Seminar 10

In this seminar you will study:

- Solutions of Ordinary Differential Equations (ODE)
- Solving ODEs of Variable-Separable Form
- Solving Initial Value Problems (IVP) of Variable-Separable Form
- Applications of Differential Equations

Solutions of Differential Equations

Definition:

A function $f(x)$ is called a solution of a differential equation if the differential equation is satisfied when $y = f(x)$ and its derivatives are substituted into the given differential equation.

Example 1: Show that $y = C_1 \sin 4x + C_2 \cos 4x$ is a solution of the ODE

$$\frac{d^2y}{dx^2} + 16y = 0.$$

Solution:

$$y = C_1 \sin 4x + C_2 \cos 4x$$

$$\Rightarrow \quad \frac{dy}{dx} = 4C_1 \cos 4x - 4C_2 \sin 4x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -16C_1 \sin 4x - 16C_2 \cos 4x = -16y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 16y = 0$$

Solutions of Differential Equations

Example 2: Show that $y = e^{-x} + ax + b$ is a solution of the ODE

$$e^x \cdot \frac{d^2y}{dx^2} - 1 = 0.$$

Find the following volumes of revolution:

1. Show that $y = C_1 e^{2x} + C_2 e^{3x}$ is a solution of the ODE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$, where C_1 and C_2 are arbitrary constants.

2. Show that $y = C_1 e^{-2x} + C_2 e^{x}$ is a solution of the ODE $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, where C_1 and C_2 are arbitrary constants.

3. Show that $y = a \cos^{-1} x + b$ is a solution of the ODE $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$, where a and b are arbitrary constants.

4. Show that $y = \frac{a}{x} + b$ is a solution of the ODE $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$, where a and b are arbitrary constants.

Solving ODEs of Variable-Separable Form

The variable-separable differential equation can be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \qquad \text{i.e.} \qquad g(y)\,dy = f(x)\,dx$$

Integrating both sides:

$$\int g(y)\,dy = \int f(x)\,dx$$

$$\Rightarrow \qquad G(y) = F(x) + C$$

where $G(y)$ and $F(x)$ denote the antiderivatives of $g(y)$ and $f(x)$, respectively.

Solving ODEs of Variable-Separable Form

Example 1: Solve the variable separable ODE: $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

Solution:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \quad y\,dy = -x\,dx$$

$$\Rightarrow \quad \int y\,dy = -\int x\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C_0$$

$$\Rightarrow \quad x^2 + y^2 = C \qquad \boxed{\text{general solution of the given ODE}}$$

Solving ODEs of Variable-Separable Form

Example 2: Solve the variable-separable ODE: $\ln(\sin x)\,\dfrac{dy}{dx} = \cot x$.

Answer:

$$y = \ln|\ln(\sin x)| + C \qquad \boxed{\text{general solution of the given ODE}}$$

Solving ODEs of Variable-Separable Form

Example 3: Solve the variable-separable ODE: $\dfrac{dy}{dx} = \dfrac{xe^x}{y\sqrt{1+y^2}}$.

Answer:

$$(1+y^2)^{\frac{3}{2}} = 3xe^x - 3e^x + C \qquad \boxed{\text{general solution of the given ODE}}$$

Solve the following Variable-Separable ODEs:

1. $\dfrac{dy}{dx} = \dfrac{y}{x}$ **Answer:** $\ln\lvert y\rvert = \ln\lvert x\rvert + C$	2. $\dfrac{dy}{dx} = \dfrac{1+x^2}{1+y^2}$ **Answer:** $y + \dfrac{y^3}{3} = x + \dfrac{x^3}{3} + C$
3. $\dfrac{dy}{dx} = x^2(1+y^2)$ **Answer:** $\tan^{-1} y = \dfrac{x^3}{3} + C$	4. $y\dfrac{dy}{dx} = (1+y^2)\tan x$ **Answer:** $\ln(1+y^2) = -2\ln\lvert\cos x\rvert + C$

Solving the following Variable-Separable ODEs:

1. $\dfrac{dy}{dx} = \dfrac{\tan y}{x\sec^2 y}$ **Answer:** $\ln\lvert\tan y\rvert = \ln\lvert x\rvert + C$	2. $(e^x + e^{-x})\dfrac{dy}{dx} = e^x - e^{-x}$ **Answer:** $y = \ln\left(e^x + e^{-x}\right) + C$
3. $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+\sin x}$ **Answer:** $\ln\lvert y\rvert = \ln(1+\sin x) + C$	4. $y\ln y dx = xdy$ **Answer:** $\ln\lvert\ln y\rvert = \ln\lvert x\rvert + C$

Solving Initial Value problem (IVP) of Variable-Separable ODE

Example 1: Solve the IVP of the variable-separable ODE:

$$\frac{dy}{dx} = \frac{x^2}{y^2}; \quad y(0) = 2$$

Solution:

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

$$\Rightarrow \quad y^2\,dy = x^2\,dx$$

$$\Rightarrow \int y^2\,dy = \int x^2\,dx$$

$$\Rightarrow \quad \frac{y^3}{3} = \frac{x^3}{3} + C$$ general solution of the given ODE

Now, $x = 0, \ \Rightarrow \ y = 2$ intial value

Solving Initial Value problem (IVP) of Variable-Separable ODE

Solution:

$$\Rightarrow \quad \frac{2^3}{3} = \frac{0^3}{3} + C$$

$$\Rightarrow \quad C = \frac{8}{3}$$

$$\Rightarrow \quad y^3 = x^3 + 8 \quad \left(\text{or } y = \sqrt[3]{x^3+8}\right)$$ particular solution of the given ODE

Solving Initial Value problem (IVP) of Variable-Separable ODE

Example 2: Solve the IVP of the variable separable ODE:

$$e^{\frac{dy}{dx}} = x + 1 \quad (x > -1); \quad y(0) = 3.$$

Answer:

$$y = (x+1)\ln|x+1| - x + 3$$ particular solution of the given ODE

Solve the following Variable-Separable ODE with the specified initial value:

1. $\frac{dy}{dx} + 4xy^2 = 0; \; y(0) = 1$

Answer: $y = \frac{1}{2x^2+1}$

2. $\frac{dy}{dx} = \frac{y \sin x}{1+y^2}; \; y(0) = 1$

Answer: $2\ln y + y^2 + 2\cos x = 3$

3. $\frac{dy}{dx} = y \tan x; \; y(0) = 1$

Answer: $y = \sec x$

4. $x + 2y\sqrt{x^2+1}\frac{dy}{dx} = 0; \; y(0) = 1$

Answer: $y = \sqrt{2 - \sqrt{x^2+1}}$

Applications of Ordinary Differential Equation (ODE)

Example: The rate of population increase of insects used in an experiment is proportional to the insect population (P).

(i) Formulate a differential equation to show that the population of the insect at time t is $P = P_0 \cdot e^{kt}$, where $k > 0$ is constant, and P_0 is the initial population.

(ii) If the population increased from 1000 to 1300 after 20 days, find the population after 35 days?

(iii) How long will it take for the population to reach 2000.

Solution:

(i) $$\frac{dP}{dt} \propto P$$

$$\Rightarrow \frac{dP}{dt} = kP \quad (k > 0)$$ The ODE is variable-separable

$$\Rightarrow \frac{dP}{dt} = k\,dt$$

Applications of Ordinary Differential Equation (ODE)

Solution:

$$\Rightarrow \int \frac{dP}{dt} = k \int dt$$

$$\Rightarrow \ln P = kt + C$$ general solution of the ODE

Now, $t = 0, \Rightarrow P = P_0$ intial value

$$\Rightarrow \ln P_0 = k(0) + C$$

$$\Rightarrow C = \ln P_0$$

$$\Rightarrow \ln P = kt + \ln P_0$$ particular solution of the ODE

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = kt$$

$$\therefore P = P_0 e^{kt}$$

Applications of Ordinary Differential Equation (ODE)

Solution:

(ii) $P_0 = 1000$

$$\Rightarrow \quad P = 1000\, e^{kt}$$

t	20	35
P	1300	?

$$\Rightarrow \quad 1300 = 1000\, e^{k\,(20)}$$

$$\Rightarrow \quad k = \frac{1}{20}\ln\left(\frac{13}{10}\right)$$

$$\Rightarrow \quad P = 1000\, e^{\left[\frac{1}{20}\ln\left(\frac{13}{10}\right)\right]t}$$

$$t = 35$$

$$\Rightarrow \quad P = 1000\, e^{\left[\frac{1}{20}\ln\left(\frac{13}{10}\right)\right](35)}$$

$$\Rightarrow \quad P \approx 1583 \text{ insects}$$

Applications of Ordinary Differential Equation (ODE)

Solution:

(iii)

t	20	35	?
P	1300	1583	2000

$$\text{Now,} \quad P = 1000\, e^{\left(\frac{1}{20}\ln\left[\frac{13}{10}\right]\right)t}$$

$$\Rightarrow \quad 2000 = 1000\, e^{\left(\frac{1}{20}\ln\left[\frac{13}{10}\right]\right)t}$$

$$\Rightarrow \quad \ln\left(\frac{2000}{1000}\right) = \left(\frac{1}{20}\ln\left[\frac{13}{10}\right]\right)t$$

$$\Rightarrow \quad t = \frac{\ln(2)}{\ln(13/10)}\cdot 20$$

$$\Rightarrow \quad t \approx 52.84 \text{ days}$$

Applications of ODE:

1. The population of a city increases at the rate of 2% per year.
   How many years will it take for the population to double.

   Answer: $\approx$ 34.657 years

All the best in the Examinations