

# Lecture 11 - Eigenvalues and Eigenvectors

COMP1046 - Maths for Computer Scientists

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By the end of this lecture we will have learned:

- ⊙ Eigenvalues and Eigenvectors
- ⊙ Eigenspaces
- ⊙ Determining Eigenvalues and Eigenvectors

Based on Section 10.5 of text book (Neri 2018).

In order to introduce the new concept of eigenvalues and eigenvectors, let us consider the following example.

## Example

Let us consider the following linear mapping  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,

$$f(x, y) = (2x - y, 3y)$$

corresponding to the matrix  $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ . Consider

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ so } f \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

*continues...*

## Example

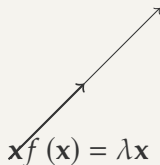
This can be graphically represented as



For analogy, we may think that a linear mapping (at least  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ ) is represented by a clock where one pointer is the input and the other is the output. Since both the vectors are applied in the origin, a linear mapping varies the input in length and rotates it around the origin.

## Example

Now suppose we find an  $\mathbf{x}$  which gives no rotation:



Then  $\mathbf{x}$  is called an eigenvector of  $f$  and scalar  $\lambda$  is the corresponding eigenvalue.

# Eigenvalues and Eigenvectors

## Definition

Let  $f : E \rightarrow E$  be an endomorphism where  $(E, +, \cdot)$  is a finite-dimensional vector space defined on the scalar field  $\mathbb{K}$  whose dimension is  $n$ .

Every vector  $\mathbf{x}$  such that  $f(\mathbf{x}) = \lambda \mathbf{x}$  with  $\lambda$  scalar and  $\mathbf{x} \in E \setminus \{\mathbf{0}_E\}$  is said *eigenvector* of the endomorphism  $f$  related to the *eigenvalue*  $\lambda$ .

Perhaps surprisingly eigenvalues and eigenvectors have a wide range of applications in physics, engineering and computer science.

## Example

Let us consider the endomorphism  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$f(x) = 5x.$$

In this case, any vector  $x$  (number in this specific case) is a potential eigenvector and  $\lambda = 5$  would be the eigenvalue.

In general, for endomorphisms  $\mathbb{R} \rightarrow \mathbb{R}$ , the detection eigenvectors and eigenvalues is trivial because the endomorphisms are already in the form  $f(x) = \lambda x$ .

# Eigenvalues and Eigenvectors

When the endomorphism is between multidimensional vector spaces, the search of eigenvalues and eigenvectors is not trivial.

## Example

Let us consider the following endomorphism  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f(x, y) = (x + y, 2x) .$$

By definition, an eigenvector  $(x, y)$  and an eigenvalue  $\lambda$ , respectively, verify the following equation

$$f(x, y) = \lambda (x, y) .$$

*continued...*



# Eigenvalues and Eigenvectors

## Example

By combining the last two equations we have

$$\begin{cases} x + y = \lambda x \\ 2x = \lambda y \end{cases} \Rightarrow \begin{cases} (1 - \lambda)x + y = 0 \\ 2x - \lambda y = 0. \end{cases}$$

A scalar  $\lambda$  with a corresponding vector  $(x, y)$  that satisfy the homogeneous system of linear equations are an eigenvalue and its eigenvector, respectively.

Since the system is homogeneous the only way for it to be determined is if  $(x, y) = (0, 0)$ . If this situation occurs, regardless of the value of  $\lambda$ , the equations of the system are verified. Since by definition an eigenvector  $\mathbf{x} \in E \setminus \{\mathbf{o}_E\}$ , it follows that  $(x, y) = (0, 0) = \mathbf{o}_E$  is not an eigenvector.

*continued...*

## Example

On the other hand, if we fix the value of  $\lambda$  such that the matrix associated with the system is singular, we have infinite eigenvectors associated with  $\lambda$ .

*What value of  $\lambda$  will achieve this?*

*We will return to this question later.*

# Eigenspace



## Theorem

*Let  $f : E \rightarrow E$  be an endomorphism.*

*The set  $V(\lambda) \subset E$  with  $\lambda \in \mathbb{K}$  defined as*

$$V(\lambda) = \{\mathbf{0}_E\} \cup \{\mathbf{x} \in E \mid f(\mathbf{x}) = \lambda \mathbf{x}\}$$

*with the composition laws is a vector subspace of  $(E, +, \cdot)$ .*

## Definition

The vector subspace  $(V(\lambda), +, \cdot)$  defined as above is said *eigenspace* of the endomorphism  $f$  related to the eigenvalue  $\lambda$ . The dimension of the eigenspace is said *geometric multiplicity* of the eigenvalue  $\lambda$  and is indicated with  $\gamma_m$ .

## Proof.

Let us prove the closure of  $V(\lambda)$  with respect to the composition laws.

Consider any  $\mathbf{x}_1, \mathbf{x}_2 \in V(\lambda)$ . Then,

$$\mathbf{x}_1 \in V(\lambda) \Rightarrow f(\mathbf{x}_1) = \lambda \mathbf{x}_1$$

$$\mathbf{x}_2 \in V(\lambda) \Rightarrow f(\mathbf{x}_2) = \lambda \mathbf{x}_2.$$

It follows that

$$f(\mathbf{x}_1 + \mathbf{x}_2) = f(\mathbf{x}_1) + f(\mathbf{x}_2) = \lambda \mathbf{x}_1 + \lambda \mathbf{x}_2 = \lambda(\mathbf{x}_1 + \mathbf{x}_2).$$

Hence, since  $(\mathbf{x}_1 + \mathbf{x}_2) \in V(\lambda)$ , the set  $V(\lambda)$  is closed with respect to the internal composition law. *continued...*

## Proof.

Let us consider a scalar  $h \in \mathbb{K}$ . From the definition of  $V(\lambda)$  we know that

$$\mathbf{x} \in V(\lambda) \Rightarrow f(\mathbf{x}) = \lambda \mathbf{x}.$$

It follows that

$$f(h\mathbf{x}) = hf(\mathbf{x}) = h(\lambda \mathbf{x}) = \lambda(h\mathbf{x}).$$

Hence, since  $(h\mathbf{x}) \in V(\lambda)$ , the set  $V(\lambda)$  is closed with respect to the external composition law.

We can conclude that  $(V(\lambda), +, \cdot)$  is a vector subspace of  $(E, +, \cdot)$ . □

## Example

Let us consider again the endomorphism  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f(x, y) = (x + y, 2x).$$

Notice that a solution is given for  $\lambda = -1$  so this is an eigenvalue (*it is not the only one!*).

Solutions are of the type  $(x, y) = (\alpha, -2\alpha) = \alpha(1, -2)$  with the parameter  $\alpha \in \mathbb{R}$ , forming a set of solutions,  $V(-1)$ . More specifically this can be interpreted as a line within the plane  $(\mathbb{R}^2)$ .

The theorem above says that  $(V(-1), +, \cdot)$  is a vector space (and referred to as eigenspace) and is a subspace of  $(\mathbb{R}^2, +, \cdot)$ .

## Exercise 1: Eigenspace

Confirm that  $(V(-1), +, \cdot)$  is indeed a vector space.



## Exercise 1: *Solution*

*To be completed.*

## Determining Eigenvalues and Eigenvectors

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# Determining Eigenvalues

This section conceptualizes in a general fashion the method for determining eigenvalues for any  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  endomorphisms.

Let  $f : E \rightarrow E$  be an endomorphism defined over  $\mathbb{K}$  and let  $(E, +, \cdot)$  be a finite-dimensional vector space having dimension  $n$ . A matrix  $\mathbf{A} \in \mathbb{R}_{n,n}$  is associated with the endomorphism:

$$\mathbf{y} = f(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

Let us impose the constraint for eigenvectors,  $f(\mathbf{x}) = \lambda\mathbf{x}$ , so then we can write

$$\begin{aligned}\mathbf{A}\mathbf{x} = \lambda\mathbf{x} &\Rightarrow \mathbf{A}\mathbf{x} - \lambda\mathbf{I}\mathbf{x} = \mathbf{o} \\ &\Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{o}\end{aligned}$$

# Determining Eigenvalues

If  $(\mathbf{A} - \lambda \mathbf{I}) \mathbf{x} = \mathbf{0}$  is solved (for  $\mathbf{x}$ ), it must have multiple solutions. Hence, by Cramer's Theorem,  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ .

The determinant can be characterized as a polynomial in  $\lambda$  with the form,

$$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} k_{n-1} \lambda^{n-1} + \dots + (-1) k_1 \lambda + k_0.$$

called the *characteristic polynomial* of the endomorphism  $f$ .

Solving for  $p(\lambda) = 0$  where  $\lambda \in \mathbb{K}$  will yield the eigenvalues.

In general, for  $n > 2$ , computational methods are needed to solve this. In practice, an iterative method such as QR

Algorithm is used to find eigenvalues and eigenvectors (not covered here).

# Determining Eigenvalues

## Example

Let us consider again the endomorphism  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f(x, y) = (x + y, 2x).$$

The identifying matrix is  $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$  and  $\lambda \mathbf{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ .

Hence we need to solve

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{pmatrix} (1 - \lambda) & 1 \\ 2 & -\lambda \end{pmatrix} = 0.$$

This means that  $(1 - \lambda)(-\lambda) - 2 = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0$ .

The solutions  $\lambda_1 = -1$  and  $\lambda_2 = 2$  are the eigenvalues.

*continued...*

# Determining Eigenvalues

## Example

- Choose  $\lambda_1 = -1$  for the homogeneous system above:

$$\begin{cases} (1 - \lambda_1)x + y = 0 \\ 2x - \lambda_1 y = 0 \end{cases} \Rightarrow \begin{cases} 2x + y = 0 \\ 2x + y = 0. \end{cases}$$

This gives eigenvectors of the type  $(\alpha, -2\alpha) = \alpha(1, -2)$  with the parameter  $\alpha \in \mathbb{R}$ .

- Choose  $\lambda_2 = 2$  for the homogeneous system above:

$$\begin{cases} (1 - \lambda_1)x + y = 0 \\ 2x - \lambda_1 y = 0 \end{cases} \Rightarrow \begin{cases} y - x = 0 \\ 2x - 2y = 0. \end{cases}$$

This gives eigenvectors of the type  $(\alpha, \alpha) = \alpha(1, 1)$  with the parameter  $\alpha \in \mathbb{R}$ .

## Exercise 2: Eigenvalues and Eigenvectors

Consider the endomorphism  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f(x, y) = (2x, x + 3y).$$

Find the eigenvalues and eigenvectors of  $f$ .

## Exercise 2: *Solutions*

*To be completed.*



## Example

Let us analyse a case with three variables, i.e. the following linear mapping  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,

$$f(x, y, z) = (x + z, 2y, -x + 3z).$$

By applying the definition of eigenvalue we write

$$\begin{cases} x + z = \lambda x \\ 2y = \lambda y \\ -x + 3z = \lambda z \end{cases} \Rightarrow \begin{cases} (1 - \lambda)x + z = 0 \\ (2 - \lambda)y = 0 \\ -x + (3 - \lambda)z = 0. \end{cases}$$

*...continued...*

## Example

This system is undetermined when

$$\det \begin{pmatrix} (1 - \lambda) & 0 & 1 \\ 0 & (2 - \lambda) & 0 \\ -1 & 0 & (3 - \lambda) \end{pmatrix} = (2 - \lambda)^3 = 0$$

that is when  $\lambda = 2$ .

*(Exercise: confirm for yourself that  $(2 - \lambda)^3$  is the determinant using the method in Lecture 3).*

This means that only one eigenvector can be calculated, i.e. the three eigenvectors are linearly dependent. ...continued...

## Example

By substituting into the system we have

$$\begin{cases} -x + z = 0 \\ 0y = 0 \\ -x + z = 0. \end{cases}$$

The second equation is always verified while the first and the third say that  $x = z$ . Equivalently, we can see that this system has rank  $\rho = 1$  and thus  $\infty^2$  solutions. If we pose  $x = \alpha$  and  $y = \beta$  for  $\alpha, \beta \in \mathbb{R}$ , we have that the generic solution  $(\alpha, \beta, \alpha) = \alpha (1, 0, 1) + \beta (0, 1, 0)$ . The eigenspace is thus spanned by the vectors  $(1, 0, 1)$  and  $(0, 1, 0)$ .

## Exercise 3: Eigenvalues and eigenspaces

Find eigenvalues and corresponding eigenspaces for the following linear mapping  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,

$$f(x, y, z) = (5x + 2y + 3z, z - 2y, 6z).$$

## Exercise 3: *Solutions*

*To be completed.*

## Exercise 3: *Solutions*

*To be completed.*

## Historical Note

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Why are they called “eigenvalues”, “eigenvectors” and “eigenspaces”?

- ⊙ The earliest known use is by Cauchy 1829, where he called eigenvalues, “variables principales” and termed the phrase “équation caractéristique” for  $p(\lambda)$ :

Les équations (6) deviendront

$$(9) \quad \begin{cases} A_{xx}x + A_{xy}y + A_{xz}z + \dots = sx, \\ A_{xy}x + A_{yy}y + A_{yz}z + \dots = sy, \\ A_{xz}x + A_{yz}y + A_{zz}z + \dots = sz, \\ \dots \end{cases}$$

et pourront s'écrire comme il suit :

$$(10) \quad \begin{cases} (A_{xx} - s)x + A_{xy}y + A_{xz}z + \dots = 0, \\ A_{xy}x + (A_{yy} - s)y + A_{yz}z + \dots = 0, \\ A_{xz}x + A_{yz}y + (A_{zz} - s)z + \dots = 0, \\ \dots \end{cases}$$



- ⊙ Poincaré followed in 1894 and called eigenvalues, “nombre caractéristique”.
- ⊙ The terms “eigen...” first appear in 1904 by David Hilbert who, working in German, translated “characteristic number” into “eigenwert”.
- ⊙ The term “eigenvalue”, which is a half-translation seems to have been popularized later by mathematics students in North America.

Reference source:

<https://www.maa.org/press/periodicals/convergence/math-origins-eigenvectors-and-eigenvalues>

## Summary

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# Summary and next lecture

## Summary

- ⊙ Eigenvalues and Eigenvectors
- ⊙ Eigenspaces
- ⊙ Determining Eigenvalues and Eigenvectors

## The next lecture

Univariate calculus and optimization.