AE1MCS: Mathematics for Computer Scientists
Counting

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 6, Section 6.1 The Basics of Counting
- Chapter 6, Section 6.2 The Pigeonhole Principle
- Chapter 6, Section 6.3 Permutations and Combinations

Counting

- Combinatorics, the study of arrangements of objects, is an important part of discrete mathematics. (17th century, study of gambling games)
- Enumeration, the counting of objects with certain properties, is an important part of combinatorics.
- We must count objects to solve many different types of problems. Counting is used
  - to determine the complexity of algorithms.
  - to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand.
  - to calculate probabilities of events.

Content

- Basic Counting Principles
  - Product Rule
  - Sum Rule
  - Subtraction Rule (Principle of Inclusion-Exclusion)
  - Division Rule
- The Pigeonhole Principle
- Permutations and Combinations

Basic Counting Principles: Product Rule

The product rule applies when a procedure is made up of separate tasks.

Task 1 $\rightarrow$ task 2.

n_1 n_2

Definition

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Example

$$\begin{array}{ccc} & n_1 & n_2 & \\ 26\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} A \\ B \\ \vdots \\ Z \end{array} & \begin{array}{c} 1 \\ 2 \\ \vdots \\ 100 \end{array} & \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} 100 \end{array}$$

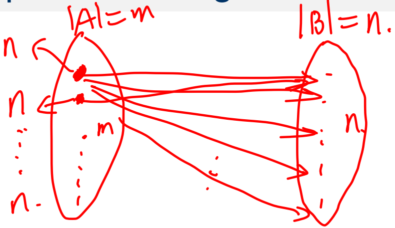
The chairs of a lecture room are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Hint: There are 26 English letters.

$n_1 \times n_2$

$26 \times 100 = 2600$

Example: Counting Functions



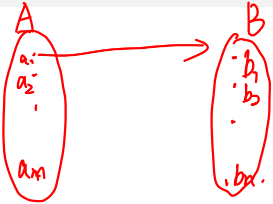
domain

How many functions are there from a set with m elements to a set with n elements?

Codomain

$$\overbrace{n \times n \times n \cdots n}^{m} = n^m$$

Example: Counting One-to-One Functions



$m \le n$

How many one-to-one functions are there from a set with *m* elements to one with *n* elements?

$n \times (n-1) \times (n-2) \times \cdots (n-m+1)$

a_1	n
a_2	$n-1$
a_3	$n-2$
$\vdots$	
a_m	$n-m+1$

Example: Counting elements in a Cartesian product

If A_1, A_2,..., A_m are finite sets, then what is the number of elements in the Cartesian product of these sets?

$$A_1 \times A_2 \times \cdots A_m = \{(a_1, a_2, \ldots a_m) \mid a_1 \in A_1, a_2 \in A_2, \vdots, a_m \in A_m\}$$

$|A_1|$ $|A_2|$

Example: Counting elements in a Cartesian product

If A_1, A_2,..., A_m are finite sets, then what is the number of elements in the Cartesian product of these sets?

Answer:
It is the product of the number of elements in each set.

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|$$

Basic Counting Principles: Sum Rule

Definition

If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: Choosing a Project

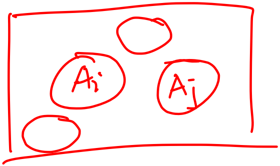
A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

$23 + 15 + 19 =$

Example: Counting elements in pairwise disjoint finite sets

$$|A_i \cup A_j| = |A_i| + |A_j|$$

$$A_i \cap A_j = \phi$$



If A_1, A_2,..., A_m are **pairwise disjoint** finite sets, then what is the number of elements in the union of these sets?

$$|A_1 \cup A_2, \cdots A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

Example: Counting elements in pairwise disjoint finite sets

If A_1, A_2,..., A_m are **pairwise disjoint** finite sets, then what is the number of elements in the union of these sets?

Answer:
It is the sum of the numbers of elements in the sets.

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

where $A_i \cap A_j = \emptyset$ for all i, j.

Exercise

$A, B, \ldots Z.$ $\quad 0, 1, \ldots 9.$

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

36 36 36 36 36 36.

26 26 26 26 26 26.

$P_6 = 36^6 - 26^6$

$+$

$P_7 = 36^7 - 26^7$

$+$

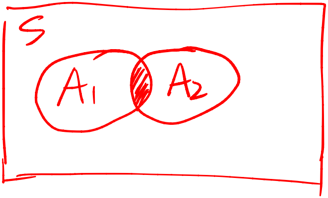
$P_8 = 36^8 - 26^8$

Basic Counting Principles: Subtraction Rule

Definition

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

Example: Counting the number of elements in the union of two sets



$$|A_1 \cup A_2| =? \ |A_1| + |A_2| - |A_1 \cap A_2|$$

$|A_1|$

$|A_2|$

Example: Counting the number of elements in the union of two sets

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Exercise

$1\underbrace{\_\,\_\,\_\,\_\,\_\,\_\,\_}_{7}$ 2^7

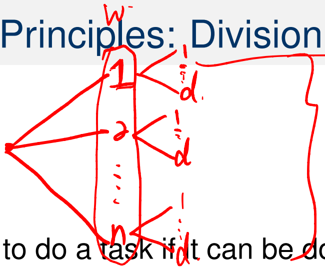
How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

$2^7 + 2^6 - 2^5$

$\underbrace{\_\,\_\,\_\,\_\,\_\,\_}\,0\,0$ 2^6

$1\underbrace{\_\,\_\,\_\,\_\,\_}_{5}\,0\,0$ 2^5

Basic Counting Principles: Division Rule



nd.

Definition

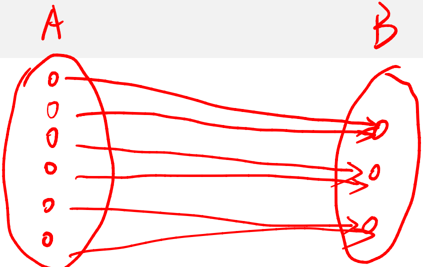
There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

$n = \frac{nd}{d} = n$

Restate the division rule in terms of sets:

If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$.

Example



two-to-one

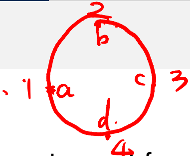
If f is a function from A to B where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d-to-one), then $|B| = |A|/d$.

$|A| = 6, \quad |B| = 3$

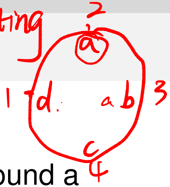
$|B| = {|A|}/{d} = 6/2 = 3$

Example

(a, b, c, d)



One seating



How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Solution: We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note that are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and one way to select the person for seat 4. Thus, there are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table. ◀

$4 \times 3 \times 2 \times 1 = 4! = 24$

$\frac{24}{4} = 6$

The Pigeonhole Principle

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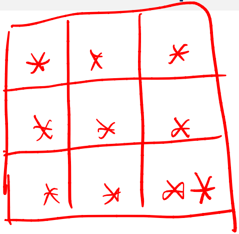

Definition

If k is a positive integer and $k+1$ **or more** objects are placed into k boxes, then there is at least one box containing two or more of the objects.

The Pigeonhole Principle

Definition

If k is a positive integer and $k + 1$ **or more** objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof.

Let us prove by contradiction. Suppose that none of the k boxes contains two or more objects. This is, every box contains at most one object. Then the total number of objects is at most k. This contradicts that there are at least $k + 1$ objects. □

Example

domain

Codomain

A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

Homework: How to prove it?

Permutations and Combinations

- Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where **the order of these elements matters.**
- Many other counting problems can be solved by finding the number of ways to select a particular number of distinct elements from a set of a particular size, where **the order of the elements selected does not matter.**

Example

5 students

Alice Bob Tony.

Bob Alice Tony.

$5 \times 4 \times 3 = 60$.

- In how many ways can we select three students from a group of five students to stand in line for a picture?
- In how many ways can we arrange all five of these students in a line for a picture?

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Permutation

- A **permutation** of a set of *distinct* objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set is called an **r-permutation**.

Example

Let $S = \{1, 2, 3\}$.

- The ordered arrangement $3, 1, 2$ is a permutation of S.
- The ordered arrangement $3, 2$ is a 2-permutation of S.
- What is the number of 2-permutations of S? 6.

1, 2; 2, 3; 1, 3.

$\{1, 2; 2, 1\}$. $\{2, 3; 3, 2\}$ $\{1, 3; 3, 1\}$

r-permutations

n elements.

r.

n n-1 n-2 . . . . . . n-r+1

rth.

The number of r-permutations of a set with n elements is denoted by $P(n,r)$. We can find $P(n,r)$ using the *product rule*.

Theorem

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n,r) = n(n-1)(n-2)...(n-r+1)$$

r-permutations of a set with n ***distinct*** *elements.*

How to prove it?

r-permutations

$$P(n,0) = 1$$

$$P(n,n) = n\cdot(n-1)\cdot(n-2)\cdots\cdot 1 = n!$$

- $P(n, 0) = 1$, whenever n is a nonnegative integer because there is exactly one way to order zero elements.
- That is, there is exactly one list with no elements in it, namely the empty list.

Theorem

$$P(n,r) = n \cdot (n-1) \cdots (n-r+1)$$
$$= \frac{n!}{(n-r)!}$$

Theorem

If n and r are integers with $0 \leq r \leq n$, then $P(n,r) = \frac{n!}{(n-r)!}$

- How to prove it?
- $P(n,n) = ?$

Exercise

8

How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

ABC _ _ _ _ _

6

$P(6,6) = 6! = 720$

Combination

$S = \{1, 2, 3, 4\}$.
3-combination.

$\{1,2,3\}$, $\{1,3,4\}$, $\{2,3,4\}$.
$\{1,2,4\}$.

- We now turn our attention to counting **unordered** selections of objects.
- An ***r*-combination** of elements of a set is an unordered selection of *r* elements from the set. S.
- Thus, an *r*-combination is simply a subset of the set with *r* elements.

Example

Let S be the set $\{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from S.

Note that $\{4, 1, 3\}$ is the same 3-combination as $\{1, 3, 4\}$, because the order in which the elements of a set are listed does not matter.

r-combinations

- The number of r-combinations of a set with n elements is denoted by $C(n, r)$.
- How to calculate $C(n, r)$?

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!} \Big/ r!$$

$$= \frac{n!}{(n-r)!\,r!}$$

r-combinations

- We can determine the number of r-combinations of a set with n elements using the formula for the number of r-permutations of a set.
- The r-permutations of a set can be obtained by first forming r-combinations and then ordering the elements in these combinations.

r-combinations

Theorem

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$\boxed{C(n,r) = \frac{n!}{r!(n-r)!}}$$

How to prove it?

Hint of the Proof

The $P(n, r)$ r-permutations of the set can be obtained by forming the $C(n, r)$ r-combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r, r)$ ways.

Theorem

$$C(n,0) = \frac{n!}{(n-0)!\,0!} = 1 \qquad C(n,0) = C(n,n)$$

$$C(n,n) = \frac{n!}{0!\,(n-0)!} = 1.$$

Theorem

Let n and r be nonnegative integers with $r \leq n$. Then $C(n,r) = C(n, n-r)$.

- How to prove it?
- $C(n,n) = ?$

Exercise

$$C(10,5) = \frac{10!}{5!\,5!} = 252.$$

- How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?
- How many bit strings of length n contain exactly r 1s?

$C(n,r)$.

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

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