AE1MCS: Mathematics for Computer Scientists

Huan Jin University of Nottingham Ningbo China

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 7, Section 7.2 Probability Theory
- Chapter 7, Section 7.3 Bayes' Theorem
- Chapter 7, Section 7.4 Expected Value and Variance

Probability Distribution

Let s be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability $p(s)$ to each outcome. We require that two conditions be met:

1. $0 \leq p(s) \leq 1$ for each $s \in S$
2. $\sum_{s \in S} p(s) = 1$.

The function p from the set of all outcomes of the sample space S is called a **probability distribution.**

Conditional Probability

Given an event F occurs, the probability that event E occurs is the **conditional probability** of E given F.

Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F, denoted by $p(E|F)$, is defined as:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

.

Conditional Probability

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

Solution: Let E be the event that a bit of length four contains at least two consecutive 0s,
Let F be the event that the first bit of a bit string of length four is a 0.
The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Conditional Probability

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

Solution: Let E be the event that a bit of length four contains at least two consecutive 0s,
Let F be the event that the first bit of a bit string of length four is a 0.
The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Because $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$, then $p(E \cap F) = \frac{5}{16}$. Because there are 8 bit strings of length four that start with a 0, we have $p(F) = \frac{8}{16} = \frac{1}{2}$.

$$p(E|F) = \frac{5/16}{1/2} = \frac{5}{8}$$

Independence

When two events are independent, the occurrence of one of the events gives no information about the probability of that the other event occurs.
The events E and F are independent **if and only if** $p(E \cap F) = p(E)p(F)$.

Independence

Suppose E is the event that a randomly generated bit string of length four begin with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

Roll the Dice

What is the probability that a die comes up an odd number when it is rolled?

Bayes' Theorem

Suppose we know p(F), the probability that an event F occurs, but we have knowledge that an event E occurs.
The conditional probability that F occurs given that E occurs, $p(F|E)$

Bayes' Theorem

Bayes' Theorem

Suppose that E and F are events from a sample space S such that $P(E) \neq 0$ and $P(F) \neq 0$. Then

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

Generalizing Bayes' Theorem

Suppose that E is an event from a sample space S and that $F_1, F_2, ..., F_n$ are mutually exclusive events such that $\cup_{i=1}^{n} F_i = S$. Assume that $p(E) \neq 0$ and $p(F_i) \neq 0$ for $i = 1, 2, ..., n$. Then

$$p(F_j|E) = \frac{p(E|F_j)p(F_j)}{\sum_{i=1}^{n} p(E|Fi)p(Fi).}$$

Expected Value and Variance

The **expected value**, also called the expectation or mean, of the random variable X on the sample space S:

$$E(X) = \sum_{s \in S} p(s)X(S)$$

If X is a random variable and $p(X = r)$ is the probability that $X = r$, so that $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$, then

$$E(X) = \sum_{r \in X(S)} p(X = r)r$$

Expectation

Corollary

If X is a random variable and $P(X = i)$ is the probability that $X = i$, then

$$E(X) = \sum_{i=1}^{\infty} iP(X = i)$$

Theorem

If $X : S \to \mathbb{N}$, then

$$E(X) = \sum_{i=0}^{\infty} P(X > i) = \sum_{i=1}^{\infty} P(X \geq i)$$

Linearity of Expectations

If X_i, $i = 1, 2, ..., n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then

1. $E(X_1 + X_2 + \cdots + Xn) = E(X_1) + E(X_2) + \cdots + E(X_n)$
2. $E(aX + b) = aE(X) + b$.

Independent Random Variables

The random variables X and Y on a sample space S are independent if

$$p(X = r1 \text{ and } Y = r2) = p(X = r1) \cdot p(Y = r2)$$

or, if the probability that $X = r1$ and $Y = r2$ equals the product of the probabilities that $X = r1 and Y = r2$, for all real numbers $r1$ and $r2$.

Corollary

If X is independent of Y, then

$$E(XY) = E(X) \cdot E(Y) \tag{1}$$

If $X_1, X_2, \ldots, X_n$ are mutually independent, then,

$$E(X_1 X_2 \ldots X_n) = E(X_1)E(X_2) \ldots E(X_n) \tag{2}$$

Variance

Variance provides a measure of how widely X is distributed about its expected value.

Definition

Let X be a random variable on a sample space S. The variance of X, denoted by $Var(X)$, is

$$Var(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

That is, Var(X) is the weighted average of the square of the deviation of X. The standard deviation of X, denoted $\sigma(X)$, is defined to be $\sqrt{Var(X)}$.

Variance

Theorem

If X is a random variable on a sample space S, then
$Var(X) = E(X^2) - E(X)^2$

Corollary

If X is a random variable on a sample space S and $E(X) = \mu$, then
$Var(X) = E((X - \mu)^2)$.

How to prove it?

Example: Rolling a Die

Let X be the number that comes up when a fair die is rolled. What is the expected value and variance of X?

Example: Rolling a Die

Let X be the number that comes up when a fair die is rolled. What is the expected value and variance of X?

Solution: The random variable X takes the values $1, 2, 3, 4, 5, 6$, each with probability 1/6. It follows that:

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 21/6 = 7/2 = 3.5$$

$$Var(X) = \frac{1}{6} \cdot (1 - 3.5)^2 + \frac{1}{6} \cdot (2 - 3.5)^2 + \frac{1}{6} \cdot (3 - 3.5)^2 + \frac{1}{6} \cdot (4 - 3.5)^2 + \frac{1}{6} \cdot (5 - 3.5)^2$$

$$+ \frac{1}{6} \cdot (6 - 3.5)^2 = 2.917$$

Example

What is the variance of the random variable $X((i, j)) = 2i$, where i is the number appearing on the first die and j is the number appearing on the second die, when two fair dice are rolled?

Example

What is the variance of the random variable $X((i,j)) = 2i$, where i is the number appearing on the first die and j is the number appearing on the second die, when two fair dice are rolled?

Solution: Note that because $p(X = k)$ is $1/6$ for $k = 2, 4, 6, 8, 10, 12$ and is 0 otherwise,
$E(X) = (2 + 4 + 6 + 8 + 10 + 12)/6 = 7$,
and

$$E(X^2) = (2^2 + 4^2 + 6^2 + 8^2 + 10^2 + 12^2)/6 = 182/3.$$

It follows that:

$$Var(X) = E(X^2) - E(X)^2 = 182/3 - 49 = 35/3.$$

Variance for the sum of random variables

If X and Y are independent variable,

$$Var(X+Y) = Var(X) + Var(Y)$$

In addition,

$$Var(aX+b) = a^2 Var(X)$$

Expected Value Homework

The expected number of successes when n mutually independent Bernoulli trials are performed, where p is the probability of success on each trial, is np.

Proof: Let X be the random variable equal to the number of successes in n trials. By Theorem 2 of Section 7.2 we see that $p(X = k) = C(n, k)p^k q^{n-k}$. Hence, we have

$$
\begin{aligned}
E(X) &= \sum_{k=1}^{n} k p(X = k) && \text{by Theorem 1} \\
&= \sum_{k=1}^{n} k C(n, k) p^k q^{n-k} && \text{by Theorem 2 in Section 7.2} \\
&= \sum_{k=1}^{n} n C(n-1, k-1) p^k q^{n-k} && \text{by Exercise 21 in Section 6.4} \\
&= np \sum_{k=1}^{n} C(n-1, k-1) p^{k-1} q^{n-k} && \text{factoring } np \text{ from each term} \\
&= np \sum_{j=0}^{n-1} C(n-1, j) p^j q^{n-1-j} && \text{shifting index of summation with } j = k-1 \\
&= np(p+q)^{n-1} && \text{by the binomial theorem} \\
&= np. && \text{because } p + q = 1
\end{aligned}
$$

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 7, Section 7.3 Bayes' Theorem
- Chapter 7, Section 7.4 Expected Value and Variance