COMP1046 Tutorial 4 : Linear Mappings

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Consider the set $E = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 3x_1 = x_2 + x_3 + x_4\}$ and the linear mapping $f : E \to \mathbb{R}^4$,

$$f(x_1, x_2, x_3, x_4) = (3x_1 + x_3 + 2x_4, 2x_1 - x_2 + 2x_3 + x_4, x_1 + x_2 - x_3 + x_4, 4x_1 + x_2 + 3x_4).$$

1. Show that $(E, +, \cdot)$ is a vector space, where the internal and external composition laws are the usual real number addition and scalar product.

   Answer:

   Since $E \subset \mathbb{R}^4$ and $(\mathbb{R}^4, +, \cdot)$ is a vector space, then we only need to prove closure of two composition laws:

   - For the internal composition law: consider two arbitrary vectors

     $$\mathbf{x} = (x_1, x_2, x_3, x_4) \in E,$$

     $$\mathbf{x}' \in (x_1', x_2', x_3', x_4') \in E.$$

     Then, $3x_1 = x_2 + x_3 + x_4$ and $3x_1' = x_2' + x_3' + x_4'$ which implies

     $$3(x_1 + x_1') = (x_2 + x_2') + (x_3 + x_3') + (x_4 + x_4')$$

     which is the condition for $\mathbf{x} + \mathbf{x}' \in E$.
   - For the external composition law, consider an arbitray scalar λ. Then, since $\mathbf{x} = (x_1, x_2, x_3, x_4) \in E$,

     $$3\lambda x_1 = \lambda x_2 + \lambda x_3 + \lambda x_4$$

     which is the condition for $\lambda\mathbf{x} \in E$.

2. Construct a basis for $(E, +, \cdot)$.

   Answer:

   There are several answers to this, but the general solution is to find a set of three vectors from E that span E and are linearly independent.
   Here is one answer:-

   - Rewrite $E = \{(x_1, x_2, x_3, 3x_1 - x_2 - x_3) \mid (x_1, x_2, x_3) \in \mathbb{R}^3\}$ so we can see that one of the variables can be written as a function of the others. Then a basis is proposed with the last term dependent on the first three:

     $$B = \{\ (1, 0, 0, 3),\ (0, 1, 0, -1),\ (0, 0, 1, -1)\ \}.$$

- Show that B spans E: Take an arbitrary $(x_1, x_2, x_3, 3x_1 - x_2 - x_3)) \in E$ and use scalars $\lambda_1 = x_1$, $\lambda_2 = x_2$ and $\lambda_3 = x_3$. Then,

  $$\lambda_1(1,0,0,3) + \lambda_2(0,1,0,-1) + \lambda_3(0,0,1,-1) = (x_1, x_2, x_3, 3x_1 - x_2 - x_3)$$

  which is the arbitrary vector in E. Hence, the basis is shown to span E.
- Show that B is linearly independent: Suppose

  $$\lambda_1(1,0,0,3) + \lambda_2(0,1,0,-1) + \lambda_3(0,0,1,-1) = (0,0,0,0)$$

  so looking at the first three components, we see that this can only be true if $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 0$. Hence B is linearly independent.

3. Compute $\dim(E)$.

Answer:
Since the basis for $(E, +, \cdot)$ has three vectors, $\dim(E) = 3$.