MCS Tutorial 4
Relations and Counting

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Relations

1. Consider a relation R on the set $\mathbb{Z}^+$ defined as

$$R = \{(x, y) | x + y \text{ is even}\}$$

Show whether R is reflexive, symmetric, antisymmetric and/or transitive.

2. Let R and S be the following relations:
   - R={(1,1), (1,2),(2,4),(3,2),(4,3)}
   - S={(1,0),(2,4),(3,1),(3,2),(4,1)}

   What is the composite of the relations R and S, $S^\circ R$?

3. Let R={(1,1), (2,4), (3,4),(4,2)}. Find the powers R^2, R^3, R^4,…

$$R = \{(x, y) | x + y \text{ is even}\}$$

1. Answer:

- Since $x + x$ is even for any x, then $(x, x) \in R$ and R is **reflexive**.
- Since $x + y = y + x$, R is **symmetrical**.
- Since $4 + 2$ and $2 + 4$ are even, but $4 \neq 2$, R is **not antisymmetric**.
- Suppose $(x, y) \in R$ and $(y, z) \in R$.
  - Then, either both x and y are odd, or both are even.
  - If x and y are odd, then z must be odd $\Rightarrow$ $x + z$ is even $\Rightarrow$ $(x, z) \in R$.
  - If x and y are even, then z must be even $\Rightarrow$ $x + z$ is even $\Rightarrow$ $(x, z) \in R$.
  - Hence R is **transitive**.

- 2

Answer:

For every $(a, b) \in R$, $(b, c) \in S$ *forms* $(a, c) \in S \circ R$.

(a, b)	(b, c)	(a, c)
$(1, 1)$	$(1, 0)$	$(1, 0)$
$(1, 2)$	$(2, 4)$	$(1, 4)$
$(2, 4)$	$(4, 1)$	$(2, 1)$
$(3, 2)$	$(2, 4)$	$(3, 4)$
$(4, 3)$	$(3, 1)$	$(4, 1)$
$(4, 3)$	$(3, 2)$	$(4, 2)$

Therefore, $S \circ R = \{(1, 0), (1, 4), (2, 1), (3, 4), (4, 1), (4, 2)\}$.

- 3 ***Answer:***

Find $R^2 = R \circ R$.

(a,b)	(b,c)	(a,c)
$(1,1)$	$(1,1)$	$(1,1)$
$(2,4)$	$(4,2)$	$(2,2)$
$(3,4)$	$(4,2)$	$(3,2)$
$(4,2)$	$(2,4)$	$(4,4)$

Then, $R^2 = \{(1,1),(2,2),(3,2),(4,4)\}$.

Find $R^3 = R^2 \circ R$. $(a,b) \in R$ *and* $(b,c) \in R^2$, *then* $(a,c) \in R^3$.

(a,b)	(b,c)	(a,c)
$(1,1)$	$(1,1)$	$(1,1)$
$(2,4)$	$(4,4)$	$(2,4)$
$(3,4)$	$(4,4)$	$(3,4)$
$(4,2)$	$(2,2)$	$(4,2)$

Then, $R^3 = \{(1,1),(2,4),(3,4),(4,2)\}$.

Find $R^4 = R^3 \circ R$. $(a,b) \in R$ *and* $(b,c) \in R^2$, *then* $(a,c) \in R^3$

(a,b)	(b,c)	(a,c)
$(1,1)$	$(1,1)$	$(1,1)$
$(2,4)$	$(4,2)$	$(2,2)$
$(3,4)$	$(4,2)$	$(3,2)$
$(4,2)$	$(2,4)$	$(4,4)$

Then, $R^4 = \{(1,1),(2,2),(3,2),(4,4)\}$.

4. On the set of real numbers R define the relation S = {(x, y) : x, y ∈ R, and x − y is an integer}

(1) Show that S is an equivalence relation on R

(2) What is the equivalence class for each x ∈ R

4. Answer:

(1). There are three things to show.

- The relation is reflexive: For every real number x, $x - x$ is the integer 0.
- The relation is symmetric: For all real numbers x and y, if $x - y$ is an integer, then $y - x$ is an integer.
- The relation is transitive: For all real numbers x, y, and z, if $x - y$ is an integer and $y - z$ is an integer, then $(x - y) + (y - z) = x - z$ is an integer.

(2). For each $x \in R$, the equivalence class for x is

$$[x] = \{x + k : k \in Z\}$$

Equivalence classes

5. Consider the power set of $X = \{a, b, c\}$ and define R on the power set as follows: URV iff U and V have the same cardinality. Find the equivalence classes of R

Equivalence classes

5. Answer:

- The equivalence classes are:
- [{Ø}] = {Ø};
- [{a}] = {{a} , {b} , {c}};
- [{a, b}] = {{a, b} , {a, c} , {b, c}};
- [{a, b, c}] = {{a, b, c}}

Counting

1. Recall that a bit string is an ordered list of characters using only the digits 0 and 1.
   a) How many bit strings of length ten are there?
   b) How many bit strings of length ten have exactly three 1s?
   c) How many bit strings of length ten have exactly three 1s and none of these 1s are adjacent to each other?

Answers: Question 1

a) There are ten positions in the string where each position must be filled with one of two possibilities. Thus, there are 2^{10}=1024 ways.

b) There are C(10,3)=120 ways to designate places to choose the positions for the 1s in the string. Fill the remaining with 0s.

c) Start with a string of seven 0s. There are eight places to put the 1s so that no two of the 1s are adjacent. We can put those three 1s into the eight places in C(8,3)=56 different ways.

2. You have a combination lock with four digits (0 to 9). You set the lock so that you do not use the same digit more than once.

a) How many ways are there to set the lock?
b) If you additionally do not use four consecutive digits (ie increasing by 1 in each place; eg 1,2,3,4 or 3,4,5,6), how many ways do you have for setting the lock?
c) You now think that you will **also** allow for **any** four-digits made from the digits 0 to 4. How many ways are there to set the lock now?

Answers: Question 2

a) Four digits, order matters: P(10,4)=5040.

b) There are 7 ways to get consecutive digits (ie 0,1,2,3 to 6,7,8,9), so 5040-7 = 5033 ways.

c) Do this using the subtraction rule:
   - Number of ways to get any four digits from 0 to 4: using product rule: 5^4=625.
   - Overlap is the number of ways that 0 to 4 can be allocated without repeat: P(5,4)=120, minus the 2 ways to get consecutive digits with 0 to 4 = 118.
   - Now use the subtraction rule: 5033+625-118=5540.

Question 3

Consider sets A and B with $|A|=10$and $|B|=17$.

- How many functions $f:A\to B$ are there?
- How many functions $f:A\to B$ are injective?

Question 3

Consider sets A and B with |A|=10and |B|=17.

- How many functions f:A→B are there?
- How many functions f:A→B are injective?

Q3 Answers:

- 17^{10} functions. There are 17 choices for the image of each element in the domain.
- P(17,10) injective functions. There are 17 choices for image of the first element of the domain, then only 16 choices for the second, and so on.

Further practice and homework

Many exercises in Rosen’s textbook:-

- For Counting, try exercises in Section 6.1;
- For Permutations and Combinations, try exercises in Section 6.3.