# Lecture 2

## Topics covered in this lecture session

1. Quadratic functions and equations.
2. Exponential function.
3. Logarithmic function.





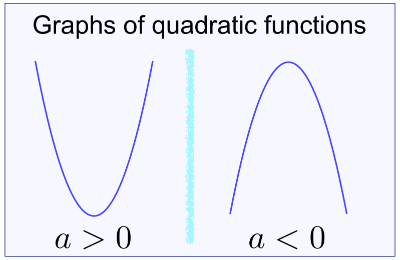
# Quadratic Equations

A general quadratic equation takes the form:

$$ax^2 + bx + c = 0 \quad ; \quad a \neq 0, \quad a, b, c \in \mathbb{R}$$

Roots are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$




Example: Solve $2x^2 + 3x - 1 = 0$.





# Quadratic Equations (Nature of roots)

The nature of roots depends on

Discriminant $\Delta = b^2 - 4ac$

| | | |
|---|---|---|
| Discriminant $\Delta = b^2 - 4ac$ | $> 0$ | Roots are real and distinct |
| | $= 0$ | Roots are real and equal (i.e. repeated roots) |
| | $< 0$ | No real roots (i.e. roots are complex numbers) |

Example: Find $k$ if roots of the equation $2x^2 + 3x + k = 0$ are equal.





# Method of completing the square

Consider sketching graph of $f(x) = x^2 + bx + c$

**Method**:

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

$$= \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$$

For $g(x) = ax^2 + bx + c$, first divide by $a$ throughout and then apply the above method.

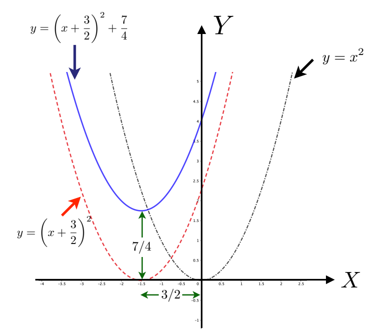
# Method of completing the square

Example: Complete the square to find the range of $f(x) = x^2 + 3x + 4$. Also sketch the graph $f$.

$$\begin{aligned} f(x) &= x^2 + 3x + 4 \\ &= \boxed{x^2 + 3x + \left(\frac{3}{2}\right)^2} - \left(\frac{3}{2}\right)^2 + 4 \\ &= \boxed{\left(x + \frac{3}{2}\right)^2} + 4 - \frac{9}{4} \\ &= \left(x + \frac{3}{2}\right)^2 + \frac{7}{4} \end{aligned}$$

Now, $\left(x + \frac{3}{2}\right)^2 \geq 0 \;\Rightarrow\; f(x) \geq \frac{7}{4}$

$\therefore$ Range of $f$ is $\left[\frac{7}{4}, \infty\right)$







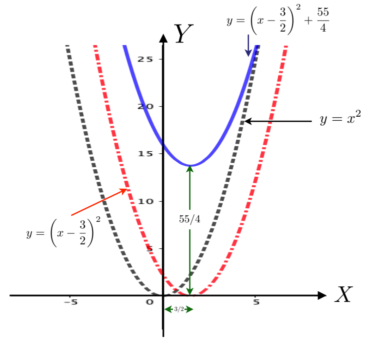

# Method of completing the square

Use method of completing the square to find the range and sketch the graph of the quadratic function $f(x) = x^2 - 3x + 16$.

$$\begin{aligned} f(x) &= x^2 - 3x + 16 \\ &= \boxed{x^2 - 3x + \left(\frac{3}{2}\right)^2} - \left(\frac{3}{2}\right)^2 + 16 \\ &= \boxed{\left(x - \frac{3}{2}\right)^2} + 16 - \frac{9}{4} \\ &= \left(x - \frac{3}{2}\right)^2 + \frac{55}{4} \end{aligned}$$

Now, $\left(x - \frac{3}{2}\right)^2 \geq 0 \;\Rightarrow\; f(x) \geq \frac{55}{4}$

$\therefore$ Range of $f$ is $\left[\frac{55}{4}, \infty\right)$








# Exponential functions

- The exponential function is the function

$$y = f(x) = a^x \quad \text{where} \quad a > 0$$

- A particularly important exponential function is

$$y = f(x) = e^x, \quad \text{where} \quad e \approx 2.718281828.$$

This is often called the exponential function.

- The exponential function is widely used in physics, chemistry, engineering, mathematical biology, economics and mathematics.

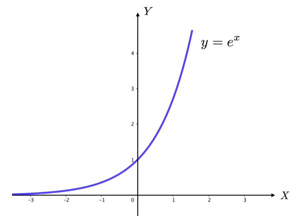


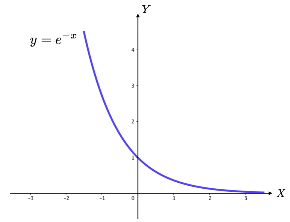


# Graph of an exponential function

Observations:

- The graph of $y = e^x$ is upward-sloping.
- It increases faster as x increases.
- It always lies above the X-axis.
- It get arbitrarily close to it for negative $x$.
- Thus, the $X$-axis is a horizontal asymptote.



# Laws of indices/exponents

$$a^0 = 1 \quad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(ab)^n = a^n \cdot b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$(a^m)^n = a^{mn} = (a^n)^m$$

$$a^{1/n} = \sqrt[n]{a}$$

In particular $a^{\frac{1}{2}} = \sqrt{a}$

and $a^{\frac{1}{3}} = \sqrt[3]{a}$

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$





# Logarithmic functions

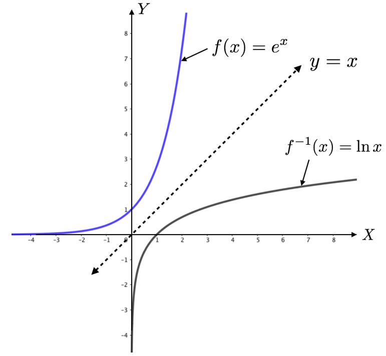
The logarithmic function is inverse of the exponential function and they are connected by the relation:

$$a^x = y \quad \Leftrightarrow \quad x = \log_a y$$

For example,

$$2^5 = 32 \quad \Leftrightarrow \quad 5 = \log_2 32$$

$$\log_3 81 = 4 \quad \Leftrightarrow \quad 81 = 3^4$$








# Bases of Logarithms

Three choices for bases of logarithms are particularly common.

$$a = 10, \quad a = e \approx 2.718281828, \quad \text{and} \quad a = 2.$$

Logarithms with base $e$ are called natural logarithms and written as $\ln x$.

In mathematical analysis, the logarithm to base $e$ is widespread.

Logarithm with base 10 is called decimal/standard logarithm and written as $\log x$.

Base 10 logarithms are easy to use for manual calculations in the decimal number system.

The logarithm to base 2 is used in Computer Science, and in Music Theory.





# Uses of Logarithms

- to quantify the loss of voltage levels in transmitting electrical signals;
- to describe power levels of sounds in acoustics;
- to determine the strength of an earthquake by measuring (on the Richter scale) the common logarithm of the energy emitted at the quake;
- to determine the brightness of stars;
- to determine pH value;
- to scale both the axes logarithmically to draw log-log graphs.

# Laws of Logarithms

$$\log_a 1 = 0 \quad (a > 0)$$

$$\log_a a = 1$$

$$\log_a (x\,y) = \log_a x + \log_a y$$

**(Product Rule)**

$$\boxed{\log_a (x + y) \;\neq\; \log_a x + \log_a y}$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

**(Quotient Rule)**

$$\boxed{\log_a (x - y) \;\neq\; \log_a x \div \log_a y}$$

$$\log_y x = \frac{\log_a x}{\log_a y}$$

**(Change of base rule)**

$$\log_b a = \frac{1}{\log_a b}$$

$$\log_a x^n = n\ \log_a x$$

**(Logarithms of Power)**

$$a^{\log_a x} = x$$





# Logarithmic functions

Example Simplify:

$$\ln 3x^2 + \ln 2x - \ln 6x^3$$

Expression = $\ln\left(\dfrac{3x^2 \,.\, 2x}{6x^3}\right)$

$= \ln 1$

$= 0$





# Logarithmic functions

Example Solve for $x$:

$$\ln(x+1) + \ln x = \ln(3x-1)$$

$$\Rightarrow \ln[x(x+1)] = \ln(3x-1)$$

$$\Rightarrow x(x+1) = 3x - 1$$

$$\Rightarrow x^2 - 2x + 1 = 0$$

$$\Rightarrow (x-1)^2 = 0 \quad \Rightarrow \quad x = 1$$





# Exponential functions

Example Solve: $e^{4-x} = 10$

$$\Rightarrow 4 - x = \ln 10$$

$$\Rightarrow 4 - x \approx 2.3026$$

$$\Rightarrow x \approx 4 - 2.3026$$

$$\Rightarrow x \approx 1.6974$$

obtained using calculator

Solve: $e^{2x} + e^x - 6 = 0$

Let $e^x = t$

$$\Rightarrow t^2 + t - 6 = 0$$

$$\Rightarrow t = 2 \quad or \quad -3$$

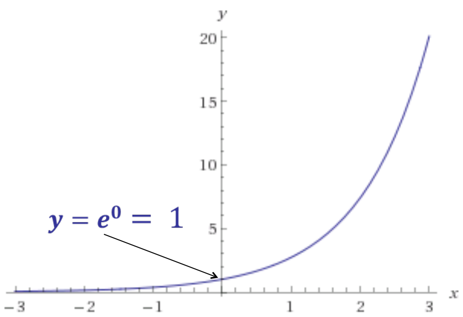
$$\Rightarrow e^x = 2 \quad or \quad \cancel{e^x = -3}$$

But, $e^x > 0$ for $x \in R$

$$\Rightarrow e^x = 2 \quad \Rightarrow x = \ln 2$$

# Sketching Log and Expo functions

Graph of $y = e^x$




Graph of $y = ln\ x$