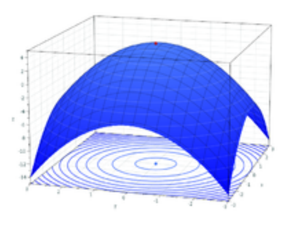
# Lecture 10 - Geometric Mappings

## COMP1046 - Maths for Computer Scientists


Dr. Ferrante Neri / Dr. Tony Bellotti

# Learning outcomes

**By the end of this lecture we will have learned:**

- Matrix Representation of Linear Mapping
- Geometric Mappings

Based on Sections 10.4 and 10.5 of text book (Neri 2018).

# Matrix Representation of Linear Mapping

# Every linear mapping has a matrix representation

## Proposition

*Every linear mapping is a multiplication of a matrix by a vector.*

We say the matrix *identifies* the linear mapping.

## Example

Let us consider the linear mapping $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $f(x, y, z) = \big(x + y - z, x - z, 3x + 2y + z\big)$. Consider a vector $(1, 2, 1)$. The mapped vector $f\,(1, 2, 1) = (2, 0, 8)$.

Calculate as the product of a matrix by the vector:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}.$$

# Every linear mapping has a matrix representation

## Proof.

Let $f : E \rightarrow F$ be a linear mapping where $(E, +, \cdot)$ and $(F, +, \cdot)$ are finite-dimensional vector spaces defined on the same field $\mathbb{K}$ and whose dimension is $n$ and $m$, respectively.
Let us consider a vector $\mathbf{x} \in E$:

$$\mathbf{x} = (x_1, x_2, \ldots x_n)$$

and a vector $\mathbf{y} \in F$: $\mathbf{y} = \left(y_1, y_2, \ldots y_m\right)$.

Let us consider now the expression $\mathbf{y} = f\left(\mathbf{x}\right)$ which can be written as

$$\left(y_1, y_2, \ldots y_m\right) = f\left(x_1, x_2, \ldots x_n\right).$$

*continued...*

# Every linear mapping has a matrix representation

## Proof.

Since $f$ is a linear mapping it can be written as

$$(y_1, y_2, \dots y_m) = \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 \dots a_{1,n}x_n, \\ a_{2,1}x_1 + a_{2,2}x_2 \dots a_{2,n}x_n, \\ \dots \\ a_{m,1}x_1 + a_{m,2}x_2 \dots a_{m,n}x_n \end{pmatrix}.$$

This means that

$$\begin{aligned} y_1 &= a_{1,1}x_1 + a_{1,2}x_2 \dots a_{1,n}x_n, \\ y_2 &= a_{2,1}x_1 + a_{2,2}x_2 \dots a_{2,n}x_n, \\ &\dots \\ y_m &= a_{m,1}x_1 + a_{m,2}x_2 \dots a_{m,n}x_n. \end{aligned}$$

*continued...*

# Every linear mapping has a matrix representation

**Proof.**

Furthermore, since all these equations need to be simultaneously verified, these equations compose a system of linear equations. This is a matrix equation

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & & & \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}.$$

□

# Geometric Mapping

# Geometric Mapping

- Let us consider a mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.
- This mapping can be interpreted as an operator that transforms a point in the plane into another point in the plane.
- Under these conditions, the mapping is said *geometric mapping in the plane*.
- Geometric transformations that can be represented are: rescaling, rotation, shearing, reflection and translation.

# Scaling

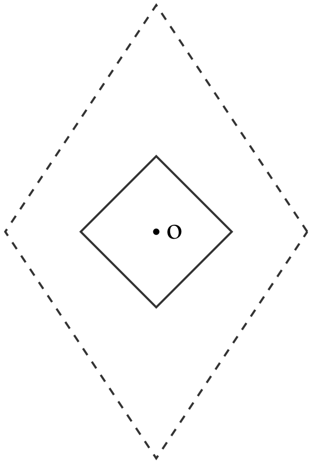
Let us now consider the following mapping $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s_1 x_1 \\ s_2 x_2 \end{pmatrix}.$$

This linear mapping is called *scaling*. If $s_1 = s_2$, it is called *uniform scaling*.

# Scaling

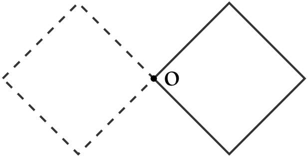
In the following figures, the basic points are indicated with a solid line while the transformed points are indicated with a dashed line ($s_1 = 2, s_2 = 3$).



# Reflection

The reflection with respect to the origin of the reference system is given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}.$$



# Exercise 1: Geometric Mapping

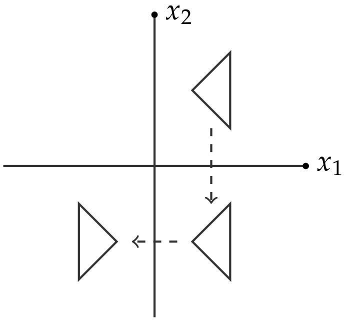
Consider the following geometric mappings:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- What geometric transformation does each of these matrices represent?
- If the two transformations are performed one after the other, what transformation does this form?
- Take the product of the two matrices: is the result consistent with your answer above?

# Exercise 1: *Solution*

- The two mappings transform $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to $\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$ and $\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$, respectively.
  That is, a reflection in the $x_1$ axis and $x_2$ axis respectively.



# Exercise 1: *Solution*

- Reflection in the origin.
- Product of matrices gives

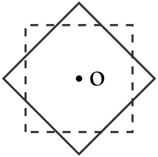
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

  which from slide 11 is the matrix representing reflection in the origin, hence this is consistent with performing each transformation after each other.

# Rotation

The following linear mapping is called *rotation* and is represented by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix}.$$



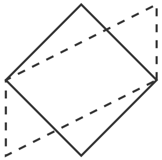

$\theta = \pi/4$

# Shearing

The following linear mapping is called *shearing* and is represented by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & s_1 \\ s_2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + s_1 x_2 \\ s_2 x_1 + x_2 \end{pmatrix}.$$

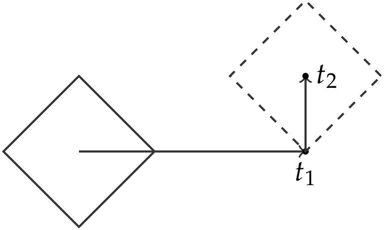
If, as in the figure below, the coefficient $s_2 = 0$ then this mapping is said *horizontal shearing*. If $s_1 = 0$ the mapping is a *vertical shearing*.



# Translation

Let us consider now the following mapping:

$$(y_1, y_2) = f(x_1, x_2) = (x_1, x_2) + (t_1, t_2)$$




This is called a *translation*.

# Translation

*Translation* moves the points a constant distance in a specific direction.

Unlike the previous geometric mappings, a translation is not a linear mapping as the linearity properties are not valid and a matrix representation by means of $\mathbb{R}_{2,2}$ matrices is not possible.

More specifically, a translation is an affine mapping.

# Translation

In order to give a matrix representation to affine mappings let us introduce the concept of *homogeneous coordinates*, i.e. we algebraically represent each point $\mathbf{x}$ of the plane by means of three coordinates where the third is identically equal to 1:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}.$$

We can now give a matrix representation to the affine mapping translation in a plane:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 + t_1 \\ x_2 + t_2 \\ 1 \end{pmatrix}.$$

# Translation

All the linear mappings can be written in homogeneous coordinates simply adding a row and a column to the matrix representing the mapping.

For example, the scaling and rotation can be respectively performed by multiplying the following matrices by a point $\mathbf{x}$:

$$\begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Translation

If we indicate with $\mathbf{M}$ the $2 \times 2$ matrix representing a linear mapping in the plane and with $\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ the translation vector of the plane, the generic geometric mapping is given by a matrix

$$\begin{pmatrix} \mathbf{M} & \mathbf{t} \\ 0 & 1 \end{pmatrix}.$$

# Translation with Reflection

## Example

Consider reflection in the $x_2$ axis, given by $\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and translation $\mathbf{t} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Construct mapping matrix so

$$\begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} -x_1 + 2 \\ x_2 - 1 \\ 1 \end{pmatrix}$$

# Exercise 2: Geometric Mapping

Consider the sequence of transformations:

1. Uniform scaling by 5;
2. Vertical shear of 2 with a translation by $(-1, 4)$.

- Construct the $3 \times 3$ matrices identifying each of these transformations.
- Compute the matrix that expresses the sequence of transformations as a single geometric mapping?
- Show that the order of the transformations matters.

# Exercise 2: *Solution*

- Express the scaling and the vertical shear with translation with $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ respectively.
- Then take the product to compute the combined geometric mapping:

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -1 \\ 10 & 5 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

# Exercise 2: *Solution*

- Take the product in a different order:

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -5 \\ 10 & 5 & 20 \\ 0 & 0 & 1 \end{pmatrix}.$$

This clearly gives a different transformation matrix (in the translation of the last column).

# Summary

# Summary and next lecture

## Summary

- Matrix Representation of Linear Mapping
- Geometric Mappings

## The next lecture

We will learn about Eigenvalues, Eigenvectors and Eigenspaces.