Lecture 5

Topics covered in this lecture session

1. Expressing $a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$
2. Remainder and Factor Theorems
3. Polynomial Division
4. Polynomial Factorisation.

Expressing $a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$ or similar forms

Sometimes it is important to express

$f(x) = a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$,

so as to

- determine the range of f
- find the period of f
- sketch the graph of the function f.

Expressing $a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$ or similar forms

Example:

Express $f(x) = \sin x - \sqrt{3}\cos x$ in the form $r\sin(x-\theta)$, where $\theta \in \left(0, \frac{\pi}{2}\right)$. Sketch the graph of $y = f(x)$. Find the range and period of f.

$$f(x) = \sin x - \sqrt{3}\cos x \equiv r\sin(x-\theta)$$

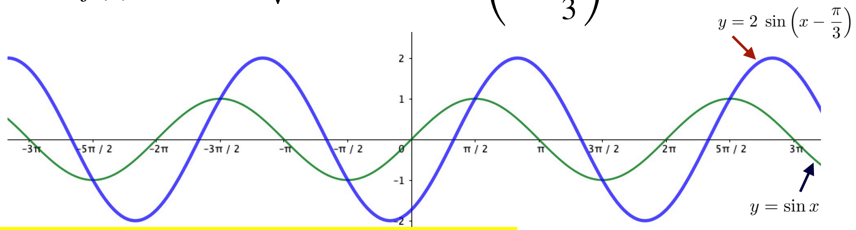
$$\Rightarrow \underline{\sin x} - \underline{\sqrt{3}\cos x} \equiv \underline{r\sin x\cos\theta} - \underline{r\cos x\sin\theta}$$

$$\Rightarrow r\cos\theta = 1 \quad \text{and} \quad r\sin\theta = \sqrt{3} \quad \Rightarrow r = 2$$

Expressing $a\cos x + b\sin x$ in the form $r\ \cos(\theta - x)$ or similar forms

Also, $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$

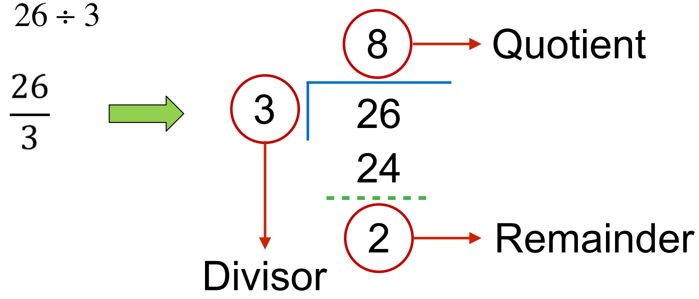
Thus $f(x) = \sin x - \sqrt{3}\cos x = 2\sin\left(x - \frac{\pi}{3}\right)$



Period of $f = 2\pi$, Range of f is $[-2, 2]$

Division process (for numbers)

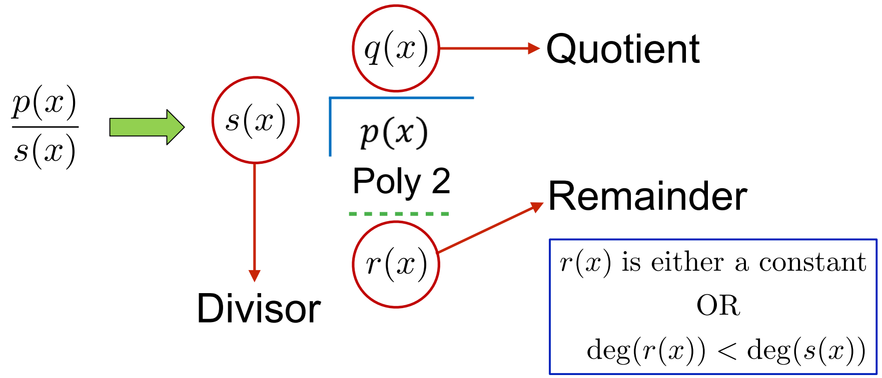
Example $26 \div 3$



$$\therefore \frac{26}{3} = 8 + \frac{2}{3} = Quotient + \frac{Remainder}{Divisor}$$

Division process (for polynomials)

e.g. $p(x) \div s(x)$ where $s(x) \neq 0$



Division of polynomials

Thus, $\dfrac{p(x)}{s(x)} = q(x) + \dfrac{r(x)}{s(x)} \quad \Rightarrow \quad p(x) = s(x)\,q(x) + r(x)$

where, $q(x)$ is the quotient, and

$r(x)$ is the remainder - which is either a constant (r)

or $\deg(r(x)) < \deg(s(x))$.

In particular, when $p(x)$ is divided by $(x-c)$, the remainder must be some constant r.

Remainder Theorem

i.e. $\dfrac{p(x)}{(x-c)} = q(x) + \dfrac{r}{(x-c)}$

$\Rightarrow \quad p(x) = (x-c)\,q(x) + r$

$\Rightarrow \quad p(c) = r$

Remainder Theorem

If a polynomial $p(x)$ is divided by $(x-c)$, then the remainder is $p(c)$.

Remainder Theorem

Example: If $x^2 - 7x + k$ has a remainder 1 when divided by $(x+1)$, find k.

Solution: $(x+1) \equiv (x-c) \Rightarrow c = -1$

By Remainder Theorem, $p(c) = r$

$$\Rightarrow \; p(-1) = 1$$

$$\Rightarrow \; (-1)^2 - 7(-1) + k = 1$$

$$\Rightarrow \; k + 8 = 1 \quad \Rightarrow \quad k = -7.$$

Factor Theorem

Factorising a polynomial $p(x)$ means to write it as a product of lower-degree polynomials - called factors of $p(x)$.

For $s(x)$ to be a factor of $p(x)$, there must be no remainder when $p(x)$ is divided by $s(x)$.

$$\text{i.e.} \; \frac{p(x)}{s(x)} = q(x) + 0 \qquad \textbf{or} \qquad p(x) = s(x)\,q(x)$$

Factor Theorem

In particular, when $(x-c)$ is a factor of the polynomial $p(x)$, $p(x)$ can be expressed as

$$p(x) = (x-c)\,q(x) \quad \text{i.e.} \quad p(c) = 0.$$

Factor Theorem

A polynomial $p(x)$ has a factor $(x-c)$, if and only if $p(c) = 0$.

Note: $p(c) = r$ is the Remainder Theorem
$p(c) = 0$ is the Factor Theorem

Factor Theorem

Example: If $(x-2)$ is a factor of $ax^2 - 12x + 4$, find a.

Solution: Here, $(x-c) = (x-2) \Rightarrow c = 2$

By Factor theorem, $p(c) = 0$.

$$\Rightarrow p(2) = 0 \;\Rightarrow\; a\,(2)^2 - 12\,(2) + 4 = 0$$

$$\Rightarrow 4a - 24 + 4 = 0$$

$$\Rightarrow 4a = 20 \quad \Rightarrow a = 5.$$

Polynomial Division

1. Method of Long Division (or actual division)

The process of long division for dividing polynomials is similar to that of division of numbers.

Suppose, we want to determine

$$\frac{27x^3+9x^2-3x-10}{3x-2}$$

$$\begin{array}{r|l}
 & 9x^2 + 9x + 5 \\
\hline
3x-2 & 27x^3 + 9x^2 - 3x - 10 \\
 & 27x^3 - 18x^2 \\
 & \quad 27x^2 - 3x - 10 \\
 & \quad 27x^2 - 18x \\
 & \qquad 15x - 10 \\
 & \qquad 15x - 10 \\
 & \qquad\qquad 0
\end{array}$$

Thus, $\dfrac{27x^3+9x^2-3x-10}{3x-2} = 9x^2+9x+5$

Polynomial Division

2. Method of Synthetic Division

The method of Synthetic Division is a powerful alternative to the Method of Long Division.

We study this method only for linear divisors of the form $(x-c)$.

To understand the method, let us consider the example:

Example: If $\dfrac{x^3-9x^2-20}{(x-3)} = q(x) + \dfrac{r(x)}{(x-3)}$, find $q(x)$ and $r(x)$.

Method of Synthetic Division

Step 1

$$\begin{array}{c|cccc}
 & 1 & -9 & 0 & -20 \\
\hline
 & & & &
\end{array}$$

Write the coefficients of the polynomial to be divided at the top. Put zero as coefficient for unseen power(s) of x.

Step 2

$$\begin{array}{c|cccc}
 & 1 & -9 & 0 & -20 \\
3 & & & & \\
\hline
 & & & &
\end{array}$$

$\dfrac{x^3-9x^2-20}{x-3}$

Negate the constant term in the divisor, and write-in on the left side, that is, if $(x-a)$ is the divisor, write a on the left side.

Method of Synthetic Division

Step 3

$$\begin{array}{c|cccc}
 & 1 & -9 & 0 & -20 \\
3 & \downarrow & & & \\
\hline
 & 1 & & &
\end{array}$$

Drop the first coefficient after the bar to the last row.

Step 4

$$\begin{array}{c|cccc}
 & 1 & -9 & 0 & -20 \\
3 & \downarrow & 3 & & \\
\hline
 & 1 & & &
\end{array}$$

Multiply the dropped number with the number before the bar, and place it in the next column.

Method of Synthetic Division

Step 5

	1	-9	0	-20
3	$\downarrow$	3		
	1	-6		

Perform addition in the next column.

Repeat the previous two steps to obtain the following.

	1	-9	0	-20
3	$\downarrow$	3	-18	-54
	1	-6	-18	**-74**

Thus,

$$\frac{x^3 - 9x^2 - 20}{(x-3)} = (x^2 - 6x - 18) + \frac{-74}{(x-3)}$$

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Method of Synthetic Division

Example: Given $p(x) = x^3 - 5x^2 + 4x + 9$ and $s(x) = x + 1$ find $q(x)$ and r when $p(x)$ is divided by $s(x)$.

Here, $s(x) = x + 1$

$\equiv x - C$

$\Rightarrow C = -1$

	1	-5	4	9
-1	$\downarrow$	-1	6	-10
	1	-6	10	-1

$\therefore\ q(x) = x^2 - 6x + 10$

and $r = -1$

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Factorising Polynomials

(with at least one integer zero)

Result: Let $p(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_1 x + c_0$ be a polynomial with integer coefficients. Then, r is an integer zero of $p(x)$, if r is a divisor of the constant term c_0.

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Factorising Polynomials

Example: Show that $s(x) = x - 1$ is a factor of $p(x) = x^3 - 2x^2 - 19x + 20$. Hence solve $p(x) = 0$.

Here,

$p(1) = 1 - 2 - 19 + 20 = 0.$

$\therefore\ (x-1)$ is one of the factor.

We use synthetic division to find the other factor.

$p(x) = x^3 - 2x^2 - 19x + 20$

	1	-2	-19	20
1	$\downarrow$	1	-1	-20
	1	-1	-20	0

Factorising Polynomials

$$
\begin{array}{c|cccc}
 & 1 & -2 & -19 & 20 \\
1 & \downarrow & 1 & -1 & -20 \\
\hline
 & 1 & -1 & -20 & 0
\end{array}
$$

$\therefore$ The other factor is $(x^2 - x - 20)$.

$$\therefore\ p(x) = (x-1)\cdot(x^2 - x - 20)$$

$$= (x-1)\cdot(x-5)\cdot(x+4)$$

$$\therefore\ p(x) = 0 \ \Rightarrow\ (x-1)\cdot(x-5)\cdot(x+4) = 0$$

$$\Rightarrow\ x = 1 \quad \text{or} \quad x = 5 \quad \text{or} \quad x = -4.$$