Seminar 6

In this seminar you will study:

- The Intermediate Value Theorem
- Numerical methods for finding the root of an equation
    - (Fixed Point) Iteration method
    - Bisection method

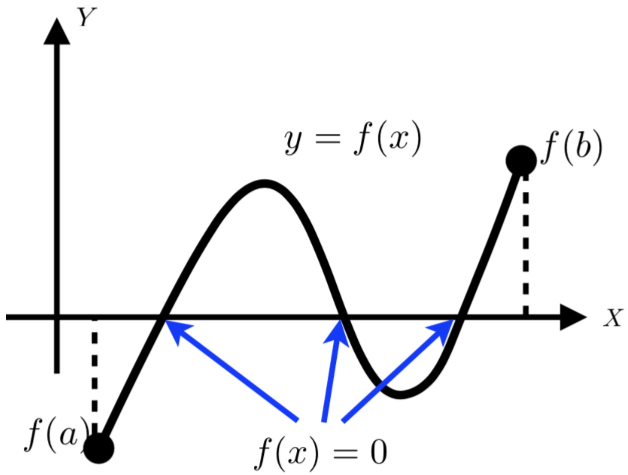
Intermediate Value Theorem

If two numbers a and b can be found such that

(i) $a < b$, and

(ii) $f(a)$ and $f(b)$ have **different** signs,

then, $f(x) = 0$ has *at least one* root in (a, b), provided that $f(x)$ is continuous in the interval $[a, b]$.



Intermediate Value Theorem (IVT)

Example: Show that $f(x) = 2x^3 - 5x^2 - 14x + 12 = 0$ has a root in $(0, 1)$.

Solution:

From the given interval $a = 0$ and $b = 1$

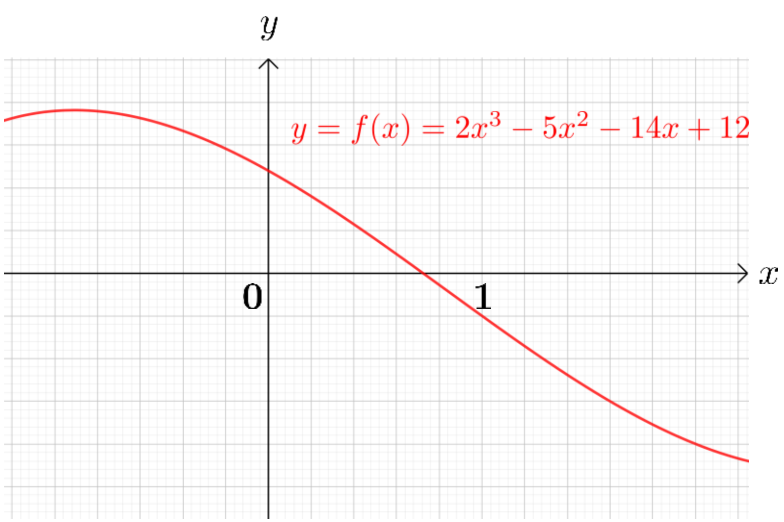
Here $\quad f(a) = f(0) = 12 \quad > 0$

and $\quad f(b) = f(1) = 2 - 5 - 14 + 12 = -5 \quad < 0$

$\therefore \quad f(0) \cdot f(1) < 0$

Thus, by the IVT $f(x) = 0$ has a root in $(0, 1)$.

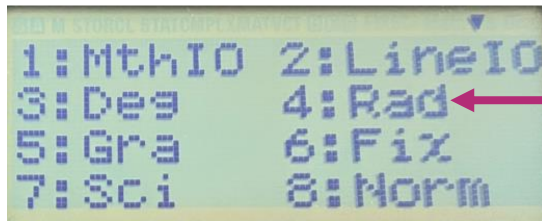
Note: in the exams, write the above steps when verifying the existence of roots in a given interval



Notes on Calculator Use

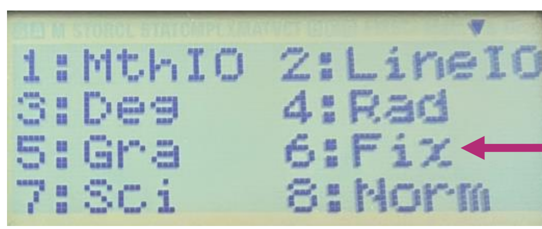
If the equation involves Trigonometric functions, set the calculator to **RADIAN** mode.

Shift Mode 4



If you are asked to obtain the root correct to n decimal places (d. p.)

Shift Mode 6 n



n is the number of decimal places required

Intermediate Value Theorem

(i). Show that $f(x) = \ln x - x + 3 = 0$

has a root in the interval $(4, 5)$.

(ii). Show that $f(x) = \dfrac{x^5}{4} - \sin x + \dfrac{1}{2} = 0$

has a root in the interval $(-1.5, -1.4)$.

Note: set calculator to radian mode

(iii). Show that $f(x) = \sin x - \dfrac{x^5}{4} - \dfrac{1}{2} = 0$

has a root in the interval $(1, 1.5)$.

Note: set calculator to radian mode

(iv). Show that $f(x) = x^3 - 3x^2 + 2x - 8 = 0$

has a root in the interval $(3, 3.5)$.

(Fixed-Point) Iteration Method

Example: Verify that $f(x) = x^2 + 4x - \sin x - 2 = 0$ has a root in $(0, 1)$.

Show that $f(x) = 0$ can be rearranged to give the iterative formula

$$x_{n+1} = \frac{\sin x_n + 2 - x_n^2}{4}.$$

Apply the Iteration method to find the root correct to 5 d. p.

Solution:

Step 1: Set calculator to RADIAN mode: $\boxed{\textbf{Shift} \quad \textbf{Mode} \quad \textbf{4}}$

Step 2: Fix calculator to 5 d. p.: $\boxed{\textbf{Shift} \quad \textbf{Mode} \quad \textbf{6} \quad \textbf{5}}$

(Fixed-Point) Iteration Method

Solution:

Step 3: Apply the Intermediate Value Theorem:

$$f(0) = (0)^2 + 4(0) - \sin(0) - 2 = -2 < 0$$

$$f(1) = (1)^2 + 4(1) - \sin(1) - 2 = 3 - \sin(1) > 0$$

$$\Rightarrow \quad f(0) \cdot f(1) < 0$$

$$\Rightarrow \quad f(x) = 0 \text{ has a root in } (0, 1).$$

Step 4: Derive the iterative formula

$$x^2 + 4x - \sin x - 2 = 0$$

$$\Rightarrow \quad 4x = \sin x + 2 - x^2$$

$$\Rightarrow \quad x = \frac{\sin x + 2 - x^2}{4}$$

$$\boxed{\Rightarrow \quad x_{n+1} = \frac{\sin x_n + 2 - x_n^2}{4}}$$

(Fixed-Point) Iteration Method

Solution:

Step 5: Set up Iterative formula on calculator

$$\boxed{x_{n+1} = \frac{\sin x_n + 2 - x_n^2}{4}}$$

On calculator:

Start with: $x_0 = \dfrac{0+1}{2} = 0.5$

Enter $\mathbf{0.5}$ and press "$=$"

Enter the iterative formula obtained in Step 4 (replace x_n with Ans)

Enter the iterative formula:

$(\sin(\text{Ans}) + 2 - \text{Ans}^2) \div 4$

(Fixed-Point) Iteration Method

Solution:

Step 6: Write down succesive approximations

n	x_n
0	0.50000
1	0.55736
2	0.55457
3	0.55476
4	$\boxed{0.55475}$
5	$\boxed{0.55475}$

Note: All approximations and the final result must be given with the required d.p.

Note: The desired root is obtained when succesive approximations are equal

$\Rightarrow$ The desired root is 0.55475

(Fixed-Point) Iteration method

(i). Verify that $f(x) = x^2 - \sin x - 2 = 0$ has a root in $(-2, -1)$.

Show that $f(x) = 0$ can be rearranged to give the iterative formula

$$x_{n+1} = \frac{\sin x_n}{x_n - \sqrt{2}} - \sqrt{2}.$$

Apply the iteration method to find the root correct to 5 d.p.

Answer: -1.06155

n	x_n
0	-1.50000
1	-1.07193
2	-1.06101
3	-1.06158
4	$\boxed{-1.06155}$
5	$\boxed{-1.06155}$

(ii). Verify that $f(x) = x^2 - \sin x - 2 = 0$ has a root in $(1, 2)$.

Show that $f(x) = 0$ can be rearranged to give the iterative formula

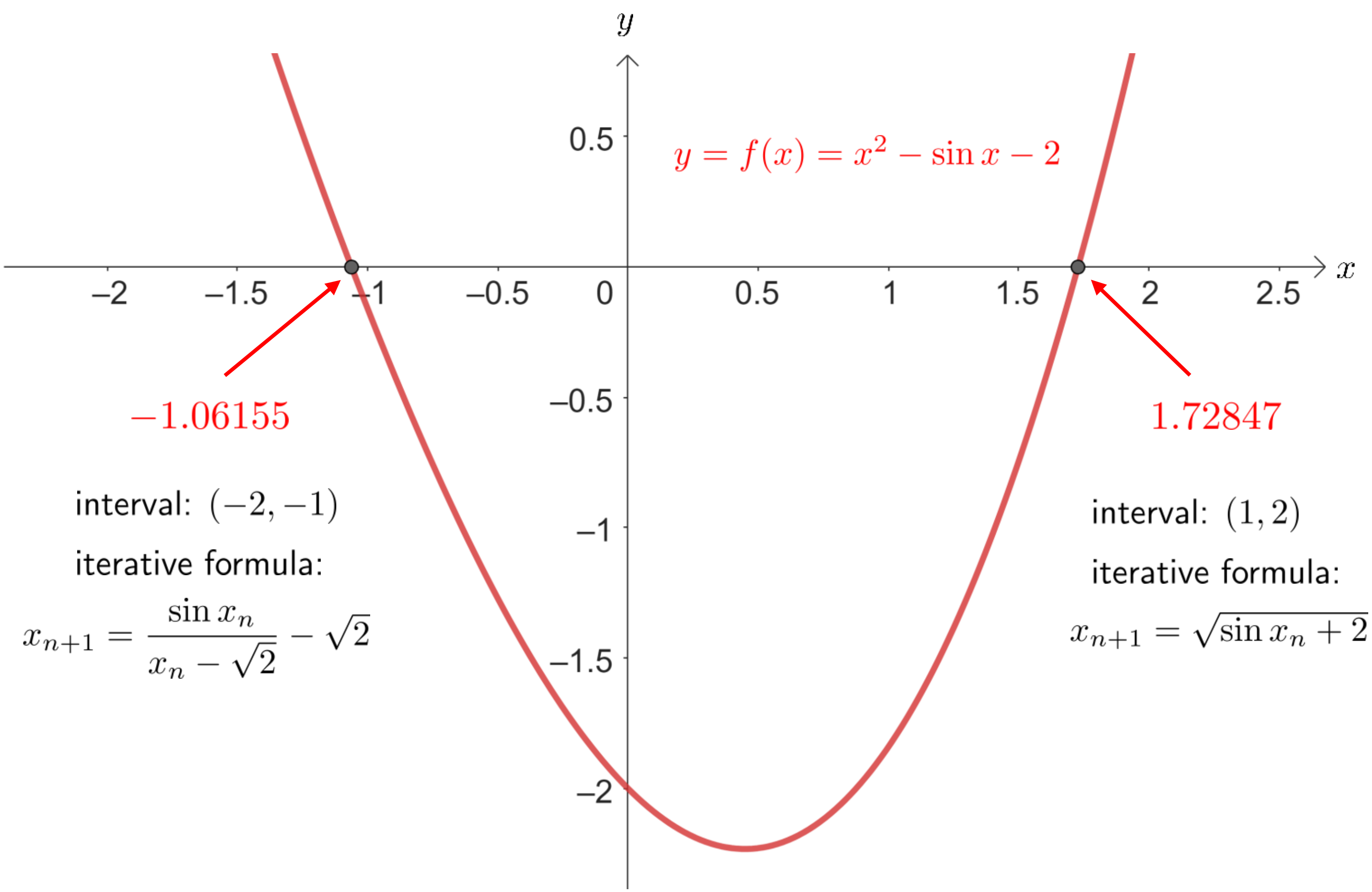
$$x_{n+1} = \sqrt{\sin x_n + 2}.$$

Apply the iteration method to find the root correct to 5 d.p.

Answer: 1.72847

n	x_n
0	1.50000
1	1.73133
2	1.72834
3	1.72847
4	1.72847

(Fixed-Point) Iteration method



(Fixed-Point) Iteration method

(i). Verify that $f(x) = x^2 + e^x - 2 = 0$ has a root in $(-2, -1)$.

Show that $f(x) = 0$ can be rearranged to give the iterative formula

$$x_{n+1} = \frac{2 - e^{x_n}}{x_n}.$$

Apply the iteration method to find the root correct to 2 d.p.

Answer: -1.32

n	x_n
0	-1.50
1	-1.18
2	-1.43
3	-1.23
4	-1.39
5	-1.26
6	-1.36
⋮	⋮
21	-1.31
22	$\boxed{-1.32}$
23	$\boxed{-1.32}$

(ii). Verify that $f(x) = x^2 + e^x - 2 = 0$ has a root in $(0, 1)$.

Show that $f(x) = 0$ can be rearranged to give the iterative formula

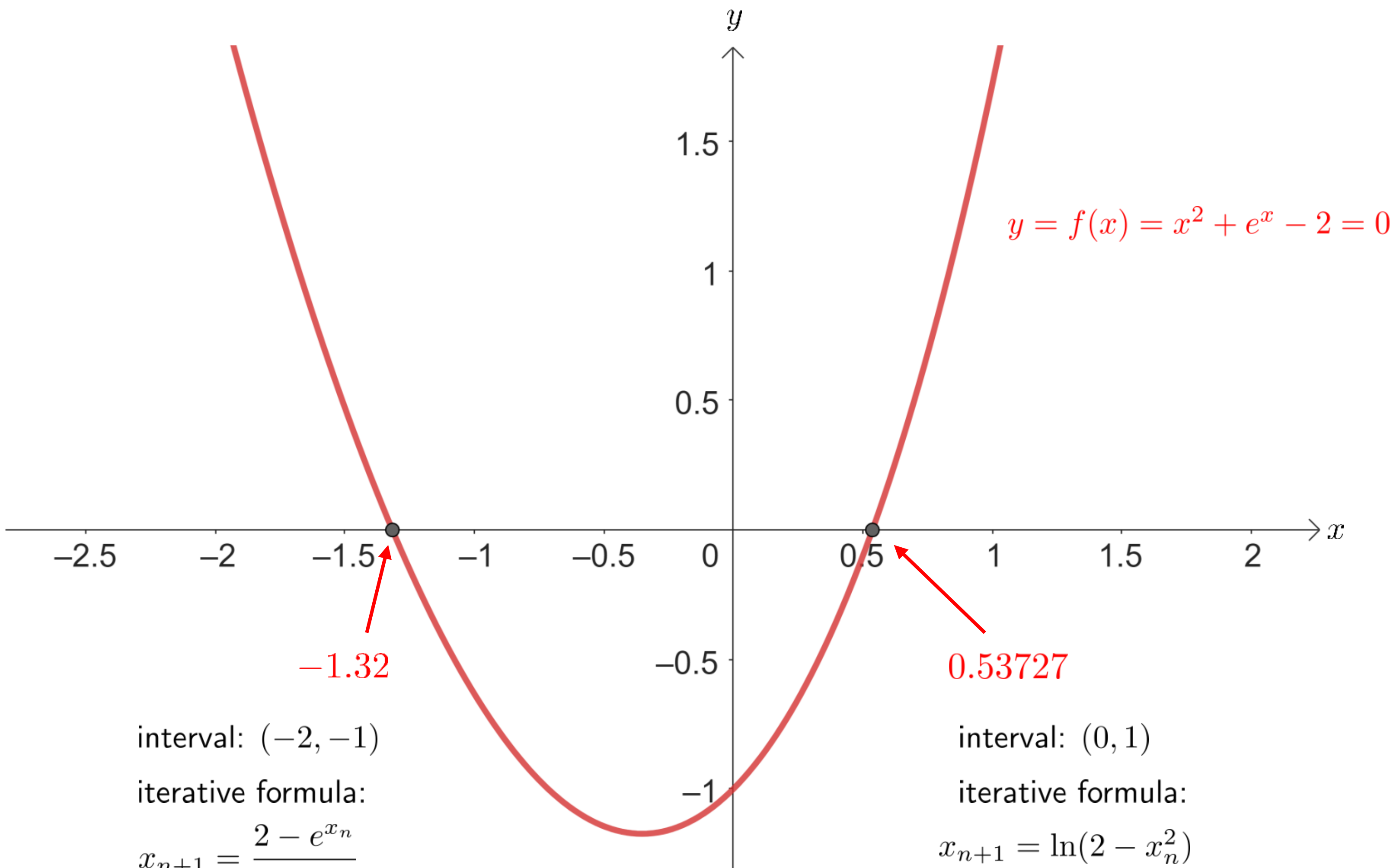
$$x_{n+1} = \ln(2 - x_n^2).$$

Apply the iteration method to find the root correct to 5 d.p.

Answer: 0.53727

n	x_n
0	0.50000
1	0.55962
2	0.52285
3	0.54617
4	0.53163
5	0.54080
6	0.53505
⋮	⋮
23	0.53728
24	$\boxed{0.53727}$
25	$\boxed{0.53727}$

(Fixed-Point) Iteration method



interval: $(-2, -1)$

iterative formula:

$$x_{n+1} = \frac{2 - e^{x_n}}{x_n}$$

interval: $(0, 1)$

iterative formula:

$$x_{n+1} = \ln(2 - x_n^2)$$

(Fixed-Point) Iteration method

IMPORTANT NOTES:

- Make sure you tabulate the obtained approximations.
- Make sure you write the result from all iterations in the required d.p.
- The desired root is obtained when $\boxed{x_{n+1} = x_n}$

Watch the [Video on calculator use for the Iteration Method](#) on Moodle

Bisection Method

Example: Show that $f(x) = 2x^3 - 5x^2 - 14x + 12 = 0$ has a root in $(0, 1)$.
Use Bisection method to numerically find the root correct to 2 d. p.
Show the steps of calculation for finding x_0, x_1, x_2, and x_3.

Solution:

Step 1: Fix calculator to 2 d.p.

Shift Mode 6 2

Step 2: Use the IVT to find the zeroth approximation of the root

Let $a = 0$ and $b = 1$

Since $f(0) > 0$ and $f(1) < 0$,

$\therefore$ root lies between $a = 0$ and $b = 1$

$$\therefore \quad \text{zeroth approximation } x_0 = c = \frac{a+b}{2} = \frac{0+1}{2}$$

$$= 0.50$$

then, $f(c) = 2(0.50)^3 - 5(0.50)^2 - 14(0.50) + 12 \quad > 0$

Bisection Method

Solution:

Step 3: Obtain successive approximations of the root

$\therefore$ Replace a by c

New variable c must always be used in the next step.

$\cancel{f(a) > 0}$

$f(b) < 0$

$f(c) > 0$

Then, proceed by entering values in the Table; continue until a root of desired accuracy is obtained.

Bisection Method:

Solution:

Step 4: Use of table to find the root

only signs required

n	a	b	$c = \frac{a+b}{2}$	$f(a)$	$f(b)$	$f(c)$	Decision: Replace __ by c
0	0.00	1.00	$x_0 = 0.50$	> 0	< 0	> 0	a by c
1	0.50	1.00	$x_1 = 0.75$	> 0	< 0	< 0	b by c
2	0.50	0.75	$x_2 = 0.63$	> 0	< 0	> 0	a by c
3	0.63	0.75	$x_3 = 0.69$	> 0	< 0	> 0	a by c

Bisection method

(i). Show that $f(x) = x^3 - 2x^2 + x - 7 = 0$ has a root in $(2.5, 3)$.

Apply the Bisection method to find the root, give your answer in 2 d. p.,

and show the steps of calculation for finding x_0, x_1, and x_2.

(ii). Show that $f(x) = x^2 + 4x - \sin x - 2 = 0$ has a root in $(0, 1)$.

Apply the Bisection method to find the root, give your answer in 2 d. p.,

and show the steps of calculation for finding x_0, x_1, and x_2.

Bisection method (Solution)

(i).

n	a	b	$c = \dfrac{a+b}{2}$	$f(a)$	$f(b)$	$f(c)$	Decision: Replace __ by c
0	2.5	3	2.75	< 0	> 0	> 0	Replace b by c
1	2.5	2.75	2.63	< 0	> 0	< 0	Replace a by c
2	2.63	2.75	2.69	< 0	> 0	> 0	Replace b by c

(ii).

n	a	b	$c = \dfrac{a+b}{2}$	$f(a)$	$f(b)$	$f(c)$	Decision: Replace __ by c
0	0	1	0.5	< 0	> 0	< 0	Replace a by c
1	0.5	1	0.75	< 0	> 0	> 0	Replace b by c
2	0.5	0.75	0.63	< 0	> 0	> 0	Replace b by c

THANKS FOR YOUR ATTENTION