

# AE1MCS: Mathematics for Computer Scientists

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# Aim and Learning Objectives

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
- To gain a good understanding of other definitions: tautology, contradiction, contingency, logical equivalence, converse, contrapositive and inverse.
- To be able to draw truth tables and use them as a tool to solve logical problems;
- To be able to apply important logical equivalences to solve logical problems.

# Proposition

## Definition

A *proposition* is a statement that is either true or false.

The area of logic that deals with propositions is called the *propositional logic* or *propositional calculus*.

# Is it a proposition?

1 Beijing is the capital of China. ✓

2  $1 + 1 = 2$ . ✓

3  $2 + 2 = 3$ . ✓

4 What time is it? ✗

5 Read this sentence carefully. ✗


6  $x + 1 = 2$ . ✗

7  $x + y = z$ . ✗

8 If  $x > 0$ , then  $x > 1$ . ✗

# Is a claimed proposition true or false?

Unfortunately, it is not always easy to decide if a claimed proposition is true or false.

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- 3  $313(x^3 + y^3) = z^3$  has no positive integer solutions.
- 4 Every map can be colored with 4 colors so that adjacent regions have different colors [Four Color Theorem].
- 5 Every even integer greater than 2 is the sum of two primes [Goldbach's conjecture, 1742].

# Propositional Variable

- a variable that represents a proposition
- denoted using a letter  $p, q, r, s, \dots$
- truth value: T (true); F (false)

# Logical Operators

- **Compound Proposition:** formed from existing propositions using logical operators

- Logical Operators

- Negation
- Conjunction
- Disjunction
- Implication
- ...



# Negation

## Definition (Negation)

Let  $p$  be a proposition. The *negation* of  $p$ , denoted by  $\neg p$ , is the statement

'It is not the case that  $p$ '.

The proposition  $\neg p$  is read 'not  $p$ '. The truth value of  $\neg p$  is the opposite of the truth value of  $p$ .

$p$ : Beijing is the capital of China.

$p$	$\neg p$
T	F
F	T

# Conjunction

## Definition (Conjunction)

Let  $p$  and  $q$  be propositions. The *conjunction* of  $p$  and  $q$ , denoted by  $p \wedge q$ , is the proposition

*'p and q'.*

The proposition  $p \wedge q$  is true when both  $p$  and  $q$  are true and is false otherwise.

*Truth table*

$p$	$q$	$p \wedge q$
T	T	T
T	F	<u>F</u>
F	T	<u>F</u>
F	F	<u>F</u>

# Disjunction

## Definition (Disjunction)

Let  $p$  and  $q$  be propositions. The *disjunction* of  $p$  and  $q$ , denoted by  $p \vee q$ , is the proposition

' $p$  or  $q$ '.

The proposition  $p \vee q$  is false when both  $p$  and  $q$  are false and is true otherwise.

$p$	$q$	$p \vee q$
<u>T</u>	<u>T</u>	<u>T</u>
<u>T</u>	F	<u>T</u>
F	<u>T</u>	<u>T</u>
F	F	F

# Exclusive Or

## Definition (Exclusive Or)

Let  $p$  and  $q$  be propositions. The *exclusive or* of  $p$  and  $q$ , denoted by  $p \oplus q$ , is the proposition that is true when *exactly one* of  $p$  and  $q$  is true and is false otherwise.

Truth table

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

# Implication

Definition (Implication) *Conditional statement.*

Let  $p$  and  $q$  be propositions. The implication  $p \rightarrow q$  is the proposition 'if  $p$ , then  $q$ '.  *$p$  implies  $q$ .*

The proposition  $p \rightarrow q$  is false when  $p$  is true and  $q$  is false, and true otherwise.  $p$  is called the hypothesis or premise and  $q$  is called the conclusion or consequence.

$p$	$q$	$p \rightarrow q$
<u>T</u>	<u>T</u>	<u>T</u>
T	F	F
<u>F</u>	<u>T</u>	<u>T</u>
F	F	T

The proposition  $p \rightarrow q$  is true, if  $p$  is false or  $q$  is true.



# Examples

- If Goldbach's Conjecture is true, <sup>P</sup> then  $x^2 \geq 0$  for every real number x. <sup>q.</sup>  $\rightarrow$  True.
- If pigs fly, then your account will not get hacked. <sup>True.</sup>  
<sub>P</sub> <sub>q</sub>

$P \rightarrow q.$   
F.

# Bi-Implication

## Definition (Bi-Implication)

Let  $p$  and  $q$  be propositions. The *bi-implication*  $p \leftrightarrow q$  is the proposition ' $p$  if and only if  $q$ '.

The bi-implication  $p \leftrightarrow q$  is true when  $p$  and  $q$  have the same truth values, and is false otherwise.

The words 'if and only if' are sometimes abbreviated 'iff'.

$q \rightarrow p$	$p \rightarrow q$	$p$	$q$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	✓
T	F	T	F	F	✗
F	T	F	T	F	✗
F	F	F	F	T	✓

$p \leftrightarrow q$  is true when both  $p \rightarrow q$  and  $q \rightarrow p$  are true, and is false otherwise.

# More Definitions

- Tautology, Contradiction and Contingency
- Logical Equivalence
- Converse, Contrapositive and Inverse

# Tautology, Contradiction and Contingency

## Definition (Tautology, Contradiction and Contingency)

A compound proposition that is *always true*, no matter what the truth values of the propositions that occur in it, is called a **tautology**. A compound proposition that is *always false* is called a **contradiction**. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

# Exercise

How to construct a tautology, a contradiction and a contingency using just one propositional variable?  $p$

$p$	$p \vee \neg p$	$p \wedge \neg p$	$\neg p$	$p \rightarrow \neg p$
T	T	F	F	F
F	T	F	T	T
	↓ tautology	↓ contradiction	✓ contingency	

## Exercise Answer

How to construct a tautology, a contradiction and a contingency using just one propositional variable?

$p$	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$	$p \rightarrow \neg p$
T	F	T	F	F
F	T	T	F	T

[There are some other ways not shown in the table above...]

# Logical Equivalence

## Definition (Equivalence)

The compound propositions  $p$  and  $q$  are logically equivalent, if they always have the same truth value (i.e.  $p \leftrightarrow q$  is a tautology). The notation  $p \equiv q$  denotes that  $p$  and  $q$  are logically equivalent.

$p$	$q$	$p \equiv q$	$p \leftrightarrow q$
T	T	T	T
F	F	T	T

# Exercise

Truth table.

- Are  $\neg(p \vee q)$  and  $\neg p \wedge \neg q$  logically equivalent? Why?
- Are  $p \rightarrow q$  and  $\neg p \vee q$  logically equivalent? Why?

[Hint: construct truth tables]

$p$	$q$	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T



## Exercise Answer

Are  $\neg(p \vee q)$  and  $\neg p \wedge \neg q$  logically equivalent? Why?

**Answer:** Yes. As shown in the truth table below,  $\neg(p \vee q)$  and  $\neg p \wedge \neg q$  always have the same truth value. Thus, they are logical equivalent.

$p$	$q$	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

## Exercise Answer

Are  $p \rightarrow q$  and  $\neg p \vee q$  are logically equivalent? Why?

**Answer:** Yes. As shown in the truth table below,  $p \rightarrow q$  and  $\neg p \vee q$  always have the same truth value. Thus, they are logical equivalent.

$p$	$q$	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

# Converse, Contrapositive and Inverse

- The **converse** of  $p \rightarrow q$  is the proposition  $q \rightarrow p$ .
- The **contrapositive** of  $p \rightarrow q$  is the proposition  $\neg q \rightarrow \neg p$ .
- The **inverse** of  $p \rightarrow q$  is the proposition  $\neg p \rightarrow \neg q$ .

Which pairs of the following propositions are equivalent? Why?

- a conditional statement and its converse  $p \rightarrow q$      $q \rightarrow p$
- ✓ ■ a conditional statement and its contrapositive  $p \rightarrow q$      $\neg q \rightarrow \neg p$
- a conditional statement and its inverse  $p \rightarrow q$      $\neg p \rightarrow \neg q$

$p$	$q$	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T	T
T	F	F	F	T	T	F
F	T	T	T	F	F	T
F	F	T	T	T	T	T

# Some Important Logical Equivalences

	Equivalence	Name
1	$p \wedge T \equiv p$	Identity laws ✓
2	$p \vee F \equiv p$	
3	$p \vee T \equiv T$	<u>Domination laws</u> ✓
4	$p \wedge F \equiv F$	
5	$p \vee p \equiv p$	Idempotent laws ✓
6	$p \wedge p \equiv p$	
7	$\neg(\neg p) \equiv p$	Double negation law ✓
8	$p \vee q \equiv q \vee p$	Commutative laws ✓
9	$p \wedge q \equiv q \wedge p$	

# Some Important Logical Equivalences

	Equivalence	Name
10	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	<u>Associative laws</u> ✓
11	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
12	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	<u>Distributive laws</u> ✓
13	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
14	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
15	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	
16	$p \vee (p \wedge q) \equiv p$	<u>Absorption laws</u> ✓
17	$p \wedge (p \vee q) \equiv p$	
18	$p \vee \neg p \equiv T$	Negation laws ✓
19	$p \wedge \neg p \equiv F$	

# Logical Equivalences involving Implications

20	$p \rightarrow q \equiv \neg p \vee q$
21	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
22	$p \vee q \equiv \neg p \rightarrow q$
23	$p \wedge q \equiv \neg(p \rightarrow \neg q)$
24	$\neg(p \rightarrow q) \equiv p \wedge \neg q$
25	$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
26	$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
27	$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
28	$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

# Logical Equivalences involving Bi-Implications

29	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
30	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
31	$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
32	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

# Using De Morgan's Laws

Use De Morgan's laws to express the negations of the following sentences.

- Tony has a cellphone and he has a laptop computer.  $p \wedge q$
- Heather will go to the concert or Steve will go to the concert.  $r \vee s$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(r \vee s) \equiv \neg r \wedge \neg s \quad \text{by De Morgan's law.}$$



# Constructing New Logical Equivalences

- A proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.
- Prove two propositions are logically equivalent by developing a series of logical equivalences.

# Exercise

- Show that  $\neg(p \rightarrow q)$  and  $p \wedge \neg q$  are logically equivalent.
- Show that  $\neg(p \vee (\neg p \wedge q))$  and  $\neg p \wedge \neg q$  are logically equivalent.
- Show that  $(p \wedge q) \rightarrow (p \vee q)$  is a tautology.

## Exercise Answer

Show that  $\neg(p \rightarrow q)$  and  $p \wedge \neg q$  are logically equivalent.

**Answer:**

$$\begin{aligned} & \neg(p \rightarrow q) \\ \equiv & \neg(\neg p \vee q) && \text{by law 20} \\ \equiv & \neg(\neg p) \wedge \neg q && \text{by De Morgan's law} \\ \equiv & p \wedge \neg q. && \text{by double negation law} \end{aligned}$$

## Exercise Answer

Show that  $\neg(p \vee (\neg p \wedge q))$  and  $\neg p \wedge \neg q$  are logically equivalent.

**Answer:**

$$\begin{aligned} & \neg(p \vee (\neg p \wedge q)) \\ \equiv & \neg p \wedge \neg(\neg p \wedge q) && \text{De Morgan's.} \\ \equiv & \neg p \wedge (\neg(\neg p) \vee \neg q) && \text{by 1st De Morgan's} \\ \equiv & \neg p \wedge (p \vee \neg q) && \text{double negation} \\ \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{distributive law} \\ \equiv & F \vee (\neg p \wedge \neg q) && \text{by } \neg p \wedge p \equiv F \\ \equiv & \neg p \wedge \neg q. && \text{identity law} \end{aligned}$$

## Exercise Answer

Show that  $(p \wedge q) \rightarrow (p \vee q)$  is a tautology.

**Answer:**

$$\begin{aligned} & (p \wedge q) \rightarrow (p \vee q) \\ \equiv & \neg(p \wedge q) \vee (p \vee q) \\ \equiv & (\neg p \vee \neg q) \vee (p \vee q) \\ \equiv & (p \vee \neg p) \vee (q \vee \neg q) \\ \equiv & T \vee T \equiv T \end{aligned}$$

# List of Symbols

<i>SYMBOL</i>	<i>MEANING</i>	<i>PAGE</i>
$\neg p$	negation of $p$	3
$p \wedge q$	conjunction of $p$ and $q$	4
$p \vee q$	disjunction of $p$ and $q$	4
$p \oplus q$	exclusive or of $p$ and $q$	6
$p \rightarrow q$	the implication $p$ implies $q$	6
$p \leftrightarrow q$	biconditional of $p$ and $q$	9
$p \equiv q$	equivalence of $p$ and $q$	23
<b>T</b>	tautology	23
<b>F</b>	contradiction	23
$P(x_1, \dots, x_n)$	propositional function	36
$\forall x P(x)$	universal quantification of $P(x)$	38
$\exists x P(x)$	existential quantification of $P(x)$	40
$\exists! x P(x)$	uniqueness quantification of $P(x)$	41
$\therefore$	therefore	64
$p\{S\}q$	partial correctness of $S$	364

# Expected Learning Outcomes

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
- To gain a good understanding of other definitions: tautology, contradiction, contingency, logical equivalence, converse, contrapositive and inverse.
- To be able to draw truth tables and use them as a tool to solve logical problems;
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Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Sections 1.1-1.3.



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