AE1MCS: Mathematics for Computer Scientists

Huan Jin
University of Nottingham Ningbo China

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 7, Section 7.1 An Introduction to Discrete Probability

Discrete Probability

- Combinatorics and probability theory share common origins (analyzing gambling games).
- The theory of probability now plays an essential role in a wide variety of disciplines (e.g. the study of genetics).
- In computer science,
  - Probability theory plays an important role in the study of the complexity of algorithms.
  - Probabilistic algorithms vs. deterministic algorithms.
  - Probability theory can help us answer questions that involve uncertainty.
  - ...

Content

Monty Hall Three-Door Puzzle

You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is be behind each door:

- whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors).
- Then he asks you whether you would like to switch doors.

Player

Sirius.

	Prize	Picked	Review	Switch	Win/Lose
Sirus.	1	1	2.	No	Win
Tony.	3	2	1.	Yes.	Win
Galway	3.	3	1	No.	Win.

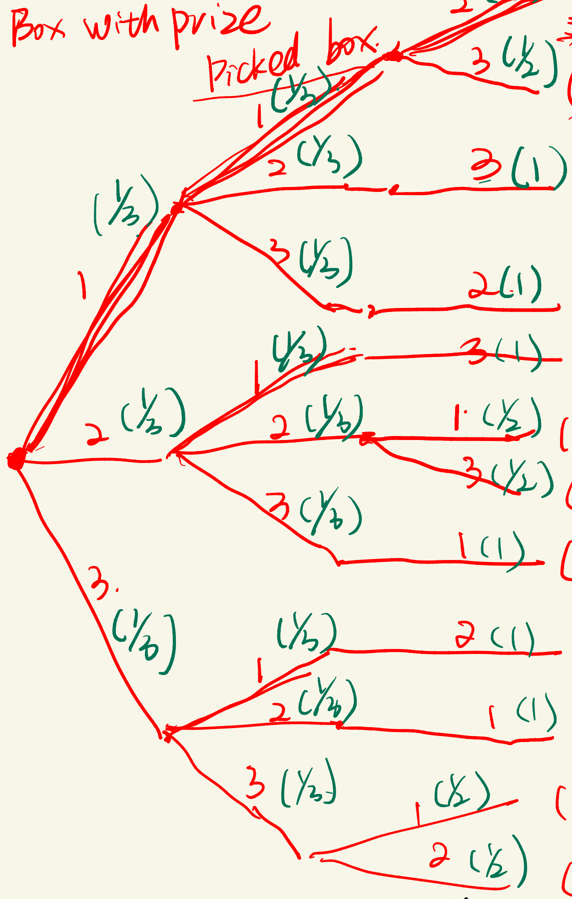
Outcomes. = all info. about the experiment after it's been performed.

Prize box, picked box, Review box,

Outcome: (1, 1, 2), (3, 2, 1), (3, 3, 1).

(1, 2, 1) X (2, 1, 1) X

Decision tree



Reviewed box		Switch	Stay	Prob
(1, 1, 2)	1/18	L	W	
(1, 1, 3)	1/18	L	W	
(1, 2, 3)	1/9	W	L	
(1, 3, 2)	1/9	W	L	
(2, 1, 3)	1/9	W	L	
(2, 2, 1)	1/18	L	W	
(2, 2, 3)	1/18	L	W	
(2, 3, 1)	1/9	W	L	
(3, 1, 2)	1/9	W	L	
(3, 2, 1)	1/9	W	L	
(3, 3, 1)	1/18	L	W	
(3, 3, 2)	1/18	L	W	

$|S| = 12$

6 wins

6 Loses

Probability:

Switching

$$\begin{cases} P(W) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{2}{3} \\ P(L) = \frac{1}{18} + \cdots + \frac{1}{18} = \frac{1}{3} \end{cases}$$

player chooses to stay.

$$\begin{cases} P(W) = \frac{1}{3} = \frac{1}{18} + \frac{1}{18} + \cdots + \frac{1}{18} \\ P(L) = \frac{2}{3} \end{cases}$$

Probability space:

Consists of a sample space and a prob function ~~that~~ map.

$$P : S \to \mathbb{R}$$

s.t.

$$\forall w \in S, \quad 0 \leq P(w) \leq 1$$

$$\sum_{w \in S} P(w) = 1$$

Finite Probability

Laplace's definition of the probability of an event with **finitely many, equally likely, possible outcomes** is as follows.

Definition

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is

$$p(E) = \frac{|E|}{|S|}$$

- An **experiment** is a procedure that yields one of a given set of possible outcomes.

Monty Hall game

- The **sample space** of the experiment is the set of possible outcomes.
- An **event** is a subset of the sample space.

Probability of an Event

In the eighteenth century, the French mathematician Laplace, who also studied gambling, defined **the probability of an event as the number of successful outcomes divided by the number of possible outcomes.**

Example (Answer)

$S = \{b_1, b_2, b_3, b_4, r_1, r_2, r_3, r_4, r_5\}$

$E = \{b_1, b_2, b_3, b_4\}$

A box contains 4 blue balls and 5 red balls. What is the probability that a ball chosen at random from the box is blue?

Example

A sample space is uniform if all the outcomes have the same probability.

A box contains 4 blue balls and 5 red balls. What is the probability that a ball chosen at random from the box is blue?

$p(b_1) = p(b_2) = p(b_3) = p(b_4) = p(r_1) = \cdots = p(r_5) = \frac{1}{9}$

$$S = \{b1, b2, b3, b4, r1, r2, r3, r4, r5\}$$ 9.

$$E = \{b1, b2, b3, b4\}$$ blue

so

$$p(E) = \frac{|E|}{|S|} = \frac{4}{9}$$

Example: Lottery

$C_4^1 \times 9$

9
⋮
0

$10 \times 10 \times 10 \times 10$

10^4

In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched.

- What is the probability that a player wins the large prize? $\frac{1}{10^4}$
- What is the probability that a player wins the small prize?

$$\frac{4 \times 9}{10^4} = \frac{36}{10^4}$$

Example: Lottery (Answer)

$$S = \{x \in \mathbb{Z} : 0 \leq x \leq 9999\} \Rightarrow |S| = 10^4$$

- What is the probability that a player wins the large prize? $E_1 = \{p\}$ where p is the lottery number picked by player, so $|E_1| = 1$ and $p(E) = 1/10^4$.
- What is the probability that a player wins the small prize?

$$E_2 = \{x : x \text{ is like } p \text{ but with just one digit change}\}$$

hence $|E_2| = C(4, 1) \times 9 = 4 \times 9$, so $P(E_2) = C(4, 1) \times 9/10^4$. That is, 9 ways to change one digit over four digits.

Example: Poker 1

$|S| = C_{52}^{5}$

Find the probability that a hand of five cards in poker contains four cards of one kind.

- A deck of cards contains 52 cards.
- There are 13 different kinds of cards, with four cards of each kind.
- These kinds are twos, threes, fours, fives, sixes, sevens, eights, nines, tens, jacks, queens, kings, and aces.
- There are 4 suits: spades, clubs, hearts, and diamonds, each containing 13 cards.

Example: Poker 1 (Answer)

Find the probability that a hand of five cards in poker contains four cards of one kind.

- S is number of ways to choose any 5 cards from 52: $|S| = C(52, 5)$.
- E is the number of ways to get four of a kind: $|E| = 13 \times (52 - 4)$, using product rule with
  - Choose one of the 13 kinds which is repeated 4 times;
  - Choose any remaining card for last card (52-4).
- Hence $p(E) = \dfrac{13 \times (52 - 4)}{C(52, 5)} \approx 0.00024$.

Example: Poker 2

2 2 2 Q Q

Q Q Q 2 2

What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

Example: Poker 2 (Answer)

What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

Same sample space: $|S| = C(52, 5)$.

$$
\begin{array}{rcll}
|E| & = & \text{\# ways to get two different kinds} & C(13,2) \\
& \times & \text{\# ways to select which is three cards} & \\
& & \text{of the same kind} & 2 \\
& \times & \text{\# ways to choose three cards of the same kind} & C(4,3) \\
& \times & \text{\# ways to choose two cards of the same kind} & C(4,2) \\
& = & C(13,2) \times 2 \times C(4,3) \times C(4,2) &
\end{array}
$$

Notice: $C(13, 2) \times 2 = P(13, 2) = C(13, 1)C(12, 1)$, *so all correct.*

$$P = \frac{|E|}{|S|} = \frac{C(13,2) \times 2 \times C(4,3) \times C(4,2)}{C(52,5)}$$

Probabilities of Complements and Unions of Events

Theorem

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

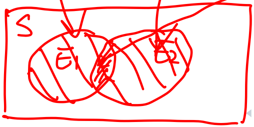
$$p(\overline{E}) = 1 - p(E)$$

Theorem

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

How to prove them?



Example

$10^{10} - 9^{10}$.

$\underline{1/0}\ \underline{1/0}\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$

$2\ 2\ 2\ 2 \cdots\cdots 2.$

- A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?
- What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

$|E| = |E_1| + |E_2| + \cdots + |E_{10}|$

1 bit is 0   2 bits are 0   10 bits are 0.

$|\bar{E}| = 1,$ $\bar{E} = \{1111111111\}$

Example 1 (Answer)

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

- Sample space is all bit strings of length 10: $|S| = 2^{10}$.
- Think about the event when bit string has *no* 0's. Then $E = \{1111111111\}$ and $|E| = 1$.
- Now the event we are interested in (at least one 0) is complement of E, so

$$p(\overline{E}) = 1 - p(E) = 1 - 2^{-10}.$$

$P(E) = 1 - P(\bar{E}) = 1 - 2^{-10} = 1 - \frac{1}{2^{10}}$

Example 2 (Answer)

What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by *either* 2 or 5?

- $S = \{x \in \mathbb{Z}^+ : x \leq 100\}$, so $|S| = 100$;
- $E_1 = \{x \in S : x \text{ is divisible by } 2\}$, so $|E_1| = 50$;
- $E_2 = \{x \in S : x \text{ is divisible by } 5\}$, so $|E_2| = 20$;
- then $E_1 \cap E_2 = \{x \in S : x \text{ is divisible by } 10\}$, so $|E_1 \cap E_2| = 10$,

so

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = 0.6$$

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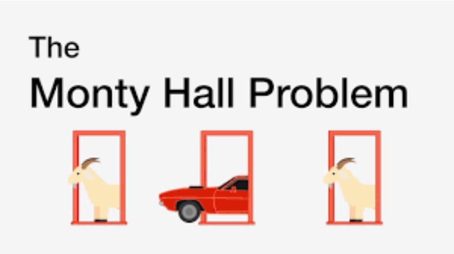

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