

Lecture 2 - Matrices

COMP1046 - MCS2

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Learning outcomes

By the end of this lecture we will have learned:

- ⊙ Numeric vectors
- ⊙ Matrices
- ⊙ Matrix operations: sum and product

Learning Resources

Chapter 2, Section 2.1-2.3: F. Neri, Linear Algebra for Computational Sciences and Engineering

Numeric vectors

Definition

Numeric Vector

- ⊙ Let $n \in \mathbb{N}$ and $n > 0$.
- ⊙ The set generated by the Cartesian product of \mathbb{R} by itself n times ($\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \dots$) is indicated with \mathbb{R}^n and is a set of ordered n -tuples of real numbers.
- ⊙ The generic element $\mathbf{a} = (a_1, a_2, \dots, a_n)$ of this set is named *numeric vector* or simply vector of order n on the real field.
- ⊙ The generic $a_i \forall i$ from 1 to n is said the i^{th} component of the vector \mathbf{a} .

Example

The n -tuple

$$\mathbf{a} = (1, 0, 56.3, \sqrt{2})$$

is a vector of \mathbb{R}^4 .

Definition

Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be two numeric vectors $\in \mathbb{R}^n$. The *sum* of these two vectors is the vector

$$\mathbf{c} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

generated by the sum of the corresponding components.

We write the sum of two vectors as $\mathbf{a} + \mathbf{b}$.

Similarly, we can subtract one vector from another, $\mathbf{a} - \mathbf{b}$, component-wise.

Example: Adding two vectors

Example

Let us consider the following vectors of \mathbb{R}^3

$$\mathbf{a} = (1, 0, 3)$$

$$\mathbf{b} = (2, 1, -2).$$

The sum of these two vectors is

$$\mathbf{a} + \mathbf{b} = (3, 1, 1).$$

Definition

Scalar A numeric vector $\lambda \in \mathbb{R}^1$ is said *scalar*.

Definition

Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ be a numeric vector $\in \mathbb{R}^n$ and λ a number $\in \mathbb{R}$. The *product of a vector by a scalar* is the vector

$$\lambda \mathbf{a} = (\lambda a_1, \lambda a_2, \dots, \lambda a_n)$$

generated by the product of λ by each corresponding component.

Example: Product of a scalar and a vector

Example

Let us consider the vector $\mathbf{a} = (1, 0, 4)$ and the scalar $\lambda = 2$.
The product of this scalar by this vector is

$$\lambda \mathbf{a} = (2, 0, 8).$$

Scalar product

Definition

Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ be two numeric vectors $\in \mathbb{R}^n$. The *scalar product* of \mathbf{a} by \mathbf{b} is a real number

$$\mathbf{a}\mathbf{b} = a_1b_1 + a_2b_2, \dots, a_nb_n$$

generated by the sum of the products of each pair of corresponding components.

Note: The scalar product is also referred to as the *dot product* written as $\mathbf{a} \cdot \mathbf{b}$ (just different notation).

Example

Let us consider again

$$\mathbf{a} = (1, 0, 3)$$

$$\mathbf{b} = (2, 1, -2).$$

The scalar product of these two vectors is

$$\mathbf{ab} = (1 \cdot 2) + (0 \cdot 1) + (3 \cdot (-2)) = 2 + 0 - 6 = -4.$$

Properties of the scalar product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. It can be proved that the following properties of the scalar product are valid.

- ⊙ symmetry: $\mathbf{a}\mathbf{b} = \mathbf{b}\mathbf{a}$
- ⊙ associativity: $\lambda (\mathbf{b}\mathbf{a}) = (\lambda\mathbf{a})\mathbf{b} = \mathbf{a}(\lambda\mathbf{b})$
- ⊙ distributivity: $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{a}\mathbf{b} + \mathbf{a}\mathbf{c}$

Matrices

Definition

Matrix

Let $m, n \in \mathbb{N}$ and both $m, n > 0$. A matrix $(m \times n)$ \mathbf{A} is a generic table of the kind:

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

where each *matrix element* $a_{i,j} \in \mathbb{R}$.

Matrix characteristics

- ⊙ The set containing all the matrices of real numbers having m rows and n columns is indicated with $\mathbb{R}_{m,n}$.
- ⊙ A matrix $\mathbf{A} \in \mathbb{R}_{n,n}$ is said *n order square matrix*.
- ⊙ If the matrix is not square ($m \neq n$), it is said *rectangular*.
- ⊙ The numeric vector $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,n})$ is said *generic i^{th} row vector*, and
- ⊙ $\mathbf{a}^j = (a_{1,j}, a_{2,j}, \dots, a_{m,j})$ is said *generic j^{th} column vector*.

Example

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 3 \\ 2 & -2 & 0 \end{pmatrix}$$

- ⊙ Then $m = 3$ and $n = 3$. Hence \mathbf{A} is a square matrix and $\mathbf{A} \in \mathbb{R}_{3,3}$.
- ⊙ The 2nd row vector is $\mathbf{a}_2 = (0, 1, 3)$ and the 3rd column vector is $\mathbf{a}^3 = (-1, 3, 0)$.

Matrix transpose

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$. The *transpose* matrix of \mathbf{A} is a matrix \mathbf{A}^T whose elements are the same as \mathbf{A} but $\forall i, j: a_{j,i} = a_{i,j}^T$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 7 & 3.4 & \sqrt{2} \\ 5 & 0 & 4 & 1 \end{pmatrix}$$

$$\mathbf{A}^T = \begin{pmatrix} 2 & 5 \\ 7 & 0 \\ 3.4 & 4 \\ \sqrt{2} & 1 \end{pmatrix}$$

It can be easily proved that the transpose of the transpose of a matrix is the matrix itself: $\mathbf{A} = (\mathbf{A}^T)^T$.

Symmetry

Definition

A matrix $\mathbf{A} \in \mathbb{R}_{n,n}$ is said *symmetric* when $\forall i, j : a_{i,j} = a_{j,i}$.

Example

The following matrix is symmetric:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

Proposition

Let \mathbf{A} be a symmetric matrix. It follows that $\mathbf{A}^T = \mathbf{A}$.

Definition

Let $\mathbf{A} \in \mathbb{R}_{n,n}$. The *diagonal of a matrix* is the ordered n -tuple that displays the same index twice: $\forall i$ from 1 to n $a_{i,i}$.

Definition

Let $\mathbf{A} \in \mathbb{R}_{n,n}$. The *trace of a matrix* $\text{tr}(\mathbf{A})$ is the sum of the diagonal elements: $\text{tr}(\mathbf{A}) = \sum_{i=1}^n a_{i,i}$.

Example

The diagonal of the matrix

$$\begin{pmatrix} 1 & 3 & 0 \\ 9 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

is $(1, 2, 2)$ and the trace is $1 + 2 + 2 = 5$.

Definition

A matrix is said *null* \mathbf{O} if all its elements are zeros.

Example

The null matrix of $\mathbb{R}_{2,3}$ is

$$\mathbf{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Identity Matrices

Definition

An *identity matrix* \mathbf{I} is a square matrix whose diagonal elements are all ones while all the other extra-diagonal elements are zeros.

Example

The identity matrix of $\mathbb{R}_{3,3}$ is

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix operations: Sum and Product

Definition

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}_{m,n}$. The *matrix sum* \mathbf{C} is defined as:

$$\forall i, j : c_{i,j} = a_{i,j} + b_{i,j}.$$

Written as $\mathbf{C} = \mathbf{A} + \mathbf{B}$.

Important: Both matrices must be the same size.

Similarly, one matrix can be subtracted from another by

$$c_{i,j} = a_{i,j} - b_{i,j}.$$

Example: Matrix addition and subtraction

Example

The sum of two matrices is shown below.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 5 & 1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 4 \\ 2 & 5 & 1 \\ 3 & 0 & 2 \end{pmatrix}$$

Example

Matrix subtraction is shown below.

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 4 & -1 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 1 \\ 1 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$$

Properties of the matrix sum

The following properties can be easily proved for the sum operation amongst matrices.

- ⊙ commutativity: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- ⊙ associativity: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- ⊙ neutral element: $\mathbf{A} + \mathbf{O} = \mathbf{A}$
- ⊙ opposite element:
 $\forall \mathbf{A} \in \mathbb{R}_{m,n} : \exists ! \mathbf{B} \in \mathbb{R}_{m,n} \text{ such that } \mathbf{A} + \mathbf{B} = \mathbf{O}$

Product of a scalar and a matrix

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$ and $\lambda \in \mathbb{R}$. The *product of a scalar by a matrix* is a matrix \mathbf{C} defined as: $\forall i, j : c_{i,j} = \lambda a_{i,j}$.

Example

The product of a scalar $\lambda = 2$ by the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 4 \end{pmatrix}$$

is

$$\lambda \mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & -2 & 8 \end{pmatrix}.$$

Properties of multiplying by a scalar

The following properties can be easily proved for the product of a scalar by a matrix.

- ⊙ associativity: $\forall \mathbf{A} \in \mathbb{R}_{m,n}$ and $\forall \lambda, \mu \in \mathbb{R}$:

$$(\lambda\mu) \mathbf{A} = (\mathbf{A}\mu) \lambda = (\mathbf{A}\lambda) \mu$$

- ⊙ distributivity of the product of a scalar by the sum of two matrices: $\forall \mathbf{A}, \mathbf{B} \in \mathbb{R}_{m,n}$ and $\forall \lambda \in \mathbb{R}$:

$$\lambda (\mathbf{A} + \mathbf{B}) = \lambda \mathbf{A} + \lambda \mathbf{B}$$

- ⊙ distributivity of the product of a matrix by the sum of two scalars: $\forall \mathbf{A} \in \mathbb{R}_{m,n}$ and $\forall \lambda, \mu \in \mathbb{R}$:

$$(\lambda + \mu) \mathbf{A} = \lambda \mathbf{A} + \mu \mathbf{A}$$

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,r}$ and $\mathbf{B} \in \mathbb{R}_{r,n}$. The *product of matrices \mathbf{A} and \mathbf{B}* is a matrix $\mathbf{C} = \mathbf{AB}$ whose generic element $c_{i,j}$ is defined in the following way:

$$c_{i,j} = \mathbf{a_i b^j} = \sum_{k=1}^n a_{i,k} b_{k,j} = a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \dots + a_{i,n} b_{n,j}.$$

- ⊙ This is just the scalar product of row vectors of \mathbf{A} with column vectors of \mathbf{B} .
- ⊙ Hence the number of columns in the first matrix must be the same as the number of rows in the second.
- ⊙ The product of a $(m \times r)$ and a $(r \times n)$ matrix always yields a $(m \times n)$ matrix. That is, $\mathbf{C} \in \mathbb{R}_{m,n}$.

Example: Matrix product

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 7 & 3 & 1 \\ 5 & 0 & 4 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 8 & 0 \\ 2 & 2 \end{pmatrix}$$

Compute by taking scalar products of rows and columns:

e.g. $c_{2,1} = \mathbf{a}_2 \mathbf{b}^1 = 5 \times 1 + 0 \times 2 + 4 \times 8 + 1 \times 2 = 39$.

Repeat for all $c_{i,j}$ then

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} \mathbf{a}_1 \mathbf{b}^1 & \mathbf{a}_1 \mathbf{b}^2 \\ \mathbf{a}_2 \mathbf{b}^1 & \mathbf{a}_2 \mathbf{b}^2 \end{pmatrix} = \begin{pmatrix} 42 & 41 \\ 39 & 12 \end{pmatrix}$$

Properties of matrix product

The following properties can be easily proved for the product between two matrices.

- ⊙ left distributivity: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- ⊙ right distributivity: $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$
- ⊙ associativity: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- ⊙ transpose of the product: $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
- ⊙ neutral element: $\forall \mathbf{A} : \mathbf{AI} = \mathbf{A}$
- ⊙ absorbing element: $\forall \mathbf{A} : \mathbf{AO} = \mathbf{O}$

Commutativity is *not* a general property

- ⊙ It must be observed that commutativity with respect to the matrix product is generally not valid.
- ⊙ It may happen in some cases that $\mathbf{AB} = \mathbf{BA}$. In these cases the matrices are said *commutable* (one with respect to the other).
- ⊙ Every matrix \mathbf{A} is commutable with \mathbf{O} (and the result is always \mathbf{O}) and with \mathbf{I} (and the result is always \mathbf{A}).

Example: Non-commutativity

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

The products \mathbf{AB} and \mathbf{BA} are

$$\mathbf{AB} = \begin{pmatrix} 4 & 3 & 4 \\ 4 & 4 & 9 \\ 6 & 6 & 13 \end{pmatrix}, \quad \mathbf{BA} = \begin{pmatrix} 18 & 6 & 2 \\ 3 & 1 & 0 \\ 4 & 1 & 2 \end{pmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$ in this case.

Summary

- ⊙ Numeric vectors
- ⊙ Matrices
- ⊙ Matrix operations: sum and product

The next lecture

We will learn more about matrices: Determinants and Inversion.

THANK
YOU