# Bayesian Synthesis Models

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Data Confidentiality

# Outline


1. Introduction
2. Preserving relationships and Bayesian models
3. Bayesian synthesis models estimation
4. Generating synthetic values for sensitive variables
5. Miscellany

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# Adding random noise

- Add noise to
  - provide privacy protection: noise.
  - preserve important relationships: signal.
- For a sensitive continuous variable, e.g. family income $Y_i$, we can add noise $Y_i^*$ from a normal centered at 0 (why 0?):

$$Y_i^* \sim \mathrm{Normal}(0, \sigma)$$

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- This approach preserves distributional characteristics of family income, but not its relationships with other variable(s).

## The CE sample

- The Consumer Expenditure Surveys (CE)
  - conducted by the U.S. Census Bureau for the U.S. Bureau of Labor Statistics.
  - contains data on expenditures, income, and tax statistics about consumer units (CU) across the country.
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```r
CEdata <- read.csv(file = "CEdata.csv")
head(CEdata)
```

```
##   UrbanRural Income Race Expenditure
## 1          1  98600    1    5972.167
## 2          1  24360    1    5854.500
## 3          1  80200    1    5506.667
## 4          1 150500    1    8968.891
## 5          1 130000    1   10092.833
## 6          1  32836    1    5520.267
```

# The CE sample cont'd

| Variable | Information |
|---|---|
| UrbanRural | Binary; the urban / rural status of CU: 1 = Urban, 2 = Rural. |
| Income | Continuous; the amount of CU income bfore taxes in past 12 months. |
| Race | Categorical; the race category of the reference person: 1 = White, 2 = Black, 3 = Native American, 4 = Asian, 5 = Pacific Islander, 6 = Multi-race. |
| Expenditure | Continuous; CU's total expenditures in last quarter. |

# The synthesis goals and questions

- Suppose `Income` is deemed the most sensitive among the four variables, and we want to add noise to it for privacy protection.
- To complete this, we can consider the following questions:
  1. What kind of relationships do you think are important to preserve in the confidential data?

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  3. How to estimate the chosen Bayesian models, and how can you generate synthetic data from them?

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- This lecture focuses on Questions 1-3.

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# Important relationships?

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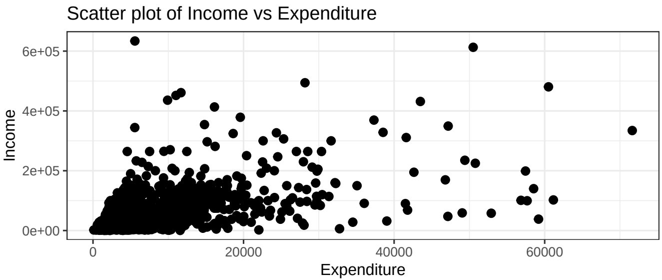
- When we say "preserve", what do we mean?
  - Inferences done on synthetic data are "close" to those done on confidential data.
  - However, too "close" could mean less privacy protection.
  - We need to strike the balance between utility and disclosure risks.

# Example: Relationship between Income and Expenditure

- Suppose we want to preserve the relationship between `Income` and `Expenditure`.
- Visualizing this relationship on the raw scale of these two variables shows fanning trend.

```
ggplot(CEdata, aes(x = Expenditure, y = Income)) +
  geom_point(size = 1) +
  labs(title = "Scatter plot of Income vs Expenditure") +
  theme_bw(base_size = 6, base_family = "")
```



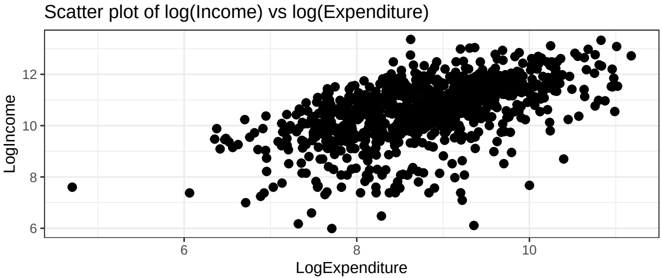
# Income and Expenditure: log scale

- On the log scale, the relationship between `log(Income)` and `log(Expenditure)` appears to be linear.

```
CEdata$LogIncome <- log(CEdata$Income)
CEdata$LogExpenditure <- log(CEdata$Expenditure)
```

```
ggplot(CEdata, aes(x = LogExpenditure, y = LogIncome)) +
  geom_point(size = 1) +
  labs(title = "Scatter plot of log(Income) vs log(Expenditure)") +
  theme_bw(base_size = 6, base_family = "")
```

# Income and Expenditure: log scale cont'd




- To capture and preserve this linear relationship, a Bayesian linear regression model of `log(Expenditure)` on `log(Income)` seems a good choice.

# A Bayesian simple linear regression model

- Let $Y_i$ be the `log(Income)` and $X_i$ be the `log(Expenditure)` for CU $i$, a Bayesian simple linear regression model can be expressed as:

$$Y_i \mid \mu_i, \sigma \overset{ind}{\sim} \mathrm{Normal}(\mu_i, \sigma) \tag{1}$$

$$\mu_i = \beta_0 + \beta_1 X_i \tag{2}$$

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$$\begin{aligned} Y_i \mid \mu_i, \sigma &\overset{ind}{\sim} \text{Normal}(\mu_i, \sigma) && (1) \\ \mu_i &= \beta_0 + \beta_1 X_i && (2) \end{aligned}$$

- The expected `log(Income)` is $\mu_i$, which is a linear function of `log(Expenditure)` $X_i$ through the intercept parameter $\beta_0$ and the slope parameter $\beta_1$.
- The intercept $\beta_0$: the expected `log(Income)` $\mu_i$ for CU $i$ that has zero `log(Expenditures)` (i.e. $X_i = 0$).
- The slope $\beta_1$: the chance in the expected `log(Income)` $\mu_i$ when the `log(Expenditures)` of CU $i$ increases by 1 unit (i.e. $X_i$ increases by $log(1)).

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- To estimate the proposed Bayesian linear regression model, we need to assign appropriate prior distributions for all parameters in the model: $\{\beta_0, \beta_1, \sigma\}$.
- If we have limited prior information about these parameters, we could use a weakly informative prior distribution. Assuming independence of the three parameters:

$$\pi(\beta_0, \beta_1, \sigma) = \pi(\beta_0)\pi(\beta_1)\pi(\sigma). \tag{3}$$

# A weakly informative prior cont'd

- We can then give individual weakly informative prior for each parameter:

$$\begin{aligned} \beta_0 &\sim \text{Normal}(\mu_0, s_0) && (4) \\ \beta_1 &\sim \text{Normal}(\mu_1, s_1) && (5) \\ 1/\sigma^2 &\sim \text{Gamma}(a, b), && (6) \end{aligned}$$

where we can use $\mu_0 = \mu_1 = 0$, $s_0 = s_1 = 100$, and $a = b = 1$.

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# MCMC simulation by JAGS

- Let's use JAGS (Just Another Gibbs Sampler) to estimate our chosen Bayesian simple linear regression model.
- We will obtain pre-specified number of posterior parameter draws from the JAGS output, which will be used for synthetic data generation through the posterior predictive distribution.
- Make sure that we require the `runjags` and `coda` libraries.

```
require(runjags)
require(coda)
```

# Using JAGS: part 1

- Describe the model by a script.

```
modelString <-"
model {
## sampling
for (i in 1:N){
y[i] ~ dnorm(beta0 + beta1*x[i], invsigma2)
}

## priors
beta0 ~ dnorm(mu0, g0)
beta1 ~ dnorm(mu1, g1)
invsigma2 ~ dgamma(a, b)
sigma <- sqrt(pow(invsigma2, -1))
}
"
```

# Using JAGS: part 2

- Define the data and prior parameters.

```
y <- as.vector(CEdata$LogIncome)
x <- as.vector(CEdata$LogExpenditure)
N <- length(y)
the_data <- list("y" = y, "x" = x, "N" = N,
                 "mu0" = 0, "g0" = 0.0001,
                 "mu1" = 0, "g1" = 0.0001,
                 "a" = 1, "b" = 1)

initsfunction <- function(chain){
  .RNG.seed <- c(1,2)[chain]
  .RNG.name <- c("base::Super-Duper",
                 "base::Wichmann-Hill")[chain]
  return(list(.RNG.seed=.RNG.seed,
              .RNG.name=.RNG.name))
}
```

# Using JAGS: part 3

- Generate samples from the posterior distribution.

```
posterior <- run.jags(modelString,
                      n.chains = 1,
                      data = the_data,
                      monitor = c("beta0", "beta1", "sigma"),
                      adapt = 1000,
                      burnin = 5000,
                      sample = 5000,
                      thin = 50,
                      inits = initsfunction)
```

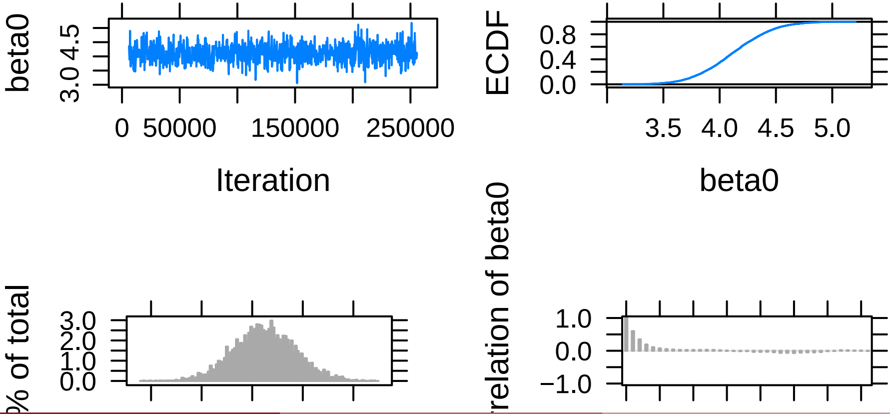
- The value of `thin` is set given MCMC diagnostics.

# Using JAGS: part 4

- MCMC diagnostics (check $\beta_1$ and $\sigma$ as well).

```
plot(posterior, vars = "beta0")
```

```
## Generating plots...
```



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- `post` contains 5000 rows and 3 columns:
  - each column corresponds to a parameter: `beta0`, `beta1`, and `sigma`.
  - each row corresponds to one of the 5000 MCMC iterations.

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- `post` contains 5000 rows and 3 columns:
  - each column corresponds to a parameter: `beta0`, `beta1`, and `sigma`.
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- Next, we will use the posterior parameter draws saved in `post` to generate synthetic values for `log(Income)`.

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# Generating one set of synthetic data

- The synthetic data generation process is no different from making prediction of future values. To predict future `log(Income)`, $\tilde{Y}_i$ for a CU given its `log(Expenditure)`, $X_i$:

$$\tilde{Y}_i \mid \beta_0, \beta_1, \sigma \overset{ind}{\sim} \mathrm{Normal}(\beta_0 + \beta_1 X_i, \sigma). \tag{7}$$

# Generating one set of synthetic data cont'd

- Given $X_i$ and one set of posterior draws of the parameters $\{\beta_0, \beta_1, \sigma\}$, we could simulate $\tilde{Y}_i$ for each all $n$ CUs:

$$
\begin{aligned}
\text{simulate } E[Y_1] = \beta_0 + \beta_1 X_1 \quad &\rightarrow \quad \text{sample } \tilde{Y}_1 \sim \mathrm{Normal}(E[Y_1], \sigma) \\
\text{simulate } E[Y_2] = \beta_0 + \beta_1 X_2 \quad &\rightarrow \quad \text{sample } \tilde{Y}_2 \sim \mathrm{Normal}(E[Y_2], \sigma) \\
&\vdots \\
\text{simulate } E[Y_n] = \beta_0 + \beta_1 X_n \quad &\rightarrow \quad \text{sample } \tilde{Y}_n \sim \mathrm{Normal}(E[Y_n], \sigma)
\end{aligned}
$$

# Generating one set of synthetic data cont'd

- Suppose we use one set of posterior draws, the function below returns a synthetic dataset with synthesized `log(Income)` and un-synthesized `log(Expenditure)`.

```
synthesize <- function(X, index, n){
  mean_Y <- post[index, "beta0"] +  X * post[index, "beta1"]
  synthetic_Y <- rnorm(n, mean_Y, post[index, "sigma"])
  data.frame(X, synthetic_Y)
}
```

- The input `X` is a vector of the un-synthesized variable, i.e. `log(Expenditure)`.
- `index` indicates which set of posterior draws to be used.

# Generating one set of synthetic data cont'd

- For example, `index = 1` of we use the first of 5000 sets. `n` is the number of observations.

```
synthesize <- function(X, index, n){
  mean_Y <- post[index, "beta0"] +  X * post[index, "beta1"]
  synthetic_Y <- rnorm(n, mean_Y, post[index, "sigma"])
  data.frame(X, synthetic_Y)
}
```

```
n <- dim(CEdata)[1]
synthetic_one <- synthesize(CEdata$LogExpenditure, 1, n)
names(synthetic_one) <- c("logExpenditure", "logIncome_syn")
```

# Generating multiple sets of synthetic data

- Typical practice generates multiple sets of synthetic data, for example, $m = 20$.
- Later, we will explore why multiple synthetic datasets are needed for data utility evaluation.

# Generating multiple sets of synthetic data cont'd

- To use the last $m = 20$ sets of the obtained 5000 MCMC iterations for generating $m = 20$ synthetic datasets:

```
n <- dim(CEdata)[1]
m <- 20
synthetic_m <- vector("list", m)
for (l in 1:m){
  synthetic_one <- synthesize(CEdata$LogExpenditure, 4980+l, n)
  names(synthetic_one) <- c("logExpenditure", "logIncome_syn")
  synthetic_m[[l]] <- synthetic_one
}
```

# Generating multiple sets of synthetic data cont'd

- To use the last $m = 20$ sets of the obtained 5000 MCMC iterations for generating $m = 20$ synthetic datasets:

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}
```

- Use a list `synthetic_m` to save all $m = 20$ synthetic datasets.
- Each synthetic dataset contains the un-synthesized `logExpenditure`, and the synthesized `logIncome_syn` from the estimated Bayesian simple linear regression model.

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# Preserve important relationships

- If `Expenditure` is deemed sensitive, what relationships do you want to preserve, why, and how?
- If `UrbanRural` is deemed sensitive, what relationships do you want to preserve, why, and how?
- If `Race` is deemed sensitive, what relationships do you want to preserve, why, and how?

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- If `Income` and `UrbanRural` are deemed sensitive, what relationships do you want to preserve, why, and how?

# Partial synthesis vs full synthesis

- Partial synthesis: only a subset of variables / attributes are deemed sensitive and to be synthesized.
- Full synthesis: all variables / attributes are deemed sensitive and to be synthesized.

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- Implications for Bayesian synthesis models?
- Implications for disclosure risks evaluation?