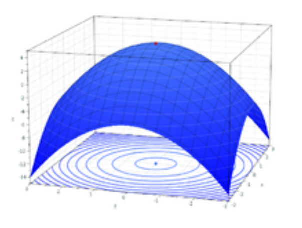
Lecture 4 - Linear Dependency and Rank

COMP1046 - Maths for Computer Scientists

Dr. Ferrante Neri / Dr. Tony Bellotti



Learning outcomes

By the end of this lecture we will have learned:

- Linear dependency
- Rank of a matrix

Based on Sections 2.4.1 and 2.7 of text book (Neri 2018).

Linear Dependency

Linear Combinations of Rows

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$ be a matrix. The i^{th} row is said *linear combination* of the other rows if each of its elements $a_{i,j}$ can be expressed as weighted sum of the other elements of the j^{th} column by means of the same scalars $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots \lambda_m$:

$$\mathbf{a_i} = \lambda_1 \mathbf{a_1} + \lambda_2 \mathbf{a_2} + \cdots + \lambda_{i-1} \mathbf{a_{i-1}} + \lambda_{i+1} \mathbf{a_{i+1}} + \ldots + \lambda_m \mathbf{a_m}.$$

Linear Combinations of Rows

Example

Let us consider the following matrix: $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & 1 \\ 6 & 5 & 3 \end{pmatrix}$.

The third row is a linear combination of the first two by means of scalars $\lambda_1, \lambda_2 = 1, 2$, the third row is equal to the weighted sum obtained by multiplying the first row by 1 and summing to it the second row multiplied by 2:

$$(6, 5, 3) = (0, 1, 1) + 2\,(3, 2, 1)$$

that is

$$\mathbf{a_3} = \mathbf{a_1} + 2\mathbf{a_2}.$$

Linear Combinations of Columns

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$ be a matrix. The j^{th} column is said *linear combination* of the other column if each of its element $a_{i,j}$ can be expressed as weighted sum of the other elements of the i^{th} row by means of the same scalars $\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots \lambda_n$:

$$\mathbf{a^j} = \lambda_1 \mathbf{a^1} + \lambda_2 \mathbf{a^2} + \cdots + \lambda_{j-1} \mathbf{a^{j-1}} + \lambda_{j+1} \mathbf{a^{j+1}} + \ldots + \lambda_n \mathbf{a^n}.$$

Linear Combinations of Columns

Example

Let us consider the following matrix: $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \\ 1 & 3 & 0 \end{pmatrix}$.

The third column is a linear combination of the first two by means of scalars $\lambda_1, \lambda_2 = 3, -1$, the third column is equal to the weighted sum obtained by multiplying the first column by 3 and summing to it the second row multiplied by -1:

$$\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}.$$

that is

$$\mathbf{a^3} = 3\mathbf{a^1} - \mathbf{a^2}.$$

Linear Dependency

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$ be a matrix. The m rows (n columns) are *linearly dependent* if a row (column) composed of all zeros $\mathbf{o} = (0, 0, \ldots, 0)$ can be expressed as the linear combination of the m rows (n columns) by means of non-null scalars (i.e. at least one is non-null).

Linear Dependency

Example

The rows in the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \\ 4 & 6 & 6 \end{pmatrix}$$

are linearly dependent since

$$\mathbf{o} = -2\mathbf{a_1} - \mathbf{a_2} + \mathbf{a_3}$$

That is a null row can be expressed as the linear combination of the row vector by means of $\lambda_1, \lambda_2, \lambda_3 = -2, -1, 1$.

Proposition

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ be a matrix and $\det \mathbf{A}$ its determinant.
The determinant of the matrix is zero if and only if the rows (columns) are linearly dependent.

Not proved here.

This proposition links linear dependency to singularity of the determinant to matrix non-invertibility.

Linear Dependency and Singularity

Example

Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

- The determinant is $1 \times 4 - 2 \times 2 = 0$.
- The rows are linearly dependent since

$$\mathbf{o} = 2\mathbf{a_1} - \mathbf{a_2}$$

  with $\lambda_1, \lambda_2 = 2, -1$.

Rank of a Matrix

Rank

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$ with $\mathbf{A}$ assumed to be different from the null matrix.

The *rank* of the matrix $\mathbf{A}$, indicated as $\rho_{\mathbf{A}}$, is the highest order of the non-singular submatrix $\mathbf{A}_\rho \subset \mathbf{A}$.

If $\mathbf{A}$ is the null matrix then its rank is taken equal to 0.

Example

The rank of the matrix $\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 0 \end{pmatrix}$ is 2 as the submatrix $\begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$ is non-singular (determinant is non-zero).

Rank and Linear Dependency

Theorem

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ and ρ its rank. The matrix $\mathbf{A}$ has ρ linearly independent rows (columns).

Proof.

Let us prove this theorem for the rows. For linearly independent columns the proof would be analogous.

For ρ, $\det \mathbf{A}_\rho \neq 0$ for at least one submatrix $\mathbf{A}_\rho$ of order ρ, by definition. Hence all rows are linearly independent (see slide 9) and so there are ρ linearly independent rows.

Consider all s such that $\rho < s \leq n$. Then all square submatrices $\mathbf{A_s}$ of order s have $\det \mathbf{A_s} = 0$, by definition. Hence the rows must be linearly dependent (see slide 9). $\square$

Rank and Linear Dependency

Example

Let us consider the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}.$$

We can easily verify that $\det \mathbf{A} = 0$ and that the rank of the matrix is $\rho = 2$. We can observe that the third row is sum of the other two rows:

$$\mathbf{a_3} = \mathbf{a_1} + \mathbf{a_2}$$

that is the rows are linearly dependent. On the other hand, any two rows are linearly independent.

Summary and next lecture

Summary

- Linear dependency
- Rank of a matrix

The next lecture

We will learn about systems of linear equations.