The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN/SPRING SEMESTER 2021-2022

Mathematics for Computer Scientists

Time allowed: 2 Hours and 0 Minutes

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced

Answer all FOUR questions. All questions are worth 25 marks each, hence the total mark is 100.

No calculators are permitted in this examination.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

ADDITIONAL MATERIAL: None.
INFORMATION FOR INVIGILATORS: None.

Question 1: This question is about predicate logic, sets, functions, and relations.

[**overall 25 marks**]

a. Translate the following two sentences from English to predicate logic. Suppose the domain that you are working over is X, the set of people. Let $F(x, y)$ denote 'x and y are friends'. Then use De Morgan's laws to express their negation such that no negation operator precedes a quantifier.

 (i) Everyone has a friend. (4 marks)

 (ii) Someone has exactly one friend. (4 marks)

b. Suppose that g is a function from A to B and f is a function from B to C. Show that if both f and g are one-to-one functions, then the composition of the functions f and g, denoted by $f \circ g$, is one-to-one. (6 marks)

c. Consider the following relations on $\{1, 3, 5\}$:

- $R_1 = \{(1,1), (1,3), (1,5)\}$
- $R_2 = \{(1,3), (3,1), (3,3)\}$
- $R_3 = \{(1,5), (5,1), (5,5)\}$
- $R_1 \cup R_3$
- $R_1 \cup R_2 \cup R_3$
- R_1^3
- R_2^2

 (i) List all members of each of the relations $R_1 \cup R_3$, $R_1 \cup R_2 \cup R_3$, R_1^3, and R_2^2. (4 marks)

 (ii) Which of the relations above are reflexive? Which of them are symmetric? Which of them are transitive? (7 marks)

Question 2: This question is about counting, probability, and proof methods.

[**overall 25 marks**]

a. You repeatedly roll two fair dice and look at the sum.

(i) What is the probability that you will roll a sum of 4 and a sum of 7 within the first 4 rolls? Express your answer as a real number. (8 marks)

(ii) What is the expected number of rolls until you get a sum of 4 or a sum of 7? (For example, if you get 7 on the first roll, the number of rolls is 1.) (4 marks)

b. Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$$h_0 = 1, h_1 = 2, h_2 = 3,$$

$$h_k = h_{k-1} + h_{k-2} + h_{k-3} \text{ for all integers } k \geq 3.$$

(i) Prove that $h_n \leq 3^n$ for all integers $n \geq 0$. (6 marks)

(ii) Suppose that s is any real number such that $s^3 \geq s^2+s+1$. (This implies that $2 > s > 1.83$.) Prove that $h_n \leq s^n$ for all $n \geq 2$. (7 marks)

The table below lists powers of 1.83 for your reference.

1.83^2	1.83^3	1.83^4	1.83^5	1.83^6
3.3489	6.1285	11.215	20.524	37.558

Question 3: This question is about vector spaces.

[**overall 25 marks**]

Consider the following sets:

- Let $U = \{(x, y) \in \mathbb{R}^2 : 3x + y = 0\}$
- Let $V = \{(x, y, z) \in \mathbb{R}^3 : 3x + y - z = 1\}$
- Let $W = \{(x, y, z) \in \mathbb{R}^3 : 3x + y = 0,\ z = 0\}$

where $+$ is the usual arithmetic addition and the scalar is the usual arithmetic scalar.

a. Show whether each of U, V and W are vector spaces or not. (5 marks)

b. Show which of U, V and W is a vector subspace of $\{(x, y, z) \in \mathbb{R}^3 : 3x + y = 0\}$. (2 marks)

c. Let $(E, +, \cdot)$ be a vector space with vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in E$.
Prove the following theorem:

The span $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ with the composition laws is a vector subspace of $(E, +, \cdot)$.

(5 marks)

d. Show that the linear span $L((1, -3, 0)) = W$. (2 marks)

e. Consider these three sets of vectors:

(i) $(1, 0, -3),\ (0, 1, -2)$
(ii) $(1, 0, -3),\ (0, 1, 2)$
(iii) $(1, 0, -3),\ (0, 1, 2),\ (2, 2, -2)$

Which of these is a basis for the vector space $\{(x, y, z) \in \mathbb{R}^3 : 3x - 2y - z = 0\}$ and which are not? Explain your reasoning in each case. (6 marks)

f. Let $(E, +, \cdot)$ be a vector space with $E \subset \mathbb{R}^4$ and

$$\{(1, 0, -1, -1), (0, -2, 1, 1), (-2, 2, -2, 0), (3, -6, 3, 1)\} \subset E.$$

Show that 3 or more vectors are required to span E. (5 marks)

Hint: You may use Steinitz's Lemma in your answer.

Question 4: This question is about linear mappings, eigenvalues, eigenvectors and eigenspaces.

[**overall 25 marks**]

Consider these four mappings:

- $f_1 : \mathbb{R}^2 \to \mathbb{R}^3, f_1(x, y) = (x^2, x + y, y^2)$
- $f_2 : \mathbb{R} \to \mathbb{R}^2, f_2(x) = (x, -2x)$
- $f_3 : \mathbb{R}^3 \to \mathbb{R}^3, f_3(x, y, z) = (x + y + z, 2y, z - x)$
- $f_4 : \mathbb{R}^2 \to \mathbb{R}, f_4(x, y) = xy$

a. Which are linear mappings? (2 marks)

b. Which are endomorphisms? (2 marks)

c. Give the formal definition of eigenvectors and eigenvalues. (3 marks)

Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = 4x$ and $g : \mathbb{R}^3 \to \mathbb{R}^3,\ g(x, y, z) = (2x + 2y + z, 5y, 3x + 6y + 4z)$.

d. Compute the eigenvalue for f. (2 marks)

e. Compute the eigenvectors for f. (2 marks)

f. Compute the eigenvalues for g. (4 marks)

g. Compute the eigenvectors for g. (4 marks)

h. Give the formal definition of eigenspace. (3 marks)

i. Compute the eigenspace for f. (1 mark)

j. Compute the eigenspaces for g. (2 marks)