COMP1046 Tutorial 6 : Eigenvalues and Eigenvectors

Anthony Bellotti

Find the eigenvalues, eigenvectors and eigenspaces for the endomorphism, $f : \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(x, y, z) = (2x + 3y - z, 4y + 2z, z - y).$$

Answer:

Eigenvalues:

Solve $f(x, y, z) = \lambda(x, y, z)$. This forms a system of three linear equations:

$$\begin{aligned} 2x + 3y - z &= \lambda x \\ 4y + 2z &= \lambda y \\ z - y &= \lambda z \end{aligned}$$

Equivalently,

$$\begin{aligned} (2-\lambda)x + 3y - z &= 0 \\ (4-\lambda)y + 2z &= 0 \\ -y + (1-\lambda)z &= 0 \end{aligned} \tag{1}$$

and as a complete matrix:

$$\mathbf{A^C} = \left(\begin{array}{ccc|c} (2-\lambda) & 3 & -1 & 0 \\ 0 & (4-\lambda) & 2 & 0 \\ 0 & -1 & (1-\lambda) & 0 \end{array} \right).$$

Since we know there will be multiple solutions, for $\mathbf{A}$ is the incomplete matrix, $\det(\mathbf{A}) = 0$. Use I Laplace Theorem, choosing the first column (since it has only one non-zero value):

$$\begin{aligned} \det(\mathbf{A}) &= (2-\lambda)((4-\lambda)(1-\lambda) - (-2)) \\ &= (2-\lambda)(\lambda^2 - 5\lambda + 6) \\ &= (2-\lambda)(\lambda-2)(\lambda-3) = 0 \end{aligned}$$

so eigenvalues are $\lambda = 2$ and 3.

Eigenvectors:

- Take $\lambda = 2$.
  Substitute into first and second linear equation of (1): $3y - z = 0$ and $2y + 2z = 0$, hence $y = z = 0$ is the only solution. There is no constraint on x, hence the general form of eigenvector is
  $$(\alpha, 0, 0)$$
  for $\alpha \in \mathbb{R}$, except $\alpha \neq 0$.
- Take $\lambda = 3$.
  Substitute into second and third linear equation of (1): $y + 2z = 0$ and $-y - 2z = 0$, hence $y = -2z$.
  Pose $z = -\alpha$, then $y = 2\alpha$.
  Then from the first linear equation of (1): $-x + 3y - z = 0 \Rightarrow x = 7\alpha$.
  Therefore, the general form of eigenvector is
  $$\alpha(7, 2, -1)$$
  for $\alpha \in \mathbb{R}$, except $\alpha \neq 0$.

Eigenspace:

- For the general form of eigenvector $(\alpha, 0, 0)$, the eigenspace is
  $$\{(\alpha, 0, 0) | \alpha \in \mathbb{R}\}.$$
- For the general form of eigenvector $\alpha(7, 2, -1)$, the eigenspace is
  $$\{\alpha(7, 2, -1) | \alpha \in \mathbb{R}\}.$$

Note that in both cases, the eigenspace contains the null vector when $\alpha = 0$.