AE1MCS: Mathematics for Computer Scientists

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 9, Section 9.1 Relations and Their Properties
- Chapter 9, Section 9.2 Database and Relations
- Chapter 9, Section 9.5 Equivalence Relations

Relations

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

- Relationships between elements of sets occur in many contexts.
- Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets.

$R \subseteq A \times B$

(a, b)

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. Hence, sets of ordered pairs are called binary relations.

Definition

Let A and B be sets. A *binary relation* from A to B is a subset of $A \times B$.

We use $a \; R \; b$ or $R(a, b)$ to denote that $(a, b) \in R$.

a is said to be related to b by R

$(a, b) \in R$

Exercise

$\{(a,b) \mid a \in A,\ b \in A\}$

$A \times A$

$(1,1), (1,2), (1,3), (1,4)$
$(2,1)\quad (2,2)\quad (2,3)\quad (2,4)$
$(3,1)\quad (3,2)\quad (3,3)\quad (3,4)$

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

a/b

$1/1,\ 1/2,\ 1/3,\ 1/4,$

$2/2\qquad 2/4$

$3/3$

$4/4$

Exercise

a/b

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Answer:
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

$$\overset{n}{|A} \times \overset{n}{A|} = \underline{n^2}$$

$|A| = n$

2^{n^2}. relations

Relations from a set A to itself are of special interest.

Definition

A *relation on a set A* is a relation from A to A.

How many relations are there on a set with n elements? 2^{n^2}.

Reflexive Relations

There are several properties that are used to classify relations on a set.

Definition

A relation R on a set A is called *reflexive*, if $(a, a) \in R$ for every element $a \in A$.

How to use quantifiers to express it?

$\forall a \in A, (a, a) \in R$

Symmetric Relations

$R = \{(1,1), (1,2), (2,1)\}$. $A = \{1,2\}$

Definition

A relation R on a set A is called *symmetric*, if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

How to use quantifiers to express it?

R on set A is symmetric if.

$\forall a \ \forall b, ((a,b) \in R \rightarrow (b,a) \in R)$

Antisymmetric Relations

R is antisymmetric if

$\forall a \forall b \, ((a,b) \in R \land (b,a) \in R \Rightarrow (a=b))$ T

Hypothesis

F

Definition

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

- The terms symmetric and antisymmetric are not opposites.
- A relation can have both of these properties or may lack both of them. Examples?

$R = \{(2,2), (3,3), (4,4)\}$

$A = \{2, 3, 4\}$

· sym & anti-sym

$(a,a) \in R$

$R = \{(0,1), (1,2), (2,1)\}$

$A = \{0, 1, 2\}$

Neither.

$(a,b) \in R, \ (b,a) \notin R$

$(c,d) \in R, \ (d,c) \in R, \ c \neq d.$

Transitive Relations

$R = \{(a,b) \mid a \mid b\}$ transitive

Definition

A relation R on a set A is called *transitive*, if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

How to use quantifiers to express it?

$\forall a \forall b \forall c \, ((a,b) \in R \wedge (b,c) \in R \to (a,c) \in R)$

Examples

$(a,b) \in R_1$, $a \le b$, $(1,2)$

$(a,b) \in R \wedge (b,a) \in R \rightarrow a = b$

$p \rightarrow q$

Consider these relations on the set of integers:

	Reflexive	Sym	Anti-Sym	Transitive
$R_1 = \{(a,b) \mid a \le b\}$,	Y	N	Y	Y
$R_2 = \{(a,b) \mid a > b\}$,	N	N	Y	Y
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,	Y	Y	N	Y
$R_4 = \{(a,b) \mid a = b\}$,	Y	Y	Y	Y
$R_5 = \{(a,b) \mid a = b + 1\}$,	N	N	Y	N
$R_6 = \{(a,b) \mid a + b \le 3\}$.	N	Y	N	N

$(a,b) \in R$ $a = b+1$

$(b,a) \in R$, $b = a+1$

$(2,1)$
$(1,2)$
$(2,2)$

Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined.

$R_1 \cup R_2 = \{(x,y) \mid x<y \text{ or } x>y\}$

$R_1 \cap R_2 = \phi$

$R_1 - R_2 = R_1$

$R_2 - R_1 = R_2$

Exercise

Let R_1 be the 'less than' relation on the set of real numbers and let R_2 be the 'greater than' relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

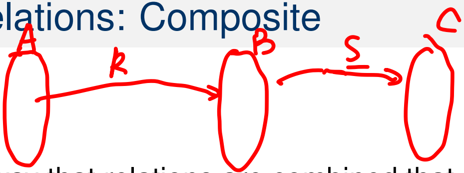
$R_1 \cup R_2 =$

$R_1 \cap R_2 =$

$R_1 - R_2 =$

$R_2 - R_1 =$

Combining Relations: Composite



There is another way that relations are combined that is analogous to the composition of functions.

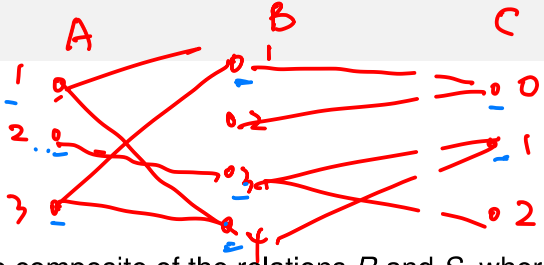
Definition

Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

$$S \circ R = \{(a,c) \mid (a,b) \in R, (b,c) \in S\}$$

common elements,

Exercise



What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

$S \circ R = \{(1,0), (1,1), (2,1), (3,0), (3,1), (2,2)\}$

Composing a Relation with Itself

Definition

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, ...,$ are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

$R^2 = R \circ R$

$R^3 = R^2 \circ R$

$\vdots$

$R^n = R^{n-1} \circ R$

Exercise

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find the powers R^n, $n = 2, 3, 4, ...$

$R^2 = \{(1,1), (2,1), (3,1), (4,2)\}$

$R^3 = \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = \{(1,1), (2,1), (3,1), (4,1)\}$

$\vdots$

$R^n = R^3$, for $n = 4, 5, 6 \ldots$

A Theorem

Theorem

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

See Rosen's textbook, p.581

More Examples

$a \equiv b \pmod{m}$: a is congruent to b modulo m if m divides $(a-b)$

- $A = \mathbb{Z}$, xRy if $x \equiv$ y (mod 5)
- $A = \mathbb{Z}^+$, xRy if $x | y$
- $A = \mathbb{N}$, xRy if $x \leq y$

	Reflexive	Sym	Anti-Sym	Transitive
$A = \mathbb{Z}$, xRy if $x \equiv$ y (mod 5)	Y	Y	N	Y
$A = \mathbb{Z}^+$, xRy if $x \mid y$	Y	N	Y	Y
$A = \mathbb{N}$, xRy if $x \leq y$	Y	N	Y	Y

$(2,7)$ $2 \sim 7$ w.r.t.

Equivalence Relations

Equivalence relation: a reflexive, symmetric, and transitive relation.

Two elements a and b that are related by an equivalence relation are called **equivalent**. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Equivalence Classes

$[a]_R$ (equivalence class of a with respect to R): the set of all elements of A that are equivalent to a $\in A$

$$[a]_R = \{s | (a, s) \in R\}.$$

Example $R = \{(a,b) \mid a \equiv b \pmod 4\}$ $a, b \in \mathbb{Z}$

What are the equivalence classes of 0 and 1 for congruence modulo 4?

$[0]_R = \{\cdots, -12, -8, -4, 0, 4, 8, 12, \cdots\}$

$[1]_R = \{\cdots -11, -7, -3, 1, 5, 9, 13, \cdots\}$

$[2]_R = \{\cdots -10, -6, -2, 2, 6, 10, 14, \cdots\}$

$[3]_R = \{\cdots -9, -5, -1, 3, 7, 11, 15, \cdots\}$

$[a]_m = \{\cdots a-3m, a-2m, a-m, a, a+m, a+2m, \cdots\}$

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