1 Staged Diamond Type Theory

1.1 Syntax

Phase 0 (e_0 pre-expansion, e post-expansion)	**Phase 1** (macro definitions)
$e_0 ::= x_0 \mid \lambda x_0.\, e_1 \mid e_0\ e_0 \mid \$\text{splice}\ e_1 \mid \$\text{let-macro}\ x_1 = e_1 \text{ in } e_0$	$e_1 ::= x_1 \mid \lambda x_1.\, e_1 \mid e_1\ e_1 \mid \text{dia}(e_0) \mid \text{let-dia}\ x_0 = e_0 \text{ in } e_1$
$e ::= x \mid \lambda x.\, e \mid e\ e$	
$\tau_0, \tau ::= \tau \to \tau$	$\tau_1 ::= \tau_1 \to \tau_1 \mid \Diamond \tau_0$
$v ::= \lambda x.\, e$	$v_1 ::= \lambda x_1.\, e_1 \mid \text{dia}(e)$
$\Gamma_0 ::= \cdot \mid \Gamma_0, x_0 : \tau_0$	$\Gamma_1 ::= \cdot \mid \Gamma_1, x_1 : \tau_1$
$\Gamma ::= \cdot \mid \Gamma, x : \tau$	

2 Typing Rules

$\boxed{\Gamma_1; \Gamma_0 \vdash_0 e_0 : \tau_0}$ $\boxed{\Gamma \vdash e : \tau}$ $\boxed{\Gamma_1; \Gamma_0 \vdash_1 e_1 : \tau_1}$

Lambda calculus fragment

$$\frac{x_0 : \tau_0 \in \Gamma_0}{\Gamma_1; \Gamma_0 \vdash_0 x_0 : \tau_0} \qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{x_1 : \tau_1 \in \Gamma_1}{\Gamma_1; \Gamma_0 \vdash_1 x_1 : \tau_1}$$

$$\frac{\Gamma_1; \Gamma_0, x_0 : \tau_0^{\text{in}} \vdash_0 e_0 : \tau_0^{\text{out}}}{\Gamma_1; \Gamma_0 \vdash_0 \lambda x_0.\, e_0 : \tau_0^{\text{in}} \to \tau_0^{\text{out}}} \qquad \frac{\Gamma_1; \Gamma, x : \tau^{\text{in}} \vdash e : \tau^{\text{out}}}{\Gamma_1; \Gamma \vdash \lambda x.\, e : \tau^{\text{in}} \to \tau^{\text{out}}} \qquad \frac{\Gamma_1, x_1 : \tau_1^{\text{in}}; \Gamma_0 \vdash_0 e_1 : \tau_1^{\text{out}}}{\Gamma_1; \Gamma_0 \vdash_1 \lambda x_1.\, e_1 : \tau_1^{\text{in}} \to \tau_1^{\text{out}}}$$

$$\frac{\Gamma_1; \Gamma_0 \vdash_0 e_0^{\text{fun}} : \tau_0^{\text{in}} \to \tau_0^{\text{out}} \quad \Gamma_1; \Gamma_0 \vdash_0 e_0^{\text{in}} : \tau_0^{\text{in}}}{\Gamma_1; \Gamma_0 \vdash_0 e_0^{\text{fun}}\ e_0^{\text{in}} : \tau_0^{\text{out}}} \qquad \frac{\Gamma_1; \Gamma \vdash e^{\text{fun}} : \tau^{\text{in}} \to \tau^{\text{out}} \quad \Gamma_1; \Gamma \vdash e^{\text{in}} : \tau^{\text{in}}}{\Gamma_1; \Gamma \vdash e^{\text{fun}}\ e^{\text{in}} : \tau^{\text{out}}} \qquad \frac{\Gamma_1; \Gamma_0 \vdash_1 e_1^{\text{fun}} : \tau_1^{\text{in}} \to \tau_1^{\text{out}} \quad \Gamma_1; \Gamma_0 \vdash_1 e_1^{\text{in}} : \tau_1^{\text{in}}}{\Gamma_1; \Gamma_0 \vdash_1 e_1^{\text{fun}}\ e_1^{\text{in}} : \tau_1^{\text{out}}}$$

Modal fragment

$$\frac{\Gamma_1; \Gamma_0 \vdash_1 e_1 : \tau_1 \quad \Gamma_1, x_1 : \tau_1; \Gamma_0 \vdash_0 e_0 : \tau_0}{\Gamma_1; \Gamma_0 \vdash_0 \$\text{let-macro}\ x_1 = e_1 \text{ in } e_0 : \tau_0} \qquad \frac{\Gamma_1; \Gamma_0 \vdash_1 e_1^{\text{x}} : \Diamond \tau_0 \quad \Gamma_1; \Gamma_0, x_0 : \tau_0 \vdash_1 e_1^{\text{body}} : \tau_1}{\Gamma_1; \Gamma_0 \vdash_1 \text{let-dia}\ x_0 = e_1^{\text{x}} \text{ in } e_1^{\text{body}} : \tau_1}$$

$$\frac{\Gamma_1; \Gamma_0 \vdash_1 e_1 : \Diamond \tau_0}{\Gamma_1; \Gamma_0 \vdash_0 \$\text{splice}\ e_1 : \tau_0} \qquad \frac{\Gamma_1; \Gamma_0 \vdash_0 e_0 : \tau_0}{\Gamma_1; \Gamma_0 \vdash_1 \text{dia}(e_0) : \Diamond \tau_0}$$

3 Big-Steps Operational Semantics

$$\boxed{e_0 \Downarrow_0 v} \qquad \boxed{e \Downarrow v} \qquad \boxed{e_1 \Downarrow_1 v_1}$$

$$\frac{e_0 \Downarrow_E e \qquad e \Downarrow v}{e_0 \Downarrow_0 v}$$

Lambda calculus fragment

$$\frac{}{\lambda x.\, e \Downarrow \lambda x.\, e} \qquad \frac{}{\lambda x_1.\, e_1 \Downarrow_1 \lambda x_1.\, e_1}$$

$$\frac{\begin{array}{c} e^{\text{fun}} \Downarrow \lambda x.\, e^{\text{out}} \qquad e^{\text{in}} \Downarrow v^{\text{in}} \\ e^{\text{out}}[v^{\text{in}}/x] \Downarrow v^{\text{out}} \end{array}}{e^{\text{fun}}\ e^{\text{in}} \Downarrow v^{\text{out}}} \qquad \frac{\begin{array}{c} e_1^{\text{fun}} \Downarrow_1 \lambda x_1.\, e_1^{\text{out}} \qquad e_1^{\text{in}} \Downarrow_1 v_1^{\text{in}} \\ e_1^{\text{out}}[v_1^{\text{in}}/x_1] \Downarrow_1 v_1^{\text{out}} \end{array}}{e_1^{\text{fun}}\ e_1^{\text{in}} \Downarrow_1 v_1^{\text{out}}}$$

Modal fragment

$$\frac{\begin{array}{c} e_1^{\text{x}} \Downarrow_1 \text{dia}(e) \\ e_1^{\text{body}}[e/x_0] \Downarrow_1 v_1 \end{array}}{\text{let-dia } x_0 = e_1^{\text{x}} \text{ in } e_1^{\text{body}} \Downarrow_1 v_1}$$

$$\frac{e_0 \Downarrow_{\text{E}} e}{\text{dia}(e_0) \Downarrow_1 \text{dia}(e)}$$

$$\boxed{\begin{array}{c} \textbf{Expansion} \\ e_0 \Downarrow_{\text{E}} e \end{array}}$$

Lambda calculus fragment

$$\frac{}{x_0 \Downarrow_{\text{E}} x} \qquad \frac{e_0 \Downarrow_{\text{E}} e}{\lambda x_0.\, e_0 \Downarrow_{\text{E}} \lambda x.\, e} \qquad \frac{e_0^{\text{fun}} \Downarrow_{\text{E}} e^{\text{fun}} \qquad e_0^{\text{in}} \Downarrow_{\text{E}} e^{\text{in}}}{e_0^{\text{fun}}\ e_0^{\text{in}} \Downarrow_{\text{E}} e^{\text{fun}}\ e^{\text{in}}}$$

Modal fragment

$$\frac{e_1 \Downarrow_1 v_1 \qquad e_0[v_1/x_1] \Downarrow_{\text{E}} e}{\$\text{let-macro } x_1 = e_1 \text{ in } e_0 \Downarrow_0 e} \qquad \frac{e_1 \Downarrow_1 \text{dia}(e)}{\$\text{splice } e_1 \Downarrow_{\text{E}} e}$$

4 Staged Box Type Theory

$$\boxed{\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash e : \tau}$$

Lambda calculus fragment

$$\frac{x : \tau \in \Gamma_1}{\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash x : \tau}$$

$$\frac{\Delta; \Gamma_n; \ldots; \Gamma_1, x : \tau^{\text{in}} \vdash e : \tau^{\text{out}}}{\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash \lambda x.\, e : \tau^{\text{in}} \to \tau^{\text{out}}} \qquad \frac{\begin{array}{c}\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash e^{\text{fun}} : \tau^{\text{in}} \to \tau^{\text{out}} \\ \Delta; \Gamma_n; \ldots; \Gamma_1 \vdash e^{\text{in}} : \tau^{\text{in}}\end{array}}{\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash e^{\text{fun}}\ e^{\text{in}} : \tau^{\text{out}}}$$

Modal fragment

$$\frac{x : \tau \in \Delta}{\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash x : \tau}$$

$$\frac{\Delta; \Gamma_n; \ldots; \Gamma_1; \cdot \vdash e : \tau}{\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash \text{box}(e) : \Box\tau} \qquad \frac{\begin{array}{c}\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash e^{\text{x}} : \Box\tau^{\text{x}} \\ \Delta, x : \tau^{\text{x}}; \Gamma_n; \ldots; \Gamma_1 \vdash e^{\text{body}} : \tau^{\text{body}}\end{array}}{\Delta; \Gamma_n; \ldots; \Gamma_1 \vdash \text{let-box}\ x = e^{\text{x}}\ \text{in}\ e^{\text{body}} : \tau^{\text{body}}}$$

4.1 Comparison

$$\frac{\Delta; \Gamma_1; \cdot \vdash e : \tau}{\Delta; \Gamma_1 \vdash \text{box}(e) : \Box\tau} \qquad \frac{\Gamma_1; \Gamma_0 \vdash_0 e_0 : \tau_0}{\Gamma_1; \Gamma_0 \vdash_1 \text{dia}(e_0) : \Diamond\tau_0}$$

$$\frac{\begin{array}{c}\Delta; \Gamma_1 \vdash e^{\text{x}} : \Box\tau^{\text{x}} \\ \Delta, x : \tau^{\text{x}}; \Gamma_1 \vdash e^{\text{body}} : \tau^{\text{body}}\end{array}}{\Delta; \Gamma_1 \vdash \text{let-box}\ x = e^{\text{x}}\ \text{in}\ e^{\text{body}} : \tau^{\text{body}}} \qquad \frac{\begin{array}{c}\Gamma_1; \Gamma_0 \vdash_1 e_1^{\text{x}} : \Diamond\tau_0 \\ \Gamma_1; \Gamma_0, x_0 : \tau_0 \vdash_1 e_1^{\text{body}} : \tau_1\end{array}}{\Gamma_1; \Gamma_0 \vdash_1 \text{let-dia}\ x_0 = e_1^{\text{x}}\ \text{in}\ e_1^{\text{body}} : \tau_1}$$