University of Nottingham
UK | CHINA | MALAYSIA

Centre for English Language Education
Module: **FCMT (CELEN037)**

Revision session Spring 2023

1. Structure of the final exam paper

Ques. No.	TOPICS
1	Differentiation (Definition, Rules, Implicit, Inverse)
2	Differentiation (Parametric, Logarithmic), Application of derivatives (Tangent and Normal lines, Increasing and decreasing functions, Related rates)
3	Application of derivatives (Stationary points, derivative tests, optimization problems, Newton Raphson method), Higher order derivatives
4	Maclaurin?s series, Simple integration (including simple substitutions)
5	Integration by substitution (including trigonometric substitution), results (e.g. $\int \frac{f?(x)}{f(x)}dx$, $\int e^x(f(x) + f?(x))dx$), etc., integrating algebraic fractions, t-substitution
6	Techniques of Integration (partial fractions, parts, including definite integrals), Applications of integration (Area, volume of solid of revolution), Numerical integration methods
8	Order and Degree of ODEs, Solving V-S form ODEs, IVPs, Formation of ODE, Solutions of ODE, Applications of ODE

- Each question carries 10 marks.
- Total time: 90 minutes (i.e. approximately 12 min. per question).
- Only $fx - 82$ series calculators are permitted.
- Standard (small) translation dictionary permitted.

2. **Differentiation**

(1) Given $y = \dfrac{3x^3 - 7x^2 + 1}{x^4}$, find $\dfrac{dy}{dx}$.

$$\Rightarrow y = \frac{3}{x} - \frac{7}{x^2} + \frac{1}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = 3(-x^{-2}) - 7(-2x^{-3}) + (-4x^{-5})$$

$$= \frac{-3}{x^2} + \frac{14}{x^3} - \frac{4}{x^5}$$

(2) Given $y = x \cdot e^x \cdot \sin x$, find $\dfrac{dy}{dx}$.

$$\Rightarrow \frac{dy}{dx} = xe^x\left[\frac{d}{dx}(\sin x)\right] + x\sin x\,\frac{d}{dx}(e^x) + e^x\sin x\,\frac{d}{dx}(x)$$

$$= xe^x(\cos x) + x\sin x\cdot(e^x) + e^x\cdot\sin x\cdot(1)$$

$$= e^x\left[x\cos x + x\sin x + \sin x\right].$$

(3) Given $y = \dfrac{5x^2 - 10x + 9}{(x-1)^2}$; $x \neq 1$, use the quotient rule for derivatives to show that

$$\frac{dy}{dx} + \frac{8}{(x-1)^3} = 0.$$

$$\Rightarrow y = \frac{(x-1)^2(10x-10) - (5x^2-10x+9)\cdot 2(x-1)}{[(x-1)^2]^2}$$

$$= \frac{\cancel{10x^3} - 10 - \cancel{30x^2} + 30x - \cancel{10x^3} + \cancel{10x^2} + \cancel{20x^2} - 20x - 18x + 18}{(x-1)^4}$$

$$= \frac{-8x+8}{(x-1)^4} = \frac{-8}{(x-1)^3} \Rightarrow \frac{dy}{dx} + \frac{8}{(x-1)^3} = 0$$

(4) Given $y = \left(\dfrac{1}{x^2}\right)^{\sin x}$, use logarithmic differentiation to find $\dfrac{dy}{dx}$.

$$\Rightarrow \ln y = \sin x \ln\left(\frac{1}{x^2}\right) = \sin x\left(-2\ln x\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = -2\sin x \cdot \frac{1}{x} - 2\ln x \cdot \cos x$$

$$\Rightarrow \frac{dy}{dx} = y\left[-\frac{2\sin x}{x} - 2\cos x \cdot \ln x\right]$$

$$\Rightarrow \frac{dy}{dx} = -2\left(\frac{1}{x^2}\right)^{\sin x} \cdot \left[\frac{\sin x}{x} + \cos x \ln x\right]$$

(5) Find the gradient of $x^2 + 2xy - 2y^2 + x = 2$ at the point $(-4, 1)$.

$$\Rightarrow 2x + 2x\frac{dy}{dx} + 2y - 4y\frac{dy}{dx} + 1 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1 - 2x - 2y}{2x - 4y}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(-4,1)} = \frac{-1 + 8 - 2}{-8 - 4}$$

$$= -\frac{5}{12} \quad \text{(Answer)}.$$

3. **Differentiation and applications**

(1) Given $x^3 + y^3 = 3xy^2$, use the method of implicit differentiation to find $\frac{dy}{dx}$.

Differentiate both sides w.r.t. x.

$$\Rightarrow \frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(3xy^2)$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3\left[y^2 + 2xy\frac{dy}{dx}\right]$$

$$\Rightarrow \frac{dy}{dx}\left[3y^2 - 6xy\right] = 3y^2 - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cancel{3}(y^2-x^2)}{\cancel{3}(y^2-2xy)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2-x^2}{y^2-2xy}$$

(2) Given $y\cos\left(x^2\right) = x\sin\left(y^2\right)$, use the method of implicit differentiation to find $\frac{dy}{dx}$.

Differentiate both sides w.r.t. x

$$\Rightarrow \frac{d}{dx}[y\cos(x^2)] = \frac{d}{dx}[x\sin(y^2)]$$

$$\Rightarrow y\cdot\frac{d}{dx}[\cos(x^2)] + \cos(x^2)\cdot\frac{d}{dx}(y) = x\frac{d}{dx}[\sin(y^2)] + \sin(y^2)\cdot\frac{d[x]}{dx}$$

$$\Rightarrow -2xy\sin(x^2) + \cos(x^2)\cdot\frac{dy}{dx} = 2xy\cos(y^2)\cdot\frac{dy}{dx} + \sin(y^2)$$

$$\Rightarrow \frac{dy}{dx}\left[\cos(x^2) - 2xy\cos(y^2)\right] = 2xy\sin(x^2) + \sin(y^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy\sin(x^2)+\sin(y^2)}{\cos(x^2)-2xy\cos(y^2)}$$

(3) Given $y = \cot^{-1}(x) + \cot^{-1}\left(\dfrac{1}{x}\right)$, find $\dfrac{dy}{dx}$.

$$\Rightarrow y = \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x^{-2}}{1+(1/x)^2} + \frac{1}{1+x^2}$$

$$= \frac{\cancel{x^2}\,-1}{x^2+1} + \frac{1}{1+x^2}$$

$$= \cancel{\frac{x^2+1}{x^2+1}} = 0$$

$$\boxed{\frac{-1}{x^2+1} + \frac{1}{x^2+1} = 0}$$

(4) Given $y = \left(\tan^{-1} x\right)^2$, prove that $(1+x^2)^2\,\dfrac{d^2y}{dx^2} + 2x(1+x^2)\,\dfrac{dy}{dx} = 2$.

$$\frac{dy}{dx} = \frac{2(\tan^{-1}x)}{1+x^2}$$

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)\cdot\frac{d}{dx}\{2\tan^{-1}x\} - 2\tan^{-1}x\cdot\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{2(1+x^2)}{1+x^2} - 4x\tan^{-1}x}{(1+x^2)^2} = \frac{2-4x\tan^{-1}x}{(1+x^2)^2}$$

$$\cancel{(1+x^2)^2}\cdot\left[\frac{2-4x\tan^{-1}x}{\cancel{(1+x^2)^2}}\right] + 2x\cancel{(1+x^2)}\cdot\frac{2(\tan^{-1}x)}{\cancel{(1+x^2)}} = 2$$

$$\frac{\sqrt{2}}{2}$$

(5) The equation of a curve is given by $x = 2\,\cos t + \sin 2t,\;\; y = \cos t - 2\,\sin t\;\;;\;\; t \in (0, \pi)$.

Find $\left.\dfrac{dy}{dx}\right|_{t=\pi/4}$.

$$\frac{dx}{dt} = -2\sin t + 2\cos 2t \quad \& \quad \frac{dy}{dt} = -\sin t - 2\cos t$$

$$\Rightarrow \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-\sin t - 2\cos t}{-2\sin t + 2\cos 2t}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\pi/4} = \frac{3}{2}$$

(6) Find the gradient of the curve given by $y = x^2 + 2x$ at the point $P(-3, 3)$.

Hence find equations of the tangent and the normal lines to the curve at point P.

$f'(x) = 2x + 2$

$\therefore \; f'(-3) = 2\cdot(-3) + 2 = -4$

tangent line: $y - 3 = -4(x+3)$

$y = -4x - 9$

slope of normal line: $n = \frac{-1}{-4} = \frac{1}{4}$

normal line: $y - 3 = \frac{1}{4}(x+3)$

$y = \frac{1}{4}x + \frac{15}{4}$

(7) Apply the Newton-Raphson iteration formula $x_{n+1} = \dfrac{4\,x_n^3 - 3\,x_n^2 + 7}{6\,x_n\,(x_n - 1)}$ with $x_0 = 2$

to obtain a root of the equation $2x^3 - 3x^2 - 7 = 0$ correct to 5 decimal places.

n	x_n
0	2
1	2.25000
2	2.21481
3	2.21401
4	2.21401

$\therefore$ desired root: 2.21401

(8) (a) Given $f(x) = x^3 + x^2 - 8x - 15 = 0$, find the stationary points of f.

$f'(x) = 3x^2 + 2x - 8$

$f'(x) = 0 \Rightarrow 3x^2 + 2x - 8 = 0$

$(3x-4)(x+2) = 0 \Rightarrow x = \frac{4}{3},\ x = -2.$

$\therefore$ stationary points: ~~$x = \frac{4}{3},\ x = -2$.~~

$\left(\frac{4}{3}, f\left(\frac{4}{3}\right)\right) = \left(\frac{4}{3}, -\frac{581}{27}\right)$

$(-2, f(-2)) = (-2, -3)$.

(b) Use the second derivative test to classify the stationary points as the points of maximum or minimum values of f.

$f''(x) = 6x + 2$.

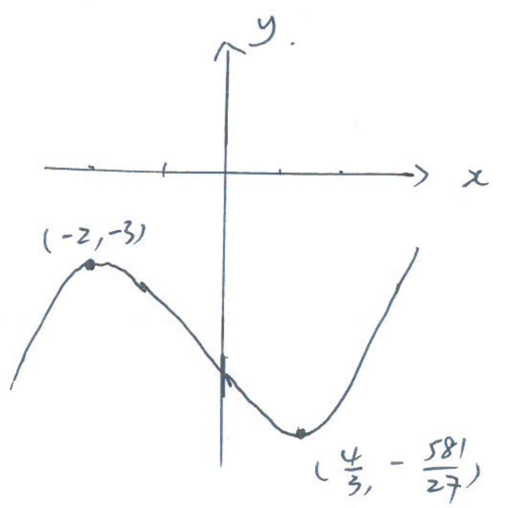
$f''\left(\frac{4}{3}\right) = 6 \cdot \frac{4}{3} + 2 = 10 > 0$

$\therefore \left(\frac{4}{3}, -\frac{581}{27}\right)$: point of minimum value.

$f''(-2) = 6 \cdot (-2) + 2 = -10 < 0$

$\therefore (-2, -3)$: point of maximum value.

(c) Draw a rough sketch of $y = f(x)$.



(9) The volume of a sphere increases at a rate of 8 cm^3/sec. Find the rate of increase of its surface area, when the radius is 4 cm.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$$

$$8 = 4\pi r^2 \cdot \frac{dr}{dt}$$

When $r = 4$ cm,

$$\frac{dr}{dt} = \frac{8}{4\pi r^2} = \frac{2}{\pi \cdot 4^2} = \frac{1}{8\pi}$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt}$$

$$= 4\pi \cdot 2 \cdot 4 \cdot \frac{1}{8\pi}$$

$$= 4 \text{ cm}^2/\text{sec.}$$

(10) The volume of a right circular cone is given by $V = \dfrac{1}{3}\,\pi\, r^2\, h,$ where r is the radius and h is the height of the cone. If the height h of the cone is increasing at a rate of 3 cm/sec, find the rate at which its volume is increasing, given that radius is 5 cm.

Given: $\dfrac{dh}{dt} = 3$ cm/sec

Find: $\left.\dfrac{dV}{dt}\right|_{r=5\text{cm}}$

When $r = 5$cm, $V = \frac{1}{3}\pi r^2 h = \frac{25\pi}{3} h$

$$\frac{dV}{dt} = \frac{25\pi}{3} \cdot \frac{dh}{dt} = \frac{25\pi}{3} \cdot 3 = 25\pi \text{ cm}^3/\text{sec}$$

4. **Maclaurin's series**

(1) Obtain the Maclaurin's series expansion of the function $f(x) = x \sin x$.

Using the Maclaurin's series expansion of $\sin x$, we obtain

$$f(x) = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+2}$$

$$= x \cdot \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots \right)$$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots\cdots$$

(2) Obtain the Maclaurin's series expansion of the function $f(x) = \dfrac{\cos x}{x^2}$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$$

$$f(x) = \frac{\cos x}{x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2(n-1)} = x^{-2} - \frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots\cdots$$

(3) Obtain the Maclaurin's series expansion of the function $f(x) = \ln(1+x)$; $x \in \mathbb{R},\ x > -1$.

Hence deduce that $\ln\sqrt{\dfrac{1+x}{1-x}} = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots\ldots \quad (-1 < x < 1)$.

$$f(x) = \ln(x+1), \qquad f(0) = 0$$

$$f'(x) = \frac{1}{x+1}, \qquad f'(0) = 1$$

$$f''(x) = -\frac{1}{(x+1)^2}, \qquad f''(0) = -1$$

$$f^{(3)}(x) = \frac{2}{(x+1)^3}, \qquad f^{(3)}(0) = 2$$

$$f^{(4)}(x) = -\frac{6}{(x+1)^4}, \qquad f^{(4)}(0) = -6$$

$$f^{(5)}(x) = \frac{24}{(x+1)^5}, \qquad f^{(5)}(0) = 24$$

$$\therefore f(x) = 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 + \frac{24}{5!}x^5 + \cdots$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots$$

$$\ln\sqrt{\frac{1+x}{1-x}} = \ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

$$\ln(1-x) = f(-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 + \cdots$$

$$\therefore \ln(1+x) - \ln(1-x) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$$

$$\therefore \ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\left(2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots\right) = x + \frac{x^3}{3} + \frac{x^5}{5}$$

(4) Obtain the Maclaurin's series expansion of the function $f(x) = e^x$. Hence deduce that

$$\frac{1}{2}\,\left(e^x - e^{-x}\right) \;=\; \sum_{1}^{\infty}\,\frac{x^{2k-1}}{(2k-1)\,!}.$$

$f(x) = e^x \quad , \quad f(0) = 1$

$f'(x) = e^x \quad , \quad f'(0) = 1$

$f''(x) = e^x , \quad f''(0) = 1$

$\therefore f(x) = 1 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$e^{-x} = f(-x) = 1 - \frac{1}{1!} + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots$

$\therefore \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left(\frac{2}{1!}x + \frac{2}{3!}x^3 + \frac{2}{5!}x^5 + \cdots\right)$

$= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$= \sum_{1}^{\infty} \frac{x^{2k-1}}{(2k-1)!}$

3. **Ques. No. 5 Integration**

(1) Evaluate $\int (\tan x - \cot x)^2 \; dx$.

$$= \int (\tan^2 x + \cot^2 x - 2) \, dx$$

$$= \int (\sec^2 x - 1 + \operatorname{cosec}^2 x - 1 - 2) \, dx$$

$$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx - \int 4 \, dx$$

$$= \underline{\tan x - \cot x - 4x + C}$$

(2) Evaluate $\int \frac{x^2}{36 - x^2} \, dx$.

$$= -\int \frac{(36 - x^2) - 36}{36 - x^2} \, dx$$

$$= -\int \left[1 - \frac{36}{36 - x^2} \right] dx$$

$$= 36 \int \frac{1}{6^2 - x^2} \, dx - \int 1 \, dx$$

$$= \cancel{36}^{3} \cdot \frac{1}{\cancel{2(6)}} \ln \left| \frac{x+6}{x-6} \right| - x + C$$

(3) Evaluate $\int \frac{e^x}{\sqrt{e^{2x} - 4}} \, dx$ by using appropriate substitution.

Let $e^x = t \Rightarrow e^x \, dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt{t^2 - 4}}$$

$$= \ln \left| t + \sqrt{t^2 - 4} \right| + C$$

$$= \underline{\ln \left| e^x + \sqrt{e^{2x} - 4} \right| + C}$$

(4) Evaluate $\displaystyle\int \frac{1}{x \cdot \left[(\ln x)^2 + 81\right]}\, dx$ by using appropriate substitution.

Let $\ln x = t \Rightarrow \frac{1}{x}\, dx = dt$

$$\therefore I = \int \frac{dt}{t^2 + 9^2}$$

$$= \frac{1}{9} \tan^{-1}\left(\frac{t}{9}\right) + C$$

$$= \frac{1}{9} \tan^{-1}\left(\frac{\ln|x|}{9}\right) + C$$

(5) Evaluate $\displaystyle\int \sin^4 x \cdot \cos^3 x \, dx$ by using appropriate substitution.

$$= \int \sin^4 x \cdot \cos^2 x \cdot \underline{\cos x \, dx}$$

Let $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\therefore I = \int t^4 \cdot (1 - t^2)\, dt$$

$$= \int (t^4 - t^6)\, dt$$

$$= \frac{1}{5} t^5 - \frac{1}{7} t^7 + C$$

$$= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$$

(6) Evaluate $\displaystyle\int \sin 4x \cdot \cos 3x \, dx$

$$= \frac{1}{2} \int 2 \sin 4x \cos 3x \, dx$$

$$= \frac{1}{2} \int (\sin 7x + \sin x)\, dx$$

$$= \frac{1}{2}\left[\frac{-\cos 7x}{7} - \cos x\right] + C$$

OR

$$= -\frac{1}{14}\left[\cos 7x + 7\cos x\right] + C$$

(7) Evaluate $\displaystyle\int \frac{\sec x\,\tan x}{\sec x + 1}\,dx.$

Let $f(x) = \sec x + 1$

so that $f'(x) = \sec x \tan x$

$$\therefore I = \int \frac{f'(x)}{f(x)}\,dx$$

$$= \ln|f(x)| + C = \ln|\sec x + 1| + C$$

(8) Evaluate $\displaystyle\int \frac{e^{2x}-1}{e^{2x}+1}\,dx.$

$$= \int \frac{\cancel{e^{x}}(e^{x}-e^{-x})}{\cancel{e^{x}}(e^{x}+e^{-x})}\,dx$$

Let $f(x) = e^{x} + e^{-x}$

$f'(x) = e^{x} - e^{-x}$

$$\therefore I = \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$= \underline{\ln|e^{x} + e^{-x}| + C}$$

(9) Evaluate $\displaystyle\int \frac{2x+3}{\sqrt{1+3x+x^2}}\,dx.$

Let $f(x) = 1 + 3x + x^2 \Rightarrow f'(x) = 3 + 2x$

$$\therefore I = \int \frac{f'(x)}{\sqrt{f(x)}}\,dx$$

$$= 2\sqrt{f(x)} + C$$

$$= \underline{2\sqrt{1+3x+x^2} + C}$$

(10) Evaluate $\int e^x \cdot \left(1 - x^2 - 2x\right)\, dx$.

Let $f(x) = 1 - x^2$

$\Rightarrow f'(x) = -2x$

$\therefore I = \int e^x \left(f(x) + f'(x) \right) dx$

$= e^x \cdot f(x) + C$

$= e^x \cdot (1 - x^2) + C$

(11) Evaluate $\int e^x \cdot \left(5 - \sec^2 x - \tan x\right)\, dx$.

Let $f(x) = 5 - \tan x$

$\therefore f'(x) = -\sec^2 x$

$\therefore \quad I = \int e^x \left(f(x) + f'(x) \right) dx$

$= e^x \cdot f(x) + C$

$= e^x \left(5 - \tan x \right) + C$

4. **Ques. No. 6, 7 Integration, Definite Integration, Area and Volume calculation**

(1) Evaluate $\displaystyle\int \frac{2x+5}{(x-2)(x+1)}\,dx$ by using the method of partial fractions.

$$\frac{2x+5}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$\Rightarrow \quad A(x+1) + B(x-2) = 2x+5$$

$$\therefore\ x=2 \Rightarrow A(3) = 9 \Rightarrow \underline{A=3}$$

$$x=-1 \Rightarrow B(-3) = 3 \Rightarrow \underline{B=-1}$$

$$\therefore \int \frac{2x+5}{(x-2)(x+1)}\,dx = \int \frac{3\,dx}{x-2} - \int \frac{1\,dx}{x+1}$$

$$= \underline{3\ln|x-2| - \ln|x+1| + C}$$

(2) Evaluate $\displaystyle\int \frac{1}{(3x-1)(x^2+1)}\,dx$ by using the method of partial fractions.

$$\frac{1}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$\therefore\ A(x^2+1) + (Bx+C)(3x-1) = 1$$

$$\therefore\ x = \frac{1}{3} \Rightarrow A\left(\frac{10}{9}\right) = 1 \Rightarrow \underline{A = \frac{9}{10}}$$

$$Ax^2 + A + 3Bx^2 - Bx + 3Cx - C = 1$$

$$\text{Coeffi. of } x^2 \Rightarrow A + 3B = 0 \Rightarrow \underline{B = \frac{-3}{10}}$$

$$\text{Coeffi. of } x \Rightarrow -B + 3C = 0 \Rightarrow \underline{C = \frac{-1}{10}}$$

$$\therefore \int \frac{1}{(3x-1)(x^2+1)}\,dx = \int \frac{9\,dx}{10(3x-1)} - \frac{3}{10\cdot 2}\int \frac{2x}{x^2+1}\,dx - \int \frac{1}{10}\frac{1}{x^2+1}\,dx$$

$$= \frac{\cancel{9}^{\,3}}{10}\,\frac{\ln|3x-1|}{\cancel{3}} - \frac{3}{20}\ln|x^2+1| - \frac{1}{10}\tan^{-1}x + C$$

(3) Evaluate $\int \frac{\ln x}{x^2}\, dx$ by using the method of integration by parts.

$$= \int \ln x \cdot \frac{1}{x^2}\, dx$$

LIATE

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{x^2} \Rightarrow v = -\frac{1}{x}$

$$\therefore I = uv - \int v \frac{du}{dx}\, dx$$

$$= \ln x \cdot \left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right) \frac{1}{x}\, dx$$

$$= -\frac{1}{x} \ln x - \frac{1}{x} + C = -\frac{1}{x}(1 + \ln x) + C$$

(4) Evaluate $\int x \sec^2 x\, dx$ by using the method of integration by parts.

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

LIATE

$$\therefore I = uv - \int v \frac{du}{dx}\, dx$$

$$= x \tan x - \int \tan x \cdot (1)\, dx$$

$$= \underline{x \tan x - \ln|\sec x| + C}$$

(5) Evaluate $\displaystyle\int_1^4 f(x)\,dx$ where $f(x) = \begin{cases} 2x+8 & ; \quad 1 \le x \le 2 \\ 6x & ; \quad 2 < x \le 4 \end{cases}$

$$= \int_1^2 (2x+8)\,dx + \int_2^4 (6x)\,dx$$

$$= \left[x^2+8x\right]_1^2 + \left[3x^2\right]_2^4$$

$$= 4+16-1-8+48-12 = \underline{47}.$$

(6) Use the property of definite integrals to evaluate Let $I = \displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx.$ — ①

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore I = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx \quad \text{— ②}$$

$$\therefore ① + ② \Rightarrow 2I = \int_0^{\pi/2} \left[\frac{\cos x}{\sin x + \cos x} + \frac{\sin x}{\sin x + \cos x}\right] dx$$

$$\therefore I = \frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{\pi}{4}.$$

(7) Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 49}\,dx$ by using appropriate substitution.

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

x	0	$\pi/4$
t	0	1

$$\therefore I = \int_0^1 \frac{dt}{t^2+7^2} = \left[\frac{1}{7}\tan^{-1}\left(\frac{t}{7}\right)\right]_0^1$$

$$= \underline{\frac{1}{7}\tan^{-1}\left(\frac{1}{7}\right)}.$$

(8) Evaluate the definite integral $\displaystyle\int_0^1 \frac{1}{e^x+e^{-x}}\,dx$ by using appropriate substitution.

$$\therefore I = \int_0^1 \frac{e^x\,dx}{e^{2x}+1}$$

Let $e^x = t \Rightarrow e^x\,dx = dt$

x	0	1
t	1	e

$$\therefore I = \int_1^e \frac{dt}{t^2+1} = \left[\tan^{-1} t\right]_1^e = \underline{\tan^{-1}(e) - \frac{\pi}{4}}$$

(9) Evaluate the integral $\displaystyle\int \frac{1}{2\cos x+3}\,dx$ by using the method of t-substitution.

Let $\tan\left(\frac{x}{2}\right) = t \Rightarrow dx = \frac{2\,dt}{1+t^2}$ ✓

& $\cos x = \frac{1-t^2}{1+t^2}$

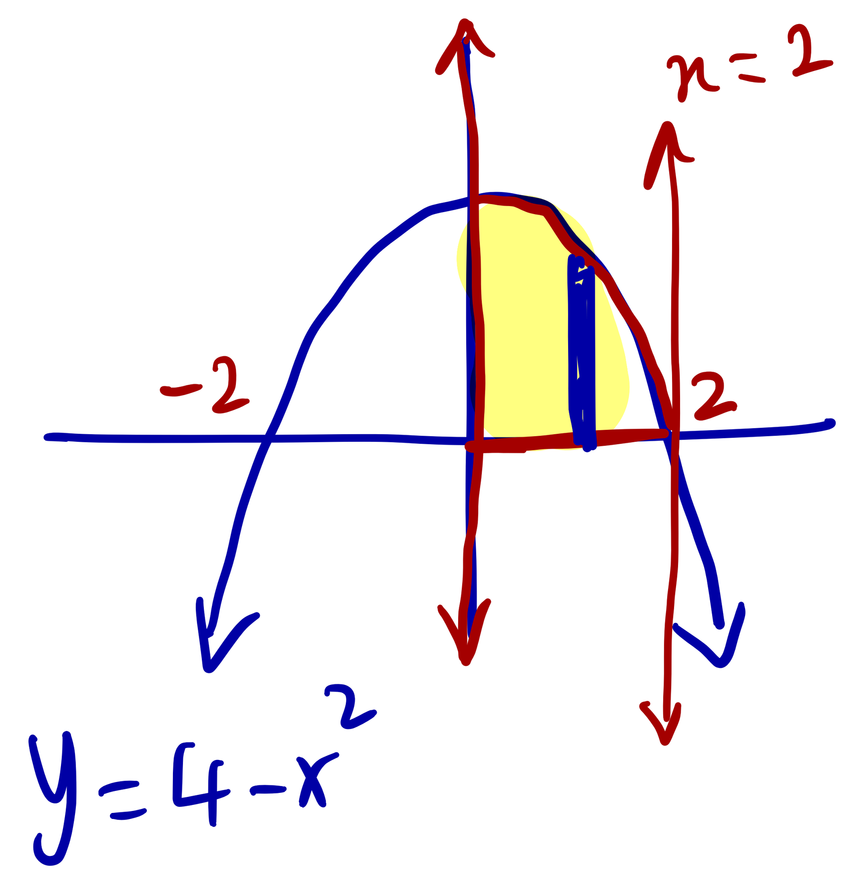
$$\therefore I = \int \frac{\frac{2\,dt}{1+t^2}}{2\left(\frac{1-t^2}{1+t^2}\right)+3}$$

$$= \int \frac{2\,dt}{2-2t^2+3+3t^2} = \int \frac{2\,dt}{t^2+5}$$ ✓

$$= 2\int \frac{dt}{t^2+(\sqrt{5})^2} = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{t}{\sqrt{5}}\right)+C$$

$$= \underline{\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{\tan(x/2)}{\sqrt{5}}\right)+C}$$ ✓

(10) Find the area of the region bounded by the curve $y = 4 - x^2$, lines $x = 0$, $x = 2$ and the X-axis.



$$A = \int_0^2 (4-x^2)\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 8 - \frac{8}{3} = \frac{16}{3}.$$

(11) Find the volume of the solid obtained when the region bounded by $xy = 4$ and $x + y = 5$ is rotated around the X-axis.

$xy = 4$ & $x + y = 5$

$\therefore x + \left(\frac{4}{x}\right) = 5$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x-1)(x-4) = 0$

$\Rightarrow x = 1$ or 4

$$V = \pi \int_1^4 \left[\left(\frac{4}{x}\right)^2 - (5-x)^2\right] dx$$

$$= \pi \int_1^4 \left[\frac{16}{x^2} - 25 + 10x - x^2\right] dx$$

$$= \pi \left[-\frac{16}{x} - 25x + 5x^2 - \frac{x^3}{3}\right]_1^4$$

$$= \pi\left[\left\{-4 - 100 + 80 - \frac{64}{3}\right\} - \left\{-16 - 25 + 5 - \frac{1}{3}\right\}\right]$$

$$= \pi\left[-24 - \frac{64}{3} + 36 + \frac{1}{3}\right]$$

$$= \left|\pi\,[12 - 21]\right| = 9\pi.$$

5. **Ques. No. 8 Differentiation equations and their applications**

(1) Solve the ODE: $\dfrac{dy}{dx} = x\, e^{y}$.

$$\int e^{-y}\, dy = \int x\, dx$$

$$\Rightarrow -e^{-y} = \frac{x^2}{2} + c$$

(2) Solve the ODE: $2y\,(e^x + 2019)\; dy = \left(y^2+1\right)\; e^x\; dx$.

$$\int \frac{2y\; dy}{y^2+1} = \int \frac{e^x\, dx}{e^x+2019}$$

$$\Rightarrow \ln|y^2+1| = \ln|e^x+2019| + \ln c$$

$$\Rightarrow y^2+1 = c\,(e^x+2019)$$

(3) Solve the ODE: $\dfrac{dy}{dx} + xy = y\; \sec^2 x$.

$$\frac{dy}{dx} = y\left(\sec^2 x - x\right)$$

$$\int \frac{dy}{y} = \int (\sec^2 x - x)\; dx$$

$$\ln|y| = \tan x - \frac{x^2}{2} + c$$

(4) Solve the IVP: $\dfrac{dy}{dx} - x\,e^y = 5\,e^y$; $y(0) = 0$.

$$\frac{dy}{dx} = e^y(x+5)$$

$$\int e^{-y}\,dy = \int (x+5)\,dx$$

$$\Rightarrow -e^{-y} = \frac{x^2}{2} + 5x + C \quad \text{(General Solution)}$$

But $y(0)=0 \Rightarrow$ when $x=0$, $y=0$.

$$\therefore -1 = 0 + 0 + C$$

$$\therefore \underline{C = -1}$$

$$\therefore \underline{-e^{-y} = \frac{x^2}{2} + 5x - 1.} \quad \text{(Particular Sol}^{n}\text{.)}$$

(6) Show that $y = \left(\sin^{-1} x\right)^2$ is a solution of the differential equation

$$(1 - x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} = 2.$$

$$y = (\sin^{-1} x)^2$$

$$\frac{dy}{dx} = 2(\sin^{-1} x)\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,\frac{dy}{dx} = 2(\sin^{-1} x)$$

Squaring

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1} x)^2 = 4y$$

Diff. $\Rightarrow$ $(1-x^2)\cdot \cancel{2}\cancel{\left(\frac{dy}{dx}\right)}\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^{\cancel{2}}\cdot(-\cancel{2}x) = \overset{2}{\cancel{4}}\cancel{\left(\frac{dy}{dx}\right)}$

$$\Rightarrow (1-x^2)\,\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right) = 2$$

(6) Show that $y = e^{-x} + ax + b$ (a, b are arbitrary constants) is a solution of the differential equation $e^x \dfrac{d^2y}{dx^2} - 1 = 0$.

$$y = e^{-x} + ax + b$$

$$\frac{dy}{dx} = -e^{-x} + a$$

$$\frac{d^2y}{dx^2} = e^{-x}$$

$$\therefore e^x \frac{d^2y}{dx^2} = 1 \qquad \therefore \underline{e^x \frac{d^2y}{dx^2} - 1 = 0}$$

(7) The rate of decay of a radioactive material is proportional to the amount (m) of material present at that time. Formulate a differential equation model to show that the amount of material at time t is $m(t) = m_0 \cdot e^{kt}$, where $k < 0$ is constant and m_0 is the initial amount of the radioactive material.

$$\frac{dm}{dt} \propto m$$

$$\therefore \frac{dm}{dt} = km \qquad (\text{where } k < 0)$$

$$\int \frac{dm}{m} = \int k \, dt$$

$$\ln m = kt + C \quad \text{—①}$$

When $t = 0$, $m = m_0$

$$\Rightarrow \ln m_0 = k(0) + C \Rightarrow C = \ln m_0$$

$$\therefore ① \Rightarrow \ln m = kt + \ln m_0$$

$$\Rightarrow \ln\left(\frac{m}{m_0}\right) = kt \Rightarrow \underline{m(t) = m_0 e^{kt}}$$