Seminar 8

In this seminar you will study:

- Algebra of Matrices
- Inverse of 2×2 matrices
- Solving 2×2 systems of linear equations

Algebra of Matrices: Equality of Matrices

Example: Given matrices $A = \begin{pmatrix} x & 2 \\ 5 & y \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ z & -1 \end{pmatrix}$,

find the constants x, y, and z, if $3A = 2B$.

Solution:

$$3A = \begin{pmatrix} 3x & 6 \\ 15 & 3y \end{pmatrix}$$

$$2B = \begin{pmatrix} 2 & 6 \\ 2z & -2 \end{pmatrix}$$

$$3A = 2B \Rightarrow \begin{pmatrix} \boxed{3x} & 6 \\ \boxed{15} & \boxed{3y} \end{pmatrix} = \begin{pmatrix} \boxed{2} & 6 \\ \boxed{2z} & \boxed{-2} \end{pmatrix}$$

$$\Rightarrow \begin{cases} 3x = 2 \\ 3y = -2 \\ 2z = 15 \end{cases}$$

$$\Rightarrow \quad x = \frac{2}{3}, \quad y = -\frac{2}{3}, \quad z = \frac{15}{2}$$

Algebra of Matrices: Matrix multiplication

Example: Given matrices $A = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 7 \\ 4 & 0 \end{pmatrix}$,

find AB and BA.

Solution:

$$AB = \begin{pmatrix} \boxed{2 \quad 5} \\ \boxed{3 \quad 1} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} 1 \\ 4 \end{matrix}} & \begin{matrix} 7 \\ 0 \end{matrix} \end{pmatrix}$$

Similarly

$$= \begin{pmatrix} \boxed{2 \times 1 + 5 \times 4} & 2 \times 7 + 5 \times 0 \\ \boxed{3 \times 1 + 1 \times 4} & 3 \times 7 + 1 \times 0 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 14 \\ 7 & 21 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 7 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 2 + 7 \times 3 & 1 \times 5 + 7 \times 1 \\ 4 \times 2 + 0 \times 3 & 4 \times 5 + 0 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 23 & 12 \\ 8 & 20 \end{pmatrix}$$

In general, $AB \neq BA$

Algebra of Matrices: The Transpose of a Matrix

Example: Given matrices $A = \begin{pmatrix} 3 & 2 \\ -4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ 2 & -1 \end{pmatrix}$, find A^T, B^T, and $(A+B)^T$.,

Hence show that $(A+B)^T = A^T + B^T$.

Solution:

$$A^T = \begin{pmatrix} 3 & 2 \\ -4 & 0 \end{pmatrix}^T = \begin{pmatrix} 3 & -4 \\ 2 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1 & 2 \\ 6 & -1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 4 & 8 \\ -2 & -1 \end{pmatrix}$$

$$\Rightarrow (A+B)^T = \begin{pmatrix} 4 & -2 \\ 8 & -1 \end{pmatrix}$$

$$A^T + B^T = \begin{pmatrix} 3 & -4 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 6 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 \\ 8 & -1 \end{pmatrix}$$

$$\Rightarrow (A+B)^T = A^T + B^T$$

Algebra of Matrices

1. Given matrices $A = \begin{pmatrix} 1 & 6 \\ 5 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 2 \\ 3 & -2 \end{pmatrix}$, find AB.

Answer: $AB = \begin{pmatrix} 25 & -10 \\ 59 & -6 \end{pmatrix}$

2. Given matrices $A = \begin{pmatrix} 5 & 4 \\ -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 5 \\ 2 & 6 \end{pmatrix}$, find BA.

Answer: $BA = \begin{pmatrix} -15 & 10 \\ -8 & 20 \end{pmatrix}$

3. Given matrices $A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 9 & 5 \end{pmatrix}$, find BA.

Answer: $BA = \begin{pmatrix} 1 & 2 \\ 32 & 18 \end{pmatrix}$

4. Given matrices $A = \begin{pmatrix} 1 & 5 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -4 \\ 1 & 10 \end{pmatrix}$, find AB.

Answer: $AB = \begin{pmatrix} 8 & 46 \\ 14 & 4 \end{pmatrix}$

Algebra of Matrices

1. Given matrices $A = \begin{pmatrix} 1 & 5 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -4 \\ 1 & 10 \end{pmatrix}$, find BA^T.

Answer: $BA^T = \begin{pmatrix} -17 & 4 \\ 51 & 24 \end{pmatrix}$

2. Given matrices $A = \begin{pmatrix} 1 & 6 \\ 5 & 8 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 2 \\ 3 & -2 \end{pmatrix}$, find $A^T B$.

Answer: $A^T B = \begin{pmatrix} 22 & -8 \\ 66 & -4 \end{pmatrix}$

3. Given matrices $A = \begin{pmatrix} 5 & 4 \\ -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 5 \\ 2 & 6 \end{pmatrix}$, find $B^T A$.

Answer: $B^T A = \begin{pmatrix} -6 & 4 \\ 7 & 32 \end{pmatrix}$

4. Given matrices $A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 9 & 5 \end{pmatrix}$, find $(AB)^T$.

Answer: $(AB)^T = \begin{pmatrix} 21 & 1 \\ 4 & -2 \end{pmatrix}$

Inverse of 2×2 Matrices

Example: Given $A = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ find the inverse matrix A^{-1}, if it exists.

Solution:

Step 1 Find the determinant of A: $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ given that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Step 2 Use the formula for the inverse: If $\det(A) \neq 0$, then $A^{-1} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Note: If $\det(A) = 0$, A is singular, and its inverse does not exist.

Here, $\det(A) = 7 \times 3 - 4 \times 5 = 1 \neq 0$

Thus, A^{-1} exists,

$$\Rightarrow A^{-1} = \frac{1}{1} \begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix}$$

Inverse of 2 × 2 Matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

1. Given a matrix $A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$, find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} 1 & -2 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

2. Given a matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$, find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix}$

3. Given a matrix $A = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$, find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} -3 & \frac{5}{2} \\ 2 & -\frac{3}{2} \end{pmatrix}$

4. Given a matrix $A = \begin{pmatrix} 6 & 2 \\ 5 & 1 \end{pmatrix}$, find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{5}{4} & -\frac{3}{2} \end{pmatrix}$

Inverse of 2 × 2 Matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

1. Given the matrix $A = \begin{pmatrix} 3 & 4 \\ 1 & -2 \end{pmatrix}$, find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{1}{10} & -\frac{3}{10} \end{pmatrix}$

2. Given the matrix $A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$, find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} 0 & -1 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$

3. Given the matrix $A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$, find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{pmatrix}$

4. Given the matrix $A = \begin{pmatrix} 6 & -1 \\ 4 & 1 \end{pmatrix}$, find A^{-1}.

Answer: $A^{-1} = \begin{pmatrix} \frac{1}{10} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{5} \end{pmatrix}$

Solving 2×2 Systems of Linear Equations

Example: Express the system of linear equations $\begin{cases} 3x + y = 4 \\ 5x - 4y = 1 \end{cases}$ into the matrix form $AX = B$, and use the matrix method ($AX = B \Rightarrow X = A^{-1}B$) to solve it.

Solution:

<u>Step 1 Matrix form $AX = B$:</u>

$$\underbrace{\begin{pmatrix} 3 & 1 \\ 5 & -4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}_{B}$$

<u>Step 2 Find the determinant of A:</u> $\det(A) = -12 - 5 = -17 \neq 0 \quad \therefore A^{-1}$ exists

<u>Step 3 Find A^{-1}:</u>

Since $\det(A) \neq 0$, then $A^{-1} = -\frac{1}{17}\begin{pmatrix} -4 & -1 \\ -5 & 3 \end{pmatrix}$

Solving 2×2 Systems of Linear Equations

Step 4 Find X:

$$X = A^{-1}B \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-17}\begin{pmatrix} -4 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$= \frac{1}{-17}\begin{pmatrix} -4 \times 4 + -1 \times 1 \\ -5 \times 4 + 3 \times 1 \end{pmatrix}$$

Always show the matrix multiplication process in the exams

$$= \frac{1}{-17}\begin{pmatrix} -17 \\ -17 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\therefore x = 1, \ y = 1}$$

Solving 2 × 2 Systems of Linear Equations

1. Use the matrix method to solve the system of linear equations:

$$\begin{cases} 2x + 3y = 6 \\ 3x - 9y = 0 \end{cases}$$

Answer: $x = 2, \ y = \frac{2}{3}$

2. Use the matrix method to solve the system of linear equations:

$$\begin{cases} 5x + 4y - 5 = 0 \\ 4x + 3y - 3 = 0 \end{cases}$$

Answer: $x = -3, \ y = 5$

3. Use the matrix method to solve the system of linear equations:

$$\begin{cases} 5x - 2y = 11 \\ 4x + 3y = 18 \end{cases}$$

Answer: $x = 3, \ y = 2$

4. Use the matrix method to solve the system of linear equations:

$$\begin{cases} 3x + 4y + 7 = 0 \\ 5x + 10y + 5 = 0 \end{cases}$$

Answer: $x = -5, \ y = 2$

Solving 2 × 2 Systems of Linear Equations

1. Use the matrix method to solve the system of linear equations:

$$\begin{cases} 2x + 3y = 7 \\ 3x - 4y = 9 \end{cases}$$

Answer: $x = \frac{55}{17},\ y = \frac{3}{17}$

2. Use the matrix method to solve the system of linear equations:

$$\begin{cases} 5x - 2y = 9 \\ 4x + 3y = 10 \end{cases}$$

Answer: $x = \frac{47}{23},\ y = \frac{14}{23}$

3. Use the matrix method to solve the system of linear equations:

$$\begin{cases} 9x + y = 17 \\ x - 10y = 8 \end{cases}$$

Answer: $x = \frac{178}{91},\ y = -\frac{55}{91}$

4. Use the matrix method to solve the system of linear equations:

$$\begin{cases} 8x + 3y = 18 \\ 5x - y = 10 \end{cases}$$

Answer: $x = \frac{48}{23},\ y = \frac{10}{23}$