AE1MCS: Mathematics for Computer Scientists

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 7, Section 7.2 Probability Theory
- Chapter 7, Section 7.3 Bayes' Theorem
- Chapter 7, Section 7.4 Expected Value and Variance

Probability Distribution

Let s be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability $p(s)$ to each outcome. We require that two conditions be met:

1. $0 \leq p(s) \leq 1$ for each $s \in S$
2. $\sum_{s \in S} p(s) = 1$.

The function p from the set of all outcomes of the sample space S is called a **probability distribution.**

Conditional Probability

Given an event F occurs, the probability that event E occurs is the **conditional probability** of E given F.

Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F, denoted by $p(E|F)$, is defined as:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

.

Conditional Probability

A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

Solution: Let E be the event that a bit of length four contains at least two consecutive 0s,
Let F be the event that the first bit of a bit string of length four is a 0.
The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

Conditional Probability

0 1 1 1 $|F| = 8.$
0 0 0

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The probability that a bit string of length four has at least two consecutive 0s, given that its first bit is a 0, equals:

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

$|E \cap F|$

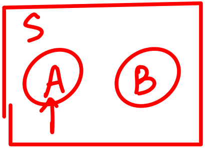
$|S| = 16$

Because $E \cap F = \{0000, 0001, 0010, 0011, 0100\}$, then $p(E \cap F) = \frac{5}{16}$.
Because there are 8 bit strings of length four that start with a 0, we have $p(F) = \frac{8}{16} = \frac{1}{2}$.

$$p(E|F) = \frac{5/16}{1/2} = \frac{5}{8}$$

Independence

are disjoint events independent? No



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = 0$$

independent $P(A|B) = P(A) \neq 0$

When two events are independent, the occurrence of one of the events gives no information about the probability of that the other event occurs.

The events E and F are independent **if and only if** $\leftrightarrow$
$p(E \cap F) = p(E)p(F)$.

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$$

Independence

Suppose E is the event that a randomly generated bit string of length four begin with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

Solution: There are eight bit strings of length four that begin with a one: 1000, 1001, 1010, 1011, 1100, 1101, 1110, and 1111. There are also eight bit strings of length four that contain an even number of ones: 0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111. Because there are 16 bit strings of length four, it follows that

$$p(E) = p(F) = 8/16 = 1/2.$$

Because $E \cap F = 1111, 1100, 1010, 1001$, we see that $p(E \cap F) = 4/16 = 1/4$.
Because $p(E \cap F) = 1/4 = (1/2)(1/2) = p(E)p(F)$,we conclude that E and F are independent.

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Flip coins:

A = 1st coin is Head

$$P(A) = \frac{1}{2}$$

B = 2nd coin is Head

$$P(B) = \frac{1}{2}.$$

$$P(A \cap B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = 1/2 = P(B)$$

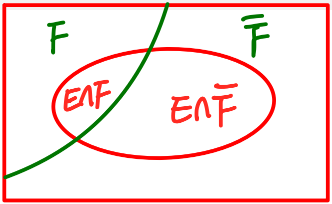
Bayes' Theorem

Suppose we know p(F), the probability that an event F occurs, but we have knowledge that an event E occurs.
The conditional probability that F occurs given that E occurs, $p(F|E)$

$P(F)$, $P(\bar{F})$ ✓

$P(E|F)$ ✓

$P(F|E)$??



Bayes' Theorem

Suppose that E and F are events from a sample space S such that $P(E) \neq 0$ and $P(F) \neq 0$. Then

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{P(E|F)P(F)}{}$$

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})}$$

$$P(E) = P(E \cap F) + P(E \cap \bar{F})$$

$$= P(E|F) \cdot P(F) + P(E|\bar{F})P(\bar{F})$$

Bayesian Spam Filters

Suppose that we have found that the word "Rolex" occurs in 250 of 2000 messages known to be spam and in 5 of 1000 messages known not to be spam. Estimate the probability that an incoming message containing the word "Rolex" is spam, assuming that it is equally likely that an incoming message is spam or not spam. If our threshold for rejecting a message as spam is 0.9, will we reject such messages?

F = the message is a spam, $\bar{F}$ = the message is not a spam

$P(F) = P(\bar{F}) = 0.5$

E = message contain "Rolex"

$P(E|F) = \frac{250}{2000} = 0.125$

$P(E|\bar{F}) = \frac{5}{1000} = 0.005$

$$P(F|E) = \frac{P(E|F)\cdot P(F)}{P(E|F)P(F)\ P(E|\bar{F})P(\bar{F})}$$

$$= \frac{0.125 \times 0.5}{0.125 \times 0.5 + 0.005 \times 0.5} = 0.962$$

Solution: We use the counts that the word "Rolex" appears in spam messages and messages that are not spam to find that $p(Rolex) = 250/2000 = 0.125$ and $q(Rolex) = 5/1000 = 0.005$. Because we are assuming that it is equally likely for an incoming message to be spam as it is not to be spam, we can estimate the probability that an incoming message containing the word "Rolex" is spam by

$$r(Rolex) = \frac{p(Rolex)}{p(Rolex) + q(Rolex)} = \frac{0.125}{0.125 + 0.005} = 0.962$$

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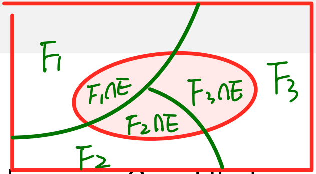
Generalizing Bayes' Theorem



Suppose that E is an event from a sample space S and that $F_1, F_2, ..., F_n$ are mutually exclusive events such that $\cup_{i=1}^{n} F_i = S$. Assume that $p(E) \neq 0$ and $p(F_i) \neq 0$ for $i = 1, 2, ..., n$. Then

$$P(E) = \sum_{i=1}^{n} p(E|Fi)p(Fi)$$

$$p(F_j|E) = \frac{p(E|F_j)p(F_j)}{\sum_{i=1}^{n} p(E|Fi)p(Fi)}.$$

$$P(F_1|E) = \frac{P(F_1 \cap E)}{P(E)} = \frac{P(E|F_1)P(F_1)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$$

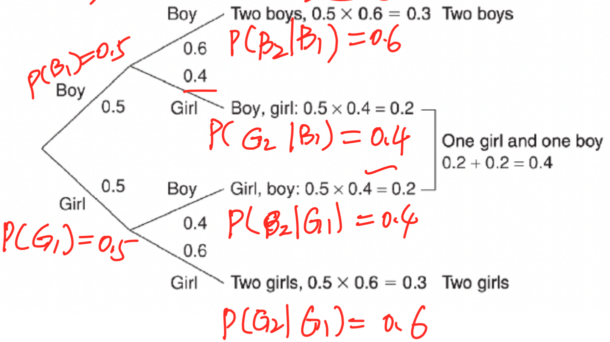
$$P(E) = P(F_1 \cap E) + P(F_2 \cap E) + P(F_3 \cap E)$$
$$= P(E|F_1)P(F_1) + P(E|F_2) \cdot P(F_2) + P(E|F_3)P(F_3)$$

Example

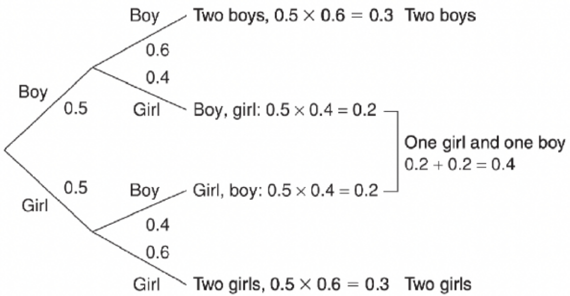
Consider a mother with two children. Let B1 be an event that the first child is a boy. Let B2 be an event that the second child is a boy. Similarly, G1 and G2 for girls.

P(the first-born is boy and the second-born is girl)=?

$P(B_1 \cap G_2) = P(B_1) \cdot P(G_2 | B_1) = 0.5 \times 0.4 = 0.2$



Example



P(second-born is a boy)=?

$$P(B_2) = P(B_2|B_1)P(B_1) + P(B_2|G_1)P(G_1) = 0.5 \cdot 0.6 + 0.4 \cdot 0.5$$

Expected Value and Variance

$s \in S$	$X(s)$	$p(s)$	$E(X)$
	r	$p(X=r)$	

The **expected value**, also called the expectation or mean, of the random variable X on the sample space S:

$$E(X) = \sum_{s \in S} p(s)X(S)$$

If X is a random variable and $p(X = r)$ is the probability that $X = r$, so that $p(X = r) = \sum_{s \in S, X(s)=r} p(s)$, then

$$E(X) = \sum_{r \in X(S)} p(X = r)r$$

Ramdam Variable: $X: S \to \mathbb{R}$

Expectation

$X(S)$	$P(S)$	$E(X)$
$X=i$	$P(X=i)$	

Corollary

If X is a random variable and $P(X = i)$ is the probability that $X = i$, then

$$E(X) = \sum_{i=1}^{\infty} iP(X = i)$$

Linearity of Expectations

If X_i, $i = 1, 2, ..., n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then

1. $E(X_1 + X_2 + \cdots + Xn) = E(X_1) + E(X_2) + \cdots + E(X_n)$
2. $E(aX + b) = aE(X) + b.$

$Y = ax + b$

$E(Y) = aE(X) + b$

Independent Random Variables

The random variables X and Y on a sample space S are independent if

$$p(X = r1 \text{ and } Y = r2) = p(X = r1) \cdot p(Y = r2)$$

or, if the probability that $X = r1$ and $Y = r2$ equals the product of the probabilities that $X = r1 and Y = r2$, for all real numbers $r1$ and $r2$.

Corollary

If X is independent of Y, then

$$E(XY) = E(X) \cdot E(Y) \tag{1}$$

If $X_1, X_2, \dots, X_n$ are mutually independent, then,

$$E(X_1 X_2 \dots X_n) = E(X_1)E(X_2)\dots E(X_n) \tag{2}$$

Variance

Betting a game

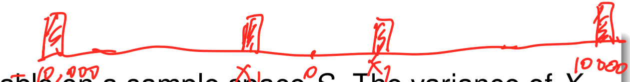
10 RMB

10,000 RMB.

$E(X_1) = \frac{1}{2} \times (10) + \frac{1}{2}(-10) = 0$

$E(X_2) = \frac{1}{2}(10{,}000) + \frac{1}{2}(-10000) = 0$

Variance provides a measure of how widely X is distributed about its expected value.



Definition

Let X be a random variable on a sample space S. The variance of X, denoted by $Var(X)$, is

$Var(X_1) = (10-0)^2 \cdot \frac{1}{2} + (-10-0)^2 \cdot \frac{1}{2}$

$$Var(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

$Var(X_2) = (10{,}000-0)^2 \frac{1}{2} + (-10000-0)^2 \cdot \frac{1}{2}$

That is, Var(X) is the weighted average of the square of the deviation of X. The standard deviation of X, denoted $\sigma(X)$, is defined to be $\sqrt{Var(X)}$.

Variance

Theorem

If X is a random variable on a sample space S, then
$Var(X) = E(X^2) - E(X)^2$

Corollary

If X is a random variable on a sample space S and $E(X) = \mu$, then
$Var(X) = E((X - \mu)^2)$.

How to prove it?

expected value.

Example: Rolling a Die

$P(X=i, Y=j)$

$P(X|Y) =$

Let X be the number that comes up when a fair die is rolled. What is the expected value and variance of X?

$X(s)$	$P(s)$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

$E(X)$

$$E(X) = \sum_{s \in S} X(s) \cdot P(s)$$

$$= \sum_{s \in S} P(X=i) \cdot i$$

$$= P(X=1) \cdot 1 + P(X=2) \cdot 2 + \cdots + P(X=6) \cdot 6$$

$$= \frac{1}{6}(1+2+3+\cdots+6) = 3.5$$

Example: Rolling a Die

Let X be the number that comes up when a fair die is rolled. What is the expected value and variance of X?

Solution: The random variable X takes the values $1, 2, 3, 4, 5, 6$, each with probability 1/6. It follows that:

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 21/6 = 7/2 = 3.5$$

$$Var(X) = \frac{1}{6} \cdot (1 - 3.5)^2 + \frac{1}{6} \cdot (2 - 3.5)^2 + \frac{1}{6} \cdot (3 - 3.5)^2 + \frac{1}{6} \cdot (4 - 3.5)^2 + \frac{1}{6} \cdot (5 - 3.5)^2$$

$$+ \frac{1}{6} \cdot (6 - 3.5)^2 = 2.917$$

Variance for the sum of random variables

If X and Y are independent variable,

$$Var(X + Y) = Var(X) + Var(Y)$$

In addition,

$$Var(aX + b) = a^2 Var(X)$$

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