AE1MCS: Mathematics for Computer Scientists

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Reading

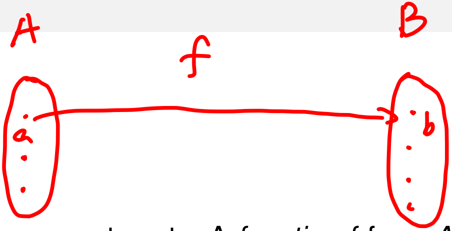
Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 2, Section 2.3. Functions
- Chapter 2, Section 2.4. Sequences and Summations

Functions

- A function assigns to each element of a first set exactly one element of a second set, where the two sets are not necessarily distinct.
- Functions play important roles throughout discrete mathematics.
- They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects, and in a myriad of other ways.
- Useful structures such as sequences and strings are special types of functions.

Functions



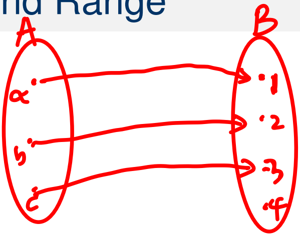
$f(a) = b$

Definition

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Remark: Functions are sometimes also called *mappings* or *transformations*.

Domain and Range



$f(a) = 1$

a. pre-image of "1"

1. image of a

Definition

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.

Domain: $A = \{a, b, c\}$

Codomain: $B = \{1, 2, 3, 4\}$.

range: $\{1, 2, 3\}$

Examples

- $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n$. The domain and codomain are both the set of integers. However, the range is only the set of integer multiples of 3.
- $g : \{1, 2, 3\} \to \{a, b, c\}$ defined by $g(1) = c$, $g(2) = a$, $g(3) = a$.

	f	g
Domain	$\mathbb{Z}$	$\{1, 2, 3\}$
Codomain	$\mathbb{Z}$	$\{a, b, c\}$
Range	$\{n \in \mathbb{Z} \mid 3n\}$	$\{a, c\}$

Equal Functions

Two functions are **equal** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.

Image of a Set

Definition

Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Remark: The notation $f(S)$ for the image of the set S under the function f is potentially ambiguous. Here, $f(S)$ denotes a set, and not the value of the function f for the set S.

One-to-One Function

Definition

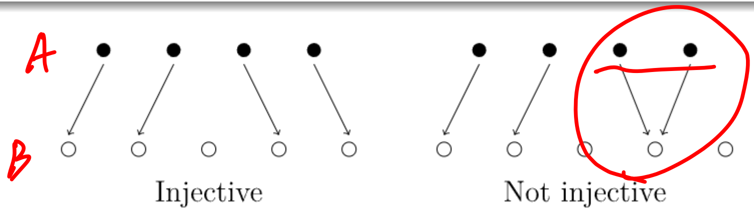
A function f is said to be *one-to-one*, or an *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. This is

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$$

or equivalently

$$\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$$

A function is said to be *injective* if it is one-to-one.



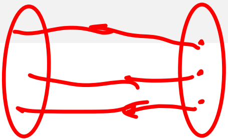
Prove or Disprove a Function is Injective

Suppose that $f : A \to B$.

To show that f is injective Show that for arbitrary $x, y \in A$, if $f(x) = f(y)$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

Onto Functions



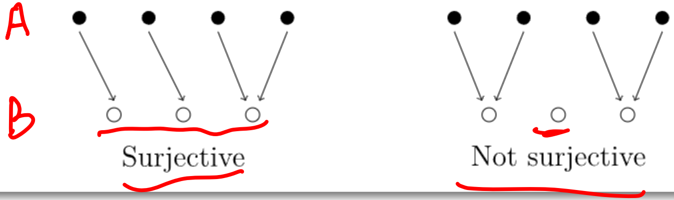
Definition

A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. This is,

$$\forall b \in B \, \exists a \in A \, (f(a) = b)$$

Range = Codomain

A function f is called *surjective* if it is onto.



Prove or Disprove a Function is Surjective

Range = Codomain

Suppose that $f : A \to B$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that for all $x \in A$, $f(x) \neq y$.

One-to-one Correspondence

Definition

The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.

Example: are the following functions one-to-one? onto? neither? or both?

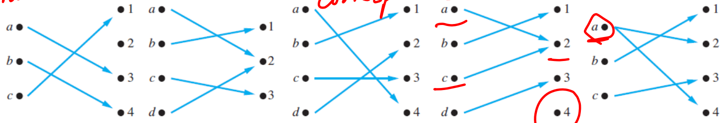
one-to-one
not onto

not one-to-one
onto

one-to-one
correspondence

not one-to-one
not onto

not a function



$a \neq d$
$f(a) = f(d)$

Inverse Functions

Definition

Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

Remark: Be sure not to confuse the function f^{-1} with the function $1/f$, which is the function that assigns to each x in the domain the value $1/f(x)$. Notice that the latter makes sense only when $f(x)$ is a non-zero real number.

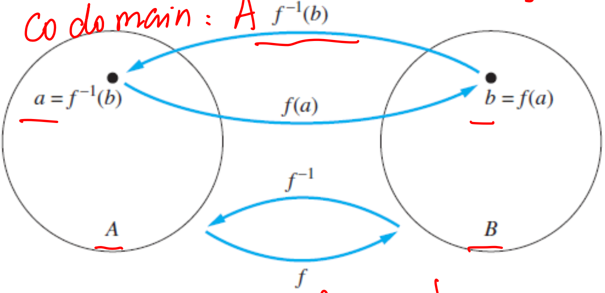
Invertible Functions

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

$f^{-1}(x)$ domain : B

codomain : A

$a = f^{-1}(b)$



$f(a) = b$

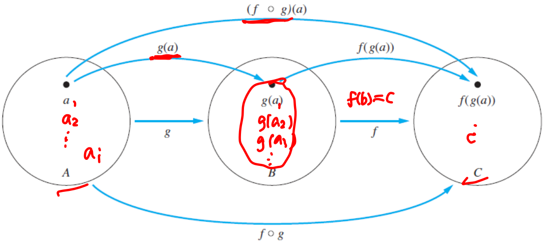
Compositions of Functions

Definition

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

Note that the *composition* $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.



Identity Function

Suppose that f is a one-to-one correspondence from the set A to the set B. $f(a) = b$.

Domain Codomain

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a \quad A \quad A$$
$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b \quad B \quad B$$

$f^{-1} \circ f$ and $f \circ f^{-1}$ are the identity functions on the sets A and B respectively. $(f^{-1})^{-1} = f$.

The Graphs of Functions

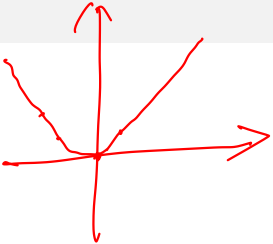
$f(x) = x^2$ $\quad \mathbb{R} \to \mathbb{R}$

$(0, 0).$

$(1, 1)$

$(-1, 1)$

$(0.1, 0.01)$



Definition

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \wedge f(a) = b\}$.

Example

$\mathbb{R} \to \mathbb{R}$

Consider f(n)=2n+3, is it bijective from $\mathbb{Z}$ to $\mathbb{Z}$?

No, because it's not surjective.

for a element $b=0$ in the codomain,

we can't find $a \in \mathbb{Z}$, s.t. $f(a)=b$

Example

For each of the following functions, is it invertible? If yes, what is its inverse?

(a) Let f(x) be a function from $\mathbb{R}$ to $\mathbb{R}$. $f(x) = 2x + 1$

(b) Let f(x) be a function from $\mathbb{R}^+$ to $\mathbb{R}$. $g(x) = log_2(2x) - 1$

(a)

$f^{-1}(x) = \frac{x-1}{2}$

$\mathbb{R} \to \mathbb{R}$

(b).

$g(x) = \log_2(2x) - 1$

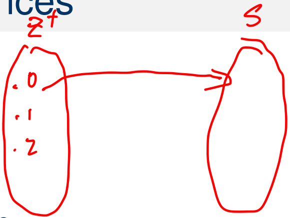
$y = \log_2(2x) - 1$

$2^{y+1} = 2x$

$x = 2^y$

$g^{-1}(x) = 2^x$. $\mathbb{R} \to \mathbb{R}^+$

Sequences



Definition

A *sequence* is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, ...\}$ or the set $\{1, 2, 3, ...\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

$\{a_n\}$: $\{a_0, a_1, a_2, \cdots\}$

$a_n = \frac{1}{n}$, $n = 1, 2, 3, \cdots$.

$\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

Geometric Progression

$\{b_n\}$, $a = 1$, $r = -1$, $n = 0, 1, 2, \ldots$

$b_0 = a \cdot r^0 = 1 \cdot (-1)^0 = 1$

$b_1 = a \cdot r^1 = 1 \cdot (-1)^1 = -1$

$b_2 = a \cdot r^2 = 1 \cdot (-1)^2 = 1$

$\{1, -1, 1, -1, \cdots\}$

Definition

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, ..., ar^n, ...$$

where the initial term a and the common ratio r are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

Arithmetic Progression

$\{s_n\}$, $a=-1$, $d=4$, $n=0,1,2,\cdots$

$s_0 = a = -1$

$s_1 = a+d = -1+4 = 3$

s_2

$s_n = a+nd = -1+4d$

$\{-1, 3, 7, \cdots\}$

Definition

An *arithmetic progression* is a sequence of the form

$$a, a+d, a+2d, ..., a+nd, ...$$

where the initial term *a* and the common difference *d* are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

Recurrence Relation

$a_0 \quad a_1$

$a_n = a_{n-1} + a_{n-2}$

Definition

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, ..., a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

e.g. $a_0 = 1$. $a_{n+1} = a_n + 1$ for $n = 0, 1, 2, ...$

$a_1 = 2$

$a_2 = 3$

$\vdots$

$\{1, 2, 3, \ldots\}$

Homework

Learn the following definitions by yourself.

- Real-valued and Integer-valued Functions
- Adding and Multiplying Real-valued Functions
- Increasing and Decreasing Functions
- Floor Function and Ceiling Function

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