Lecture 4

Topics covered in this lecture session

1. Formulae for addition, factor and multi-angle.
2. Inverse Trigonometric functions.

Addition and factor formulae

Note: $x(A+B) = xA + xB$, but $\sin(A+B) \neq \sin A + \sin B$.

$$
\boxed{\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A \sin B \\
\sin(A-B) &= \sin A\cos B - \cos A \sin B \\
\cos(A+B) &= \cos A\cos B - \sin A \sin B \\
\cos(A-B) &= \cos A\cos B + \sin A \sin B \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B}
\end{aligned}}
$$

Addition and factor formulae

Prove: $\sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

LHS = $\sin 75^\circ = \sin(45^\circ + 30^\circ)$

$$\boxed{\sin(A+B) = \sin A\cos B + \cos A \sin B}$$

$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\cdot\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$ = RHS

Addition and factor formulae

$$
\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cancel{\cos A \sin B} \\
\sin(A-B) &= \sin A\cos B - \cancel{\cos A \sin B}
\end{aligned}
$$

Adding

$$\boxed{\sin(A+B) + \sin(A-B) = 2\sin A\cos B}$$

$$
\begin{aligned}
\sin(A+B) &= \sin \cancel{A\cos B} + \cos A \sin B \\
-\sin(A-B) &= -\sin \cancel{A\cos B} + \cos A \sin B
\end{aligned}
$$

Subtracting

$$\boxed{\sin(A+B) - \sin(A-B) = 2\cos A\sin B}$$

Similarly, it can be proved that

$$
\begin{aligned}
\cos(A+B) + \cos(A-B) &= 2\cos A\cos B \\
\cos(A+B) - \cos(A-B) &= -2\sin A\sin B
\end{aligned}
$$

Addition and factor formulae

Writing $A+B=C$ and $A-B=D$ $\Rightarrow$ $A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$

$$\boxed{\begin{aligned}\sin(A+B)+\sin(A-B) &= 2\sin A\cos B\\ \sin(A+B)-\sin(A-B) &= 2\cos A\sin B\\ \cos(A+B)+\cos(A-B) &= 2\cos A\cos B\\ \cos(A+B)-\cos(A-B) &= -2\sin A\sin B\end{aligned}} \Rightarrow \boxed{\begin{aligned}\sin C+\sin D &= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\ \sin C-\sin D &= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\ \cos C+\cos D &= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\ \cos C-\cos D &= -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\end{aligned}}$$

Example Prove that $\sin 50^\circ + \sin 10^\circ = \sin 70^\circ$

Multi-angle formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\boxed{\begin{aligned}&= 2\cos^2\theta - 1\\ &= 1-2\sin^2\theta\end{aligned}}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\boxed{\begin{aligned}\sin^2\theta &= \frac{1}{2}(1-\cos 2\theta)\\ \cos^2\theta &= \frac{1}{2}(1+\cos 2\theta)\end{aligned}}$$

$$\begin{aligned}\sin 3\theta &= 3\sin\theta - 4\sin^3\theta\\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta\\ \tan 3\theta &= \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\end{aligned}$$

Worked Examples

1. Given $\cos\theta = -\dfrac{3}{5}$; $180^\circ < x < 270^\circ$. Find the values of $\sin 2\theta$ and $\tan 2\theta$.
2. Prove that $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta} = \tan\theta$.
3. Prove that $\dfrac{\sin 3\theta}{1+2\cos 2\theta} = \sin\theta$. Hence deduce the value of $\sin 15^o$.

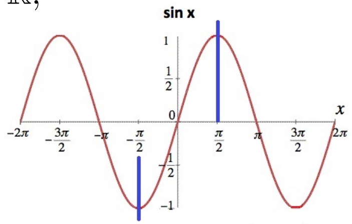
With $t = \tan\left(\dfrac{\theta}{2}\right)$,

useful formulae in Calculus

$\sin\theta = \dfrac{2t}{1+t^2}$	$\cos\theta = \dfrac{1-t^2}{1+t^2}$	$\tan\theta = \dfrac{2t}{1-t^2}$

Inverse Trigonometric Functions

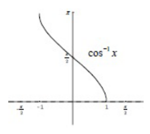
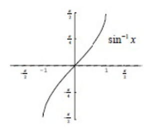
The graph of the sine function over $\mathbb{R}$, indicates that it is not one-one however, if we restrict the domain to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, then sine function is one-one and its inverse exists.



It is denoted by $\sin^{-1}$ **or** $\arcsin$ and is defined by

$$y = \sin x \iff x = \sin^{-1} y \quad ; \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Inverse Trigonometric Functions

Inverse Function	Domain of Inverse function $\equiv$ Range of Trigonometric function	Range of Inverse function *i.e.* Restricted Domain for Trigonometric function	Graph of Inverse Trigonometric function
$\cos^{-1} x$ or $\arccos$	$[-1, 1]$	$[0, \pi]$	
$\sin^{-1} x$ or $\arcsin$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	

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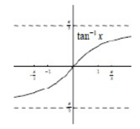
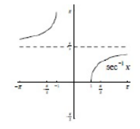
Inverse Trigonometric Functions

Inverse Function	Domain of Inverse function $\equiv$ Range of Trigonometric function	Range of Inverse function *i.e.* Restricted Domain for Trigonometric function	Graph of Inverse Trigonometric function
$\tan^{-1} x$ or $\arctan$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	
$\sec^{-1} x$ or arcsec	$\mathbb{R} - (-1, 1)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	

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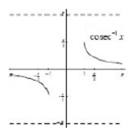
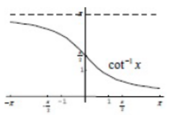
Inverse Trigonometric Functions

Inverse Function	Domain of Inverse function $\equiv$ Range of Trigonometric function	Range of Inverse function *i.e.* Restricted Domain for Trigonometric function	Graph of Inverse Trigonometric function
$\operatorname{cosec}^{-1} x$ or $\operatorname{arccosec}$	$\mathbb{R} - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	
$\cot^{-1} x$ or arccot	$\mathbb{R}$	$(0, \pi)$	

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Inverse Trigonometric Functions

Find the values of:

$(i)\quad \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$(ii)\quad \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$(iii)\quad \sin^{-1}\left(\sin\left(\frac{7\pi}{4}\right)\right)$