Lecture 9

Topics covered in this lecture session

1. Complex numbers - Introduction
2. Algebra of complex numbers
3. Square root of a complex number
4. Modulus and argument of a complex number
5. Polar form of a complex number
6. Algebraic operations on Argand diagram.

Complex Numbers - Introduction

In solving quadratic equations $ax^2+bx+c=0$ using the formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$, if the discriminant $\Delta<0$, then no real root exist.

With Complex Numbers, we can explore further into the possibility of finding roots even when $\Delta=b^2-4ac<0$.

For that, we first define *Imaginary Numbers*.

Complex Numbers - Introduction

An imaginary number is the one whose square is a negative real number.

e.g. $\sqrt{-1},\quad \sqrt{-7},\quad \sqrt{-8},\quad \sqrt{-25},\quad \sqrt{-1.21},$ etc.

are all imaginary numbers, because their squares $-1,\ -7,\ -8,\ -25,\ -1.21$ are all negative real numbers.

We use the notation $\sqrt{-1}=i$ to represent imaginary numbers. e.g. $\sqrt{-7}=\sqrt{7}\,i,\ \ \sqrt{-25}=5\,i,$ and so on.

Imaginary numbers

Note: $i=\sqrt{-1}\ \Rightarrow\ i^2=-1.$

$\begin{aligned} i^3 &= i^2\cdot i \\ &= (-1)\cdot i \\ &= -i \end{aligned}$	$\begin{aligned} i^4 &= i^2\cdot i^2 \\ &= (-1)\cdot(-1) \\ &= 1 \end{aligned}$
$\begin{aligned} i^5 &= i^4\cdot i \\ &= (1)\cdot i \\ &= i \end{aligned}$	$\begin{aligned} \frac{1}{i} &= \frac{i}{i^2} = \frac{i}{-1} \\ &= -i \end{aligned}$

Imaginary numbers

$i = \sqrt{-1}$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$	$i^5 = i$

Simplify: $(1+i)^{10} - (1-i)^{10}$

$$\begin{aligned}(1+i)^{10} - (1-i)^{10} &= [(1+i)^2]^5 - [(1-i)^2]^5 \\ &= (1+2i+i^2)^5 - (1-2i+i^2)^5 \\ &= (1+2i-1)^5 - (1-2i-1)^5 \\ &= (2i)^5 - (-2i)^5 \\ &= 32\,i^5 - (-32i^5) \\ &= 32\,i + 32\,i \;\; = 64\,i\end{aligned}$$

Complex Numbers - Introduction

Using the notation $i = \sqrt{-1}$, it is now possible to solve quadratic equations with negative discriminants.

e.g. $x^2 - 2x + 2 = 0 \;\Rightarrow\; x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2}$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\,i}{2}$$

$$= 1 \pm i$$

$$= \boxed{(1) \pm i \cdot (1)}$$

Form: $a + i\,b$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

Complex Numbers - Introduction

A Complex Number is of the form: $a + i\,b$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

a is called the Real part of the complex number z and is denoted by $Re(z)$.

b is called the Imaginary part of the complex number z and is denoted by $Im(z)$.

Thus, $\boxed{z = Re(z) + i\,Im(z).}$ **Cartesian form** of complex number

Algebra of Complex Numbers

1. Equality

Two complex numbers are equal if and only if their real and imaginary parts are equal.

i.e. $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ are equal

$\Leftrightarrow$ $x_1 = x_2$ and $y_1 = y_2$

Algebra of Complex Numbers

Example: Given $(2+i)\,x-(1+3\,i)\,y-7=0$, find $x, y \in \mathbb{R}$.

$$(2+i)\,x-(1+3\,i)\,y-7=0$$

$$\Rightarrow\ 2\,x-y-7+i\,(x-3\,y)=0=0+i\,(0)$$

$$\Rightarrow\ 2\,x-y-7=0 \quad \text{and} \quad x-3\,y=0$$

$$\Rightarrow\ 6\,y-y-7=0\ \Rightarrow\ y=\frac{7}{5}\quad \Rightarrow\ x=\frac{21}{5}$$

Algebra of Complex Numbers

2. Addition and Subtraction

For two complex numbers z_1 and z_2, the operations of addition and subtraction are defined by

Addition
$$z_1+z_2=(x_1+i\,y_1)+(x_2+i\,y_2)$$
$$=(x_1+x_2)+i\,(y_1+y_2)$$

Subtraction
$$z_1-z_2=(x_1+i\,y_1)-(x_2+i\,y_2)$$
$$=(x_1-x_2)+i\,(y_1-y_2)$$

Algebra of Complex Numbers

3. Multiplication

Multiplication of complex numbers is carried out in a similar way to expanding brackets, and then replacing i^2 by -1.

Multiplication
$$z_1\cdot z_2=(x_1+i\,y_1)\cdot(x_2+i\,y_2)$$
$$=x_1\,x_2+i\,x_1\,y_2+i\,x_2\,y_1+i^2\,y_1\,y_2$$
$$=(x_1\,x_2-y_1\,y_2)+i\,(x_1\,y_2+x_2\,y_1)$$

Algebra of Complex Numbers

4. Division

To define division of complex numbers, we first need to define the conjugate complex number.

Conjugate Complex Number

For $z=a+i\,b$, the conjugate complex number, denoted by $\overline{z}$, is defined by $\overline{z}=a-i\,b$.

Clearly, $\overline{\overline{z}}=\overline{a-i\,b}=a+i\,b=z\ \Rightarrow$ z and $\overline{z}$ are conjugates of each other.

Algebra of Complex Numbers

Division $\dfrac{z_1}{z_2} = \dfrac{x_1 + i\,y_1}{x_2 + i\,y_2} = \left(\dfrac{x_1 + i\,y_1}{x_2 + i\,y_2}\right) \cdot \left(\dfrac{x_2 - i\,y_2}{x_2 - i\,y_2}\right)$

(Multiply and Divide by the Conjugate of the Denominator)

$$= \frac{x_1\,x_2 - i\,x_1\,y_2 + i\,x_2\,y_1 - i^2\,y_1\,y_2}{x_2^2 - i^2\,y_2^2}$$

$$= \frac{x_1\,x_2 - i\,x_1\,y_2 + i\,x_2\,y_1 - i^2\,y_1\,y_2}{x_2^2 + y_2^2} \quad (\because\ i^2 = -1)$$

$$= \left(\frac{x_1\,x_2 + y_1\,y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2\,y_1 - x_1\,y_2}{x_2^2 + y_2^2}\right)$$

Square root of a complex number

Example: Find $\sqrt{5 - 12\,i}$

Suppose $\sqrt{5 - 12\,i} = a + i\,b$

$$\Rightarrow 5 - 12\,i = a^2 + 2\,i\,ab + i^2\,b^2$$

$$\Rightarrow 5 - 12\,i = (a^2 - b^2) + i\,(2ab)$$

Equating real and imaginary parts $\Rightarrow a^2 - b^2 = 5$ and $2ab = -12$

which upon solving gives: $a = 3,\ b = -2$ **or** $a = -3,\ b = 2$

Thus, $\sqrt{5 - 12\,i} = 3 - 2\,i$ **or** $-3 + 2\,i$

Argand diagram

A complex number can be represented on the Argand diagram; where

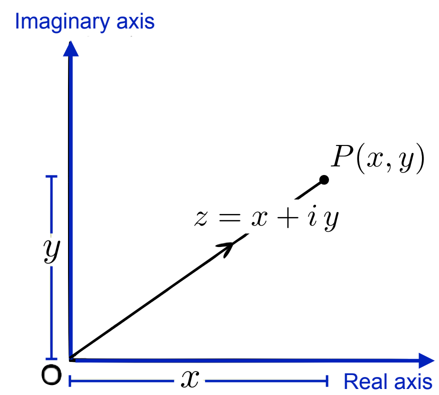
- Real numbers are represented on the X-axis (called real axis);

and

- Imaginary numbers are represented on the Y–axis (called imaginary axis).

Argand diagram

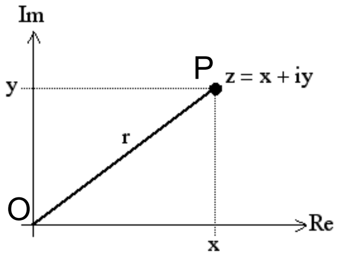
Thus, a general complex number $z = x + i\,y$ is represented by the vector $\overrightarrow{OP}$ where $P(x, y)$ is the point (x, y) in the XY-plane (called the Argand plane or complex plane).



Modulus of a complex number

The length of $\overline{OP}$ is called the modulus of the complex number $z = x + i\,y$ and is denoted by:

$$r = |\,z\,| = |\,x + i\,y\,|$$
$$= \sqrt{x^2 + y^2}$$



i.e. $|\,z\,| = \sqrt{[Re(z)]^2 + [Im(z)]^2}$

Properties of Modulus

1) $|\,z_1 \cdot z_2\,| = |\,z_1\,| \cdot |\,z_2\,|$

2) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|\,z_1\,|}{|\,z_2\,|}$

3) $|\,z_1 + z_2\,| \leq |\,z_1\,| + |\,z_2\,|$

4) $|\,z_1 - z_2\,| \geq |\,z_1\,| \sim |\,z_2\,|$ ($\sim$ denotes positive difference)

Properties of Modulus

1) Find $|\,-4 + 7\,i\,|$

$$|\,-4 + 7\,i\,| = \sqrt{(-4)^2 + (7)^2} = \sqrt{65}$$

2) Find $\left|\dfrac{2 - 3\,i}{4 + \sqrt{2}\,i}\right|$

$$\left|\frac{2 - 3\,i}{4 + \sqrt{2}\,i}\right| = \frac{|\,2 - 3\,i\,|}{|\,4 + \sqrt{2}\,i\,|} = \frac{\sqrt{2^2 + (-3)^2}}{\sqrt{4^2 + (\sqrt{2})^2}} = \sqrt{\frac{13}{18}}$$

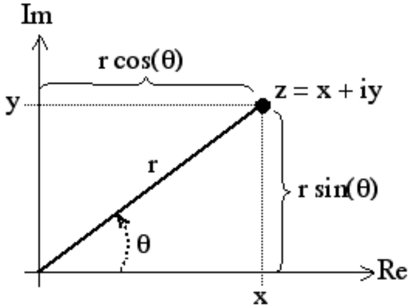
Polar form of a complex number

There is another way of representing a complex number using Polar coordinates (r, θ), and is called the Polar form of a complex number.

Suppose, the complex number $z = x + i\,y$ is represented in Cartesian form on the Argand diagram, by the point $P(x, y)$.

Polar form of a complex number

The same point P can be located by using its distance r from the origin O, and the angle θ made by the line $\overrightarrow{OP}$ with the real axis (X-axis).



Polar form of a complex number

Thus, $P(x, y)$ becomes the point

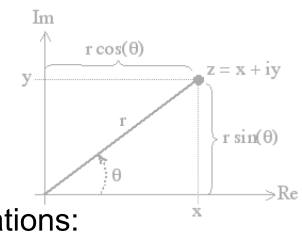
$$P(r, \theta) \equiv (r\cos\theta, r\sin\theta)$$

where, $r = \sqrt{x^2 + y^2}$

and θ is found from the set of equations:

$$\cos\theta = \frac{x}{r} \quad ; \quad \sin\theta = \frac{y}{r}.$$

Thus, $z = x + i\,y = r\cos\theta + i\,r\sin\theta$



Argument of a complex number

The angle θ is called the argument of the complex number

$$z = x + i\,y = r\cos\theta + i\,r\sin\theta$$

It is written as $Arg(x + i\,y)$, and obtained from the set of equations: $\cos\theta = \frac{x}{r} \quad ; \quad \sin\theta = \frac{y}{r}.$

As there are infinite number of angles that satisfy the above set of equations, the definition needs to be tightened so that everyone gets the same answer.

(Principal) Argument of a complex number

We denote the principal value of the argument by

$$\arg(z) = \theta \qquad \text{if} \qquad \underline{-\pi < \theta \leq \pi}.$$

Example

Express the following complex numbers in polar form and show them on the Argand diagram:

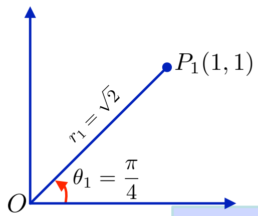
$$(i) \quad z_1 = 1 + i \qquad\qquad (iii) \quad z_3 = -1 - i$$
$$(ii) \quad z_2 = -1 + i \qquad\qquad (iv) \quad z_4 = 1 - i$$

Polar form of a complex number

$(i)\quad z_1 = 1 + i \;\equiv\; x + i\,y \;\Rightarrow\; x = 1,\; y = 1$

$$\therefore\; r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{array}{l} \cos\theta = \dfrac{x}{r} = \dfrac{1}{\sqrt{2}} \\[2ex] \sin\theta = \dfrac{y}{r} = \dfrac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow\; \theta = \frac{\pi}{4}$$



$-\pi < \theta \le \pi$

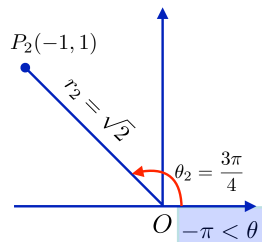
Thus, $z_1 = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)\right]$

Polar form of a complex number

$(ii)\quad z_2 = -1 + i \;\equiv\; x + i\,y \;\Rightarrow\; x = -1,\; y = 1$

$$\therefore\; r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{array}{l} \cos\theta = \dfrac{x}{r} = \dfrac{-1}{\sqrt{2}} \\[2ex] \sin\theta = \dfrac{y}{r} = \dfrac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow\; \theta = \frac{3\pi}{4}$$



$-\pi < \theta \le \pi$

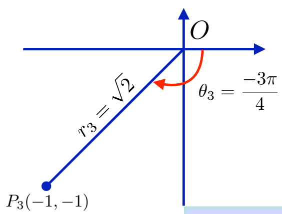
Thus, $z_2 = \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\,\sin\left(\frac{3\pi}{4}\right)\right]$

Polar form of a complex number

$(iii)\quad z_3 = -1 - i \;\equiv\; x + i\,y \;\Rightarrow\; x = -1,\; y = -1$

$$\therefore\; r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{array}{l} \cos\theta = \dfrac{x}{r} = \dfrac{-1}{\sqrt{2}} \\[2ex] \sin\theta = \dfrac{y}{r} = \dfrac{-1}{\sqrt{2}} \end{array}\right\} \Rightarrow\; \theta = \frac{-3\pi}{4}$$



$-\pi < \theta \le \pi$

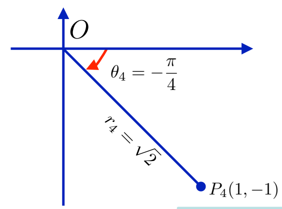
Thus, $z_3 = \sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\,\sin\left(\frac{-3\pi}{4}\right)\right]$

Polar form of a complex number

$(iv)\quad z_4 = 1 - i \;\equiv\; x + i\,y \;\Rightarrow\; x = 1,\; y = -1$

$$\therefore\; r = \sqrt{x^2 + y^2} = \sqrt{2}$$

$$\left.\begin{array}{l} \cos\theta = \dfrac{x}{r} = \dfrac{1}{\sqrt{2}} \\[2ex] \sin\theta = \dfrac{y}{r} = \dfrac{-1}{\sqrt{2}} \end{array}\right\} \Rightarrow\; \theta = -\frac{\pi}{4}$$



$-\pi < \theta \le \pi$

Thus, $z_4 = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\,\sin\left(-\frac{\pi}{4}\right)\right]$

Polar form of a complex number

Cartesian form

$z = x + i\,y$

where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.

Polar form

$z = r\,(\cos\theta + i\sin\theta)$

where $r > 0$ and $-\pi < \theta \le \pi$.

$$r = |\,z\,| = \sqrt{x^2 + y^2}$$

and $\theta = \arg(z)$ is obtained from

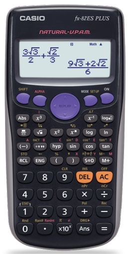
$x < 0$ and $y > 0$ $\theta = \pi - \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$x > 0$ and $y > 0$ $\theta = \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$
$x < 0$ and $y < 0$ $\theta = -\pi + \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$x > 0$ and $y < 0$ $\theta = -\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$

Finding arg(z) using a calculator

Quadrant	First	Second	Third	Fourth
Interval	$\left(0, \frac{\pi}{2}\right)$	$(0, \pi)$	$\left(\pi, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Signs of x and y	$x > 0,\ y > 0$	$x < 0,\ y > 0$	$x < 0,\ y < 0$	$x > 0,\ y < 0$
Principal argument $\theta = \arg(z)$	$\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$\pi - \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$-\pi + \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$-\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$

Finding arg(z) using a calculator

$x < 0$ and $y > 0$ $\theta = \pi - \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$x > 0$ and $y > 0$ $\theta = \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$
$x < 0$ and $y < 0$ $\theta = -\pi + \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$x > 0$ and $y < 0$ $\theta = -\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$



Calculator need to be set to **RADIAN** mode

Polar form of a complex number

Express the complex number $z = -5 + 6\,i$ in polar form $z = r\,(\cos\theta + i\sin\theta)$ where $r > 0$ and $-\pi < \theta \le \pi$.

$$z = -5 + 6\,i$$

$\therefore\ x = -5$ and $y = 6$

$\therefore\ r = \sqrt{x^2 + y^2} = \sqrt{61}$

and $\theta = \pi - \tan^{-1}\left\lvert \frac{6}{-5} \right\rvert$

$= \pi - 0.876$

$= 2.2655$ radians

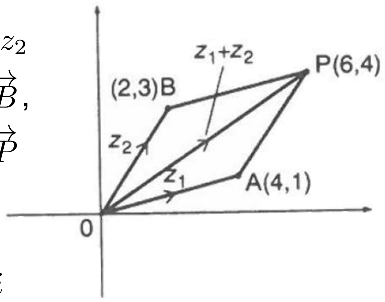
$x < 0$ and $y > 0$ $\theta = \pi - \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$x > 0$ and $y > 0$ $\theta = \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$
$x < 0$ and $y < 0$ $\theta = -\pi + \tan^{-1}\left\lvert \frac{y}{x} \right\rvert$	$x > 0$ and $y < 0$ $\theta = -\tan^{-1}\left\lvert \frac{y}{x} \right\rvert$

Thus, $z = \sqrt{61}\,[\cos(2.2655) + i\sin(2.2655)]$

Algebraic operations on Argand diagram

1. Addition

If the complex numbers z_1 and z_2 are shown by sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then $z_1 + z_2$ is the diagonal $\overrightarrow{OP}$ of the parallelogram $OAPB$.



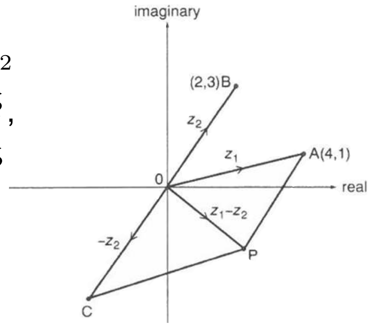
e.g. $z_1 = 4 + i$ and $z_2 = 2 + 3\,i$

then, $z = z_1 + z_2 = 6 + 4\,i$ is the point $P(6,4)$.

Algebraic operations on Argand diagram

2. Subtraction

If the complex numbers z_1 and z_2 are shown by sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$, then $z_1 - z_2$ is the diagonal $\overrightarrow{OP}$ of the parallelogram $OAPC$.



e.g. $z_1 = 4 + i$ and $z_2 = 2 + 3\,i$

then, $z = z_1 - z_2 = 2 - 2\,i$ is the point $P(2,-2)$.

Algebraic operations on Argand diagram

3. Multiplication

To show the product of complex numbers on the Argand plane, it is useful to first represent them in polar form.

Let $z_1 = r_1\,(\cos\theta_1 + i\,\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\,\sin\theta_2)$

be two complex numbers in polar form.

Then, $z_1 \cdot z_2 = r_1\,(\cos\theta_1 + i\,\sin\theta_1) \cdot r_2\,(\cos\theta_2 + i\,\sin\theta_2)$

Algebraic operations on Argand diagram

$$\begin{aligned}
\therefore\ z_1 \cdot z_2 &= r_1 \cdot r_2\,(\underline{\cos\theta_1\ \cos\theta_2} + \underline{i\,\cos\theta_1\ \sin\theta_2} \\
&\qquad + \underline{i\,\sin\theta_1\ \cos\theta_2} + \underline{i^2\,\sin\theta_1\ \sin\theta_2}) \\
&= r_1 \cdot r_2\,(\underline{\cos\theta_1\ \cos\theta_2 - \sin\theta_1\ \sin\theta_2} \\
&\qquad \underline{+ i\,(\sin\theta_1\ \cos\theta_2 + \cos\theta_1\ \sin\theta_2)} \\
&= r_1 \cdot r_2\ [\cos(\theta_1 + \theta_2) + i\,\sin(\theta_1 + \theta_2)]
\end{aligned}$$

Thus,

$$\boxed{z_1 \cdot z_2 \equiv R\ (\cos\theta + i\,\sin\theta) \text{ where } R = r_1 \cdot r_2\,,\ \theta = \theta_1 + \theta_2}$$

Algebraic operations on Argand diagram

e.g. $z_1 = 2 + 2\,i$

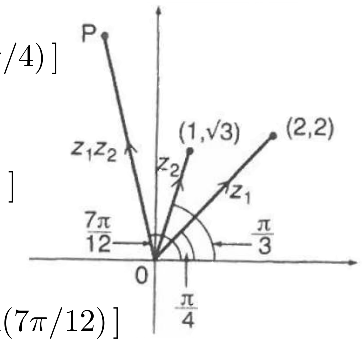
$$= 2\sqrt{2}\,[\cos(\pi/4) + i\,\sin(\pi/4)\,]$$

$z_2 = 1 + \sqrt{3}\,i$

$$= 2\,[\cos(\pi/3) + i\,\sin(\pi/3)\,]$$

then, $z = z_1 \cdot z_2$

$$= 4\sqrt{2}\,[\cos(7\pi/12) + i\,\sin(7\pi/12)\,]$$



Algebraic operations on Argand diagram

4. Division

In a similar way, it can be shown that:

$$\frac{z_1}{z_2} = \frac{r_1\,(\cos\theta_1 + i\,\sin\theta_1)}{r_2\,(\cos\theta_2 + i\,\sin\theta_2)} = \frac{r_1}{r_2}\,[\cos(\theta_1 - \theta_2) + i\,\sin(\theta_1 - \theta_2)\,]$$

Thus,

$$\boxed{\frac{z_1}{z_2} \equiv R\,(\cos\theta + i\,\sin\theta) \text{ where } R = \frac{r_1}{r_2} \text{ and } \theta = \theta_1 - \theta_2.}$$