AE1MCS: Mathematics for Computer Scientists

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Aim and Learning Objectives

- To be able to understand different methods of proving theorems.
- To be able to apply different methods to construct proofs.

Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 1, Section 1.8. Introduction to Proofs
- Chapter 5, Section 5.1. Mathematical Induction
- Chapter 5, Section 5.2. Strong Induction

Methods of Proving Theorems

- Direct Proof
- Proof by Contraposition
- Proof by Contradiction
- Proof by Induction
- ...

Direct Proof and Indirect Proof

If a proof leads from the premises of a theorem to the conclusion, then it is a direct proof, otherwise, it is an indirect proof.

- Important true propositions are called *theorems*.
- A *lemma* is a preliminary proposition useful for proving later propositions.
- A *corollary* is a proposition that follows in just a few logical steps from a theorem.

Direct Proof

A direct proof shows that a conditional statement $p \to q$ is true by showing that

$$\text{if } p \text{ is true, then } q \text{ must also be true,}$$

so that the combination p true and q false never occurs.

Direct Proof

In a direct proof,

1. we assume that p is true,
2. then, we use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true.

Exercise

Give a direct proof of the theorem 'If n is an odd integer, then n^2 is odd'.

Even and Odd

Definition (Even and Odd)

The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k such that $n = 2k + 1$.

Exercise Answer

Give a direct proof of the theorem ‘If n is an odd integer, then n^2 is odd’.

Proof.

Suppose n is an odd integer. Then there exists an integer k such that $n = 2k + 1$. $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer, n^2 is odd. $\square$

Proof by Contraposition

- An extremely useful type of indirect proof is known as proof by contraposition.
- Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.
- The conditional statement $p \rightarrow q$ is proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true.

Exercise

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Exercise Answer

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Proof.

To prove that if n is an integer and $3n + 2$ is odd, then n is odd, we show if n is an even integer, then $3n + 2$ is even.

Suppose n is an even integer. Then there exists an integer k such that $n = 2k$. $3n + 2 = 3 \times 2k + 2 = 2 \times (3k + 1)$. Since $3k + 1$ is an integer, $3n + 2$ is even.

By contraposition, we showed that if n is an integer and $3n + 2$ is odd, then n is odd. $\square$

Proof by Contradiction

- Suppose we want to prove that a statement p is true.
- Furthermore, suppose that we can find a contradiction q such that $\neg p \rightarrow q$ is true.
- Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true.

Proof by Contradiction

- Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \rightarrow (r \wedge \neg r)$ is true for some proposition r.
- Proofs of this type are called **proofs by contradiction.**
- A proof by contradiction is another type of indirect proof.

Exercise

Prove that $\sqrt{2}$ is irrational.

Exercise Answer

Prove that $\sqrt{2}$ is irrational.

Proof.

Suppose $\sqrt{2}$ is rational. Then there exist integers p and q with $q \neq 0$ such that $\sqrt{2} = p/q$ and p and q do not have any common factor. Thus, $2 = p^2/q^2$. $p^2 = 2q^2$. Thus, p^2 is even. Since if n is odd, then n^2 is odd (proved in previous slides), p is even. Hence there exists an integer k such that $p = 2k$. Then $p^2 = (2k)^2 = 2q^2$. $q^2 = 2k^2$. Thus q^2 is even, hence q is even. Thus, p and q are both even, which contradicts the fact that p and q do not have any common factor. $\square$

Proof of Equivalence

To prove a theorem that is a biconditional statement or a bi-implication, that is, a statement of the form $p \leftrightarrow q$, we show that $p \rightarrow q$ and $q \rightarrow p$ are both true.

Proof of Equivalence

Sometimes a theorem states that several propositions are equivalent. Such a theorem states that propositions $p_1, p_2, p_3, ..., p_n$ are equivalent. This can be written as

$$p_1 \leftrightarrow p_2 \leftrightarrow ... \leftrightarrow p_n,$$

which states that all n propositions have the same truth values, and consequently, that for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$, p_i and p_j are equivalent. One way to prove these mutually equivalent is to use the tautology

$$[p_1 \leftrightarrow p_2 \leftrightarrow ... \leftrightarrow p_n] \leftrightarrow [(p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_3) \wedge ... \wedge (p_n \rightarrow p_1)].$$

This shows that if the n conditional statements $p_1 \rightarrow p_2$, $p_2 \rightarrow p_3$,..., $p_n \rightarrow p_1$ can be shown to be true, then the propositions p_1, p_2,..., p_n are all equivalent.

Counterexamples

To show that a statement of the form $\forall x\, P(x)$ is false, we need only find a counterexample, that is, an example x for which $P(x)$ is false.

Exercise

Show that the statement ‘Every positive integer is the sum of the squares of two integers’ is false.

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Proof.

3 is a positive integer but is not the sum of the squares of two integers. Note that the only perfect squares not exceeding 3 are $0^2 = 0$ and $1^2 = 1$. Therefore, the statement ‘Every positive integer is the sum of the squares of two integers’ is false. $\square$

Proof by Cases

- Sometimes we cannot prove a theorem using a single argument that holds for all possible cases.
- Need to consider different cases separately.
- **Rationale:** To prove a conditional statement of the form

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$$

the tautology

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$$

can be used as a rule of inference.

Proof by Cases

- The original conditional statement with a hypothesis made up of a disjunction of the propositions p_1, p_2,..., p_n can be proved by proving each of the n conditional statements $p_i \to q$, $i = 1, 2, ..., n$, individually. Such an argument is called a **proof by cases.**
- A proof by cases must cover **all possible cases** that arise in a theorem.

Exercise

Prove that if n is an integer, then $n^2 \geq n$.

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Proof.

Let us prove by cases.

- If $n = 0$, then $0^2 \geq 0$.
- If $n \geq 1$, we multiply both sides of the inequality $n \geq 1$ by the positive integer n, then we have $n^2 \geq n$.
- If $n \leq -1$, $n^2 \geq n$ holds, since $n^2 \geq 0$.

Thus, in each case, $n^2 \geq n$. □

Induction

In general, mathematical induction can be used to prove statements that assert that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function.

Proofs using mathematical induction have two parts.

1. **Basis Step:** We show that the statement holds for the positive integer 1 (i.e. $P(1)$ is true).
2. **Inductive Step:** We show that if the statement holds for a positive integer then it must also hold for the next larger integer (i.e. for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true).

Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis Step: We verify that $P(1)$ is true.

Inductive Step: We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

Inductive Step

To complete the inductive step of a proof using the principle of mathematical induction,

- we assume that $P(k)$ is true for an *arbitrary* positive integer k
- we show that under this assumption, $P(k+1)$ must also be true.

The assumption that $P(k)$ is true is called the **inductive hypothesis.**

Mathematical Induction

Mathematical induction can be expressed as the following rule of inference

$$
\begin{array}{rl}
 & P(1) \\
 & \forall k\,(P(k) \rightarrow P(k+1)) \\
\hline
\therefore & \forall n\,P(n)
\end{array}
$$

where the domain is the set of positive integers.

Detailed Explanations about Mathematical Induction

- The first thing we do to prove that $P(n)$ is true for all positive integers n is to show that $P(1)$ is true.
- This amounts to showing that the particular statement obtained when n is replaced by 1 in $P(n)$ is true.
- Then we must show that $P(k) \rightarrow P(k+1)$ is true for every positive integer k.
- To prove that this conditional statement is true for every positive integer k, we need to show that $P(k+1)$ cannot be false when $P(k)$ is true.
- This can be accomplished by assuming that $P(k)$ is true and showing that *under this hypothesis* $P(k+1)$ must also be true.

Mathematical Induction: Remark

In a proof by mathematical induction it is **not** assumed that $P(k)$ is true for all positive integers!

It is only shown that if it is assumed that $P(k)$ is true, then $P(k+1)$ is also true.

Explanations about Mathematical Induction

When we use mathematical induction to prove a theorem,

- we first show that $P(1)$ is true.
- Then we know that $P(2)$ is true, because $P(1)$ implies $P(2)$.
- Further, we know that $P(3)$ is true, because $P(2)$ implies $P(3)$.
- ...

Continuing along these lines, we see that $P(n)$ is true for every positive integer n.

Using Mathematical Induction

- Mathematical induction can be used to prove statements of the form $\forall n\, P(n)$, where the domain is the set of positive integers.
- Mathematical induction can be used to prove an extremely wide variety of theorems, each of which is a statement of this form.
  - summation formulae, inequalities, identities for combinations of sets, divisibility results, theorems about algorithms, and some other creative results
  - the correctness of computer programs and algorithms

Proving Summation Formulae: Examples

- Show that if n is a positive integer, then $1 + 2 + ... + n = \frac{n(n+1)}{2}$.
- Use mathematical induction to show that $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.
- Use mathematical induction to prove the inequality $n < 2^n$ for all positive integers n.

Strong Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

Basis Step We verify that the proposition $P(1)$ is true.

Inductive Step We show that the conditional statement $[P(1) \wedge P(2) \wedge ... \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Strong Induction

- The difference between mathematical induction and strong induction is the inductive step.
  - Mathematical induction: $\forall k\,(P(k) \to P(k+1))$.
  - Strong induction: $\forall k\,((P(1) \wedge P(2) \wedge ... \wedge P(k)) \to P(k+1))$.
- Mathematical induction and strong induction are actually equivalent.
- This is, each can be shown to be a valid proof technique assuming that the other is valid.

Expected Learning Outcomes

- To be able to understand different methods of proving theorems.
- To be able to apply different methods to construct proofs.

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