Continuous probability and Statistics Exercises

In the following questions you may use a calculator (note, however, calculators are not allowed in the exam, but then you will not got complex computations to perform). Please also make use of this table where required:-

t	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$P(Z \leq t)$	0.023	0.067	0.159	0.309	0.5	0.691	0.841	0.933	0.977

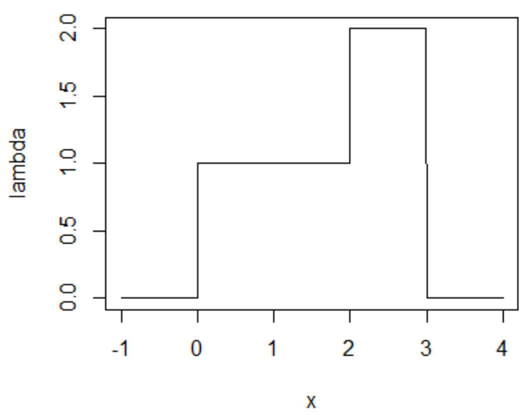
Table of probabilities for standard Z-scores

1. Consider the function over real number x,

$$f(x \mid \lambda) = \begin{cases} \lambda & \text{if} \quad 0 \leq x \leq 2 \\ 2\lambda & \text{if} \quad 2 < x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(i) Draw a graph of this function.

ANSWER:



(ii) For which value of λ is this a probability density?
Hint: work out area under the graph.

ANSWER: Area $=2 \times \lambda + \times 2\lambda = 4\lambda$. Since the total area under density f is 1, then $\lambda = 1/4$.
Also, for $\lambda = 1/4$, $f(x) \geq 0$ is true for all x as required for the density.

(iii) Compute $P(3/2 \leq X \leq 3)$.

ANSWER: Area from 3/2 to 3 $= (2 - 3/2)\lambda + (3 - 2) \times 2\lambda = 5/8$.

(iv) Compute $F(5/2)$.

ANSWER: Area from 0 to 5/2 $= 2\lambda + 1/2 \times 2\lambda = 3/4$.

2. Consider the function over real number x,

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ x(6-x)/9 & \text{if} \quad 0 \le x \le 3 \\ 1 & \text{if} \quad x > 3 \end{cases}$$

(i) Show that F is a CDF.
Do not use differential calculus and carefully consider all possible cases.

ANSWER:
First, prove $0 \le F(x) \le 1$ for all x. (*)
This is clearly true for $x < 0$ and $x > 3$.
For $0 \le x \le 3$, $F(x) = x(6-x)/9 \ge 0$ since $x \ge 0$ and $6 - x > 0$.
Also $x(6-x)/9 \le 1$ since $x \le 3$ and $6 - x \le 3$.

Secondly, need to prove that for any a, b where $a < b$, $F(a) \le F(b)$. There are the following combinations to consider:-

$a < 0$	$F(a) = 0 \le F(b)$ for any b from the first result (*)
$0 \le a < 3$, $a < b \le 3$	Write $b = a + \delta$ for some $\delta > 0$. Then $$\begin{aligned} F(b) &= b(6-b)/9 \\ &= (a+\delta)(6-a-\delta)/9 \\ &= a(6-a)/9 - a\delta/9 + \delta(6-a-\delta)/9 \\ &= F(a) + \delta(6-a-b)/9 \\ &> F(a) \quad \text{since } \delta > 0, a < 3, b \le 3. \end{aligned}$$
$b > 3$	$F(b) = 1 \ge F(a)$ for any a from the first result (*)

Thirdly, $F(-\infty) = 0$ follows from $F(x) = 0$ for $x < 0$, and $F(+\infty) = 1$ follows from $F(x) = 1$ for $x > 3$.

(ii) Compute $P(x \le 9/4)$.

ANSWER:

$$P(x \le 9/4) = F(9/4) = 9/4 \times (6 - 9/4)/9 = 15/16$$

(iii) Compute $P(-1 \le x \le 2)$.

ANSWER:

$$P(-1 \le x \le 2) = F(2) - F(-1) = 2 \times (6-2)/9 - 0 = 8/9$$

(iv) Compute $P(x > 1)$.

ANSWER:

$$P(x > 1) = 1 - F(1) = 1 - 1 \times (6 - 1)/9 = 4/9$$

3. Suppose a javelin thrower Paul is able to throw a javelin with average (mean) distance of 62 meters and each throw is governed by a normal distribution with standard deviation of 4 meters. In a competition, Paul has one throw. He will come last if he throws less than 54 meters. His best opponent throws 66 meters.

   (i) What is the probability that Paul will come last?

   ANSWER:
   Compute Z-score $= (x - \mu)/\sigma = (54 - 62)/4 = -2$.
   Now work out $P(Z < -2) = 0.023$ from the table above. So probability of winning is 2.3%.

   (ii) What is the probability that Paul will beat his best opponent and win?

   ANSWER:
   Compute Z-score $= (x - \mu)/\sigma = (66 - 62)/4 = 1$.
   Now work out $P(Z > 1) = 1 - P(Z \leq 1) = 1 - 0.841 = 0.159$ from the table above. So probability of winning is 15.9%.

4. A data scientist Clare is working on a project to predict whether a client will buy a particular product. The current machine learning model achieves an accuracy of 90% on this task. However, after 1 week of data analysis and coding, Clare is able to achieve 94%. She excitedly reports this to her manager who tells her to go back and test her model on 10 randomly selected data samples. When she does this she gets the following accuracy results on each of the ten samples:-

Accuracy (%)	94	96	93.5	91	90.5	93	88.5	93	88	90

(i) What do these trials tell us about Clare's original result of 94%: was it an over- or under-estimate?

ANSWER:

$$\text{Mean} = \frac{1}{10}(94 + 96 + 93.5 + 91 + 90.5 + 93 + 88.5 + 93 + 88 + 90) = 91.75$$

This figure is much less than 94 and closer to the original 90, so Clare's original measurement was an over-estimate.

(ii) Conduct a statistical test to show whether or not Clare's new model improves on the old one. Use a significance level of 5% and assume the results in the table follow a normal distribution.

ANSWER:
Use CLT to compute distribution of sample mean

$$N(\mu, \sigma/\sqrt{n}) = N(91.75, 2.56/\sqrt{10}) = N(91.75, 0.811)$$

The Null Hypothesis is that the sample mean of the current model (90%) comes from this distribution (i.e the accuracy of the current and new model are no different).
Now calculate the p-value, the probability that the current model fits this distribution,

$$P(X \leq 90) = F_Z((90{-}91.75)/0.811) = F_Z(-2.158) < F_Z(-2) = 0.023 < 0.05$$

where F_Z is the CDF for the standard Z-score.
This shows that probability of achieving at most 90% with Clare's model is less than significance level, and hence Clare's model is *evidently* better than the current one.

5. A scientist Dr Bird claims that the average (mean) height of an adult Norwegian blue parrot is 32 centimeters. A scientific study of 100 Norwegian blue parrots finds that the mean height is 30.5 centimeters with standard deviation 10 centimeters. At a significance level of 5% is it possible to prove Dr Bird wrong?

   ANSWER:
   Use CLT to compute distribution of sample mean

   $$N(\mu, \sigma/\sqrt{n}) = N(30.5, 10/\sqrt{100}) = N(30.5, 1)$$

   The Null Hypothesis is that a sample mean of 32cm could come from this distribution (i.e Dr. Bird's claim could be measured as a sample mean).

   Now calculate the p-value, the probability that Dr Bird's estimate fits this distribution,

   $$P(X >= 32) = 1 - P(X < 32) = 1 - F_Z((32 - 30.5)/1) = 1 - F_Z(1.5) = 0.067 > 0.05$$

   Hence there is insufficient evidence to reject the null hypothesis. Hence, Dr Bird's claim is a plausible estimate.