

# AE1MCS: Mathematics for Computer Scientists

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Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 2, Section 2.1. Sets
- Chapter 2, Section 2.2. Set Operations

# Discrete Structures

- Much of discrete mathematics is devoted to the study of **discrete structures**, used to represent discrete objects..
- Many important discrete structures are built using **sets**, which are collections of objects.
  - **combinations**: unordered collections of objects used extensively in counting;
  - **relations**: sets of ordered pairs that represent relationships between objects;
  - **graphs**: sets of vertices and edges that connect vertices;
  - **finite state machines**, used to model computing machines;
  - ...

# Set

An intuitive definition (not part of a formal theory of sets)

## Definition

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to contain its elements. We write  $a \in A$  to denote that  $a$  is an element of the set  $A$ . The notation  $a \notin A$  denotes that  $a$  is not an element of the set  $A$ .

Uppercase letters are usually used to denote sets. Lowercase letters are usually used to denote elements of sets.

# Describe a Set

There are several ways to describe a set.

- 1 List all the members of a set (if it is possible):  
e.g.  $\{a, b, c\}$ ,  $\{1, a\}$ ,  $\{1, 2, 3, \dots, 99\}$  (positive integers  $< 100$ )
- 2 Use *set builder* notation: characterize all elements in a set by stating the property or properties they must have.
  - $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$
  - or  $O = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10\}$

# Important Sets

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ , the set of **natural numbers**
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ , the set of **integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ , the set of **positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0\}$ , the set of **rational numbers**
- $\mathbb{R}$ , the set of **real numbers**
- $\mathbb{R}^+$  or  $\mathbb{R}_{>0}$ , the set of **positive real numbers**

# Equal Sets

## Definition

Two sets are equal if and only if they have the same elements. Therefore, if  $A$  and  $B$  are sets, then  $A$  and  $B$  are equal if and only if  $\forall x (x \in A \leftrightarrow x \in B)$ . We write  $A = B$  if  $A$  and  $B$  are equal sets.

- $\{1, 2, 3\}$
- $\{3, 2, 1\}$
- $\{1, 2, 2, 3, 3, 3\}$

# Empty Set and Singleton Set

- Empty set: a set that has no element. It is denoted by  $\emptyset$  or  $\{\}$ .
- Singleton set: a set that has only one element.
- $\emptyset$  vs.  $\{\emptyset\}$ ?



# Venn Diagram

- Sets can be represented graphically using Venn diagrams <sup>1</sup>.
- The universal set  $U$ , which contains all the objects under consideration, is represented by a rectangle.
- Inside this rectangle, circles or other geometrical figures are used to represent sets.
- Sometimes points are used to represent the particular elements of the set.
- Venn diagrams are often used to indicate the relationships between sets.

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<sup>1</sup>named after the English mathematician John Venn, who introduced their use in 1881.

# Subsets

## Definition

The set  $A$  is a *subset* of  $B$  if and only if every element of  $A$  is also an element of  $B$ . We use the notation  $A \subseteq B$  to indicate that  $A$  is a subset of the set  $B$ .

$$A \subseteq B \text{ iff } \forall x (x \in A \rightarrow x \in B)$$

# Prove or Disprove $A$ is a Subset of $B$

Showing that  $A$  is a Subset of  $B$  To show that  $A \subseteq B$ , show that if  $x$  belongs to  $A$  then  $x$  also belongs to  $B$ .

Showing that  $A$  is Not a Subset of  $B$  To show that  $A \not\subseteq B$ , find a single  $x \in A$  but  $x \notin B$ .

# Proper Subset

*A* is a *proper subset* of *B* ( $A \subset B$ ) if and only if

$$\forall x (x \in A \rightarrow x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

# Equal Sets

$$A = B \text{ iff } A \subseteq B \text{ and } B \subseteq A.$$

# The Size of a Set

## Definition

Let  $S$  be a set. If there are exactly  $n$  distinct elements in  $S$  where  $n$  is a nonnegative integer, we say that  $S$  is a *finite set* and that  $n$  is the *cardinality* of  $S$ . The cardinality of  $S$  is denoted by  $|S|$ .

## Definition

A set is said to be infinite if it is not finite.

# Power Sets

## Definition

Given a set  $S$ , the *power set* of  $S$  is the set of all subsets of the set  $S$ . The power set of  $S$  is denoted by  $\mathcal{P}(S)$ .

- What is the power set of the set  $\{0, 1, 2\}$ ?
- What is the power set of the empty set?
- What is the power set of the set  $\{\emptyset\}$ ?

If a set has  $n$  elements, then its power set has  $2^n$  elements.

# Ordered $n$ -tuples

## Definition

The *ordered  $n$ -tuple*  $(a_1, a_2, \dots, a_n)$  is the ordered collection that has  $a_1$  as its first element,  $a_2$  as its second element, ..., and  $a_n$  as its  $n$ th element.

- We say that two ordered  $n$ -tuples are equal if and only if each corresponding pair of their elements is equal.
- Ordered 2-tuples are called *ordered pairs*.



# Cartesian products

## Definition

Let  $A$  and  $B$  be sets. The *Cartesian product* of  $A$  and  $B$ , denoted by  $A \times B$ , is the set of all ordered pairs  $(a, b)$ , where  $a \in A$  and  $b \in B$ . Hence,  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$ .

- What is the Cartesian product of  $A = \{1, 2\}$  and  $B = \{a, b, c\}$ ?
- $A \times B = B \times A$ ?

# Cartesian products

## Definition

The *Cartesian product* of the sets  $A_1, A_2, \dots, A_n$ , denoted by  $A_1 \times A_2 \times \dots \times A_n$ , is the set of ordered  $n$ -tuples  $(a_1, a_2, \dots, a_n)$ , where  $a_i$  belongs to  $A_i$  for  $i = 1, 2, \dots, n$ . In other words,

$$A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

$$A^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \dots, n\}$$

# Using Set Notation with Quantifiers

- Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation.
- For example,  $\forall x \in S P(x)$  denotes the universal quantification of  $P(x)$  over all elements in the set  $S$ .
- $\forall x \in S P(x) \equiv \forall x (x \in S \rightarrow P(x))$
- $\exists x \in S P(x)$  denotes the existential quantification of  $P(x)$  over all elements in  $S$ .
- $\exists x \in S P(x) \equiv \exists x (x \in S \wedge P(x))$

# Truth Sets and Quantifiers

- We will now tie together concepts from set theory and from predicate logic.
- Given a predicate  $P$ , and a domain  $D$ , we define the truth set of  $P$  to be the set of elements  $x$  in  $D$  for which  $P(x)$  is true.
- The truth set of  $P(x)$  is denoted by  $\{x \in D \mid P(x)\}$ .
- $\forall x P(x)$  is true over the domain  $U$  if and only if the truth set of  $P$  is the set  $U$ .
- $\exists x P(x)$  is true over the domain  $U$  if and only if the truth set of  $P$  is nonempty.

# Set Operations

- Union
- Intersection
- Difference
- Complement

# Union

## Definition

Let  $A$  and  $B$  be sets. The *union* of the sets  $A$  and  $B$ , denoted by  $A \cup B$ , is the set that contains those elements that are either in  $A$  or in  $B$ , or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

# Intersection

## Definition

Let  $A$  and  $B$  be sets. The *intersection* of the sets  $A$  and  $B$ , denoted by  $A \cap B$ , is the set containing those elements in both  $A$  and  $B$ .

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

# Disjoint

## Definition

Two sets are called *disjoint* if their intersection is the empty set.



# Difference

## Definition

Let  $A$  and  $B$  be sets. The *difference* of  $A$  and  $B$ , denoted by  $A - B$ , is the set containing those elements that are in  $A$  but not in  $B$ . The difference of  $A$  and  $B$  is also called the complement of  $B$  with respect to  $A$ .

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Remark: The difference of sets  $A$  and  $B$  is sometimes denoted by  $A \setminus B$ .

# Complement

Once the universal set  $U$  has been specified, the complement of a set can be defined.

## Definition

Let  $U$  be the universal set. The *complement* of the set  $A$ , denoted by  $\overline{A}$ , is the complement of  $A$  with respect to  $U$ . Therefore, the complement of the set  $A$  is  $U - A$ .

$$\overline{A} = \{x \in U \mid x \notin A\}$$

# Difference and Complement

$$A - B = A \cap \overline{B}$$

# Set Identities

	Identity	Name
1	$A \cap U = A$	Identity laws
2	$A \cup \emptyset = A$	
3	$A \cup U = U$	Domination laws
4	$A \cap \emptyset = \emptyset$	
5	$A \cup A = A$	Idempotent laws
6	$A \cap A = A$	
7	$\overline{(\overline{A})} = A$	Complementation law
8	$A \cup B = B \cup A$	Commutative laws
9	$A \cap B = B \cap A$	

# Set Identities

	Identity	Name
10	$A \cup (B \cup C) = (A \cup B) \cup C$	Associative laws
11	$A \cap (B \cap C) = (A \cap B) \cap C$	
12	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
13	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
14	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
15	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	
16	$A \cup (A \cap B) = A$	Absorption laws
17	$A \cap (A \cup B) = A$	
18	$A \cup \overline{A} = U$	Complement laws
19	$A \cap \overline{A} = \emptyset$	

# Exercise

Let  $A$ ,  $B$  and  $C$  be sets. Show that

- $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .
- $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

# Generalized Unions and Intersections

## Definition

The *union* of a collection of sets is the set that contains those elements that are members of *at least one* set in the collection.

## Definition

The *intersection* of a collection of sets is the set that contains those elements that are members of *all* the sets in the collection.

# Homework: Proving a Theorem

## Theorem

*For every set  $S$ ,  $\emptyset \subseteq S$  and  $S \subseteq S$ .*



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