Seminar 9

In this seminar you will study:

- Complex numbers
- Solving quadratic equations with negative discriminant ($\Delta < 0$)
- Algebra of complex numbers and their cartesian form: $z = x + iy$
- Properties of modulus
- Polar form of a complex number: $z = r(\cos\theta + i\sin\theta)$

Complex numbers

Complex number: Cartesian Form

$$z = x + i\,y$$

where $x,\, y \in \mathbb{R}$, and the imaginary number $i = \sqrt{-1} \quad \Rightarrow \quad i^2 = -1.$

Real part of z: $\quad \mathrm{Re}(z) = x$

Imaginary part of z: $\quad \mathrm{Im}(z) = y$

Conjugate of a Complex number: Cartesian Form

If $z = x + i\,y$ is a complex number then its conjugate is defined and denoted by:

$$\overline{z} = x - i\,y$$

Real part of $\overline{z}$: $\quad \mathrm{Re}(\overline{z}) = x$

Imaginary part of $\overline{z}$: $\quad \mathrm{Im}(\overline{z}) = -y$

Complex numbers

Example: Solve the quadratic equation $2x^2 - 10x + 17 = 0.$

Solution: On comparing $2x^2 - 10x + 17 = 0$ with $ax^2 + bx + c = 0$

$$a = 2, \qquad b = -10, \qquad c = 17$$

$$\Delta = b^2 - 4ac = (-10)^2 - 4(2)(17) = -36 < 0 \Rightarrow \Delta = 36i^2$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-10) \pm \sqrt{36i^2}}{2(2)} = \frac{10 \pm 6i}{4} = \frac{5 \pm 3i}{2}$$

$x = \left(\frac{5}{2}\right) + i\left(\frac{3}{2}\right)$ or $x = \left(\frac{5}{2}\right) - i\left(\frac{3}{2}\right)$

Solving quadratic equations with negative discriminant

1. Solve the quadratic equation:

$$4x^2 + 5x + 3 = 0$$

Answer: $x = \dfrac{-5 \pm i\sqrt{23}}{8}$

2. Solve the quadratic equation:

$$3x^2 - 4x + 9 = 0$$

Answer: $x = \dfrac{2 \pm i\sqrt{23}}{3}$

3. Solve the quadratic equation:

$$9x^2 - 8x + 5 = 0$$

Answer: $x = \dfrac{4 \pm i\sqrt{29}}{9}$

4. Solve the quadratic equation:

$$5x^2 + 7x + 4 = 0$$

Answer: $x = \dfrac{-7 \pm i\sqrt{31}}{10}$

Simplification of expressions involving i

$$i = \sqrt{-1} \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = -i^2 = 1 \qquad i^5 = i$$

Example: Simplify: $(i^2 + i + 2)^{10} + (i^2 - i + 2)^{10}$.

Solution: $(i^2 + i + 2)^{10} + (i^2 - i + 2)^{10}$

$$= (-1 + i + 2)^{10} + (-1 - i + 2)^{10}$$

$$= (1 + i)^{10} + (1 - i)^{10}$$

$$(1 + i)^{10} = [\,(1 + i)^2\,]^5$$

$$= [1 + 2i + i^2]^5 = [1 + 2i - 1]^5$$

$$\therefore (1 + i)^{10} = [2i]^5 = 32i^5 = 32i$$

Similarly,

$$(1 - i)^{10} = [-2i]^5 = -32i$$

$$\therefore (i^2 + i + 2)^{10} + (i^2 - i + 2)^{10}$$

$$= 32i - 32i = 0$$

Simplification of expressions involving i

$i=\sqrt{-1} \quad i^2=-1 \quad i^3=-i \quad i^4=-i^2=1 \quad i^5=i$

1. Simplify: $\left(1+i^4\right)^6+\left(i+i^5\right)^6$

   Answer: 0

2. Simplify: $\left(i^2+i+2\right)^3+\left(i^2-i+2\right)^3$

   Answer: -4

3. Simplify: $\left(i^3+i-1\right)^4-\left(i^3+i^2+i\right)^5$

   Answer: 2

4. Simplify: $\left(i^4+i^3+2i\right)^3+(i-1)^2$

   Answer: -2

Algebra of Complex numbers (Equality)

1. Find the constants p and q if the complex numbers $z_1 = (p+q) - 2\,i$ and $z_2 = 4 + i\,q$ are equal.

   Answer: $p = 6,\ q = -2$

2. Find the constants p and q if the complex numbers $z_1 = (p-q) + 3\,i$ and $z_2 = 4 + i\,(p+q)$ are equal.

   Answer: $p = 3.5,\ q = -0.5$

3. Find the constants p and q if the complex numbers $z_1 = p + 5\,i$ and $z_2 = (2-p) + i\,q$ are equal.

   Answer: $p = 1,\ q = 5$

4. Find the constants p and q if the complex numbers $z_1 = (p-2q) - i$ and $z_2 = 8 + i\,(p+q)$ are equal.

   Answer: $p = 2,\ q = -3$

Algebra of Complex numbers

1. Given $z_1 = 3 + 5\,i$ and $z_2 = 4 - i$, find:

- $z_1 + z_2 \quad = \quad 7 + 4\,i$
- $z_1 - z_2 \quad = \quad -1 + 6\,i$
- $z_1 \cdot z_2 \quad = \quad 17 + 17\,i$
- $\dfrac{z_1}{z_2} \quad = \quad \dfrac{7}{17} + \dfrac{23\,i}{17}$

2. Given $z_1 = 4 - 3\,i$ and $z_2 = 7 + 5\,i$, find:

- $z_1 + z_2 \quad = \quad 11 + 2\,i$
- $z_1 - z_2 \quad = \quad -3 - 8\,i$
- $z_1 \cdot z_2 \quad = \quad 43 - i$
- $\dfrac{z_1}{z_2} \quad = \quad \dfrac{13}{74} - \dfrac{41\,i}{74}$

Algebra of Complex numbers

1. Given $z_1 = 3 + 5\,i$, $z_2 = 2 - i$, and $z_3 = 5 + 4\,i$, express $z = \dfrac{\overline{z_1} \cdot z_2}{z_3}$ in the Cartesian form $a + i\,b$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.

   Answer: $z = -\dfrac{47}{41} - \dfrac{69\,i}{41}$

2. Given $z_1 = 4 - 3\,i$, $z_2 = 1 + 2\,i$, and $z_3 = 5 - i$, express $z = \dfrac{z_1 \cdot \overline{z_2}}{z_3}$ in the Cartesian form $a + i\,b$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$.

   Answer: $z = \dfrac{1}{26} - \dfrac{57\,i}{26}$

Properties of Modulus

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2| \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad |\overline{z}| = |z|$$

1. Given $z_1 = 2 - 7\,i$ and $z_2 = 3 + 8\,i$, find $|z_1 \cdot z_2|$ and $\left|\dfrac{z_1}{z_2}\right|$.

Answer: $|z_1 \cdot z_2| = \sqrt{3869}$, $\left|\dfrac{z_1}{z_2}\right| = \sqrt{\dfrac{53}{73}}$

2. Given $z_1 = 7 + 3\,i$ and $z_2 = 2 - i$, find $|z_1 \cdot z_2|$ and $\left|\dfrac{z_1}{z_2}\right|$.

Answer: $|z_1 \cdot z_2| = \sqrt{290}$, $\left|\dfrac{z_1}{z_2}\right| = \sqrt{\dfrac{58}{5}}$

3. Given $z_1 = 1 + 4\,i$ and $z_2 = -3 - 2\,i$, find $|z_1 \cdot z_2|$ and $\left|\dfrac{z_1}{z_2}\right|$.

Answer: $|z_1 \cdot z_2| = \sqrt{221}$, $\left|\dfrac{z_1}{z_2}\right| = \sqrt{\dfrac{17}{13}}$

4. Given $z_1 = 1 - 8\,i$ and $z_2 = -4 - i$, find $|z_1 \cdot z_2|$ and $\left|\dfrac{z_1}{z_2}\right|$.

Answer: $|z_1 \cdot z_2| = \sqrt{1105}$, $\left|\dfrac{z_1}{z_2}\right| = \sqrt{\dfrac{65}{17}}$

Properties of Modulus

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2| \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad |\overline{z}| = |z|$$

1. Given $z_1 = 4 - 7\,i$, $z_2 = 9 + 2\,i$, and $z_3 = 3 - 8\,i$, find $\left|\dfrac{\overline{z_1} \cdot z_2}{z_3}\right|$.

   Answer: $\sqrt{\dfrac{5525}{73}}$

2. Given $z_1 = 3 - 5\,i$, $z_2 = 2 + i$, and $z_3 = 9 - 5\,i$, find $\left|\dfrac{z_1 \cdot \overline{z_2}}{z_3}\right|$.

   Answer: $\sqrt{\dfrac{85}{53}}$

3. Given $z_1 = 7 - i$, $z_2 = 9 + 2\,i$, and $z_3 = 5 - 2\,i$, find $\left|\dfrac{\overline{z_1 \cdot z_2}}{z_3}\right|$.

   Answer: $\sqrt{\dfrac{4250}{29}}$

4. Given $z_1 = 6 + 3\,i$, $z_2 = 2 + 3\,i$, and $z_3 = 3 - i$, find $\left|\dfrac{z_1^2 \cdot z_3}{z_2}\right|$.

   Answer: $45 \cdot \sqrt{\dfrac{10}{13}}$

Polar form of Complex numbers

Cartesian form

$$z = x + i\,y$$

where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.

Polar form

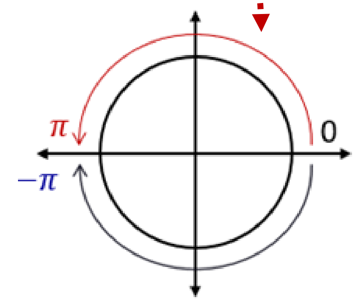
$$z = r\,(\cos\theta + i\sin\theta)$$

where $r > 0$ and $-\pi < \theta \leq \pi$.

$$r = |\,z\,| = \sqrt{x^2 + y^2}$$

and $\theta = \arg(z)$ is obtained from

$x < 0$ and $y > 0$ $\theta = \pi - \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$x > 0$ and $y > 0$ $\theta = \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$
$x < 0$ and $y < 0$ $\theta = -\pi + \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$x > 0$ and $y < 0$ $\theta = -\tan^{-1}\left\lvert\frac{y}{x}\right\rvert$



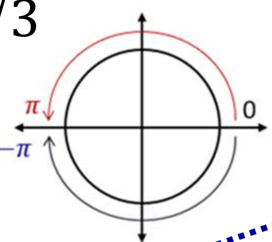
Polar form of Complex numbers

Example: Express the complex number $z = 1 + \sqrt{3}\, i$ in the polar form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $\theta \in (-\pi, \pi]$.

Solution: $z = 1 + \sqrt{3}\, i = x + iy$

$\therefore\ x = 1 \quad \text{and} \quad y = \sqrt{3}$

$\Rightarrow r = \sqrt{x^2 + y^2} = 2$



and $\theta = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right|$

$= \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$x < 0$ and $y > 0$ $\theta = \pi - \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$x > 0$ and $y > 0$ $\theta = \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$
$x < 0$ and $y < 0$ $\theta = -\pi + \tan^{-1}\left\lvert\frac{y}{x}\right\rvert$	$x > 0$ and $y < 0$ $\theta = -\tan^{-1}\left\lvert\frac{y}{x}\right\rvert$

$\therefore\ z = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

Polar form of a Complex number

1. Express the complex number $z = -2 - 3i$ in polar form $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$.

Answer: $\sqrt{13}\,[\cos(-2.1588) + i\sin(-2.1588)]$

2. Express the complex number $z = 2 + 3i$ in polar form $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$.

Answer: $\sqrt{13}\,[\cos(0.9828) + i\sin(0.9828)]$

3. Express the complex number $z = 2 - 3i$ in polar form $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$.

Answer: $\sqrt{13}\,[\cos(-0.9828) + i\sin(-0.9828)]$

4. Express the complex number $z = -2 + 3i$ in polar form $z = r(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$.

Answer: $\sqrt{13}\,[\cos(2.1588) + i\sin(2.1588)]$

Polar form of a Complex number

1. Express the complex number

$z = 7 - 9\,i$ in polar form $z = r(\cos\theta + i\sin\theta)$

where $r > 0$ and $-\pi < \theta \leq \pi$.

Answer: $\sqrt{130}\,[\cos(-0.9097) + i\sin(-0.9097)]$

2. Express the complex number

$z = -5 + 2\,i$ in polar form $z = r(\cos\theta + i\sin\theta)$

where $r > 0$ and $-\pi < \theta \leq \pi$.

Answer: $\sqrt{29}\,[\cos(2.7611) + i\sin(2.7611)]$

3. Express the complex number

$z = 9 + 7\,i$ in polar form $z = r(\cos\theta + i\sin\theta)$

where $r > 0$ and $-\pi < \theta \leq \pi$.

Answer: $\sqrt{130}\,[\cos(0.6610) + i\sin(0.6610)]$

4. Express the complex number

$z = -3 - 4\,i$ in polar form $z = r(\cos\theta + i\sin\theta)$

where $r > 0$ and $-\pi < \theta \leq \pi$.

Answer: $5\,[\cos(-2.2143) + i\sin(-2.2143)]$

Algebraic operations with Polar form of Complex numbers

Given $z_1 = r_1(\cos\theta_1 + \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \sin\theta_2)$,

$$z_1 \cdot z_2 = r_1 \cdot r_2\,[\,\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)\,].$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,[\,\cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2)\,]$$

Note: In the above results, $(\theta_1 \pm \theta_2)$ only represent the arguments of $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$ respectively.

The principal argument can be obtained by using $(\theta_1 \pm \theta_2) \pm 2\pi$

Algebraic operations with Polar form of Complex numbers

Example: Given $z_1 = 1 + \sqrt{3}\,i$ and $z_2 = \sqrt{3} + i$. Find the polar form of

$z_1 \cdot z_2$ and $\dfrac{z_1}{z_2}$.

Solution: $z_1 = 1 + \sqrt{3}\,i = 2\left[\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)\right] = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

$$z_2 = \sqrt{3} + i = 2\left[\left(\frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2}\right)\right] = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

$$z_1 \cdot z_2 = 2 \times 2\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)\right] = 4\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{2}{2}\left[\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)\right] = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$$

Algebraic operations with polar form of Complex numbers

1. Given $z_1 = -2$ and $z_2 = 1 + \sqrt{3}\, i$.

   Find the polar forms of $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$.

   $$z_1 \cdot z_2 = 4\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$$

   $$\frac{z_1}{z_2} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

2. Given $z_1 = i$ and $z_2 = -2 + 2i$.

   Find the polar forms of $z_1 \cdot z_2$ and $\frac{z_1}{z_2}$.

   $$z_1 \cdot z_2 = 2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

   $$\frac{z_1}{z_2} = \frac{1}{2\sqrt{2}}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

3. Given $z_1 = 1 + \sqrt{3}\, i$ and $z_2 = 1 - \sqrt{3}\, i$.

   Find the polar form of $z_1 \cdot z_2$.

   Hence, verify that $z_1 \cdot z_2 = 4$.

   $$z_1 \cdot z_2 = 4[\cos 0 + i\sin 0]$$

4. Given $z_1 = i$ and $z_2 = 1 - \sqrt{3}\, i$ and $z_3 = \sqrt{3} + i$.

   Find the polar form of $z_1 \cdot z_2$.

   Hence, verify that $z_1 \cdot z_2 = z_3$.

   $$z_1 \cdot z_2 = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

THANKS FOR YOUR ATTENTION