AE1MCS: Mathematics for Computer Scientists

Huan Jin and HeShan Du
University of Nottingham Ningbo China

Oct 2021

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 9, Section 9.1 Relations and Their Properties
- Chapter 9, Section 9.2 ~~Database and Relations~~
- Chapter 9, Section 9.5 Equivalence Relations
- Chapter 9, Section 9.6 Partial Orderings

Relations

- Relationships between elements of sets occur in many contexts.
- Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets.

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

Binary Relations

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. Hence, sets of ordered pairs are called binary relations.

Definition

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

We use $a\,R\,b$ or $R(a, b)$ to denote that $(a, b) \in R$.

aRb

Exercise

$|A| = n$

$|A \times A| = n^2$

2^{n^2} Relations

$A \times A = \{(1,1), (1,2), (1,3), (1,4),$
$(2,1), (2,2), (2,3), (2,4),$
$(3,1) \quad \cdots \quad (3,4),$
$(4,1), \quad \cdots \quad (4,4)\}$

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

$R_1 = \{(1,1), (1,2)\}$

$R = \{(1,1), (1,2), (1,3), (1,4),$
$(2,2), \quad (2,4),$
$(3,3),$
$(4,4)\}$

Exercise

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \textit{ divides } b\}$?

Answer:
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

Relations on a Set

Relations from a set A to itself are of special interest.

Definition

A *relation on a set A* is a relation from A to A.

How many relations are there on a set with n elements?

2^{n^2}.

Reflexive Relations

There are several properties that are used to classify relations on a set.

Definition

A relation R on a set A is called *reflexive*, if $(a, a) \in R$ for every element $a \in A$.

How to use quantifiers to express it?

$\forall a \in A, \quad (a,a) \in R.$

$R_1 = \{(1,1), (1,2), (2,2)\} \quad A = \{1,2\}$

Symmetric Relations

$R = \{(1,1), (1,2), (2,1)\}$

Definition

A relation R on a set A is called *symmetric*, if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

How to use quantifiers to express it?

$\forall a \ \forall b \ (\underbrace{(a,b) \in R}_{\text{False}} \to (b,a) \in R)$

Antisymmetric Relations

T

$$\forall a \forall b\,(\underbrace{(a,b)\in R \wedge (b,a)\in R}_{F} \rightarrow \underbrace{(a=b)}_{T\ F}).$$

Definition

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

- The terms symmetric and antisymmetric are not opposites.
- A relation can have both of these properties or may lack both of them. Examples?

$R = \{(1,1)\}$.

$R = \{(2,2), (3,3), (4,4)\}$.

$(a,a) \in R$.

$R = \{(0,1), (1,2), (2,1)\}$

$R = \{(0,0), (1,5), (2,10), (10,2)\}$

$(a,b) \in R$, $(b,a) \in R$,

(c,d) & $(d,c) \in R$, $c \neq d$.

Transitive Relations

$\forall a \forall b \forall c. ((a,b) \in R \land (b,c) \in R \to (a,c) \in R).$

Definition

A relation R on a set A is called *transitive*, if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

How to use quantifiers to express it?

$R = \{(a,b) \mid a | b\}$, $a, b \in \mathbb{Z}^+$

Reflexive. Y

Symmetric N

Antisymmetric. Y
$a|b$, $b|a$
$\to a = b$

transitive. Y
$a|b$, $b|c$, $\to a|c$

Examples

Consider these relations on the set of integers:

	reflexive	sym	anti-symm	transitive
$R_1 = \{(a,b) \mid a \leq b\}$,	Y	N	Y	Y
$R_2 = \{(a,b) \mid a > b\}$,	N	N	Y	Y
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,	Y	Y	N	Y
$R_4 = \{(a,b) \mid a = b\}$,	Y	Y	Y	Y
$R_5 = \{(a,b) \mid a = b + 1\}$,	N	N	Y	N
$R_6 = \{(a,b) \mid a + b \leq 3\}$.	N	Y	N	N

(2,1) (1,2); (2,2) X

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

$R_1 \cup R_2$

$R_1 \cap R_2$

$R_1 - R_2$

$R_2 - R_1$

Exercise

Let R_1 be the 'less than' relation on the set of real numbers and let R_2 be the 'greater than' relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

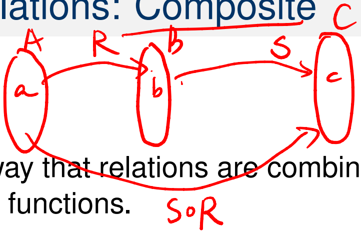
$R_1 \cup R_2 = \{(x,y) \mid x<y, \text{or } x>y\}.$

$R_1 \cap R_2 = \phi.$

$R_1 - R_2 = R_1$

$R_2 - R_1 = R_2$

Combining Relations: Composite

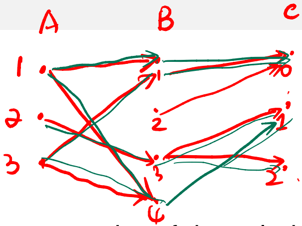


There is another way that relations are combined that is analogous to the composition of functions.

Definition

Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

Exercise



What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

$S \circ R = \{(1,0), (1,1), (2,1), (2,2), (3,0), (3,1)\}$

Composing a Relation with Itself

Definition

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, ...$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

$$R^2 = R \circ R$$

$$R^3 = R \circ R \circ R$$

Exercise

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find the powers R^n, $n = 2,3,4,....$

$R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$

$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

$R^4 = R^3 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

$\vdots$

$R^n = R^3$ for $n = 4, 5, 6, \cdots$

A Theorem

Theorem

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

See Rosen's textbook, p.581

Databases and Relations

Let $A_1, A_2, \ldots, A_n$ be sets.An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the domains of the relation, and n is called its degree.

A **relational database model** is a model for representing databases using n-ary relations.

Primary Key: a domain of an n-ary relation such that an n-tuple is uniquely determined by its value for this domain.

Composite Key: the Cartesian product of domains of an n-ary relation such that an n-tuple is uniquely determined by its values in these domains

Primary Key and Composite Key

Major X GPA

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

- Which domains are primary keys for the n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?
- Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the n-ary relation from Table 1, assuming that no n-tuples are ever added?

Projections

The projection $P_{i_1,i_2,...,i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the n-tuple $(a_1, a_2, ..., a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, ..., a_{i_m})$, where $m \leq n$.

What results when the projection $P_{1,3}$ is applied to the 4-tuples $(2, 3, 0, 4)$, (Jane Doe, 234111001, Geography, 3.14), and (a_1, a_2, a_3, a_4)?

(2, 0). (Jane Doe, Geography), (a_1, a_3)

More Examples

$a \equiv b \pmod m$ iff $m \mid (a-b)$. a is congruent to b modulo m

$2 \equiv 7 \pmod 5$ $\quad 5 \mid (2-7)$

$7 \equiv 12 \pmod 5$ $\quad 5 \mid (7-12)$

$2 \equiv -3 \pmod 5$ $\quad 5 \mid (2-(-3))$

Reflexive, sym, not anti-sym, transitive

- $A = \mathbb{Z}$, xRy if $x \equiv y \pmod 5$ Y, N, Y, Y
- $A = \mathbb{Z}^+$, xRy if $x|y$ Y, N, Y, Y
- $A = \mathbb{N}$, xRy if $x \leq y$ Y N Y Y

Equivalence Relations

Equivalence relation: a reflexive, symmetric, and transitive relation.

Two elements a and b that are related by an equivalence relation are called **equivalent**. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

$\{(a,b) \mid a \equiv b \pmod 5\}$. $2 \sim 7$

$\{(a,b) \mid a = b, \text{ or } a = -b\}$

Equivalence Classes

$[a]_R$ (equivalence class of a with respect to R): the set of all elements of A that are equivalent to a

$R = \{(a,b) \mid a \equiv b \pmod 4\}$

$$[a]_R = \{s \mid (a,s) \in R\}.$$

$-4, 0, 4$

Example

What are the equivalence classes of 0 and 1 for congruence modulo 4?

$[0]_R = \{\cdots, -8, -4, 0, 4, 8, 12, 16, \cdots\}$

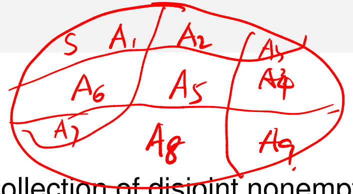
$[1]_R = \{\cdots, -7, -3, 1, 5, 9, 13, 17, \cdots\}$

$[2]_R$

$[a]_m$ the congruence class of an integer a modulo m.
equivalence

$[a]_m = \{\cdots a-2m, a-m, a, a+m, a+2m, \cdots\}$

Partition



A partition of a set S is a collection of disjoint nonempty subsets of S t hat have S as their union.

- $A_i \neq \phi$ for $i \in I$
- $A_i \cap A_j = \phi$ when$i \neq q$
- $\cup_{i \in I} A_i = S$

What are the sets in the partition of the integers $\mathbb{Z}$ arising from congruence modulo 4?

$[0]_4$

$[1]_4$

$[2]_4 = \{ \cdots -6, -2, 2, 6, 10, 14, \cdots \}$

$[3]_4 = \{ \cdots -5, -1, 3, 7, 11, 15, \cdots \}$

Partial Orderings

$R = \{(x,y) \mid x \geq y\}$, $x, y \in \mathbb{Z}$. Partial ordering

$\geq$ is a partial ordering on the set of $\mathbb{Z}$.

$(\mathbb{Z}, \geq)$

$(\mathbb{Z}, \leq)$

$(\mathbb{Z}, >)$

Partial ordering: a relation that is reflexive, antisymmetric, and transitive

A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R). Members of S are called elements of the poset.

$(\mathbb{Z}^+, |)$

$(P(S), \subseteq)$

$\{(A,B) \mid A \subseteq B\}$, $A, B \in P(S)$

Partial Orderings

In different posets different symbols such as $\leq$, $\subseteq$, and $|$, are used for a partial ordering.

- $\leq$
- $|$
- $\subseteq$

Comparable V.S. Incomparable

$A = \{1, 2\}$, $B = \{1, 3\}$

$(A, B) \notin R'$

Comparable: the elements a and b in the poset $(A, \preceq)$ are comparable if $a \preceq b$ or $b \preceq a$

Incomparable: When a and b are elements of S such that neither $a \preceq b$ nor $b \preceq a$, a and b are called incomparable.

- $(P(\mathbb{Z}), \subseteq)$
- $(\mathbb{Z}^+, |)$

$a = 3$, $b = 9$ $a | b = \frac{b}{a} = \frac{9}{3} = 3$.

$(a, b) \in R = \{(x, y) \mid x | y\}$, $a \preceq b$, $b \npreceq a$

$c = 5$, $d = 7$.

$(c, d) \notin R$

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 9, Section 9.1 Relations and Their Properties
- Chapter 9, Section 9.2 Database and Relations
- Chapter 9, Section 9.5 Equivalence Relations
- Chapter 9, Section 9.6 Partial Orderings

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 9, Section 9.1 Relations and Their Properties
- Chapter 9, Section 9.2 Database and Relations
- Chapter 9, Section 9.5 Equivalence Relations
- Chapter 9, Section 9.6 Partial Orderings