AE1MCS: Mathematics for Computer Scientists

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 7, Section 7.1 An Introduction to Discrete Probability

Discrete Probability

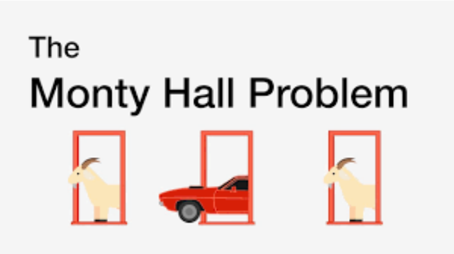
- Combinatorics and probability theory share common origins (analyzing gambling games).
- The theory of probability now plays an essential role in a wide variety of disciplines (e.g. the study of genetics).
- In computer science,
    - Probability theory plays an important role in the study of the complexity of algorithms.
    - Probabilistic algorithms vs. deterministic algorithms.
    - Probability theory can help us answer questions that involve uncertainty.
    - ...

Content

Monty Hall Three-Door Puzzle

You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is be behind each door:

- whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors).
- Then he asks you whether you would like to switch doors.



Group 1.	# Choose	Review.	Switch	Win?
	1	3	Y	N
Group 2	1	3	Y	Y
Group	3	1	N	Y

Finite Probability

Laplace's definition of the probability of an event with **finitely many, equally likely, possible outcomes** is as follows.

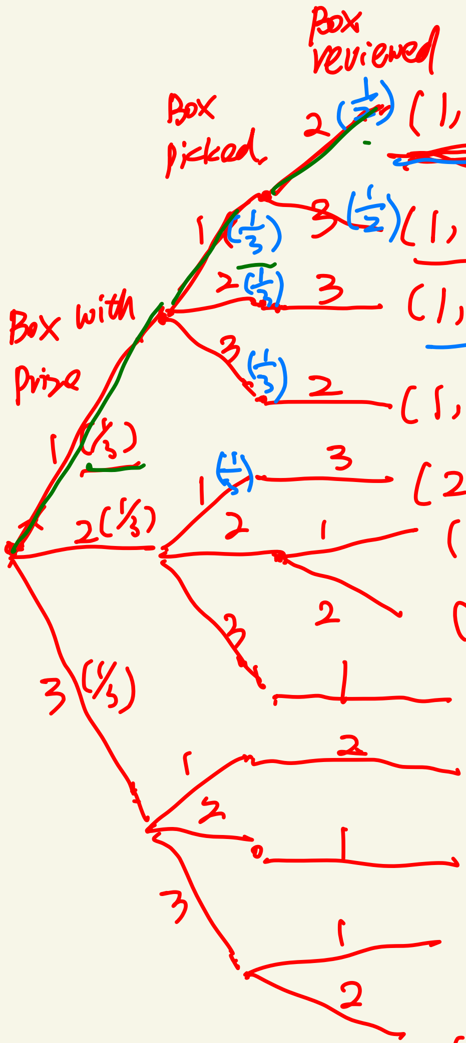
Definition

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is

$$p(E) = \frac{|E|}{|S|}$$

- An **experiment** is a procedure that yields one of a given set of possible outcomes.
- The **sample space** of the experiment is the set of possible outcomes.
- An **event** is a subset of the sample space.

Prob tree: a tree diagram representing the prob and outcome



Box reviewed	Switch	Win/Lose	Prob.	
(1, 1, 2)		L	$\frac{1}{18}$	.
(1, 1, 3)		L	$\frac{1}{18}$	.
(1, 2, 3)		W	$\frac{1}{9}$	✓
(1, 3, 2)		W	$\frac{1}{9}$	✓
(2, 1, 3)		W	$\frac{1}{9}$	✓
(2, 2, 1)		L	$\frac{1}{18}$	.
(2, 2, 2)		L	$\frac{1}{18}$	.
(2, 3, 1)		W	$\frac{1}{9}$	✓
(3, 1, 2)		W	$\frac{1}{9}$	✓
(3, 2, 1)		W	$1/9$	✓
(3, 3, 1)		L	$1/18$	.
(3, 3, 2)		L	$1/18$	..

sample space: $S = \{(1,1,2), (1,1,3), \cdots\}$

$|S| = 12$

(1, 2, 1) ✗

(2, 1, 2) ✗

Switch:

$P(W) = \frac{6}{9} = \frac{2}{3}$, $P(L) = \frac{6}{18} = \frac{1}{3}$

Probability Distribution :

"P" : $S \to \mathbb{R}$

$[0,1]$

S $\mathbb{R}$

0.1
0.2
$\vdots$

Let S be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability $p(s)$ to each outcome $s \in S$. We require that two conditions be met:

$s \in S \to [0,1]$

1. $0 \leq p(s) \leq 1$ for each $s \in S$
2. $\sum_{s \in S} p(s) = 1$.

The function p from the set of all outcomes of the sample space S is called a **probability distribution.**

Probability of an Event

In the eighteenth century, the French mathematician Laplace, who also studied gambling, defined **the probability of an event as the number of successful outcomes divided by the number of possible outcomes.**

Example: Poker 1

Find the probability that a hand of five cards in poker contains four cards of one kind.

- A deck of cards contains 52 cards.
- There are 13 different kinds of cards, with four cards of each kind.
- These kinds are twos, threes, fours, fives, sixes, sevens, eights, nines, tens, jacks, queens, kings, and aces.
- There are 4 suits: spades, clubs, hearts, and diamonds, each containing 13 cards.

Example: Poker 1 (Answer)

Find the probability that a hand of five cards in poker contains four cards of one kind.

- S is number of ways to choose any 5 cards from 52: $|S| = C(52, 5)$.
- E is the number of ways to get four of a kind: $|E| = 13 \times (52 - 4)$, using product rule with
  - Choose one of the 13 kinds which is repeated 4 times;
  - Choose any remaining card for last card (52-4).
- Hence $p(E) = \dfrac{13 \times (52 - 4)}{C(52, 5)} \approx 0.00024$.

Example: Poker 2

What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

Example: Poker 2 (Answer)

What is the probability that a poker hand contains a full house, that is, three of one kind and two of another kind?

Same sample space: $|S| = C(52, 5)$.

$$
\begin{array}{rcll}
|E| & = & \text{\# ways to get two different kinds} & C(13,2) \\
 & \times & \text{\# ways to select which is three cards} & \\
 & & \text{of the same kind} & 2 \\
 & \times & \text{\# ways to choose three cards of the same kind} & C(4,3) \\
 & \times & \text{\# ways to choose two cards of the same kind} & C(4,2) \\
 & = & C(13,2) \times 2 \times C(4,3) \times C(4,2) &
\end{array}
$$

Notice: $C(13, 2) \times 2 = P(13, 2) = C(13, 1)C(12, 1)$, *so all correct.*

$P = \frac{|E|}{|S|} =$

Probabilities of Complements and Unions of Events

Theorem

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E)$$

Theorem

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

How to prove them?

Example

$|S| = 100$

$E_1 =$ divisible by 2 $= \{2, 4, 8, \cdots 100\}$ $|E_1| = 50$

$E_2 =$ divisible by 5 $= \{5, 10, 15, \cdots 100\}$ $|E_2| = 20$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100}$

- A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?
- What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

$\{1, 2, 3, \cdots 100\}$

$E_1 \cap E_2 =$ divisible by 2 and 5 $= \{10, 20, \cdots 100\}$

$|E_1 \cap E_2| = 10$

Example 1 (Answer)

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

- Sample space is all bit strings of length 10: $|S| = 2^{10}$.
- Think about the event when bit string has *no* 0's. Then $E = \{1111111111\}$ and $|E| = 1$.
- Now the event we are interested in (at least one 0) is complement of E, so

$$p(\overline{E}) = 1 - p(E) = 1 - 2^{-10}.$$

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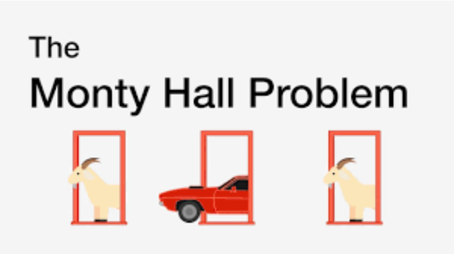

Monty Hall Three-Door Puzzle

Choose to stay

$$P(W) = \frac{1}{3}$$

$$P(L) = \frac{2}{3}$$

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