Lecture 9 - Linear Mappings Part 2

COMP1046 - Maths for Computer Scientists

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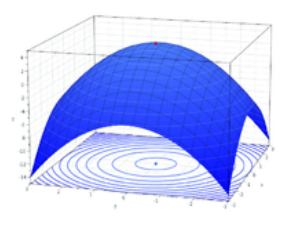

Learning outcomes

By the end of this lecture we will have learned:

- Endomorphisms and Kernel
- Injectivity
- Rank and Nullity of Linear Mappings

Based on Sections 10.3 and 10.4 of textbook (Neri 2018).

Endomorphism and Kernel

Endomorphism

Definition

Let f be a linear mapping $E \longrightarrow F$. If $E = F$, i.e. $f : E \longrightarrow E$, the linear mapping is said *endomorphism*.

Example

The linear mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 2x$ is an endomorphism since both the sets are $\mathbb{R}$.

Null mapping

Definition

A *null mapping* $O : E \longrightarrow F$ is a mapping defined in the following way:

$$\forall \mathbf{v} \in E : O(\mathbf{v}) = \mathbf{o_F}.$$

It can easily be proved that a null mapping is linear.

Example

The linear mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 0$ is a null mapping.

Identity mapping

Definition

An *identity mapping* $I : E \longrightarrow F$ is a mapping defined in the following way:

$$\forall \mathbf{v} \in E : I(\mathbf{v}) = \mathbf{v}.$$

It can easily be proved that an identity mapping is linear and is an endomorphism.

Example

The linear mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x$ is an identity mapping.

Kernel

Definition

Let $f : E \longrightarrow F$ be a linear mapping. The *kernel* of f is the set

$$\ker\left(f\right) = \left\{\mathbf{v} \in E \quad | \quad f\left(\mathbf{v}\right) = \mathbf{0_F}\right\}.$$

Example

Let us consider the linear mapping $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = 5x - y$.
To find the kernel means to find the (x, y) values such that $f(x, y) = 0$, i.e. those (x, y) values that satisfy the equation

$$5x - y = 0.$$

This is an equation in two variables. For the Rouché Capelli Theorem this equation has ∞^1 solutions. These solutions are $(\alpha, 5\alpha)$ for any $\alpha \in \mathbb{R}$.
Therefore the kernel is

$$\ker(f) = \{(\alpha, 5\alpha) \mid \quad \alpha \in \mathbb{R}\}.$$

Kernel

Example

Consider now the linear mapping $f: \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f(x, y, z) = (x + y + z, x - y - z, 2x + 2y + 2z).$$

To find the kernel means to solve the following system of linear equations:

$$\begin{cases} x + y + z = 0 \\ x - y - z = 0 \\ 2x + 2y + 2z = 0. \end{cases}$$

Continued...

Example

We can easily verify that

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 2 & 2 \end{pmatrix} = 0$$

and the rank of the system is $\rho = 2$. Thus, this system is undetermined and has ∞^1 solutions. If we pose $y = \alpha$ we find out that the infinite solutions of the system are $\alpha\,(0, 1, -1)$, $\forall \alpha \in \mathbb{R}$. Thus, the kernel of the mapping is

$$\ker\left(f\right) = \{\alpha\,(0, 1, -1) \mid \quad \alpha \in \mathbb{R}\}.$$

Kernel is a vector subspace

Theorem

Let $f : E \longrightarrow F$ be a linear mapping.
The triple $\left(\ker\left(f\right), +, \cdot\right)$ is a vector subspace of $(E, +, \cdot)$.

Proof.

Let us consider two vectors $\mathbf{v}, \mathbf{v}' \in \ker\left(f\right)$. If a vector $\mathbf{v} \in \ker\left(f\right)$ then $f\left(\mathbf{v}\right) = \mathbf{o_F}$. Thus,

$$f\left(\mathbf{v} + \mathbf{v}'\right) = f\left(\mathbf{v}\right) + f\left(\mathbf{v}'\right) = \mathbf{o_F} + \mathbf{o_F} = \mathbf{o_F}$$

and $\mathbf{v} + \mathbf{v}' \in \ker\left(f\right)$. Thus, $\ker\left(f\right)$ is closed with respect to the first composition law.

continued...

Kernel is a vector subspace

Proof.

Let us consider a generic scalar $\lambda \in \mathbb{K}$ and calculate

$$f(\lambda \mathbf{v}) = \lambda f(\mathbf{v}) = \lambda \mathbf{o_F} = \mathbf{o_F}.$$

Hence, $\lambda \mathbf{v} \in \ker(f)$ and $\ker(f)$ is closed with respect to the second composition law.

This means that $(\ker(f), +, \cdot)$ is a vector subspace of $(E, +, \cdot)$. $\square$

Kernel

Theorem

Let $f : E \longrightarrow F$ be a linear mapping and $\mathbf{u}, \mathbf{v} \in E$.
It follows that $f(\mathbf{u}) = f(\mathbf{v})$ if and only if $\mathbf{u} - \mathbf{v} \in \ker(f)$.

Proof.

If $f(\mathbf{u}) = f(\mathbf{v})$ then

$$f(\mathbf{u}) - f(\mathbf{v}) = \mathbf{o_F} \Rightarrow f(\mathbf{u}) + f(-\mathbf{v}) = \mathbf{o_F} \Rightarrow f(\mathbf{u} - \mathbf{v}) = \mathbf{o_F}.$$

From the definition of kernel $\mathbf{u} - \mathbf{v} \in \ker(f)$.

If $\mathbf{u} - \mathbf{v} \in \ker(f)$ then

$$f(\mathbf{u} - \mathbf{v}) = \mathbf{o_F} \Rightarrow f(\mathbf{u}) - f(\mathbf{v}) = \mathbf{o_F} \Rightarrow f(\mathbf{u}) = f(\mathbf{v}).$$

□

Exercise 1: Kernels

Consider $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 : f(x, y, z) = (2z, x + 2y)$.

1. Compute $\ker(f)$.
2. Show that $\ker(f)$ is a vector space by showing closure with respect to the internal and external composition laws.

Exercise 1: *Solution*

1. The kernel is the set of all (x, y, z) such that
   $f(x, y, z) = (0, 0)$;
   i.e. $2z = 0$ and $x + 2y = 0$.
   This can easily be solved with solutions in the form
   $\alpha(-2, 1, 0)$.
   Hence, $\ker(f) = \{\alpha(-2, 1, 0) : \alpha \in \mathbb{R}\}$.
2. Consider any $\alpha_1(-2, 1, 0) \in \ker(f)$ and $\alpha_2(-2, 1, 0) \in \ker(f)$.
   Then $\alpha_1(-2, 1, 0) + \alpha_2(-2, 1, 0) = (\alpha_1 + \alpha_2)(-2, 1, 0)$ which must also be in $\ker(f)$.
   Consider any $\alpha \in \ker(f)$. Then
   $\lambda(\alpha(-2, 1, 0)) = (\lambda\alpha)(-2, 1, 0)$ which must also be in $\ker(f)$.
   Hence we have proved closure and since $\ker(f) \subset \mathbb{R}^3$,
   $\ker(f)$ is a vector (sub)space.

Injectivity

Linear independence and injectivity

Theorem

Let $f : E \longrightarrow F$ be a linear mapping. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be n linearly independent vectors $\in E$.

If f is injective then $f(\mathbf{v_1}), f(\mathbf{v_2}), \ldots, f(\mathbf{v_n})$ are also linearly independent vectors $\in F$.

Proof.

Let us assume, by contradiction that
$\exists \lambda_1, \lambda_2, \ldots, \lambda_n \neq 0, 0, \ldots, 0$ such that

$$\mathbf{o_F} = \lambda_1 f(\mathbf{v_1}) + \lambda_2 f(\mathbf{v_2}) + \cdots + \lambda_n f(\mathbf{v_n}).$$

continued...

Linear independence and injectivity

Proof.

From the Proposition on Slide 16 of Lecture 8 and the linearity of f we can write this expression as

$$f(\mathbf{o_E}) = f(\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \cdots + \lambda_n \mathbf{v_n}).$$

Since for hypothesis f is injective, it follows that

$$\mathbf{o_E} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \cdots + \lambda_n \mathbf{v_n}$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n \neq 0, 0, \ldots, 0$.

This is impossible because $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are linearly independent. Hence we reached a contradiction and $f(\mathbf{v_1}), f(\mathbf{v_2}), \ldots, f(\mathbf{v_n})$ must be linearly independent. $\square$

Linear independence and injectivity

Example

Let us consider the injective mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f(x, y, z) = (x + y + z, x - y - z, x + y + 2z)$$

and the following linearly independent vectors of $\mathbb{R}^3$ with transformations:

$$\begin{aligned} \mathbf{u} &= (1, 0, 0) \qquad & f(\mathbf{u}) &= (1, 1, 1) \\ \mathbf{v} &= (0, 1, 0) \qquad & f(\mathbf{v}) &= (1, -1, 1) \\ \mathbf{w} &= (0, 0, 1) \qquad & f(\mathbf{w}) &= (1, -1, 2). \end{aligned}$$

Continued...

Example

Let us check their linear dependence by finding, if they exist, the values of λ, μ, ν such that

$$\mathbf{o} = \lambda f(\mathbf{u}) + \mu f(\mathbf{v}) + \nu f(\mathbf{w}).$$

This is equivalent to solving the following homogeneous system of linear equations:

$$\begin{cases} \lambda + \mu + \nu = 0 \\ \lambda - \mu - \nu = 0 \\ \lambda + \mu + 2\nu = 0. \end{cases}$$

The system is determined; thus, its only solution is $(0, 0, 0)$. It follows that the vectors are linearly independent.

Injectivity and the Kernel

Theorem

Let $f : E \longrightarrow F$ be a linear mapping. The mapping f is injective if and only if

$$\ker\left(f\right) = \{\mathbf{o_E}\}.$$

Proof.

Let us assume that f is injective and, by contradiction, let us assume that $\exists \mathbf{v} \in \ker\left(f\right)$ with $\mathbf{v} \neq \mathbf{o_E}$.

For definition of kernel

$$\forall \mathbf{v} \in \ker\left(f\right) : f\left(\mathbf{v}\right) = \mathbf{o_F}.$$

On the other hand, $f\left(\mathbf{o_E}\right) = \mathbf{o_F}$. *...continued*

Injectivity and the Kernel

Proof.

Thus,

$$f(\mathbf{v}) = f(\mathbf{o_E}).$$

Since f is injective, for definition of injective mapping this means that $\mathbf{v} = \mathbf{o_E}$. We have reached a contradiction.

Hence, every vector $\mathbf{v}$ in the kernel is $\mathbf{o_E}$, i.e.

$$\ker(f) = \{\mathbf{o_E}\}.$$

Let us assume that $\ker(f) = \{\mathbf{o_E}\}$ and let us consider two vectors $\mathbf{u}, \mathbf{v} \in E$ such that $f(\mathbf{u}) = f(\mathbf{v})$. It follows that

$$f(\mathbf{u}) = f(\mathbf{v}) \Rightarrow f(\mathbf{u}) - f(\mathbf{v}) = \mathbf{o_F}.$$

...continued

Injectivity and the Kernel

Proof.

It follows from the linearity of f that $f(\mathbf{u}-\mathbf{v}) = \mathbf{o_F}$.

For the definition of kernel, $\mathbf{u}-\mathbf{v} \in \ker(f)$.

However, since for hypothesis, $\ker(f) = \{\mathbf{o_E}\}$, $\mathbf{u}-\mathbf{v} = \mathbf{o_E}$. Hence, $\mathbf{u} = \mathbf{v}$.

Since, $\forall \mathbf{u}, \mathbf{v} \in E$ such that $f(\mathbf{u}) = f(\mathbf{v})$ it follows that $\mathbf{u} = \mathbf{v}$ then f is injective. $\square$

The curious null vector space

- Notice that $\{\mathbf{o_E}\}$ is a vector subspace, since $\mathbf{o_E} + \mathbf{o_E} = \mathbf{o_E}$ and $\lambda \mathbf{o_E} = \mathbf{o_E}$ shows closure.
- The basis of $\{\mathbf{o_E}\}$ is empty since the only vector in $\{\mathbf{o_E}\}$ in $\mathbf{o_E}$ and this is linearly dependent (by definition).
- It immediately follows that $\dim(\{\mathbf{o_E}\}) = 0$.

Rank and Nullity of Linear Mappings

Rank and Nullity

Definition

Let $f : E \rightarrow F$ be a linear mapping and $\text{Im}\left(f\right)$ its image.
The dimension of the image, $\dim\left(\text{Im}\left(f\right)\right)$ is said *rank* of a mapping.

Definition

Let $f : E \rightarrow F$ be a linear mapping and $\ker\left(f\right)$ its kernel.
The dimension of the kernel, $\dim\left(\ker\left(f\right)\right)$ is said *nullity* of a mapping.

Rank-Nullity Theorem

Theorem

Let $f : E \longrightarrow F$ be a linear mapping where $(E, +, \cdot)$ and $(F, +, \cdot)$ are vector spaces defined on the same scalar field $\mathbb{K}$. Let $(E, +, \cdot)$ be a finite-dimensional vector space whose dimension is $\dim(E) = n$.

Under these hypotheses the sum of rank and nullity of a mapping is equal to the dimension of the vector space $(E, +, \cdot)$:

$$\dim\left(\ker\left(f\right)\right) + \dim\left(Im(f)\right) = \dim\left(E\right).$$

Usually, $\dim(\mathrm{Im}(f))$ is the hardest to calculate directly. This theorem allows an easy way to compute it as

$$\dim(E) - \dim(\ker(f)).$$

Proof.

The proof is long (11 slides!) and structured into three parts:

- Well-posedness of the equality
- Special (degenerate) cases
- General case

Well-posedness of the equality.

At first, let us prove that the equality considers only finite numbers. In order to prove this fact, since

$$\dim(E) = n$$

is a finite number we have to prove that also $\dim(\ker(f))$ and $\dim(\mathrm{Im}(f))$ are finite numbers. □

Rank-Nullity Theorem

Proof.

Since, by definition of kernel, the $\ker(f)$ is a subset of E, then

$$\dim(\ker(f)) \leq \dim(E) = n.$$

Hence, $\dim(\ker(f))$ is a finite number.

Since $(E, +, \cdot)$ is finite-dimensional,

$$\exists \text{ a basis } B = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$$

such that every vector $\mathbf{v} \in E$ can be expressed as

$$\mathbf{v} = \lambda_1 \mathbf{e_1} + \lambda_2 \mathbf{e_2} + \ldots + \lambda_n \mathbf{e_n}.$$

□

Proof.

Let us apply the linear transformation f to both the terms in the equation

$$f(\mathbf{v}) = f(\lambda_1\mathbf{e_1} + \lambda_2\mathbf{e_2} + \ldots + \lambda_n\mathbf{e_n}) = \\ = \lambda_1 f(\mathbf{e_1}) + \lambda_2 f(\mathbf{e_2}) + \ldots + \lambda_n f(\mathbf{e_n}).$$

Thus, remembering L denotes linear span,

$$\mathrm{Im}\left(f\right) = L\left(f(\mathbf{e_1}), f(\mathbf{e_2}), \ldots, f(\mathbf{e_n})\right).$$

It follows (from Steinitz's Lemma) that

$$\dim\left(\mathrm{Im}\left(f\right)\right) \le n.$$

Hence, the equality contains only finite numbers.

□

Proof.

Special cases.

Let us consider now two special cases:

1. $\dim\left(\ker\left(f\right)\right) = 0$
2. $\dim\left(\ker\left(f\right)\right) = n$

If $\dim\left(\ker\left(f\right)\right) = 0$, i.e. $\ker\left(f\right) = \{\mathbf{o_E}\}$, then f injective. Hence, if a basis of $(E, +, \cdot)$ is $B = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\}$, also the vectors

$$f\left(\mathbf{e_1}\right), f\left(\mathbf{e_2}\right), \ldots, f\left(\mathbf{e_n}\right) \in \mathrm{Im}\left(f\right)$$

are linearly independent from Theorem on Slide 43. Since these vectors also span $\left(\mathrm{Im}\left(f\right), +, \cdot\right)$, they compose a basis.

□

Rank-Nullity Theorem

Proof.

It follows that $\dim\left(\mathrm{Im}\left(f\right)\right) = n$ and $\dim\left(\ker\left(f\right)\right) + \dim\left(\mathrm{Im}\left(f\right)\right) = \dim\left(E\right)$.

If $\dim\left(\ker\left(f\right)\right) = n$, i.e. $\ker\left(f\right) = E$. Hence,

$$\forall \mathbf{v} \in E : f\left(\mathbf{v}\right) = \mathbf{o_F}$$

and

$$\mathrm{Im}\left(f\right) = \left\{\mathbf{o_F}\right\}.$$

Thus,

$$\dim\left(\mathrm{Im}\left(f\right)\right) = 0$$

and $\dim\left(\ker\left(f\right)\right) + \dim\left(\mathrm{Im}\left(f\right)\right) = \dim\left(E\right)$.

□

Rank-Nullity Theorem

Proof.

General case.

In the remaining cases, $\dim\left(\ker\left(f\right)\right) \neq 0$ and $\neq n$.
We can write

$$\dim\left(\ker\left(f\right)\right) = r \Rightarrow \exists B_{\ker} = \{\mathbf{u_1}, \mathbf{u_2}, \ldots \mathbf{u_r}\}$$
$$\dim\left(\mathrm{Im}\left(f\right)\right) = s \Rightarrow \exists B_{\mathrm{Im}} = \{\mathbf{w_1}, \mathbf{w_2}, \ldots \mathbf{w_s}\}$$

with $0 < r < n$ and $0 < s < n$ where $B_{\ker}$ and B_{Im} are bases for $\ker\left(f\right)$ and $\mathrm{Im}\left(f\right)$ respectively.

We have $\mathbf{w_i} = f\left(\mathbf{v_i}\right)$ for some $\mathbf{v_i} \in E$.

□

Proof.

$\forall \mathbf{x} \in E$, express the linear mapping $f(\mathbf{x})$ as linear combination of the elements of B_{Im} by means of the scalars $h_1, h_2, \ldots, h_s$,

$$\begin{aligned} f(\mathbf{x}) &= h_1\mathbf{w_1} + h_2\mathbf{w_2} + \cdots + h_s\mathbf{w_s} = \\ &= h_1 f(\mathbf{v_1}) + h_2 f(\mathbf{v_2}) + \cdots + h_s f(\mathbf{v_s}) = \\ &= f(h_1\mathbf{v_1} + h_2\mathbf{v_2} + \cdots + h_s\mathbf{v_s}). \end{aligned}$$

We know that f is not injective because $r \neq 0$. On the other hand, for the Theorem on Slide 40,

$$\mathbf{u} = \mathbf{x} - h_1\mathbf{v_1} - h_2\mathbf{v_2} - \cdots - h_s\mathbf{v_s} \in \ker\left(f\right).$$

□

Rank-Nullity Theorem

Proof.

If we express $\mathbf{u}$ as a linear combination of the elements of B_{ker} by means of the scalars $l_1, l_2, \ldots, l_r$, we can rearrange the equality as

$$\mathbf{x} = h_1\mathbf{v_1} + h_2\mathbf{v_2} + \cdots + h_s\mathbf{v_s} + l_1\mathbf{u_1} + l_2\mathbf{u_2} + \cdots + l_r\mathbf{u_r}.$$

Since $\mathbf{x}$ has been arbitrarily chosen, we can conclude that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}, \mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$ span E:

$$E = L\left(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}, \mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\right).$$

□

Proof.

Let us check the linear independence of these vectors.

Consider scalars $a_1, a_2, \ldots a_s, b_1, b_2, \ldots, b_r$ and let us express the null vector as linear combination of the other vectors

$$\mathbf{o_E} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_s\mathbf{v_s} + b_1\mathbf{u_1} + b_2\mathbf{u_2} + \cdots + b_r\mathbf{u_r}.$$

Then apply the linear properties, $f(\mathbf{o_E}) = \mathbf{o_F} =$

$$\begin{aligned}
&= f(a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_s\mathbf{v_s} + b_1\mathbf{u_1} + b_2\mathbf{u_2} + \cdots + b_r\mathbf{u_r}) \\
&= a_1 f(\mathbf{v_1}) + a_2 f(\mathbf{v_2}) + \cdots + a_s f(\mathbf{v_s}) \\
&\quad + b_1 f(\mathbf{u_1}) + b_2 f(\mathbf{u_2}) + \cdots + b_r f(\mathbf{u_r}) \\
&= a_1\mathbf{w_1} + a_2\mathbf{w_2} + \cdots + a_s\mathbf{w_s} \\
&\quad + b_1 f(\mathbf{u_1}) + b_2 f(\mathbf{u_2}) + \cdots + b_r f(\mathbf{u_r}).
\end{aligned}$$

□

Rank-Nullity Theorem

Proof.

We know that since $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r} \in \ker\left(f\right)$ then

$$f\left(\mathbf{u_1}\right) = \mathbf{o_F}, \quad f\left(\mathbf{u_2}\right) = \mathbf{o_F}, \quad \ldots \quad f\left(\mathbf{u_r}\right) = \mathbf{o_F}.$$

It follows that $f\left(\mathbf{o_E}\right) = \mathbf{o_F} = a_1\mathbf{w_1} + a_2\mathbf{w_2} + \cdots + a_s\mathbf{w_s}$.

Since $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_s}$ compose a basis, they are linearly independent. It follows that $a_1, a_2, \ldots, a_s = 0, 0, \ldots, 0$ and that

$$\mathbf{o_E} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_s\mathbf{v_s} + b_1\mathbf{u_1} + b_2\mathbf{u_2} + \cdots + b_r\mathbf{u_r} =$$
$$= b_1\mathbf{u_1} + b_2\mathbf{u_2} + \cdots + b_r\mathbf{u_r}.$$

Since $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$ compose a basis, they are linearly independent. Hence, also $b_1, b_2, \ldots, b_r = 0, 0, \ldots, 0$. $\square$

Proof.

It follows that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_s}, \mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}$ are linearly independent.
Since these vectors also span E, they compose a basis.
We know, for the hypothesis, that $\dim(E) = n$ and we know that this basis is composed of $r + s$ vectors, that is $\dim(\ker(f)) + \dim(\mathrm{Im}(f))$.
Hence,

$$\dim(\ker(f)) + \dim(\mathrm{Im}(f)) = r + s = n = \dim(E).$$

□

Rank-Nullity Theorem

Example

Consider the following mapping $f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$f_2\left(x, y\right) = x + y.$$

The kernel is calculated as

$$x + y = 0 \Rightarrow \left(x, y\right) = \alpha\left(1, -1\right), \alpha \in \mathbb{R}$$

so $\ker\left(f_2\right) = \alpha\left(1, -1\right) \Rightarrow \dim\left(\ker\left(f_2\right)\right) = 1$.

Since $\dim\left(\mathbb{R}^2\right) = 2$, it follows that $\dim\left(\mathrm{Im}\left(f_2\right)\right) = 1$.
This means that the mapping f_2 transforms the points of the plane ($\mathbb{R}^2$) into the points of a line in the plane.

Rank-Nullity Theorem

Example

Let us consider the linear mapping $\mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f(x, y, z) = (x + 2y + z, 3x + 6y + 3z, 5x + 10y + 5z).$$

The kernel of this linear mapping is the set of points (x, y, z) such that

$$\begin{cases} x + 2y + z = 0 \\ 3x + 6y + 3z = 0 \\ 5x + 10y + 5z = 0. \end{cases}$$

It can be checked that the rank of this homogeneous system of linear equations is $\rho = 1$. Thus ∞^2 solutions exists.

Continued...

Rank-Nullity Theorem

Example

If we pose $x = \alpha$ and $z = \gamma$ with $\alpha, \gamma \in \mathbb{R}$ we have that the solution of the system of linear equations is

$$(x, y, z) = \left(\alpha, -\frac{\alpha + \gamma}{2}, \gamma\right),$$

that is also the kernel of the mapping:

$$\ker(f) = \left(\alpha, -\frac{\alpha + \gamma}{2}, \gamma\right).$$

It follows that $\dim(\ker(f), +, \cdot) = 2$. Since $\dim(\mathbb{R}^3, +, \cdot) = 3$, it follows from the rank-nullity theorem that $\dim(\mathrm{Im}(f)) = 1$. We can conclude that the mapping f transforms the points of the space $(\mathbb{R}^3)$ into the points of a line of the space.

Exercise 2: Rank-Nullity Theorem

Consider $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (2x - y + z, x + y + z, x - 2y)$.

Compute $\dim(\ker(f))$ and $\dim(\operatorname{Im}(f))$.

Exercise 2: *Solution*

The kernel forms the system of linear equations,

$$\begin{aligned} 2x - y + z &= 0 \\ x + y + z &= 0 \\ x - 2y &= 0 \end{aligned}$$

Since they are linearly dependent, there are ∞^1 solutions. Pose $y = \alpha$. Then, solutions are given by $\alpha(2, 1, -3)$.
This shows the basis of the kernel is 1 (i.e. only one vector is needed to span all elements): $\dim(\ker(f)) = 1$.

Since $\dim(\mathbb{R}^3) = 3$, use the Rank-Nullity Theorem to show $\dim(\mathrm{Im}(f)) = 3 - 1 = 2$.

Summary and next lecture

Summary

- Endomorphisms and Kernel
- Injectivity
- Rank and Nullity of Linear Mappings

The next lecture

We will learn about Geometric Mappings.