AE1MCS: Mathematics for Computer Scientists

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Aim and Learning Objectives

- To be able to understand different methods of proving theorems.
- To be able to apply different methods to construct proofs.

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 1, Section 1.8. Introduction to Proofs
- Chapter 5, Section 5.1. Mathematical Induction
- Chapter 5, Section 5.2. Strong Induction

Methods of Proving Theorems

- Direct Proof
- Proof by Contraposition
- Proof by Contradiction
- Proof by Induction
- Proof by Cases

Direct Proof and Indirect Proof

If a proof leads from the premises of a theorem to the conclusion, then it is a direct proof, otherwise, it is an indirect proof.

- Important true propositions are called *theorems*.
- A *lemma* is a preliminary proposition useful for proving later propositions.
- A *corollary* is a proposition that follows in just a few logical steps from a theorem.

Direct Proof

A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that

if p is true, then q must also be true,

so that the combination p true and q false never occurs.

Exercise

Give a direct proof of the theorem ‘If n is an odd integer, then n^2 is odd’.

p $\qquad\qquad$ q

Exercise Answer

Give a direct proof of the theorem ‘If n is an odd integer, then n^2 is odd’.

Proof.

p

Suppose n is an odd integer. Then there exists an integer k such that $n = 2k + 1$. $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(\underbrace{2k^2 + 2k}_{l}) + 1$. Since $2k^2 + 2k$ is an integer, n^2 is odd. □

Good Proofwriting Tips

- Proofs should be composed of sentences that include verbs, nouns, and grammar.
- Never start a sentence with a mathematical symbol. In other words, always start a sentence with a word. This is to avoid confusion, as "." can also be a mathematical symbol, so you don't want people to believe you are performing multiplication when you are simply ending a sentence and beginning another.
- When drawing a conclusion, it is generally good form to give a reason for that conclusion. You see above things like "by definition", "by arithmetic rules," etc. This can help explain the intermediary conclusions of the proof. If you can't come up with a reason like this for something to be true, it may not be a fair conclusion to draw.
- If you introduce a new symbol, you should clearly define what kind of thing it is.

Proof by Contraposition

$p \to q$ ✓

$\neg q \to \neg p$ ✓

- An extremely useful type of indirect proof is known as proof by contraposition.
- Proofs by contraposition make use of the fact that the conditional statement $p \to q$ is equivalent to its contrapositive, $\neg q \to \neg p$.
- The conditional statement $p \to q$ is proved by showing that its contrapositive, $\neg q \to \neg p$, is true.

Exercise

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

p q

$\neg q \to \neg p$

Exercise Answer

Let p be " n is an integer and $3n+2$ is odd.

Let q be " n is odd "

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Proof.

To prove that if n is an integer and $3n + 2$ is odd, then n is odd, we show if n is an even integer, then $3n + 2$ is even.

Suppose n is an even integer. Then there exists an integer k such that $n = 2k$. $3n + 2 = 3 \times 2k + 2 = 2 \times (3k + 1)$. Since $3k + 1$ is an integer, $3n + 2$ is even.

By contraposition, we showed that if n is an integer and $3n + 2$ is odd, then n is odd. □

Proof by Contradiction

- Suppose we want to prove that a statement p is true
- Furthermore, suppose that we can find a contradiction q such that $\neg p \to q$ is true. Because q is false, but $\neg p \to q$ is true, then p must be true.
- The statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \to (r \wedge \neg r)$ is true for some proposition r.
- Proofs of this type are called **proofs by contradiction.**
- A proof by contradiction is another type of indirect proof.

Proof of p by Contradiction

$\neg p \rightarrow r \wedge \neg r$

- 1. Assume p is false.
- 2. Follow the method of Direct Proof to conclude that q must be true (for some q that is observably false).
- 3. Conclude that p cannot be false.
- 4. Conclude that p is therefore true.

Proof of $p \to q$ by Contradiction

$$\neg(p \to q) \to r$$

$$\neg p \to \neg q$$

- 1. Assume p is true, and q is false.
- 2. Follow the method of Direct Proof to conclude that r must be true (for some r that is observably false). → Contradiction
- 3. Conclude that if p is true, q cannot be false.
- 4. Conclude that anytime p is true, q is also true, and thus $p \to q$.

Exercise

Rational number: $\frac{p}{q}$

Prove that $\sqrt{2}$ is irrational.

Exercise Answer

assume $\neg p$

Prove that $\sqrt{2}$ is irrational. $\neg p \to r \wedge \neg r$

$\neg p$ false.

p.

Proof.

Suppose $\sqrt{2}$ is rational. Then there exist integers p and q with $q \neq 0$ such that $\sqrt{2} = p/q$ and p and q do not have any common factor. Thus, $2 = p^2/q^2$. $p^2 = 2q^2$. Thus, p^2 is even. Since if n is odd, then n^2 is odd (proved in previous slides), p is even. Hence there exists an integer k such that $p = 2k$.
Then $p^2 = (2k)^2 = 2q^2$. $q^2 = 2k^2$. Thus q^2 is even, hence q is even. Thus, p and q are both even, which contradicts the fact that p and q do not have any common factor. This is impossible (Contradiction). Therefore, it cannot be the case that the proposition is false, so it must be true.

$\neg r$

□

Good Proofwriting Tips

- When proving a statement with the method of contradiction, inform your reader that you are planning to achieve contradiction with an introductory clause such as "Suppose, for the sake of contradiction" or "Suppose the proposition is false" followed by the assumptions you wish to make.

by contradiction

Proof of Equivalence

To prove a theorem that is a biconditional statement or a bi-implication, that is, a statement of the form $p \leftrightarrow q$, we show that $p \rightarrow q$ and $q \rightarrow p$ are both true.

Good Proofwriting Tips

- When proving a biconditional statement, clearly communicate when you are proving each direction.

Proof of Equivalence

Sometimes a theorem states that several propositions are equivalent. Such a theorem states that propositions $p_1, p_2, p_3, ..., p_n$ are equivalent. This can be written as

$p_1 \to p_2$ $\quad$ $p_2 \to p_1$ $\quad$ n

$p_2 \to p_3$ $\quad$ $p_3 \to p_2$

$$p_1 \leftrightarrow p_2 \leftrightarrow ... \leftrightarrow p_n,$$

$p_{n-1} \to p_n$ $\quad$ $p_n \to p_{n-1}$

which states that all n propositions have the same truth values, and consequently, that for all i and j with $1 \le i \le n$ and $1 \le j \le n$, p_i and p_j are equivalent. One way to prove these mutually equivalent is to use the tautology

$p_1 \to p_2 \to p_3$

$$[p_1 \leftrightarrow p_2 \leftrightarrow ... \leftrightarrow p_n] \leftrightarrow [(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge ... \wedge (p_n \to p_1)].$$

$p_n \quad p_{n-1} \quad n+1$

This shows that if the n conditional statements $p_1 \to p_2$, $p_2 \to p_3$,..., $p_n \to p_1$ can be shown to be true, then the propositions p_1, p_2,..., p_n are all equivalent.

Counterexamples

To show that a statement of the form $\forall x\, P(x)$ is false, we need only find a counterexample, that is, an example x for which $P(x)$ is false.

$p \not\Leftrightarrow q$

Exercise

Show that the statement ‘Every positive integer is the sum of the squares of two integers’ is false.

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Proof.

3 is a positive integer but is not the sum of the squares of two integers. Note that the only perfect squares not exceeding 3 are $0^2 = 0$ and $1^2 = 1$. Therefore, the statement 'Every positive integer is the sum of the squares of two integers' is false. $\square$

Proof by Cases

- Sometimes we cannot prove a theorem using a single argument that holds for all possible cases.
- Need to consider different cases separately. $p \to q$
- **Rationale:** To prove a conditional statement of the form

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \to q$$

the tautology

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \to q] \leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \ldots \wedge (p_n \to q)]$$

can be used as a rule of inference.

Proof by Cases

- The original conditional statement with a hypothesis made up of a disjunction of the propositions p_1, p_2,..., p_n can be proved by proving each of the n conditional statements $p_i \rightarrow q$, $i = 1, 2, ..., n$, individually. Such an argument is called a **proof by cases.**
- A proof by cases must cover **all possible cases** that arise in a theorem.

Exercise

Prove that if n is an integer, then $n^2 \geq n$.

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$p \rightarrow q$

Proof.

Let us prove by cases.

- If $n = 0$, then $0^2 \geq 0$.
  $p_1 \rightarrow q$
- If $n \geq 1$, we multiply both sides of the inequality $n \geq 1$ by the positive integer n, then we have $n^2 \geq n$.
  $p_2 \rightarrow q$
- If $n \leq -1$, $n^2 \geq n$ holds, since $n^2 \geq 0$.
  $p_3 \rightarrow q$

Thus, in each case, $n^2 \geq n$. □

Good Proofwriting Tips

Proof by cases.

- When proving by cases, clearly communicate to the reader that cases will be considered, and label the cases as they occur. Tell the reader how you will split by cases before you do it.

Induction

In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.

Proofs using mathematical induction have two parts.

1. **Basis Step:** We show that the statement holds for the positive integer 1 (i.e. $P(1)$ is true). $P(0)$
2. **Inductive Step:** We show that if the statement holds for a positive integer then it must also hold for the next larger integer (i.e. for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true).

Mathematical Induction

Mathematical induction can be expressed as the following rule of inference

$$\begin{array}{l} P(1) \\ \forall k\,(P(k) \to P(k+1)) \\ \hline \therefore \forall n\, P(n) \end{array}$$

where the domain is the set of positive integers.

Mathematical Induction: Remark

In a proof by mathematical induction it is **not** assumed that $P(k)$ is true for all positive integers!

It is only shown that if it is assumed that $P(k)$ is true, then $P(k+1)$ is also true.

Using Mathematical Induction

- Mathematical induction can be used to prove statements of the form $\forall n\, P(n)$, where the domain is the set of positive integers.
- Mathematical induction can be used to prove an extremely wide variety of theorems, each of which is a statement of this form.
  - summation formulae, inequalities, identities for combinations of sets, divisibility results, theorems about algorithms, and some other creative results
  - the correctness of computer programs and algorithms

Proving Summation Formulae: Examples

- Show that if n is a positive integer, then $1 + 2 + ... + n = \frac{n(n+1)}{2}$.
- Use mathematical induction to show that $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.
- Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers n.

An example

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Strong Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis Step We verify that the proposition $P(1)$ is true.

Inductive Step We show that the conditional statement $[P(1) \wedge P(2) \wedge ... \wedge P(k)] \to P(k+1)$ is true for all positive integers k.

Strong Induction

- The difference between mathematical induction and strong induction is the inductive step.
  - Mathematical induction: $\forall k\,(P(k) \rightarrow P(k+1))$.
  - Strong induction: $\forall k\,((P(1) \land P(2) \land ... \land P(k)) \rightarrow P(k+1))$.
- Mathematical induction and strong induction are actually equivalent.
- This is, each can be shown to be a valid proof technique assuming that the other is valid.

Expected Learning Outcomes

- To be able to understand different methods of proving theorems.
- To be able to apply different methods to construct proofs.

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