CS111 Assignment Deep Dive 1: Limits, Continuity, and Single Variable Differentiation

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Deep Dives

For every positive value ϵ, there exists another value, δ, such that the following is true: if the distance between the x value and our point of interest, a, is less than δ, then the distance between our function at that point f(x) and the limit value, L, is less than ϵ.

1. (a) Consider the function f(x) = 4x+ 1. Find the value of lim x→2 f(x) using the techniques we discussed in class. We will call that value L.

$$L = lim_{x \to 2} 4x + 1 \quad (1)$$

$$L = 4 * 2 + 1 \quad (2)$$

$$L = 9 \quad (3)$$

(b) Now, we will prove that the lim x→2 f(x) = L, where L is the value you found in part a, by verifying that the definition above holds. Say you are given a value of $\epsilon = 1$. What value of δ would guarantee us that if the distance between x and our point of interest, x = 2, is less than δ, then the distance between f(x) and L is less than 1?

To find this, we need to see if what is the max constant we can take off of x at the limit(2). This is because a distance is destination minus current coordinate, and we want it to be a module, since negative distance doesn't exist (we are looking for an absolute value), so we're looking for $|x - c|$ when it's smaller than δ. c here is the approach point: the value of x when f(x) = L. As long as this condition is true, we know the distance between the function's value at x and its value at the limit will hold true, IF THE LIMIT EXISTS! In summary, this was an interpretation of the definition given so that I could deduce this formula: when the limit exists, $|x - c| < \delta$ should imply $|f(x) - L| < \epsilon$ For this, let's solve the second inequality since we know all of its values, for an unknown x. The easiest way to get to a relationship with the first inequality and therefore verify both as true is to somehow transform the second inequality into one defined in the terms of the first. Therefore, it should take some form of $|x-2|$ for this case.

$$|4 * x + 1 - 9| < 1 \tag{4}$$

$$|4 * x - 8| < 1 \tag{5}$$

Here, we can bring out the 4 as a factor

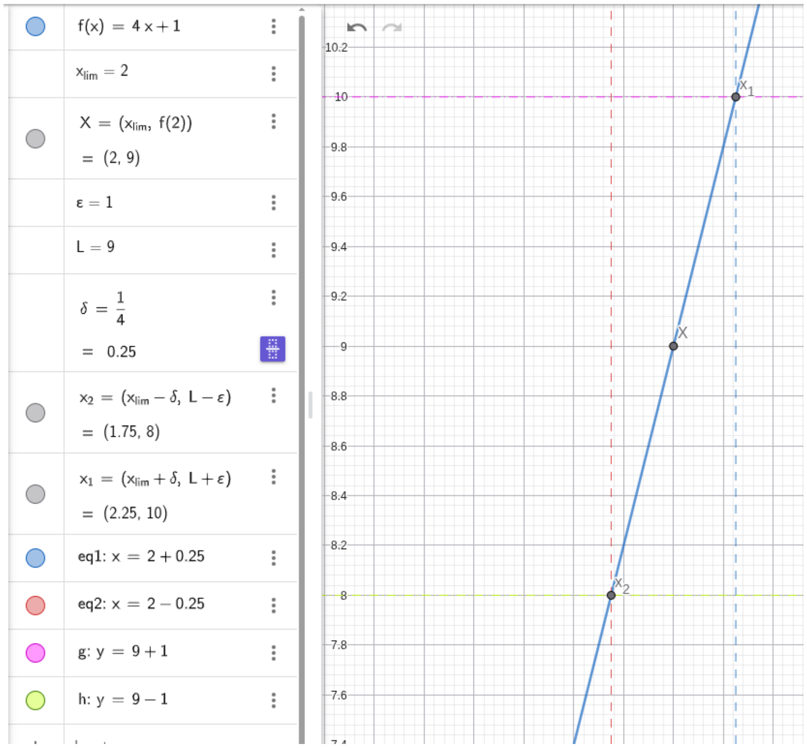
$$4|x-2| < 1 \tag{6}$$

$$|x-2| < 1/4 \tag{7}$$

This achieved a form exactly equal to our first inequality, so we can say that for this $\epsilon = 1$, the δ value is $1/4$.

(c) Draw a diagram that illustrates the scenario you analyzed in part (b). Make sure to clearly indicate where x, δ, and ϵ are in your diagram.

This graphic reflects the graph of f(x), with the points that we are trying to approach the limit from delimited as x_2 and x_1. This graph shows that for the bound that ϵ creates, there is a bound that δ creates that contains all values of x between it that are approaching x. Therefore, x, δ, and ϵ aren't points on the diagram, but rather, the values that determine this bound, within which x is tending to 2.



(d) Try to generalize this: make a table with various values of ϵ and the corresponding value of δ that would satisfy the conditions above. Use the observations from your table to come up with a formula that relates δ and ϵ.

$4 * \|x-2\| < 1$	$\delta = 1/4$	$\epsilon = 1$
$4 * \|x-2\| < 2$	$\delta = 1/2$	$\epsilon = 2$
$4 * \|x-2\| < 3$	$\delta = 3/4$	$\epsilon = 3$
$4 * \|x-2\| < 4$	$\delta = 1$	$\epsilon = 4$

What we understand from this table is that the value of δ is always 1/4 of the value of ϵ. So,

$$\delta = \frac{\epsilon}{4} \tag{8}$$

2. (a) Consider the function $g(x) = x^2 - 5x + 6$, Find the value of lim x→1 g(x) using

the techniques we discussed in class. We will call that value L.

$$L = lim_{x \to 2} x^2 - 5x + 6 \tag{9}$$

$$L = 1^2 - 5 * 1 + 6 \tag{10}$$

$$L = 2 \tag{11}$$

(b) Now, we will prove that the lim x→1 g(x) = L, where L is the value you found in part a, by verifying that the definition above holds. Say you are given a value of $\epsilon = 1$. What value of δ would guarantee us that if the distance between x and our point of interest, x = 1, is less than δ, then the distance between f(x) and L is less than 1?

We can use the same formula as before,

$|x - c| < \delta$ should imply $|g(x) - L| < \epsilon$

but here we can't write the function in terms of —x-1—. Why? Since we have a term that is x squared, we have more than 1 x term. Therefore, I'll approach the problem a different way. Let's say that a is the distance $|x - c|$, which is $|x - 1|$ for this case. Given that, we can say that $x = 1 + a$ now, and we can use that in the second inequality,

$$|(1 + a)^2 - 5(1 + a) + 6 - 2| < 1 \tag{12}$$

Now, to offset this module, we can just bound the value by the negative and positive versions of the right hand side of the inequality, so that the condition still holds true.

$$-1 < (1 + a)^2 - 5(1 + a) + 6 - 2 < 1 \tag{13}$$

$$1 < (1 + a)^2 - 5(1 + a) + 6 < 3 \tag{14}$$

$$1 < 1^2 + 2a + a^2 - 5 * 1 - 5 * a + 6 < 3 \tag{15}$$

$$1 < -3a + a^2 + 2 < 3 \tag{16}$$

$$-1 < -3a + a^2 < 1 \tag{17}$$

Now, we have two quadratic inequalities. Let's solve each of them:

$$-3a + a^2 - 1 < 0 \tag{18}$$

The roots of this inequality are $\frac{3 \pm \sqrt{3^2 - 4*1*-1}}{2} = \frac{3 \pm \sqrt{13}}{2} = -0.3028, 3.3028$

$$-3a + a^2 + 1 < 0 \tag{19}$$

The roots of this inequality are $\frac{3 \pm \sqrt{3^2 - 4*1*1}}{2} = \frac{3 \pm \sqrt{5}}{2} = 0.3819, 2.6180$

Those values are "possible values" for δ bounds. In practice, it means that a solution must be the one of these that contains the module of a. Then, we go back to the equations we had found:

$|a^2 - 3a| < 1$ holds when a is between the boundaries we just defined. However, it doesn't reach all boundaries. It reaches the most restrictive one from whatever side (positive or negative), and its boundary on the other side is that as well. Why? Because our core equation that makes this condition relate to δ is $|a| < \delta$, so we need for one value of delta to be the bound of —a— (or $|x - 1|$ from both sides).

Therefore, the most restrictive one is the one which module is smaller. This would be $\frac{3-\sqrt{13}}{2}$, since its value is -0.3028

Therefore, the value of δ so that $|g(x) - 2| < 1$ holds true as long as $|x - 1| < \delta$ is $\delta = 0.3028$.

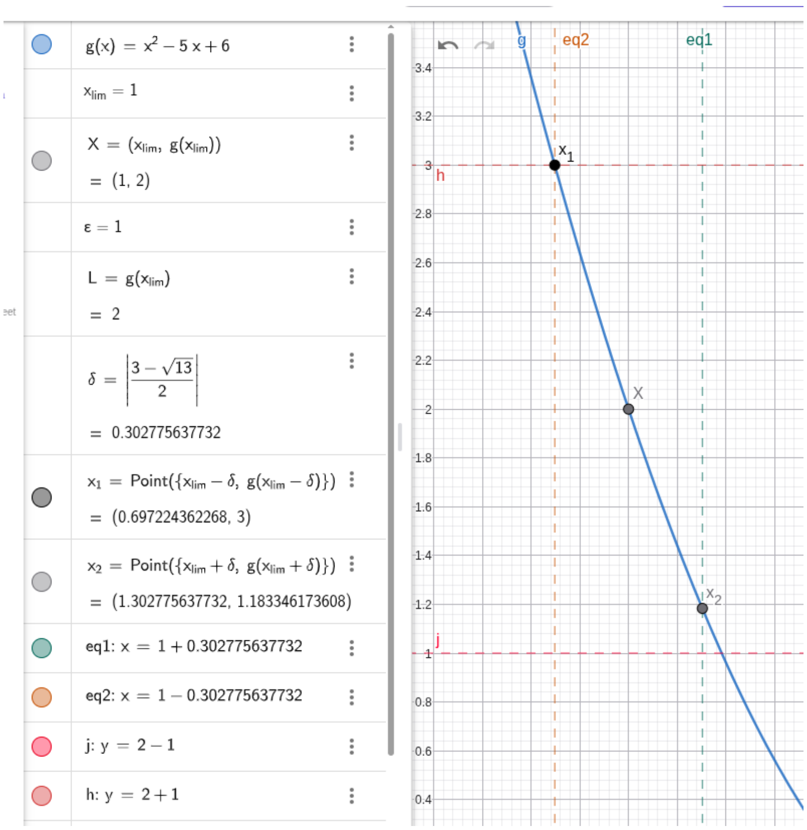
(c) Draw a diagram that illustrates the scenario you analyzed in part (b). Make sure to clearly indicate where x, δ, and ϵ are in your diagram.

This graphic reflects the graph of g(x), with the points that we are trying to approach the limit from delimited as x_2 and x_1. This graph shows that for the bound that ϵ creates, there is a bound that δ creates that contains all values of x between it that are approaching x.

The main difference here between the two function is how the coordinates of the points of distance δ from x_{lim} are defined. In f, the linear function, the maximum value that the function could take would be its value at the limit minus epsilon. That made sense because it's a linear function, and any value that held true for one side would hold true for the other, since behavior was not expected to change.

Since this one isn't, we will have some asymmetry on the graph, meaning that only one of the endpoints defined by a δ distance will be hugged by the ϵ bound, in this case x_1, which is where we found to be the most restrictive bound, and the other endpoint must be the mirror of that x value, lying on the function, due to the nature of the formula as I described already.

Still, the relationship holds, since for any x that satisfies this $|x - 1| < 0.302$, it is true that $|g(x) - 2| < 1$.



(d) Try to generalize this: make a table with various values of ϵ and the corresponding value of δ that would satisfy the conditions above. Use the observations from your table to come up with a formula that relates δ and ϵ.

$-3a + a^2 < 1$	$\delta = \lvert\frac{3-\sqrt{13}}{2}\rvert = 0.302$	$\epsilon = 1$
$-3a + a^2 < 2$	$\delta = \lvert\frac{3-\sqrt{17}}{2}\rvert = 0.561$	$\epsilon = 2$
$-3a + a^2 < 3$	$\delta = \lvert\frac{3-\sqrt{21}}{2}\rvert = 0,791$	$\epsilon = 3$
$4 * \lvert x - 2\rvert < 4$	$\delta = \lvert\frac{3-\sqrt{25}}{2}\rvert = 1$	$\epsilon = 4$

Since from testing we found that the bound that is always the most restrictive is the one that subtracts a bigger square root from 3, we can use that as a formula, and as we observe that ϵ plays the role of the constant C in this quadratic inequality, we can just say the formula for δ is just the following for this specific function, in terms of ϵ:

$$\delta = |\frac{3 - \sqrt{9 - 4 * \epsilon}}{2}| \tag{20}$$