Revision session Spring 2023

1. Structure of the final exam paper

Ques. No.	**TOPICS**
1	Differentiation (Definition, Rules, Implicit, Inverse)
2	Differentiation (Parametric, Logarithmic), Application of derivatives (Tangent and Normal lines, Increasing and decreasing functions, Related rates)
3	Application of derivatives (Stationary points, derivative tests, optimization problems, Newton Raphson method), Higher order derivatives
4	Maclaurin?s series, Simple integration (including simple substitutions)
5	Integration by substitution (including trigonometric substitution), results (e.g. $\int \frac{f?(x)}{f(x)}dx$, $\int e^x(f(x) + f?(x))dx$), etc., integrating algebraic fractions, t-substitution
6	Techniques of Integration (partial fractions, parts, including definite integrals), Applications of integration (Area, volume of solid of revolution), Numerical integration methods
7	Order and Degree of ODEs, Solving V-S form ODEs, IVPs, Formation of ODE, Solutions of ODE, Applications of ODE

- Each question carries 10 marks.
- Total time: 90 minutes (i.e. approximately 12 min. per question).
- Only $fx - 82$ series calculators are permitted.
- Standard (small) translation dictionary permitted.

2. **Differentiation**

(1) Given $y = \dfrac{3x^3 - 7x^2 + 1}{x^4}$, find $\dfrac{dy}{dx}$.

(2) Given $y = x \cdot e^x \cdot \sin x$, find $\dfrac{dy}{dx}$.

(3) Given $y = \dfrac{5x^2 - 10x + 9}{(x-1)^2}$; $x \neq 1$, use the quotient rule for derivatives to show that

$$\frac{dy}{dx} + \frac{8}{(x-1)^3} = 0.$$

(4) Given $y = \left(\dfrac{1}{x^2}\right)^{\sin x}$, use logarithmic differentiation to find $\dfrac{dy}{dx}$.

(5) Find the gradient of $x^2 + 2xy - 2y^2 + x = 2$ at the point $(-4, 1)$.

3. **Differentiation and applications**

(1) Given $x^3 + y^3 = 3xy^2$, use the method of implicit differentiation to find $\dfrac{dy}{dx}$.

(2) Given $y\cos\left(x^2\right) = x\sin\left(y^2\right)$, use the method of implicit differentiation to find $\dfrac{dy}{dx}$.

(3) Given $y = \cot^{-1}(x) + \cot^{-1}\left(\dfrac{1}{x}\right)$, find $\dfrac{dy}{dx}$.

(4) Given $y = \left(\tan^{-1} x\right)^2$, prove that $(1 + x^2)^2 \, \dfrac{d^2y}{dx^2} + 2x(1 + x^2) \, \dfrac{dy}{dx} = 2$.

(5) The equation of a curve is given by $x = 2\,\cos t + \sin 2t, \quad y = \cos t - 2\,\sin t \;\; ; \;\; t \in (0, \pi)$.

Find $\left.\dfrac{dy}{dx}\right|_{t=\pi/4}$.

(6) Find the gradient of the curve given by $y = x^2 + 2x$ at the point $P(-3, 3)$.

Hence find equations of the tangent and the normal lines to the curve at point P.

(7) Apply the Newton-Raphson iteration formula $x_{n+1} = \dfrac{4\,x_n^3 - 3\,x_n^2 + 7}{6\,x_n\,(x_n - 1)}$ with $x_0 = 2$

to obtain a root of the equation $2x^3 - 3x^2 - 7 = 0$ correct to 5 decimal places.

(8) (a) Given $f(x) = x^3 + x^2 - 8\,x - 15 = 0$, find the stationary points of f.

(b) Use the second derivative test to classify the stationary points as the points of maximum or minimum values of f.

(c) Draw a rough sketch of $y = f(x)$.

(9) The volume of a sphere increases at a rate of 8 cm^3/sec. Find the rate of increase of its surface area, when the radius is 4 cm.

(10) The volume of a right circular cone is given by $V = \dfrac{1}{3}\,\pi\, r^2\, h$, where r is the radius and h is the height of the cone. If the height h of the cone is increasing at a rate of 3 cm/sec, find the rate at which its volume is increasing, given that radius is 5 cm.

4. **Maclaurin's series**

(1) Obtain the Maclaurin's series expansion of the function $f(x) = x \sin x$.

(2) Obtain the Maclaurin's series expansion of the function $f(x) = \dfrac{\cos x}{x^2}$.

(3) Obtain the Maclaurin's series expansion of the function $f(x) = \ln(1 + x)\ ;\ x \in \mathbb{R},\ x > -1$.

Hence deduce that $\ln\sqrt{\dfrac{1+x}{1-x}} = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots\ldots \quad (-1 < x < 1)$.

(4) Obtain the Maclaurin's series expansion of the function $f(x) = e^x$. Hence deduce that

$$\frac{1}{2}\,\left(e^x - e^{-x}\right) \,=\, \sum_{1}^{\infty}\,\frac{x^{2k-1}}{(2k-1)\,!}.$$

5. **Integration**

(1) Evaluate $\displaystyle\int (\tan x - \cot x)^2 \, dx.$

(2) Evaluate $\displaystyle\int \frac{x^2}{36 - x^2} \, dx.$

(3) Evaluate $\displaystyle\int \frac{e^x}{\sqrt{e^{2x} - 4}} \, dx$ by using appropriate substitution.

(4) Evaluate $\displaystyle\int \frac{1}{x \cdot \Big[\,(\ln x)^2 + 81\,\Big]}\,dx$ by using appropriate substitution.

(5) Evaluate $\displaystyle\int \sin^4 x \;\cdot\; \cos^3 x \, dx$ by using appropriate substitution.

(6) Evaluate $\displaystyle\int \sin 4x \;\cdot\; \cos 3x \, dx$

(7) Evaluate $\displaystyle\int \frac{\sec x \, \tan x}{\sec x + 1}\, dx.$

(8) Evaluate $\displaystyle\int \frac{e^{2x} - 1}{e^{2x} + 1}\, dx.$

(9) Evaluate $\displaystyle\int \frac{2x + 3}{\sqrt{1 + 3x + x^2}}\, dx.$

(10) Evaluate $\displaystyle\int e^x \cdot \left(1 - x^2 - 2x\right)\, dx.$

(11) Evaluate $\displaystyle\int e^x \cdot \left(5 - \sec^2 x - \tan x\right)\, dx.$

6. **Integration, Definite Integration, Area and Volume calculation**

(1) Evaluate $\displaystyle\int \frac{2x+5}{(x-2)(x+1)}\,dx$ by using the method of partial fractions.

(2) Evaluate $\displaystyle\int \frac{1}{(3x-1)(x^2+1)}\,dx$ by using the method of partial fractions.

(3) Evaluate $\displaystyle\int \frac{\ln x}{x^2}\, dx$ by using the method of integration by parts.

(4) Evaluate $\displaystyle\int x\ \sec^2 x\, dx$ by using the method of integration by parts.

(5) Evaluate $\displaystyle\int_1^4 f(x)\,dx$ where $f(x) = \begin{cases} 2x+8 & ; \quad 1 \le x \le 2 \\ 6x & ; \quad 2 < x \le 4 \end{cases}$

(6) Use the property of definite integrals to evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx.$

(7) Evaluate the definite integral $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{\tan^2 x + 49}\,dx$ by using appropriate substitution.

(8) Evaluate the definite integral $\int\limits_0^1 \frac{1}{e^x + e^{-x}}\,dx$ by using appropriate substitution.

(9) Evaluate the integral $\int \frac{1}{2\ \cos x + 3}\,dx$ by using the method of t-substitution.

(10) Find the area of the region bounded by the curve $y = 4 - x^2$, lines $x = 0$, $x = 2$ and the X-axis.

(11) Find the volume of the solid obtained when the region bounded by $xy = 4$ and $x + y = 5$ is rotated around the X-axis.

7. **Differentiation equations and their applications**

(1) Solve the ODE: $\dfrac{dy}{dx} = x\, e^y$.

(2) Solve the ODE: $2y\,(e^x + 2019)\; dy = \left(y^2 + 1\right)\; e^x\; dx$.

(3) Solve the ODE: $\dfrac{dy}{dx} + xy = y\; \sec^2 x$.

(4) Solve the IVP: $\dfrac{dy}{dx} - x\,e^{y} = 5\,e^{y}$; $y(0) = 0$.

(6) Show that $y = \left(\sin^{-1} x\right)^2$ is a solution of the differential equation

$$(1 - x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} = 2.$$

(6) Show that $y = e^{-x} + ax + b$ (a, b are arbitrary constants) is a solution of the differential equation $e^x \dfrac{d^2y}{dx^2} - 1 = 0.$

(7) The rate of decay of a radioactive material is proportional to the amount (m) of material present at that time. Formulate a differential equation model to show that the amount of material at time t is $m(t) = m_0 \cdot e^{kt}$, where $k < 0$ is constant and m_0 is the initial amount of the radioactive material.