

# Lecture 8 - Linear Mappings Part 1

COMP1046 - Maths for Computer Scientists

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Dr. Ferrante Neri / Dr. Tony Bellotti

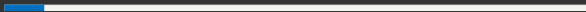


By the end of this lecture we will have learned:

- ⊙ Mappings
- ⊙ Linear Mappings
- ⊙ Linear Mappings and Vector Spaces

Based on Sections 10.1 and 10.2 of text book (Neri 2018).

# Mappings



## Definition

Let  $(E, +, \cdot)$  and  $(F, +, \cdot)$  be two vector spaces defined over the scalar field  $\mathbb{K}$ . Let  $f : E \rightarrow F$  be a relation.

Let  $U$  be a set such that  $U \subseteq E$ .

The relation  $f$  is said *mapping* when

$$\forall \mathbf{u} \in U : \exists ! \mathbf{w} \in F \text{ such that } f(\mathbf{u}) = \mathbf{w}.$$

The set  $U$  is said *domain* and is indicated with  $\text{dom}(f)$ .

A vector  $\mathbf{w}$  such that

$$\mathbf{w} = f(\mathbf{u})$$

is said to be the *mapped* (or *transformed*) of  $\mathbf{u}$  through  $f$ .

## Definition

Let  $f$  be a mapping  $E \rightarrow F$ , where  $E$  and  $F$  are sets associated with the vector spaces  $(E, +, \cdot)$  and  $(F, +, \cdot)$ . The *image* of  $f$ , indicated with  $\text{Im}(f)$ , is a set defined as

$$\text{Im}(f) = \{ \mathbf{w} \in F \mid \exists \mathbf{u} \in E \text{ such that } f(\mathbf{u}) = \mathbf{w} \}.$$

## Example

Let  $(\mathbb{R}, +, \cdot)$  be a vector space.

An example of mapping  $\mathbb{R} \rightarrow \mathbb{R}$  is  $f(x) = x^2 + 2x + 2$ .

- ⦿ The domain of the mapping  $\text{dom}(f) = \mathbb{R}$ .
- ⦿ The image  $\text{Im}(f) = [1, \infty[$ .

## Example

Let  $(\mathbb{R}^2, +, \cdot)$  and  $(\mathbb{R}, +, \cdot)$  be two vector spaces.

An example of mapping  $\mathbb{R}^2 \rightarrow \mathbb{R}$  is  $f(x, y) = x + 2y + 2$ .

- ⊙ The domain of the mapping  $\text{dom}(f) = \mathbb{R}^2$ .
- ⊙ The image  $\text{Im}(f) = \mathbb{R}$ .

## Example

From the vector spaces  $(\mathbb{R}^2, +, \cdot)$  and  $(\mathbb{R}^3, +, \cdot)$  an example of mapping  $\mathbb{R}^3 \rightarrow \mathbb{R}^2$  is

$$f(x, y, z) = (x + 2y + -z + 2, 6y - 4z + 2).$$

- ⊙ The domain of the mapping  $\text{dom}(f) = \mathbb{R}^3$ .
- ⊙ The image  $\text{Im}(f) = \mathbb{R}^2$ .

## Definition

Let  $f$  be a mapping  $E \rightarrow F$ , where  $E$  and  $F$  are sets associated with the vector spaces  $(E, +, \cdot)$  and  $(F, +, \cdot)$ .

- ⊙ The mapping  $f$  is said *surjective* if the image of  $f$  coincides with  $F$ :  $\text{Im}(f) = F$ .
- ⊙ The mapping  $f$  is said *injective* if

$$\forall \mathbf{u}, \mathbf{v} \in E \text{ with } \mathbf{u} \neq \mathbf{v} \Rightarrow f(\mathbf{u}) \neq f(\mathbf{v}).$$

- ⊙ The mapping  $f$  is said *bijective* if  $f$  is injective and surjective.



## Example

Consider these mappings:

- ⊙  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ :
  - Not surjective since  $\text{Im}(f) = \mathbb{R}^+$ .
  - Not injective because  $\exists x_1, x_2$  with  $x_1 \neq x_2$  such that  $x_1^2 = x_2^2$ .  
For example if  $x_1 = 3$  and  $x_2 = -3$ , thus  $x_1 \neq x_2$ , it occurs that  $x_1^2 = x_2^2 = 9$ .
  - Hence not bijective.
- ⊙  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = e^x$  is injective but not surjective.  
Hence, this mapping is not bijective.
- ⊙  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 2$  is injective and surjective.  
Hence, this mapping is bijective.

## Linear Mappings



## Definition

Let  $f$  be a mapping  $E \rightarrow F$ , where  $E$  and  $F$  are sets associated with the vector spaces  $(E, +, \cdot)$  and  $(F, +, \cdot)$ .

The mapping  $f$  is said *linear mapping* if the following properties are valid:

- ⊙ additivity:  $\forall \mathbf{u}, \mathbf{v} \in E : f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$
- ⊙ homogeneity:  $\forall \lambda \in \mathbb{K} \text{ and } \forall \mathbf{v} \in E : f(\lambda \mathbf{v}) = \lambda f(\mathbf{v})$

## Example

Let us check the linearity of the mapping  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\forall x : f(x) = e^x.$$

Consider two vectors (numbers in this case)  $x_1$  and  $x_2$ .

Calculate  $f(x_1 + x_2) = e^{x_1 + x_2}$ .

From basic calculus we know that

$$e^{x_1 + x_2} \neq e^{x_1} + e^{x_2}.$$

Therefore, additivity is not verified. Hence the mapping is not linear.

## Example

Let us consider the linearity of the mapping  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\forall x : f(x) = 2x.$$

Consider two vectors (numbers in this case)  $x_1$  and  $x_2$ . Then,

$$\begin{aligned} f(x_1 + x_2) &= 2(x_1 + x_2) \\ f(x_1) + f(x_2) &= 2x_1 + 2x_2. \end{aligned}$$

It follows that  $f(x_1 + x_2) = f(x_1) + f(x_2)$ .

Hence, this mapping is additive.

*Continued...*

## Example

Let us check the homogeneity by considering a generic scalar  $\lambda$ . We have that

$$f(\lambda x) = 2\lambda x$$

$$\lambda f(x) = \lambda 2x.$$

It follows that  $f(\lambda x) = \lambda f(x)$ .

Hence, since also homogeneity is verified this mapping is linear.

## Definition

Let  $f$  be a mapping  $E \rightarrow F$ , where  $E$  and  $F$  are sets associated with the vector spaces  $(E, +, \cdot)$  and  $(F, +, \cdot)$ .

The mapping  $f$  is said *affine mapping* if the mapping

$$g(\mathbf{v}) = f(\mathbf{v}) - f(\mathbf{o})$$

is linear.

# Affine Mapping

## Example

Consider the mapping  $f : \mathbb{R} \rightarrow \mathbb{R}, \forall x : f(x) = x + 2$ .

Consider two vectors (numbers in this case)  $x_1$  and  $x_2$ . Then

$$f(x_1 + x_2) = x_1 + x_2 + 2$$

$$f(x_1) + f(x_2) = x_1 + 2 + x_2 + 2 = x_1 + x_2 + 4.$$

It follows that  $f(x_1 + x_2) \neq f(x_1) + f(x_2)$ . Hence, this mapping is not linear. Still,  $f(0) = 2$  and

$$g(x) = f(x) - f(0) = x$$

which is a linear mapping. This means that  $f(x)$  is an affine mapping.



## Proposition

*Let  $f$  be a linear mapping  $E \rightarrow F$ .*

*Let us indicate with  $\mathbf{o}_E$  and  $\mathbf{o}_F$  the null vectors of the vector spaces  $(E, +, \cdot)$  and  $(F, +, \cdot)$ , respectively.*

*It follows that*

$$f(\mathbf{o}_E) = \mathbf{o}_F.$$

## Proof.

$$f(\mathbf{o}_E) = f(0\mathbf{o}_E) = 0f(\mathbf{o}_E) = \mathbf{o}_F.$$



## Exercise 1: Linear and Affine Mappings

Which of these mappings are linear and which are affine (or both or neither)?:

⊙  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x^2 + 2.$

⊙  $f : \mathbb{R}^2 \rightarrow \mathbb{R}, f(x, y) = 2x - 3y.$

⊙  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, f(x, y) = (2x - y + 1, 2y - x - 2).$

## Exercise 1: *Solution*

*To be completed.*

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*To be completed.*

# Linear Mappings and Vector Spaces

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# Linear Mappings and Vector Spaces

## Definition

Let  $f$  be a mapping  $U \subset E \rightarrow F$ .

The *image of  $U$  through  $f$* , indicated with  $f(U)$ , is the set

$$f(U) = \{ \mathbf{w} \in F \mid \exists \mathbf{u} \in U \text{ such that } f(\mathbf{u}) = \mathbf{w} \}.$$

## Theorem

Let  $f : E \rightarrow F$  be a linear mapping and  $(U, +, \cdot)$  be a vector subspace of  $(E, +, \cdot)$ .

It follows that the triple  $(f(U), +, \cdot)$  is a vector subspace of  $(F, +, \cdot)$ .

## Proof.

In order to prove that  $(f(U), +, \cdot)$  is a vector subspace of  $(F, +, \cdot)$  we have to show that the set  $f(U)$  is closed with respect to the two composition laws.

By definition, the fact that a vector  $\mathbf{w} \in f(U)$  means that  $\exists \mathbf{v} \in U$  such that  $f(\mathbf{v}) = \mathbf{w}$ .

Thus, if we consider two vectors  $\mathbf{w}, \mathbf{w}' \in f(U)$  then

$$\mathbf{w} + \mathbf{w}' = f(\mathbf{v}) + f(\mathbf{v}') = f(\mathbf{v} + \mathbf{v}').$$

Since for hypothesis  $(U, +, \cdot)$  is a vector space, then  $\mathbf{v} + \mathbf{v}' \in U$ . Hence,  $f(\mathbf{v} + \mathbf{v}') \in f(U)$ . The set  $f(U)$  is therefore closed with respect to the internal composition law. *continued...*



## Proof.

Let us now consider a generic scalar  $\lambda \in \mathbb{K}$  and calculate

$$\lambda \mathbf{w} = \lambda f(\mathbf{v}) = f(\lambda \mathbf{v}).$$

Since for hypothesis  $(U, +, \cdot)$  is a vector space, then  $\lambda \mathbf{v} \in U$ .

Hence,  $f(\lambda \mathbf{v}) \in f(U)$ . The set  $f(U)$  is closed with respect to the external composition law.

Since the set  $f(U)$  is closed with respect to both the composition laws the triple  $(f(U), +, \cdot)$  is a vector subspace of  $(F, +, \cdot)$ . □

## Definition

Let  $f$  be a mapping  $E \rightarrow W \subset F$ .

The *inverse image* of  $W$  through  $f$ , indicated with  $f^{-1}(W)$ , is a set defined as

$$f^{-1}(W) = \{\mathbf{u} \in E \mid f(\mathbf{u}) \in W\}.$$

## Theorem

Let  $f : E \rightarrow F$  be a linear mapping.

If  $(W, +, \cdot)$  is a vector subspace of  $(F, +, \cdot)$ ,

then  $(f^{-1}(W), +, \cdot)$  is a vector subspace of  $(E, +, \cdot)$ .

## Proof.

In order to prove that  $(f^{-1}(W), +, \cdot)$  is a vector subspace of  $(E, +, \cdot)$  we have to prove the closure of  $f^{-1}(W)$  with respect to the two composition laws.

If a vector  $\mathbf{v} \in f^{-1}(W)$  then  $f(\mathbf{v}) \in W$ .

We can write for the linearity of  $f$

$$f(\mathbf{v} + \mathbf{v}') = f(\mathbf{v}) + f(\mathbf{v}').$$

Since  $(W, +, \cdot)$  is a vector space,  $f(\mathbf{v}) + f(\mathbf{v}') \in W$ .

Hence,  $f(\mathbf{v} + \mathbf{v}') \in W$  and  $\mathbf{v} + \mathbf{v}' \in f^{-1}(W)$ .

Thus, the set  $f^{-1}(W)$  is closed with respect to the first composition law.

*continued...*

## Proof.

Let us consider a generic scalar  $\lambda \in \mathbb{K}$  and calculate

$$f(\lambda \mathbf{v}) = \lambda f(\mathbf{v}).$$

Since  $(W, +, \cdot)$  is a vector space,  $\lambda f(\mathbf{v}) \in W$ . Since  $f(\lambda \mathbf{v}) \in W$  the  $\lambda \mathbf{v} \in f^{-1}(W)$ . Thus, the set  $f^{-1}(W)$  is closed with respect to the second composition law.

Hence,  $(f^{-1}(W), +, \cdot)$  is a vector subspace of  $(E, +, \cdot)$ . □

## Example

Let us consider the linear mapping  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f(x, y) = (x + y, x - y)$$

and the set  $U = \{(x, y) \in \mathbb{R}^2 \mid 2x - y = 0\}$ .

It can be easily checked that  $(U, +, \cdot)$  is a vector space. This set can be represented by means of the vector

$$U = \alpha (1, 2)$$

with  $\alpha \in \mathbb{R}$ .

*Continued...*

Note: Here we use the shorthand  $\alpha \mathbf{v}$  for  $\{\alpha \mathbf{v} : \alpha \in \mathbb{R}\}$ .

## Example

Let us now calculate  $f(U)$  by replacing  $(x, y)$  with  $(\alpha, 2\alpha)$ :

$$f(U) = (\alpha + 2\alpha, \alpha - 2\alpha) = (3\alpha, -\alpha) = \alpha(3, -1)$$

that is a line passing through the origin. Hence also

$$(f(U), +, \cdot)$$

is a vector space.

# Summary and next lecture

## Summary

- ⊙ Mappings
- ⊙ Linear Mappings
- ⊙ Linear Mappings and Vector Spaces

## The next lecture

We will learn about Endomorphisms, Kernels and Rank-Nullity Theorem.