# Limits, Derivatives and Optimization Exercises

1. Show that each of these equations are true, using the definition of a limit. Do not use any of the rules or theorems about limits.

   (a)
   $$\lim_{x\to 2} 4x + 1 = 9$$
   ANSWER:
   Here $f(x) = 4x + 1, a = 2, L = 9$ so, for any $\varepsilon > 0$, taking $0 < \delta \leq \varepsilon/4$,
   $$\begin{aligned} |x - a| < \delta \quad &\Rightarrow |4x - 8| < 4\delta \leq \varepsilon \\ &\Rightarrow |(4x + 1) - 9| < \varepsilon \\ &\Rightarrow |f(x) - L| < \varepsilon \end{aligned}$$

   (b)
   $$\lim_{x\to 3} \frac{3x - x^2}{3 - x} = 3$$
   ANSWER:
   $$\lim_{x\to 3} \frac{3x - x^2}{3 - x} = \lim_{x\to 3} \frac{(3 - x)x}{3 - x} = \lim_{x\to 3} x$$
   Here $f(x) = x, a = 3, L = 3$ so, for any $\varepsilon > 0$, taking $0 < \delta \leq \varepsilon$,
   $$\begin{aligned} |x - a| < \delta \quad &\Rightarrow |x - 3| < \varepsilon \\ &\Rightarrow |f(x) - L| < \varepsilon \end{aligned}$$

   (c)
   $$\lim_{x\to 1} \frac{(x + 1)(x - 2x^2 + 1)}{1 - x^2} = 3$$
   ANSWER:
   $$\lim_{x\to 1} \frac{(x + 1)(x - 2x^2 + 1)}{1 - x^2} = \lim_{x\to 1} \frac{(x + 1)(1 - x)(2x + 1)}{(1 - x)(1 + x)} = \lim_{x\to 1} 2x + 1$$
   Here $f(x) = 2x + 1, a = 1, L = 3$ so, for any $\varepsilon > 0$, taking $0 < \delta \leq \varepsilon/2$,
   $$\begin{aligned} |x - a| < \delta \quad &\Rightarrow |2x - 2| < 2\delta \leq \varepsilon \\ &\Rightarrow |(2x + 1) - 3| < \varepsilon \\ &\Rightarrow |f(x) - L| < \varepsilon \end{aligned}$$

2. Work out the derivative for these functions using the definition of derivative. Do not use any of the rules or theorems.

   (a)
   $$f(x) = 2x^2 - x + 1$$

   ANSWER:

   $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\left[2(x+h)^2-(x+h)+1\right]-\left[2x^2-x+1\right]}{h} \\ &= \lim_{h\to 0} \frac{4xh+2h^2-h}{h} \\ &= \lim_{h\to 0} 4x+2h-1 \\ &= 4x-1 \end{aligned}$$

   (b) For $x > 0$,
   $$f(x) = \frac{1}{\sqrt{x}}$$

   ANSWER:

   $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\ &= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{hx\sqrt{x+h}+h(x+h)\sqrt{x}} \\ &= \lim_{h\to 0} \frac{-h}{hx\sqrt{x+h}+h(x+h)\sqrt{x}} \\ &= \lim_{h\to 0} \frac{-1}{x\sqrt{x+h}+(x+h)\sqrt{x}} \\ &= \frac{-1}{x\sqrt{x}+x\sqrt{x}} \\ &= \frac{-1}{2x\sqrt{x}} \end{aligned}$$

3. Work out the derivative for these functions. Assume $x > 1$ for all cases. You may use any of the rules or theorems introduced in the lecture.

   (a) $f(x) = 4x^3 - 2x^2 + 5$.
   ANSWER:
   Using the rule for power terms, $f'(x) = 12x^2 - 4x$.

   (b) $f(x) = 3(\ln(x))^2$.
   ANSWER:

   $$\begin{aligned} f'(x) &= 3 \times 2\ln(x) \times \tfrac{1}{x} \quad \text{using the chain rule} \\ &= 6\ln(x)/x \end{aligned}$$

   (c) For $f(x) = (3x-1)(\ln(x)-1)$, show that $f'(x) = 3\ln(x) - \frac{1}{x}$.
   ANSWER:

   $$\begin{aligned} f'(x) &= (3x-1)\tfrac{1}{x} + 3(\ln(x)-1) \quad \text{using the chain rule} \\ &= 3\ln(x) - \tfrac{1}{x} \end{aligned}$$

(d) For $f(x) = 2\ln(\ln(x))x^4$, show that $f'(x) = 2x^3\left[4\ln(\ln x)) + \frac{1}{\ln x}\right]$.

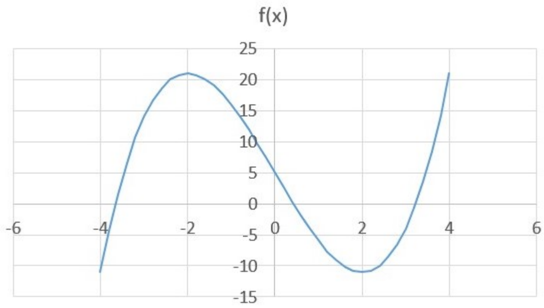
ANSWER:

$$\begin{aligned} f'(x) &= x^4 \tfrac{d}{dx}[2\ln(\ln(x))] + 2\ln(\ln(x)) \times 4x^3 && \text{using the product rule} \\ &= x^4 \tfrac{2}{\ln(x)} \times \tfrac{1}{x} + 2\ln(\ln(x)) \times 4x^3 && \text{using the chain rule} \\ &= 2x^3\left[4\ln(\ln x)) + \tfrac{1}{\ln x}\right] \end{aligned}$$

4. Consider the function $f(x) = x^3 - 12x + 5$.

   (a) Find all critical numbers for $f$.
   ANSWER:
   The function is well-defined for all real numbers $x$, so look for cases where $f'(x) = 0$.
   $$f'(x) = 3x^2 - 12 = 0.$$
   This is solved for $x = -2$ and $x = 2$ so these are the critical numbers.

   (b) Sketch the graph of $f$ to determine whether each critical number is a minima, maxima or neither.
   ANSWER:




   Therefore $x = -2$ is a maxima and $x = 2$ is a minima.

5. Prove the following statement for all positive integers $n$ using mathmatical induction:
$$\frac{d}{dx}x^n = nx^{n-1}.$$
ANSWER:
Base case $n = 1$:
$$\frac{d}{dx}x = 1 = x^0 = nx^{n-1}.$$
Inductive step, for $n > 1$, assume true for $n - 1$: $\frac{d}{dx}x^{n-1} = (n-1)x^{n-2}$.
Then
$$\begin{aligned} \tfrac{d}{dx}x^n &= \tfrac{d}{dx}(x.x^{n-1}) \\ &= x\tfrac{d}{dx}x^{n-1} + \tfrac{dx}{dx}x^{n-1} \quad \text{using the product rule} \\ &= x(n-1)x^{n-2} + x^{n-1} \\ &= (n-1)x^{n-1} + x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

6. Find two real numbers that sum to 100 and whose product is a maximum.

ANSWER:
Let $x, y$ be the two real numbers. Then $x + y = 100$ and maximize $xy$.
Find critical numbers,

$$\frac{d}{dx}xy = \frac{d}{dx}x(100 - x) = 100 - 2x = 0.$$

Hence $x = 50$ and $y = 50$.
This is a maximum since any deviation $\delta \neq 0$ from the solution gives a smaller product:

$$(x + \delta)(y - \delta) = xy + (y - x)\delta - \delta^2 = xy - \delta^2 < xy.$$

7. A vase manufacturer need a container to carry the vases on his truck. The container has to be 1 meter wide to fit on the truck and needs to be $3m^3$ in volume. The roofing material costs 50 CNY per $m^2$, the side material costs 40 CNY per $m^2$ and the floor material costs 10 CNY per $m^2$. The *standard* container has length 3 meters and height 1 meter. However, the analytics team figure out how to build the container at minimum cost.

   (a) How much will the standard container cost to build?
   (b) What are the dimensions of the container that gives minimum cost?
   (c) How much saving does the new container provide?

   ANSWER:
   If $x$ is the length and $h$ the height of the container, the volume is given by $xh = 3$ since the width is 1 meter.
   The cost is given by

$$C = 50x + 40 \times (2xh + 2h) + 10x = 20(3x + 4xh + 4h).$$

   (a) Therefore, the standard container has $x = 3$ and $h = 1 \Rightarrow C = 20(9+12+4) = 500$ CNY.
   (b) What are the dimensions of the container that gives minimum cost? Substitute volume equation $xh = 3$ into the cost equation to get it just in terms of $h$:

$$C = 20(9/h + 12 + 4h).$$

   Differentiate $C$ to find extemum, using the result from part 2:

$$\frac{dC}{dh} = 20(-9/h^2 + 4) = 0$$

$$\Rightarrow h^2 = \frac{9}{4} \Rightarrow h = \frac{3}{2}$$

   since $h$ must be a positive number and cannot be zero since then the volume would be zero. Then $x = 3/h = 2$.
   So, overall, the optimal container has height 1.5 meters and length 2 meters.

(c) Cost with this design is $C = 20(6 + 12 + 6) = 480$ CNY.
Therefore the saving is 20 CNY.

8. Consider designing a particular hard disk. This hard disk can run up to speeds where a block of data can be read in 4ms. So time to read a block is $t > 4$. However, the faster that the hard disk runs, the more prone to error in reading a block, and the probability of error is given by

$$p = \frac{1}{1 + \exp(t/2 - 3.2)}.$$

If an error occurs, it needs to be read again and this takes $t$ ms. If this also fails with an error, then an attempt to read the block is made again, and so on, until the block is read without error. We want to set the time $t$ in order to minimize the *expected value* of time to read without error.

(a) Show that the expected value of time to read without error, as a function of $t$, taking into account the probability of error, is

$$T_e(t) = t\left(\frac{1}{1-p}\right).$$

Hint: you will need the formula for an infinite geometric progression: for $0 \leq r < 1$,

$$1 + r + r^2 + r^3 + \cdots = \sum_{i=1}^{\infty} r^i = \frac{1}{1-r}.$$

ANSWER:
The expected time is the time for the first read plus probability of error times time for next read and so on:

$$\begin{aligned} T_e(t) &= t + p(t + p(t + p(t + \cdots))) \\ &= t + pt + p^2 t + p^3 t + \cdots \\ &= t\left(\tfrac{1}{1-p}\right) \end{aligned}$$

(b) Compute the gradient $\frac{d}{dt}T_e(t)$.
Hint: $\frac{d}{dt}\exp(t) = \exp(t)$.

ANSWER:
Substituting the formula for $p$,

$$\begin{aligned} T_e(t) &= t\left(\tfrac{1}{1-\frac{1}{1+\exp(t/2-3.2)}}\right) \\ &= t\left(\tfrac{1+\exp(t/2-3.2)}{1+\exp(t/2-3.2)-1}\right) \\ &= t\left(\tfrac{1+\exp(t/2-3.2)}{\exp(t/2-3.2)}\right) \\ &= t\left(\tfrac{1}{\exp(t/2-3.2)} + 1\right) \end{aligned}$$

Then, using the product rule and then the chain rule,

$$\begin{aligned}
\frac{d}{dt}T_e(t) &= \frac{d}{dt}t\left(\frac{1}{\exp(t/2-3.2)}+1\right)+t\frac{d}{dt}\left(\frac{1}{\exp(t/2-3.2)}+1\right)\\
&= \left(\frac{1}{\exp(t/2-3.2)}+1\right)+t\frac{-1}{\exp(t/2-3.2)^2}\exp(t/2-3.2)\times\frac{1}{2}\\
&= \left(\frac{1}{\exp(t/2-3.2)}+1\right)-\frac{t/2}{\exp(t/2-3.2)}\\
&= \frac{1-t/2}{\exp(t/2-3.2)}+1
\end{aligned}$$

(c) Compute the first 10 iterations of gradient descent to approximate the value of $t$ that minimized $T_e(t)$.
Hint: You can use Excel with a row for each iteration of the algorithm. Use the EXP function in Excel for the exponent.
ANSWER:
See the separate Excel spreadsheet, available with these answers.
Approximate solution is $t = 8.868$ ms, which gives $T_e(t) = 11.45$ ms.

(d) Based on your gradient descent, what is the minimum value of $T_e(t)$?
ANSWER:
$T_e(8.868) = 11.45$ ms.