# Lecture 4 - Linear Dependency and Rank

## COMP1046 - Maths for Computer Scientists

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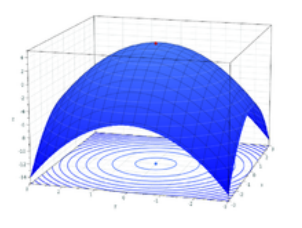

# Learning outcomes

**By the end of this lecture we will have learned:**

- Linear dependency
- Rank of a matrix

Based on Sections 2.4.1 and 2.7 of text book (Neri 2018).

# Linear Dependency

# Linear Combinations of Rows

## Definition

Let $\mathbf{A}$ be a matrix. The $i^{th}$ row is said *linear combination* of the other rows if each of its elements $a_{i,j}$ can be expressed as weighted sum of the other elements of the $j^{th}$ column by means of the same scalars $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots \lambda_n$:

$$\mathbf{a_i} = \lambda_1 \mathbf{a_1} + \lambda_2 \mathbf{a_2} + \cdots + \lambda_{i-1} \mathbf{a_{i-1}} + \lambda_{i+1} \mathbf{a_{i+1}} + \ldots + \lambda_n \mathbf{a_n}.$$

# Linear Combinations of Rows

## Example

Let us consider the following matrix: $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & 1 \\ 6 & 5 & 3 \end{pmatrix}$.

The third row is a linear combination of the first two by means of scalars $\lambda_1, \lambda_2 = 1, 2$, the third row is equal to the weighted sum obtained by multiplying the first row by 1 and summing to it the second row multiplied by 2:

$$(6, 5, 3) = (0, 1, 1) + 2\,(3, 2, 1)$$

that is

$$\mathbf{a_3} = \mathbf{a_1} + 2\mathbf{a_2}.$$

# Linear Combinations of Columns

## Definition

Let $\mathbf{A}$ be a matrix. The $j^{th}$ column is said *linear combination* of the other column if each of its element $a_{i,j}$ can be expressed as weighted sum of the other elements of the $i^{th}$ row by means of the same scalars $\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots \lambda_n$:

$$\mathbf{a^j} = \lambda_1 \mathbf{a^1} + \lambda_2 \mathbf{a^2} + \cdots + \lambda_{j-1} \mathbf{a^{j-1}} + \lambda_{j+1} \mathbf{a^{j+1}} + \ldots + \lambda_n \mathbf{a^n}.$$

# Linear Combinations of Columns

## Example

Let us consider the following matrix: $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \\ 1 & 3 & 0 \end{pmatrix}$.

The third column is a linear combination of the first two by means of scalars $\lambda_1, \lambda_2 = 3, -1$, the third column is equal to the weighted sum obtained by multiplying the first column by 3 and summing to it the second row multiplied by $-1$:

$$\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}.$$

that is

$$\mathbf{a^3} = 3\mathbf{a^1} - \mathbf{a^2}.$$

# Linear Dependency

**Definition**

Let $\mathbf{A} \in \mathbb{R}_{m,n}$ be a matrix. The $m$ rows ($n$ columns) are *linearly dependent* if a row (column) composed of all zeros $\mathbf{o} = (0, 0, \ldots, 0)$ can be expressed as the linear combination of the $m$ rows ($n$ columns) by means of non-null scalars (i.e. at least one is non-null).

# Linear Dependency

## Example

The rows in the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \\ 4 & 6 & 6 \end{pmatrix}$$

are linearly dependent since

$$\mathbf{o} = -2\mathbf{a_1} - \mathbf{a_2} + \mathbf{a_3}$$

That is a null row can be expressed as the linear combination of the row vector by means of $\lambda_1, \lambda_2, \lambda_3 = -2, -1, 1$.

# Linear Dependency and Singularity

**Proposition**

*Let* $\mathbf{A} \in \mathbb{R}_{n,n}$ *be a matrix and* $\det \mathbf{A}$ *its determinant.*
*The determinant of the matrix is zero if and only if the rows (columns) are linearly dependent.*

Not proved here.

This proposition links linear dependency to singularity of the determinant to matrix non-invertibility.

# Linear Dependency and Singularity

## Example

Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

- The determinant is $1 \times 4 - 2 \times 2 = 0$.
- The rows are linearly dependent since

$$\mathbf{o} = 2\mathbf{a_1} - \mathbf{a_2}$$

  with $\lambda_1, \lambda_2 = 2, -1$.

# Rank of a Matrix

# Rank

## Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$ with $\mathbf{A}$ assumed to be different from the null matrix.
The *rank* of the matrix $\mathbf{A}$, indicated as $\rho_{\mathbf{A}}$, is the highest order of the non-singular submatrix $\mathbf{A}_\rho \subset \mathbf{A}$.
If $\mathbf{A}$ is the null matrix then its rank is taken equal to 0.

## Example

The rank of the matrix $\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 0 \end{pmatrix}$ is 2 as the submatrix $\begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$ is non-singular (determinant is non-zero).

# Rank and Linear Dependency

**Theorem**

*Let $\mathbf{A} \in \mathbb{R}_{n,n}$ and $\rho$ its rank. The matrix $\mathbf{A}$ has $\rho$ linearly independent rows (columns).*

**Proof.**

Let us prove this theorem for the rows. For linearly independent columns the proof would be analogous.

For $\rho$, $\det \mathbf{A}_\rho \neq 0$ for at least one submatrix $\mathbf{A}_\rho$ of order $\rho$, by definition. Hence all rows are linearly independent (see slide 9) and so there are $\rho$ linearly independent rows.

Consider all $s$ such that $\rho < s \leq n$. Then all square submatrices $\mathbf{A_s}$ of order $s$ have $\det \mathbf{A_s} = 0$, by definition. Hence the rows must be linearly dependent (see slide 9). □

# Rank and Linear Dependency

## Example

Let us consider the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix}.$$

We can easily verify that $\det \mathbf{A} = 0$ and that the rank of the matrix is $\rho = 2$. We can observe that the third row is sum of the other two rows:

$$\mathbf{a_3} = \mathbf{a_1} + \mathbf{a_2}$$

that is the rows are linearly dependent. On the other hand, any two rows are linearly independent.

# Summary and next lecture

## Summary

- Linear dependency
- Rank of a matrix

## The next lecture

We will learn about systems of linear equations.