

# Lecture 3 - Determinants and Matrix Inversion

COMP1046 - Maths for Computer Scientists

---

Dr. Ferrante Neri / Dr. Tony Bellotti



By the end of this lecture we will have learned:

- ⊙ Introduction to Matrix Inversion
- ⊙ Adjugate Matrix
- ⊙ Determinants
- ⊙ Invertible Matrices

## Introduction to Matrix inversion

---

# Inverting a matrix

- ⊙ In lecture 2, we saw how we could add, subtract or take the product of matrices. It would be really useful to be able to divide matrices: something like  $\mathbf{A}/\mathbf{B}$ .
- ⊙ We can do that through the matrix inverse, written  $\mathbf{B}^{-1}$ . The equivalent of a “divide” is then  $\mathbf{AB}^{-1}$ .

## Definition

For a square matrix  $\mathbf{A}$ , the *inverse matrix*  $\mathbf{A}^{-1}$  is defined as the matrix for which

$$\mathbf{AA}^{-1} = \mathbf{I}$$

If we take the special case of  $\mathbf{A} = (a_{1,1})$  and  $\mathbf{A}^{-1} = (b_{1,1})$ , this makes sense since it is only true if  $b_{1,1} = \frac{1}{a_{1,1}}$ .

# What is the Inverse?

The definition on the previous slide raises a number of questions:-

1. Can we calculate the inverse  $\mathbf{A}^{-1}$  exactly, in the general case?
2. Does an inverse exist for all square matrices?
3. Is there a unique inverse where an inverse does exist? (if no, then there are multiple solutions).

The short answers to these questions are “Yes”, “No”, “Yes”.  
This lecture is about proving these answers.

# Inverting an order 2 square matrix

Consider the special case,  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

Then the inverse is given as  $\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

## Example

Take  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ . Then the inverse is  $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$  since

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

*Exercise:* Prove the result for any order 2 matrix.

# Formula for matrix inverse

- ⊙ There is a formula to compute the inverse for any square matrix, in general:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj}(\mathbf{A})$$

where  $\det$  is the determinant and  $\text{adj}$  is the adjugate matrix of  $\mathbf{A}$ .

- ⊙ The remainder of the lecture we will define and explore these concepts and finally we can prove this result.
- ⊙ The formula shows us when an inverse does not exist: i.e. when  $\det \mathbf{A} = 0$ .

# Determinants and adjugate matrices

- ⊙ There is a precise definition of the determinant  $\det \mathbf{A}$  based on permutations of elements of  $\mathbf{A}$  (see Neri 2019, section 2.4).
- ⊙ However, in this lecture we will work with the Laplace Theorem which expresses the determinant in terms of the adjugate matrix.
- ⊙ However, the adjugate matrix is defined in terms of the determinant.
- ⊙ Hence  $\det$  and  $\text{adj}$  form a *recursive* relationship.



# Adjugate Matrices

---

## Definition

### **Submatrices.**

Let us consider a matrix  $\mathbf{A} \in \mathbb{R}_{m,n}$ . Let  $r, s$  be two positive integer numbers such that  $1 \leq r \leq m$  and  $1 \leq s \leq n$ .

A *submatrix* is a matrix obtained from  $\mathbf{A}$  by cancelling  $m - r$  rows and  $n - s$  columns.

## Example

Let us consider the following matrix:

$$\mathbf{A} = \begin{pmatrix} 3 & 3 & 1 & 0 \\ 2 & 4 & 1 & 2 \\ 5 & 1 & 1 & 1 \end{pmatrix}.$$

The submatrix obtained by cancelling the second row, the second and fourth columns is

$$\begin{pmatrix} 3 & 1 \\ 5 & 1 \end{pmatrix}.$$

## Definition

Let us consider a matrix  $\mathbf{A} \in \mathbb{R}_{m,n}$  and one of its square submatrices.

The determinant of this submatrix is said *minor*.

If the submatrix is the largest square submatrix of the matrix  $\mathbf{A}$ , its determinant is said *major determinant* or simply *major*.

It must be observed that a matrix can have multiple majors.

## Example

Let us consider the following matrix  $\mathbf{A} \in \mathbb{R}_{4,3}$ :

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

- ⊙ An example of minor is  $\det \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$  obtained after cancelling the  $2^{nd}$  column as well as the  $3^{rd}$  and  $4^{th}$  rows.
- ⊙ Several minors can be calculated, including several majors which must all be  $3 \times 3$  matrices.

## Definition

Let us consider a matrix  $\mathbf{A} \in \mathbb{R}_{n,n}$ .

The submatrix obtained by cancelling only the  $i^{th}$  row and the  $j^{th}$  column from  $\mathbf{A}$  is said *complement submatrix* to the element  $a_{i,j}$  and its determinant is here named *complement minor* and indicated with  $M_{i,j}$ .

# Complement submatrices and minors

## Example

Let us consider a matrix  $\mathbf{A} \in \mathbb{R}_{3,3}$ :

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}.$$

The complement submatrix to the element  $a_{1,2}$  is

$$\begin{pmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{pmatrix}$$

and the complement minor  $M_{1,2} = \det \begin{pmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{pmatrix}.$

## Definition

Let us consider a matrix  $\mathbf{A} \in \mathbb{R}_{n,n}$ , its generic element  $a_{i,j}$  and corresponding complement minor  $M_{i,j}$ . The *cofactor*  $A_{i,j}$  of the element  $a_{i,j}$  is defined as  $A_{i,j} = (-1)^{i+j} M_{i,j}$ .

## Example

From the matrix of the previous example, the cofactor  $A_{1,2} = (-1)M_{1,2}$ .



# Adjugate Matrices

## Definition

**Adjugate Matrix.** Let us consider a matrix  $\mathbf{A} \in \mathbb{R}_{n,n}$ :

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}.$$

Let  $A_{i,j}$  be the cofactor for  $a_{i,j}$ .

The *adjugate matrix* (or adjunct or adjoint)  $\mathbf{A}$  is

$$\text{adj}(\mathbf{A}) = \begin{pmatrix} A_{1,1} & A_{2,1} & \dots & A_{n,1} \\ A_{1,2} & A_{2,2} & \dots & A_{n,2} \\ \dots & \dots & \dots & \dots \\ A_{1,n} & A_{2,n} & \dots & A_{n,n} \end{pmatrix}.$$

## Determinants

---

## Theorem

**I Laplace Theorem** *Let  $\mathbf{A} \in \mathbb{R}_{n,n}$ . The determinant of  $\mathbf{A}$  can be computed as the sum of each row (element) multiplied by the corresponding cofactor:*

$$\det \mathbf{A} = \sum_{j=1}^n a_{i,j} A_{i,j} \text{ for any arbitrary } i \text{ and}$$

$$\det \mathbf{A} = \sum_{i=1}^n a_{i,j} A_{i,j} \text{ for any arbitrary } j.$$

Not proved here.

With this theorem and knowing that the determinant of a one-element matrix is just that element ( $\det(a) = a$ ), we can compute the determinant of any square matrix.

# Determinant of an order 2 square matrix

Consider  $\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ .

From I Laplace Theorem, taking  $i = 1$ ,

$$\begin{aligned} \det \mathbf{A} &= a_{1,1}A_{1,1} + a_{1,2}A_{1,2} \\ &= a_{1,1} \cdot (-1)^{1+1}M_{1,1} + a_{1,2} \cdot (-1)^{1+2}M_{1,2} \\ &= a_{1,1} \det(a_{2,2}) - a_{1,2} \det(a_{2,1}) \\ &= a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \end{aligned}$$

## Example

Let us consider the following  $\mathbf{A} \in \mathbb{R}_{3,3}$ :

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

If we consider the second row, it follows that

$$\begin{aligned} \det \mathbf{A} &= a_{2,1}(-1)M_{2,1} + a_{2,2}M_{2,2} + a_{2,3}(-1)M_{2,3} \\ &= 0 + 1 \times \det \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix} - 1 \times \det \begin{pmatrix} 1 & 2 \\ 4 & 2 \end{pmatrix} \\ &= 0 - 4 - 1 \times -6 = 2 \end{aligned}$$

## Matrix inversion

---

## Definition

Let  $\mathbf{A} \in \mathbb{R}_{n,n}$ . The matrix  $\mathbf{A}$  is said *invertible* if  $\exists$  a matrix  $\mathbf{B} \in \mathbb{R}_{n,n} | \mathbf{AB} = \mathbf{I} = \mathbf{BA}$ . The matrix  $\mathbf{B}$  is said *inverse* matrix of the matrix  $\mathbf{A}$ .

# Uniqueness of invertability

## Theorem

*If  $\mathbf{A} \in \mathbb{R}_{n,n}$  is an invertible matrix and  $\mathbf{B}$  is its inverse, it follows that the inverse matrix is unique:  $\exists! \mathbf{B} \in \mathbb{R}_{n,n} | \mathbf{AB} = \mathbf{I} = \mathbf{BA}$ .*

## Proof.

Let us assume by contradiction that the inverse matrix is not unique. Thus, besides  $\mathbf{B}$ , there exists another inverse of  $\mathbf{A}$ , indicated as  $\mathbf{C} \in \mathbb{R}_{n,n}$ .

This would mean that for the hypothesis  $\mathbf{B}$  is inverse of  $\mathbf{A}$  and thus  $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ .

For the contradiction hypothesis also  $\mathbf{C}$  is inverse of  $\mathbf{A}$  and thus  $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$ . *continued...*



## Proof.

Considering that  $\mathbf{I}$  is the neutral element with respect to the product of matrices ( $\forall \mathbf{A} : \mathbf{AI} = \mathbf{IA} = \mathbf{A}$ ) and that the product of matrices is associative, it follows that

$$\mathbf{C} = \mathbf{CI} = \mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B}.$$

In other words, if  $\mathbf{B}$  is an inverse matrix of  $\mathbf{A}$  and another inverse matrix  $\mathbf{C}$  exists, then  $\mathbf{C} = \mathbf{B}$ . Thus, the inverse matrix is unique. □

# II Laplace Theorem

## Theorem

### II Laplace Theorem

*Let  $\mathbf{A} \in \mathbb{R}_{n,n}$  with  $n > 1$ . The sum of the elements of a row (column) multiplied by the corresponding cofactor related to another row (column) is always zero:*

$$\sum_{j=1}^n a_{i,j} A_{k,j} = 0 \text{ for any arbitrary } k \neq i \text{ and}$$

$$\sum_{i=1}^n a_{i,j} A_{i,k} = 0 \text{ for any arbitrary } k \neq j.$$

Not proved here.

We now have everything in place to show what the inverse matrix is.

## II Laplace Theorem

### Example

Consider  $\mathbf{A} = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 2 & 1 \\ 4 & 2 & 5 \end{pmatrix}$ .

Take rows  $i = 3$  and  $k = 2$ :

$$\begin{aligned} \sum_{j=1}^n a_{i,j} A_{k,j} &= 4 \times (-1) \det \begin{pmatrix} 0 & 2 \\ 2 & 5 \end{pmatrix} + 2 \times (+1) \det \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \\ &\quad + 5 \times (-1) \det \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix} \\ &= 4 \times 4 + 2 \times 7 - 5 \times 6 \\ &= 0 \end{aligned}$$

# The Inverse Matrix

## Theorem

Let  $\mathbf{A} \in \mathbb{R}_{n,n}$  and  $A_{i,j}$  its generic cofactor.

The inverse matrix  $\mathbf{A}^{-1}$  is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj}(\mathbf{A}).$$

## Proof.

Let us consider the matrix  $\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix}$ .

*continued...*

# The Inverse Matrix

Proof.

Its adjugate matrix is  $\text{adj}(\mathbf{A}) = \begin{pmatrix} A_{1,1} & A_{2,1} & \dots & A_{n,1} \\ A_{1,2} & A_{2,2} & \dots & A_{n,2} \\ \dots & \dots & \dots & \dots \\ A_{1,n} & A_{2,n} & \dots & A_{n,n} \end{pmatrix}.$

Let us compute the product matrix  $\mathbf{P} = \mathbf{A} \text{adj}(\mathbf{A})$ .

$$\mathbf{P}_{i,k} = a_{i,1}A_{k,1} + a_{i,2}A_{k,2} + \dots + a_{i,n}A_{k,n} = \sum_{j=1}^n a_{i,j}A_{k,j}$$

*continued...*

# The Inverse Matrix

## Proof.

For the I Laplace Theorem, the sum of the diagonal elements of  $\mathbf{P}$  are equal to  $\det \mathbf{A}$ :

$$\mathbf{P}_{i,i} = \sum_{j=1}^n a_{i,j} A_{i,j} = \det \mathbf{A} \quad \text{for all the rows } i.$$

For the II Laplace Theorem, the sum of the extra-diagonal elements of  $\mathbf{P}$  are equal to zero:

$$\mathbf{P}_{i,k} = \sum_{j=1}^n a_{i,j} A_{k,j} = 0 \quad \text{with } i \neq k.$$

*continued...*

# The Inverse Matrix

Proof.

The result of the multiplication is then

$$\mathbf{A} \operatorname{adj}(\mathbf{A}) = \mathbf{P} = \begin{pmatrix} \det \mathbf{A} & 0 & \dots & 0 \\ 0 & \det \mathbf{A} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \det \mathbf{A} \end{pmatrix}.$$

Thus,

$$\mathbf{A} \operatorname{adj}(\mathbf{A}) = (\det \mathbf{A}) \mathbf{I}$$

and

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj}(\mathbf{A}).$$

□

## Example

Let us now invert a matrix  $\mathbf{A} \in \mathbb{R}_{3,3}$ :

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$$

The determinant of this matrix is  $\det \mathbf{A} = 2 - 1 = 1$ .

The transpose of this matrix is

$$\mathbf{A}^T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}.$$

*continued...*



## Example

The adjugate matrix is

$$\text{adj}(\mathbf{A}) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & -5 & 2 \end{pmatrix}$$

and the corresponding inverse matrix is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \text{adj}(\mathbf{A}) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & -5 & 2 \end{pmatrix}.$$

# Inversion and non-singularity

## Definition

Let  $\mathbf{A} \in \mathbb{R}_{n,n}$ . If  $\det \mathbf{A} = 0$  the matrix is said *singular*. If  $\det \mathbf{A} \neq 0$  the matrix is said *non-singular*.

## Theorem

Let  $\mathbf{A} \in \mathbb{R}_{n,n}$ . The matrix  $\mathbf{A}$  is invertible if and only if  $\mathbf{A}$  is non-singular.

The proof is given in Neri (2019). However, the intuition is that the inverse can be computed except when  $\det \mathbf{A} = 0$ .

## Proposition

*Let  $\mathbf{A}$  and  $\mathbf{B}$  be two square and invertible matrices. it follows that*

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$

## Proof.

Let us calculate  $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$ .

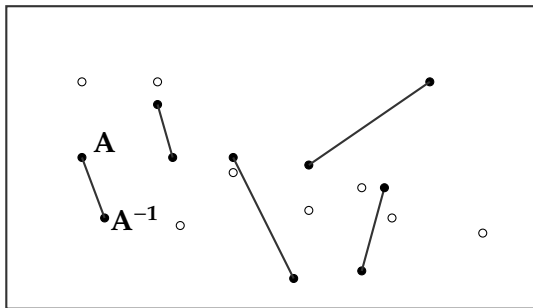
Thus, the inverse of  $(\mathbf{AB})$  is  $(\mathbf{B}^{-1}\mathbf{A}^{-1})$ , i.e.

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}.$$



# Summary

This illustration represents and summarizes our matrix theory studies so far.



- ⊙ The empty circles represent singular matrices which are not linked to other matrices since they have no inverse.
- ⊙ Conversely, filled circles represent non-singular matrices, each with just one inverse.

## Lecture 3: Exercises

Work on these problems with reference to the definitions given in Lecture 3.

$$\text{Let } \mathbf{A} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & 0 & -1 & 2 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

- Q1. What is the submatrix of  $\mathbf{B}$  when the 2nd row and the 2nd and 4th columns are cancelled?
- Q2. Compute the minor for this submatrix.
- Q3. What is the complement submatrix of  $a_{2,3}$  from  $\mathbf{A}$ ?
- Q4. Compute the complement minor of  $a_{2,3}$  from  $\mathbf{A}$ .
- Q5. Compute the cofactor of  $a_{2,3}$  from  $\mathbf{A}$ .
- Q6. Compute the adjugate matrix  $\text{adj}(\mathbf{A})$ .
- Q7. Compute  $\mathbf{A}(\text{adj}(\mathbf{A}))$ .
- Q8. What do you think the value on the leading diagonal is?

# Summary and next lecture

## Summary

- ⊙ Introduction to Matrix Inversion
- ⊙ Adjugate Matrix
- ⊙ Determinants
- ⊙ Invertible Matrices

## The next lecture

We will learn about linear dependencies.