Maths for Computer Science: Tutorial 5 Solutions

1. Let A and B be sets. Prove:

$$A \cap B = \emptyset \text{ iff } A \subseteq \overline{B} \tag{1}$$

Answer:

We split task into two subtasks.

*The **first** is to prove that* $A \cap B = \emptyset \to A \subseteq \overline{B}$.

By way of contradiction, suppose $A \cap B = \emptyset$ *and* $A \not\subseteq \overline{B}$.

If $A \not\subseteq \overline{B}$, *we can find an* x, *such that* $x \in A$ *and* $x \notin \overline{B}$.

But,

$x \in A \wedge x \notin \overline{B}$

therefore, $x \in A \wedge x \in B$ *(by definition of complement)*

therefore, $x \in A \cap B$ *(by definition of intersection)* $\to A \cap B \neq \emptyset$

This leads to a contradiction.

*The **second** task is to prove that* $A \subseteq \overline{B} \to A \cap B = \emptyset$

Again, by way of contradiction, suppose $A \subseteq \overline{B}$ *and* $A \cap B \neq \emptyset$

If $A \cap B \neq \emptyset$ *there exists an* x, *such that,* $x \in A \cap B$.

But,

$x \in A \cap B$

therefore, $x \in A \wedge x \in B$ *(by definition of intersection)*

therefore, $x \in A \wedge x \notin \overline{B}$ *(by definition of complement)*$\to A \not\subseteq \overline{B}$

This leads to a contradiction.

Finally, we have proved: $A \cap B = \emptyset$ *if and only if* $A \not\subseteq \overline{B}$

2. Let A,B and C be any sets. Show that:

$$A \times (B \cup C) = (A \times B) \cup (A \times C) \tag{2}$$

Answer:

We can prove $X = Y$ *by showing that if* $a \in X$ *then* $a \in Y$*, and if* $a \notin X$ *then* $a \notin Y$.

Let $a = (x, y)$*, and suppose* $a \in A \times (B \cup C)$.

$(x, y) \in A \times (B \cup C)$

therefore, $x \in A \wedge y \in (B \cup C)$ *(by definition of Cartesian Products)*

therefore, $x \in A \wedge (y \in B \vee y \in C)$ *(by definition of union)*

therefore, $(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$ *(by distributive law)*

therefore, $(x, y) \in A \times B \vee (x, y) \in A \times C$ *(by definition of Cartesian Products)*

therefore, $(x, y) \in (A \times B) \cup (A \times C)$ *(by definition of union)*

Therefore, $a \in (A \times B) \cup (A \times C)$

Let $a = (x, y)$*, and suppose* $a \notin A \times (B \cup C)$.

$= (x, y) \notin A \times (B \cup C)$

therefore, $x \notin A \vee y \notin (B \cup C)$ *(by definition of Cartesian Products)*

therefore, $x \notin A \vee (y \notin B \wedge y \notin C)$ *(by definition of union)*

therefore, $(x \notin A \vee y \notin B) \wedge (x \notin A \vee y \notin C)$ *(by distributive law)*

therefore, $(x, y) \notin A \times B \wedge (x, y) \notin A \times C$ *(by definition of Cartesian Products)*

therefore, $(x, y) \notin (A \times B) \cup (A \times C)$ *(by definition of union)*

Therefore, $a \notin (A \times B) \cup (A \times C)$

We have finally proved that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

3. Let A and B be sets. Show that:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \tag{3}$$

Answer:

Let $P = \overline{A \cup B}$ *and* $Q = \overline{A} \cap \overline{B}$

Let x *be an arbitrary element of* P *then* $x \in P \Rightarrow x \in \overline{(A \cup B)}$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ *and* $x \notin B$

$\Rightarrow x \in \overline{A}$ *and* $x \in \overline{B}$

$\Rightarrow x \in \overline{A} \cap \overline{B}$

$\Rightarrow x \in Q$

Therefore, $P \subseteq Q$ *... (i)*

Again let y *be an arbitrary element of* Q *then* $y \in Q \Rightarrow y \in \overline{A} \cap \overline{B}$

$\Rightarrow y \in \overline{A}$ *and* $y \in \overline{B}$

$\Rightarrow y \notin A$ *and* $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in \overline{(A \cup B)}$

$\Rightarrow y \in P$

Therefore, $Q \subseteq P$ *... (ii)*

Now, combine (i) and (ii), we get:

$P = Q$

$\overline{A \cup B} = \overline{A} \cap \overline{B}$