Seminar 4

In this seminar you will study:

- Solving Trigonometric equations
- Addition and factor formulae
- Multi-angle and half-angle formulae
- Inverse trigonometric functions

Solving Trigonometric equations

Example 1: Solve $\sin\theta = \frac{1}{2}, \quad \theta \in [0, \pi]$.

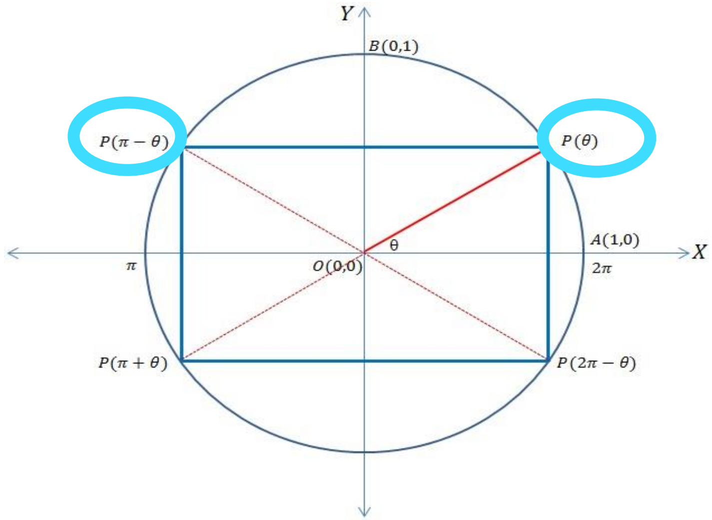
Solution:

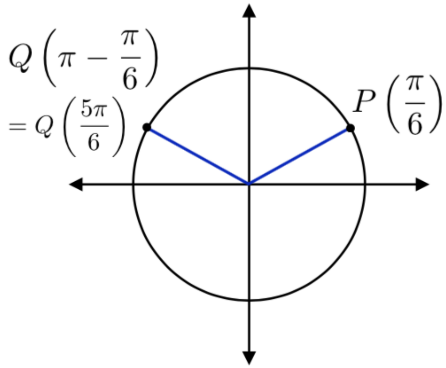
$$\sin\theta = \frac{1}{2}$$

$$\therefore \text{reference angle in quadrant I is } \alpha = \frac{\pi}{6}$$

$$\text{But } \theta \in [0, \pi]$$

$$\therefore \ \theta = \begin{cases} \dfrac{\pi}{6} \\ \\ \dfrac{5\pi}{6} \end{cases}$$



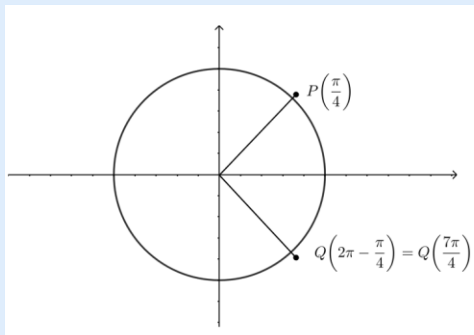


α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Solving Trigonometric equations

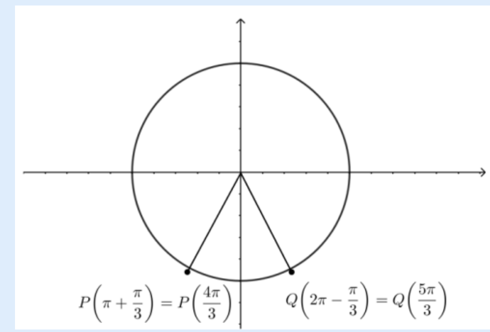
(i). Solve $\cos\theta = \dfrac{1}{\sqrt{2}}, \quad \theta \in [0, 2\pi].$

Answer: $\theta = \begin{cases} \dfrac{\pi}{4} \\ \dfrac{7\pi}{4} \end{cases}$



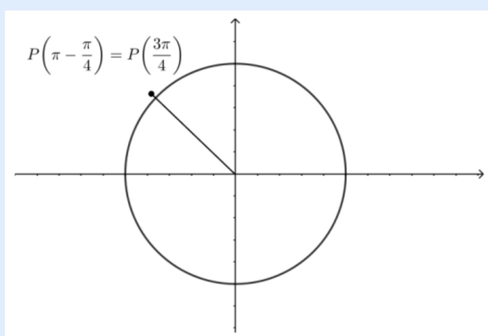
(ii). Solve $\sin\theta = -\dfrac{\sqrt{3}}{2}, \quad \theta \in [0, 2\pi]$

Answer: $\theta = \begin{cases} \dfrac{4\pi}{3} \\ \dfrac{5\pi}{3} \end{cases}$



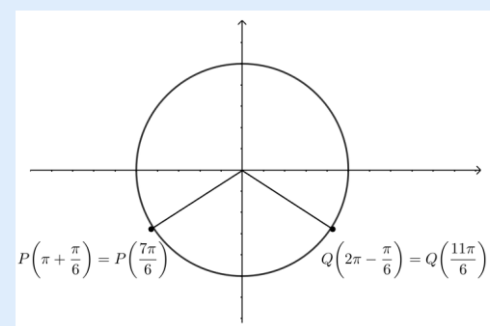
(iii). Solve $\tan\theta = -1, \quad \theta \in [0, \pi].$

Answer: $\theta = \dfrac{3\pi}{4}$



(iv). Solve $\operatorname{cosec}\theta = -2, \quad \theta \in (0, 2\pi].$

Answer: $\theta = \begin{cases} \dfrac{7\pi}{6} \\ \dfrac{11\pi}{6} \end{cases}$



Solving Trigonometric equations

Example 2: Solve for $\theta \in [0, 2\pi]$, $\sin^2\theta + 2\sin\theta - 3 = 0$.

Solution:

Let $\sin\theta = t$

$\therefore\ t^2 + 2t - 3 = 0$

$\Rightarrow (t+3)(t-1) = 0$

$\Rightarrow t = -3 \text{ or } t = 1$

But $\sin\theta \in [-1, 1]$

$\therefore\ \sin\theta \neq -3$

$\Rightarrow\ \sin\theta = 1$

$\therefore$ reference angle in quadrant I is $\alpha = \dfrac{\pi}{2}$

since $\theta \in [0, 2\pi]$

$\therefore\ \theta = \dfrac{\pi}{2}$

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Solving Trigonometric equations

(i). Solve for $\theta \in [0, 2\pi]$,

$$2\sin^2\theta + \sin\theta - 1 = 0.$$

Answer: $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ or $\dfrac{3\pi}{2}$

(ii). Solve for $\theta \in [0, \pi]$,

$$\tan^2\theta + 2\sec\theta + 1 = 0.$$

Answer: $\theta = \dfrac{2\pi}{3}$

(iii). Solve for $\theta \in [0, 2\pi]$,

$$\sin^2\theta + 5\cos\theta - 7 = 0.$$

Answer: $\varnothing$ (The empty set. Such θ does not exist).

(iv). Solve for $\theta \in [0, \pi]$,

$$4\cos^2\theta - 2(\sqrt{2}+1)\cos\theta + \sqrt{2} = 0.$$

Answer: $\theta = \dfrac{\pi}{3}$ or $\dfrac{\pi}{4}$

Addition formulae

$$
\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\sin(A-B) &= \sin A\cos B - \cos A\sin B \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B \\
\cos(A-B) &= \cos A\cos B + \sin A\sin B \\
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A\tan B} \\
\tan(A-B) &= \frac{\tan A - \tan B}{1 + \tan A\tan B}
\end{aligned}
$$

Addition formulae

Example: Prove that $\tan 29^\circ = \dfrac{\cos 16^\circ - \sin 16^\circ}{\cos 16^\circ + \sin 16^\circ}$.

Solution:

$$\begin{aligned}
\text{LHS} &= \tan 29^\circ \\
&= \tan(45^\circ - 16^\circ) \\
&= \frac{\tan 45^\circ - \tan 16^\circ}{1 + \tan 45^\circ \cdot \tan 16^\circ} \\
&= \frac{1 - \dfrac{\sin 16^\circ}{\cos 16^\circ}}{1 + \dfrac{\sin 16^\circ}{\cos 16^\circ}} \\
&= \frac{\dfrac{\cos 16^\circ - \sin 16^\circ}{\cos 16^\circ}}{\dfrac{\cos 16^\circ + \sin 16^\circ}{\cos 16^\circ}} = \frac{\cos 16^\circ - \sin 16^\circ}{\cos 16^\circ + \sin 16^\circ} = \text{RHS}
\end{aligned}$$

$\left[\text{using } \tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}\right]$

Addition formulae

(i). Prove that

$$\tan 74^\circ = \frac{\cos 29^\circ + \sin 29^\circ}{\cos 29^\circ - \sin 29^\circ}$$

(ii). Prove that

$$\sec x \cdot \sec y \cdot \sin(x+y) = \tan x + \tan y$$

(iii). Given $3\cos(x-y) = \cos(x+y)$,

prove that $2\tan x \cdot \tan y + 1 = 0$.

(iv). Prove that

$$\sin 135^\circ + \cos 30^\circ = \frac{\sqrt{3}+\sqrt{2}}{2}$$

Factor formulae

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

Allied angle formulae:

$$\sin(90^\circ - \theta) = \cos\theta$$

$$\cos(90^\circ - \theta) = \sin\theta$$

Factor formulae

Example: Prove that $\sin 20^\circ + \cos 50^\circ = \sin 80^\circ$

Solution:

$$\begin{aligned}
\text{LHS} &= \sin 20^\circ + \cos 50^\circ \\
&= \cos(90^\circ - 20^\circ) + \cos 50^\circ \\
&= \cos 70^\circ + \cos 50^\circ \\
&= 2\cos\left(\frac{70^\circ + 50^\circ}{2}\right)\cos\left(\frac{70^\circ - 50^\circ}{2}\right) \\
&= 2\cos 60^\circ \cos 10^\circ \\
&= 2\cdot\frac{1}{2}\sin(90^\circ - 10^\circ) \\
&= \sin 80^\circ = \text{RHS}
\end{aligned}$$

$$\left[\text{using } \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

Factor formulae

(i). Prove that

$$\cos 80^\circ + \cos 40^\circ = \sin 70^\circ$$

(ii). Prove that

$$\frac{\cos 6\theta - \cos 4\theta}{\sin\theta} + 2\sin 5\theta = 0$$

(iii). Prove that

$$\frac{\cos 70^\circ + \cos 10^\circ}{\sin 70^\circ - \sin 10^\circ} = \sqrt{3}$$

(iv). Prove that

$$\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \tan 4\theta$$

Multi-angle and half-angle formulae

Multi-angle formulae

$$1 + \cos 2\theta = 2\cos^2\theta$$

$$1 - \cos 2\theta = 2\sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Half-angle formulae

$$1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$$

$$1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

Use $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Multi-angle and half-angle formulae

Example: Prove that $\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}} = -\tan\theta, \quad \theta \in \left(\dfrac{\pi}{2}, \pi\right)$

Solution:

$$\begin{aligned}
\text{LHS} &= \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}} \\
&= \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}} \qquad \text{using } \boxed{\begin{array}{c} 1-\cos 2\theta = 2\sin^2\theta \\ 1+\cos 2\theta = 2\cos^2\theta \end{array}} \\
&= \sqrt{\tan^2\theta} \\
&= |\tan\theta| \\
&= -\tan\theta \qquad \left[\text{since } \tan\theta < 0 \text{ for } \theta \in \left(\frac{\pi}{2}, \pi\right)\right] \\
&= \text{RHS}
\end{aligned}$$

Multi-angle and half-angle formulae

(i). Prove that

$$\frac{1+\cos 2\theta+\sin 2\theta}{1-\cos 2\theta+\sin 2\theta}=\cot\theta$$

(ii). Prove that

$$\sqrt{2-2\cos 2\theta}=2\sin\theta,$$

where $0\leq\theta\leq\pi$.

(iii). Prove that

$$\sqrt{\frac{1}{4}+\frac{1}{4}\cos 4\theta}=\frac{\cos 2\theta}{\sqrt{2}},$$

where $0\leq\theta\leq\frac{\pi}{4}$.

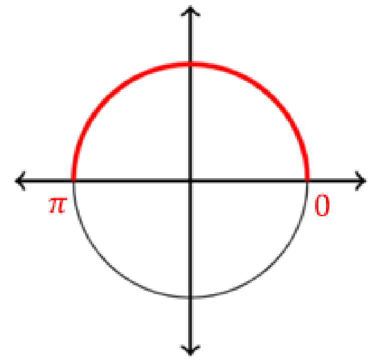
(iv). Prove that

$$\frac{1+\sin 2\theta}{1-\sin 2\theta}=\left(\frac{1+\tan\theta}{1-\tan\theta}\right)^2$$

Inverse Trigonometric functions

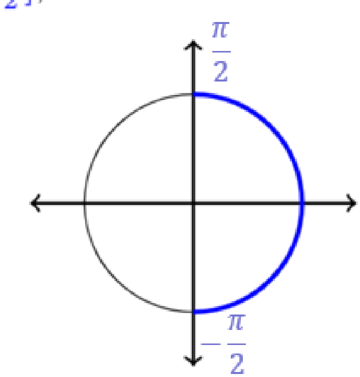
- The restrcited domain of cos function is $[0, \pi]$.

$[0, \pi]$, used for $\cos^{-1}$



- The restrcited domain of sin function is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

$[-\frac{\pi}{2}, \frac{\pi}{2}]$, used for $\sin^{-1}$



- The restrcited domain of tan function is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, used for $\tan^{-1}$

Inverse Trigonometric functions

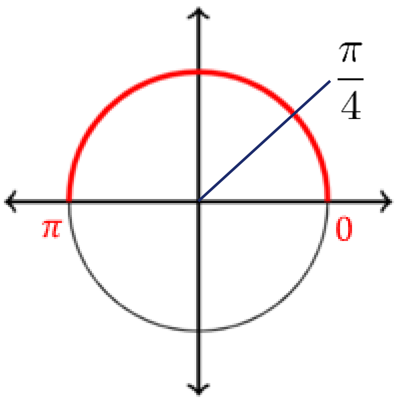
Example 1: Find $\cos^{-1}\left(\sin\dfrac{3\pi}{4}\right)$

Solution:

$$\cos^{-1}\left(\sin\frac{3\pi}{4}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{4} \qquad \boxed{\because \ \frac{\pi}{4} \in [0,\pi]}$$



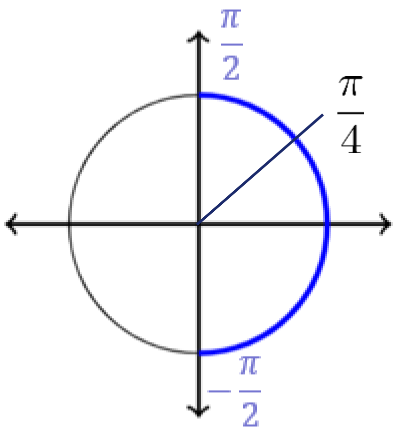
Inverse Trigonometric functions

Example 2: Find $\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$

Solution:

$$\sin^{-1}\left(\sin\frac{3\pi}{4}\right) \neq \frac{3\pi}{4} \quad \text{since} \quad \boxed{\frac{3\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$

But
$$\sin^{-1}\left(\sin\frac{3\pi}{4}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$
$$= \sin^{-1}\left(\sin\frac{\pi}{4}\right)$$
$$= \frac{\pi}{4} \qquad \boxed{\because \ \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$$



Inverse Trigonometric functions

(i). Find the value of

$$\tan^{-1}\left(\tan\frac{7\pi}{4}\right)$$

Answer: $-\dfrac{\pi}{4}$

(ii). Find the value of

$$\cos^{-1}\left(-\frac{1}{2}\right)$$

Answer: $\dfrac{2\pi}{3}$

(iii). Find the value of

$$\sin^{-1}\left(\cos\frac{3\pi}{4}\right)$$

Answer: $-\dfrac{\pi}{4}$

(iv). Find the value of

$$\cos^{-1}\left(\cos\frac{3\pi}{4}\right)$$

Answer: $\dfrac{3\pi}{4}$

THANKS FOR YOUR ATTENTION