The University of Nottingham Ningbo China

Centre for English Language Education

MID-SEMESTER Exam, Spring 2022

FOUNDATION CALCULUS & MATHEMATICAL TECHNIQUES

Time allowed 60 Minutes

Candidates may complete the information required on the front page of this booklet but must NOT write anything else until the start of the examination period is announced.

This paper comprises TWENTY questions.

Answers must be written (with necessary steps) in this booklet.

Figures enclosed by square brackets, eg. [3] *indicate marks for that question.*

Only CELE approved calculator (with university logo) is allowed during this exam.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

Do NOT turn examination paper over until instructed to do so.

ADDITIONAL MATERIAL: *Useful formulae on* **Page 2** *of this booklet.*

INFORMATION FOR INVIGILATORS:

1) Please advise students at the start of the exam that they can do rough work on the last page.

2) Please give a 15 minutes warning before the end of the exam.

3) Please collect this booklet at the end of the exam.

Student ID: ________________

Seminar Group: ____________ (e.g. A23 or B13 or C17) **Marks (out of 60):** _______

Useful formulae:

- **Differentiation: Useful results**

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}(u \cdot v \cdot w) = u\,v \cdot \frac{dw}{dx} + v\,w \cdot \frac{du}{dx} + u\,w \cdot \frac{dv}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$$

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{\left(\dfrac{dx}{dy}\right)}$$

- **Derivatives of standard functions**

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

- **Maclaurin's series**

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}\,f''(0) + \frac{x^3}{3!}\,f'''(0) + \ldots\ldots + \frac{x^n}{n!}\,f^{(n)}(0) + \ldots\ldots$$

- **Integration**

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \qquad (a > 0)$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \frac{1}{x^2+a^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{|x|\sqrt{x^2-a^2}} \, dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

$$\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

- **Trigonometry**

$$\left[\begin{array}{l} \cos^2\theta + \sin^2\theta = 1 \\ \tan^2\theta + 1 = \sec^2\theta \\ \cot^2\theta + 1 = \operatorname{cosec}^2\theta \end{array}\right.$$

$$\left[\begin{array}{lcl} 2\sin A\cos B & = & \sin(A+B) + \sin(A-B) \\ 2\cos A\sin B & = & \sin(A+B) - \sin(A-B) \\ 2\cos A\cos B & = & \cos(A+B) + \cos(A-B) \\ -2\sin A\sin B & = & \cos(A+B) - \cos(A-B) \end{array}\right.$$

1

Given $y = \sqrt{x+3}$, use the definition of derivative, to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h + \cancel{3} - \cancel{x} - \cancel{3}}{h\left[\sqrt{x+h+3} + \sqrt{x+3}\right]}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}$$

$$= \frac{1}{2\sqrt{x+3}}$$

[2]

2

Given $y = e^x \cdot \left(x^{2022} - 2022\,x + 2022\right)$, use the product rule for derivatives to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(x^{2022} - 2022x + 2022)$$
$$+ (x^{2022} - 2022x + 2022) \cdot \frac{d}{dx} e^x$$
$$= e^x (2022x^{2021} - 2022)$$
$$+ e^x (x^{2022} - 2022x + 2022)$$
$$= e^x \left[x^{2022} + 2022x^{2021} - 2022x\right]$$

[2]

3

Given $y = \dfrac{1-x^4}{1+x^4}$, use the quotient rule for derivatives to show that $\dfrac{dy}{dx} + \dfrac{8\,x^3}{\left(x^4+1\right)^2} = 0.$

$$\frac{dy}{dx} = \frac{(1+x^4)\frac{d}{dx}(1-x^4) - (1-x^4)\frac{d}{dx}(1+x^4)}{(1+x^4)^2}$$

$$= \frac{(1+x^4)(-4x^3) - (1-x^4)(4x^3)}{(1+x^4)^2}$$

$$= \frac{(-4x^3) - \cancel{4x^3(x^4)} - 4x^3 + \cancel{4x^3(x^4)}}{(1+x^4)^2}$$

$$\therefore \frac{dy}{dx} = \frac{-8x^3}{(1+x^4)^2}$$

[3]

4

Given $y = \tan\left[e^{\left(x^2+3\right)}\right]$, use the substitutions $u = x^2+3$ and $v = e^u$, and then apply the chain rule for derivatives to find $\dfrac{dy}{dx}$. Write necessary steps.

$$y = \tan(v) \Rightarrow \frac{dy}{dv} = \sec^2(v) = \sec^2\left[e^{(x^2+3)}\right]$$

$$v = e^u \Rightarrow \frac{dv}{du} = e^u = e^{(x^2+3)}$$

$$u = x^2+3 \Rightarrow \frac{du}{dx} = 2x$$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dv} \cdot \dfrac{dv}{du} \cdot \dfrac{du}{dx}$

$$\therefore \frac{dy}{dx} = \sec^2\left[e^{(x^2+3)}\right] \cdot e^{(x^2+3)} \cdot 2x$$

[3]

5

Given $x^3y + x\,y^3 = \sin\left(x^3\,y\right)$, use the method of implicit differentiation to find $\dfrac{dy}{dx}$.

$$\frac{d}{dx}\left[x^3y + xy^3\right] = \frac{d}{dx}\left[\sin(x^3y)\right]$$

$$\left[3x^2y + x^3\frac{dy}{dx} + y^3 + x\,3y^2\frac{dy}{dx}\right] = \cos(x^3y)\left[3x^2y + x^3\frac{dy}{dx}\right]$$

$$\frac{dy}{dx}\left[x^3 + 3xy^2 - x^3\cdot\cos(x^3y)\right] = 3x^2y\cos(x^3y) - 3x^2y - y^3$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{3x^2\cos(x^3y) - 3x^2 - y^2}{x^2 + 3y^2 - x^2\cos(x^3y)}\right]$$

[4]

6

Given $x = \tan\theta - \sec\theta$ and $y = \tan\theta + \sec\theta$; $\theta \in (0, \pi) - \left\{\dfrac{\pi}{2}\right\}$, use parametric differentiation to evaluate $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}$.

$$\frac{dx}{d\theta} = \sec^2\theta - \sec\theta\tan\theta$$

$$\frac{dy}{d\theta} = \sec^2\theta + \sec\theta\tan\theta$$

$$\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec^2\theta - \sec\theta\tan\theta}$$

$$\therefore \frac{dy}{dx} = \frac{\sec\theta + \tan\theta}{\sec\theta - \tan\theta}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\pi/4} = \frac{\sec(\pi/4) + \tan(\pi/4)}{\sec(\pi/4) - \tan(\pi/4)} = \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

[3]

7

Given $y = (\tan x)^{e^x}$, use logarithmic differentiation to find $\frac{dy}{dx}$.

$\ln y = \ln(\tan x)^{e^x}$

$\ln y = e^x \ln(\tan x)$

$\frac{dy}{dx} = y\left[e^x \cdot \frac{\sec^2 x}{\tan x} + e^x \cdot \ln(\tan x)\right]$

$\therefore \frac{dy}{dx} = (\tan x)^{e^x} \cdot e^x \left[\frac{\sec^2 x}{\tan x} + \ln(\tan x)\right]$

[4]

8

Given $f(x) = \sin 3x - \cos 3x$, find the equation of the tangent line to the curve $y = f(x)$ at $P\left(\frac{\pi}{2}, -1\right)$.

$\frac{dy}{dx} = 3\cos 3x + 3\sin 3x$

$\therefore m$ (slope of tangent line at $(\pi/2, -1)$) $= \left.\frac{dy}{dx}\right|_{(\pi/2, -1)} = -3$

$y - (-1) = -3\left(x - \frac{\pi}{2}\right)$

$y + 1 = -3x + \frac{3\pi}{2}$

$\therefore y + 3x + \frac{2 - 3\pi}{2} = 0$

[3]

9

Given $f(x) = x^3 + 4x^2 - 3x + 1$, find the stationary points of f.

Stationary points are at: $f'(x) = 0$

$\therefore\ 3x^2 + 8x - 3 = 0$

$\Rightarrow\ (x+3)(3x-1) = 0$

$\therefore\ x = -3$ or $x = 1/3$

at $x = -3$, $f(-3) = 19$

at $x = 1/3$, $f(1/3) = 13/27$

$\therefore\ (-3, 19)$ and $(1/3, 13/27)$ are stationary points.

[3]

10

Use the second derivative test to classify the stationary points obtained in Q.9 as the points of maximum or minimum values.

$f''(x) = 6x + 8$

$f''(x)\big|_{(-3,19)} = -10 < 0$

$\therefore\ (-3, 19)$ is a point of maximum value

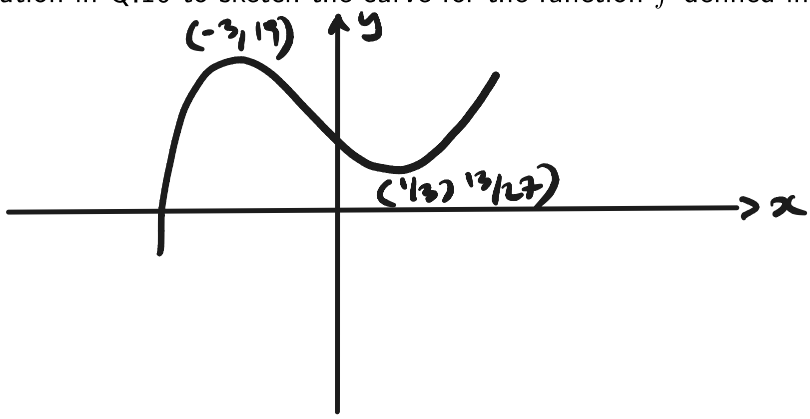
$f''(x)\big|_{(1/3,\ 13/27)} = 10 > 0$

$\therefore\ (1/3, 13/27)$ is a point of minimum value

[2]

11

Use the information in Q.10 to sketch the curve for the function f defined in Q.9.



[1]

12

A circular disc of radius 2 cm is being heated. Due to expansion, its radius increases at the rate of 0.025 cm/sec. Find the rate at which its area is increasing when its radius is 2.1 cm.

(Area of circular disk is $A = \pi r^2$)

$$A = \pi r^2$$

$$\frac{dA}{dt} = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt}$$

$$= 2\pi r \frac{dr}{dt}$$

$$\therefore \left.\frac{dA}{dt}\right|_{r=2.1} = 2 \cdot \pi \cdot \frac{21}{10} \cdot \frac{25}{1000} = 0.105\pi \text{ cm}^2/\text{s}$$

[2]

13

Given $y = A\sin mx + B\cos mx$, where A, B are constants, show that $\dfrac{d^2y}{dx^2} + m^2y = 0$.

$$\frac{dy}{dx} = mA\cos mx - mB\sin x$$

$$= -m^2 A\sin mx - m^2 B\cos mx$$

$$= -m^2 (A\sin mx + B\cos mx$$

$$= -m^2 y$$

$$\therefore \frac{dy}{dx^2} = -m^2 y$$

$$\Rightarrow \frac{dy}{dx^2} + m^2 y = 0$$

[2]

14

The Newton-Raphson iteration formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ; \quad (n = 0,\, 1,\, 2,\, 3,\, \ldots).$$

For the function f defined by $f(x) = 4\,x^3 + x^2 - 3\,x - 10$,

(i) show that the Newton-Raphson scheme is given by

$$x_{n+1} = \frac{8\,x_n^3 + x_n^2 + 10}{12\,x_n^2 + 2\,x_n - 3}. \tag{14.1}$$

$$f'(x) = 12x^2 - 2x - 3$$

$$x_{n+1} = x_n - \frac{4x_n^3 + x_n^2 - 3x_n - 10}{12x_n^2 + 2x_n - 3}$$

$$x_{n+1} = \frac{12x_n^3 + 2x_n^2 - 3x_n - 4x_n^3 - x_n^2 + 3x_n + 10}{12x_n^2 + 2x_n - 3}$$

$$x_{n+1} = \frac{8x_n^3 + x_n^2 + 10}{12x_n^2 + 2x_n - 3}$$

(ii) Given that the root lies in the interval (1,2), use (14.1) to obtain a root of $f(x) = 0$, correct to 6 d.p. Write values in the table below.

n	x_n
0	1.5
1	1.453704
2	1.452108
3	1.452106
4	1.452106

The desired root (to 6 d.p.) is $x^* =$ 1.452106 [4]

15

Given $f(x) = \sqrt{1+x}$; $|x| < 1$, find the values of $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

Hence, show that the Maclaurin's series expansion of $f(x)$ is

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}.$$

$f(x) = (1+x)^{1/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2} \Rightarrow f'(0) = 1/2$

$f''(x) = \frac{-1}{4}(1-x)^{-3/2} \Rightarrow f''(0) = -1/4$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2} \Rightarrow f'''(0) = 3/8$

$\therefore \sqrt{1+x} = 1 + x(1/2) + \frac{x^2}{2 \times 1}(-1/4) + \frac{x^3}{\cancel{3} \times 2}(\cancel{3}/8) + \ldots$

$\Rightarrow \sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}.$

[5]

16

(i) Evaluate $\displaystyle\int \frac{x^3+x^2-6x}{x+3}\,dx$.

Let $I = \int \frac{x(x^2+x-6)}{(x+3)}\,dx$

$I = \int \frac{x(x-2)\cancel{(x+3)}}{\cancel{x+3}}\,dx = \int x^2\,dx - 2\int x\,dx$

$\therefore I = \frac{x^3}{3} - x^2 + c$

(ii) Prove that $\displaystyle\int \left(\frac{\sec^2 x - \tan^2 x}{\sec x}\right) dx = \sin x + C.$

Let $I = \displaystyle\int \frac{\sec^2 x(1-\sin^2 x)}{\sec x}\, dx$

$I = \displaystyle\int \frac{1}{\cos x} \cdot \cos^2 x \, dx = \int \cos x \, dx$

$\therefore\ I = \sin x + C$

(iii) Use the substitution $\ln x = t$ to evaluate the integral $\displaystyle\int \frac{(\ln x)^n}{x}\, dx\ ;\ n \in \mathbf{N}.$

Let $I = \displaystyle\int \frac{(\ln x)^n}{x}\, dx$

and let $\ln x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore\ I = \displaystyle\int t^n \cdot dt = \frac{t^{n+1}}{n+1} + C$

$\Rightarrow I = \dfrac{(\ln x)^{n+1}}{n+1} + C$

[5]

17

By completing the square for the term $(x^2 + 4x - 21)$, evaluate the integral

$$\int \frac{1}{\sqrt{x^2 + 4x - 21}}\, dx.$$

Let $I = \displaystyle\int \frac{1}{\sqrt{x^2+4x-21}}\, dx$

$= \displaystyle\int \frac{1}{\sqrt{x^2+4x+2^2-21-2^2}}\, dx$

$= \displaystyle\int \frac{1}{\sqrt{x^2+4x+4-25}}\, dx$

$= \displaystyle\int \frac{1}{\sqrt{(x+2)^2-5^2}}\, dx$

$\therefore\ I = \ln\left|(x+2) + \sqrt{(x+2)^2-5^2}\right| + C$

[3]

18

Use the substitution $\cos x = t^2$ to evaluate the integral $\int \sin 2x \sqrt{\cos x}\ dx$.

Let $I = \int \sin 2x \sqrt{\cos x}\, dx = 2\int \sqrt{\cos x} \cdot \sin x \cos x\, dx$

$\cos x = t^2 \Rightarrow \sqrt{\cos x} = t$

also $-\sin x\, dx = 2t\, dt$

$\therefore\ I = -2\int t \cdot t^2 \cdot (-2t\, dt) = -4\int t^4\, dt$

$I = -\frac{4}{5} t^5 + C$

$\Rightarrow I = -\frac{4}{5} (\cos x)^{5/2} + C$

[3]

19

Evaluate the integral $\int \cos 5x \cos 2x\ dx$.

Let $I = \int \cos 5x \cos 2x\, dx$

$I = \frac{1}{2}\left[\int \cos 7x\, dx + \int \cos 3x\, dx\right]$

$= \frac{1}{2}\left[\frac{\sin 7x}{7} + \frac{\sin 3x}{3}\right] + C$

[3]

20

Use the substitution $x\,e^{x} = t$ to evaluate the integral $\displaystyle\int \frac{e^{x}\,(x+1)}{\cos^{2}(x\,e^{x})}\,dx$.

Let $I = \displaystyle\int \frac{e^{x}(x+1)}{\cos^{2}(xe^{x})}\,dx$

let $xe^{x} = t \Rightarrow e^{x}(x+1)\,dx = dt$

$\therefore I = \displaystyle\int \frac{1}{\cos^{2}(t)}\,dt = \int \sec^{2}(t)\,dt$

$\therefore I = \tan(t) + C$

$I = \tan(xe^{x}) + C$

[3]

You may use this space for rough work