Lecture 6 - Vector Spaces

COMP1046- Maths for Computer Scientists

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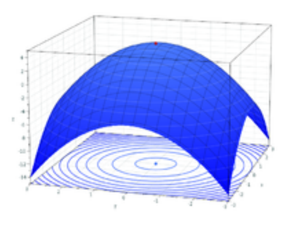

Learning outcomes

By the end of this lecture we will have learned:

- Vector Spaces
- Linear Dependence
- Linear Span

Based on Sections 8.1 to 8.4 of text book (Neri 2018).

Vector Spaces

Definition

- Let E to be a non-null set ($E \neq \emptyset$) and $\mathbb{K}$ to be a *scalar set* (typically $\mathbb{K}$=$\mathbb{R}$).
- Let us name *vectors* the elements of the set E.
- Let "+" be an internal composition law, $E \times E \longrightarrow E$.
- Let "$\cdot$" be an external composition law, $\mathbb{K} \times E \longrightarrow E$.

The triple $(E, +, \cdot)$ is said *vector space* of the vector set E over the *scalar field* $(\mathbb{K}, +,)$ if and only if the ten *vector space axioms* are verified.

continued...

Vector space axioms (1 to 5)

Definition

- E is closed with respect to the internal composition law:
  $\forall \mathbf{u}, \mathbf{v} \in E : \mathbf{u} + \mathbf{v} \in E$
- E is closed with respect to the external composition law:
  $\forall \mathbf{u} \in E$ and $\forall \lambda \in \mathbb{K} : \lambda \mathbf{u} \in E$
- commutativity for the internal composition law:
  $\forall \mathbf{u}, \mathbf{v} \in E : \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- associativity for the internal composition law:
  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in E \times E : \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- neutral element for the internal composition law:
  $\forall \mathbf{u} \in E : \exists! \mathbf{o} \in E | \mathbf{u} + \mathbf{o} = \mathbf{u}$

where $\mathbf{o}$ is the null vector.

Vector space axioms (6 to 10)

Definition

- opposite element for the internal composition law:
  $\forall \mathbf{u} \in E : \exists ! -\mathbf{u} \in E | \mathbf{u} + -\mathbf{u} = \mathbf{o}$
- associativity for the external composition law: $\forall \mathbf{u} \in E$ and $\forall \lambda, \mu \in \mathbb{K} : \lambda \left(\mu \mathbf{u}\right) = \left(\lambda \mu\right) \mathbf{u} = \lambda \mu \mathbf{u}$
- distributivity 1: $\forall \mathbf{u}, \mathbf{v} \in E$ and $\forall \lambda \in \mathbb{K} : \lambda \left(\mathbf{u} + \mathbf{v}\right) = \lambda \mathbf{u} + \lambda \mathbf{v}$
- distributivity 2: $\forall \mathbf{u} \in E$ and $\forall \lambda, \mu \in \mathbb{K} : \left(\lambda + \mu\right) \mathbf{u} = \lambda \mathbf{u} + \mu \mathbf{u}$
- neutral elements for the external composition law:
  $\forall \mathbf{u} \in E : \exists ! 1 \in \mathbb{K} | 1 \mathbf{u} = \mathbf{u}$

where $\mathbf{o}$ is the null vector.

Example

The set of numeric vectors $E = \mathbb{R}^3$ with scalar set $E = \mathbb{R}$, vector sum and scalar product form a vector space.

- Example of commutativity:
  $(2,3,1) + (0,-1,2) = (0,-1,2) + (2,3,1) = (2,2,3)$.
- Example of distributivity 1:
  $2 \times ((1.5,3,-1.4) + (0,-1.5,2)) =$
  $2 \times (1.5,3,-1.4) + 2 \times (0,-1.5,2) = (3,3,3.2)$.

Example

The set of matrices $\mathbb{R}_{m,n}$, the sum between matrices and the product of a scalar by a matrix, $(\mathbb{R}_{m,n}, +,)$.

Vector Subspace

Definition

Let $(E, +, \cdot)$ be a vector space, $U \subset E$, and $U \neq \emptyset$.

The triple $(U, +, \cdot)$ is a *vector subspace* of $(E, +, \cdot)$ if $(U, +, \cdot)$ is a vector space over the same field $\mathbb{K}$ with respect to both the composition laws.

Vector Subspace

Proposition

Let $(E, +, \cdot)$ be a vector space, $U \subset E$, and $U \neq \emptyset$.

The triple $(U, +, \cdot)$ is a vector subspace of $(E, +, \cdot)$ if and only if U is closed with respect to both the composition laws $+$ and $\cdot$, i.e.

- $\forall \mathbf{u}, \mathbf{v} \in U : \mathbf{u} + \mathbf{v} \in U$
- $\forall \lambda \in \mathbb{K}$ *and* $\forall \mathbf{u} \in U : \lambda \mathbf{u} \in U$.

This proposition shows that we do not need to prove all 10 axioms to show $(U, +, \cdot)$ is a vector subspace of another. We just need to prove closure of the two composition laws.

Vector Subspace

Need to prove "if and only if" both ways:

Proof.

Since the elements of U are also elements of E, they are vectors that satisfy the eight axioms regarding internal and external composition laws. If U is also closed with respect to the composition laws then $(U, +, \cdot)$ is a vector space and since $U \subset E$, U is vector subspace of $(E, +, \cdot)$.

If $(U, +, \cdot)$ is a vector subspace of $(E, +, \cdot)$, then it is a vector space. Thus, the ten axioms, including the closure with respect of the composition laws, are valid. $\square$

Vector Subspace

Example

Let us consider the vector space $(\mathbb{R}^3, +, \cdot)$ and its subset $U \subset \mathbb{R}^3$:

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 4y - 5z = 0\}$$

and let us prove that $(U, +, \cdot)$ is a vector subspace of $(\mathbb{R}^3, +, \cdot)$.

We have to prove the closure with respect to the two composition laws.

continued...

Example

1. Let us consider two arbitrary vectors belonging to U, $\mathbf{u_1} = (x_1, y_1, z_1)$ and $\mathbf{u_2} = (x_2, y_2, z_2)$. These two vectors are such that

$$3x_1 + 4y_1 - 5z_1 = 0 \text{ and } 3x_2 + 4y_2 - 5z_2 = 0.$$

Let us calculate $\mathbf{u_1} + \mathbf{u_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$.
In correspondence to the vector $\mathbf{u_1} + \mathbf{u_2}$,

$$3(x_1 + x_2) + 4(y_1 + y_2) - 5(z_1 + z_2) =$$
$$= 3x_1 + 4y_1 - 5z_1 + 3x_2 + 4y_2 - 5z_2 = 0 + 0 = 0.$$

This means that $\forall \mathbf{u_1}, \mathbf{u_2} \in U : \mathbf{u_1} + \mathbf{u_2} \in U$.

continued...

Example

2. Let us consider an arbitrary vector $\mathbf{u} = (x, y, z) \in U$ and an arbitrary scalar $\lambda \in \mathbb{R}$. We know that $3x + 4y - 5z = 0$.
   Let us calculate $\lambda \mathbf{u} = (\lambda x, \lambda y, \lambda z)$.
   In correspondence to the vector $\lambda \mathbf{u}$,

$$3\lambda x + 4\lambda y - 5\lambda z =$$
$$= \lambda (3x + 4y - 5z) = \lambda 0 = 0.$$

   This means that $\forall \lambda \in \mathbf{K}$ and $\forall \mathbf{u} \in U : \lambda \mathbf{u} \in U$.

Thus, we proved that $(U, +, \cdot)$ is a vector subspace $(\mathbb{R}^3, +, \cdot)$.

Exercise 1

Consider the vector space $\left(\mathbb{R}^2, +, \cdot\right)$ and its subsets $V \subset \mathbb{R}^2$ and $W \subset \mathbb{R}^2$:

$$V = \{\left(x, y\right) \in \mathbb{R}^2 \mid x + 2y > 1\}$$

$$W = \{\left(x, y\right) \in \mathbb{R}^2 \mid 2x = y\}.$$

Show whether or not $(V, +, \cdot)$ and $(W, +, \cdot)$ are vector subspaces of $\left(\mathbb{R}^2, +, \cdot\right)$.

Exercise 1: Solution

Due to Proposition on slide 8, we only need to prove closure for "+" and ".".

- For V, take scalar $\lambda = -1$ and $\mathbf{v} = (x, y) \in V$.
  Then $\lambda\mathbf{v} = (-x, -y)$. Since $x + 2y > 1 \Rightarrow (-x) + 2(-y) < -1$,
  hence $\lambda\mathbf{v} \notin V$. So this counter-example shows that $(V, +, \cdot)$
  is **not** a vector subspace.
- For W, take $\mathbf{w}_1 = (x_1, y_1) \in W$ and $\mathbf{w}_2 = (x_2, y_2) \in W$.
  Hence $2x_1 = y_1$ and $2x_2 = y_2$
  $\Rightarrow \quad 2(x_1 + x_2) = y_1 + y_2 \quad \Rightarrow \quad \mathbf{w}_1 + \mathbf{w}_2 \in W$.
  This proves the case for "+".
  Then, for any $\lambda \in \mathbb{R}$, $2\lambda x_1 = \lambda y_1$ proves the case for ".".
  Hence $(W, +, \cdot)$ is a vector subspace.

Linear Dependence

Linear Dependence in n Dimensions

Definition

Let $(E, +, \cdot)$ be a vector space. Let the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in E$ and the scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$.

The *linear combination* of the n vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ by means of n scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the vector $\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_n \mathbf{v_n}$.

Definition

Let $(E, +, \cdot)$ be a vector space. Let the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in E$. These vectors are said to be *linearly dependent* if the null vector $\mathbf{o}$ can be expressed as linear combination by means of the scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \neq 0, 0, \ldots, 0$.

Note: this means at least one $\lambda_i \neq 0$ (not necessarily all).

Linear Dependence in n Dimensions

Definition

Let $(E, +, \cdot)$ be a vector space. Let the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in E$.

These vectors are said to be *linearly independent* if the null vector $\mathbf{o}$ can be expressed as linear combination only by means of the scalars $0, 0, \ldots, 0$.

Linear Dependence in 3 Dimensions

Example

Let us consider the following vectors $\in \mathbb{R}^3$

$$\begin{aligned}
\mathbf{v_1} &= (4, 2, 0) \\
\mathbf{v_2} &= (1, 1, 1) \\
\mathbf{v_3} &= (6, 4, 2)\,.
\end{aligned}$$

These vectors are linearly dependent since

$$(0, 0, 0) = (4, 2, 0) + 2\,(1, 1, 1) - (6, 4, 2)\,;$$

that is, $\mathbf{v_3}$ as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$

$$(6, 4, 2) = (4, 2, 0) + 2\,(1, 1, 1)\,.$$

Linear Dependence in n Dimensions

Theorem

Let $(E, +, \cdot)$ be a vector space. Let the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in E$.

If the n vectors are linearly dependent while $n-1$ are linearly independent, there is a unique way to express one vector as linear combination of the others:

$$\forall \mathbf{v_k} \in E, \exists! \lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n \neq 0, 0, \ldots, 0 \textit{ such that}$$
$$\mathbf{v_k} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_{k-1} \mathbf{v_{k-1}} + \lambda_{k+1} \mathbf{v_{k+1}} + \ldots + \lambda_n \mathbf{v_n}$$

Linear Dependence in n Dimensions

Proof.

Let us assume by contradiction that the linear combination is not unique:

- $\exists \lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n \neq 0, 0, \ldots, 0$ such that

$$\mathbf{v_k} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} \ldots + \lambda_{k-1} \mathbf{v_{k-1}} + \lambda_{k+1} \mathbf{v_{k+1}} + \ldots + \lambda_n \mathbf{v_n}$$

- $\exists \mu_1, \mu_2, \ldots, \mu_{k-1}, \mu_{k+1}, \ldots, \mu_n \neq 0, 0, \ldots, 0$ such that

$$\mathbf{v_k} = \mu_1 \mathbf{v_1} + \mu_2 \mathbf{v_2} \ldots + \mu_{k-1} \mathbf{v_{k-1}} + \mu_{k+1} \mathbf{v_{k+1}} + \ldots + \mu_n \mathbf{v_n}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n \neq \mu_1, \mu_2, \ldots, \mu_{k-1}, \mu_{k+1}, \ldots, \mu_n \neq 0, 0, \ldots, 0.$

continued...

Proof.

Under this hypothesis, we can write that

$$\begin{aligned}
\mathbf{o} &= \left(\lambda_1 - \mu_1\right)\mathbf{v_1} \\
&+ \left(\lambda_2 - \mu_2\right)\mathbf{v_2} \\
&+ \ldots \\
&+ \left(\lambda_{k-1} - \mu_{k-1}\right)\mathbf{v_{k-1}} \\
&+ \left(\lambda_{k+1} - \mu_{k-1}\right)\mathbf{v_{k+1}} \\
&+ \ldots \\
&+ \left(\lambda_n - \mu_n\right)\mathbf{v_n}
\end{aligned}$$

continued...

Linear Dependence in n Dimensions

Proof.

Since the $n-1$ vectors are linearly independent

$$\begin{cases} \lambda_1 - \mu_1 = 0 \\ \lambda_2 - \mu_2 = 0 \\ \dots \\ \lambda_{k-1} - \mu_{k-1} = 0 \\ \lambda_{k+1} - \mu_{k+1} = 0 \\ \dots \\ \lambda_n - \mu_n = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \mu_1 \\ \lambda_2 = \mu_2 \\ \dots \\ \lambda_{k-1} = \mu_{k-1} \\ \lambda_{k+1} = \mu_{k+1} \\ \dots \\ \lambda_n = \mu_n. \end{cases}$$

Thus, the linear combination is unique. □

Linear Dependence in n Dimensions

Proposition

Let $(E, +, \cdot)$ be a vector space and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be its n vectors. If one of these vectors is equal to the null vector $\mathbf{o}$, these vectors are linearly dependent.

Proof.

Let us assume that $\mathbf{v_n} = \mathbf{o}$ and let us pose

$$\mathbf{o} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_{n-1} \mathbf{v_{n-1}} + \lambda_n \mathbf{o}.$$

Even if $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} = 0, 0, \ldots, 0$, the equality is verified for any scalar $\lambda_n \in \mathbb{K}$. Thus, the vectors are linearly dependent. $\square$

Exercise 2

1. Consider the vector space $\left(\mathbb{R}^3, +, \cdot\right)$ and the vectors $(1, 0, 2),\ (2, -1, 1),\ (3, x, 0)$ from this vector space.
   What value(s) of x will make these three vectors linearly dependent?
2. Consider the vector space $\left(\mathbb{R}^2, +, \cdot\right)$ and the vectors $(1, 0),\ (0, 2)$ from this vector space.
   Write down a third vector $\mathbf{v}$ such that $(1, 0),\ (0, 2),\ \mathbf{v}$ are linearly independent. Otherwise, if it is not possible, explain why.

Exercise 2: Solution

1. We need to find scalars $\lambda_1, \lambda_2, \lambda_3$ on these three vectors, such that their linear combination with these scalars is $(0, 0, 0)$ and at least one scalar is non-zero.

Let us suppose $\lambda_1 = 1$ [1].

- From the third component of the vectors,
  $\lambda_1 \times 2 + \lambda_2 \times 1 + \lambda_3 \times 0 = 0 \Rightarrow \lambda_2 = -2.$
- From the first component, $\lambda_1 \times 1 + \lambda_2 \times 2 + \lambda_3 \times 3 = 0$
  $\Rightarrow \lambda_3 = 1.$
- From the second component, $\lambda_1 \times 0 + \lambda_2 \times -1 + \lambda_3 x = 0$
  $\Rightarrow x = -2.$

[1] This is fine since λ_2 and λ_3 can be rescaled to match this. The only other possibility is that $\lambda_1 = 0$ but we will deal with that if we need to.

Exercise 2: Solution

2. Suppose $\mathbf{v} = (x, y)$ for some values of x and y.
Then we can choose non-zero scalars $\lambda_1 = x,\ \lambda_2 = \frac{y}{2},\ \lambda_3 = -1$ such that

$$\lambda_1(1, 0) + \lambda_2(0, 2) + \lambda_3\mathbf{v} = (0, 0).$$

Hence, it is not possible to find any (x, y) such that all three vectors are linearly independent.

Linear Span

Linear Span

Definition

Let $(E, +, \cdot)$ be a vector space. The set containing the totality of all the possibly linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in E$ by means of n scalars is named *linear span* (or simply span) and is indicated with $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}) \subset E$ or synthetically with L:

$$L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}) = \{\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_n \mathbf{v_n} | \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}\}.$$

In the case $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}) = E$, the vectors are said to span the set E or, equivalently, are said to span the vector space $(E, +, \cdot)$.

Linear Span

Example

The vectors $\mathbf{v_1} = (1, 0)$, $\mathbf{v_2} = (0, 2)$, $\mathbf{v_3} = (1, 1)$ span the entire $\mathbb{R}^2$ since any point $(x, y) \in \mathbb{R}^2$ can be generated from

$$\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \lambda_3 \mathbf{v_3}$$

with

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

We think of the vectors forming the span as building blocks for the vector space.

Theorem

The span $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ *with the composition laws is a vector subspace of* $(E, +, \cdot)$.

Proof.

In order to prove that $(L, +, \cdot)$ is a vector subspace, using Proposition on Slide 8, it is enough to prove the closure of L with respect to the composition laws. *continued...*

Proof.

1. Let $\mathbf{u}$ and $\mathbf{w}$ be two arbitrary distinct vectors $\in L$. Thus,

$$\mathbf{u} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_n \mathbf{v_n}$$
$$\mathbf{w} = \mu_1 \mathbf{v_1} + \mu_2 \mathbf{v_2} + \ldots + \mu_n \mathbf{v_n}.$$

Let us compute $\mathbf{u} + \mathbf{w}$,

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_n \mathbf{v_n} \\ &\quad + \mu_1 \mathbf{v_1} + \mu_2 \mathbf{v_2} + \ldots + \mu_n \mathbf{v_n} \\ &= (\lambda_1 + \mu_1) \mathbf{v_1} + (\lambda_2 + \mu_2) \mathbf{v_2} + \ldots + (\lambda_n + \mu_n) \mathbf{v_n}. \end{aligned}$$

Hence $\mathbf{u} + \mathbf{w} \in L$.

continued...

Proof.

1. Let $\mathbf{u}$ be an arbitrary vector $\in L$ and μ an arbitrary scalar $\in \mathbb{K}$. Thus,

$$\mathbf{u} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_n \mathbf{v_n}.$$

Let us compute $\mu \mathbf{u}$:

$$\begin{aligned} \mu \mathbf{u} &= \mu \left(\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_n \mathbf{v_n} \right) = \\ &= \mu \lambda_1 \mathbf{v_1} + \mu \lambda_2 \mathbf{v_2} + \ldots + \mu \lambda_n \mathbf{v_n}. \end{aligned}$$

Hence, $\mu \mathbf{u} \in L$.

$\square$

Exercise 3

Consider the vector space $\left(\mathbb{R}^3, +, \cdot\right)$ and its subset $U \subset \mathbb{R}^3$:

$$U = \{\left(x, y, z\right) \in \mathbb{R}^3 \mid 3x + 4y - 5z = 0\}$$

from slide 10.
Show that $\left(1, 0, \frac{3}{5}\right)$, $\left(0, 1, \frac{4}{5}\right)$ span U.

Exercise 3: Solution

Take any $\mathbf{u} = (x, y, z) \in U$.

Therefore $5z = 3x + 4y$, so write as $\mathbf{u} = (x, y, \frac{3x+4y}{5})$.

Take scalars $\lambda_1 = x,\ \lambda_2 = y$.

Then $\lambda_1 \left(1, 0, \frac{3}{5}\right) + \lambda_2 \left(0, 1, \frac{4}{5}\right) = \left(x, y, \frac{3x+4y}{5}\right) = \mathbf{u}$.

We have shown that any element from U is a linear combination of the two vectors, hence they span U.

Summary and next lecture

Summary

- Vector Spaces
- Linear Dependence
- Linear Span

The next lecture

We will learn about the Basis and Dimension of a Vector Space.