AE1MCS: Mathematics for Computer Scientists

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Aim and Learning Objectives

- To gain a good understanding of the definitions of universal quantification, existential quantification and logical equivalence involving quantifiers;
- To be able to translate between English expressions and quantified expressions.
- To be able to apply De Morgan's laws to negate quantified expressions.
- To be able to apply important logical equivalences to solve logical problems.
- To be able to use predicate logic as a tool to solve problems.

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Section 1.4. Predicates and Quantifiers
- Section 1.5. Nested Quantifiers

Predicate Logic

Predicate Logic: the area of logic that deals with *predicates* and *quantifiers*.

Statements involving variables

- $x > 3$
- $x = y + 3$
- $x + y = z$
- Student x likes mathematics.
- ...

Are they propositions?

Statements involving variables

$x = 5$

- $x > 3$
- $x = y + 3$
- $x + y = z$
- Student x likes mathematics.
- ...

Are they propositions?

No. These statements are neither true nor false when the values of the variables are not specified.

How to produce propositions from such statements?

Predicates

Consider the statement 'x is greater than 3'.

- The subject is the variable x.
- **Predicate:** a property that the subject of a statement can have.
- The predicate is 'is greater than 3'.
- Let P denote the predicate 'is greater than 3'.
- The statement can be denoted as $P(x)$.
- Is $P(x)$ a proposition?

x

$P(x)$

Predicates

P(x) T F

Consider the statement 'x is greater than 3'.

- The subject is the variable x.
- **Predicate:** a property that the subject of a statement can have.
- The predicate is 'is greater than 3'.
- Let P denote the predicate 'is greater than 3'.
- The statement can be denoted as $P(x)$.
- Is $P(x)$ a proposition?

Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition.

P(4) T

P(2) F

Predicates

In general, a statement involving the n variable $x_1, x_2, ..., x_n$ can be denoted by

$$P(x_1, x_2, ..., x_n).$$

A statement of the form $P(x_1, x_2, ..., x_n)$ is the value of the **propositional function** P at the n-tuple $(x_1, x_2, ..., x_n)$, and P is also called a ***n*-place predicate** or a ***n*-ary predicate.**

Quantification

$P(x)$

- **Quantification** expresses the extent to which a predicate is true over a range of elements. x
- In English, the words 'all', 'some', 'many', 'none' and 'few' are used in quantifications. one.
- We will focus on two types of quantification here:
  - universal quantification: a predicate is true for every element under consideration;
  - existential quantification: there is one or more element under consideration for which a predicate is true.

Universal Quantification

Definition (Universal Quantification)

The *universal quantification* of $P(x)$ is the statement

'$P(x)$ for all values of x in the domain.'

The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$.
Here $\forall$ is called the **universal quantifier**.

We read $\forall x\, P(x)$ as 'for all x, $P(x)$' or 'for every x, $P(x)$'.
A domain must be specified when a statement $\forall x\, P(x)$ is used.

Exercise X:

1. Let $P(x)$ be the statement '$x + 1 > x$'. What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers? T
2. Let $Q(x)$ be the statement '$x < 2$'. What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers? F
3. What is the truth value of $\forall x\, P(x)$, where $P(x)$ is the statement '$x^2 < 10$' and the domain consists of the positive integers not exceeding 4? $\{1, 2, 3, 4\}$ F
4. What is the truth value of $\forall x\, (x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers? $\left(\frac{1}{2}\right)^2 < \frac{1}{2}$ F

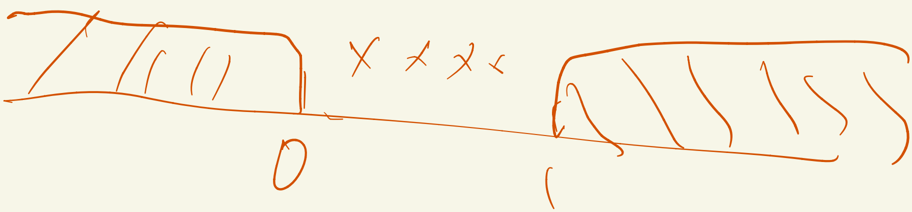
T

$x^2 \geq x$

$x^2 - x \geq 0$

$x(x-1) \geq 0$

$x \leq 0$ or $x \geq 1$



Universal Quantification

$$\underbrace{\forall x P(x_1, x_2, \cdots x_n)}_{T} \equiv \underbrace{\forall x (P(x_1) \land P(x_2) \cdots \land P(x_n))}_{T}$$

When all the elements in the domain can be listed – say, $x_1, x_2, ..., x_n$ – it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction

$$P(x_1) \land P(x_2) \land ... \land P(x_n),$$

because this conjunction is true if and only if $P(x_1)$, $P(x_2)$, ..., $P(x_n)$ are all true.

Existential Quantification

Definition (Existential Quantification)

The *existential quantification* of $P(x)$ is the proposition

'There exists an element x in the domain such that $P(x)$'.

We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the **existential quantifier**.

We read $\exists x\, P(x)$ as 'there exists an x such that $P(x)$' or 'for some x, $P(x)$'. A domain must be specified when a statement $\exists x\, P(x)$ is used.

Exercise

1. Let $P(x)$ be the statement '$x < 2$'. What is the truth value of the quantification $\exists x\, P(x)$, where the domain consists of all real numbers?

   $x < x+1$

2. Let $Q(x)$ be the statement '$x = x + 1$'. What is the truth value of the quantification $\exists x\, Q(x)$, where the domain consists of all real numbers?

   F

3. What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement '$x^2 > 10$' and the domain consists of the positive integers not exceeding 4?

   $\{1, 2, 3, \underline{4}\}$

   $\exists x\, P(x)$ T

Existential Quantification

When all elements in the domain can be listed – say, $x_1, x_2, ..., x_n$ – the existential quantification $\exists x\, P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee ... \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1)$, $P(x_2)$,..., $P(x_n)$ is true.

Universal Quantification and Existential Quantification

Statement	When True?
$\forall x\, P(x)$	$P(x)$ is true for every x
$\exists x\, P(x)$	?

Statement	When False?
$\forall x\, P(x)$	?
$\exists x\, P(x)$	?

Universal Quantification and Existential Quantification

Statement	When True?
$\forall x\, P(x)$	$P(x)$ is true for every x
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.

Statement	When False?
$\forall x\, P(x)$	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	$P(x)$ is false for every x

Quantifiers with Restricted Domains

- An abbreviated notation is often used to restrict the domain of a quantifier.
- In this notation, a condition a variable must satisfy is included after the quantifier.
- This is illustrated in the example shown in the next slide.
- We will also describe other forms of this notation involving set membership later.

Quantifiers with Restricted Domains: Exercise

$p \to q$

$\forall x < 0\ (x^2 > 0)$

$\Leftrightarrow \forall x\,(x < 0 \to x^2 > 0)$

$P(x)$ $Q(x)$

What do the statements $\forall x < 0\,(x^2 > 0)$, $\forall y \neq 0\,(y^3 \neq 0)$ and $\exists z > 0\,(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

$\exists z > 0\ (z^2 = 0)$

$\Leftrightarrow \exists z\,(z > 0 \wedge z^2 = 2)$

Quantifiers with Restricted Domains

$\forall$

- The restriction of a universal quantification is the same as the universal quantification of a conditional statement. $\rightarrow$
- The restriction of an existential quantification is the same as the existential quantification of a conjunction. $\exists$

$\wedge$

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
- $\forall x\, P(x) \lor Q(x)$
- $\forall x\, (P(x) \lor Q(x))$

$(\forall x\, P(x)) \lor Q(x)$

or

$\forall x\, (P(x) \lor Q(x))$

Binding Variables

∀ ∃

- When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**.
- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.
- All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition.
- The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.

$\exists x\,(x + y = 1)$

bound free.

Exercise

1. In the statement $\exists x\,(x + y = 1)$, which variables are bound?
2. In the statement $\exists x\,(P(x) \wedge Q(x)) \vee \forall x\, R(x)$, which variables are bound? What is the scope of the existential quantifier? What is the scope of the universal quantifier?

$\exists x\,(P(x) \wedge Q(x)) \quad \vee \quad \forall x\, R(x)$

Logical Equivalences Involving Quantifiers

$x > 3$

$x = 2$

$$\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

Definition (Logical Equivalences Involving Quantifiers)

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Exercise

if ① is true, ② is true

if ② is true, ① is true

Show that $\forall x\,(P(x) \land Q(x))$ ① and $\forall x\, P(x) \land \forall x\, Q(x)$ ② are logically equivalent (where the same domain is used throughout).

Negating Quantified Expressions

$\forall x P(x)$ $\quad \neg \forall x P(x) \equiv \exists x \neg P(x)$

We will often want to consider the negation of a quantified expression. For instance, consider the negation of the statement

Every student in your class has taken a course in calculus.

It's not the case that every studen ....

P(x) to be: x has taken a course in calculs

x: students in ur class

Negating Quantified Expressions

$$\neg \forall x \, P(x) \equiv \exists x \, \neg P(x)$$

Negating Quantified Expressions

$\neg \exists x\, Q(x) \equiv \forall \neg Q(x)$

every student has not taken a course in Calculus.

all students

Suppose we wish to negate an existential quantification. For instance, consider the proposition

There is a student in this class who has taken a course in calculus.

It's not the case that there's a student ...

Negating Quantified Expressions

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Exercise

De Morgan's

Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

$\neg\forall x\,(P(x) \rightarrow Q(x))$

$\equiv \exists x \neg(P(x) \rightarrow Q(x))$ by De Morgan's with quantifiers

$\equiv \exists x \neg(\neg P(x) \vee Q(x))$ by law 20

$\equiv \exists x\,(P(x) \wedge \neg Q(x))$ by De Morgan's

Exercise Answer

Show that $\neg\forall x\,(P(x) \rightarrow Q(x))$ and $\exists x\,(P(x) \wedge \neg Q(x))$ are logically equivalent.

Answer:

$$\begin{aligned}
&\neg\forall x\,(P(x) \rightarrow Q(x)) \\
\equiv\ &\exists x\,\neg(P(x) \rightarrow Q(x)) \\
\equiv\ &\exists x\,\neg(\neg P(x) \vee Q(x)) \\
\equiv\ &\exists x\,(P(x) \wedge \neg Q(x))
\end{aligned}$$

Nested Quantifiers

- Two quantifiers are nested if one quantifier is within the scope of another, such as
$$\forall x\, \exists y\, (x + y = 0).$$

Exercise

1. Let $Q(x, y)$ denote '$x + y = 0$'. What are the truth values of the quantifications $\exists y \forall x\, Q(x, y)$ and $\forall x \exists y\, Q(x, y)$, where the domain for all variables consists of all real numbers? F T
2. Let $Q(x, y, z)$ be the statement '$x + y = z$'. What are the truth values of the statements $\forall x \forall y \exists z\, Q(x, y, z)$ and $\exists z \forall x \forall y\, Q(x, y, z)$, where the domain of all variables consists of all real numbers? T F

Quantifications of Two Variables

Statement	When True?	When False?
$\forall x \, \forall y \, P(x,y)$ $\forall y \, \forall x \, P(x,y)$	$P(x,y)$ is true for every pair x, y.	? one pair of (x, y) that P(x,y) is false
$\forall x \, \exists y \, P(x,y)$	?	?
$\exists x \, \forall y \, P(x,y)$	?	?
$\exists x \, \exists y \, P(x,y)$ $\exists y \, \exists x \, P(x,y)$	? there's one pair of (x, y), p(x,y) is true.	? p(x,y) is false for all pair of x, y.

Quantifications of Two Variables

Statement	When True?
$\forall x \, \forall y \, P(x, y)$ $\forall y \, \forall x \, P(x, y)$	$P(x, y)$ is true for every pair x, y.
$\forall x \, \exists y \, P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.
$\exists x \, \forall y \, P(x, y)$	There is an x for which $P(x, y)$ is true for every y.
$\exists x \, \exists y \, P(x, y)$ $\exists y \, \exists x \, P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.

Quantifications of Two Variables

Statement	When False?
$\forall x \, \forall y \, P(x, y)$ $\forall y \, \forall x \, P(x, y)$	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \, \exists y \, P(x, y)$	There is an x such that $P(x, y)$ is false for every y.
$\exists x \, \forall y \, P(x, y)$	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \, \exists y \, P(x, y)$ $\exists y \, \exists x \, P(x, y)$	$P(x, y)$ is false for every pair x, y.

Negating Nested Quantifiers

Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.

Express the negation of the statement $\forall x \, \exists y \, (xy = 1)$ so that no negation precedes a quantifier. $\neg \forall x \exists y (xy = 1)$

It's not the case that . . .

$\neg\forall x\,\exists y\,(xy=1)$

$\equiv \exists x\,\neg\exists y\,(xy=1)$ De Morgan's Law with quantifiers

$\equiv \exists x\,\forall y\,\neg(xy=1)$ De Morgan's Law with — — —

$\equiv \exists x\,\forall y\,(xy\neq 1)$ ~~De Morgan's Law~~

Expected Learning Outcomes

- To gain a good understanding of the definitions of universal quantification, existential quantification and logical equivalence involving quantifiers;
- To be able to translate between English expressions and quantified expressions.
- To be able to apply De Morgan's laws to negate quantified expressions.
- To be able to apply important logical equivalences to solve logical problems.
- To be able to use predicate logic as a tool to solve problems.

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Section 1.4. Predicates and Quantifiers
- Section 1.5. Nested Quantifiers

Homework

1. Show that $\forall x\,(P(x) \wedge Q(x))$ and $\forall x\,P(x) \wedge \forall x\,Q(x)$ are logically equivalent (where the same domain is used throughout).
2. Show that $\neg\forall x\,P(x)$ and $\exists x\,\neg P(x)$ are logically equivalent no matter what the propositional function $P(x)$ is and what the domain is.
3. Show that $\neg\exists x\,Q(x)$ and $\forall x\,\neg Q(x)$ are logically equivalent no matter what $Q(x)$ is and what the domain is.

Exercise Answer

Show that $\forall x\,(P(x) \wedge Q(x))$ and $\forall x\,P(x) \wedge \forall x\,Q(x)$ are logically equivalent (where the same domain is used throughout).

Proof.

Suppose $\forall x\,(P(x) \wedge Q(x))$ is true. Then for any element c in the domain, $P(c) \wedge Q(c)$ is true. Hence $P(c)$ is true and $Q(c)$ is true. Since $P(c)$ is true for any element c in the domain and $Q(c)$ is true for any element c in th domain, $\forall x\,P(x)$ and $\forall x\,Q(x)$ are both true, i.e. $\forall x\,P(x) \wedge \forall x\,Q(x)$ is true.

Proof by natural language

Suppose $\forall x\,P(x) \wedge \forall x\,Q(x)$ is true. Then $\forall x\,P(x)$ and $\forall x\,Q(x)$ are both true. This means for any element c in the domain, $P(c)$ is true, and for any element c in the domain, $Q(c)$ is true. Thus, for any element c in the domain, $P(c) \wedge Q(c)$ is true. Hence, $\forall x\,(P(x) \wedge Q(x))$ is true.

Therefore, $\forall x\,(P(x) \wedge Q(x))$ and $\forall x\,P(x) \wedge \forall x\,Q(x)$ are logically equivalent. $\square$

Negating Quantified Expressions

Show that $\neg \forall x\, P(x)$ and $\exists x\, \neg P(x)$ are logically equivalent no matter what the propositional function $P(x)$ is and what the domain is.

Proof.

- $\neg \forall x\, P(x)$ is true if and only if $\forall x\, P(x)$ is false.
- $\forall x\, P(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false.
- This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true.
- There is an element x in the domain for which $\neg P(x)$ is true if and only if $\exists x\, \neg P(x)$ is true.
- Putting these steps together, we can conclude that $\neg \forall x\, P(x)$ is true if and only if $\exists x\, \neg P(x)$ is true.
- It follows that $\neg \forall x\, P(x)$ and $\exists x\, \neg P(x)$ are logically equivalent.

$\square$

Negating Quantified Expressions

Show that $\neg \exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent no matter what $Q(x)$ is and what the domain is.

Proof.

- $\neg \exists x\, Q(x)$ is true if and only if $\exists x\, Q(x)$ is false.
- This is true if and only if no x exists in the domain for which $Q(x)$ is true.
- No x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain.
- $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all x in the domain, which holds if and only if $\forall x\, \neg Q(x)$ is true.
- Putting these steps together, we see that $\neg \exists x\, Q(x)$ is true if and only if $\forall x\, \neg Q(x)$ is true.
- We conclude that $\neg \exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent.

$\square$

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Show that $\neg\exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent no matter what $Q(x)$ is and what the domain is.

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- This is true if and only if no x exists in the domain for which $Q(x)$ is true.
- No x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain.
- $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all x in the domain, which holds if and only if $\forall x\, \neg Q(x)$ is true.
- Putting these steps together, we see that $\neg\exists x\, Q(x)$ is true if and only if $\forall x\, \neg Q(x)$ is true.
- We conclude that $\neg\exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent.

$\square$

Negating Quantified Expressions

Show that $\neg\exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent no matter what $Q(x)$ is and what the domain is.

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- This is true if and only if no x exists in the domain for which $Q(x)$ is true.
- No x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain.
- $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all x in the domain, which holds if and only if $\forall x\, \neg Q(x)$ is true.
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- We conclude that $\neg\exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent.

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- This is true if and only if no x exists in the domain for which $Q(x)$ is true.
- No x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain.
- $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all x in the domain, which holds if and only if $\forall x\, \neg Q(x)$ is true.
- Putting these steps together, we see that $\neg\exists x\, Q(x)$ is true if and only if $\forall x\, \neg Q(x)$ is true.
- We conclude that $\neg\exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent.

$\square$

Negating Quantified Expressions

Show that $\neg\exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent no matter what $Q(x)$ is and what the domain is.

Proof.

- $\neg\exists x\, Q(x)$ is true if and only if $\exists x\, Q(x)$ is false.
- This is true if and only if no x exists in the domain for which $Q(x)$ is true.
- No x exists in the domain for which $Q(x)$ is true if and only if $Q(x)$ is false for every x in the domain.
- $Q(x)$ is false for every x in the domain if and only if $\neg Q(x)$ is true for all x in the domain, which holds if and only if $\forall x\, \neg Q(x)$ is true.
- Putting these steps together, we see that $\neg\exists x\, Q(x)$ is true if and only if $\forall x\, \neg Q(x)$ is true.
- We conclude that $\neg\exists x\, Q(x)$ and $\forall x\, \neg Q(x)$ are logically equivalent.

$\square$