed math, then so did Jill. What can you conclude if you know that:
    i. Jill passed math?
    ii. Jill did not pass math?

a. $P \wedge Q$.

b. $P \to \neg Q$.

c. Jack passed math or Jill passed math (or both).

d. If Jack and Jill did not both pass math, then Jill did.

e.

   i. Nothing else.

   ii. Jack did not pass math either.

Truth table

- Make a truth table for the statement $\neg P \vee Q$.

Truth table

- Make a truth table for the statement ¬P∨Q.

Note that this statement is not $\neg(P \vee Q)$, the negation belongs to P alone. Here is the truth table:

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We added a column for $\neg P$ to make filling out the last column easier. The entries in the $\neg P$ column were determined by the entries in the P column. Then to fill in the final column, look only at the column for Q and the column for $\neg P$ and use the rule for $\vee$.

Logical Equivalence

- Two statements P and Q are ***logically equivalent*** provided P is true precisely when Q is true. That is, P and Q have the same truth value under any assignment of truth values to their atomic parts.
- To verify that two statements are logically equivalent, you can make a truth table for each and check whether the columns for the two statements are identical.
- Recognizing two statements as logically equivalent can be very helpful. Rephrasing a mathematical statement can often lends insight into what it is saying, or how to prove or refute it.

Are the statements, “it will not rain or snow” and “it will not rain and it will not snow” logically equivalent?

We want to know whether $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$. Make a truth table which includes both statements:

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Since in every row the truth values for the two statements are equal, the two statements are logically equivalent.

Examples

- 1. Prove that the statements $\neg(P \to Q)$ and $P \wedge \neg Q$ are logically equivalent without using truth tables.
- 2. Are the statements $(P \vee Q) \to R$ and $(P \to R) \vee (Q \to R)$ logically equivalent?

- 1. Prove that the statements ¬(P→Q)and P∧¬Q are logically equivalent without using truth tables.

We want to start with one of the statements, and transform it into the other through a sequence of logically equivalent statements. Start with $\neg(P \to Q)$. We can rewrite the implication as a disjunction this is logically equivalent to

$$\neg(\neg P \vee Q).$$

Now apply DeMorgan's law to get

$$\neg\neg P \wedge \neg Q.$$

Finally, use double negation to arrive at $P \wedge \neg Q$

- 2. Are the statements $(P \vee Q) \rightarrow R$ and $(P \rightarrow R) \vee (Q \rightarrow R)$ logically equivalent?

Note that while we could start rewriting these statements with logically equivalent replacements in the hopes of transforming one into another, we will never be sure that our failure is due to their lack of logical equivalence rather than our lack of imagination. So instead, let's make a truth table:

P	Q	R	$(P \vee Q) \rightarrow R$	$(P \rightarrow R) \vee (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

Look at the fourth (or sixth) row. In this case, $(P \rightarrow R) \vee (Q \rightarrow R)$ is true, but $(P \vee Q) \rightarrow R$ is false. Therefore the statements are not logically equivalent.

Examples

3. Simplify the following statements (so that negation only appears right before variables).

a. $\neg(P \rightarrow \neg Q)$
b. $(\neg P \vee \neg Q) \rightarrow \neg(\neg Q \wedge R)$.
c. $\neg((P \rightarrow \neg Q) \vee \neg(R \wedge \neg R))$.
d. It is false that if Sam is not a man then Chris is a woman, and that Chris is not a woman.

a. $P \wedge Q$.

b. $(\neg P \vee \neg R) \rightarrow (Q \vee \neg R)$ or, replacing the implication with a disjunction first: $(P \wedge Q) \vee (Q \vee \neg R)$.

c. $(P \wedge Q) \wedge (R \wedge \neg R)$. This is necessarily false, so it is also equivalent to $P \wedge \neg P$.

d. Either Sam is a woman and Chris is a man, or Chris is a woman.

Beyond Propositions

- Not every statement can be analyzed using logical connectives alone. For example, we might want to work with the statement:

  All primes greater than 2 are odd.
- To write this statement symbolically, we must use quantifiers. We can translate as follows

$$\forall x((P(x) \land x > 2) \to O(x))$$

- Use P(x) denote “x is prime” and O(x) to denote “x is odd.” These are not propositions, since their truth value depends on the input x. Better to think of P and O as denoting *properties* of their input. The technical term for these is ***predicates*** and when we study them in logic, we need to use ***predicate logic***.

- It is important to stress that predicate logic *extends* propositional logic (much in the way quantum mechanics extends classical mechanics).
- You will notice that our statement above still used the (propositional) logical connectives. Everything that we learned about logical equivalence and deductions still applies.
- However, predicate logic allows us to analyze statements at a higher resolution, digging down into the individual propositions P, Q, etc.

Predicates

- Suppose we claim that there is no smallest number. We can translate this into symbols as

$$\neg\exists x\forall y\ (x\leq y)$$

(literally, "it is not true that there is a number x such that for all numbers y, x is less than or equal to y").

- However, we know how negation interacts with quantifiers: we can pass a negation over a quantifier by switching the quantifier type (between universal and existential). So the statement above should be *logically equivalent* to

$$\forall x\exists y\ (y<x)$$

- Notice that $y<x$ is the negation of $x\leq y$. This literally says, "for every number x there is a number y which is smaller than x." We see that this is another way to make our original claim.

Examples

1. Use quantifiers to express the statement that "There is a woman who has taken a flight on every airline in the world."

Solution: Let $P(w, f)$ be "w has taken f" and $Q(f, a)$ be "f is a flight on a." We can express the statement as

$$\exists w \forall a \exists f (P(w, f) \wedge Q(f, a)),$$

where the domains of discourse for w, f, and a consist of all the women in the world, all airplane flights, and all airlines, respectively.

The statement could also be expressed as

$$\exists w \forall a \exists f\, R(w, f, a),$$

where $R(w, f, a)$ is "w has taken f on a." Although this is more compact, it somewhat obscures the relationships among the variables. Consequently, the first solution is usually preferable.

Question 1

2. Use quantifiers to express the statement that “There does not exist a woman who has taken a flight on every airline in the world.”

Solution: This statement is the negation of the statement “There is a woman who has taken a flight on every airline in the world” from Example 13. By Example 13, our statement can be expressed as $\neg\exists w \forall a \exists f(P(w, f) \wedge Q(f, a))$, where $P(w, f)$ is “w has taken f” and $Q(f, a)$ is “f is a flight on a.” By successively applying De Morgan’s laws for quantifiers in Table 2 of Section 1.4 to move the negation inside successive quantifiers and by applying De Morgan’s law for negating a conjunction in the last step, we find that our statement is equivalent to each of this sequence of statements:

$$\begin{aligned} \forall w \neg \forall a \exists f(P(w, f) \wedge Q(f, a)) &\equiv \forall w \exists a \neg \exists f(P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f \neg(P(w, f) \wedge Q(f, a)) \\ &\equiv \forall w \exists a \forall f(\neg P(w, f) \vee \neg Q(f, a)). \end{aligned}$$

This last statement states “For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline.” ◀

Can you switch the order of quantifiers?

- For example, consider the two statements:

  $\forall x \exists y P(x,y)$ and $\exists y \forall x P(x,y)$.

  Are these logically equivalent?

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These statements are NOT logically equivalent. To see this, we should provide an interpretation of the predicate $P(x, y)$ which makes one of the statements true and the other false.

Let $P(x, y)$ be the predicate $x < y$. It is true, in the natural numbers, that for all x there is some y greater than it (since there are infinitely many numbers). However, there is not a natural number y which is greater than every number x. Thus it is possible for $\forall x \exists y P(x, y)$ to be true while $\exists y \forall x P(x, y)$ is false.

More Exercises (Not required)!!

- No solution will be provided!
- Textbook
  - Section 1.4
    - 7-10, 43, 44, 46-51, 59-62
  - Section 1.5
    - 3, 4, 7-9, 14-17, 29-32, 48*, 49*, 52*
- Supplementary questions: Q1-Q19