AE1MCS: Mathematics for Computer Scientists

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 2, Section 2.3. Functions
- Chapter 2, Section 2.4. Sequences and Summations

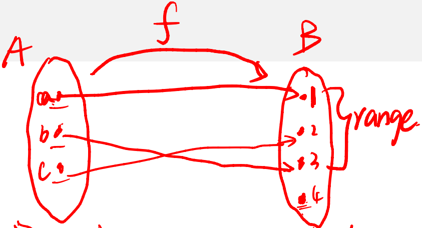
Functions

$y = f(x)$

- The concept of a function is extremely important in discrete mathematics.
- A function assigns to each element of a first set exactly one element of a second set, where the two sets are not necessarily distinct.
- Functions play important roles throughout discrete mathematics.
- They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects, and in a myriad of other ways.
- Useful structures such as sequences and strings are special types of functions.

$S = \{a, b\}$

$f(S) = \{1, 3\}$



a is preimage of 1.

1 is image of a

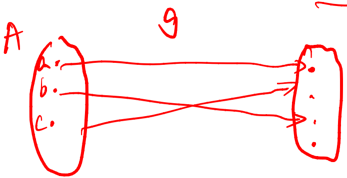
Domain

codomain

Definition

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f: A \to B$.

Remark: Functions are sometimes also called *mappings* or *transformations*.



Domain and Range

Definition

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.

Equal Functions

Two functions are **equal** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.

Image of a Set

Definition

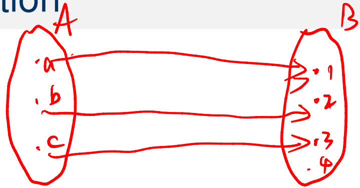
Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Remark: The notation $f(S)$ for the image of the set S under the function f is potentially ambiguous. Here, $f(S)$ denotes a set, and not the value of the function f for the set S.

One-to-One Function



Definition

A function f is said to be *one-to-one*, or an *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. This is

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$$

or equivalently

$$\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$$

A function is said to be *injective* if it is one-to-one.

Prove or Disprove a Function is Injective

Suppose that $f : A \to B$.

To show that f is injective: Show that for arbitrary $x, y \in A$, if $f(x) = f(y)$, then $x = y$.

To show that f is not injective: Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

Onto Functions

Definition

A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. This is,

$$\forall b \in B \, \exists a \in A \, (f(a) = b)$$

A function f is called *surjective* if it is onto.

Prove or Disprove a Function is Surjective

Suppose that $f : A \to B$.

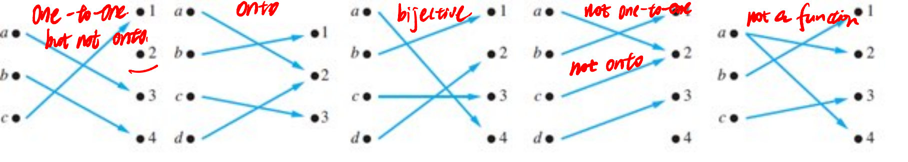
To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that for all $x \in A$, $f(x) \neq y$.

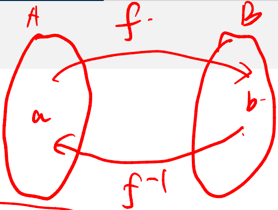
One-to-one Correspondence

Definition

The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.


One-to-one
but not onto
onto
bijective
not one-to-one
not onto
not a function
a
b
c
d
1
2
3
4

Inverse Functions



Definition

Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

B A

Remark: Be sure not to confuse the function f^{-1} with the function $1/f$, which is the function that assigns to each x in the domain the value $1/f(x)$. Notice that the latter makes sense only when $f(x)$ is a non-zero real number.

$A \to B$

$\frac{1}{f(a)} = \frac{1}{b}$

Invertible Functions

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Compositions of Functions

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

a

Definition

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

Note that the *composition* $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

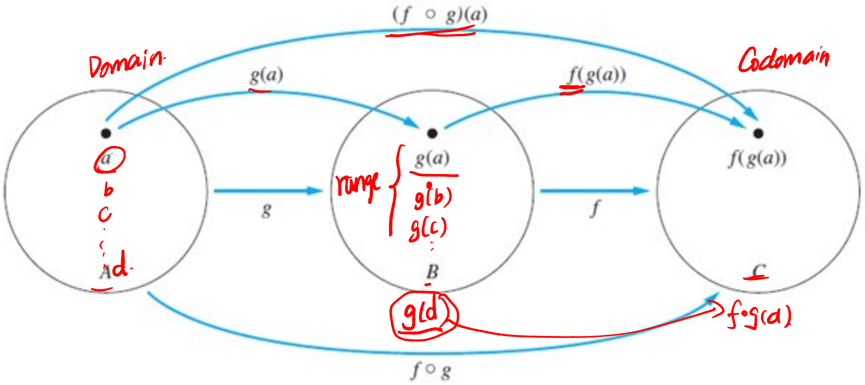
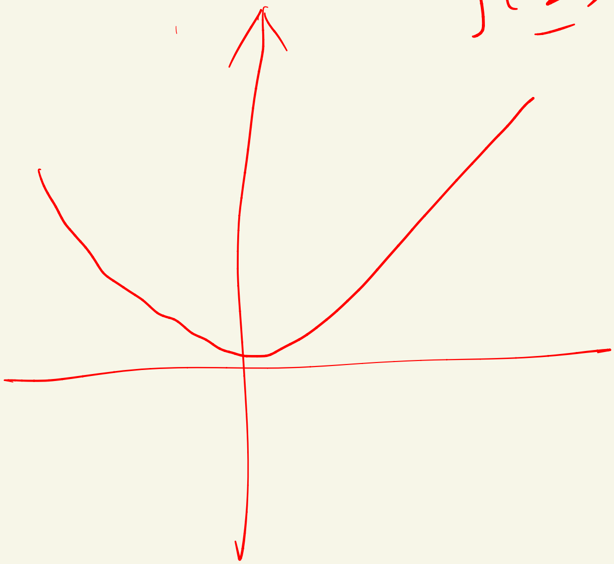


FIGURE 7 The Composition of the Functions f and g.

$f(x) = x^2$; $\quad \mathbb{R} \to \mathbb{R}$

$f(-2) = f(2)$



Identity Function

$f: A \to A$

$f(a) = a$

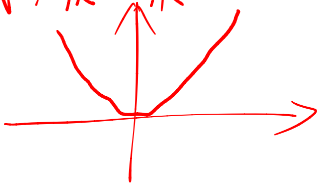
Suppose that f is a one-to-one correspondence from the set A to the set B. $f(a) = b$.

$$\begin{cases} (f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a & A \to A \\ (f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b & B \to B \end{cases}$$

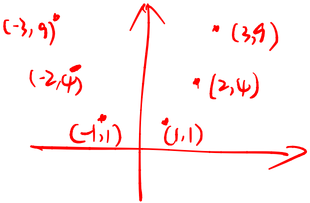
$f^{-1} \circ f$ and $f \circ f^{-1}$ are the identity functions on the sets A and B respectively. $(f^{-1})^{-1} = f$.

The Graphs of Functions

$f(x) = x^2$, $\mathbb{R} \to \mathbb{R}$



$f(x) = x^2$ $\mathbb{Z} \to \mathbb{Z}$.



Definition

Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \wedge f(a) = b\}$.

Example

Consider f(n)=2n+3, is it bijective from $\mathbb{Z}$ to $\mathbb{Z}$?

$\mathbb{R}$ $\mathbb{R}$.

One-to-one

onto :

Example

For each of the following functions, is it invertible? If yes, what is its inverse?

(a) Let f(x) be a function from $\mathbb{R}$ to $\mathbb{R}$. $f(x) = 2x + 1$

(b) Let f(x) be a function from $\mathbb{R}^+$ to $\mathbb{R}$. $g(x) = log_2(2x) - 1$

(a) $y = 2x + 1$

$x = \frac{y-1}{2}$

$f^{-1}(x) = \frac{x-1}{2}$

$R \to R$

(b) $y = \log_2(2x) - 1$

$2x = 2^{y+1}$

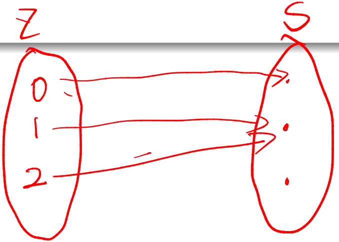
$x = 2^y$

$f^{-1}(x) = 2^x$ $\quad R \to R^+$

Sequences

Definition

A *sequence* is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, ...\}$ or the set $\{1, 2, 3, ...\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.



$a_n = \frac{1}{n}$

$\{a_1, a_2, \cdots a_n, \cdots\}$

$\{1, \frac{1}{2}, \frac{1}{3}, \cdots \cdots\}$

Geometric Progression

Definition

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the initial term a and the common ratio r are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

$\{b_n\}$ $b_n = (-1)^n$

$\{-1, 1, -1, 1, \ldots\}$

$a = 1$

$r = -1$

Arithmetic Progression

Definition

An *arithmetic progression* is a sequence of the form

$$a, a+d, a+2d, ..., a+nd, ...$$

where the initial term a and the common difference d are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

$\{S_n\}$ $S_n = -1 + 4n$, $n = 0, 1, 2 \cdots$

$\{S_n\} = \{-1, 3, 7, \cdots\}$

$a = -1$, $d = 4$

Recurrence Relation

$a_n = a_{n-1} + 1$

$a_0 = 1$
$a_1 = a_0 + 1 = 2.$
$a_2 = a_1 + 1 = 3.$

$a_n = a_{n-1} + 1 = n + 1$

Definition

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, ..., a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

e.g. $a_0 = 1$. $a_{n+1} = a_n + 1$ for $n = 0, 1, 2, ...$

Homework

Learn the following definitions by yourself.

- Real-valued and Integer-valued Functions
- Adding and Multiplying Real-valued Functions
- Increasing and Decreasing Functions
- Floor Function and Ceiling Function

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