Lecture 9

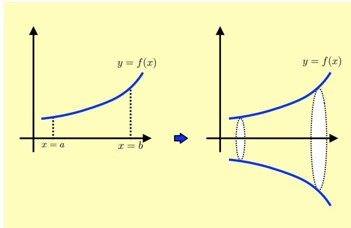
Topics covered in this lecture session

1. Application of Integration (Volume calculation)
   a) Solid of revolution
   b) Finding volume of solid of revolution
   4 results/formulae
2. Numerical methods for integration
   a) Trapezoidal rule
   b) Simpson's rule
3. Introduction to Differential equations

Applications of Integration
(**Volume** calculation)

Solid of revolution

Imagine rotating the curve $y = f(x)$ between the points $x = a$ and $x = b$ by one complete revolution ($360°$ or 2π radians) around the X-axis.

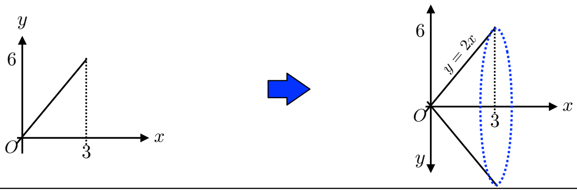


The three dimensional solid so formed is called a **solid of revolution**.

Applications of Integration
(**Volume** calculation)

Solid of revolution

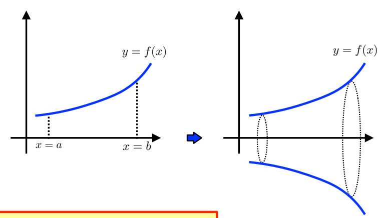
e.g. when the graph of the function $y = 2x$ between the points $x = 0$ and $x = 3$ is rotated by one complete revolution about the X-axis, the solid of revolution formed is a cone.



Applications of Integration
(**Volume** calculation)

Result 1

If the region R bounded by the curve $y = f(x)$, lines $x = a$, $x = b$, and the X-axis is revolved about the X-axis, then the volume of the solid of revolution is:



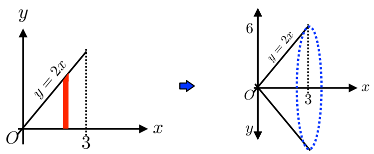
$$\boxed{V = \pi \int_a^b y^2 \, dx = \pi \int_a^b [f(x)]^2 \, dx}$$

Applications of Integration

(**Volume** calculation)

Example 1: Find the volume of the solid that is generated when the region bounded by the line $y = 2x$, and $x \in [0, 3]$ is revolved about the X-axis.

$$V = \pi \int_0^3 y^2 \, dx = \pi \int_0^3 4x^2 \, dx$$
$$= 4\pi \left[\frac{x^3}{3} \right]_0^3$$
$$= 36\pi$$



Applications of Integration

(**Volume** calculation)

Example 8: Find the volume of a sphere of radius r.

A sphere is obtained when a semi-circle is revolved about the X-axis. (one complete revolution)

Also, the equation of a semi-circle is: $x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

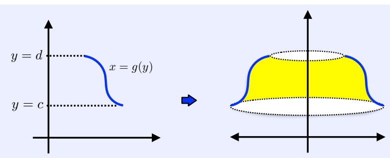
$$\therefore \; V = \pi \int_{-r}^{r} y^2 \, dx = \pi \int_{-r}^{r} \left(r^2 - x^2 \right) \, dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} = \frac{4\pi r^3}{3}$$

Applications of Integration

(**Volume** calculation)

Result 2

If the region R bounded by the curve $x = g(y)$, lines $y = c$, $y = d$, and the Y-axis is revolved about the Y-axis, then the volume of the solid of revolution is:



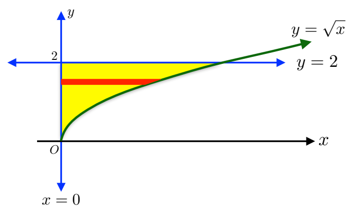
$$\boxed{V = \pi \int_c^d x^2 \, dy = \pi \int_c^d [g(y)]^2 \, dy}$$

Applications of Integration

(**Volume** calculation)

Example 9: Find the volume of the solid generated when the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is rotated about the Y-axis.

$$V = \pi \int_0^2 x^2 \, dy = \pi \int_0^2 y^4 \, dy$$
$$= \pi \left[\frac{y^5}{5} \right]_0^2 = \frac{32\pi}{5}$$

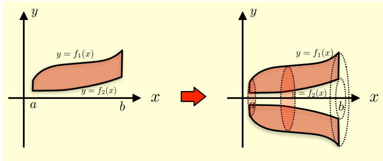


Applications of Integration

(**Volume** calculation)

Result 3

If the region R bounded by two curves $y = f_1(x)$, $y = f_2(x)$ lines $x = a$, $x = b$ is revolved about the X-axis, then the volume of the solid of revolution is:



$$\boxed{V = \pi \left| \int_a^b \left\{ [f_1(x)]^2 - [f_2(x)]^2 \right\} dx \right|}$$

Applications of Integration

(**Volume** calculation)

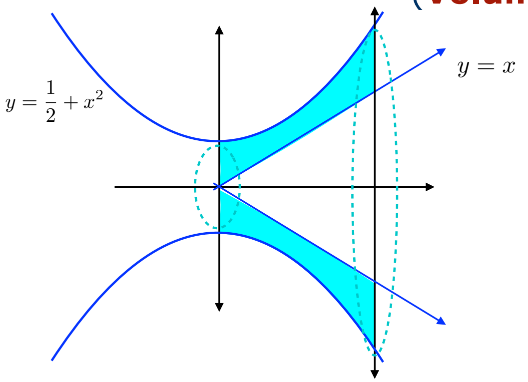
Example 10: Find the volume of the solid generated when the region bounded by $y = \frac{1}{2} + x^2$ and $y = x$ over $[0, 2]$ is rotated about the X-axis.

$$V = \pi \int_0^2 \left[\left(\frac{1}{2} + x^2 \right)^2 - (x)^2 \right] dx = \pi \int_0^2 \left[\frac{1}{4} + x^2 + x^4 - x^2 \right] dx$$

$$= \pi \left[\frac{x}{4} + \frac{x^5}{5} \right]_0^2 = \frac{69\,\pi}{10}$$

Applications of Integration

(**Volume** calculation)

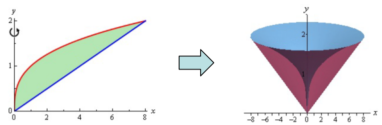


Applications of Integration

(**Volume** calculation)

Result 4

If the region R bounded by two curves $x = g_2(y)$, $x = g_1(y)$ lines $y = c$, $y = d$ is revolved about the Y-axis, then the volume of the solid of revolution is:



$$\boxed{V = \pi \left| \int_c^d \left\{ [g_1(y)]^2 - [g_2(y)]^2 \right\} dy \right|}$$

Evaluate (Please complete SEM Survey)

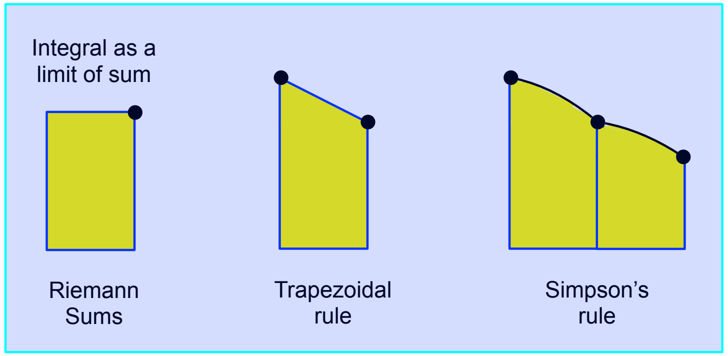
Log on to https://bluecastle-cn-surveys.nottingham.ac.uk

OR

Scan the QR code:

- Enter your username and password
- Click on My Surveys
- Find CELEN037 module
- Click on Complete Survey.

Numerical Integration



Numerical Integration

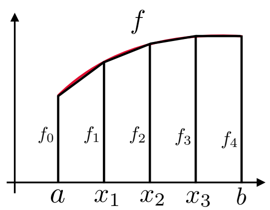
(Trapezoidal Rule)

The general idea is to use trapezoids instead of rectangles to approximate the area under the curve.

We subdivide the interval $[a, b]$ in to n subintervals of equal width

$$h = \frac{b-a}{n} \quad \text{so that}$$

$$a = x_0 \leq x_1 \leq x_2 \leq \ldots\ldots \leq x_n = b$$



Numerical Integration

(Trapezoidal Rule)

Now, area of trapezoid is

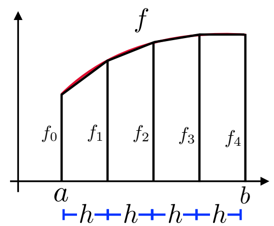
$$\frac{1}{2}\,(\text{sum of parallel sides}) \times (\underline{\text{width of the subinterval}})$$

$$\therefore\ T_1 = \frac{h}{2}(f_0 + f_1)$$

$$T_2 = \frac{h}{2}(f_1 + f_2)$$

$$\vdots \quad \vdots \quad \vdots$$

$$T_n = \frac{h}{2}(f_{n-1} + f_n)$$



Numerical Integration

(Trapezoidal Rule)

Now, total area under the curve

$= $ sum of areas of trapezoids

$$\therefore \int_a^b f(x)\ dx \approx \frac{h}{2}\ [\ f_0 + f_1 + f_1 + f_2 + f_2 + f_3 + f_3 + f_4 + \ldots + f_{n-2} + f_{n-1} + f_{n-1} + f_n\ ]$$

$$\boxed{\therefore \int_a^b f(x)\ dx \approx \frac{h}{2}\ [\ f_0 + 2\,(f_1 + f_2 + f_3 + \ldots + f_{n-1}) + f_n\ ]}$$

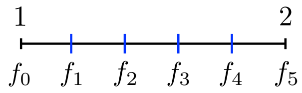
Numerical Integration

(Trapezoidal Rule)

Example 5: Evaluate the definite integral $\int_1^2 \frac{1}{x}\ dx$ using Trapezoidal rule, by dividing $[\,1, 2\,]$ in to 5 sub-intervals of equal width.

Here, $h = \frac{1-0}{5} = \frac{1}{5} = 0.2$

and $f(x) = \frac{1}{x}$



Numerical Integration

(Trapezoidal Rule)

$$\therefore \int_1^2 \frac{1}{x}\ dx \approx \frac{h}{2}\ [\ f_0 + 2\,(f_1 + f_2 + f_3 + f_4) + f_5\ ]$$

x	1	1.2	1.4	1.6	1.8	2
$f(x) = \frac{1}{x}$	1	0.8333	0.7143	0.625	0.5556	0.5
f_n	f_0	f_1	f_2	f_3	f_4	f_5

$$\therefore I \approx \frac{0.2}{2}\ [1 + 2\,(0.8333 + 0.7143 + 0.625 + 0.5556) + 0.5] \approx 0.6956$$

Numerical Integration

(Simpson's Rule)

Another technique for approximating the value of a definite integral is called Simpson's rule.

Simpson's method replaces the slanted-line with parabolas.

Source: Wikipedia

Thomas Simpson
1710-1761

Numerical Integration

(Simpson's Rule)

We state the formula without proof.

$$\int_a^b f(x)\ dx \approx \frac{h}{3}\ [\ f_0 + 2\ (\ f_2 + f_4 + f_6 + \ldots + f_{n-2}\ )$$
$$+ 4\ (\ f_1 + f_3 + f_5 + \ldots + f_{n-1}\ ) + f_n\ ]$$

Note: The method works only for even number of sub-intervals.

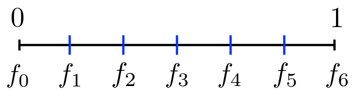
Numerical Integration

(Simpson's Rule)

Example 6: Evaluate the definite integral $\int\limits_0^1 e^{x^2}\ dx$ using Simpson's rule, by dividing $[\,0, 1\,]$ in to 6 sub-intervals of equal width.

Here, $h = \frac{1-0}{6} = \frac{1}{6}$

and $f(x) = e^{x^2}$



Numerical Integration

(Simpson's Rule)

$$\therefore \int_0^1 e^{x^2}\ dx \approx \frac{h}{3}\ [\ f_0 + 2\,(f_2 + f_4) + 4\,(f_1 + f_3 + f_5) + f_6\ ]$$

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$	$e^{0^2} = 1$	$e^{(1/6)^2}$	$e^{(2/6)^2}$	$e^{(3/6)^2}$	$e^{(4/6)^2}$	$e^{(5/6)^2}$	$e^{(6/6)^2}$
f_n	f_0	f_1	f_2	f_3	f_4	f_5	f_6

$$= \frac{1}{18}\left[1 + 2\left(e^{(2/6)^2} + e^{(4/6)^2}\right) + 4\left(e^{(1/6)^2} + e^{(3/6)^2} + e^{(5/6)^2}\right) + e^{(6/6)^2}\right] \approx 1.46287$$

Numerical Integration

Evaluate the definite integral $\int\limits_{\pi/4}^{\pi/2} \frac{\cos x}{x}\ dx$ by dividing $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ into 8 sub-intervals of equal width.

Obtain the answer correct to 4 decimal places using

(i) Trapezoidal rule

(ii) Simpson's rule.

Differential equations: Introduction

What is a Differential Equation?

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

e.g. $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 + xy = 0$ is a differential equation.

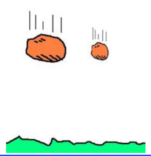
Differential equations: Introduction

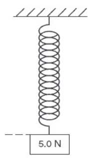
Differential equations occur when we model the change or process which occurs between two quantities.

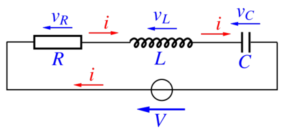
e.g.
- velocity (distance and time)
- acceleration (velocity and time)
- pressure (force and area)
- charge (current and time).

Differential equations: Introduction

What do these pictures have in common?

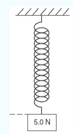
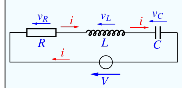






Differential equations: Introduction

Differential equations govern these processes (that changes with time).

	$\dfrac{d^2s}{dt^2} = -g$ g gravitational acceleration s distance t time travelled		$m\,\dfrac{d^2y}{dt^2} = -k\,y$ y vertical displacement m mass k spring constant ($k > 0$) t time
	$L\,\dfrac{d^2q}{dt^2} + R\,\dfrac{dq}{dt} + \dfrac{1}{C}\,q = E$ q charge on capacitor L inductance C capacitance R resistance E voltage t time		$\dfrac{dm}{dt} = k\,m \quad (k < 0)$ m mass of radioactive substance k constant t time

Differential equations: Introduction

The topic of differential equations can be treated in two ways:

A theoretical perspective:
i.e. studying only the methods of solving ODEs.

An applied perspective:
i.e. using differential equations to model real world problems.

Differential equations: Introduction

Differential Equations are of two main types:

Ordinary Differential Equations (ODEs)
An equation involving derivatives of one dependent variable with respect to one independent variable.

Partial Differential Equations (PDEs)
An equation involving derivatives of more than one dependent variable with respect to one or more independent variable.

We shall focus only on ODEs in this module.