COMP1046 Mathematics for Computer Scientists Part 2: Exercise 2 for Lecture 7

These additional questions are based on Lecture 7 and especially the method of proof for Steinitz's Lemma.

Consider the vector space $(\mathbb{R}^3, +, \cdot)$ with vectors

$$\mathbf{v_1} = (1, 0, 0), \quad \mathbf{v_2} = (0, 1, 0), \quad \mathbf{v_3} = (0, 0, 1)$$

and

$$\mathbf{w_1} = (1, -1, 0), \quad \mathbf{w_2} = (-1, 2, 1), \quad \mathbf{w_3} = (2, 1, -1)$$

in $\mathbb{R}^3$.

Q1. Show that $L(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \mathbb{R}^3$.

ANSWER:
Consider any $(x, y, z) \in \mathbb{R}^3$. Set $\lambda_1 = x, \lambda_2 = y, \lambda_3 = z$.
Then the linear combination $\lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \lambda_3\mathbf{v_3} = (x, y, z)$.
Hence, $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ spans all vectors in $\mathbb{R}^3$.

Q2. Show that $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ are linearly independent.

ANSWER:
Solve $\lambda_1\mathbf{w_1} + \lambda_2\mathbf{w_2} + \lambda_3\mathbf{w_3} = (0, 0, 0)$.
This has at least one solution: $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
Consider the complete matrix for this system of linear equations:

$$\left(\begin{array}{rrr|r} 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right).$$

The incomplete matrix has determinant

$$= (-1)^5 \times 1 \times (1 + 2) + (-1)^6 \times -1 \times (2 + 1) = 6,$$

hence its rank $\rho = 3 = n$ so by Rouchè-Capelli Theoerem, this is case 1 with a single solution.
This single solution must be $\lambda_1 = \lambda_2 = \lambda_3 = 0$, so the vectors are independent.

Q3. Using the substitution method used in the proof of Steinitz's Lemma, show that each of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ can be expressed as a linear combination of $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$.

ANSWER:
Firstly,

$$\mathbf{w_1} = \mathbf{v_1} - \mathbf{v_2} \Rightarrow \mathbf{v_1} = \mathbf{w_1} + \mathbf{v_2}.$$

Secondly,

$$\begin{aligned} & \mathbf{w_2} && = -\mathbf{v_1} + 2\mathbf{v_2} + \mathbf{v_3} \\ & && = -\mathbf{w_1} - \mathbf{v_2} + 2\mathbf{v_2} + \mathbf{v_3} \\ \Rightarrow\ & \mathbf{v_2} && = \mathbf{w_1} + \mathbf{w_2} - \mathbf{v_3}. \end{aligned}$$

Thirdly,

$$\begin{aligned} & \mathbf{w_3} && = 2\mathbf{v_1} + \mathbf{v_2} - \mathbf{v_3} \\ & && = 2\mathbf{w_1} + 2\mathbf{v_2} + \mathbf{v_2} - \mathbf{v_3} \\ & && = 2\mathbf{w_1} + 3\mathbf{v_2} - \mathbf{v_3} \\ & && = 2\mathbf{w_1} + 3(\mathbf{w_1} + \mathbf{w_2} - \mathbf{v_3}) - \mathbf{v_3} \\ & && = 5\mathbf{w_1} + 3\mathbf{w_2} - 4\mathbf{v_3} \\ \Rightarrow\ & \mathbf{v_3} && = \tfrac{5}{4}\mathbf{w_1} + \tfrac{3}{4}\mathbf{w_2} - \tfrac{1}{4}\mathbf{w_3} \\ \Rightarrow\ & \mathbf{v_2} && = -\tfrac{1}{4}\mathbf{w_1} + \tfrac{1}{4}\mathbf{w_2} + \tfrac{1}{4}\mathbf{w_3} \\ \Rightarrow\ & \mathbf{v_1} && = \tfrac{3}{4}\mathbf{w_1} + \tfrac{1}{4}\mathbf{w_2} + \tfrac{1}{4}\mathbf{w_3} \end{aligned}$$

Q4. Use the answers to Q1 and Q3 to show $L(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}) = \mathbb{R}^3$.

ANSWER:
Since $L(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \mathbb{R}^3$, for any $\mathbf{u} \in \mathbb{R}^3$, there exist some scalars a_1, a_2, a_3 such that

$$\begin{aligned} \mathbf{u} & = a_1\mathbf{v_1} + a_2\mathbf{v_2} + a_3\mathbf{v_3} \\ & = a_1\left(\tfrac{5}{4}\mathbf{w_1} + \tfrac{3}{4}\mathbf{w_2} - \tfrac{1}{4}\mathbf{w_3}\right) + a_2\left(-\tfrac{1}{4}\mathbf{w_1} + \tfrac{1}{4}\mathbf{w_2} + \tfrac{1}{4}\mathbf{w_3}\right) + a_3\left(\tfrac{3}{4}\mathbf{w_1} + \tfrac{1}{4}\mathbf{w_2} + \tfrac{1}{4}\mathbf{w_3}\right) \\ & = \tfrac{5a_1 - a_2 + 3a_3}{4}\mathbf{w_1} + \tfrac{3a_1 + a_2 + a_3}{4}\mathbf{w_2} + \tfrac{-a_1 + a_2 + a_3}{4}\mathbf{w_3} \end{aligned}$$

which is a linear combination of $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$, hence these three span all vectors in $\mathbb{R}^3$.

Q5. Suppose some $\mathbf{w_4} \in \mathbb{R}^3$. Use the answer to Q4 to show that $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ must be linearly dependent.

ANSWER: Since, $L(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}) = \mathbb{R}^3$, any vector in $\mathbb{R}^3$ can be expressed as a linear combination of $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$. In particular,

$$\mathbf{w_4} = \lambda_1\mathbf{w_1} + \lambda_2\mathbf{w_2} + \lambda_3\mathbf{w_3}$$

$$\Rightarrow \lambda_1\mathbf{w_1} + \lambda_2\mathbf{w_2} + \lambda_3\mathbf{w_3} - \mathbf{w_4} = (0, 0, 0).$$

At least one coefficient is non-zero (i.e. -1 on $\mathbf{w_4}$) or $\mathbf{w_4} = (0, 0, 0)$.
Hence $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ are linearly dependent.