Lecture 4 – Boolean Arithmetic

Dr Tianxiang Cui

Outline

- Binary Arithmetic
- Representing Negative Numbers
- Overflow
- Adder

Learning Outcome

- To be able to perform binary arithmetic
- To be able to understand different signed number representations
- To be able to understand overflow and its conditions
- To be able to implement half adder and full adder in HDL

Numbers

- Various symbols have been used over the ages to represent numbers
  - Roman numerals (I,II,III,…)
  - Arabic numbers (1,2,3,…)
- All encode a quantity
- We use the decimal system for counting
- But we also have counting systems using other bases
  - Time, eggs…

- Computers just use a different encoding based on two symbols

Binary Counting

Decimal	Binary
0	0
1	1
2	10
3	11
4	100
5	101

Decimal	Binary
6	110
7	111
8	1000
9	1001
10	1010
11	1011

Decimal	Binary
12	1100
13	1101
14	1110
15	1111
16	10000
17	10001

- The 1 in binary behaves like 9 in decimal
- Result of 1 + 1 is 0, carrying a 1 to the next digit

Binary to Decimal

- Each binary digit corresponds to a power of 2:

Place	7^{th}	6^{th}	5^{th}	4^{th}	3^{rd}	2^{nd}	1^{st}	0^{th}
Weight	2^7 $= 128$	2^6 $= 64$	2^5 $= 32$	2^4 $= 16$	2^3 $= 8$	2^2 $= 4$	2^1 $= 2$	2^0 $= 1$

- Where the digit is 1, we add the corresponding weight
- Example: convert 1100 1010_2 into decimal

$$
\begin{aligned}
1100\,1010_2 &= 1 \times 128 + 1 \times 64 + 0 \times 32 + 0 \times 16 \\
&\quad + 1 \times 8 + 0 \times 4 + 1 \times 2 + 0 \times 1 \\
&= 128 + 64 + 8 + 2 = 202_{10}
\end{aligned}
$$

Decimal to Binary

- Repeatedly divide by 2, until we reach 0
- The **right**/**left**-most binary digit is the **first**/**last** remainder
- E.g. $101_{10} = 1100101_2$

101	Remainder
50	1
25	0
12	1
6	0
3	0
1	1
0	1

- Example: convert 163_{10} into binary
- 10100011_2

Decimal to Binary (look-up table)

- 87 = 64 (64 = 2^6 , the biggest 2^n that 87 is divisible by) + 23 (reminder)
- 87 = 64 + 16 (16 = 2^4 , the biggest 2^n that 23 is divisible by) + 7 (reminder)
- 87 = 64 + 16 + 4 (4 = 2^2 , the biggest 2^n that 7 is divisible by) + 3 (reminder)
- 87 = 64 + 16 + 4 + 2 (2 = 2^1 , the biggest 2^n that 3 is divisible by) + 1 (reminder)
- 87 = 64 + 16 + 4 + 2 + 1(1 = 2^0 , the biggest 2^n that 1 is divisible by) + 0 (reminder)
- Stop when reminder = 0

Decimal to Binary (look-up table)

$$87 = 2^6 + 2^4 + 2^2 + 2^1 + 2^0$$

$$87 = \mathbf{1}*2^6 + \mathbf{0}*2^5 + \mathbf{1}*2^4 + \mathbf{0}*2^3 + \mathbf{1}*2^2 + \mathbf{1}*2^1 + \mathbf{1}*2^0$$

$$87 = (\mathbf{1010111})_2$$

- Usually faster than recursive division by 2
- 2 **low cost** processes [lookup and subtract] **better than** 1 **high cost** process [divide]
- Very important principle in CS

Binary Numbers

- Often written out with leading zeros, up to a certain number of bits usually a multiple of eight (one byte)
- If a computer receives 10011 = 19, in an 8 bit system, this is what gets stored in a register:

00010011

- In a 32 bit system, this is what gets stored (32 binary digits):

00000000000000000000000000010011

Binary, Octal and Hexadecimal

- **Octal** is a **base 8** system, **Hexadecimal** is a **base 16** system
- Both of these are powers of two – often used to compress binary
- Each octal digit equates to **three** consecutive bits of a binary number
- Each hex digit equates to **four** consecutive bits
- Binary 00111011 | Decimal 59 |Hex 3B | Octal 73

Binary, Octal and Hexadecimal

BINARY	HEXADECIMAL	OCTAL	DECIMAL
0000	0	0	0
0001	1	1	1
0010	2	2	2
0011	3	3	3
0100	4	4	4
0101	5	5	5
0110	6	6	6
0111	7	7	7
1000	8	10	8
1001	9	11	9
1010	A	12	10
1011	B	13	11
1100	C	14	12
1101	D	15	13
1110	E	16	14
1111	F	17	15

Binary, Octal and Hexadecimal

- Binary numbers are founded on base 2:

$$(10011)_2 = 1\cdot 2^4 + 0\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 = 19$$

- In general, let the following be a string of digits

$$\mathbf{x} = x_n x_{n-1} \ldots x_0$$

- The decimal value of **x** in base *b*, denoted $(\mathbf{x})_b$ is defined as follows:

$$(xnxn_{-1\ldots}x_0) = \sum_{i=0}^{n} x_i . bi$$

Binary Addition

- First recap how decimal addition works
  - Add each column together from right
  - If bigger than 9 [biggest decimal digit] , we carry over into the next column
- Binary addition is the same, except we carry if the value is greater than 1 [biggest binary digit]

Decimal Addition

$$
\begin{array}{r} \;\;\; \\ 5\;7\;8\;3 \\ +\;2\;4\;5\;6 \\ \hline \end{array}
\quad
\begin{array}{r} \\ 5\;7\;8\;3 \\ +\;2\;4\;5\;6 \\ \hline 9 \end{array}
\quad
\begin{array}{r} 1\;\; \\ 5\;7\;8\;3 \\ +\;2\;4\;5\;6 \\ \hline 3\;9 \end{array}
\quad
\begin{array}{r} 1\;1\;\; \\ 5\;7\;8\;3 \\ +\;2\;4\;5\;6 \\ \hline 2\;3\;9 \end{array}
$$

$$
\begin{array}{r} 1\;1\;\; \\ 5\;7\;8\;3 \\ +\;2\;4\;5\;6 \\ \hline 8\;2\;3\;9 \end{array}
$$

Binary Addition

```
  0 0 0 1 0 1 0 1
+ 0 1 0 1 1 1 0 0
  ---------------
```

```
  0 0 0 1 0 1 0 1
+ 0 1 0 1 1 1 0 0
  ---------------
                1
```

```
  0 0 0 1 0 1 0 1
+ 0 1 0 1 1 1 0 0
  ---------------
              0 1
```

```
        1
  0 0 0 1 0 1 0 1
+ 0 1 0 1 1 1 0 0
  ---------------
            0 0 1
```

Binary Addition

```
        1 1
  0 0 0 1 0 1 0 1
+ 0 1 0 1 1 1 0 0
  ---------------
          0 0 0 1
```

```
      1 1 1
  0 0 0 1 0 1 0 1
+ 0 1 0 1 1 1 0 0
  ---------------
        1 0 0 0 1
```

```
      1 1 1
  0 0 0 1 0 1 0 1
+ 0 1 0 1 1 1 0 0
  ---------------
      1 1 0 0 0 1
```

```
      1 1 1
  0 0 0 1 0 1 0 1
+ 0 1 0 1 1 1 0 0
  ---------------
    1 1 1 0 0 0 1
```

Representing Negative Numbers

- So far, unsigned numbers
  - How are negative numbers represented on a computer?
- What we use in decimal notation
  - +/− and 0, 1, 2, · · ·
- Such a representation is called **sign and magnitude**
- For binary numbers – define **leftmost** bit to be the **sign**
  - $0 \Rightarrow +$, $1 \Rightarrow -$
  - Rest of bits can be numerical value of number
  - Hence, only seven bits are left in a byte (apart from the sign bit), the magnitude can range from 0000000 (0) to 1111111 (127)
- Problems?

One's Complement

- Alternatively, a system known as **one's complement** can be used to represent negative numbers
- A negative binary number is the bitwise **NOT** applied to it — the "**complement**" of its positive counterpart
- E.g. the ones' complement form of 00101011 (43_{10}) becomes 11010100 (-43_{10})
- Still has two representations of 0: 00000000 (+0) and 11111111 (−0)
- The range of signed numbers using one's complement is represented by $-(2^{N-1} - 1)$ to $(2^{N-1} - 1)$ and ± 0
  - A conventional eight-bit byte is -127_{10} to $+127_{10}$ with zero being either 00000000 (+0) or 11111111 (−0)

Excess-n

- **Excess-n**, also called offset binary or biased representation, uses a pre-specified number *n* as a biasing value
- A value is represented by the unsigned number which is *n* greater than the intended value
- Therefore 0 is represented by *n*, and −*n* is represented by the all-zeros bit pattern
- E.g. Excess-3
  - 0 is represented by 0011 (3)
  - +1 is represented by 0100 (4), +2 is represented by 0101(5)…
  - -1 is represented by 0010 (2), -2 is represented by 0001 (1)
  - -3 is represented by 0000 (0)

Two’s Complement

- The **two's complement** of an N-bit binary number is defined as the complement with respect to 2^N
  - It is the result of subtracting the number from 2^N
  - -x is represented as 2^N-x
- There’s a quicker way to calculate 2^N-x:
  - x + (1’s complement of x) = 2^N-1 (all 1 bits)
  - 2^N-x = (1’s complement of x) +1
  - Take the bitwise inverse (**NOT**) of x, then add 1 to result
- An N-bit two's-complement numeral system can represent every integer in the range $-(2^{N-1})$ to $+(2^{N-1} - 1)$
  - One’s complement: $-(2^{N-1} - 1)$ to $(2^{N-1} - 1)$
- The sum of a number and its two's complement will always equal 0 (the last digit is ignored)
  - The sum of a number and its one's complement will always equal -0 (all 1 bits)

Two's Complement

- To get the negative version of a number
  - Invert the bits
  - Add 1
- So, if we want -29
  - 29 = 0001 1101
  - Invert 1110 0010
  - Add 1 1110 0011
- Try -30
- 1110 0010

Two’s Complement: Alternative View

- Assume an 8-bit two's-complement numeral system

8-Bit Two’s Complement ($-128 \leq x < 127$)

Bit	MSB 7^{th}	6^{th}	5^{th}	4^{th}	3^{rd}	2^{nd}	1^{st}	LSB 0^{th}
Weight	-2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

- What does 0000 0001 represent? 1
- What does 1111 1111 represent? -1
- What does 0101 1011 represent? 91
- How do we represent -2 in binary? 1111 1110

Example of 4-Bit Signed Encodings

Sign and Mag.	
1111	-7
1110	-6
1101	-5
1100	-4
1011	-3
1010	-2
1001	-1
1000	-0
0000	$+0$
0001	$+1$
0010	$+2$
0011	$+3$
0100	$+4$
0101	$+5$
0110	$+6$
0111	$+7$

Ones' Comp.	
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0
0000	$+0$
0001	$+1$
0010	$+2$
0011	$+3$
0100	$+4$
0101	$+5$
0110	$+6$
0111	$+7$

Excess-3	
0000	-3
0001	-2
0010	-1
0011	0
0100	$+1$
0101	$+2$
0110	$+3$
0111	$+4$
1000	$+5$
1001	$+6$
1010	$+7$
1011	$+8$
1100	$+9$
1101	$+10$
1110	$+11$
1111	$+12$

Two's Comp.	
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1
0000	0
0001	$+1$
0010	$+2$
0011	$+3$
0100	$+4$
0101	$+5$
0110	$+6$
0111	$+7$

Signed Extension

- Sign extension is the operation of increasing the number of bits of a binary number while preserving the number's sign (positive/negative) and value
  - This is done by appending digits to the **most significant** side of the number, following a procedure dependent on the particular signed number representation used

- For example, if six bits are used to represent the number "00 1010" (decimal +10) and the sign extend operation increases the word length to 16 bits, then the new representation is simply "0000 0000 0000 1010“ – padding the left side with 0s

- If ten bits are used to represent the value "11 1111 0001" (decimal -15) using two's complement, and this is sign extended to 16 bits, the new representation is "1111 1111 1111 0001 – padding the left side with 1s

Overflow

- 0111 0110 + 1101 0101

Long Addition in Binary

		0	1	1	1	0	1	1	0	= 118
+		1	1	0	1	0	1	0	1	= 213
	1	1	1	1	0	1	0	0		Carry
	1	0	1	0	0	1	0	1	1	= 331

Overflow

- One issue in computer arithmetic is dealing with finite amounts of storage, such as 8-bit register
- Overflow occurs when the result of an operation is too large to be stored
- For an unsigned number, overflow happens when the last carry (1) cannot be accommodated
- For a signed number, overflow occurs when
  - Adding two positives gives a negative
  - Or, adding two negatives gives a positive
  - Or, subtract a negative from a positive gives a negative
  - Or, subtract a positive from a negative gives a positive

Overflow Conditions

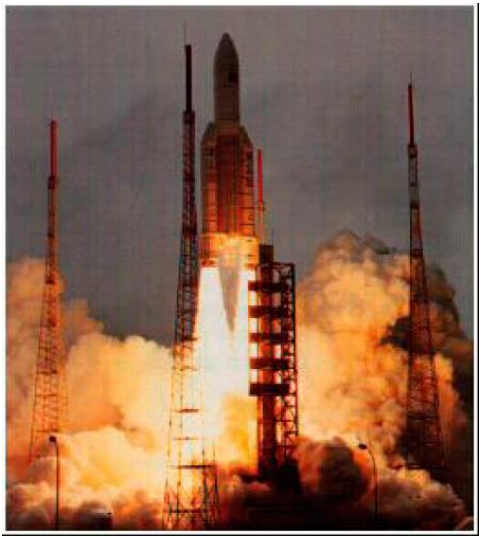
- One way to detect overflow is to check whether the sign bit is consistent with the sign of the inputs when the two inputs are of the same sign – if you added two positive numbers and got a negative number, something is wrong, and vice versa
- Overflow conditions for addition and subtraction are summarized as:

Operation	Operand A	Operand B	Result
A + B	+ve	+ve	-ve
A + B	-ve	-ve	+ve
A - B	+ve	-ve	-ve
A - B	-ve	+ve	+ve

Overflow conditions for addition and subtraction

Ariane 5

- In 1996, the European Space Agency's Ariane5 rocket was launched for the first time... and it exploded 40 seconds after liftoff
- It turns out the Ariane5 used software designed for the older Ariane4
  - The Ariane4 stored its horizontal velocity as a 16-bit signed integer
  - But the Ariane5 reaches a much higher velocity, which caused an overflow in the 16-bit quantity
- The overflow error was never caught, so incorrect instructions were sent to the rocket boosters and main engine



Binary Multiplication by Base

- Take advantage of the fact that any time you multiply a number by it's base you just add 0 to the end
- Decimal 12*10 =12**0**
- Octal 14* 10 = 14**0**
- Binary 1100 * 10 =1100**0**

Shift Operations

- Shift operations shift a word a number of places to the left or right
- Bits which are shifted out, just disappear
- E.g. on 4 bits: 1011 shifted left results in 0110 and shifted right results in 0101
- For unsigned numbers if no bit disappears, shift left corresponds to multiplication by 2
  - E.g. 0011 (3) shifted left is 0110 (6)
- For unsigned numbers, shift right corresponds to division by 2, ignoring the remainder
  - E.g. 0101 (5) shifted right is 0010 (2)

Shift Operations

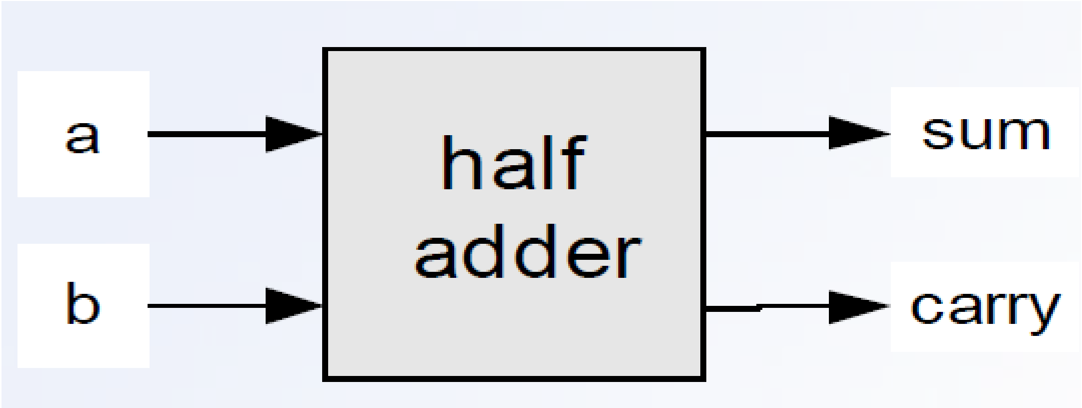
- On two's complement shift left is multiplication by 2, if there is no overflow
  - E.g. 1100 (-4) shifted left results in 1000 (-8)
- Shift right doesn't correspond to division by 2 on negative numbers
  - E.g. 1100 (-4) shifted right results in 0110 (6)
- Arithmetical shift right performs sign extension when shifting
  - E.g. 1100 (-4) shifted right arithmetical results in 1110 (-2)

Adder

- Build an Adder:
  - Half adder: adds two bits
  - Full adder: adds three bits
  - Adder: adds two integers

Half Adder

- Add **two** single binary digits and provide the **output** plus a **carry value**
- It has two inputs, called A(a) and B(b), and two outputs S (sum) and C (carry)



Half Adder

- Least significant bit in the addition is called sum (a+b)
- Most significant bit is called carry (carry of a+b)

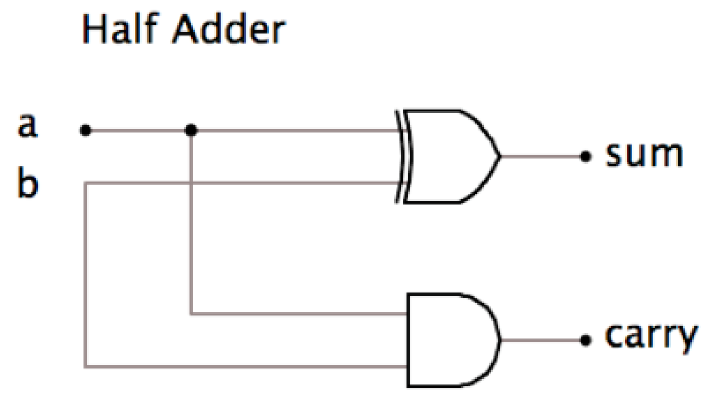
a	b	Carry	Sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

- **Never has a situation when sum and carry are both 1**

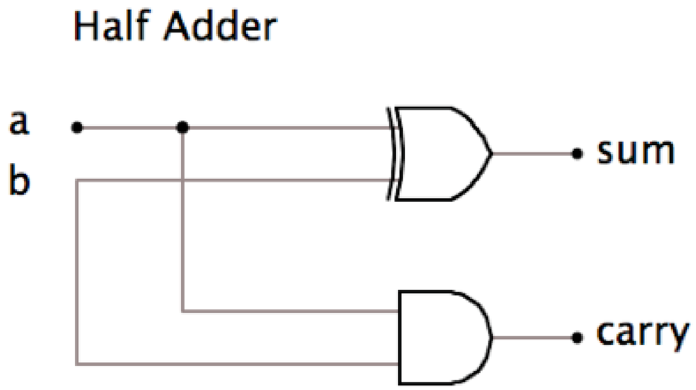
Half Adder

a	b	Carry	Sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

- The common representation uses a XOR and a AND gate



Half Adder in HDL




```
CHIP HalfAdder {
IN a, b; // 1-bit inputs
OUT sum, // Right bit of a + b
carry; // Left bit of a + b
PARTS:
Xor(a=a, b=b, out=sum);
And(a=a, b=b, out=carry);
}
```

Full Adder

- Add **three** single binary digits and provide the **output** plus a **carry value**
- It has three inputs, called A, B and Carry(in), and two outputs S (sum) and Carry(out)

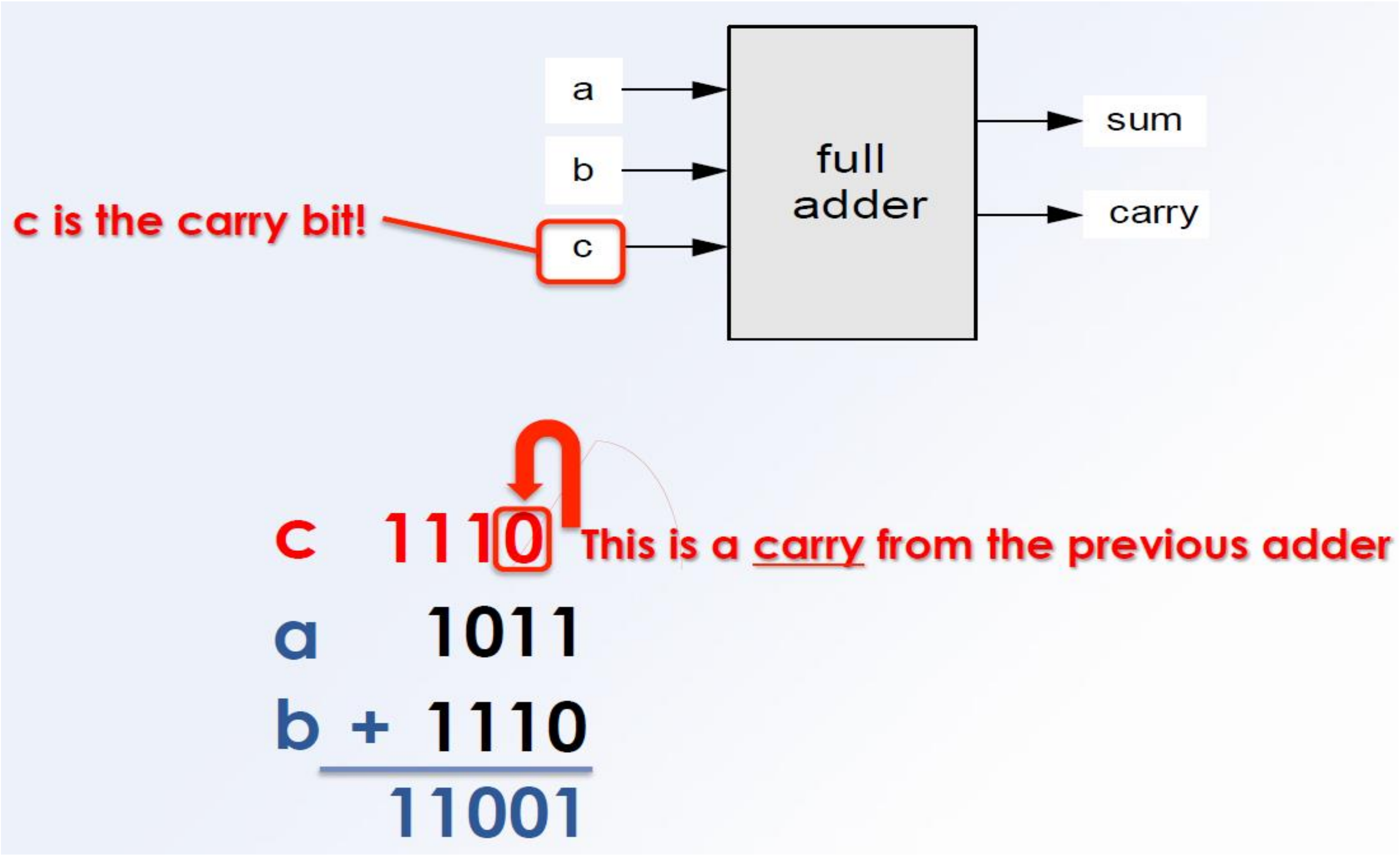
Full Adder

- Least significant bit in the addition is called sum (a+b+c_in)
- Most significant bit is called carry(out) (carry of a+b+c_in)

a	b	Carry(in)	Carry(out)	Sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

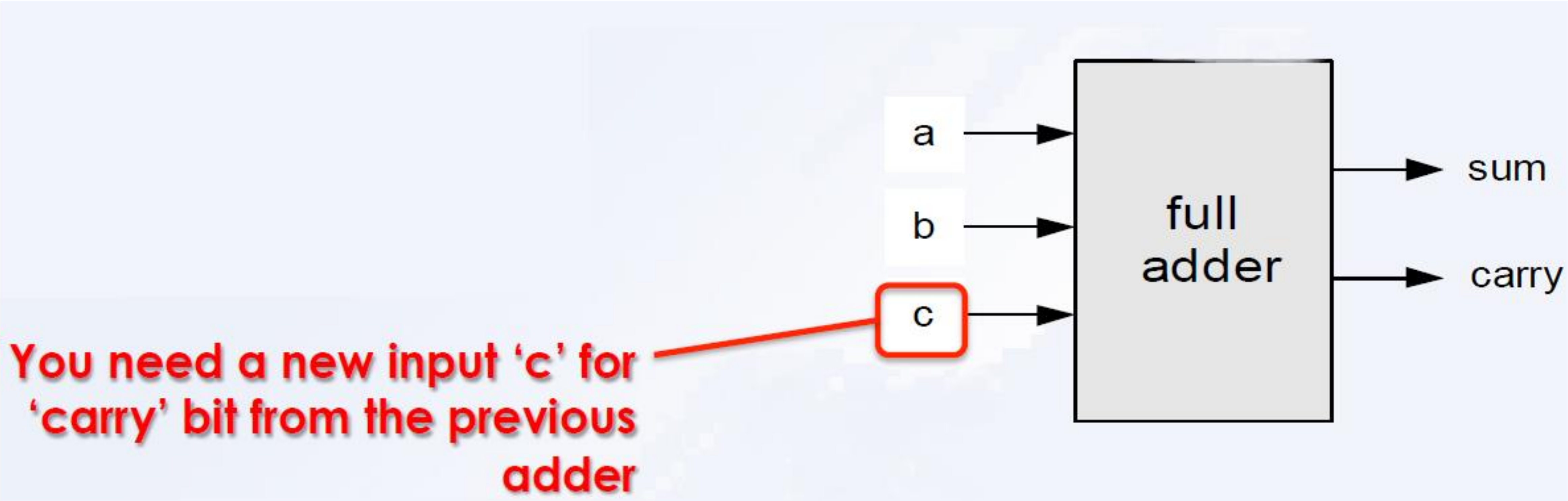
Full Adder

- Computes sum – the least significant bit of a + b + c
- Carry the most significant bit of a + b + c



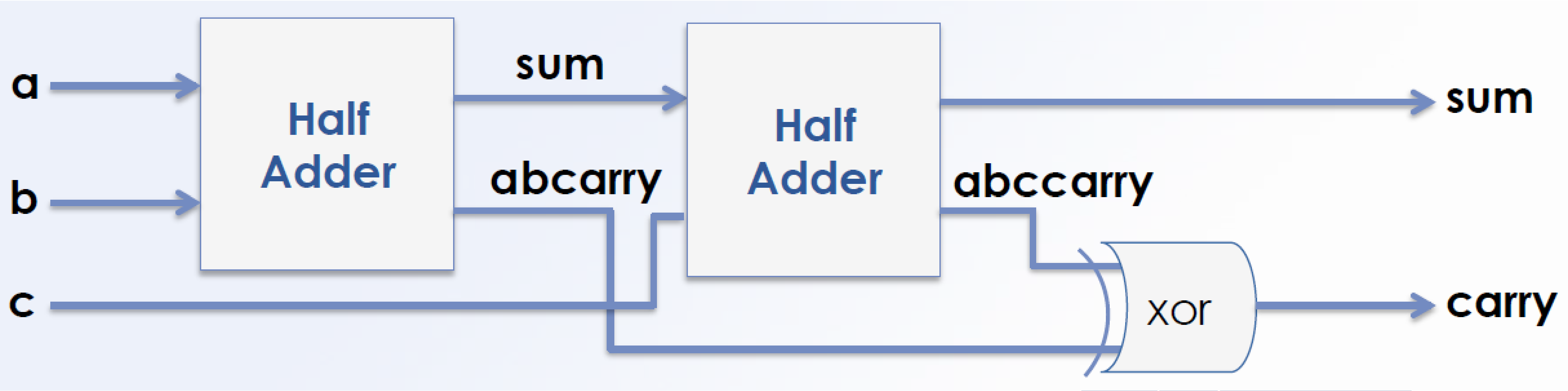
Full Adder

- Carry(in) from the previous adder is needed to bring the carry to the next bit, which makes a full adder



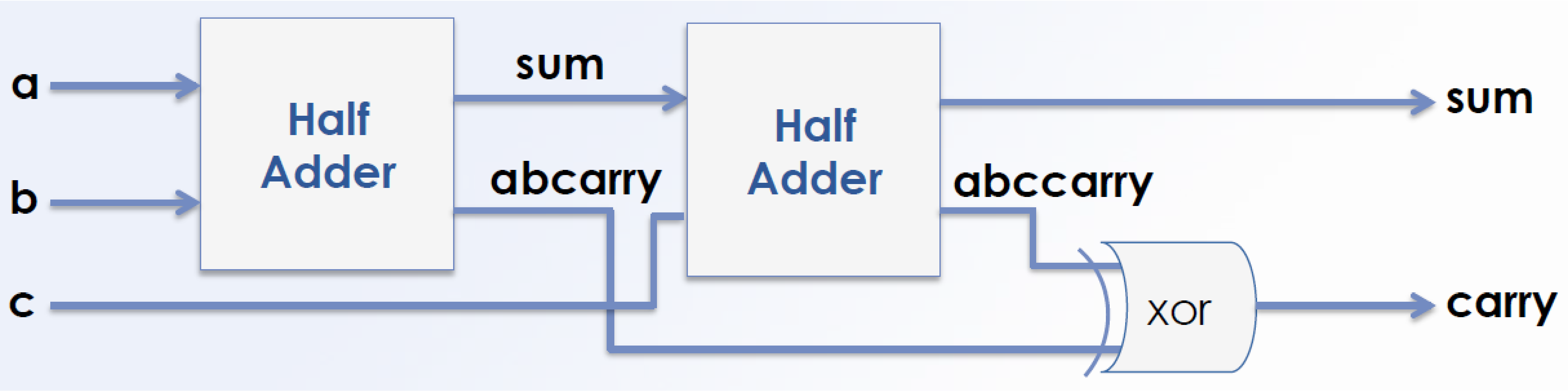
Full Adder: Implementation

- Use two half adders to build a full adder



A	B	$A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

Full Adder in HDL




```
CHIP FullAdder {
IN a, b, c; // 1-bit inputs
OUT sum, // Right bit of a + b + c
carry; // Left bit of a + b + c
PARTS:
HalfAdder(a=a, b=b, sum=absum, carry=abcarry);
HalfAdder(a=absum, b=c, sum=sum, carry=abccarry);
Xor(a=abcarry, b=abccarry, out=carry);
}
```

Summary

- Binary Arithmetic
  - Convert binary to decimal, decimal to binary, etc.
- Representing Negative Numbers
  - Sign and magnitude
  - One’s complement
  - Excess-n
  - Two’s complement
- Overflow
  - Overflow conditions
- Adder
  - Half adder
  - Full adder

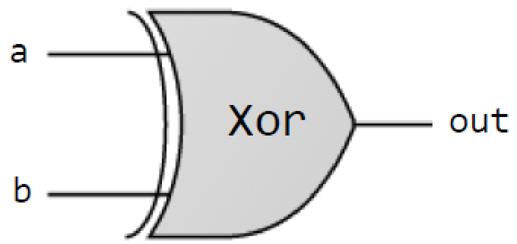
Lab 2

Given: `Nand`

Goal: Build the following gates:

Elementary logic gates	16-bit variants	Multi-way variants
`Not`	`Not16`	`Or8Way`
`And`	`And16`	`Mux4Way16`
`Or`	`Or16`	`Mux8Way16`
`Xor`	`Mux16`	`DMux4Way`
`Mux`		`DMux8Way`
`DMux`		

Chip Building Materials



outputs 1 if a != b

Xor.cmp

```
a b out
0 0  0
0 1  1
1 0  1
1 1  0
```

The contract:

> When running your Xor.hdl on the supplied Xor.tst, your Xor.out should be the same as the supplied Xor.cmp

Xor.hdl

```
CHIP Xor {
     IN  a, b;
     OUT out;

     PARTS:
     // Put your code here.
}
```

Xor.tst

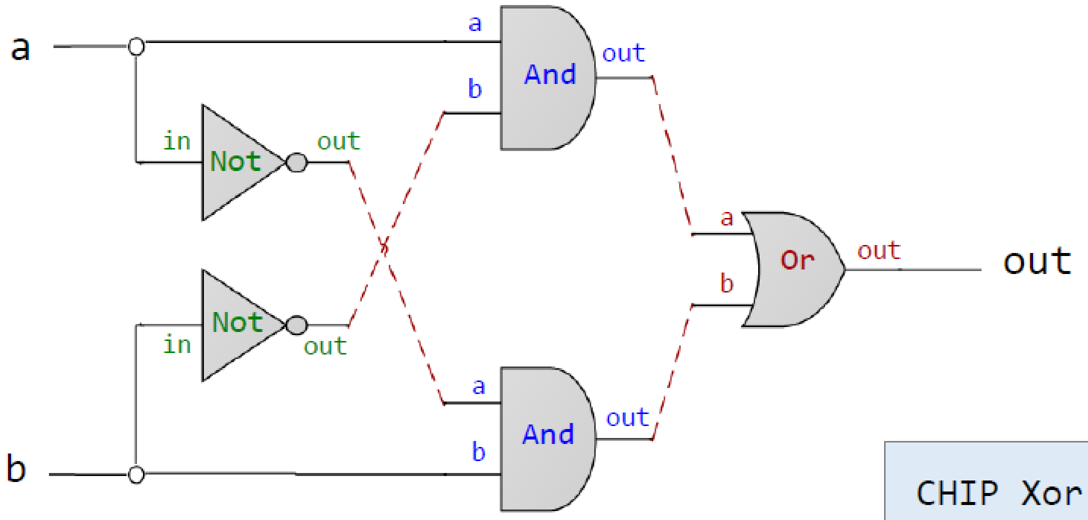
```
load Xor.hdl,
output-file Xor.out,
compare-to Xor.cmp,
output-list a b out;
set a 0, set b 0, eval, output;
set a 0, set b 1, eval, output;
set a 1, set b 0, eval, output;
set a 1, set b 1, eval, output;
```

More Resources

- Text editor (for writing your HDL files)
- HDL Survival Guide
- Hardware Simulator Tutorial
- nand2tetris Q&A forum

All available in: `www.nand2tetris.org`

Hack Chipset API




```
CHIP Xor {
    IN a, b;
    OUT out;

    PARTS:
    Not (in= , out=);
    Not (in= , out=);
    And (a= , b= , out=);
    And (a= , b=b , out=);
    Or  (a= , b= , out=);
}
```

Hack Chipset API

```
Add16 (a= ,b= ,out= );
ALU (x= ,y= ,zx= ,nx= ,zy= ,ny= ,f= ,no= ,out= ,zr= ,ng= );
And16 (a= ,b= ,out= );
And (a= ,b= ,out= );
Aregister (in= ,load= ,out= );
Bit (in= ,load= ,out= );
CPU (inM= ,instruction= ,reset= ,outM= ,writeM= ,addressM= ,pc= );
DFF (in= ,out= );
DMux4Way (in= ,sel= ,a= ,b= ,c= ,d= );
DMux8Way (in= ,sel= ,a= ,b= ,c= ,d= ,e= ,f= ,g= ,h= );
Dmux (in= ,sel= ,a= ,b= );
Dregister (in= ,load= ,out= );
FullAdder (a= ,b= ,c= ,sum= ,carry= );
HalfAdder (a= ,b= ,sum= , carry= );
Inc16 (in= ,out= );
Keyboard (out= );
Memory (in= ,load= ,address= ,out= );
Mux16 (a= ,b= ,sel= ,out= );
Mux4Way16 (a= ,b= ,c= ,d= ,sel= ,out= );
Mux8Way16 (a= ,b= ,c= ,d= ,e= ,f= ,g= ,h= ,sel= ,out= );
```

```
Mux8Way (a= ,b= ,c= ,d= ,e= ,f= ,g= ,h= ,sel= ,out= );
Mux (a= ,b= ,sel= ,out= );
Nand (a= ,b= ,out= );
Not16 (in= ,out= );
Not (in= ,out= );
Or16 (a= ,b= ,out= );
Or8Way (in= ,out= );
Or (a= ,b= ,out= );
PC (in= ,load= ,inc= ,reset= ,out= );
PCLoadLogic (cinstr= ,j1= ,j2= ,j3= ,load= ,inc= );
RAM16K (in= ,load= ,address= ,out= );
RAM4K (in= ,load= ,address= ,out= );
RAM512 (in= ,load= ,address= ,out= );
RAM64 (in= ,load= ,address= ,out= );
RAM8 (in= ,load= ,address= ,out= );
Register (in= ,load= ,out= );
ROM32K (address= ,out= );
Screen (in= ,load= ,address= ,out= );
Xor (a= ,b= ,out= );
```

(see *HDL Survival Guide* @ www.nand2tetris.org)

Built-in Chips

```
CHIP Foo {
    IN ...;
    OUT ...;

    PARTS:
    ...
    Mux16(...)
    ...
}
```

Q: What happens if there is no `Mux16.hdl` file in the current directory?

A: The simulator invokes, and evaluates, the built-in version of `Mux16` (if such exists).

- The supplied simulator software features built-in chip implementations of all the chips in the Hack chip set
- If you don't implement some chips from the Hack chipset, you can still use them as chip-parts of other chips:
  - Just rename their given stub files to, say, `Mux16.hdl1`
  - This will cause the simulator to use the built-in chip implementation.

Best Practice Advice

- Try to implement the chips in the given order
- If you don't implement some chips, you can still use them as chip parts in other chips (the built in implementations will kick in)
- You can invent new, "helper chips"; however, this is not required: you can build any chip using previously built chips only
- Try to use as few chip parts as possible