Week5-6-Single-Variable-Optimization_and_Multivariable-Differentiation

October 20, 2025

2.5.1 *The Traveling Minervan Problem*

Minerva University students during their Taiwan semester often explore the scenic beauty of the island. A popular destination is a small island located 2 miles north of a specific point along the shore-line. A student staying in the residence hall 6 miles west of that point wants to travel to the island. The student can run at a speed of 8 mph and swim at a speed of 3 mph. To minimize the total travel time to the island, determine how far the student should run before swimming and the corresponding distance they would need to swim.

The problem wasn't very clear on this, but I'll assume that both the reshall and the point are along the shoreline, meaning that there is an option of going along the hypotenuse and ONLY swim, as well as there is an option of going along the shoreline for most of the path and swimming for 2 miles at the end to reach the island.

(a) (b) Disconsidering the two extreme cases with max running and max swimming, every other case is composed by: Some amount of running + Some amount of swimming, where the swimming distance is defined by the hypothenuse of the catets: 6 (distance to the point) - the distance that was ran, and 2, which is the distance from the point to the island.

Therefore, we can, if we are creating a rule for the running distance, see this as $a * x + b * \sqrt{(6-x)^2 + 4} = T(x)$

Where $\sqrt{(6-x)^2 + 4}$ is the swimming time relative to the running time

Now, the constants a and b are the time cost of the distance, so they're the opposite of the relation we were given, which is the distance cost of the time. Therefore, we need to put the inverse of that relationship as the constants:

$x/8 + \frac{\sqrt{(6-x)^2+4}}{3} = T(x)$ This is the relationship.

(c) $T'(x) = d/dx(x/8) + d/dx(\frac{\sqrt{(6-x)^2+4}}{3})$ - sum rule $T'(x) = d/dx(1 * x^0/8) + 1/3 * d/dx(\sqrt{(6-x)^2 + 4})$ - power rule and constant multiple rule $T'(x) = 1/8 + 1/3 * d/dx(((6-x)^2 + 4)^{1/2})$ - sqrt becomes a power so we can apply power rule $T'(x) = 1/8 + 1/3 * d/dx * ((6^2 - 12x + x^2 + 4)^{1/2})$ - make a polynomial

Since we are taking the derivative of a square root of a function, that means we have a function-like operation (square root) applied to two functions (x^2 and 12x), we need to apply chain rule.

$T'(x) = 1/8 + 1/3 * 1/2 * (40 - 12x + x^2)^{-1/2} * d/dx(40 - 12x + x^2)$ - chain rule and then power rule for the first term $T'(x) = 1/8 + 1/3 * 1/2 * (40 - 12x + x^2)^{-1/2} * (-1 * 12x^0 + 2 * x^1)$ - power rule and constant rule

$T'(x) = 1/8 + \frac{-12+2x}{2*3*(40-12x+x^2)^{1/2}}$ - simplify

$T'(x) = 1/8 + \frac{-6+x}{3*(40-12x+x^2)^{1/2}}$ - This is the first derivative function of the total time in terms of the distance running

Now to find critical points:

$0 = 1/8 \pm \frac{-6+x}{3*(40-12x+x^2)^{1/2}}$

$-1/8 = \pm \frac{-6+x}{3*(40-12x+x^2)^{1/2}}$

$-1 = \pm \frac{-48+8x}{3*(40-12x+x^2)^{1/2}}$

$\pm 3 * (40 - 12x + x^2)^{1/2} = -48 + 8x$

$(\pm 3 * (40 - 12x + x^2)^{1/2})^2 = (48 + 8x)^2$ (square both sides, we can do this because both sides are equal)

$9 * x^2 - 108 * x + 360 = 64 * x^2 + 768 * x + 2304$ unwrap polynomials (we ignored the plus minus sign here because we squared it, making it so that no matter what was the sign of the term, it would be positive in the end)

[1]:
```
var('x')
print((-48+8*x)^2)
print((-3*(40-12*x+x^2)^(1/2))^2)
print(64*(x^2-12*x+36))
print((64*x^2 - 768*x + 2304)-(9*x^2 - 108*x + 360))
```

```
64*(x - 6)^2
9*x^2 - 108*x + 360
64*x^2 - 768*x + 2304
55*x^2 - 660*x + 1944
```

$55 * x^2 - 660 * x + 1944 = 0$ This is the simplified equation for the critical points of the original function. It's important to hihglight that these functions aren't equivalent, but their x values at y = 0 are the same, that's why we can do this.

$x_1 = \frac{660+\sqrt{660^2-4*55*1944}}{2*55}$

$x_2 = \frac{660-\sqrt{660^2-4*55*1944}}{2*55}$

[2]:
```
print((660+sqrt(660^2-4*55*1944))/(2*55))
print((660-sqrt(660^2-4*55*1944))/(2*55))
```

```
6/55*sqrt(55) + 6
-6/55*sqrt(55) + 6
```

The critical points of this function are when $x = 6/55 * sqrt(55) + 6$ and when $x = -6/55 * sqrt(55) + 6$.

(d) However, it doesn't make sense for the walking distance to be more than 6, as that would necessarily be more than one of the paths I previously mentioned (when you walk 6 miles then swim 2), since you are walking more than 6 miles and then swimming more than 2.

Therefore, the feasible domain for x is more or equal to 0 (you can't and it wouldn't make sense to walk a negative distance, even if that meant going the opposite direction) and less or equal to 6 (since you don't want to increase both your walking and swimming distance, you want to reduce one and augment the other.)

$0 \leq x \leq 6$

Since we determined a reasonable domain for x, and we know that x_1 is discarded for being outside that domain, we can calculate the travel time for x_2.

[3]:
```
T(x) = x/8 + (sqrt((6-x)^2 + 4))/(3)
print(T(6/(55*sqrt(55)) + 6).simplify_full())
```

```
8/9075*sqrt(10399)*sqrt(55) + 3/12100*sqrt(55) + 3/4
```

Evaluating this value, we get approximately 1.42

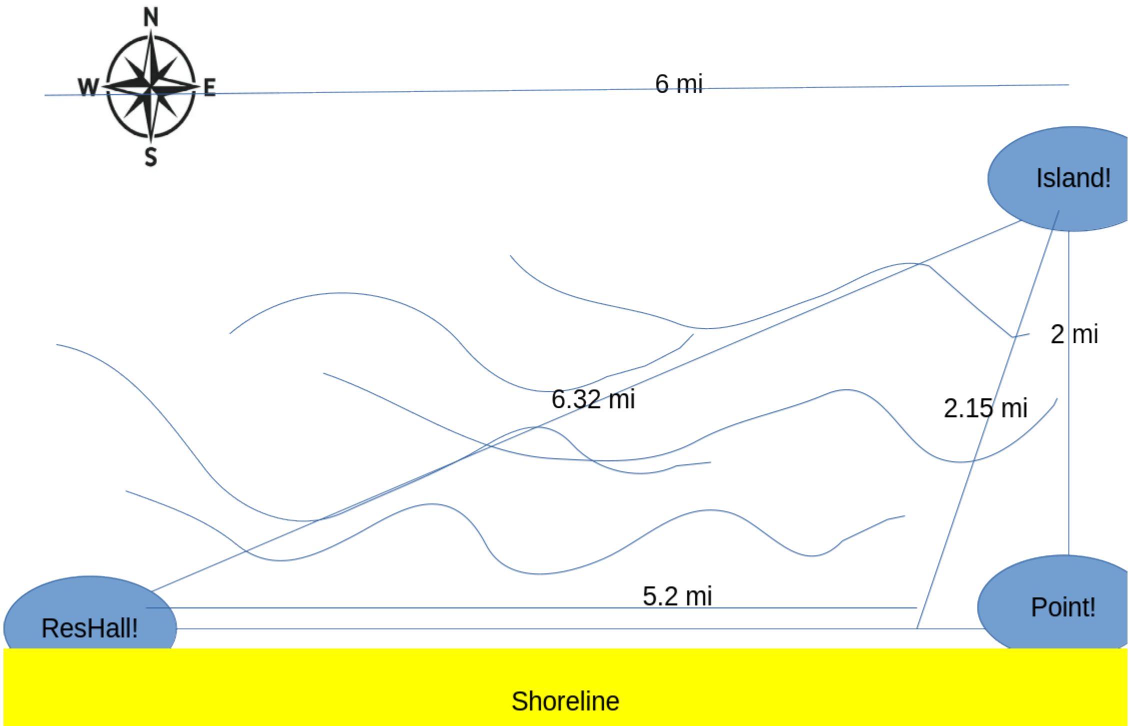
(e) Interpreting these results, we can say that the optimal time for this travel route given the rate of change of time for each mode, is approximately 85 minutes or 1.42 hours. This is given by running for 5.2 miles, which is the value of x at the critical point that's within the domain, and then swimming for 2.15 miles, which is the value of the $\sqrt{(6-x)^2+4}$ relation between swimming and running time, when x is 5.2 miles.

Optimal Swimming Distance: 2.15 miles. Optimal Running Distance: 5.2 miles. Optimal Travel Time: 1.42 hours or 85 minutes.

We can visualize this like this:

[4]:
```
import os
from IPython.display import Image, display

# Display image from file
display(Image(filename=os.path.expanduser('~/VS_Code_Projects/CS111-Assignments/
↪Illustration_Minervan_Problem.png')))  # SageMath Image class
```



2.5.2 *Partial Differentiation*

(a)

$\frac{\partial f}{\partial x}(x,y) = [d/dx(xy(x^2-y^2))*(x^2+y^2) - d/dx((x^2+y^2))*xy(x^2-y^2)]/(x^2+y^2)^2$

keeping y constant while deriving for x, and applying quotient rule

$\frac{\partial f}{\partial x}(x,y) = [(y*(d/dx(x)*(x^2-y^2)+(x)*d/dx(x^2-y^2)))*(x^2+y^2) - d/dx((x^2+y^2))*xy(x^2-y^2)]/(x^2+y^2)^2$ product rule and constant multiple rule

$\frac{\partial f}{\partial x}(x,y) = [y((x^2-y^2)+2x^2))*(x^2+y^2) - (2x)*xy(x^2-y^2)]/(x^2+y^2)^2$ constant rule and constant multiple rule and power rule. Since all the derivatives left were simple, such as a squared variable + or - a constant, or a first order variable multiplied by a constant, I just needed to calculate all of them.

$\frac{\partial f}{\partial x}(x,y) = [(y(3x^2-y^2))*(x^2+y^2) - (2yx^2)*(x^2-y^2)]/(x^2+y^2)^2$ simplified all the operations

$\frac{\partial f}{\partial x}(x,y) = [(y(3x^4+3x^2y^2-x^2y^2-y^4) - 2yx^4 - 2y^3x^2]/(x^2+y^2)^2$ expanded the terms

$\frac{\partial f}{\partial x}(x,y) = [3x^4y + 2x^2y^3 - y^5 - 2yx^4 - 2y^3x^2]/(x^2+y^2)^2$ simplified operations

$\frac{\partial f}{\partial x}(x,y) = [x^4y + 4x^2y^3 - y^5]/(x^2+y^2)^2$ simplified operations

This is the most simplified version of the derivative. Let's verify it:

[5]:
```
var('x, y')
f = x*y*(x^2 - y^2)/(x^2 + y^2)
fx = diff(f, x).simplify_full()
print(fx)
```

```
(x^4*y + 4*x^2*y^3 - y^5)/(x^4 + 2*x^2*y^2 + y^4)
```

The calculation was right, the only difference is the denominator is unfolded.

Now, the partial derivative with x constant and y variable is not that different:

$\frac{\partial f}{\partial y}(x, y) = [x * d/dy(y(x^2 - y^2)) * (x^2 + y^2) - d/dy((x^2 + y^2)) * xy(x^2 - y^2)]/(x^2 + y^2)^2$ derive with x constant and apply quotient rule

$\frac{\partial f}{\partial y}(x, y) = [x*(d/dy(y)*(x^2-y^2)+y*d/dy(x^2-y^2))*(x^2+y^2)-d/dy((x^2+y^2))*xy(x^2-y^2)]/(x^2+y^2)^2$ product rule

$\frac{\partial f}{\partial y}(x, y) = [x * ((x^2 - y^2) - y * 2 * y)) * (x^2 + y^2) - 2 * y * xy(x^2 - y^2)]/(x^2 + y^2)^2$ solve all simple derivatives

$\frac{\partial f}{\partial y}(x, y) = [(x^3 - 3xy^2) * (x^2 + y^2) - 2xy^2(x^2 - y^2)]/(x^2 + y^2)^2$ simplify operations

$\frac{\partial f}{\partial y}(x, y) = [x^5 - 3x^3y^2 + x^3y^2 - 3xy^4 - 2x^3y^2 + 2xy^4]/(x^2 + y^2)^2$ expand terms

$\frac{\partial f}{\partial y}(x, y) = [x^5 - 4x^3y^2 - xy^4]/(x^2 + y^2)^2$ simplify and cancel out

[6]:
```
fy = diff(f, y).simplify_full()
print(fy)
```

```
(x^5 - 4*x^3*y^2 - x*y^4)/(x^4 + 2*x^2*y^2 + y^4)
```

Again we got the same result. This leaves us with:

$\frac{\partial f}{\partial y}(x, y) = [x^5 - 4x^3y^2 - xy^4]/(x^2 + y^2)^2$

and

$\frac{\partial f}{\partial x}(x, y) = [x^4y + 4x^2y^3 - y^5]/(x^2 + y^2)^2$

When x, y are not 0, 0

(b)

Now, to find the partial derivatives when x, y = 0, 0, having h as the increment we add to the variable to reach the derivative at the point we want (0, 0):

$\frac{\partial f}{\partial x}(0, 0) = lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$

$\frac{\partial f}{\partial x}(0, 0) = lim_{h \to 0} \frac{\frac{0*h*(h^2-0^2)}{h^2+0^2} - 0}{h}$ substitute functions

$\frac{\partial f}{\partial x}(0, 0) = lim_{h \to 0} \frac{\frac{0}{h^2} - 0}{h} = 0$ simplify

$\frac{\partial f}{\partial y}(0, 0) = lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$

$\frac{\partial f}{\partial y}(0, 0) = lim_{h \to 0} \frac{\frac{0*h*(0^2-h^2)}{0^2+h^2} - 0}{h}$ substitute functions

$\frac{\partial f}{\partial y}(0,0) = lim_{h \to 0} \frac{\frac{0}{-h2} - 0}{h} = 0$ simplify

Both derivatives exist at f(0, 0), and they are equal to zero.

(c)

To know if the partial derivatives are continuous, we need to know if their value is approaching 0 when they are near x, y = 0, 0

$lim_{(x,y) \to (0,0)} f_x(x,y) = f_x(0,0)$

$lim_{(x,y) \to (0,0)} f_y(x,y) = f_y(0,0)$

Both of these need to hold true, since it means that as x, y approach 0, 0, the limit is the same as when it is equal to 0, 0.

This requires us to bound the derivative functions to see what is their maximum and minimum value. If we are able to confidently bound the derivative function between two functions, we can say because of the squeeze theorem, that there is a limit, and it is equal to the value we found for the partial derivative at 0, 0 (because that itself is the value of a limit fixing one of the variables).

Additionally, we want to show that the function doesn't become indefined close to zero, which is what would happen if the denominator grew faster than the numerator. Otherwise, the partial derivatives are not continuous.

For that, we need to explore some properties of working with powers. We know that $x^2 \geq 0$, because a squared number is always positive. Therefore, it can also be true that:

$x^2 \leq x^2 + y^2$, because $y^2 \geq 0$ as well.

Also, we can take square roots and arrive to this, for both y and x:

$|x| \leq \sqrt{x^2 + y^2}$ and $|y| \leq \sqrt{x^2 + y^2}$

Now, we can start by bounding the numerator of the partial derivative $f_x(x,y)$. We can do that by using the triangle inequality rule: The module of a sum is less than or equal to the sum of the module of the values.

$|x^4 y + 4x^2 y^3 - y^5| \leq |x^4 y| + 4|x^2 y^3| + |y^5|$

Because the term $(x^2 + y^2)$ is going to be so common in this calculation, I'll write it like this for better reading and understanding: $t = (x^2 + y^2)$

Now we bound each term separately:

$|x^4 y| = x^{2^2} |y|$

Considering the inequalities I put there before, we can bound it like this:

$|x^4 y| \leq t^2 * \sqrt{t}$

$|x^4 y| \leq t^{5/2}$

Next term: $4|x^2 y^3|$

$4|x^2 y^3| = 4x^2 |y|^3$

$4|x^2 y^3| \leq 4 * t * t^{3/2}$

$4|x^2y^3| \leq 4 * t^{5/2}$

Next term: $|y^5|$

$|y^5| \leq (\sqrt{t})^5$

$|y^5| \leq t^{5/2}$

What we end up with is a relationship like this (we can't bound the denominator in a meaningful way because it can never be equal to zero so we just assume its max value is its own value):

$|\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}| \leq \frac{t^{5/2}+4t^{5/2}+t^{5/2}}{t^2}$

Simplifying this:

$|\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}| \leq \frac{6t^{5/2}}{t^{4/2}}$

$|\frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}| \leq 6t^{1/2}$

since x and y tend to zero, we can also say that this term tends to zero. $6t^{1/2}$

Therefore, the partial derivatives of f are continuous!

The squeeze theorem proves that if a function is bounded from both sides, its limit exists, and it is true for the function of this partial derivative.

I proved that because the numerator of the fraction goes to zero faster than the denominator does as both approach zero, that the function is continuous and approaches zero. I would write the full explanation for the other partial derivative too, but at this point it makes sense to assume that they're the same for both, especially considering that since we are taking absolute values and using them as bounds, the negative sign (which was the only difference that y had) wouldn't make a difference.

[ ]: