Probability Theory

CSGE602013- Statistics and Probability

Credits

These course slides were prepared by Alfan F. Wicaksono. The content was based on previous semester's course slides created by **all previous lecturers**.

References

- Introduction to Probability and Statistics for Engineers & Scientists, 4th ed.,
  - Sheldon M. Ross, Elsevier, 2009.
- Probability and Statistics for Engineers & Scientists, 4^{th} Edition
  - Anthony J. Hayter, Thomson Higher Education
- Probability, Statistics, and Queueing Theory with Computer Science Applications
  - Arnold O. Allen
- Probability and Statistics for Engineers & Scientists, 4th Edition
  - Ronald E. Walpole, Raymond H. Myers

Fate laughs at probabilities …

from “Eugene Aram (1832) by Edward Bulwer-Lytton, Book I., Chapter X.

Outline

- Introduction
- Sample Space and Events
- Complement, Combinations, & Algebra of Events
- Axiom of Probability
- Sample Spaces Having Equally Likely Outcomes
- Conditional Probability
- Bayes Rule & Law of Total Probability
- Independent Events

Introduction

Probability theory provides a basis for the science of statistical inference from data.

The usual process:

- **First**, a sample (of size n) is obtained from a population – usually, we assume the underlying (population) probability distribution.
- **Second**, description of sample (descriptive statistics)
- **Third**, making a decision from a sample for our problem (Inferential statistics)

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Sample Space and Events

Sample Space

- **Experiment**: any process or procedure for which more than one outcome is possible.

- **Sample space** (***S***): Set of all possible outcomes of an experiment.

Sample Space

If the experiment consists of the **tossing of a coin**, then

$$S = \{\text{Head, Tail}\}$$

If the experiment consists of the **running of a race** among the six horses having post positions 1, 2, 3, 4, 5, 6, then

$$S = \{\text{all orderings of } (1, 2, 3, 4, 5, 6)\}$$

Experiment consists of determining **the amount of dosage** that must be given to a patient until that patients reacts positively, then

$$S = \{0, 1, 2, 3, 4, \ldots\}$$

Sample Space

Games of Chance

Games of chance commonly involve the toss of a coin, the roll of a die, or the use of a pack of cards.

The roll of a die:

- A usual six-sided die has a sample space, ***S*** = {1, 2, 3, 4, 5, 6}
- If two dice are rolled, the sample space is...

Sample space for rolling two dice

S

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

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Events

- **Event (*E*)**: a subset of the sample space S.
- If the outcome of the experiment is contained in ***E***, then we say that ***E* has occured.**

Example:

E = {an **even** score is recorded on the roll of die} = {2,4,6}

E = {an event that we get Head on the toss of coin} = {H}

Event that the number 3 horse wins the race

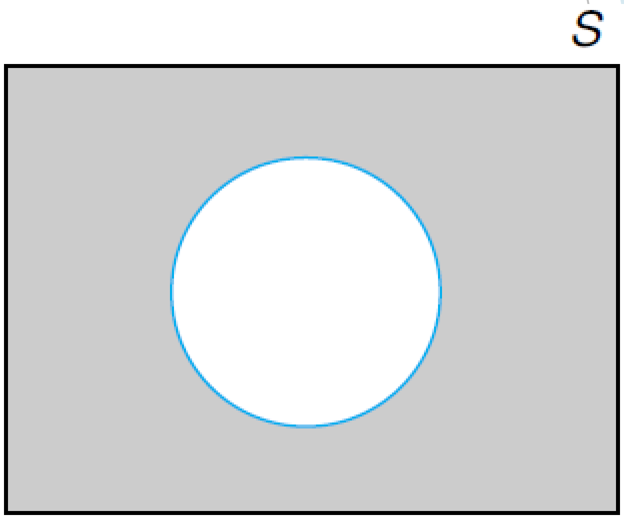
E = {all outcomes in S starting with a 3}

Complement, Combinations, & Algebra of Events

Complement of *E*

DEFINITION

For any event E, we define the event E^c, referred to as the complement of E, to consist of all outcomes in the sample space S that are not in E.



(c) Shaded region: E^c

Example:

$S = \{1, 2, 3, 4, 5, 6\}$

$E = \{1, 3, 5\}$

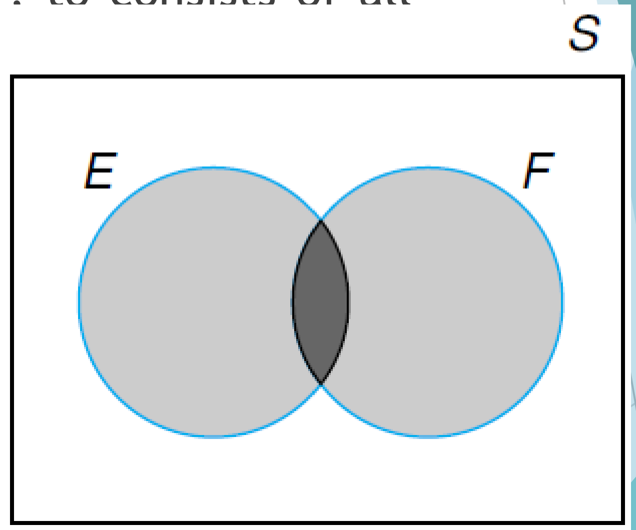
$E^c = \{2, 4, 6\}$

Question S^C = ?

Union of *E* and *F*

DEFINITION

For any event ***E*** and ***F***, we define the new event ***E*** ∪ ***F***, called the union of the events ***E*** and ***F***, to consists of all outcomes that are either in ***E*** or in ***F*** or



(a) Shaded region: $E \cup F$

Example:

E = {1, 3, 5}

F = {2, 4, 6}

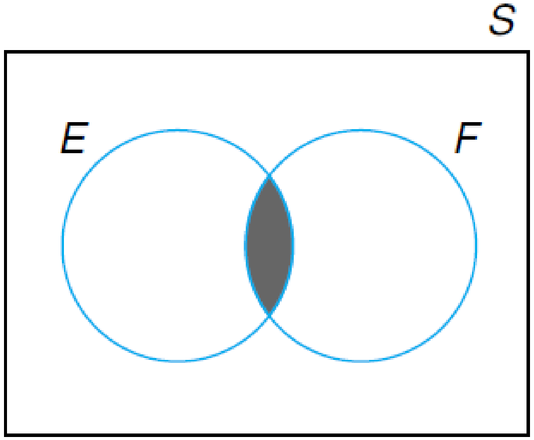
E ∪ ***F*** = {1, 2, 3, 4, 5, 6} = ***S***

$$\bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup \ldots \cup E_n$$

Intersection of *E* and *F*

DEFINITION

For any event ***E*** and ***F***, we define the new event ***EF***, called the intersection of the events ***E*** and ***F***, to consists of all outcomes that are both in ***E*** and ***F***.



(b) Shaded region: *EF*

Example: event regarding required dosage

E = (0, 5), dosage < 5

F = (2, 10), dosage is between 2 and 10

EF = (2, 5), dosage is between 2 and 5

If event ***A*** = ∅, ***A*** is a **null event.**

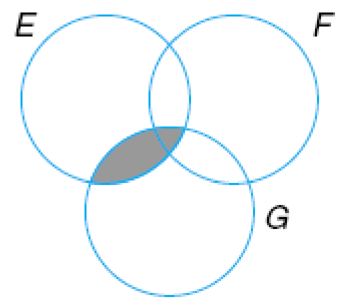
If ***EF*** = ∅, ***E*** and ***F*** are **mutually exclusive**

$E_1E_2E_3 \ldots E_n$ denotes intersection between n events

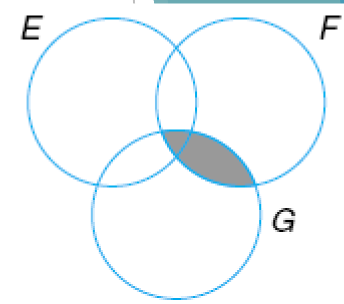
Subset & Proper Subset

- Subset (⊆) and proper subset (⊂)
  - $\{a, b\} \subseteq \{a, b, c\}$
  - $\{a, b\} \subset \{a, b, c\}$
  - $\{a, b, c\} \subseteq \{a, b, c\}$
  - $\{a, b, c\} \subset \{a, b, c\}$ Wrong!
- If $\boldsymbol{E} \subseteq \boldsymbol{F}$ and $\boldsymbol{F} \subseteq \boldsymbol{E}$, we say ***E*** and ***F*** are **equal**, or $\boldsymbol{E} = \boldsymbol{F}$

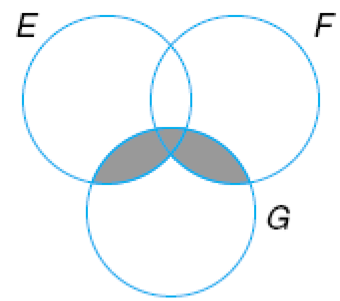
Algebra of Events



(a) Shaded region: EG



(b) Shaded region: FG



(c) Shaded region: $(E \cup F)G$

$(E \cup F)G = EG \cup FG$

- Commutative Law
  - $E \cup F = F \cup E$
  - $EF = FE$
- Associative law
  - $(E \cup F) \cup G = E \cup (F \cup G)$
  - $(EF)G = E(FG)$
- Distributive law
  - $(E \cup F)G = EG \cup FG$
  - $EF \cup G = (E \cup G)(F \cup G)$
- *DeMorgan's laws.*
  - $(E \cup F)^c = E^c F^c$
  - $(EF)^c = E^c \cup F^c$

Axioms of Probability

Probability

- Duh, nanti sore hujan nggak ya ?
- Apakah bensin mobil saya cukup hingga 4 jam kedepan ?
- Apakah saya perlu menerima pekerjaan itu ?
- Duh, hari ini KRL akan bermasalah nggak ya ?
- Berapa peluang mahasiswa FASILKOM UI menikah tepat 3 tahun setelah lulus ?

Probability

Empirical Fact

If an experiment is continually repeated under the exact same conditions, then for any event ***E***, the **proportion of time** that ***E*** occurs **approaches some constant value** as the number of repetitions increases.

This proportion is called **probability of an event *E***.

How likely the event ***E*** will occur ?

Axioms

For each event ***E*** of an experiment having a sample space ***S***, there is a number ***P*(*E*)**, where ***P*(*E*)** follows three axioms:

AXIOM 1

$$0 \le P(E) \le 1$$

AXIOM 2

$$P(S) = 1$$

AXIOM 3

For any sequence of **mutually exclusive** events $\mathbf{E_1}$, $\mathbf{E_2}$, …

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \qquad n = 1, 2, \ldots, \infty$$

We call ***P*(*E*)** the probability of the event ***E***

Axioms

An experiment with sample space $\boldsymbol{S} = \{O_1, O_2, \ldots, O_n\}$

Set of probability of outcome $\mathbf{O_i}$, denoted by $\boldsymbol{P}(\mathbf{O_i})$, satisfies

$$0 \le P(O_1) \le 1, \quad 0 \le P(O_2) \le 1, \quad \ldots, \quad 0 \le P(O_n) \le 1$$

and

$$P(O_1) + P(O_2) + \ldots + P(O_n) = 1$$

Axioms

Proposition 1

$$1 = P(S) = P(E \cup E^C) = P(E) + P(E^C)$$

Then, we obtain $P(E^C) = 1 - P(E)$

Proposition 2

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Prove it !

$$P(\varnothing) = 0$$

$$P(E^C) = 1 - P(E)$$

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

$$If\ \ A \subset B\ then\ P(A) \le P(B)$$

Example

Consider the event A that there are no more than two errors in a software product.

A = {0 errors, 1 errors, 2 errors} $\subset$ S

and

$\mathbf{P}(A) = \mathbf{P}(\text{0 errors}) + \mathbf{P}(\text{1 errors}) + \mathbf{P}(\text{2 errors})$

$= 0.05 + 0.08 + 0.35 = 0.48$

$\mathbf{P}(A^C) = 1 - \mathbf{P}(A) = 1 - 0.48 = 0.52$

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Example

Peluang seorang mahasiswa lulus matematika adalah 2/3 dan peluangnya lulus biologi adalah 4/9. Bila peluangnya lulus kedua mata kuliah adalah 1/4.

Berapakah peluangnya lulus paling sedikit satu mata kuliah ?

M : kejadian lulus matematika

B : kejadian lulus biologi

$$
\begin{aligned}
P(M \cup B) &= P(M) + P(B) - P(MB) \\
&= 2/3 + 4/9 - 1/4 \\
&= 31/36
\end{aligned}
$$

Example

A total of **28%** of American males smoke **cigarettes**, **7%** smoke **cigars**, and **5%** smoke **both cigars and cigarettes.** What percentage of males smoke **neither** cigars **nor** cigarettes ?

E: event that a randomly chosen male is a **cigarette smoker**

F: event that a randomly chosen male is a **cigar smoker**

$$P(E \cup F) = P(E) + P(F) - P(EF) = 0.28 + 0.07 - 0.05 = 0.3$$

$$P(E \cup F)^C = 1 - P(E \cup F) = 1 - 0.3 = 0.7$$

Latihan

A random experiment can result in one of the outcomes **{a, b, c, d}** with probabilities 0.1, 0.3, 0.5 and 0.1 respectively.

Let **A** denote the event **{a, b}**, **B** the event **{b, c, d}** and **C** the event **{d}**. Find :

(1) $P(A)$, $P(B)$, and $P(C)$

(2) $P(A^C)$, $P(B^C)$, and $P(C^C)$

(3) $P(A \cap B)$, $P(A \cup B)$, and $P(A \cap C)$

Sample Spaces Having Equally Likely Outcomes

Question

If a die is rolled, what is the probability that its face will equal 6 ?

1/6

Are you sure ?

Actually, we cannot directly answer that question.

But, if we **assume** that all possible outcomes are **equally likely** to occur, then **1/6** is a correct answer ☺

In many experiments, it is natural to **assume** that each point in the sample space is **equally likely** to occur.

Suppose, ***S*** = {1, 2, 3, …, N}, it is natural to **assume**

$$P(\{1\}) = P(\{2\}) = \ldots = P(\{N\}) = p \qquad (say)$$

Using axiom 2 & 3, we have

$$1 = P(S) = P(\{1\}) + P(\{2\}) + \ldots + P(\{N\}) = Np$$

and

$$P(\{i\}) = p = 1/N$$

So,

$$P(E) = \frac{\text{Number of Points in } \boldsymbol{E}}{N}$$

Fair Coin Tossing

One toss:

$\boldsymbol{S}$ = {H, T}

So, **P**({H}) = **P**({T}) = 0.5

Defect coin may result **P**({H}) ≠ **P**({T})

Toss twice:

$\boldsymbol{S}$ = {HH, HT, TH, TT}

P({HH}) = 0.25

$\boldsymbol{E}$ = {at least one head} = {HH, HT, TH}

P($\boldsymbol{E}$) = P({HH}) + P({HT}) + P({TH}) = 3 x (0.25) = 0.75

even = { an *even* score is recorded on the roll of a die }

= { 2,4,6 }

For a **fair** die, $P(even) = P(\{2\}) + P(\{4\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

A = { the sum of the scores of two dice is equal to 6 }

= { (1,5), (2,4), (3,3), (4,2), (5,1) }

$$P(A) = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36}$$

A sum of 6 will be obtained with two fair dice roughly 5 times out o 36.

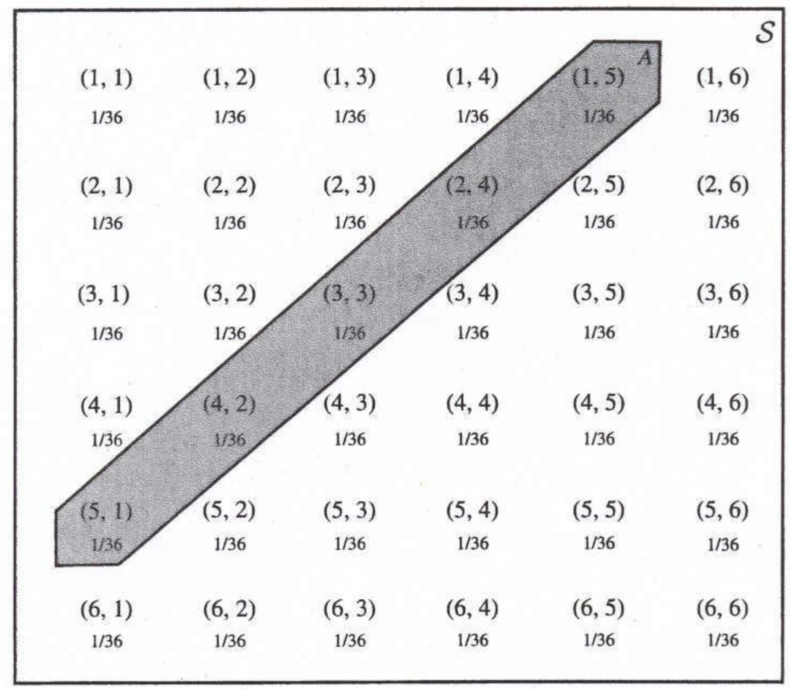

FIGURE 1.18 •
Event *A*: sum equal to 6

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Rolling a two **fair** dice

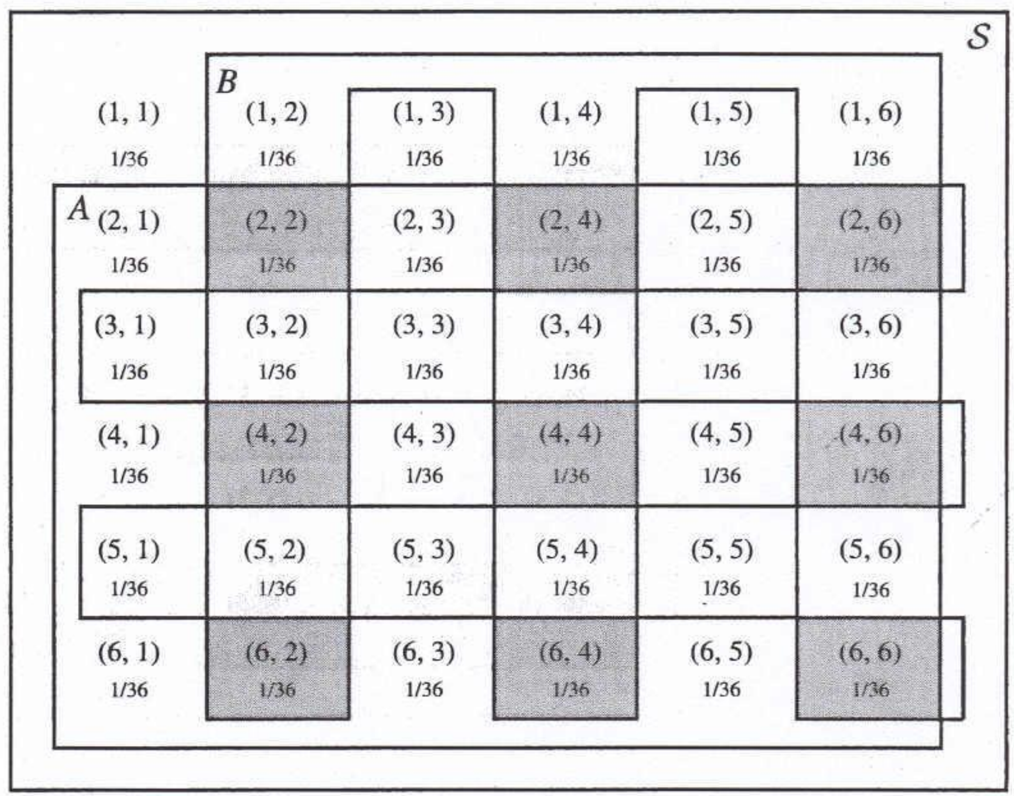
If we assume that all outcomes are considered **equally likely**,

What is the probability that **both dice have even scores** ?

A : event that even score is obtained on the First die
B : event that even score is obtained on the Second die

FIGURE 1.46 ●
Event $A \cap B$

$$P(AB) = \frac{9}{36} = \frac{1}{4}$$



Rolling a two **fair** dice

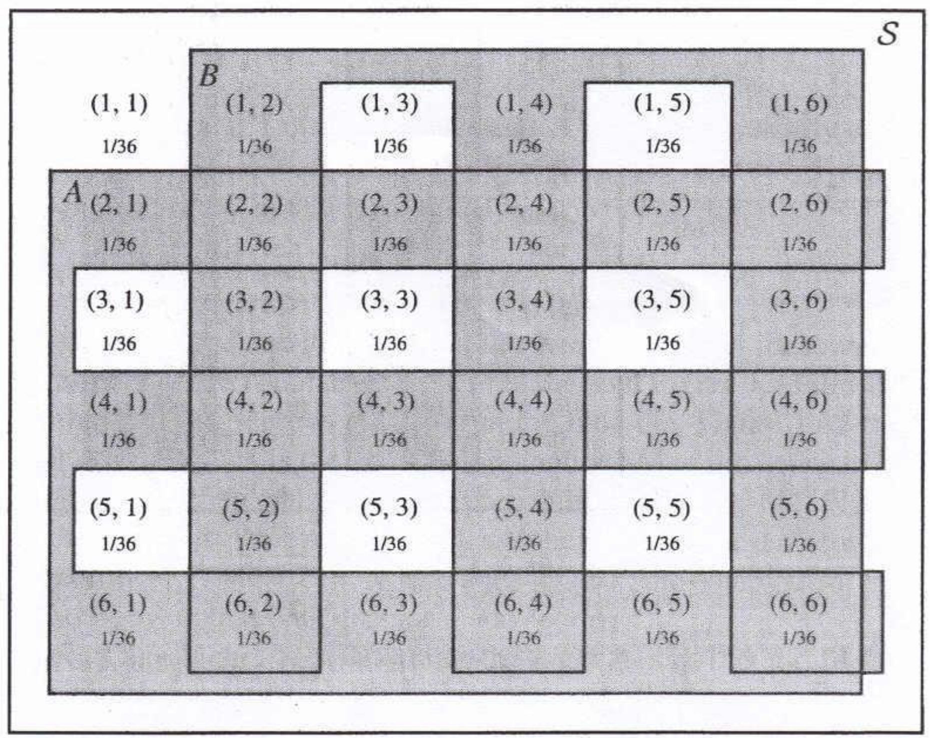
If we assume that all outcomes are considered **equally likely**,

What is the probability that **at least one die has even score**?

A : event that even score is obtained on the First die
B : event that even score is obtained on the Second die

FIGURE 1.47 ● Event $A \cup B$

$$P(A \cup B) = \frac{27}{36} = \frac{3}{4}$$



If a family has **three children**, find the probability **that two of the three** children are girls !

Suppose all outcomes are considered **equally likely**.

S **= {BBB, BBG, …, GGG}**

There are 8 outcomes ! Each has probability of **1/8**.

E **= event that we found that 2 of 3 are girls**

= {GGB, BGG, GBG}

There are 3 outcomes in the event.

So ***P*****(E)** = 3 x 0.125 = **3/8**

Basic Principle of Counting

Product Rule

In a sequence of ***r* experiments** in which the first one has $\mathbf{k_1}$ possibilities and the second event has $\mathbf{k_2}$ and the third has $\mathbf{k_3}$, and so forth, then there are a total of

$$\mathbf{k_1 .\ k_2 .\ k_3 \ldots\ k_n}$$

possible outcomes of the ***r* experiments**.

Example:

How many possible outcomes if we **toss a coin**, and subsequently **roll a die** ?

Permutation

Arrangement of **n** objects in a **specific order**.

$$P_r^n = \frac{n!}{(n-r)!}$$

P_r^n

: number of permutations of **n** objects taken **r**, in a specific order at a time.

Combination

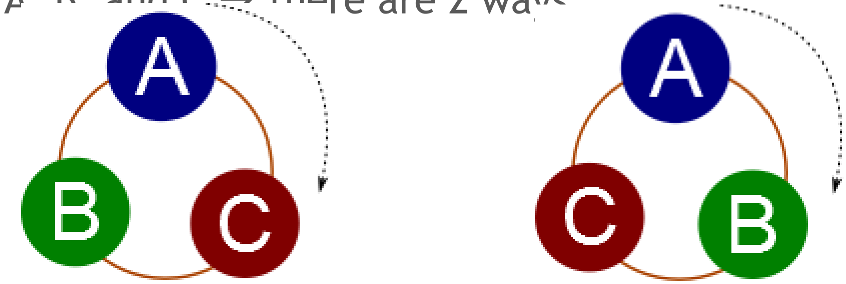
The number of different groups of size **r** that can be selected from a set of size **n** when the order of selection is not considered.

$$C(n,r)=\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

$\binom{n}{r}$: number of combinations of **n** objects taken **r**, at a time.

Circular & Ring Permutation

- **Circular Permutation**
  - is an arrangement of *n* objects in a circular order.
  - Formula: $(n-1)!$
  - Example: A, B, and C → there are 2 ways
  -



- **Ring Permutation**
  - Can be flipped over.
  - Formula: $\frac{1}{2}(n-1)!$ For $n \geq 3$ and 1 for $n = 1\ and\ 2$
  - Example: A. B and C → there is a way.

Problem I

A committee of size 5 is to be selected from a group of 6 men and 9 women. The selection is made randomly.

What is the probability that the committee consists of **3 men & 2 women** ?

Assume that "randomly selected" means that each of the **C(15, 5)** possible combinations is **equally likely** to be selected.

The probability is $$\frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$

Problem II

A class consists of 6 men and 4 women. An exam is given and the students are ranked according to their performance (no two students obtain the same score).

If all rankings are considered **equally likely**, what is the probability that women receive the top 4 scores ?

In total, there **10!** possible rankings. There are **4!** possible rankings of the women among themselves, and **6!** for men.

The probability is $$\frac{4!\,6!}{10!} = \frac{1}{210}$$

Conditional Probability

The probability of event ***E*** given that the event ***F*** has occured is called the conditional probability, is denoted by

$$P(E \mid F)$$

In this case, ***F*** **becomes our new sample space**, so

$$P(E \mid F) = \frac{P(EF)}{P(F)} \qquad \text{for } P(F) > 0$$

F is also called the ***conditioning event***

- $EF = \phi$

$$P(E \mid F) = \frac{P(EF)}{P(F)} = 0$$

- $F \subset E$

$$P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Another identity...

$$P(E \mid F) + P(E^C \mid F) = 1$$

$$
\begin{aligned}
&\textit{proof}: \\
&\quad P(E \mid F) + P(E^C \mid F) \\
&\qquad = \frac{P(EF)}{P(F)} + \frac{P(E^C F)}{P(F)} \\
&\qquad = \frac{1}{P(F)} \left(P(EF) + P(E^C F) \right) \\
&\qquad = \frac{1}{P(F)} P(F) \\
&\qquad = 1
\end{aligned}
$$

A **red die** and **blue die** are thrown:

E = {The red die = 5}

= {**(5,1)**, (5,2), (5,3), (5,4), (5,5), (5,6)}

F = {Sum of scores of two dices is 6}

= {(1,5), (2,4), (3, 3), (4, 2), **(5, 1)**}

Given ***F*** has occured, what is the probability of ***E*** ?

$$P(E) = \frac{6}{36} = \frac{1}{6} \qquad P(F) = \frac{5}{36} \qquad P(EF) = \frac{1}{36}$$

$$P(E \mid F) = \frac{P(EF)}{P(F)} = \frac{1/36}{5/36} = 0.2$$

Each employee is invited to attend the party along with **his youngest son**. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner ?

b = boy, g = girl

S = {(b,b), (b,g), (g,b), (g,g)}

Assume that all outcomes are **equally likely**.

B = event that both children are boys

A = event that at least one of them is a boy

$$P(B\mid A)=\frac{P(BA)}{P(A)}=\frac{P(\{(b,b)\})}{P(\{(b,b),(b,g),(g,b)\})}=\frac{1/4}{3/4}=\frac{1}{3}$$

General Multiplication Rule

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$P(AB) = P(B)P(A|B) = P(A)P(B|A)$$

$$P(C|AB) = \frac{P(ABC)}{P(AB)}$$

$$P(ABC) = P(AB)P(C|AB) = P(A)P(B|A)P(C|AB)$$

Then, probability of the intersection of a series of events:

$$P(A_1A_2\ldots A_n) = P(A_1)P(A_2|A_1)\ldots P(A_n|A_1A_2\ldots A_{n-1})$$

This is also called **Chain Rule**

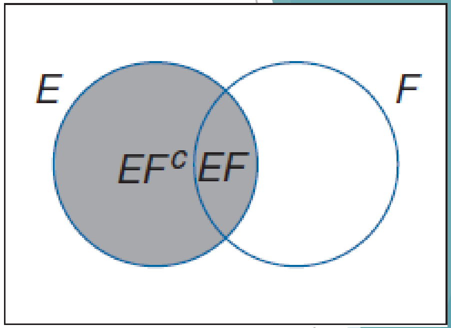
Sebuah kotak berisi **10 bola merah** dan **10 bola biru**. Jika 3 buah bola dipilih secara acak, **tanpa pengembalian**, berapa probabilitas bahwa ketiga bola tersebut berwarna merah ?

Asumsi bahwa setiap bola pada kotak mempunyai peluang yang sama untuk terpilih (**equally likely**).

Bayes Rule

Let ***E*** and ***F*** be events. We can express ***E*** as

$$E = EF \cup EF^C$$



Since, ***EF*** and ***EF^C*** are mutually exclusive,

$$\begin{aligned} P(E) &= P(EF) + P(EF^C) \\ &= P(E \mid F)P(F) + P(E \mid F^C)P(F^C) \\ &= P(E \mid F)P(F) + P(E \mid F^C)[1 - P(F)] \end{aligned}$$

It enables us to determine the probability of an event by first "conditioning" on **whether or not some second event has occured**.

An insurance company believes that people can be divided into two classes:

- Accident-prone person & Non-accident-prone person

Their statistics show that an accident-prone person will have an accident with probability **0.4**, whereas this probability decreases to **0.2** for a non-accident-prone person. If we assume 30% of the population is accident prone.

What is the probability that new holder will have an accident ?

A : event that accident will happen

F : event that a holder is accident prone

$$\begin{aligned} P(A) &= P(A \mid F)P(F) + P(A \mid F^C)P(F^C) \\ &= (0.4)(0.3) + (0.2)(0.7) \\ &= 0.26 \end{aligned}$$

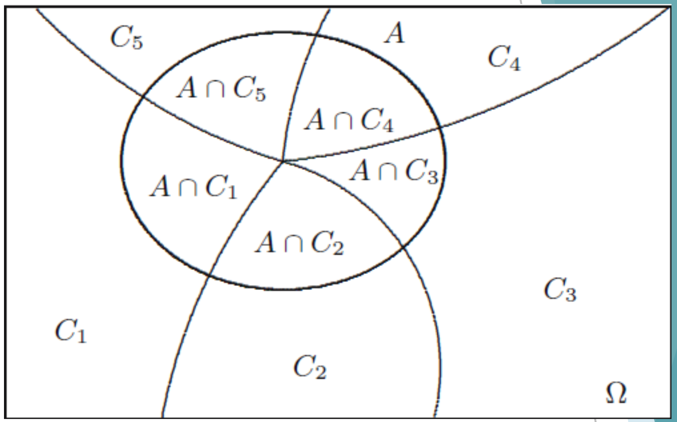
Law of Total Probability

Generalization of the previous notion...

$$S = C_1 \cup C_2 \cup \ldots \cup C_n$$

C_i : mutually exclusive

$$P(C_i) > 0$$



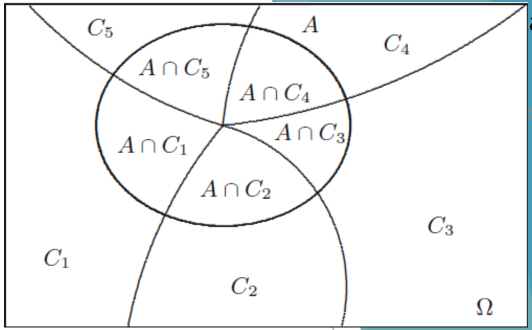
Then, the following equations hold:

$$A = (A \cap C_1) \cup (A \cap C_2) \cup \ldots \cup (A \cap C_n)$$

$$\begin{aligned} P(A) &= P(A \cap C_1) + P(A \cap C_2) + \ldots + P(A \cap C_n) \\ &= P(A \mid C_1)P(C_1) + P(A \mid C_2)P(C_2) + \ldots + P(A \mid C_n)P(C_n) \end{aligned}$$

The later is called by **Law of total probability**

Bayes' Theorem



Suppose we know …

$P(C_1), P(C_2), \ldots, P(C_n)$ **: Prior probabilities**

$P(A|C_1), P(A|C_2), \ldots, P(A|C_n)$ **: Likelihoods**

We want to compute …

$P(C_1 | A), P(C_2 | A), \ldots, P(C_n | A)$ **: Posterior probabilities**

$$P(C_i | A) = \frac{P(C_i A)}{P(A)}$$

$$= \frac{P(A|C_i)P(C_i)}{P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + \ldots + P(A|C_n)P(C_n)}$$

On a multiple-choice test, the probability that a student knows the answer is 0.4. Assume that a student who guesses at the answer will be correct with probability 0.2.

What is the conditional probability that a student knew the answer to a question given that he answered it correctly ?

C: events that the student answers correctly

K: events that the student knows the answer

$$P(K \mid C) = \frac{P(C \mid K)P(K)}{P(C \mid K)P(K) + P(C \mid K^C)P(K^C)}$$

$$= \frac{(1)(0.4)}{(1)(0.4) + (0.2)(0.6)} = 0.71$$

Independent Events

Definition

Event ***E*** and ***F*** are said to be **independent if and only if**

$$P(EF) = P(E)P(F)$$

Meaning:
The fact whether the event **E occurred** or **not** does not change the probability of the event **F** occurring (and vice versa).

Example 1:

Tossing a coin & subsequently rolling a die

E: even outcome on rolling a die

H: head outcome

T: tail outcome

$$P(E) = 0.5 \qquad P(T) = 0.5 \qquad P(E \mid T) = P(E) = 0.5$$

$$P(ET) = P(E)P(T) = 0.25$$

So, based on the definition, **E & T are independent.**

No surprise since the two events are “**physically**” independent !

Example 2: Rolling a die

E: The outcome of a die is **even**

F: The outcome is **≤ 4**

EF: an even outcome is **≤ 4**

$$P(E) = \frac{3}{6} \qquad P(F) = \frac{4}{6} \qquad P(EF) = \frac{2}{6}$$

$$P(EF) = P(E)P(F) = \frac{3}{6} \cdot \frac{4}{6} = \frac{12}{36} = \frac{2}{6}$$

E and F are independent !

So, The "independent events" **do not have to be** related to "independent **physical** process".

What is the intuition being used here ?

Using the previous definition, the following propositions hold:

Prove these statements !

$$\text{if } P(F) > 0 \text{ then}$$

$$E \text{ and } F \text{ are independent} \Leftrightarrow P(E \mid F) = P(E)$$

$$\text{if } P(E) > 0 \text{ then}$$

$$E \text{ and } F \text{ are independent} \Leftrightarrow P(F \mid E) = P(F)$$

$$\text{if } E \text{ and } F \text{ are independent then}$$

- E and F are independent
- E and F^C are independent
- E^C and F are independent
- E^C and F^C are independent

Two fair dice are thrown.

E_7 : event that the sum of the dice is 7

F : event that the first die equals 4

T : event that the second die equals 3

E_7 and F are independent

$$P(E_7) = \frac{6}{36} \qquad P(E_7 \mid F) = \frac{1}{6} \qquad P(E_7 \mid F) = P(E_7)$$

$$P(E_7 F) = \frac{1}{36} \qquad P(E_7 F) = P(E_7)P(F)$$

E_7 and T are also independent, show it !

E_7 and FT are **not** independent, show it !

The three events ***E***, ***F***, and ***G*** are said to be independent if **all** of the following conditions hold:

$$P(EFG) = P(E)P(F)P(G)$$
$$P(EF) = P(E)P(F)$$
$$P(EG) = P(E)P(G)$$
$$P(FG) = P(F)P(G)$$

If the events ***E***, ***F***, and ***G*** are independent, then ***E*** will be independent of **any event formed** from ***F*** and ***G***.

For example, ***E*** is independent of ***F*** ∪ ***G***.

$$\begin{aligned} P(E(F \cup G)) &= P(EF \cup EG) \\ &= P(EF) + P(EG) - P(EFG) \\ &= P(E)P(F) + P(E)P(G) - P(E)P(F)P(G) \\ &= P(E)[P(F) + P(G) - P(F)P(G)] \\ &= P(E)P(F \cup G) \end{aligned}$$

Definition

The definition of independence to more than three events.

The events E_1, E_2, E_3, …, E_n are said to be independent **if and only if** for every subset $E_{1'}$, $E_{2'}$, …, $E_{r'}$, **r ≤ n**, of these events:

$$P(E_{1'}E_{2'}\ldots E_{r'}) = P(E_{1'})P(E_{2'})\ldots P(E_{r'})$$

- It should be noted, though pairwise independent **does not** imply mutually independent. The following example illustrates this situation.

- Perform two independent tosses of a coin.
  - ***A*** = head on toss 1
  - ***B***= head on toss 2
  - ***C***=both tosses are equal

- It's easily seen that the three events are pairwise independent. But they are not independent since $P(ABC) \neq P(A)P(B)P(C)$.

Misal, **A**, **B**, **C** adalah kejadian-kejadian sehingga **P(A) = 0.2**, **P(B) = 0.3**, dan **P(C) = 0.4**.

Carilah probabilitas bahwa **paling tidak satu diantara A dan B terjadi** jika

(1) A dan B *mutually exclusive*

(2) A dan B *independent*

Carilah probabilitas bahwa **semua A, B, dan C terjadi** jika

(1) A, B, dan C *independent*

(2) A, B, dan C *mutually exclusive*