COMP1046
Mathematics for Computer Science

Continuous Probability and Basic Statistics

Topic Outline

- Continuous Probability Distributions
- Central Limit Theorem
- Populations and Samples
- Statistical Hypothesis Tests

Recommended textbook:

Johnson R.A. and Bhattacharyya G.K.,
Statistics: Principles and Methods, 7th ed. (Wiley)

Chapters 6, 7, 8. Copies are available in the library.

Look out for * references in these notes.

Continuous Random Variable

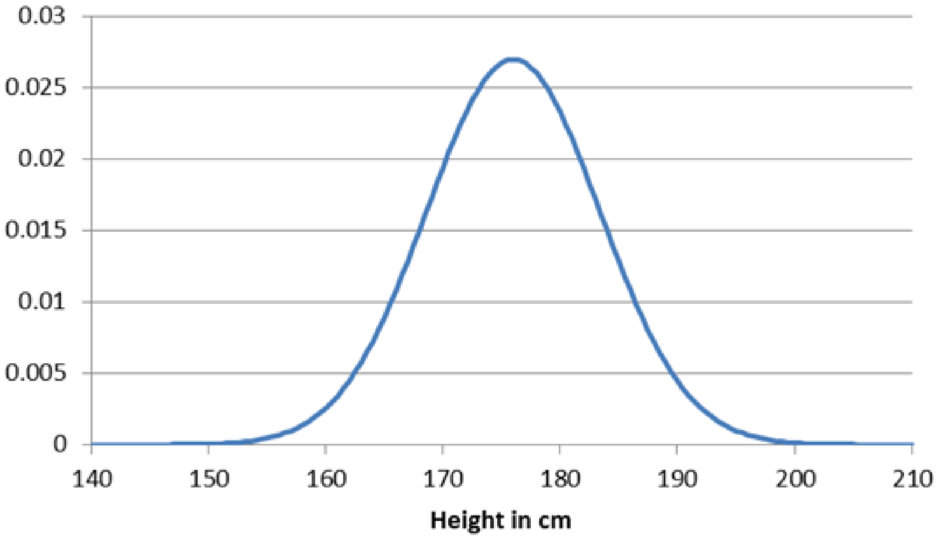
- A discrete random variable is across a finite number of possible values.
- For example, a die gives one of only 6 possible outcomes: 1, 2, 3, 4, 5, 6.

- A continuous random variable is a real number.
- Examples: Height, weight, distance.

* Textbook ref: Chapter 6, sections 1, 2 & 3.

Continuous Random Variable: Example

- Suppose X is the "Height of a person, in meters, in the COMP1046 class".
- We can draw a distribution, something like:



- What is $P(X > 190)$? How would you measure this?
- What is $P(X = 150)$? Can we measure this?

Exercise

Which of these measured quantities do you think are discrete and which continuous random variables, or neither?

1. The rainfall in Ningbo in any one week.
2. The number of slides in a lecture slide set.
3. The distance between two stars in the sky.
4. The distance between two clouds in the sky.
5. Money.
6. The speed of a car.
7. The number of ways three people could form a queue.

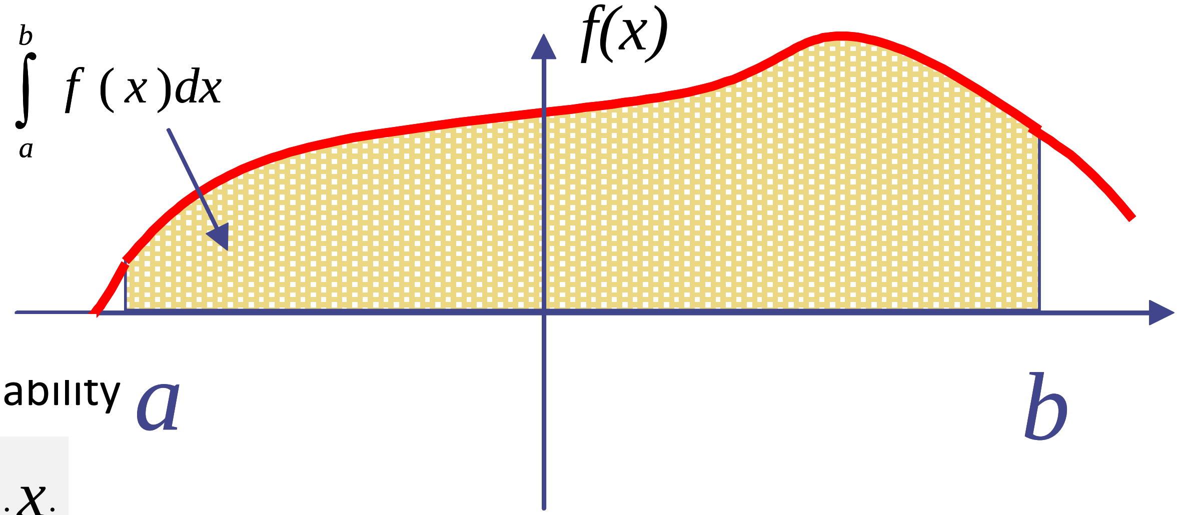
Continuous Probability Distributions

- Differences from Discrete Random Variables
  - Probability of specific value outcomes make no sense.
  - Probability of values within an interval is more helpful.
  - Cannot list all possible outcomes – instead we need to use a function.

- **Probability Density Function** (PDF)
  - Write as density function of random variable X: $f(x)$.
  - Rule: $f(x) \geq 0$ for all x.
  - Probability that X lies between values *a* and *b* is equal to the area under the curve between *a* and *b.*
  - Area under the curve sums to 1.

Continuous vs. Discrete Distributions

- Population parameters
  - Mean = μ
    - Expected Value of X or E(X)
  - Standard Deviation = σ



- Discrete
  - Multiply each value by its probability

$$\mu = E(X) = \sum_{i=1}^{n} p_i x_i$$

$$\sigma^2 = \sum_{i=1}^{n} p_i (x_i - \mu)^2$$

- Continuous
  - Requires integration to calculate μ and σ^2

$$\mu = \int_a^b t \times f(t)dt$$

$$\sigma^2 = \int_a^b (t-\mu)^2 \times f(t)dt$$

Integration across density function

- The area under the density function can be taken across whole range of the random variable, so mean and variance across the whole range of values are

$$\mu = \int_{-\infty}^{\infty} t\, f(t)\, dt\,, \qquad \sigma^2 = \int_{-\infty}^{\infty} (t-\mu)^2\, f(t)\, dt$$

- Notice that (by definition),

$$\int_{-\infty}^{\infty} f(t)\, dt = 1$$

- Relationship between density and probability:

$$F(x) = P(X < x) = \int_{-\infty}^{x} f(t)\, dt$$

The function F is called the cumulative probability distribution.

Note: You will not be required to perform any integration as part of assessment for this module.

Continuous Probability Distributions

- **Cumulative Distribution Function** (CDF)
  - $F(t) = P(X \leq t)$ or the probability that random variable X does not exceed t.
  - $F(t) = \int_{-\infty}^{t} f(x)\mathrm{d}x$
  - $0.0 \leq F(t) \leq 1.0$
  - $F(b) \geq F(a)$ if $b > a$ (increasing)

- Simple Rules
  - $P(X \leq t) = F(t)$
  - $P(X > t) = 1 – F(t)$
  - $P(c \leq X \leq d) = F(d) – F(c)$
  - $P(X = t) = 0$

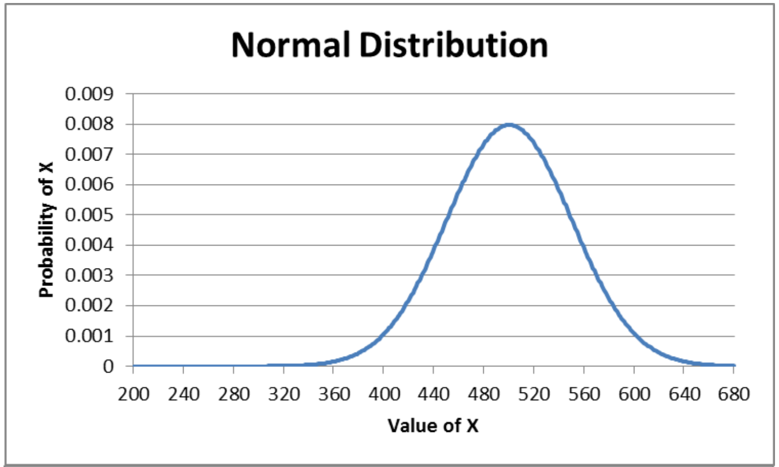
Continuous Probability Distributions

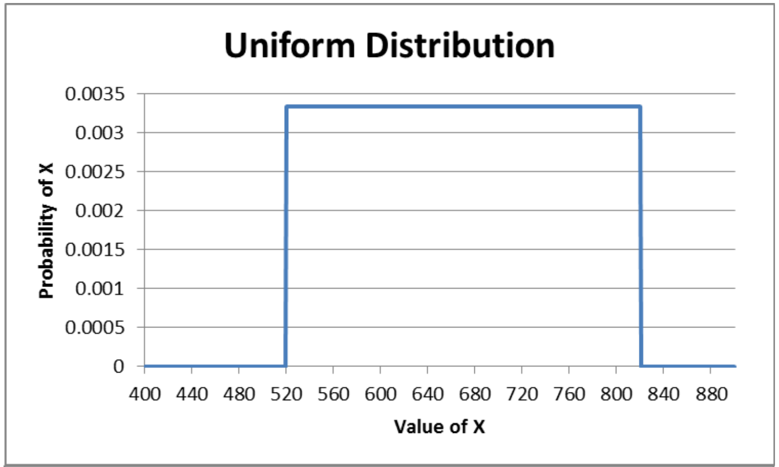
- Uniform distribution
- Normal distribution
- Many more…

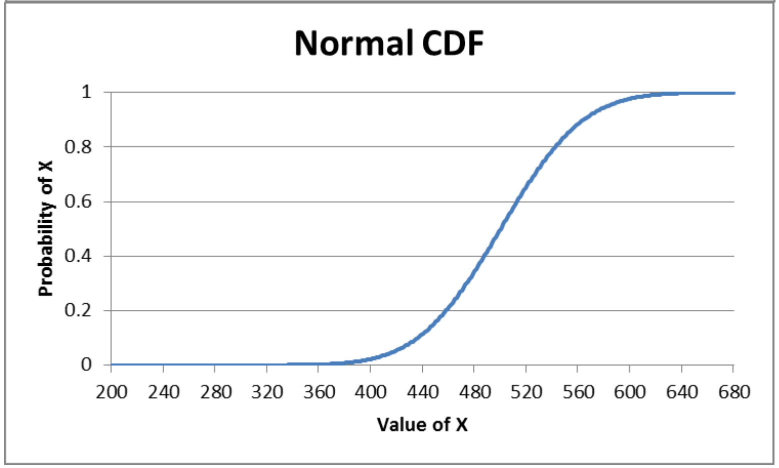
Parametric distributions:

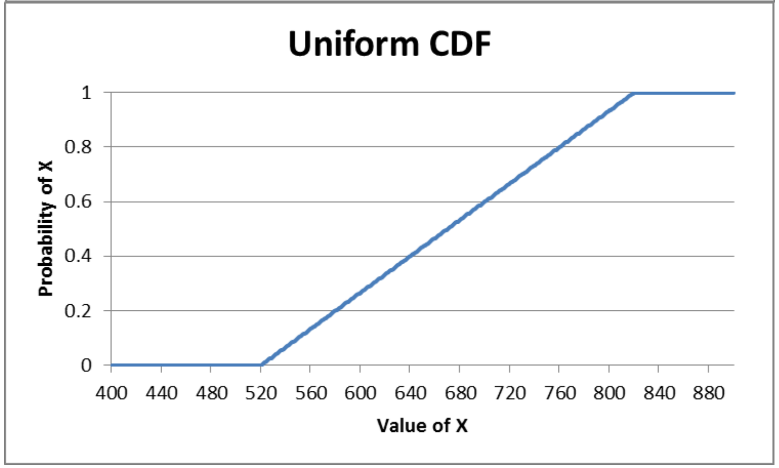
- The distribution is characterized by a fixed number of parameters.
- For example, with two parameters a, b we write $f(x \mid a, b)$ for the PDF.

Continuous Probability Functions









Uniform Distribution

- X is uniformly distributed over the range a to b, where $b > a$, or $X \sim U(a, b)$:

$$f(t \mid a,b) = \begin{cases} \dfrac{1}{b-a} & if \;\; a \le t \le b \\ 0 & otherwise \end{cases} \qquad F(t \mid a,b) = \begin{cases} 0 & if \;\; t < a \\ \dfrac{t-a}{b-a} & if \;\; a \le t \le b \\ 1 & if \;\; t > b \end{cases}$$

- Characteristics
  - Rectangular distribution with constant probability and implies the fact that each range of values that has the same length on the distributions support has equal probability of occurrence.
  - The density function integrates to unity.
  - Each of the inputs that go in to form the function have equal weighting.

Exercise 1

You want to generate a uniformly distributed random number between 1 and 10.

1. What the probability that it is between 3 and 5?

2. Supposing that $F(t \mid a, b) = 0.5$. What is the value of t in this case?

Exercise 1 Solutions

1. You want to generate a random number (Uniformly distributed) between 1 and 10. What the probability that it is between 3 and 5?

In this case a=1 and b=10, then

$$P(3 \leq X \leq 5) = F(5) - F(3) = \frac{5-1}{10-1} - \frac{3-1}{10-1} = \frac{2}{9}$$

2. Supposing that $F(t \mid a, b) = 0.5$. What is the value of t in this case?

$$F(t \mid a, b) = \frac{t-a}{b-a} = 0.5 \quad \Rightarrow \quad t = \frac{a+b}{2} = \frac{11}{2}$$

Exercise 2

为什么第一题的条件可以给其他三题用?

Consider the function for some parameter $a > 0$,

$$F(t \mid a) = \begin{cases} 0 & \text{if } t < 0 \\ \gamma t^2 & \text{if } 0 \leq t \leq a \\ 1 & \text{if } t > a \end{cases}$$

1. Show that F is a CDF when $\gamma = 1/a^2$.

2. Compute $F(a/2 \mid \ a)$.

3. Draw the shape of this CDF for $a = 5$.

4. What is $P(1 \leq X \leq 2)$? Write your answer as an expression.

Exercise 2 Solution

1. Show that F is a CDF when $\gamma = 1/a^2$.

- Firstly, prove $0 \le F(x \mid a) \le 1$ for all x.
  - This is true for case $t < 0$ and $t > a$.
  - For case $0 \le t \le a, \gamma = 1/a^2$ means $0 \le \gamma t^2 \le 1$.

- Clearly, $F(-\infty \mid a)$=0 and $F(+\infty \mid a) = 1$

- For $s < t, F(s \mid a) \le F(t \mid a)$: consider these cases:-

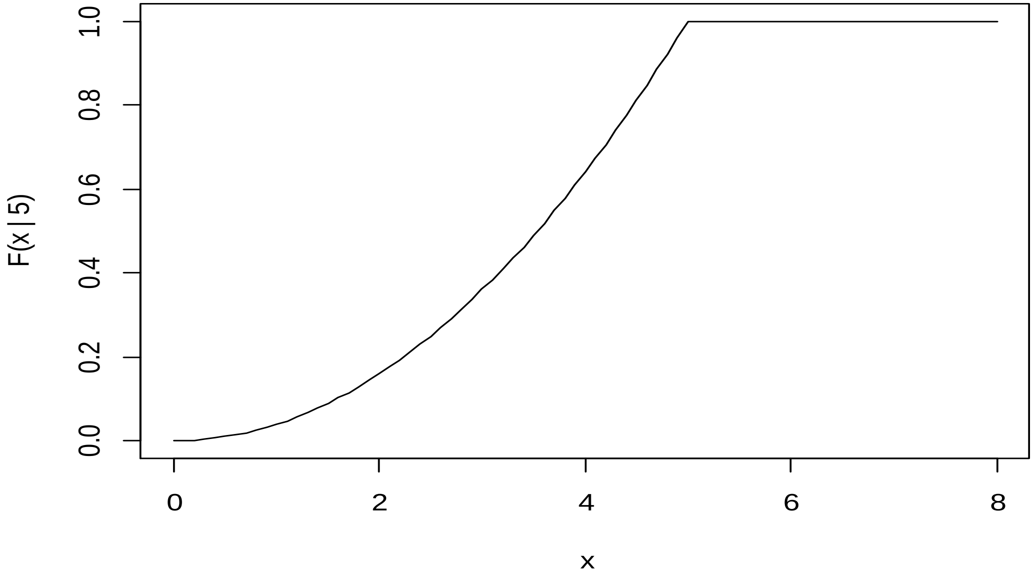
$s < t < 0$	$F(s \mid a) = F(t \mid a) = 0$
$s < 0, 0 \le t \le a$	$F(s \mid a) = 0 \le \gamma t^2$
$0 \le s < t \le a$	$\gamma s^2 < \gamma t^2$ means $F(s \mid a) < F(t \mid a)$
$t > a$	$F(t \mid a) = 1$, therefore $F(s \mid a) \le F(t \mid a)$, since $0 \le F(s \mid a) \le 1$ is already proved above.

Exercise 2 Solution

2. Compute $F(a/2 \mid a)$.

   $F(a/2 \mid a) = \gamma(a/2)^2 = 1/4$

3. Draw the shape of this CDF for $a = 5$.



Exercise 2 Solution

4. What is $P(1 \leq X \leq 2)$? Write your answer as an expression.

$$P(1 \leq X \leq 2) = F(2\ |a) - F(1\ |a)$$

and

$$F(2 \mid a) = \begin{cases} 4/a^2 & \text{if } 2 \leq a \\ 1 & \text{if } 2 > a \end{cases}$$

$$F(1 \mid a) = \begin{cases} 1/a^2 & \text{if } 1 \leq a \\ 1 & \text{if } 1 > a \end{cases}$$

So

$$P(1 \leq X \leq 2) = \begin{cases} 0 & \text{if } a < 1 \\ 1 - 1/a^2 & \text{if } 1 \leq a \leq 2 \\ 3/a^2 & \text{if } a > 2 \end{cases}$$

The Normal (Gaussian) Distribution

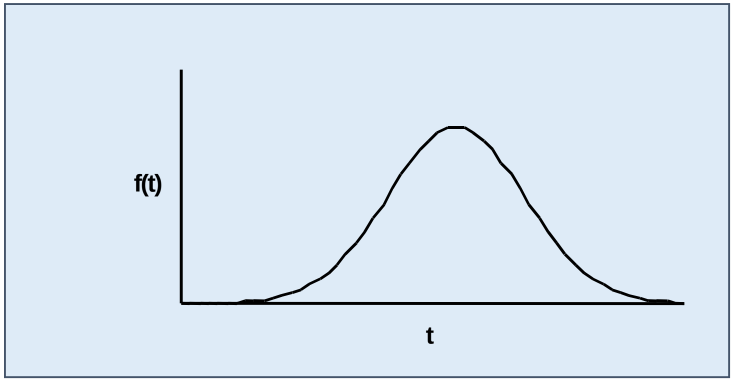
- X is normally distributed with mean **μ** and standard deviation **σ**, or X~N(μ, σ):

$$f(x \mid \mu \ ,\sigma \ ) = \frac{1}{(2\pi)^{1/2}\sigma} e^{\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma} \right)^2 \right]}$$

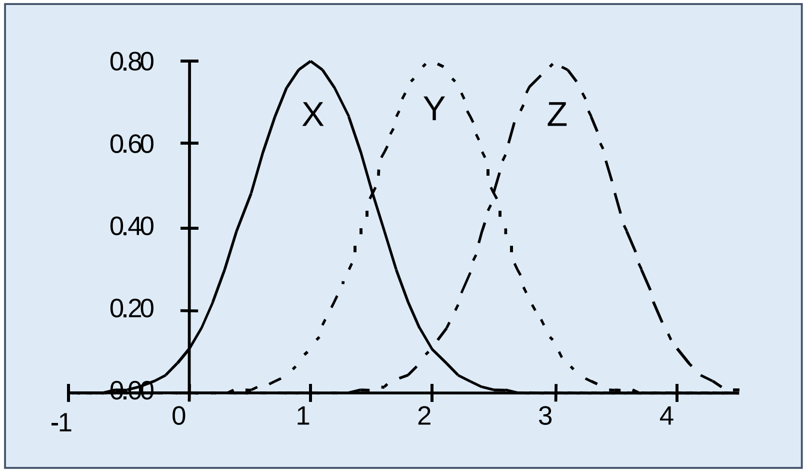
- Characteristics
  - Most commonly used distribution – many analysis assume ~ N
  - High point in “bell curve” occurs at mean.
  - Symmetric about the mean.
  - The mean “shifts” the distribution – but not the “shape”.
  - The standard deviation changes the “shape” but does not “shift” it.

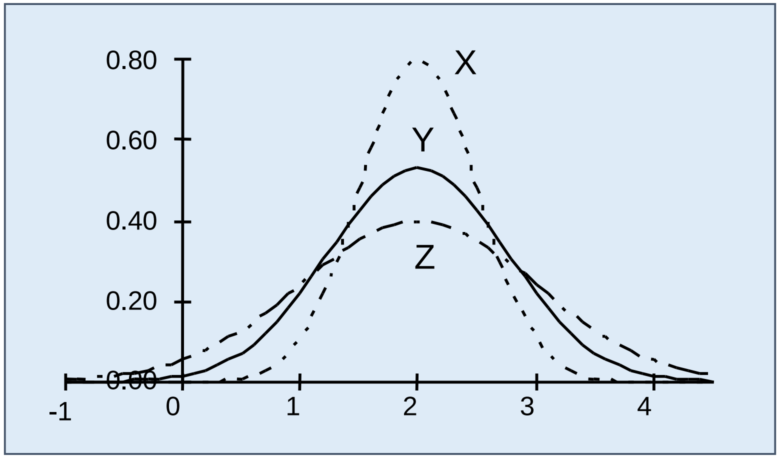
The Normal Distribution

- Density function is the familiar “bell-shaped” curve



- Completely described by mean μ and standard deviation s: N(μ,s)





Z scores and Standard Normal Distribution

- Z score given by $Z = (X - \mu)/\sigma$.

Then:

- $Z \sim N(0,1)$
- Allows for use of standard tables
- Area under the curve is 1
- Able to assess the probability of an event
- A z score can be positive or negative

Example

- Griffin Inc. ships products to an area where the distance traveled $\sim N(650, 100)$.

- You will need a calculator or standard tables to complete this:

  - What is the z-score for X = 575 miles?

  - What is the probability that distance >600?

  - What distance can I expect 50% of the shipments to be shorter than? What about 90%? 95%?

Example (Solutions)

- Griffin Inc. ships products to an area where the distance traveled $\sim N(650, 100)$.
  - What is the z-score for X = 575 miles?

$$Z = \frac{X-\mu}{\sigma} = \frac{575-650}{100} = -0.75$$

  - What is the probability that distance >600?

$$Z = \frac{600-650}{100} = -0.5$$

Look up on a scientific calculator,

$$P(Z > -0.5) = 1 - \ P(Z < -0.5) = 0.691$$

Example (Solutions)

- Griffin Inc. ships products to an area where the distance traveled $\sim N(650, 100)$.
  - What distance can I expect 50% of the shipments to be shorter than?

For standard normal,

$$P(Z < 0) = 0.5$$

So

$$\frac{X - \mu}{\sigma} = \frac{X - 650}{100} < 0$$

Hence $X < 650$

Central Limit Theorem (CLT)

- Suppose that
  - X_i,..X_n are **independently and identically distributed (iid)** with mean=μ and standard deviation=σ
  - The sum of the n random variables is $S_n=\Sigma X_i$
  - The sample mean of the n random variables is $M_n = S_n /n$

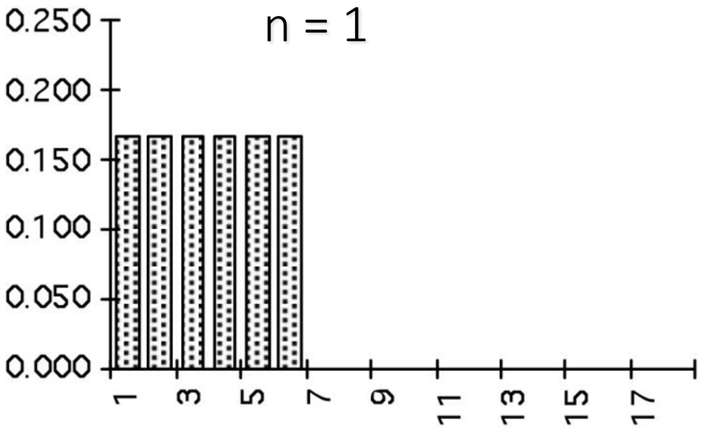
- Then, if **n** is "large",
  - S_n is Normally distributed with mean=$n\mu$ and standard deviation $\sigma\sqrt{n}$
  - M_n is Normally distributed with mean=μ and standard deviation $\boldsymbol{\sigma/\sqrt{n}}$

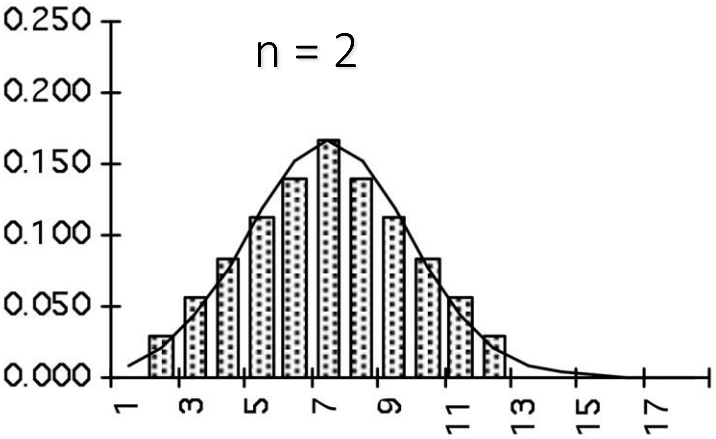
* Textbook ref: Chapter 7

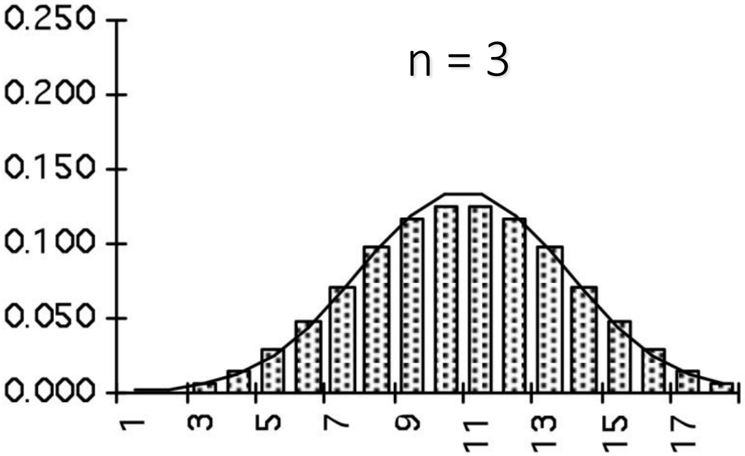
Central Limit Theorem (CLT)

- Why is CLT is so important?
    - It does not matter what distributions X follows!
    - The distribution of the sum does not reflect the distribution of its terms.
    - Requires that n is greater than 30.
    - Example: Roll die n times – approximates ~Normal.
    - CLT is the basis of much inference in statistics.

Example: Cast a die n times







Population versus Sample

- The Population is the totality of all events we are interested in.
- A Sample is some finite collection of events from the Population.
- Examples:

Population	Sample
All possible rolls of a die	10 specific dice rolls.
All people in China	100 people from Ningbo
All people in China	1000 people randomly selected from across China
All people in this lecture theatre	All people in this lecture theatre.

- Some Samples are the same as the Population, like the last example.

Descriptive Statistics versus Inference

- Descriptive Statistics is about the Sample (e.g. mean, variance).
- If the Sample is the Population, the Descriptive Statistics is about the Population.

- Inference is when we want to say something about the Population, when we only have the Sample.

- For example, suppose that we have heights of 100 Brazilian women. What can we say about all Brazilian women?

CLT for Inference

Example: 36 Brazilian women have been measured and the sample mean height of 160cm with sample standard deviation of 18cm.

So, distribution of sample mean is N(mean, $\sigma/\sqrt{n}$) = N(160, 18/ $\sqrt{36}$) = N(160, 3)

Then, **P(155 < mean height < 165)** = F(165 | 160,3) – F(155 | 160,3)

= 0.95 – 0.05 **= 0.9** ….Need a calculator here

What does this result mean?

* Textbook ref: Chapter 8

Statistical Hypothesis Test

- In a Statistical Hypothesis Test, we test a Hypothesis about a Population based on a Sample.
- Suppose that 10 years ago we know that Brazilian women had mean height of 158cm.
- Hypothesis: Brazilian women now are on average taller than they were 10 years ago.
- How can we test this hypothesis, based on our sample?

Statistical Hypothesis Test

Statistical Test Procedure:

1. Set up a Null Hypothesis (e.g. Brazilian women now are on average the same height as, or less than, 10 years ago).
2. Set a predefined significance level (e.g. 0.05).
3. Work out the probability the sample we got is from the distribution under the Null Hypothesis. This is called the ***p-value***.
4. If the p-value is small, i.e. p-value < significance level, then the Null Hypothesis is implausible, so accept the original (Alternative) Hypothesis.
5. If the p-value is large, i.e. p-value >= significance level, then insufficient evidence to support the original (Alternative) Hypothesis (but no claim to reject it either).

Example

- Alternative hypothesis: Brazilian women now are on average taller than they were 10 years ago.
- Null hypothesis: Brazilian women now are on average the same height as, or less than, 10 years ago.
- Set significance level to 0.05.
- Under the Null Hypothesis, the sampling mean is known to be 158cm and sampling standard deviation is 18/ √36 = 3 (as before).
- Work out probability of getting mean 160 or taller from this distribution, P(sample mean>=160) = 1-F(160 | 158, 3) = 0.25.
- P-value = 0.25 > 0.05 (significance level), hence we have insufficient evidence to support the hypothesis.

Statistical Hypothesis Test: Exercise

- Suppose we get further samples, and out of 100 Brazilian women, the mean height is 159.5cm with standard deviation 20cm.
- Conduct the statistical hypothesis test again with this new data.

Statistical Hypothesis Test: Exercise Answer

- Suppose we get further samples, and out of 400 Brazilian women, the mean height is 159.5cm with standard deviation 20cm.

- Conduct the statistical hypothesis test again with this new data.

- Distribution under Null is N(158, 20/ √400) = N(158, 1).

- P(sample mean>=160) = 1-F(160 | 158, 2) = 0.25.

- P-value = 0.023 < 0.05 (significance level), hence we have sufficient evidence to reject the null, and support the Alternative Hypothesis.

Think about it…

- Even though the two tests in the example and exercise above give different conclusions, why are they not contradictory?

Z Test

- The previous example and exercise was an example of a Z-test.
- In general, test if a sample with mean M_n and standard deviation σ follows a different distribution to one where the population mean is μ.
- Compute test statistic $z = (M_n - \mu)/ (\sigma/\sqrt{n})$.
- Compute p-value of z within standard normal distribution N(0,1).
  - For left-hand test ($M_n < \mu$), p-value = $F(z \mid 0,1)$
  - For right-hand test ($M_n > \mu$), p-value = $1-F(z \mid 0,1)$.
  - For two-sided ($M_n \neq \mu$), p-value = $2F(-|z| \mid 0,1)$.

Test p-value against significance level.

Conclusion

- This lecture has introduced you to the idea of continuous probability distributions, the normal distribution, samples, statistical inference and statistical hypothesis testing.

- However, this is the “tip of the iceberg” – there is much more:-

  - ➢Many more forms of continuous probability distributions;
  - ➢Many more types of statistical inference;
  - ➢Many more statistical hypothesis tests.

This is just the beginning!