### **AE1MCS: Mathematics for Computer Scientists**

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## Aim and Learning Objectives

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
- To gain a good understanding of other definitions: tautology, contradiction, contingency, logical equivalence, converse, contrapositive and inverse.
- To be able to draw truth tables and use them as a tool to solve logical problems;
- To be able to apply important logical equivalences to solve logical problems.

### Proposition

#### Definition

A proposition is a statement that is either true or false.

The area of logic that deals with propositions is called the *propositional logic* or *propositional calculus*.

## Is it a proposition?

- Beijing is the capital of China.
- √ What time is it? 

  X
  - 5 Read this sentence carefully. X
- $\begin{cases} 6 & x + 1 = 2. \\ 7 & x + y = z. \\ 8 & \text{If } x > 0, \text{ then } x > 1. \end{cases}$

Unfortunately, it is not always easy to decide if a claimed proposition is true or false.

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- 4 Every map can be colored with 4 colors so that adjacent regions have different colors [Four Color Theorem].
- 5 Every even integer greater than 2 is the sum of two primes [Goldbach's conjecture, 1742].



## Propositional Variable

- a variable that represents a proposition
- $\blacksquare$  denoted using a letter p, q, r, s, ...
- truth value: T (true); F (false)

## **Logical Operators**

■ Compound Proposition: formed from existing propositions using logical operators

■ Logical Operators

( ■ Negation
 ■ Conjunction
 ■ Disjunction
 ■ Implication

## Negation

#### **Definition (Negation)**

Let p be a proposition. The *negation* of p, denoted by p is the statement

'It is not the case that p'.

The proposition  $\neg p$  is read 'not p'. The truth value of  $\neg p$  is the opposite of the truth value of p.

P: Beijing is the capital of China

P	¬P
I	F
F	T

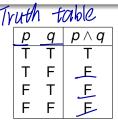
## Conjunction

#### **Definition** (Conjunction)

Let p and q be propositions. The *conjunction* of p and q, denoted by  $p \wedge q$ , is the proposition

'p and q'.

The proposition  $p \land q$  is true when both p and q are true and is false otherwise.



### Disjunction

#### Definition (Disjunction)

Let p and q be propositions. The *disjunction* of p and q, denoted by  $p \vee q$ , is the proposition

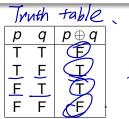
The proposition  $p \lor q$  is false when both p and q are false and is true otherwise.

р	q	$p \lor q$
I	<u>T</u>	⊣
ヿ	F	工
F	T	工
F	F	F

### **Exclusive Or**

#### Definition (Exclusive Or)

Let p and q be propositions. The *exclusive* or of p and q, denoted by  $p \oplus q$ , is the proposition that is true when *exactly* one of p and q is true and is false otherwise.



## **Implication**

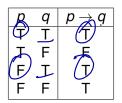
Definition (Implication) andicional statement

Let p and q be propositions. The *implication*  $p \rightarrow q$  is the proposition

'if p, then q'. p implies q.

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The proposition  $p \to q$  is false when p is true and q is false, and true otherwise. p is called the hypothesis or premise and q is called the conclusion or consequence.



The proposition  $p \rightarrow q$  is true, if p is false or q is true.

# Examples

- If Goldbach's Conjecture is true, then  $x^2 \ge 0$  for every real number x.
- If pigs fly, then your account will not get hacked.

### Bi-Implication

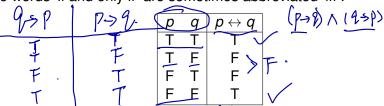
#### Definition (Bi-Implication)

Let p and q be propositions. The *bi-implication*  $p \leftrightarrow q$  is the proposition

'p if and only if 
$$q$$
'.

The bi-implication  $p \leftrightarrow q$  is true when p and q have the same truth values, and is false otherwise.

The words 'if and only if' are sometimes abbreviated 'iff'.



 $p \leftrightarrow q$  is true when both  $p \rightarrow q$  and  $q \rightarrow p$  are true, and is false otherwise.

#### More Definitions

- Tautology, Contradiction and Contingency
- Logical Equivalence
- Converse, Contrapositive and Inverse

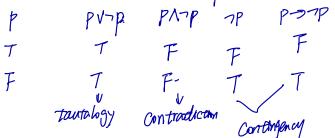
## Tautology, Contradiction and Contingency

Definition (Tautology, Contradiction and Contingency)

A compound proposition that is *always true*, no matter what the truth values of the propositions that occur in it, is called a **tautology**.. A compound proposition that is *always false* is called a **contradiction**. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

### **Exercise**

How to construct a tautology, a contradiction and a contingency using just one propositional variable? p



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How to construct a tautology, a contradiction and a contingency using just one propositional variable?

р	$\neg p$	$p \lor \neg p$	$p \wedge \neg p$	p  o  eg p
Т	F	Т	F	F
F	Т	T	F	Т

[There are some other ways not shown in the table above...]

### Logical Equivalence

#### Definition (Equivalence)

The compound propositions p and q are logically equivalent, if they always have the same truth value (i.e.  $p \leftrightarrow q$  is a tautology). The notation  $p \equiv q$  denotes that p and q are logically equivalent.

### **Exercise**

# Truth table.

- Are  $\neg (p \lor q)$  and  $\neg p \land \neg q$  logically equivalent? Why?
- Are  $p \rightarrow q$  and  $\neg p \lor q$  are logically equivalent? Why?

[Hint: construct truth tables]							
р	9-	pv9.	7(PV9)	70	79	77179	
TTFF	TFTF	イ イ て て 下	FFFT	FFTT	FT F	F. F.	llil

Are  $\neg(p \lor q)$  and  $\neg p \land \neg q$  logically equivalent? Why?

**Answer:** Yes. As shown in the truth table below,  $\neg(p \lor q)$  and  $\neg p \land \neg q$  always have the same truth value. Thus, they are logical equivalent.

р	q	$p \lor q$	$\neg (p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$
T	Т	Т	F	F	F	F
T	F	Т	F	F	Т	F
F	Τ	Т	F	T	F	F
F	F	F	T	Т	Т	Т

Are  $p \rightarrow q$  and  $\neg p \lor q$  are logically equivalent? Why?

**Answer:** Yes. As shown in the truth table below,  $p \to q$  and  $\neg p \lor q$  always have the same truth value. Thus, they are logical equivalent.

р	q	$\neg p$	$\neg p \lor q$	p  o q
Т	Т	F	Т	Т
T	F	F	F	F
F	Т	Т	Т	Т
F	F	Т	Т	Т

## Converse, Contrapositive and Inverse

- The **converse** of  $p \rightarrow q$  is the proposition  $q \rightarrow p$ .
- The **contrapositive** of  $p \rightarrow q$  is the proposition  $\neg q \rightarrow \neg p$ .
- The **inverse** of  $p \rightarrow q$  is the proposition  $\neg p \rightarrow \neg q$ .

Which pairs of the following propositions are equivalent? Why?

- - a conditional statement and its converse

a conditional statement and its contrapositive P-9

- a conditional statement and its inverse q  $p \rightarrow q$   $7p \rightarrow 7q$   $7p \rightarrow 7q$   $7q \rightarrow 7q$

# Some Important Logical Equivalences

	Equivalence	Name	]
1	$p \wedge T \equiv p$	Identity laws \	
2	$ oldsymbol{ ho} ee \mathcal{F} \equiv \mathcal{p}$		
3	$p \lor T \equiv T$	Domination laws	/
4	p <u>∧(F)</u> ≡(F)		
5	$p \lor p \equiv p$	Idempotent laws	/
6	$p \wedge p \equiv p$	·	
7	$\neg(\neg p) \equiv p$	Double negation law	]
8	$p \lor q \equiv q \lor p$	Commutative laws	V
9	$p \wedge q \equiv q \wedge p$		

## Some Important Logical Equivalences

	Equivalence	Name
10	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	Associative laws
11	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
12	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	Distributive laws
13	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
14	$ eg (p \land q) \equiv \neg p \lor \neg q $	De Morgan's laws
15	$\neg (p \lor q) \equiv \neg p \land \neg q$	
16	$p \lor (p \land q) \equiv p$	Absorption laws/
17	$\rho \wedge (\rho \vee q) \equiv \rho$	
18	$p \lor \neg p \equiv \overline{T}$	Negation laws
19	$\rho \wedge \neg \rho \equiv F$	

# Logical Equivalences involving Implications

## Logical Equivalences involving Bi-Implications

$$\begin{array}{c|c} 29 & p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p) \\ 30 & p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q \\ 31 & p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q) \\ 32 & \neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q \\ \end{array}$$

# Using De Morgan's Laws

Use De Morgan's laws to express the negations of the following sentences.  $\nearrow$ 

- Tony has a cellphone and he has a laptop computer. P^9
- Heather will go to the concert or Steve will go to the concert. YV\$

$$\frac{7(P \land 9)}{7(Y \lor S)} = \frac{7P \lor 79}{7(Y \lor S)} = \frac{1}{1} \land \frac{1}{1}$$

## Constructing New Logical Equivalences

- A proposition in a compound proposition can be replaced by a compound proposition that is logically equivalent to it without changing the truth value of the original compound proposition.
- Prove two propositions are logically equivalent by developing a series of logical equivalences.

#### Exercise

- Show that  $\neg(p \rightarrow q)$  and  $p \land \neg q$  are logically equivalent.
- Show that  $(p \lor (\neg p \land q))$  and  $\neg p \land \neg q$  are logically equivalent. Show that  $(p \land q) \to (p \lor q)$  is a tautology.

Show that  $\neg(p \to q)$  and  $p \land \neg q$  are logically equivalent.

#### Answer:

$$\begin{array}{ll} = \neg(p \rightarrow q) \\ = \neg(\neg p \lor q) & \text{by law 20} \\ = \neg(\neg p) \land \neg q & \text{by De Morgan's low} \\ \equiv p \land \neg q. & \text{by clouble negation Law} \end{array}$$

Show that  $\neg(p \lor (\neg p \land q))$  and  $\neg p \land \neg q$  are logically equivalent.

#### Answer:

$$\begin{array}{l} \underline{\neg}(p\vee(\neg p\wedge q))\\ \equiv \neg p\wedge f(\neg p\wedge q) \quad \text{Pe Morgan's}\\ \equiv \neg p\wedge(\neg p)\vee \neg q) \quad \text{by 1st De Morgan's}\\ \equiv \underline{\neg p\wedge(p\vee \neg q)} \quad \text{double negation}\\ \equiv \underline{(\neg p\wedge p)\vee(\neg p\wedge \neg q)} \quad \text{distributive law'}\\ \equiv F\vee(\neg p\wedge \neg q)\text{ by } \neg p\wedge p \equiv F\\ \equiv \neg p\wedge \neg q. \text{ identity law} \end{array}$$

Show that  $(p \land q) \rightarrow (p \lor q)$  is a tautology.

#### Answer:

$$(p \land q) \rightarrow (p \lor q)$$

$$\equiv \neg(p \land q) \lor (p \lor q)$$

$$\equiv (\neg p \lor \neg q) \lor (p \lor q)$$

$$\equiv (p \lor \neg p) \lor (q \lor \neg q)$$

$$= T \lor T = T$$

# List of Symbols

SYMBOL	MEANING	PAGE
$\neg p$	negation of p	3
$p \wedge q$	conjunction of $p$ and $q$	4
$p \vee q$	disjunction of $p$ and $q$	4
$p \oplus q$	exclusive or of $p$ and $q$	6
$p \rightarrow q$	the implication $p$ implies $q$	6
$p \leftrightarrow q$	biconditional of p and q	9
$p \equiv q$	equivalence of $p$ and $q$	23
ceiling talk or To	tautology	23
term of [u] w	contradiction	23
$P(x_1,\ldots,x_n)$	propositional function	36
$\forall x P(x)$	universal quantification of $P(x)$	38
$\exists x P(x)$	existential quantification of $P(x)$	40
$\exists !x P(x)$	uniqueness quantification of $P(x)$	41
$\exists : x F(x)$	therefore	64
$p\{S\}q$	partial correctness of S	364

## **Expected Learning Outcomes**

- To gain a good understanding of the definitions of proposition, propositional variable and logical operators;
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## Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

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