Tutorial on Probability

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Probability

Three dies with six sides but only three possible numbers:

- Die A: $2, 2, 6, 6, 7, 7$
- Die B: $1, 1, 5, 5, 9, 9$
- Die C: $3, 3, 4, 4, 8, 8$

Which one do you pick?

Bayesian Theorem

Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

Answer

Let $P(s)$ = prob of using steroid
Let $P(+|s)$ = prob of a bicyclist who uses steroids tests positive
$P(+|\bar{s})$ = prob of a bicyclist who does not use steroids tests positive
We know that $P(s) = 0.08$, $P(+|s) = 0.96$, $P(+|\bar{s}) = 0.09$
so $p(\bar{s}) = 1 - P(s) = 0.92$
We want to find $P(s|+)$
By Bay's theorem,

$$p(s|+) = \frac{p(+|s)p(s)}{p(+|s)p(s) + p(+|\bar{s})p(\bar{s})} = \frac{0.96 * 0,08}{0.96 * 0,08 + 0.09 * 0.92} = 0.481$$

Bayesian Theorem

There are 3 boxes I, II and III. There are 4 white balls and 2 black balls in box I, 2 white balls and 1 black balls in box II, and 3 white balls and 3 black balls in box III. Now one roll a six-sided die with the points 1 through 6 on its six different sides to decide which box to choose. If points 1, 2 and 3 appear, select box I; if points 4 appear, select box II; if points 5 and 6 appear, select box III. Take a ball from the selected box.

(1) Calculate the probability of taking out a white ball;
(2) If a white ball is taken out, calculate the conditional probability that the ball comes from the box III.

Answer

Let B_1 ={the ball is taken from box I} B_2 ={the ball is taken from box II} B_3 ={the ball is taken from box III} A={the ball is white}
then we have:
$P(B_1) = \frac{1}{2}$, $P(B_2) = \frac{1}{6}$, $P(B_3) = \frac{1}{3}$
$P(A|B_1) = \frac{2}{3}$, $P(A|B_2) = \frac{2}{3}$, $P(A|B_3) = \frac{1}{2}$
(1) The probability of taking out a white ball:

$$\begin{aligned} P(A) &= P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3) \\ &= \frac{1}{3} + \frac{1}{9} + \frac{1}{6} \\ &= \frac{11}{18} \end{aligned}$$

(2)By Bayesian theorem

$$P(B_3|A) = \frac{P(B_3)P(A|B_3)}{P(A)} = \frac{3}{11}$$

Rolling two dice. Assume that we are using fair, regular dice (six-sided with values 1, 2, 3, 4, 5, 6 appearing equally likely). Furthermore, assume that all dice rolls are mutually independent events.
(a) You roll two dice and look at the sum of the faces that come up. What is the expected value of this sum? Express your answer as a real number.

$$E(X+Y) = E(X) + E(Y) = 7/2 + 7/2 = 7$$

(b) Assuming that the two dice are independent, calculate the variance of their sum. Express your answer as a real number.

$$Var(X+Y) = Var(X) + Var(Y) = 2.917 + 2.917 = 5.834$$

Example

What is the variance of the random variable $X((i,j)) = 2i$, where i is the number appearing on the first die and j is the number appearing on the second die, when two fair dice are rolled?

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Solution: Note that because $p(X = k)$ is $1/6$ for $k = 2, 4, 6, 8, 10, 12$ and is 0 otherwise,
$E(X) = (2 + 4 + 6 + 8 + 10 + 12)/6 = 7$,
and

$$E(X^2) = (2^2 + 4^2 + 6^2 + 8^2 + 10^2 + 12^2)/6 = 182/3.$$

It follows that:

$$Var(X) = E(X^2) - E(X)^2 = 182/3 - 49 = 35/3.$$

Additional exercises

Section 7.1: 1, 3, 5, 7, 9, 11, 13
Section 7.2: 1, 3, 5, 7, 9, 11, 13, 15
Section 7.3: 1, 5, 7, 9
Section 7.4: 1, 3, 5, 7, 9, 11, 15, 17