AE1MCS: Mathematics for Computer Scientists

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Aim and Learning Objectives

- To be able to understand inference rules.
- To be able to use inference rules to solve logical problems.

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Section 1.6. Rules of Inference

Mathematical Proof

Definition

A *mathematical proof* of a *proposition* is a chain of *logical deductions* leading to the proposition from a base set of *axioms*.

- An *axiom* is a proposition that is assumed to be true, e.g. if $a = b$ and $b = c$, then $a = c$.
- *Logical deductions* or *inference rules* are used to prove new propositions using previously proved ones.

Proofs

- **Proofs:** valid arguments that establish the truth of mathematical statements.
- **Argument**, a sequence of statements that end with a conclusion.
- By **valid**, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or **premises**, of the argument.
- An argument is **valid** if and only if it is impossible for all the premises to be true and the conclusion to be false.

Argument

Definition (Argument)

An **argument** in propositional logic is a sequence of propositions. All but the final proposition in the argument are called **premises** and the final proposition is called the **conclusion**. An argument is valid if the truth of all its premises implies that the conclusion is true.

Argument Form

Definition (Argument Form)

An **argument form** in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is valid no matter which particular propositions are substituted for the propositional variables in its premises, the conclusion is true if the premises are all true.

The argument form with premises $p_1, p_2, ..., p_n$ and conclusion q is valid, when $(p_1 \land p_2 \land ... \land p_n) \to q$ is a tautology.

Rules of Inference for Propositional Logic

- To show that an argument form is valid, we could use a truth table.
- When an argument form involves 10 different propositional variables, to use a truth table to show this argument form is valid requires $2^{10} = 1024$ different rows.
- Alternatively, we can first establish the validity of some relatively simple argument forms, called rules of inference.
- These rules of inference can be used as building blocks to construct more complicated valid argument forms.

Inference Rules for Propositional Logic

- Modus ponens
- Modus tollens
- Hypothetical syllogism
- Disjunctive syllogism
- Addition
- Simplification
- Conjunction
- Resolution

Modus Ponens

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

The tautology $(p \land (p \to q)) \to q$ is the basis of the inference rule modus ponens.

Example

Suppose that the statements

'If it snows today, then we will go skiing'

p q

and

'It is snowing today'

p

are true. Then, by modus ponens, it follows that the statement

'We will go skiing'

is true.

Modus Tollens

false False

$$p \to q$$
$$\neg q$$
$$\overline{\qquad\qquad}$$
$$\therefore \neg p$$

Hypothetical syllogism

$$\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Disjunctive syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Addition

$$\frac{p}{\therefore p \vee q}$$

Simplification

$$\frac{p \wedge q}{\therefore p}$$

Conjunction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Resolution

$$\frac{\begin{array}{c} p \vee q \\ \neg p \vee r \end{array}}{\therefore\ q \vee r}$$

Exercise

Conclusion

Prove r, assuming the following:

Premises

$$t$$

$$\neg(\neg q \vee \neg p) \wedge q$$

$$p \vee \neg q$$

$$\neg(\neg q \wedge \neg p) \wedge (t \to (\neg(p \wedge \neg r) \wedge (p \to r)))$$

	Step	Reason
1.	$\neg(\neg p \wedge \neg q) \wedge (t \to ((\neg p \wedge \neg r) \wedge (p \to r)))$	Premise.
2.	$\neg(\neg p \wedge \neg q)$	Simplification.
3.	$q \vee p$	De Morgan's
4.	$p \vee \neg q$	Premise
5.	p	Resolution from (3) & (4)
6.	t	Premise
7.	$t \to ((\neg p \wedge \neg r) \wedge (p \to r))$	Simplification from (1)
8.	$\neg(p \wedge \neg r) \wedge (p \to r)$	Modus Ponens from (6) & (7)
9.	$\neg p \vee r$	Simplification.
10	r	Disjunctive Syllogism from (9) & (5)

Exercise Answer

Rules of Inference for Quantified Statements

- Universal instantiation
- Universal generalization
- Existential instantiation
- Existential generalization

Universal Instantiation

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is an *arbitrary* member of the domain, given the premise $\forall x\, P(x)$.

$$\begin{array}{l} \quad \forall x\, P(x) \\ \hline \therefore P(c) \end{array}$$

c any arbitrary element in the domain

Universal Generalization

Universal generalization is the rule of inference that states that $\forall x\, P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain.

$$\begin{array}{l} \quad P(c) \text{ for an arbitrary } c \\ \hline \therefore \forall x\, P(x) \end{array}$$

Existential Instantiation

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x\, P(x)$ is true.

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some element } c}$$

Existential Generalization

Existential generalization is the rule of inference that is used to conclude that $\exists x\, P(x)$ is true when a *particular* element c with $P(c)$ true is known.

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x\, P(x)}$$

Exercise

Let $D(x)$ be x is in this discrete math class
Let $C(x)$ be x has taken a course in CS

Show that the premises

- Everyone in this discrete mathematics class has taken a course in computer science. $\forall x\,(D(x) \rightarrow C(x))$.
- Marla is a student in this class. $D(Marla)$

imply the conclusion

- Marla has taken a course in computer science. $C(Marla)$

Exercise Answer

Step		Reson.
1.	$\forall x (P(x) \to C(x))$	Premise
2.	$D(Marla) \to C(marla)$	Universal instantiation.
3.	$D(Marla)$	Premise.
4.	$C(Marla)$	Modus Ponens. from (2) & (3)

Exercise

$C(x)$: x is in this class

$B(x)$: x has read the book

$P(x)$: x has passed 1st exam.

Show that the premises

- A student in this class has not read the book. $\exists x\,(C(x) \wedge \neg B(x))$
- Everyone in this class passed the first exam. $\forall x\,(C(x) \to P(x))$

imply the conclusion

- Someone who passed the first exam has not read the book.

$\exists x\,(P(x) \wedge \neg B(x))$

Exercise Answer

Step	Reasons
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$ for some a	Existenstial instan...
3. $C(a)$	Simplification
4. $\forall x(C(x) \rightarrow P(x))$	Premise.
5. $C(a) \rightarrow P(a)$ for a.	Universal instan...
6. $P(a)$	modus ponens from (3) & (5)
7. $\neg B(a)$	Simplification (2)
8. $P(a) \land \neg B(a)$	Conjunction from (6) & (7)
9. $\exists x(P(x) \land \neg B(x))$	Existantial general...

A Taste of coursework

(2 marks) Prove $\neg q$, assuming the following:

$$\neg p \vee q \to r$$
$$s \vee \neg q$$
$$\neg t$$
$$p \to t$$
$$\neg p \wedge r \to \neg s$$

Solution

1. $\neg t$ Premise.
2. $p \to t$ premise
3. $\neg p$ Modus tollens
4. $\neg p \vee q$ Addition.
5. $\neg p \vee q \to r$ Premise.
6. r. Modus ponens
7. $\neg p \wedge r$ Conjunction from 3 & 6
8. $\neg p \wedge r \to \neg s$ Premise.
9. $\neg s$. Modus ponens
10. $s \vee \neg q$ Premise.
11. $\neg q$ Disjunctive syllogism

A Taste of coursework

(4 marks) Prove $\forall x(C(x) \rightarrow \neg S(x))$, assuming the following:

$$\left\{\begin{array}{l}\forall x \exists y(C(x) \rightarrow (D(y) \wedge L(y, x))) \\ \forall x \forall y(D(x) \wedge S(y) \rightarrow \neg L(x, y))\end{array}\right.$$

Not required!!!

Please come to my office if you need the answer.

Solution

Combining Rules of Inference for Propositions and Quantified Statements: Universal Modus Ponens

As universal instantiation and modus ponens are used so often together, this combination of rules is sometimes called **universal modus ponens.**

$$\begin{array}{l} \forall x\,(P(x) \rightarrow Q(x)) \\ P(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore Q(a) \end{array}$$

Combining Rules of Inference for Propositions and Quantified Statements: Universal Modus Tollens

Universal modus tollens combines universal instantiation and modus tollens and can be expressed in the following way:

$$\begin{array}{rl} & \forall x\,(P(x) \rightarrow Q(x)) \\ & \neg Q(a), \text{ where } a \text{ is a particular element in the domain} \\ \hline \therefore & \neg P(a) \end{array}$$

Expected Learning Outcomes

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