AE1MCS: Mathematics for Computer Scientists

Heshan Du and Huan Jin
University of Nottingham Ningbo China

Oct 2020

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 7, Section 7.1 An Introduction to Discrete Probability

Discrete Probability

- Combinatorics and probability theory share common origins (analyzing gambling games).
- The theory of probability now plays an essential role in a wide variety of disciplines (e.g. the study of genetics).
- In computer science,
  - Probability theory plays an important role in the study of the complexity of algorithms.
  - Probabilistic algorithms vs. deterministic algorithms.
  - Probability theory can help us answer questions that involve uncertainty.
  - ...

Content

- Probability of an Event
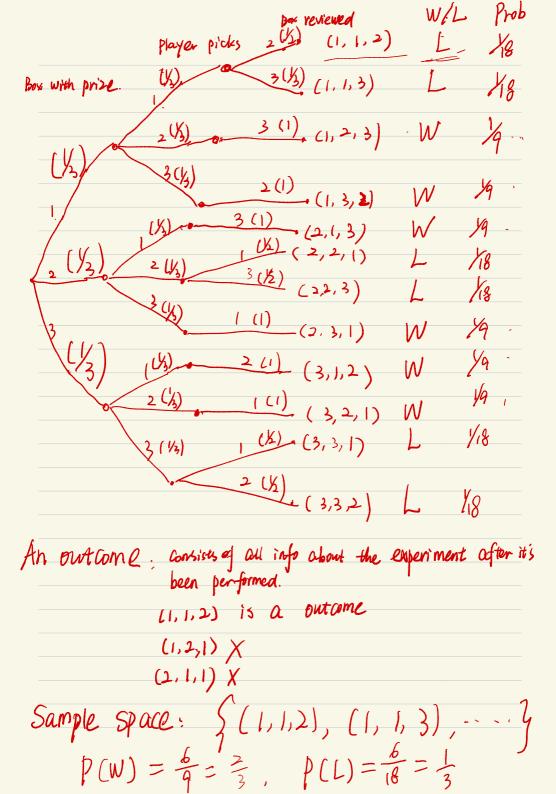- Probabilities of Complements and Unions of Events

Monty Hall Three-Door Puzzle

You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is be behind each door:

- whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors).
- Then he asks you whether you would like to switch doors.

The
Monty Hall Problem

	1st chose.	Review	Switch?	W/L
A	3	2	Y	W
Rony	2	3	N	W
Patrick	3	2	N	L

Box with prize / **Player picks** / **Box reviewed** / **W/L** / **Prob**

Tree diagram:

- $(1/3)$ 1:
  - $2 (1/2)$ → $(1,1,2)$ — L — $1/18$
  - $3 (1/2)$ → $(1,1,3)$ — L — $1/18$
  - $2 (1/3)$:
    - $3 (1)$ → $(1,2,3)$ — W — $1/9$
  - $3 (1/3)$:
    - $2 (1)$ → $(1,3,2)$ — W — $1/9$

- $(1/3)$ 2:
  - $1 (1/3)$:
    - $3 (1)$ → $(2,1,3)$ — W — $1/9$
  - $2 (1/3)$:
    - $1 (1/2)$ → $(2,2,1)$ — L — $1/18$
    - $3 (1/2)$ → $(2,2,3)$ — L — $1/18$
  - $3 (1/3)$:
    - $1 (1)$ → $(2,3,1)$ — W — $1/9$

- $(1/3)$ 3:
  - $1 (1/3)$:
    - $2 (1)$ → $(3,1,2)$ — W — $1/9$
  - $2 (1/3)$:
    - $1 (1)$ → $(3,2,1)$ — W — $1/9$
  - $3 (1/3)$:
    - $1 (1/2)$ → $(3,3,1)$ — L — $1/18$
    - $2 (1/2)$ → $(3,3,2)$ — L — $1/18$

An outcome: consists of all info about the experiment after it's been performed.

$(1,1,2)$ is a outcome

$(1,2,1)$ ✗

$(2,1,1)$ ✗

Sample space: $\{(1,1,2), (1,1,3), \cdots \}$

$$P(W) = \frac{6}{9} = \frac{2}{3}, \qquad P(L) = \frac{6}{18} = \frac{1}{3}$$

Finite Probability

Laplace's definition of the probability of an event with **finitely many, equally likely, possible outcomes** is as follows.

Definition

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is

$$p(E) = \frac{|E|}{|S|}$$

- An **experiment** is a procedure that yields one of a given set of possible outcomes.
- The **sample space** of the experiment is the set of possible outcomes.
- An **event** is a subset of the sample space.

Probability of an Event

In the eighteenth century, the French mathematician Laplace, who also studied gambling, defined **the probability of an event as the number of successful outcomes divided by the number of possible outcomes**.

A box contains 4 blue balls and 5 red balls. What is the probability that a ball chosen at random from the box is blue?

$$\frac{4}{9}$$

Example

A box contains 4 blue balls and 5 red balls. What is the probability that a ball chosen at random from the box is blue?

$$S = \{b1, b2, b3, b4, r1, r2, r3, r4, r5\}$$

$$E = \{b1, b2, b3, b4\}$$

so

$$p(E) = \frac{|E|}{|S|} = \frac{4}{9}$$

Example: Lottery

In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched.

- What is the probability that a player wins the large prize?
- What is the probability that a player wins the small prize?

Example: Lottery (Answer)

$$S = \{x \in \mathbb{Z} : 0 \le x \le 9999\} \Rightarrow |S| = 10^4$$

- What is the probability that a player wins the large prize?
  $E_1 = \{p\}$ where p is the lottery number picked by player,
  so $|E_1| = 1$ and $p(E) = 1/10^4$.
- What is the probability that a player wins the small prize?

$$E_2 = \{x : x \text{ is like } p \text{ but with just one digit change}\}$$

hence $|E_2| = C(4, 1) \times 9 = 4 \times 9$, so $P(E_2) = C(4, 1) \times 9/10^4$.
That is, 9 ways to change one digit over four digits.

Example: Poker 1

Find the probability that a hand of five cards in poker contains four cards of one kind.

- A deck of cards contains 52 cards.
- There are 13 different kinds of cards, with four cards of each kind.
- These kinds are twos, threes, fours, fives, sixes, sevens, eights, nines, tens, jacks, queens, kings, and aces.
- There are 4 suits: spades, clubs, hearts, and diamonds, each containing 13 cards.

Example: Poker 1 (Answer)

Find the probability that a hand of five cards in poker contains four cards of one kind.

$$|S| = C(52, 5)$$

- S is number of ways to choose any 5 cards from 52: $|S| = C(52, 5)$.

13   52-4

- E is the number of ways to get four of a kind: $|E| = 13 \times (52 - 4)$, using product rule with
  - Choose one of the 13 kinds which is repeated 4 times;
  - Choose any remaining card for last card (52-4).
- Hence $p(E) = \dfrac{13 \times (52 - 4)}{C(52, 5)} \approx 0.00024$.

What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

Same sample space: $|S| = C(52, 5)$.

$$
\begin{aligned}
|E| \quad = \quad & \text{\# ways to get two different kinds} & C(13, 2) \\
\times \quad & \text{\# ways to select which is three cards} & \\
& \text{of the same kind} & 2 \\
\times \quad & \text{\# ways to choose three cards of the same kind} & C(4, 3) \\
\times \quad & \text{\# ways to choose two cards of the same kind} & C(4, 2) \\
= \quad & C(13, 2) \times 2 \times C(4, 3) \times C(4, 2) \; .
\end{aligned}
$$

Notice: $C(13, 2) \times 2 = P(13, 2) = C(13, 1)C(12, 1)$, so all correct.

$P = \dfrac{|E|}{|S|}$

Probabilities of Complements and Unions of Events

Theorem

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E)$$

Theorem

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

How to prove them?

Example

$$|S| = 2^{10}$$

- A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?
- What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

$$S = \{1, 2, 3, \cdots, 100\}$$

$$E_1 = \{2, 4, 6, 8, 10, 12, \cdots 100\} \qquad |E_1| = 50$$

$$E_2 = \{5, 10, 15, 20, \cdots \cdots 100\}, \quad |E_2| = 20$$

$$\{10, 20, 30, \cdots, 100\}$$

Example 1 (Answer)

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

- Sample space is all bit strings of length 10: $|S| = 2^{10}$.
- Think about the event when bit string has *no* 0's. Then $E = \{1111111111\}$ and $|E| = 1$. $\qquad p(E) = \frac{1}{2^{10}} = 2^{-10}$
- Now the event we are interested in (at least one 0) is complement of E, so

$$p(\overline{E}) = 1 - p(E) = 1 - 2^{-10}.$$

$$\frac{1}{2^{10}} = 2^{-10}$$

Example 2 (Answer)

What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by *either* 2 or 5?

- $S = \{x \in \mathbb{Z}^+ : x \leq 100\}$, so $|S| = 100$;
- $E_1 = \{x \in S : x \text{ is divisible by } 2\}$, so $|E_1| = 50$;
- $E_2 = \{x \in S : x \text{ is divisible by } 5\}$, so $|E_2| = 20$;
- then $E_1 \cap E_2 = \{x \in S : x \text{ is divisible by } 10\}$, so $|E_1 \cap E_2| = 10$,

so

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = 0.6$$

Monty Hall Three-Door Puzzle

You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is be behind each door:

- whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors).
- Then he asks you whether you would like to switch doors.

Monty Hall Three-Door Puzzle

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 7, Section 7.1 An Introduction to Discrete Probability