COMP1046 Tutorial 1 : Matrices

Anthony Bellotti

Let $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \\ 2 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 2 & -1 & 0 & 2 \\ 4 & 0 & 1 & -1 \\ 1 & 0 & 2 & -2 \end{pmatrix}$.

1. Based on $\mathbf{A}$, what is $\mathbf{a_2}$ and $\mathbf{a^1}$?

   Answer: This is the second row and first column of $\mathbf{A}$ respectively, given as vectors: $(-1, 0)$ and $(3, -1, 2)$.

2. Compute $2\mathbf{B} + \mathbf{A}^T$.

   Answer: $\begin{pmatrix} 4 & 0 & -4 \\ 2 & 6 & 2 \end{pmatrix} + \begin{pmatrix} 3 & -1 & 2 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -1 & -2 \\ 4 & 6 & 5 \end{pmatrix}$.

3. Suppose $\mathbf{A} + \mathbf{D} = \mathbf{0}$. Compute $\mathbf{D}$.

   Answer: $\mathbf{D} = -\mathbf{A} = \begin{pmatrix} -3 & -2 \\ 1 & 0 \\ -2 & -3 \end{pmatrix}$.

4. Compute $\mathbf{AB}$.

   Answer: $\begin{pmatrix} 3 \times 2 + 2 \times 1 & 3 \times 0 + 2 \times 3 & 3 \times -2 + 2 \times 1 \\ -1 \times 2 + 0 \times 1 & -1 \times 0 + 0 \times 3 & -1 \times -2 + 0 \times 1 \\ 2 \times 2 + 3 \times 1 & 2 \times 0 + 3 \times 3 & 2 \times -2 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 8 & 6 & -4 \\ -2 & 0 & 2 \\ 7 & 9 & -1 \end{pmatrix}$.

5. Compute $\mathbf{BA}$.

   Answer: $\begin{pmatrix} 2 \times 3 + 0 \times -1 + -2 \times 2 & 2 \times 2 + 0 \times 0 + -2 \times 3 \\ 1 \times 3 + 3 \times -1 + 1 \times 2 & 1 \times 2 + 3 \times 0 + 1 \times 3 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 5 \end{pmatrix}$.

6. What is the submatrix of $\mathbf{C}$ when the 1st row and 2nd and 3rd columns are cancelled?

   Answer: $\begin{pmatrix} 4 & -1 \\ 1 & -2 \end{pmatrix}$.

7. What is the minor for this submatrix?

Answer: $\det\begin{pmatrix} 4 & -1 \\ 1 & -2 \end{pmatrix} = 4 \times -2 - 1 \times -1 = -7.$

8. Compute the complement minor $M_{1,3}$ and cofactor $A_{1,3}$ of $\mathbf{AB}$.

   Answer: Cancel row 1 and column 3 of $\mathbf{AB} = \begin{pmatrix} 8 & 6 & -4 \\ -2 & 0 & 2 \\ 7 & 9 & -1 \end{pmatrix}$ and compute its determinant: $M_{1,3} = \det\begin{pmatrix} -2 & 0 \\ 7 & 9 \end{pmatrix} = -2 \times 9 - 7 \times 0 = -18.$
   $A_{1,3} = (-1)^{3-1} M_{1,3} = M_{1,3} = -18.$

9. Compute $\det(\mathbf{BA})$ and $(\mathbf{BA})^{-1}$.

   Answer: $\det(\mathbf{BA}) = \det\begin{pmatrix} 2 & -2 \\ 2 & 5 \end{pmatrix} = 2 \times 5 - 2 \times -2 = 14.$
   $(\mathbf{BA})^{-1} = \frac{1}{\det(\mathbf{BA})}\begin{pmatrix} 5 & 2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 5/14 & 1/7 \\ -1/7 & 1/7 \end{pmatrix}.$

10. Confirm that your answer is correct by taking the product of $\mathbf{BA}$ with its inverse.

Let $\mathbf{D} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 4 & 0 \\ -4 & 9 & -1 \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 6 & -2 \\ -3 & -9 & 3 \end{pmatrix}$.

11. Compute $\det(\mathbf{D})$.

    Answer: Convenient to work on column 3 with I Laplace Theorem:-
    $\det(\mathbf{D}) = \sum_{i=1}^{3} a_{i,3}A_{i,3} = 1 \times \det\begin{pmatrix} -1 & 4 \\ -4 & 9 \end{pmatrix} - 0 + (-1)\det\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} = 7 - 7 = 0.$

12. Compute $\mathbf{D}^{-1}$, or explain if it cannot be computed.

    Answer: Since $\det(\mathbf{D}) = 0$, the inverse $\mathbf{D}^{-1}$ cannot be computed.

13. Compute the ranks of $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$.

    Answer: The largest square matrix for $\mathbf{C}$ is 3×3. Consider the first three columns to form square matrix $\begin{pmatrix} 2 & -1 & 0 \\ 4 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$. Using I Laplace Theorem with column 2, the determinant is non-zero, $-1 \times -1 \times (4 \times 2 - 1 \times 1) = 7$ hence the rank $\rho_{\mathbf{C}} = 3$.
    The largest square matrix for $\mathbf{D}$ is 3×3, i.e. itself. However from Q11, we know its determinant is zero, so $\rho_{\mathbf{D}} < 3$. Try order 2: top left 2×2

matrix, $\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$ has non-zero determinant, $2 \times 4 - (-1) \times (-1) = 7$, hence $\rho_{\mathbf{D}} = 2$.

For $\mathbf{E}$, observe that all three rows are linearly dependent. Even picking any two rows are linearly dependent. Hence, $\rho = 1$ (see Theorem on slide 13 of Lecture 4).

14. Suppose that matrix $\mathbf{X}$ has an inverse $\mathbf{X}^{-1}$. Prove that the inverse of $\mathbf{X}^{\mathrm{T}}$ is $\left(\mathbf{X}^{-1}\right)^{\mathrm{T}}$.

Hint: You should use a property of the matrix product from Lecture 2.

Answer:

$$\begin{aligned} \left(\mathbf{X}^{-1}\right)^{\mathrm{T}}\mathbf{X}^{\mathrm{T}} &= \left(\mathbf{X}\mathbf{X}^{-1}\right)^{\mathrm{T}} && \text{using transpose of the product;} \\ &= \mathbf{I}^{\mathrm{T}} = \mathbf{I} && \text{since the identity matrix is symmetrical.} \end{aligned}$$

Hence $\left(\mathbf{X}^{-1}\right)^{\mathrm{T}}$ must be the inverse of $\mathbf{X}^{\mathrm{T}}$ since their product is the identity matrix.