

# AE1MCS: Mathematics for Computer Scientists

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Oct 2020

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 2, Section 2.3. Functions
- Chapter 2, Section 2.4. Sequences and Summations

# Functions

- The concept of a function is extremely important in discrete mathematics.
- A function assigns to each element of a first set exactly one element of a second set, where the two sets are not necessarily distinct.
- Functions play important roles throughout discrete mathematics.
- They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects, and in a myriad of other ways.
- Useful structures such as sequences and strings are special types of functions.

# Functions

## Definition

Let  $A$  and  $B$  be nonempty sets. A *function*  $f$  from  $A$  to  $B$  is an assignment of exactly one element of  $B$  to each element of  $A$ . We write  $f(a) = b$  if  $b$  is the unique element of  $B$  assigned by the function  $f$  to the element  $a$  of  $A$ . If  $f$  is a function from  $A$  to  $B$ , we write  $f : A \rightarrow B$ .

**Remark:** Functions are sometimes also called *mappings* or *transformations*.

# Domain and Range

## Definition

If  $f$  is a function from  $A$  to  $B$ , we say that  $A$  is the *domain* of  $f$  and  $B$  is the *codomain* of  $f$ . If  $f(a) = b$ , we say that  $b$  is the *image* of  $a$  and  $a$  is a *preimage* of  $b$ . The *range*, or *image*, of  $f$  is the set of all images of elements of  $A$ . Also, if  $f$  is a function from  $A$  to  $B$ , we say that  $f$  *maps*  $A$  to  $B$ .

# Equal Functions

Two functions are **equal** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain.

# Image of a Set

## Definition

Let  $f$  be a function from  $A$  to  $B$  and let  $S$  be a subset of  $A$ . The *image* of  $S$  under the function  $f$  is the subset of  $B$  that consists of the images of the elements of  $S$ . We denote the image of  $S$  by  $f(S)$ , so

$$f(S) = \{t \mid \exists s \in S (t = f(s))\}.$$

We also use the shorthand  $\{f(s) \mid s \in S\}$  to denote this set.

**Remark:** The notation  $f(S)$  for the image of the set  $S$  under the function  $f$  is potentially ambiguous. Here,  $f(S)$  denotes a set, and not the value of the function  $f$  for the set  $S$ .

# One-to-One Function

## Definition

A function  $f$  is said to be *one-to-one*, or an *injection*, if and only if  $f(a) = f(b)$  implies that  $a = b$  for all  $a$  and  $b$  in the domain of  $f$ . This is

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$$

or equivalently

$$\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$$

A function is said to be *injective* if it is one-to-one.



# Prove or Disprove a Function is Injective

Suppose that  $f : A \rightarrow B$ .

To show that  $f$  is injective Show that for arbitrary  $x, y \in A$ , if  $f(x) = f(y)$ , then  $x = y$ .

To show that  $f$  is not injective Find particular elements  $x, y \in A$  such that  $x \neq y$  and  $f(x) = f(y)$ .

# Onto Functions

## Definition

A function  $f$  from  $A$  to  $B$  is called *onto*, or a *surjection*, if and only if for every element  $b \in B$  there is an element  $a \in A$  with  $f(a) = b$ . This is,

$$\forall b \in B \exists a \in A (f(a) = b)$$

A function  $f$  is called *surjective* if it is onto.

# Prove or Disprove a Function is Surjective

Suppose that  $f : A \rightarrow B$ .

To show that  $f$  is surjective Consider an arbitrary element  $y \in B$  and find an element  $x \in A$  such that  $f(x) = y$ .

To show that  $f$  is not surjective Find a particular  $y \in B$  such that for all  $x \in A$ ,  $f(x) \neq y$ .

# One-to-one Correspondence

## Definition

The function  $f$  is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.

# Inverse Functions

## Definition

Let  $f$  be a one-to-one correspondence from the set  $A$  to the set  $B$ . The *inverse function* of  $f$  is the function that assigns to an element  $b$  belonging to  $B$  the unique element  $a$  in  $A$  such that  $f(a) = b$ . The inverse function of  $f$  is denoted by  $f^{-1}$ . Hence,  $f^{-1}(b) = a$  when  $f(a) = b$ .

**Remark:** Be sure not to confuse the function  $f^{-1}$  with the function  $1/f$ , which is the function that assigns to each  $x$  in the domain the value  $1/f(x)$ . Notice that the latter makes sense only when  $f(x)$  is a non-zero real number.

# Invertible Functions

- A one-to-one correspondence is called **invertible** because we can define an inverse of this function.
- A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

# Compositions of Functions

## Definition

Let  $g$  be a function from the set  $A$  to the set  $B$  and let  $f$  be a function from the set  $B$  to the set  $C$ . The *composition* of the functions  $f$  and  $g$ , denoted for all  $a \in A$  by  $f \circ g$ , is defined by

$$(f \circ g)(a) = f(g(a)).$$

Note that the *composition*  $f \circ g$  cannot be defined unless the range of  $g$  is a subset of the domain of  $f$ .

# Identity Function

Suppose that  $f$  is a one-to-one correspondence from the set  $A$  to the set  $B$ .  $f(a) = b$ .

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$$

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$$

$f^{-1} \circ f$  and  $f \circ f^{-1}$  are the identity functions on the sets  $A$  and  $B$  respectively.  $(f^{-1})^{-1} = f$ .



# The Graphs of Functions

## Definition

Let  $f$  be a function from the set  $A$  to the set  $B$ . The *graph* of the function  $f$  is the set of ordered pairs  $\{(a, b) \mid a \in A \wedge f(a) = b\}$ .

# Example

Consider  $f(n)=2n+3$ , is it bijective from  $\mathbb{Z}$  to  $\mathbb{Z}$ ?

## Example

For each of the following functions, is it invertible? If yes, what is its inverse?

(a) Let  $f(x)$  be a function from  $\mathbb{R}$  to  $\mathbb{R}$ .  $f(x) = 2x + 1$

(b) Let  $f(x)$  be a function from  $\mathbb{R}^+$  to  $\mathbb{R}$ .  $g(x) = \log_2(2x) - 1$

# Sequences

## Definition

A *sequence* is a function from a subset of the set of integers (usually either the set  $\{0, 1, 2, \dots\}$  or the set  $\{1, 2, 3, \dots\}$ ) to a set  $S$ . We use the notation  $a_n$  to denote the image of the integer  $n$ . We call  $a_n$  a term of the sequence.

# Geometric Progression

## Definition

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \dots, ar^n, \dots$$

where the initial term  $a$  and the common ratio  $r$  are real numbers.

**Remark:** A geometric progression is a discrete analogue of the exponential function  $f(x) = ar^x$ .

# Arithmetic Progression

## Definition

An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \dots, a + nd, \dots$$

where the initial term  $a$  and the common difference  $d$  are real numbers.

**Remark:** An arithmetic progression is a discrete analogue of the linear function  $f(x) = dx + a$ .

# Recurrence Relation

## Definition

A *recurrence relation* for the sequence  $\{a_n\}$  is an equation that expresses  $a_n$  in terms of one or more of the previous terms of the sequence, namely,  $a_0, a_1, \dots, a_{n-1}$ , for all integers  $n$  with  $n \geq n_0$ , where  $n_0$  is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

e.g.  $a_0 = 1$ .  $a_{n+1} = a_n + 1$  for  $n = 0, 1, 2, \dots$

# Homework

Learn the following definitions by yourself.

- Real-valued and Integer-valued Functions
- Adding and Multiplying Real-valued Functions
- Increasing and Decreasing Functions
- Floor Function and Ceiling Function



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