Tutorial 3
Sets and Functions

Huan Jin

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Sets

Power sets

How many elements does each of these sets have where a and b are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$

b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$

c) $\mathcal{P}(\mathcal{P}(\emptyset))$

Let A be a set, and the elements of A be sets.

Define $\bigcup A = \{x \mid \exists y \in A, x \in y\}$.

1) Calculate $\bigcup \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\}$;

2) Prove that $\bigcup P(A) = A$;

3) Whether $P(\bigcup A) = A$?

$$\bigcup A = \{x \mid \exists y \in A, x \in y\}.$$

Answer:

1). Calculate $\bigcup\{\{a,b,c\},\{a,d,e\},\{a,f\}\}$;

Solution: $\bigcup\{\{a,b,c\},\{a,d,e\},\{a,f\}\}$

$$= \{a,b,c\} \cup \{a,d,e\} \cup \{a,f\}$$

$$= \{a,b,c,d,e,f\}$$

$$\bigcup A = \{x \mid \exists y \in A, x \in y\}.$$

2). Prove that $\bigcup P(A) = A$;

Proof: $\subseteq$: $\forall x \in \bigcup P(A)$, there exists y such that $y \in P(A)$ and $x \in y$;

Since $y \in P(A)$,

$y \subseteq A$ (by definition of power set)

Given $x \in y, x \in A$, we have $\bigcup P(A) \subseteq A$;

$\supseteq$: $\forall x \in A$, let $y = \{x\}$, then $y \subseteq A$ by definition of subset

y $\in P(A)$ by definitoin of power set

so $x \in \bigcup P(A)$ by definitoin of $\bigcup$,

we prove $A \subseteq \bigcup P(A)$.

$$\bigcup A = \left\{ x \mid \exists y \in A, x \in y \right\}.$$

3). Whether $P(\bigcup A) = A$?

Answer: No!

Counter example:

$$A = \left\{ \left\{ a, b, c \right\}, \left\{ a, d, e \right\}, \left\{ a, f \right\} \right\}$$

$$\bigcup A = \{a, b, c, d, e, f\}$$

$$\{a, e\} \in P(\bigcup A), \quad \text{but} \quad \{a, e\} \notin A$$

$$\text{so } P(\bigcup A) \neq A$$

- Let A and B be sets, prove

$$A \cap B = A \subseteq \overline{B}$$

We split task into two subtasks.

The ***first*** *is to prove that* $A \cap B = \emptyset \to A \subseteq \overline{B}$.

By way of contradiction, suppose $A \cap B = \emptyset$ *and* $A \not\subseteq \overline{B}$.

If $A \not\subseteq \overline{B}$, *we can find an* x, *such that* $x \in A$ *and* $x \notin \overline{B}$.

But,

$x \in A \wedge x \notin \overline{B}$

therefore, $x \in A \wedge x \in B$ *(by definition of complement)*

therefore, $x \in A \cap B$ *(by definition of intersection)* $\to A \cap B \neq \emptyset$

This leads to a contradiction.

The ***second*** *task is to prove that* $A \subseteq \overline{B} \to A \cap B = \emptyset$

Again, by way of contradiction, suppose $A \subseteq \overline{B}$ *and* $A \cap B \neq \emptyset$

If $A \cap B \neq \emptyset$ *there exists an* x, *such that,* $x \in A \cap B$.

But,

$x \in A \cap B$

therefore, $x \in A \wedge x \in B$ *(by definition of intersection)*

therefore, $x \in A \wedge x \notin \overline{B}$ *(by definition of complement)*$\to A \not\subseteq \overline{B}$

This leads to a contradiction.

Finally, we have proved: $A \cap B = \emptyset$ *if and only if* $A \not\subseteq \overline{B}$

- Let A, B, and C be any sets, show that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

We can prove $X = Y$ *by showing that if* $a \in X$ *then* $a \in Y$*, and if* $a \notin X$ *then* $a \notin Y$.

Let $a = (x, y)$*, and suppose* $a \in A \times (B \cup C)$.

$(x, y) \in A \times (B \cup C)$

therefore, $x \in A \wedge y \in (B \cup C)$ *(by definition of Cartesian Products)*

therefore, $x \in A \wedge (y \in B \vee y \in C)$ *(by definition of union)*

therefore, $(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$ *(by distributive law)*

therefore, $(x, y) \in A \times B \vee (x, y) \in A \times C$ *(by definition of Cartesian Products)*

therefore, $(x, y) \in (A \times B) \cup (A \times C)$ *(by definition of union)*

Therefore, $a \in (A \times B) \cup (A \times C)$

Let $a = (x, y)$*, and suppose* $a \notin A \times (B \cup C)$.

$= (x, y) \notin A \times (B \cup C)$

therefore, $x \notin A \vee y \notin (B \cup C)$ *(by definition of Cartesian Products)*

therefore, $x \notin A \vee (y \notin B \wedge y \notin C)$ *(by definition of union)*

therefore, $(x \notin A \vee y \notin B) \wedge (x \notin A \vee y \notin C)$ *(by distributive law)*

therefore, $(x, y) \notin A \times B \wedge (x, y) \notin A \times C$ *(by definition of Cartesian Products)*

therefore, $(x, y) \notin (A \times B) \cup (A \times C)$ *(by definition of union)*

Therefore, $a \notin (A \times B) \cup (A \times C)$

We have finally proved that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Functions and Sequences

Functions

- Show that the function $f(x) = |x|$ from the set of real numbers to the set of nonnegative real numbers is not invertible, but if the domain is restricted to the set of non-negative real numbers, the resulting function is invertible.

Answer:

For a function to be invertible, it needs to be bijective. Therefore, we need to check if this is one-to-one (injective) and onto (surjective).

(a) Injective: No:

For some $x_1 \neq x_2$, say $x_1 = -x_2$, we have $f(x_1) = f(x_2)$, by definition of injective, the function is not injective.

If the domain is restricted to the set of nonnegative real numbers, f(x) is injective because

Assume $f(x_1) = f(x_2)$, we have $x_1 = x_2$

Therefore, on the restricted domain f(x) is injective.

(b).Surjective

For some element $b \in rng(f)$, that $b = |a|$, with $a \in R$, b must be positive. Thus, the range is the set of all nonnegative real numbers. Because the range and codomain are the same, we can conclude that f is surjective.

(c) Bijective: No, because it is not injective. Though, on the restricted domain, it is bijective because it is both injective and surjective.

d) Invertible: Again, only on the restricted domain.

- How can we produce the terms of a sequence if the first 10 terms are 1,3,4,7,11,18,29,47,76,123?

- How can we produce the terms of a sequence if the first 10 terms are 1,3,4,7,11,18,29,47,76,123?

- Solution:

$L_n = L_{n-1} + L_{n-2}$, with initial condition $L_1 = 1$, $L_2 = 3$

Additional exercises on the textbook

- Section 2.1 : 7 11 13 21-27 33 39 43
- Section 2.2 : 5-10 21-23 29-31 37-43
- Section 2.3: 7,15,23-27, 33,35,41,45-47,53,59,71,73
- Section 2.4: 1-4,6-7,9