Lecture 7

Topics covered in this lecture session

1. The Binomial Theorem
2. Generalised Binomial Theorem
3. Applications in approximation problems.

Binomial Theorem - Introduction

Consider the expansion formulae:

$$\begin{aligned}(1+x) &= 1 + x \\ (1+x)^2 &= 1 + 2x + x^2 \\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\ (1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4\end{aligned}$$

Binomial coefficients

In the above expansions, the numbers

$$\underline{1\,,\,1} \qquad \underline{1\,,\,2\,,\,1} \qquad \underline{1\,,\,3\,,\,3\,,\,1} \qquad \underline{1\,,\,4\,,\,6\,,\,4\,,\,1}$$

are coefficients of powers of x, are called Binomial coefficients.

These numbers are in a fixed pattern.

If we go on writing them, the pattern so formed is the Pascal's Triangle.

Binomial coefficients

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
$\boxed{1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1}$
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

e.g. $(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

Binomial coefficients

What if we want to expand further? (i.e. higher order terms)

e.g. $(1+x)^{10}$

It is definitely not meaningful to continue writing rows of the Pascal's Triangle.

In such cases, we rely on a useful formula based on factorial function/notation.

Factorial Function

By definition,

$$0\,! = 1,$$

and

$$\begin{aligned} n\,! &= n\,(n-1)\,(n-2)\,.....\times 3\times 2\times 1 \\ &= 1\times 2\times 3\times\ .....\,(n-2)\,(n-1)\,n. \end{aligned}$$

For example, $5\,! = 5\times 4\times 3\times 2\times 1 = 120.$

Combinations - An important formula

$$nC_k \equiv \binom{n}{k} = \frac{n\,!}{k\,!\,(n-k)\,!}$$

For example,

$$\binom{10}{2} = \frac{10\,!}{2\,!\,(10-2)\,!}$$

$$= \frac{10\times 9\times 8\,!}{2\times 8\,!} = \frac{90}{2} = 45$$

Because of their appearance as coefficients in a Binomial expansion, the numbers

$$\binom{n}{k}$$

are called Binomial coefficients.

Binomial Theorem

Using this notation, we expand $(1+x)^n$ as:

$$\boxed{(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \dots\dots\dots\dots + x^n}$$

By writing $x = \dfrac{b}{a}$ and simplifying, we get a general formulation for Binomial Theorem as:

$$\boxed{(a+b)^n \;=\; a^n + \binom{n}{1}a^{n-1}\,b + \binom{n}{2}a^{n-2}\,b^2 + \binom{n}{3}a^{n-3}\,b^3 + \dots\dots + b^n}$$

Binomial Theorem (Formula 1)

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \binom{n}{4} x^4 + \ldots\ldots\ldots\ldots + x^n$$

Expand $\left(1+\frac{x}{2}\right)^4$ using the Binomial Theorem.

$$\left(1+\frac{x}{2}\right)^4 = 1 + \binom{4}{1}\cdot\left(\frac{x}{2}\right) + \binom{4}{2}\cdot\left(\frac{x}{2}\right)^2 + \binom{4}{3}\cdot\left(\frac{x}{2}\right)^3 + \binom{4}{4}\cdot\left(\frac{x}{2}\right)^4$$

$$= 1 + 4\cdot\left(\frac{x}{2}\right) + 6\cdot\left(\frac{x}{2}\right)^2 + 4\cdot\left(\frac{x}{2}\right)^3 + 1\cdot\left(\frac{x}{2}\right)^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16}$$

Binomial Theorem (Formula 2)

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots.. + b^n$$

Examples: 1. Expand $(x+7)^5$ using Binomial Theorem.

Here, $a = x$, $b = 7$, and $n = 5$.

Using $(a+b)^n$

$$= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots.. + b^n$$

Binomial Theorem (Formula 2)

$$\Rightarrow (x+7)^5 = x^5 + \binom{5}{1} x^4 (7) + \binom{5}{2} x^3 (7)^2 + \binom{5}{3} x^2 (7)^3 + \binom{5}{4} x (7)^4 + \binom{5}{5} x^0 (7)^5$$

$$= x^5 + 5x^4(7) + 10x^3(49) + 10x^2(343) + 5x(2401) + (1)x^0(16807)$$

$$= x^5 + 35x^4 + 490x^3 + 3430x^2 + 12005x + 16807$$

Binomial Theorem (Formula 2)

2. Expand $(1-3x)^4$ using Binomial Theorem.

Here, $a = 1$, $b = -3x$, and $n = 4$.

Using $(a+b)^n$

$$= a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots.. + b^n$$

$\Rightarrow (1-3x)^4$

$$= 1^4 + \binom{4}{1} 1^4(-3x) + \binom{4}{2} 1^3(-3x)^2 + \binom{4}{3} 1^2(-3x)^3 + \binom{4}{4}(-3x)^4$$

$$= 1 + 4(-3x) + 6(9x^2) + 4(-27x^3) + (1)(81x^4)$$

$$= 1 - 12x + 54x^2 - 108x^3 + 81x^4.$$

Binomial Theorem (Formula 2)

3. Expand $\left(3+\frac{2}{x}\right)^4$ using the Binomial Theorem.

$$\left(3+\frac{2}{x}\right)^4 = 3^4 + \binom{4}{1}\cdot 3^3\cdot\left(\frac{2}{x}\right) + \binom{4}{2}\cdot 3^2\cdot\left(\frac{2}{x}\right)^2 + \binom{4}{3}\cdot 3^1\cdot\left(\frac{2}{x}\right)^3 + \binom{4}{4}\cdot 3^0\cdot\left(\frac{2}{x}\right)^4$$

$$= 81 + 4\cdot 27\cdot\left(\frac{2}{x}\right) + 6\cdot 9\cdot\left(\frac{4}{x^2}\right) + 4\cdot 3\cdot\left(\frac{8}{x^3}\right) + 1\cdot 1\cdot\left(\frac{16}{x^4}\right)$$

$$= 81 + \frac{216}{x} + \frac{216}{x^2} + \frac{96}{x^3} + \frac{16}{x^4}$$

Finding coefficient of x^n

Find the coefficient of x^3 in the expansion of $\left(3-\frac{2x}{5}\right)^5$.

$$\left(3-\frac{2x}{5}\right)^5 = 3^5 + \binom{5}{1}\cdot 3^4\cdot\left(\frac{-2x}{5}\right) + \binom{5}{2}\cdot 3^3\cdot\left(\frac{-2x}{5}\right)^2 + \boxed{\binom{5}{3}\cdot 3^2\cdot\left(\frac{-2x}{5}\right)^3} + \binom{5}{4}\cdot 3^1\cdot\left(\frac{-2x}{5}\right)^4 + \binom{5}{5}\cdot 3^0\cdot\left(\frac{-2x}{5}\right)^5$$

∴ Term with x^3 is: $\binom{5}{3}\cdot 3^2\cdot\left(\frac{-2x}{5}\right)^3$

∴ The coefficient of x^3 is: $\binom{5}{3}\cdot 3^2\cdot\left(\frac{-2}{5}\right)^3 = 10\cdot 9\cdot\left(\frac{-8}{125}\right)$

$$= -\frac{144}{25}$$

Generalised Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \ldots\ldots\ldots.$$

where $n \in \mathbb{R}$ and $|\,x\,| < 1$.

Note: $\binom{n}{2} = \frac{n\,!}{2\,!\cdot(n-2)\,!}$

$$= \frac{n\cdot(n-1)\cdot(n-2)\,!}{2\,!\cdot(n-2)\,!} = \frac{n\cdot(n-1)}{2\,!}$$

Similarly other terms can be obtained.

Generalised Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \ldots\ldots\ldots.$$

where $n \in \mathbb{R}$ and $|\,x\,| < 1$.

Ex.1 Expand $(1+x)^{-3}$

$$(1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-3-1)}{2\,!}x^2 + \frac{(-3)(-3-1)(-3-2)}{3\,!}x^3 + \cdots$$

$$= 1 - 3x + \frac{(-3)(-4)}{2\cdot 1}x^2 + \frac{(-3)(-4)(-5)}{3\cdot 2}x^3 + \frac{(-3)(-4)(-5)(-6)}{4\cdot 3\cdot 2}x^4 + \cdots$$

$$= 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \ldots$$

Approximation using Generalised Binomial Thm.

Approximate $(1.02)^{-1}$ using Binomial Theorem.

$$(1.02)^{-1} = (1 + 0.02)^{-1}$$

So, $a = 1$, $x = 0.02$, and $n = -1$ **NOT a positive Integer**

General expansion formula for $n \in \mathbb{R}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \dots\dots\dots$$

apply where $n \in \mathbb{R}$ and $|\,x\,| < 1.$

Approximation using Generalised Binomial Thm.

As, $|\,x\,| = |\,0.02\,| < 1.$

Using $(1+x)^n$

$$= 1 + nx + \frac{n(n-1)}{2\,!}x^2 + \frac{n(n-1)(n-2)}{3\,!}x^3 + \dots\dots\dots$$

$$\Rightarrow (1+0.02)^{-1}$$

$$= 1 + (-1)(0.02) + \frac{(-1)(-1-1)}{2\,!}(0.02)^2$$

$$+ \frac{(-1)(-1-1)(-1-2)}{3\,!}(0.02)^3 + \dots.$$

Approximation using Generalised Binomial Thm.

$$\approx 1 - 0.02 + 0.0004 - 0.000008$$

Approximate sign is introduced because we are terminating the infinite series.

$$= 0.980392.$$

Thus, $(1.02)^{-1} \approx 0.980392.$

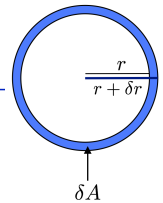
Error analysis using Binomial Theorem

Example:

The radius of a circle is measured as r, with an error of δr = 1.5% of r.

The area of the circle $A = \pi\, r^2$ is then calculated using the measured r.

Find the resulting error, δA, in the area calculated.



Note:

Using approximation, $(1+x)^n \approx 1 + nx.$

Error analysis using Binomial Theorem

given that $\delta r = 1.5\%$ of $r \quad \Rightarrow \quad \delta r = 0.015r$.

Now, $A \;=\; \pi r^2$

$$\begin{aligned}
\Rightarrow \cancel{A} + \delta A &= \pi\,(r + \delta r)^2 = \pi\,(r + 0.015r)^2 \\
&= \pi\, r^2\,(1 + 0.015)^2 \\
&\approx A\,(1 + 2 \times 0.015) \quad \text{using approximation } (1+x)^n \approx 1 + nx \\
&= A(1 + 0.03) \\
&= \cancel{A} + 0.03\,A
\end{aligned}$$

$$\boxed{\begin{aligned} \therefore\ \delta A &\approx 0.03\,A \\ \text{i.e. } \delta A &\approx 3\% \text{ of } A. \end{aligned}}$$