The University of Nottingham

SCHOOL OF COMPUTER SCIENCE

A LEVEL 1 MODULE, AUTUMN/SPRING SEMESTER 2020-2021

Mathematics for Computer Scientists

Time allowed: 2 Hours and 0 Minutes

Candidates may complete the front cover of their answer book and sign their desk card but must NOT write anything else until the start of the examination period is announced

Answer all FOUR questions. All questions are worth 25 marks each, hence the total mark is 100.

No calculators are permitted in this examination.

Dictionaries are not allowed with one exception. Those whose first language is not English may use a standard translation dictionary to translate between that language and English provided that neither language is the subject of this examination. Subject specific translation dictionaries are not permitted.

No electronic devices capable of storing and retrieving text, including electronic dictionaries, may be used.

DO NOT turn examination paper over until instructed to do so

ADDITIONAL MATERIAL: None.
INFORMATION FOR INVIGILATORS: None.

Question 1: This question is about logic.

[**overall 25 marks**]

a. Use a truth table to prove or disprove the following statements:

(i) $\neg(p \lor (q \land r)) \equiv (\neg p) \land (\neg q \lor \neg r)$

(ii) $\neg(p \land (q \lor r)) \equiv (\neg p) \lor (\neg q \lor \neg r)$

[6 Marks]

b. Translate the following sentences from English to predicate logic. Suppose the domain that you are working over is X, the set of people. Let $S(x), A(x), T(x)$ be the statements "x has been a student of MCS", "x has got an 'A' in MCS", "x is a Teaching Assistant of MCS", respectively.

(i) There are people who have taken MCS and have got A's in MCS.

(ii) All people who are MSC Teaching Assistants and have taken MCS have got A's in MCS.

(iii) There are no people who are MCS Teaching Assistants who did not get A's in MCS.

[9 Marks]

c. The binary logical connectives $\land$ (and), $\lor$ (or), and $\rightarrow$ (implies) appear often in not only computer programs, but also everyday speech. In computer chip designs, however, it is considerably easier to construct these out of another operation, nand. This is the truth table for nand:

P	Q	P nand Q
T	T	F
T	F	T
F	T	T
F	F	T

(i) For each of the following expressions, find an equivalent expression using only nand and $\neg$ (not), as well as parentheses. You may use A, B and the operators any number of times.

- $A \land B$
- $A \lor B$

(ii) It is actually possible to express each of the above using only nand, without using $\neg$. Find an equivalent expression for the above expression using only nand and parentheses.
(Hint: First find an equivalent expression for $\neg A$ using nand)

(iii) Construct an expression using an arbitrary statement A and nand that evaluates to true regardless of whether A is true.

[10 Marks]

Question 2: This question is about sets, functions, relations.

[**overall 25 marks**]

a. Let $f(x)$ be a function from $\mathbb{R}$ to $\mathbb{R}$. Is the function $f(x) = \frac{x}{x^2+1}$ invertible? If yes, what is its inverse? If not, justify your answer.

[4 Marks]

b. For all sets A and B, prove or disprove $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$, where $\mathcal{P}(X)$ is the power set of X.

[5 Marks]

c. R is a relation defined on $\mathbb{R}$ sych that $R = \{(x, y) \mid xy \geq 0\}$. Is R reflexive, symmetric, antisymmetric or transitive? Justify your answer.

[8 Marks]

d. We define the sequence of numbers

$$a_n = \begin{cases} 1, & \text{for } 0 \leq n \leq 3 \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4. \end{cases}$$

Prove that $a_n \equiv 1$(mod 3) for all $n \geq 0$

[8 Marks]

Question 3: This question is about systems of linear equations.

[**overall 25 marks**]

Consider the following system of linear equations:

$$2x_1 + 3x_2 = 4$$
$$4x_1 + 6x_2 = 8$$

a. Is the incomplete matrix for this system of linear equations singular? Explain your answer.

[4 Marks]

b. Write this system of linear equations as a complete matrix $\mathbf{A}^c$.

[2 Marks]

c. What is the rank of $\mathbf{A}^c$?

[3 Marks]

d. Is this system of linear equations compatible?
State any theorems you use to answer this question.

[3 Marks]

e. If a system of n linear equations in n variables does not have a unique solution, will the determinant of the incomplete matrix be zero or non-zero?
State any theorems you use to answer this question.

[2 Marks]

f. Describe the algorithm for Gaussian Elimination.

[4 Marks]

g. Use Gaussian Elimination to solve the following system of linear equations:

$$x_1 + x_2 + x_3 = 2$$
$$2x_1 + x_2 - x_3 = 5$$
$$3x_1 - 2x_2 + x_3 = -2$$

[7 Marks]

Question 4: This question is about linear mappings.

[**overall 25 marks**]

a. Consider these three linear mappings. Which are endomorphisms?

(i) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x$;
(ii) $f : \mathbb{R} \to \mathbb{R}^2, \quad f(x) = (2x,\ -3x)$;
(iii) $f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = (x + y,\ y + z,\ z + x)$.

[3 Marks]

b. Consider these two linear mappings. Which are null mappings?

(i) $f : \mathbb{R}^3 \to \mathbb{R}^3, \quad f(x, y, z) = (z, 0, 0)$;
(ii) $f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (0, 0)$;

[2 Marks]

Consider the linear mapping

$$f_1 : \mathbb{R}^3 \to \mathbb{R}^2, \quad f_1(x, y, z) = (3x - 2y + z,\ x - y)$$

c. Show that ker$(f_1) = \left\{(x, y, z) \in \mathbb{R}^3 :\ x = y = -z\right\}$.

[3 Marks]

d. Assuming $(\mathbb{R}^3, +, .)$ is a vector space, show that (ker$(f_1), +, .)$ is also a vector space.

[3 Marks]

e. Compute dim(ker(f_1)).

[2 Marks]

f. Compute Im(f_1).

[2 Marks]

g. Compute dim(Im(f_1)).

[3 Marks]

h. Are answers to question parts 4(e) and 4(g) consistent with the Rank-Nullity Theorem? Explain why.

[2 Marks]

i. Let $f : E \to F$ be a linear mapping where $(E, +, \cdot)$ and $(F, +, \cdot)$ are vector spaces defined on the same scalar field $\mathbb{K}$. Let $(E, +, \cdot)$ be a finite-dimensional vector space whose dimension is dim $(E) = n$.
Prove the Rank-Nullity Theorem for just the special case when dim(ker$(f)) = 0$.

[5 Marks]