AE1MCS: Mathematics for Computer Scientists

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 7, Section 7.2 Probability Theory
- Chapter 7, Section 7.3 Bayes' Theorem

Probability Distribution

Let s be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability $p(s)$ to each outcome. We require that two conditions be met:

1. $0 \leq p(s) \leq 1$ for each $s \in S$
2. $\sum_{s \in S} p(s) = 1$.

The function p from the set of all outcomes of the sample space S is called a **probability distribution.**

Conditional Probability

Given an event F occurs, the probability that event E occurs is the **conditional probability** of E given F.

Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F, denoted by $p(E|F)$, is defined as:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

.

$P(E \cap F) = P(E|F) \cdot P(F)$

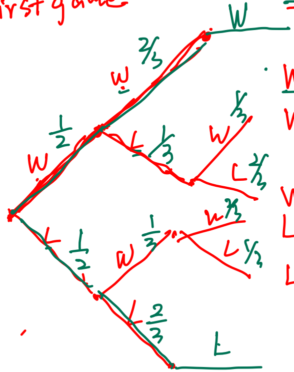
$= P(F|E) \cdot P(E)$

Conditional Probability

A = Prob of winning
B = Prob of the 1st win.

In a best 2 out of 3 serious, the probability of wining first game is 1/2, the probability of winning a game following a win is 2/3, the probability of winning after a loss is 1/3. What is the probability of winning the series given a first win?

First game



		A	B	A∩B
WWW	2/9	✓	✓	✓
WWL	1/9	✓	✓	✓
WLW	1/18	✓	✓	✓
WLL	1/9		✓	
LWW	1/9	✓		
LWL	1/18			
LLW	1/9			
LLL	2/9			

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{7/18}{9/18} = \frac{7}{9}$$

$$P(A) = \frac{2}{9} + \frac{1}{9} + \frac{1}{18} + \frac{1}{9} = \frac{9}{18}.$$

$$P(B) = \frac{9}{18}$$

$$P(A \cap B) = \frac{2}{9} + \frac{1}{9} + \frac{1}{18} = \frac{7}{18}$$

Conditional Probability

2^4

Uniform sample space

$|S| = 2^4$

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

$$P(E|E) = \frac{P(E \cap F)}{P(F)} = \frac{5/16}{8/16} = 5/8.$$

F = the event that the 1st bit is a 0.

E = ... it contains at least 2 consecutive 0s

$E \cap F = \{0010,\ 0000,\ 0001,\ 0100,\ 0011\}$

$P(E \cap F) = \frac{5}{16}$

$|F| = 2^3$

$P(F) = \frac{|F|}{|S|} = \frac{8}{16}$

Independence

E and F are independent iff.

$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)}$

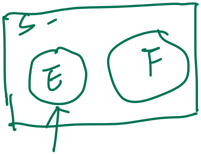
$= P(E)$

When two events are independent, the occurrence of one of the events gives no information about the probability of that the other event occurs.

The events E and F are independent **if and only if**

$$p(E \cap F) = p(E)p(F)$$

.



E and F are disjoint.

$P(E\cap F) = 0$ $\quad P(E|F) = \frac{P(E\cap F)}{P(F)} = 0$

$P(E|F) = P(E) \neq 0$ independent

Independence

Roll two fair independent coins, let A be the event that both coins match, let B be the event that the first coin is Head. Are A and B independent? In what circumstances, these two events are independent?

$P(H) = p \quad P(T) = 1-p$

$P(A|B) = P(\text{both coins match} \mid \text{the 1st coin is Head})$

$= P(\text{2nd coin is Head}) = p$

$P(A) = P(HH) + P(TT) = p \cdot p + (1-p)(1-p) = p^2 + (1-p)^2$

$P(A|B) = P(A)$

$p = p^2 + (1-p)^2$

$p = 1$ or $\frac{1}{2}$.

Independence

D_1 = 1st Die. D_2 = 2nd die

Rolling two fair regular dice (six sided with values 1, 2,. . . , 6 appearing equally likely), let D_1 and D_2 be the face that comes up for the first and second die respectively. Let S be the sum of the two dice.

	D_1	D_2.	S.
$\frac{1}{6}$	1	1	2.
$\frac{1}{6}$	2	2	3
$\frac{1}{6}$	3	3	4
$\frac{1}{6}$	4	4	5
$\frac{1}{6}$	5	5	6
$\frac{1}{6}$	6	6.	⋮
			12.

$P(S=2) = P(D_1=1, D_2=1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

S V.S. D_1.

S V.S. D_2

Independence

$0, 2, 4.$ $\quad 1\ \underbrace{\_\ \_\ \_}_{2^3}$

Suppose E is the event that a randomly generated bit string of length four begin with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

$F = \{0000, 0011, 0110, 1100, 0101, 1001, 1010, 1111\}$

$|F| = 8.$ $\quad E \cap F = \{$ $\quad 3$

$|S| = 16.$

$|E| = 2^3 = 8.$

$P(E \cap F) = 4/16$

$P(E) = P(F) = 8/16 = 1/2.$

$P(E \cap F) = P(E)P(F) = \frac{1}{4}$ independent.

Bernoulli Trials and Binomial Distribution

Random variable, R is a function $R: S \to \mathbb{R}$

Random Variable Sample space Real #

Each performance of an experiment with two possible outcomes is called a **Bernoulli trial.**

In general, a possible outcome of a Bernoulli trial is called a **success** or a **failure**.

- Generate a bit, $\{0, 1\}$. $\Rightarrow \mathbb{R}$.
- Flip a coin, $\{$Heads, tails$\}$. $\longrightarrow$ $R(\text{Head}) = \frac{1}{2}$, $R(\text{Tail}) = \frac{1}{2}$

If p is the probability of a success and q is the probability of a failure, it follows that $p + q = 1$.

$R(0) = \frac{1}{2}$.

$R(1) = \frac{1}{2}$

Binomial Distribution

$n, k.$

- The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is $C(n, k)p^k q^{n-k}$.
- We denote by $b(k; n, p)$ the probability of k successes in n independent Bernoulli trials with probability of success p and probability of failure $q = 1 - p$.
- Considered as a function of k, we call this function the **binomial distribution.**

$$b(k; n, p) = C(n, k)p^k q^{n-k}.$$

$(1-p)$

Binomial Distribution

Example

$n=7, \quad k=4, \quad p=\frac{2}{3}$

A coin is biased so that the probability of heads is 2/3. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

$$b(n,k,p) = C(n,k)\cdot p^{k}(1-p)^{n-k}$$

$$= C(7,4)\cdot\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{3}.$$

Example

An airline on average assumes that just 95% of all ticket purchasers actually show up for a flight. If the airline sells 105 tickets for a 100 seat flight, what is the probability that a flight is overbooked?

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