Lecture 3 - Determinants and Matrix Inversion

COMP1046 - Maths for Computer Scientists

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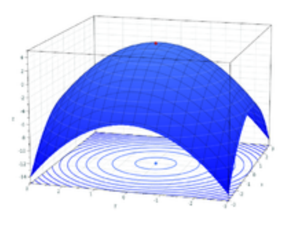


Learning outcomes

By the end of this lecture we will have learned:

- Introduction to Matrix Inversion
- Adjugate Matrix
- Determinants
- Invertible Matrices

Introduction to Matrix inversion

Inverting a matrix

- In lecture 2, we saw how we could add, subtract or take the product of matrices. It would be really useful to be able to divide matrices: something like $\mathbf{A}/\mathbf{B}$.
- We can do that through the matrix inverse, written $\mathbf{B}^{-1}$. The equivalent of a "divide" is then $\mathbf{AB}^{-1}$.

Definition

For a square matrix $\mathbf{A}$, the *inverse matrix* $\mathbf{A}^{-1}$ is defined as the matrix for which

$$\mathbf{AA}^{-1} = \mathbf{I}$$

If we take the special case of $\mathbf{A} = (a_{1,1})$ and $\mathbf{A}^{-1} = (b_{1,1})$, this makes sense since it is only true if $b_{1,1} = \frac{1}{a_{1,1}}$.

What is the Inverse?

The definition on the previous slide raises a number of questions:-

1. Can we calculate the inverse $\mathbf{A}^{-1}$ exactly, in the general case?
2. Does an inverse exist for all square matrices?
3. Is there a unique inverse where an inverse does exist? (if no, then there are multiple solutions).

The short answers to these questions are “Yes”, “No”, “Yes”. This lecture is about proving these answers.

Inverting an order 2 square matrix

Consider the special case, $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then the inverse is given as $\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example

Take $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then the inverse is $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ since

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise: Prove the result for any order 2 matrix.

Formula for matrix inverse

- There is a formula to compute the inverse for any square matrix, in general:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj}(\mathbf{A})$$

  where det is the determinant and adj is the adjugate matrix of $\mathbf{A}$.
- The remainder of the lecture we will define and explore these concepts and finally we can prove this result.
- The formula shows us when an inverse does not exist: i.e. when $\det \mathbf{A} = 0$.

Determinants and adjugate matrices

- There is a precise definition of the determinant det $\mathbf{A}$ based on permutations of elements of $\mathbf{A}$ (see Neri 2019, section 2.4).
- However, in this lecture we will work with the I Laplace Theorem which expresses the determinant in terms of the adjugate matrix.
- However, the adjugate matrix is defined in terms of the determinant.
- Hence det and adj form a *recursive* relationship.

Adjugate Matrices

Submatrices

Definition

Submatrices.

Let us consider a matrix $\mathbf{A} \in \mathbb{R}_{m,n}$. Let r, s be two positive integer numbers such that $1 \leq r \leq m$ and $1 \leq s \leq n$.

A *submatrix* is a matrix obtained from $\mathbf{A}$ by cancelling $m - r$ rows and $n - s$ columns.

Submatrices

Example

Let us consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 3 & 3 & 1 & 0 \\ 2 & 4 & 1 & 2 \\ 5 & 1 & 1 & 1 \end{pmatrix}.$$

The submatrix obtained by cancelling the second row, the second and fourth columns is

$$\begin{pmatrix} 3 & 1 \\ 5 & 1 \end{pmatrix}.$$

Minors and Majors

Definition

Let us consider a matrix $\mathbf{A} \in \mathbb{R}_{m,n}$ and one of its square submatrices.
The determinant of this submatrix is said *minor*.
If the submatrix is the largest square submatrix of the matrix $\mathbf{A}$, its determinant is said *major determinant* or simply *major*.

It must be observed that a matrix can have multiple majors.

Minors and Majors

Example

Let us consider the following matrix $\mathbf{A} \in \mathbb{R}_{4,3}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

- An example of minor is $\det \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$ obtained after cancelling the 2^{nd} column as well as the 3^{rd} and 4^{th} rows.
- Several minors can be calculated, including several majors which must all be 3×3 matrices.

Complement submatrices and minors

Definition

Let us consider a matrix $\mathbf{A} \in \mathbb{R}_{n,n}$.

The submatrix obtained by cancelling only the i^{th} row and the j^{th} column from $\mathbf{A}$ is said *complement submatrix* to the element $a_{i,j}$ and its determinant is here named *complement minor* and indicated with $M_{i,j}$.

Complement submatrices and minors

Example

Let us consider a matrix $\mathbf{A} \in \mathbb{R}_{3,3}$:

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

The complement submatrix to the element $a_{1,2}$ is

$$\begin{pmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{pmatrix}$$

and the complement minor $M_{1,2} = \det \begin{pmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{pmatrix}$.

Cofactors

Definition

Let us consider a matrix $\mathbf{A} \in \mathbb{R}_{n,n}$, its generic element $a_{i,j}$ and corresponding complement minor $M_{i,j}$. The *cofactor* $A_{i,j}$ of the element $a_{i,j}$ is defined as $A_{i,j} = (-1)^{i+j} M_{i,j}$.

Example

From the matrix of the previous example, the cofactor $A_{1,2} = (-1) M_{1,2}$.

Adjugate Matrices

Definition

Adjugate Matrix. Let us consider a matrix $\mathbf{A} \in \mathbb{R}_{n,n}$:

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}.$$

Let $A_{i,j}$ be the cofactor for $a_{i,j}$.
The *adjugate matrix* (or adjunct or adjoint) $\mathbf{A}$ is

$$\mathrm{adj}(\mathbf{A}) = \begin{pmatrix} A_{1,1} & A_{2,1} & \dots & A_{n,1} \\ A_{1,2} & A_{2,2} & \dots & A_{n,2} \\ \dots & \dots & \dots & \dots \\ A_{1,n} & A_{2,n} & \dots & A_{n,n} \end{pmatrix}.$$

Determinants

I Laplace Theorem

Theorem

I Laplace Theorem *Let* $\mathbf{A} \in \mathbb{R}_{n,n}$. *The determinant of* $\mathbf{A}$ *can be computed as the sum of each row (element) multiplied by the corresponding cofactor:*

$\det \mathbf{A} = \sum_{j=1}^{n} a_{i,j} A_{i,j}$ *for any arbitrary* i *and*

$\det \mathbf{A} = \sum_{i=1}^{n} a_{i,j} A_{i,j}$ *for any arbitrary* j.

Not proved here.

With this theorem and knowing that the determinant of a one-element matrix is just that element ($\det(a) = a$), we can compute the determinant of any square matrix.

Determinant of an order 2 square matrix

Consider $\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$.

From I Laplace Theorem, taking $i = 1$,

$$\begin{aligned} \det \mathbf{A} &= a_{1,1}A_{1,1} + a_{1,2}A_{1,2} \\ &= a_{1,1}.(-1)^{1+1}M_{1,1} + a_{1,2}.(-1)^{1+2}M_{1,2} \\ &= a_{1,1}\det(a_{2,2}) - a_{1,2}\det(a_{2,1}) \\ &= a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \end{aligned}$$

Example

Let us consider the following $\mathbf{A} \in \mathbb{R}_{3,3}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

If we consider the second row, it follows that

$$\begin{aligned} \det \mathbf{A} &= a_{2,1}(-1)M_{2,1} + a_{2,2}M_{2,2} + a_{2,3}(-1)M_{2,3} \\ &= -0 \times \det \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} + 1 \times \det \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} - 1 \times \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \\ &= 0 - 4 - 1 \times -6 = 2 \end{aligned}$$

Matrix inversion

Invertible Matrices

Definition

Let $\mathbf{A} \in \mathbb{R}_{n,n}$. The matrix $\mathbf{A}$ is said *invertible* if $\exists$ a matrix $\mathbf{B} \in \mathbb{R}_{n,n} | \mathbf{AB} = \mathbf{I} = \mathbf{BA}$. The matrix $\mathbf{B}$ is said *inverse* matrix of the matrix $\mathbf{A}$.

Uniqueness of invertability

Theorem

If $\mathbf{A} \in \mathbb{R}_{n,n}$ *is an invertible matrix and* $\mathbf{B}$ *is its inverse, it follows that the inverse matrix is unique:* $\exists!\mathbf{B} \in \mathbb{R}_{n,n} | \mathbf{AB} = \mathbf{I} = \mathbf{BA}$.

Proof.

Let us assume by contradiction that the inverse matrix is not unique. Thus, besides $\mathbf{B}$, there exists another inverse of $\mathbf{A}$, indicated as $\mathbf{C} \in \mathbb{R}_{n,n}$.

This would mean that for the hypothesis $\mathbf{B}$ is inverse of $\mathbf{A}$ and thus $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

For the contradiction hypothesis also $\mathbf{C}$ is inverse of $\mathbf{A}$ and thus $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$.

continued...

Uniqueness of invertability

Proof.

Considering that $\mathbf{I}$ is the neutral element with respect to the product of matrices ($\forall \mathbf{A} : \mathbf{AI} = \mathbf{IA} = \mathbf{A}$) and that the product of matrices is associative, it follows that

$$\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}.$$

In other words, if $\mathbf{B}$ is an inverse matrix of $\mathbf{A}$ and another inverse matrix $\mathbf{C}$ exists, then $\mathbf{C} = \mathbf{B}$. Thus, the inverse matrix is unique. $\square$

Theorem

II Laplace Theorem

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ *with* $n > 1$. *The sum of the elements of a row (column) multiplied by the corresponding cofactor related to another row (column) is always zero:*

$\sum_{j=1}^{n} a_{i,j} A_{k,j} = 0$ *for any arbitrary* $k \neq i$ *and*
$\sum_{i=1}^{n} a_{i,j} A_{i,k} = 0$ *for any arbitrary* $k \neq j$.

Not proved here.

We now have everything in place to show what the inverse matrix is.

Example

Consider $\mathbf{A} = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 2 & 1 \\ 4 & 2 & 5 \end{pmatrix}$.

Take rows $i = 3$ and $k = 2$:

$$\begin{aligned} \sum_{j=1}^{n} a_{i,j} \underline{A_{k,j}} &= (+1) \times 4 \det \begin{pmatrix} 0 & 2 \\ 2 & 5 \end{pmatrix} + (-1) \times 2 \det \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \\ &\quad + (+1) \times 5 \det \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix} \\ &= 4 \times -4 - 2 \times 7 + 5 \times 6 \\ &= 0 \end{aligned}$$

The Inverse Matrix

Theorem

Let $\mathbf{A} \in \mathbb{R}_{n,n}$ and $A_{i,j}$ its generic cofactor.
The inverse matrix $\mathbf{A}^{-1}$ is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj}(\mathbf{A}).$$

Proof.

Let us consider the matrix $\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}$.

continued...

Proof.

Its adjugate matrix is $\operatorname{adj}(\mathbf{A}) = \begin{pmatrix} A_{1,1} & A_{2,1} & \dots & A_{n,1} \\ A_{1,2} & A_{2,2} & \dots & A_{n,2} \\ \dots & \dots & \dots & \dots \\ A_{1,n} & A_{2,n} & \dots & A_{n,n} \end{pmatrix}$.

Let us compute the product matrix $\mathbf{P} = (\mathbf{A})\left(\operatorname{adj}(\mathbf{A})\right)$.

$$\mathbf{P}_{i,k} = a_{i,1}A_{k,1} + a_{i,2}A_{k,2} + \dots + a_{i,n}A_{k,n} = \sum_{j=1}^{n} \underline{a_{i,j}A_{k,j}}$$

continued...

Proof.

For the I Laplace Theorem, the diagonal elements are equal to $\det \mathbf{A}$:

$$\mathbf{P}_{i,i} = \sum_{j=1}^{n} a_{i,j} A_{i,j} = \det \mathbf{A} \quad \text{for all the rows } i.$$

For the II Laplace Theorem, the extra-diagonal elements are equal to zero:

$$\mathbf{P}_{i,k} = \sum_{j=1}^{n} a_{i,j} A_{k,j} = 0 \quad \text{with } i \neq k.$$

continued...

Proof.

The result of the multiplication is then

$$(\mathbf{A})\left(\operatorname{adj}(\mathbf{A})\right) = \mathbf{P} = \begin{pmatrix} \det \mathbf{A} & 0 & \dots & 0 \\ 0 & \det \mathbf{A} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \det \mathbf{A} \end{pmatrix}.$$

Thus,

$$(\mathbf{A})\left(\operatorname{adj}(\mathbf{A})\right) = (\det \mathbf{A})\,\mathbf{I}$$

and

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj}(\mathbf{A}).$$

$\square$

Matrix inversion

Example

Let us now invert a matrix $\mathbf{A} \in \mathbb{R}_{3,3}$:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$$

The determinant of this matrix is $\det \mathbf{A} = 2 - 1 = 1$.

The transpose of this matrix is

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}.$$

continued...

Matrix inversion

Example

The adjugate matrix is

$$\operatorname{adj}(\mathbf{A}) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & -5 & 2 \end{pmatrix}$$

and the corresponding inverse matrix is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj}(\mathbf{A}) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & -5 & 2 \end{pmatrix}.$$

Inversion and non-singularity

Definition

Let $\mathbf{A} \in \mathbb{R}_{n,n}$. If $\det \mathbf{A} = 0$ the matrix is said *singular*. If $\det \mathbf{A} \neq 0$ the matrix is said *non-singular*.

Theorem

Let $\mathbf{A} \in \mathbb{R}_{n,n}$. *The matrix* $\mathbf{A}$ *is invertible if and only if* $\mathbf{A}$ *is non-singular.*

The proof is given in Neri (2019). However, the intuition is that the inverse can be computed except when $\det \mathbf{A} = 0$.

Inverse of matrix product

Proposition

Let $\mathbf{A}$ and $\mathbf{B}$ be two square and invertible matrices. it follows that

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

Proof.

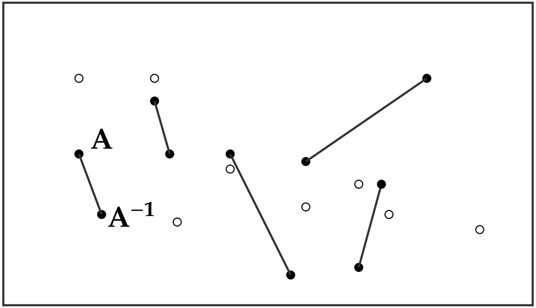
Let us calculate $(\mathbf{AB})\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right) = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$.
Thus, the inverse of $(\mathbf{AB})$ is $\left(\mathbf{B}^{-1}\mathbf{A}^{-1}\right)$, i.e.

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

□

Summary

This illustration represents and summarizes our matrix theory studies so far.



- The empty circles represent singular matrices which are not linked to other matrices since they have no inverse.
- Conversely, filled circles represent non-singular matrices, each with just one inverse.

Lecture 3: Exercises

Work on these problems with reference to the definitions given in Lecture 3.

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 0 & -1 & 2 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$.

Q1. What is the submatrix of $\mathbf{B}$ when the 2nd row and the 2nd and 4th columns are cancelled?

Q2. Compute the minor for this submatrix.

Q3. What is the complement submatrix of $a_{2,3}$ from $\mathbf{A}$?

Q4. Compute the complement minor of $a_{2,3}$ from $\mathbf{A}$.

Q5. Compute the cofactor of $a_{2,3}$ from $\mathbf{A}$.

Q6. Compute the adjugate matrix $\text{adj}(\mathbf{A})$.

Q7. Compute $\mathbf{A}(\text{adj}(\mathbf{A}))$.

Q8. What do you think the value on the leading diagonal is?

Summary and next lecture

Summary

- Introduction to Matrix Inversion
- Adjugate Matrix
- Determinants
- Invertible Matrices

The next lecture

We will learn about linear dependencies.