Tutorial on Probability

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Probability of Complements

- A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?
- What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

Example 1 (Answer)

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

- Sample space is all bit strings of length 10: $|S| = 2^{10}$.
- Think about the event when bit string has *no* 0's. Then $E = \{1111111111\}$ and $|E| = 1$.
- Now the event we are interested in (at least one 0) is complement of E, so
$$p(\overline{E}) = 1 - p(E) = 1 - 2^{-10}.$$

Independence

Suppose E is the event that a randomly generated bit string of length four begin with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

Solution: There are eight bit strings of length four that begin with a one: 1000, 1001, 1010, 1011, 1100, 1101, 1110, and 1111. There are also eight bit strings of length four that contain an even number of ones: 0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111. Because there are 16 bit strings of length four, it follows that

$$p(E) = p(F) = 8/16 = 1/2.$$

Because $E \cap F = 1111, 1100, 1010, 1001$, we see that $p(E \cap F) = 4/16 = 1/4$.
Because $p(E \cap F) = 1/4 = (1/2)(1/2) = p(E)p(F)$,we conclude that E and F are independent.

Bayesian Theorem

Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

Answer

Let $P(s) =$ prob of using steroid
Let $P(+|s) =$ prob of a bicyclist who uses steroids tests positive
$P(+|\bar{s}) =$ prob of a bicyclist who does not use steroids tests positive
We know that $P(s) = 0.08$, $P(+|s) = 0.96$, $P(+|\bar{s}) = 0.09$
so $p(\bar{s}) = 1 - P(s) = 0.92$
We want to find $P(s|+)$
By Bay's theorem,

$$p(s|+) = \frac{p(+|s)p(s)}{p(+|s)p(s) + p(+|\bar{s})p(\bar{s})} = \frac{0.96 * 0,08}{0.96 * 0,08 + 0.09 * 0.92} = 0.481$$

Expected value and Variance

Rolling two dice. Assume that we are using fair, regular dice (six-sided with values 1, 2, 3, 4, 5, 6 appearing equally likely). Furthermore, assume that all dice rolls are mutually independent events.

(a) You roll two dice and look at the sum of the faces that come up. What is the expected value of this sum? Express your answer as a real number.

$E(X+Y)=E(X)+E(Y)=7/2+7/2=7$

(b) Assuming that the two dice are independent, calculate the variance of their sum. Express your answer as a real number.

$Var(X+Y)=Var(X)+Var(Y)=2.917+2.917=5.834$

Example

What is the variance of the random variable $X((i, j)) = 2i$, where i is the number appearing on the first die and j is the number appearing on the second die, when two fair dice are rolled?

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Solution: Note that because $p(X = k)$ is $1/6$ for $k = 2, 4, 6, 8, 10, 12$ and is 0 otherwise,
$E(X) = (2 + 4 + 6 + 8 + 10 + 12)/6 = 7$,
and

$$E(X^2) = (2^2 + 4^2 + 6^2 + 8^2 + 10^2 + 12^2)/6 = 182/3.$$

It follows that:

$$Var(X) = E(X^2) - E(X)^2 = 182/3 - 49 = 35/3.$$

Additional exercises

Section 7.1: 1, 3, 5, 7, 9, 11, 13
Section 7.2: 1, 3, 5, 7, 9, 11, 13, 15
Section 7.3: 1, 5, 7, 9
Section 7.4: 1, 3, 5, 7, 9, 11, 15, 17