

# FCMT SAMPLE EXAM

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## SOLUTIONS

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$$\begin{aligned}
 1. (a) \quad \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x-1} - \sqrt{x+h-1}}{h(\sqrt{x+h-1})(\sqrt{x-1})} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{x-1} - \sqrt{x+h-1}}{h(\sqrt{x+h-1})(\sqrt{x-1})} \cdot \frac{(\sqrt{x-1} + \sqrt{x+h-1})}{(\sqrt{x-1} + \sqrt{x+h-1})} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{h[\sqrt{x+h-1} + \sqrt{x-1}] [\sqrt{x-1} + \sqrt{x+h-1}]} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{x+h-1} \cdot \sqrt{x-1} (\sqrt{x-1} + \sqrt{x+h-1})} \\
 &= \frac{-1}{2(x-1) \cdot \sqrt{x-1}} \\
 \therefore \quad \frac{dy}{dx} &= -\frac{1}{2} (x-1)^{-3/2}
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad \frac{dy}{dx} &= e^x \cdot \frac{d}{dx}(3x^3 - 4x) + (3x^3 - 4x) \cdot \frac{d}{dx} e^x \\
 &= e^x \cdot (9x^2 - 4) + (3x^3 - 4x) \cdot e^x \\
 &= e^x (3x^3 + 9x^2 - 4x - 4)
 \end{aligned}$$

(c)

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{e^{3x} \cdot \frac{d}{dx}(\sin^2 x) - \sin^2 x \cdot \frac{d}{dx}(e^{3x})}{(e^{3x})^2} \\
 &= \frac{e^{3x}(2\sin x \cos x) - 3e^{3x} \cdot \sin^2 x}{(e^{3x})^2} \\
 &= \frac{\sin 2x - 3\sin^2 x}{e^{3x}}
 \end{aligned}$$

(d)

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{\sin(\sqrt{3x^3-4x+1})} \cdot \frac{d}{dx}[\sin(\sqrt{3x^3-4x+1})] \\
 &= \frac{1}{\sin(\sqrt{3x^3-4x+1})} \cdot \cos(\sqrt{3x^3-4x+1}) \cdot \frac{d}{dx}\sqrt{3x^3-4x+1} \\
 &= \cot(\sqrt{3x^3-4x+1}) \cdot \frac{1}{2\sqrt{3x^3-4x+1}} \cdot \frac{d}{dx}(3x^3-4x+1) \\
 &= \frac{\cot(\sqrt{3x^3-4x+1})}{2\sqrt{3x^3-4x+1}} \cdot (9x^2-4) \\
 &= \frac{(9x^2-4) \cdot \cot(\sqrt{3x^3-4x+1})}{2\sqrt{3x^3-4x+1}}
 \end{aligned}$$

$$(e) \frac{d}{dx} \{ \tan(xy) \} = \frac{d}{dx} (xy + xy^2)$$

$$\sec^2(xy) \left[ x \frac{dy}{dx} + y \right] = \left[ x \frac{dy}{dx} + y + 2xy \frac{dy}{dx} + y^2 \right]$$

$$\frac{dy}{dx} \left[ x \cdot \sec^2(xy) - x - 2xy \right] = y^2 + y - y \sec^2(xy)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(y - \sec^2(xy) + 1)}{x(\sec^2(xy) - 2y - 1)}$$

2 (a)

$$\ln y = \ln(x^2 - x + 3)^3 - \ln(x^3 + 2)^{1/2} - \ln(x-2)^4$$

$$\ln y = 3\ln(x^2 - x + 3) - \frac{1}{2}\ln(x^3 + 2) - 4\ln(x-2)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{x^2 - x + 3} \cdot (2x-1) - \frac{1}{2} \frac{1}{(x^3+2)} \cdot (3x^2) - \frac{4}{x-2}$$

$$= \frac{3(2x-1)}{x^2 - x + 3} - \frac{3x^2}{2(x^3+2)} - \frac{4}{x-2}$$

$$\therefore \frac{dy}{dx} = \frac{(x^2 - x + 3)}{\sqrt{x^3 + 2} \cdot (x-2)^4} \left[ \frac{3(2x-1)}{x^2 - x + 3} - \frac{3x^2}{2(x^3+2)} - \frac{4}{x-2} \right]$$

(b) (i)

$$\frac{dx}{dt} = \cos t + \sin t$$

$$\frac{dy}{dt} = \sec^2 t$$

$$\therefore \frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{\sec^2 t}{\cos t + \sin t}$$

$$(i) \text{ at } t = \frac{\pi}{4} : x = 0 \text{ and } y = 1$$

$$\left. \frac{dy}{dx} \right|_{t=\pi/4} = \frac{\sec^2\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$$

$$\therefore (y-1) = \sqrt{2}(x-0)$$

$\Rightarrow y - \sqrt{2}x - 1 = 0$  : equation of tangent line.

(c)

$$C = 2\pi r$$

$$\frac{dc}{dt} = \frac{d}{dr}(2\pi r) \cdot \frac{dr}{dt}$$

$$\frac{dc}{dt} = 2\pi \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{dc}{dt} / 2\pi = \frac{1}{4} / 2\pi$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{8\pi}$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt}$$

$$= 2\pi r \cdot \left( \frac{dr}{dt} \right)$$

$$\left. \frac{dA}{dt} \right|_{r=2} = 2\pi (2) \cdot \frac{1}{8\pi}$$

$$= \frac{1}{2} \text{ cm}^2/\text{s}$$

(d)

$$f'(x) = 15x^4 + 12x^2$$

for all  $x \in \mathbb{R}$   $f'(x) \geq 0$

$\therefore f(x)$  is always increasing

3 (a)

(i) stationary points are at  $f'(x) = 0$

$$3x^2 + 6x - 9 = 0 \Rightarrow (x+3)(x-1) = 0$$

$$\therefore x = -3 \text{ or } x = 1$$

$$\text{Now, } x = -3 \Rightarrow f(-3) = 28$$

$$x = 1 \Rightarrow f(1) = -4$$

$\therefore (-3, 28) \text{ & } (1, -4)$  are stationary points

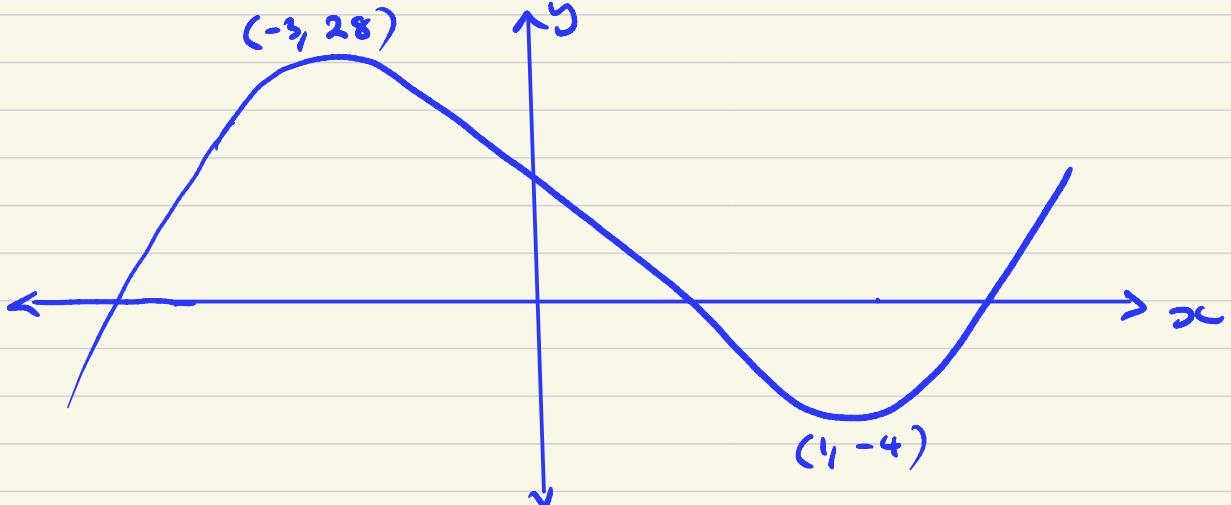
(ii)  $f''(x) = 6x+6$

$$f''(x) \Big|_{x=-3} = -12 < 0$$

$\Rightarrow (-3, 28)$  is a point of maximum value

$$f''(x) \Big|_{x=1} = 12 > 0$$

$\Rightarrow (1, -4)$  is a point of minimum value.



(b)

$$f'(x) = 3x^2 - 4$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left( \frac{x_n - 4x_n + 1}{3x_n^2 - 4} \right)$$

$$\Rightarrow x_{n+1} = \frac{3x_n^3 - 4x_n - x_n^3 + 4x_n - 1}{3x_n^2 - 4}$$

$$= \frac{2x_n^3 - 1}{3x_n^2 - 4}$$

| n | $x_n$   |
|---|---------|
| 0 | 2       |
| 1 | 1.87500 |
| 2 | 1.86098 |
| 3 | 1.86081 |
| 4 | 1.86081 |

$$\therefore x_{\text{root}} = 1.86081$$

4(a) (i)

$$f(x) = \ln(1+x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$$

$$f^{(4)}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{(4)}(0) = -6$$

$$f^{(5)}(x) = \frac{24}{(1+x)^5} \Rightarrow f^{(5)}(0) = 24$$

$$\therefore f(x) = \ln(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) \\ + \frac{x^5}{5!}(24) + \dots$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

$$g(x) = \ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} + \dots$$

$$\therefore \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \dots$$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) \\ = 2 \left\{ x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right\}$$

$$\Rightarrow \ln\left(\frac{1+x}{1-x}\right) = 2 \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}$$

(b) (i)

$$I = \int (e^{2x} - e^x) dx$$

$$= \frac{1}{2} e^{2x} - e^x + C$$

(ii)

$$I = \int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\cos 2x}{(\frac{1}{2} \sin 2x)^2} dx$$

$$= 4 \int \frac{\cos 2x}{\sin^2 2x} dx$$

$$= 4 \int \frac{\cos 2x}{\sin 2x} \cdot \frac{1}{\sin 2x} dx$$

$$= 4 \int \cot 2x \cdot \csc 2x dx$$

$$= 4 \cdot \frac{1}{2} (-\csc 2x) + C$$

$$= -2 \csc 2x + C \quad \left[ \begin{array}{l} \text{or any other} \\ \text{equivalent form} \end{array} \right]$$

(iii)

$$I = \int \frac{\csc^2 x}{4 - \csc^2 x} dx$$

$$\text{let } \csc x = t \Rightarrow -\csc x \csc x dx = dt$$

$$\therefore I = - \int \frac{(-\csc x \csc x)}{4 - \csc^2 x} dx$$

$$= - \int \frac{dt}{4 - t^2}$$

$$= -\frac{1}{4} \ln \left| \frac{t+2}{t-2} \right| + C$$

$$\Rightarrow I = -\frac{1}{2} \ln \left| \frac{\cot x + 2}{\cot x - 2} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\cot x - 2}{\cot x + 2} \right| + C$$

$$5(a) (i) \quad I = \int \sin^4 x \cdot \cos^5 x \, dx$$

$$= \int \sin^4 x \cdot \cos^4 x \cdot \cos x \, dx$$

let  $\sin x = t \Rightarrow \cos x \, dx = dt$

$$I = \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int t^4 (1 - t^2)^2 \, dt$$

$$= \int t^4 (1 - 2t^2 + t^4) \, dt$$

$$= \int (t^4 - 2t^6 + t^8) \, dt$$

$$= \frac{t^5}{5} - \frac{2t^7}{7} + \frac{t^9}{9} + C$$

$$\therefore I = \frac{( \sin x )^7}{9} - \frac{2( \sin x )^7}{7} + \frac{( \sin x )^5}{5} + C$$

(ii)

$$I = \int x^5 \sqrt{x^3 - 4} \, dx$$

$$\text{let } x^3 - 4 = t^2$$

$$\Rightarrow x^3 = t^2 + 4$$

$$\Rightarrow 3x^2 \, dx = 2t \, dt$$

$$\Rightarrow x^2 \, dx = \frac{2}{3} t \, dt$$

$$I = \int x^3 \cdot \sqrt{x^3 - 4} \cdot x^2 \, dx$$

$$= \int (t^2 + 4) \cdot t \cdot \frac{2}{3} t \, dt$$

$$= \frac{2}{3} \int (t^2 + 4) t^2 dt$$

$$= \frac{2}{3} \int (t^4 + 4t^2) dt$$

$$= \frac{2}{3} \left[ \frac{t^5}{5} + \frac{4t^3}{3} \right] + C$$

$$= \frac{2}{3} \left[ \frac{(x^3 - 4)^{5/2}}{5} + \frac{4(x^3 - 4)^{3/2}}{3} \right] + C$$

(iii)

$$I = \int e^x (3x^3 + 9x^2) dx$$

$$\text{Here } f(x) = 3x^3 \Rightarrow f'(x) = 9x^2$$

$$\therefore I = e^x (3x^3) + C$$

$$= 3e^x \cdot x^3 + C$$

(b)

$$I = \int \frac{1}{x^2+2x+2} dx$$

$$= \int \frac{1}{(x^2+2x+1)+1} dx$$

$$= \int \frac{1}{(x+1)^2+1} dx$$

$$= \tan^{-1}(x+1) + C$$

(c)

$$I = \int \frac{1}{2\sin^2 x + 3\cos^2 x} dx$$

$$= \int \frac{\frac{1}{(\cos x)}}{2\left(\frac{\sin x}{\cos x}\right)^2 + 3\left(\frac{\cos x}{\sin x}\right)} dx$$

$$= \int \frac{\sec^2 x}{2\tan^2 x + 3} dx$$

$$\text{let } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int \frac{dt}{2t^2 + 3}$$

$$= \frac{1}{2} \int \frac{dt}{t^2 + \sqrt{\frac{3}{2}}^2}$$

$$= \frac{1}{\sqrt{6}} \tan^{-1} \left[ \sqrt{\frac{2}{3}} t \right] + C$$

$$= \frac{1}{\sqrt{6}} \tan^{-1} \left[ \sqrt{\frac{2}{3}} \cdot \tan x \right] + C$$

$$6(a) \quad I = \int \frac{2x^2+3}{x(x-1)^2} dx$$

$$\frac{2x^2+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$2x^2+3 = A(x-1)^2 + Bx(x-1) + Cx \quad \dots \dots \dots (1)$$

$$\text{put } x=1 \Rightarrow C=5$$

$$\text{put } x=0 \Rightarrow A=3$$

Equate coefficients of  $x^2$  from (1)

$$\therefore 2 = A+B \Rightarrow B=-1$$

$$\Rightarrow \frac{2x^2+3}{x(x-1)^2} = \frac{3}{x} - \frac{1}{x-1} + \frac{5}{(x-1)^2}$$

$$\begin{aligned} \Rightarrow I &= \int \left( \frac{3}{x} - \frac{1}{x-1} + \frac{5}{(x-1)^2} \right) dx \\ &= 3 \ln|x| - \ln|x-1| - \frac{5}{(x-1)} + C \end{aligned}$$

$$(b) \quad I = \int_{-1}^1 \ln(x+2) dx$$

$$\text{let } x+2=t \Rightarrow dx=dt$$

change limits of integration:

|     |    |   |
|-----|----|---|
| $x$ | -1 | 1 |
| $t$ | 1  | 3 |

$$\therefore I = \int_1^3 \ln(t) dt$$

$$= \int_1^3 (1) \cdot \ln(t) dt$$

$$I = \int_a^b u \cdot \frac{dv}{dt} dt = [uv]_a^b - \int_a^b v \cdot \frac{du}{dt} dt$$

$$\text{let } u = \ln(t) \Rightarrow \frac{du}{dt} = \frac{1}{t}$$

$$\text{let } \frac{dv}{dt} = 1 \Rightarrow v = t$$

$$\begin{aligned}\therefore I &= [\ln(t) \cdot t]^3_1 - \int_1^3 t dt \\ &= [3\ln(3) - 0] - \left[ \frac{t^2}{2} \right]^3_1 \\ &= 3\ln(3) - \frac{1}{2}(3^2 - 1) \\ &= 3\ln(3) - 2\end{aligned}$$

$$(c) V = \pi \int_a^b [f(x)]^2 dx$$

$$= \pi \int_0^1 (e^{4x})^2 dx$$

$$= \pi \int_0^1 e^{4x} dx$$

$$= \pi \left[ \frac{e^{4x}}{4} \right]_0^1$$

$$= \pi \left[ \frac{e^4}{4} - \frac{1}{4} \right]$$

$$= \frac{\pi}{4} (e^4 - 1)$$

$$(d) h(\text{width of sub-interval}) = \frac{b-a}{n}$$

$$\Rightarrow h = \frac{1-0}{10} = 0.1$$

$$\text{Here, } f(x) = \frac{4}{1+x^2}$$

| $x_n$ | $f_n$    | $f(x_n)$ |
|-------|----------|----------|
| 0     | $f_0$    | 4.00000  |
| 0.1   | $f_1$    | 3.96040  |
| 0.2   | $f_2$    | 3.84615  |
| 0.3   | $f_3$    | 3.66972  |
| 0.4   | $f_4$    | 3.44828  |
| 0.5   | $f_5$    | 3.20000  |
| 0.6   | $f_6$    | 2.94118  |
| 0.7   | $f_7$    | 2.68456  |
| 0.8   | $f_8$    | 2.43902  |
| 0.9   | $f_9$    | 2.20994  |
| 1.0   | $f_{10}$ | 2.00000  |

$$\int_a^b f(x) dx \approx \frac{h}{3} [f_0 + 2(f_2 + f_4 + f_6 + f_8) + 4(f_1 + f_3 + f_5 + f_7 + f_9) + f_{10}]$$

$$\therefore I = \int_0^1 \frac{4}{1+x^2} dx \approx \frac{1}{30} \left[ 4 + 2(3.84615 + 3.44828 + 2.94118 + 2.43902) + 4(3.96040 + 3.66972 + 2.68456 + 2.20994) + 2 \right]$$

$$\Rightarrow I \approx \frac{1}{30} [4 + 25.34926 + 62.87848 + 2]$$

$$\approx \frac{1}{30} (94.24774)$$

$$\approx 3.14159$$

$$7(a) \quad y = \frac{\sin x + a}{x} - \cos x$$

$$\frac{dy}{dx} = \frac{x \cos x - (\sin x + a)}{x^2} + \sin x$$

$$\text{Now } x \frac{dy}{dx} + y = x \sin x$$

$$\text{LHS} = x \cdot \left[ \frac{x \cos x - \sin x - a}{x^2} + \sin x \right]$$

$$+ \frac{\sin x + a}{x} - \cos x$$

$$\Rightarrow \cancel{\cos x} - \frac{\cancel{(\sin x + a)}}{x} + x \sin x + \frac{\cancel{(\sin x + a)}}{x} - \cancel{\cos x}$$

$$= x \sin x = \text{RHS}$$

$$(b) \quad (1 - \cos y) dy = (1 + \sin x) dx$$

$$\int (1 - \cos y) dy = \int (1 + \sin x) dx$$

$$y - \sin y = x - \cos x + C$$

$$(c) \quad x + 2y \sqrt{x^2 + 1} \frac{dy}{dx} \Rightarrow 2y \frac{dy}{dx} = \frac{-x}{\sqrt{x^2 + 1}} dx$$

$$\Rightarrow 2 \int y dy = - \int \frac{x}{\sqrt{x^2 + 1}} dx$$

$$\Rightarrow 2 \int y dy = -\frac{1}{2} \int \frac{2x}{\sqrt{x^2 + 1}} dx$$

$$\Rightarrow y = -\sqrt{x^2 + 1} + C \quad (\text{general soln})$$

$$\text{Now, } x \rightarrow 0 \Rightarrow y = 1$$

$$\therefore 1 = -\sqrt{0+1} + C \Rightarrow -1 + C = 1$$

$$\Rightarrow C = 2$$

$$\Rightarrow y = 2 - \sqrt{x^2 + 1} \quad (\text{particular solution})$$

(d) (i)  $\frac{dP}{dt} = kP$

$$\Rightarrow \int \frac{dP}{P} = k \int dt$$

$$\Rightarrow \ln|P| = kt + C \quad (\text{general solution})$$

$$\text{Now } t=0 \Rightarrow P(0) = P_0$$

$$\therefore \ln|P_0| = k(0) + C$$

$$\Rightarrow C = \ln|P_0|$$

$$\therefore \ln|P| = kt + \ln|P_0| \quad (\text{particular solution})$$

$$\ln\left|\frac{P}{P_0}\right| = kt \Rightarrow \frac{P}{P_0} = e^{kt}$$

$$\Rightarrow P = P_0 e^{kt}$$

(ii)  $P_0 = 1100$

$$\therefore P = 1100 e^{kt}$$

|     |      |      |
|-----|------|------|
| $t$ | 15   | ?    |
| $P$ | 1450 | 1820 |

$$\Rightarrow 1450 = 1100 e^{15k}$$

$$\therefore k = \frac{1}{15} \ln\left(\frac{29}{22}\right)$$

$$\therefore P = 1100 e^{\left[ \frac{1}{15} \ln \left( \frac{29}{22} \right) \right] \cdot t}$$

when  $P = 1820$

$$\frac{1820}{1100} = e^{\frac{1}{15} \ln \left( \frac{29}{22} \right) \cdot t}$$

$$\Rightarrow t = 15 \cdot \frac{\ln (1820/1100)}{\ln (29/22)}$$

$$\Rightarrow t \approx 27.34 \text{ days}$$