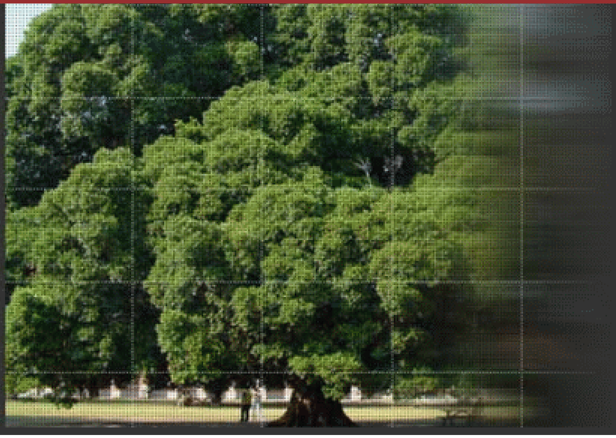

Chap 2 Trees and Distance

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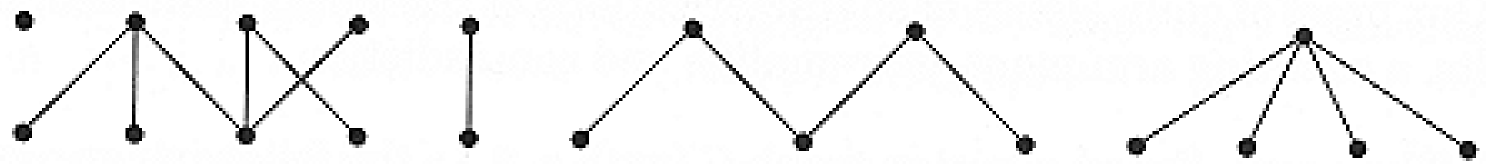
Def:

- A graph with no cycle is *acyclic*.
- A *forest* is an acyclic graph.
- A *tree* is a connected acyclic graph.
- A *leaf* (or *pendant vertex*) is a vertex of degree 1.
- A *spanning subgraph* of G is a subgraph with vertex set $V(G)$.
- A *spanning tree* is a spanning subgarph that is a tree.



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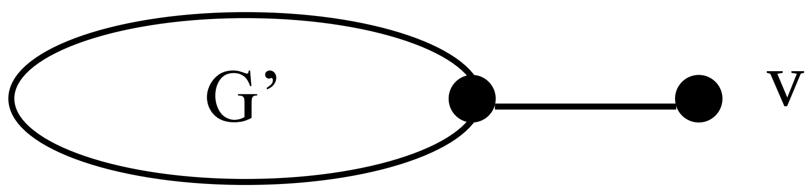
Lemma. Every tree with at least two vertices has at least two leaves. Deleting a leaf from an n-vertex tree produces a tree with n-1 vertices.

Proof:

- A connected graph with at least two vertices has an edge.

  In an acyclic graph, the endpoints of a maximal nontrivial path are leaves.
- Every u, w-path in G is also in G' ⇨ G' is connected.

  Since deleting a vertex cannot create a cycle ⇨ G' is acyclic.



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Theorem. For an n-vertex graph G (with $n\geq 1$), the following are equivalent.

A) G is connected and no cycles.

B) G is connected and has n-1 edges.

C) G has n-1 edges and no cycles.

D) G has no loops and has, for exactly one u,v-path

Proof:

A$\Rightarrow${B,C}. By induction,

n=1. (trivial)

n>1. Given an acyclic connected graph G, $\exists$ a leaf v s.t.

$G'=G$-v also is acyclic and connected.

By I.H., $e(G') = n-2 \Rightarrow e(G) = n-1.$

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A) G is connected and no cycles.

B) G is connected and has n-1 edges.

C) G has n-1 edges and no cycles.

Proof: B$\Rightarrow${A,C}.

Delete edges from cycles (no cut edges) of G one by one until the resulting graph G' is acyclic $\Rightarrow$ G' is connected $\Rightarrow$ $e(G')=n-1=e(G)$ $\Rightarrow$ G is acyclic.

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A) G is connected and has no cycles.

B) G is connected and has n-1 edges.

C) G has n-1 edges and no cycles.

Proof. C⇒{A,B}

Let $G_1, G_2, \ldots, G_k$ be the components of G.

$e(G_i) = n(G_i) - 1 \Rightarrow e(G) = \sum_i [n(G_i) - 1] = n - k.$

Since $e(G) = n - 1$, k=1 and G is connected.

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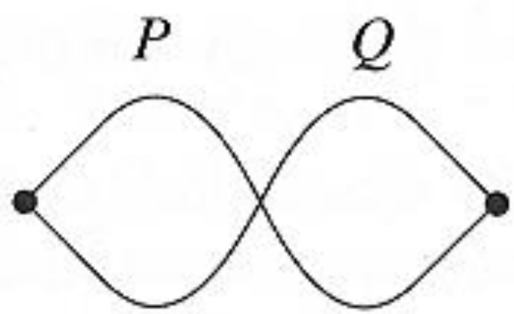
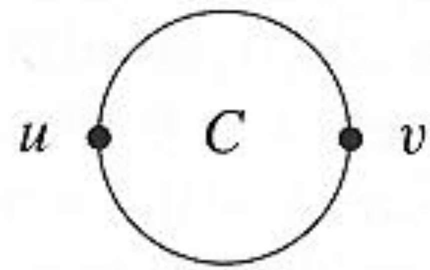
A⇒D.

Since G is connected, each pair of vertices is connected by a path.





D ⇒A.

Clearly, G is connected.

If G has a cycle C,

then G has two u,v-paths for $u,v \in V(G) \Rightarrow G$ is acyclic.

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Corollary.

A) Every edge of a tree is a cut-edge.

B) Adding one edge to a tree forms exactly one cycle.

C) Every connected graph contains a spanning tree.

Proof.

A) By the fact that a tree has no cycles.

B) By the fact that a tree has a unique path linking each pair of vertices.

C) Iteratively deleting edges from cycles in a connected graph yields a connected acyclic subgraph.

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Proposition.

If T and T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.

Proof.

Let U and U' be the two components of T-e.

Since T' is connected, T' has an edge e' with endpoints in U and U'.

T-e+e' is connected, has $n(G)$-1 edges, and is a spanning tree of G.

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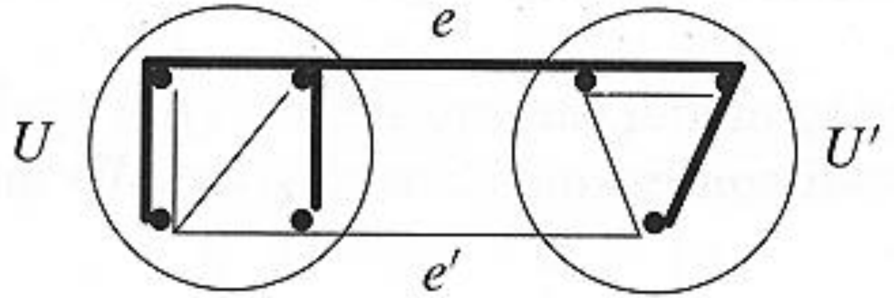
Proposition.

If T and T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree of G.

Proof.

1. $T'+e$ contains a unique cycle C.
2. T is acyclic,
   Deleting e' breaks the only cycle in $T'+e$. Then,
   $T'+e-e'$ is connected and acyclic and is a spanning tree of G.



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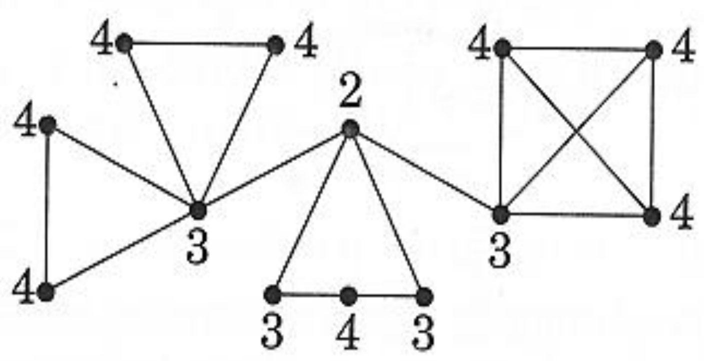
Def:

- *distance from u to v*, $d_G(u,v)$, is the least length of a u,v-path.
- *diameter* (diam G) is $\max_{u,v \in V(G)} d(u,v)$.
- *eccentricity of a vertex u*, $\varepsilon(u)$, is $\max_{v \in V(G)} d(u,v)$.
- radius of a graph G, rad G, is $\min_{u \in V(G)} \varepsilon(u)$.
- In the graph below, each vertex is labeled with its eccentricity. The radius is 2, the diameter is 4, and the length of the longest path is 7.



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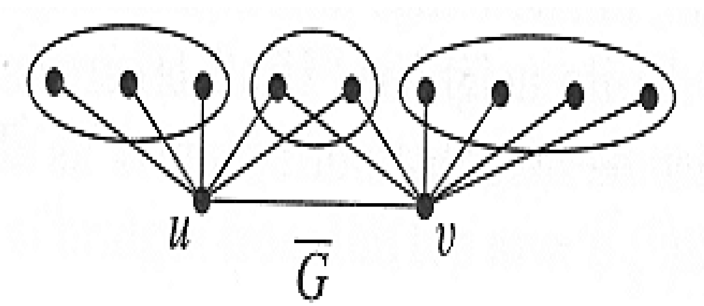
Theorem. If G is a simple graph, then $diamG \geq 3 \Rightarrow diam\overline{G} \leq 3.$

Proof:

When $diamG > 2$,there exist nonadjacent vertices $u, v \in V(G)$ with no common neighbor.

Every $x \in V(G) - \{u, v\}$ has at least one of $\{u,v\}$ as a nonneighbor$\Rightarrow$

x adjacent in $\overline{G}$ to at least one of $\{u, v\}$ in $\overline{G}$



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- The *center* of a graph G is the subgraph induced by the vertices of minimum eccentricity.
- *Wiener index* of G is $D(G) = \sum_{u,v \in V(G)} d_G(u,v)$.

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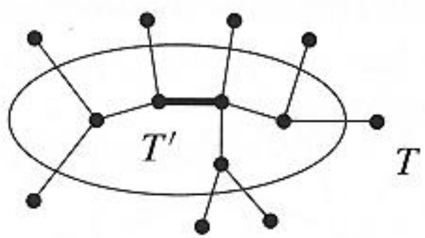
Theorem. The center of a tree is a vertex or an edge.

Proof: By induction on the number of vertices in a tree *T*.

- Basis: $n(T) \le 2$. The center is the entire tree.
- Induction step: $n(T)>2$.

Form a tree *T'* by deleting every leaf of *T*.
The internal vertices on paths between leaves of *T* remain $\Rightarrow n(T') \ge 1$



$\varepsilon_{T'}(u) = \varepsilon_T(u) - 1, \forall u \in V(T')$

Also, the eccentricity of a leaf in *T* > the eccentricity of its neighbor in *T*.

$\therefore T$ and *T'* have the same center. By I.H., the result holds.

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Theorem. Among trees with n vertices, the Wiener index $D(T) = \sum_{u,v} d(u,v)$ is minimized by stars and maximized by paths, both uniquely.

Proof:

- A tree has n-1 edges, it has n-1 pairs of vertices at distance 1, and all other pairs have distance at least 2$\Rightarrow$ star minimizes $D(T)$.
- Uniqueness: Consider a leaf x in T with its neighbor v. If all other vertices have distance 2 from x, $x \in N(v) \Rightarrow T$ is a star.

$$D(K_{1,n-1}) = (n-1) + 2\binom{n-1}{2} = (n-1)^2$$

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u: an endpoint of P_n

$\sum_{v \in V(P_n)} d(u,v) = \sum_{i=0}^{n-1} i = C(n,2)$

$D(P_n) = D(P_{n-1}) + C(n,2) \Rightarrow D(P_n) = C(n+1,3) \left(\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \right)$

Uniqueness: By induction on n

- Basis: n=1. The only tree with one vertex is P_1
- Induction step: n>1. Let u be a leaf.

$D(T) = D(T-u) + \sum_{v \in V(T)} d(u,v).$

By I.H., $D(T-u) \le D(P_{n-1})$, with equality iff T-u is a path

Consider the list of distance from u in P_n: 1,2,…,n-1

Any repetition makes $\sum_{v \in V(T)} d(u,v)$ smaller than when u is a leaf of a path.

T is a path and u is an endpoint of T.

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Lemma. If H is a subgraph of G, then $d_G(u, v) \le d_H(u, v)$

Corollary. If G is a connected n-vertex graph, then
$D(G) \le D(P_n)$

Proof:
Let T be a spanning tree of G. Then,
$D(G) \le D(T) \le D(P_n)$

2.2. Spanning Trees and Enumeration

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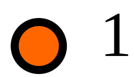
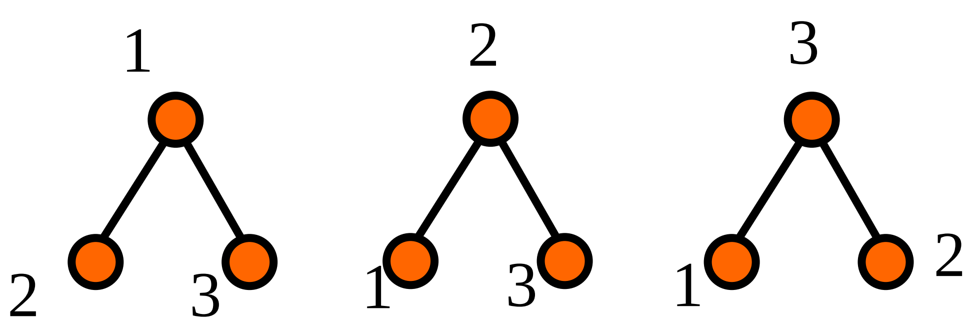


one vertex

two vertices



three vertices

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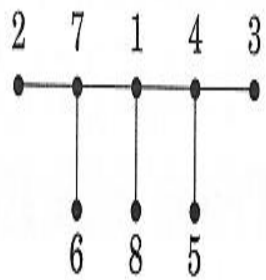
(Prüfer code) Production of $f(T)=(a_1,\ldots,a_{n-2})$

Input: A tree T with vertex set $S \subseteq N$

Iteration: At the ith step, delete the least remaining leaf, and let a_i be the neighbor of this leaf



Full code (744171)

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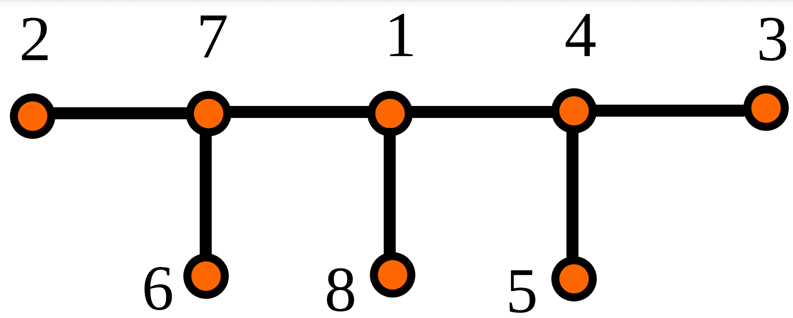
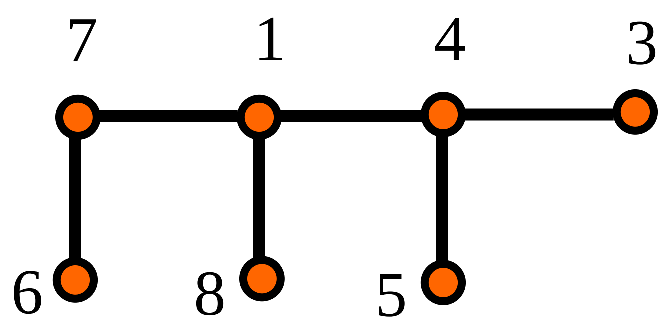
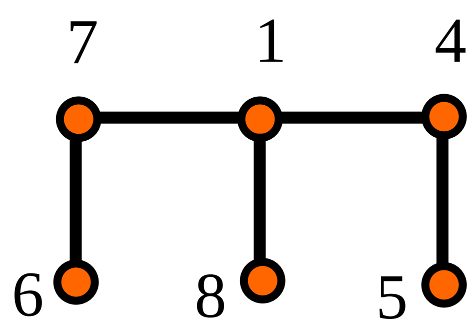




$a=(7)$



$a=(74)$

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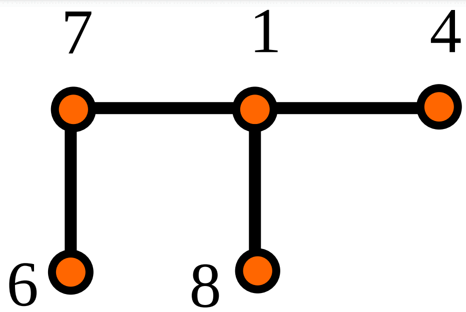
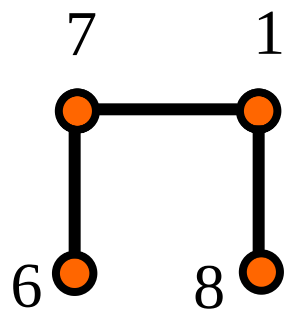
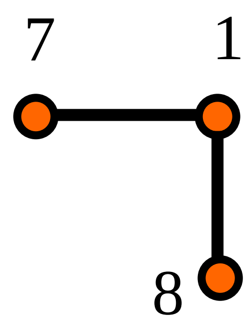


$a=(744)$



$a=(7441)$



$a=(74417)$

$a=(744171)$

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$a=(744171)$
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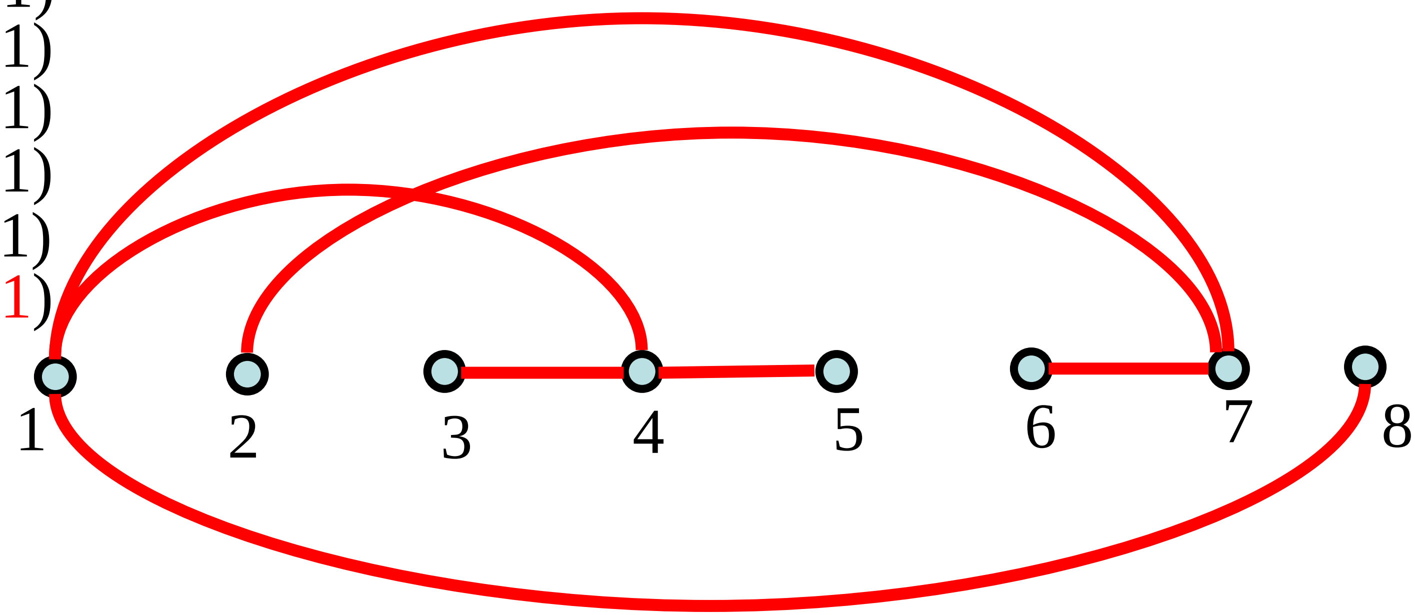

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Theorem. (Cayley's Formula [1889]). For a set $S \subseteq N$ of size n, there are n^{n-2} trees with vertex set S.

Proof: Assume

- Algorithm 2.2.1 d efines a bijection f from the set of trees with vertex set S to the set S^{n-2} of lists of length n-2 from $S \Rightarrow$ Show for each $a=(a_1,a_2,\ldots,a_{n-2})\in S^{n-2}$ that exactly one tree with vertex set S with $f(T)=a$.
- Basis: n=2. The 2-vertex tree corresponding to a length-0 list.
  I. S.: For $a\in S^{n-2}$, find all solutions to $f(T)=a$. Every such a tree has x as its least leaf and has edge xa_1

  By I.H., there is exactly one tree T' having vertex set $S'=S-\{x\}$, and prufer code $a'=(a_2, a_3,\ldots,a_{n-2})$.
- Therefore, there is exactly one tree T with $f(T)=a$.

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Corollary.

Given positive integers $d_1,\ldots,d_n$ summing to $2n$-2,there are exactly $\frac{(n-2)!}{\Pi(d_i-1)!}$ trees with vertex set $[n]$ such that vertex i has degree d_i, for each i.

Proof:

IDEA: each vertex x appears $d_T(x)-1$ times in the Prufer code.

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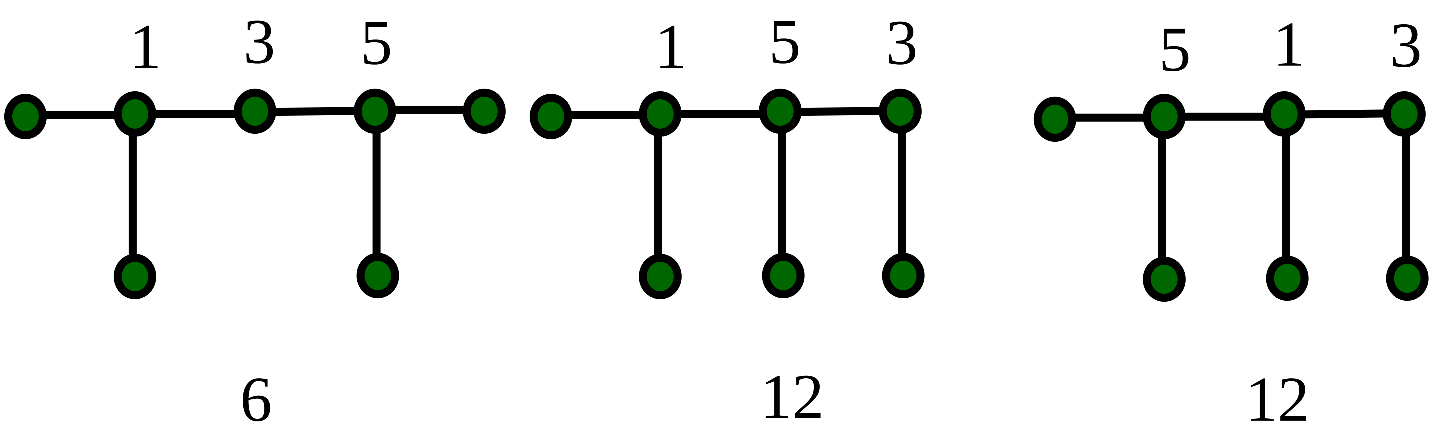
- Eg. Consider trees with vertices {1, 2, 3, 4, 5, 6, 7} that have degrees (3, 1, 2, 1, 3, 1, 1), respectively.
- Totally $\frac{(n-2)!}{\prod (d_i - 1)!} = 30$ trees.



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Eg.

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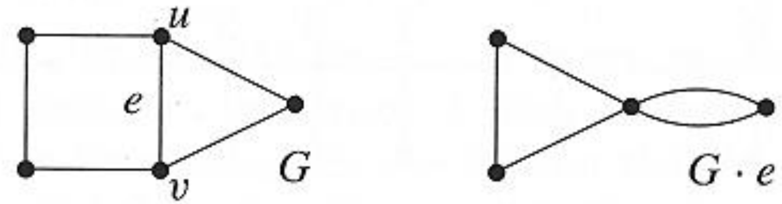
Def:

In a graph G, contraction of edge e with endpoints u, v is the replacement of u and v with a single vertex whose incident edges are the edges other than e that were incident to u or v.



Proposition. Let $\tau(G)$ denote the number of spanning trees of a graph G. If $e \in E(G)$ is not a loop, then

$\tau(G) = \tau(G - e) + \tau(G \cdot e)$.

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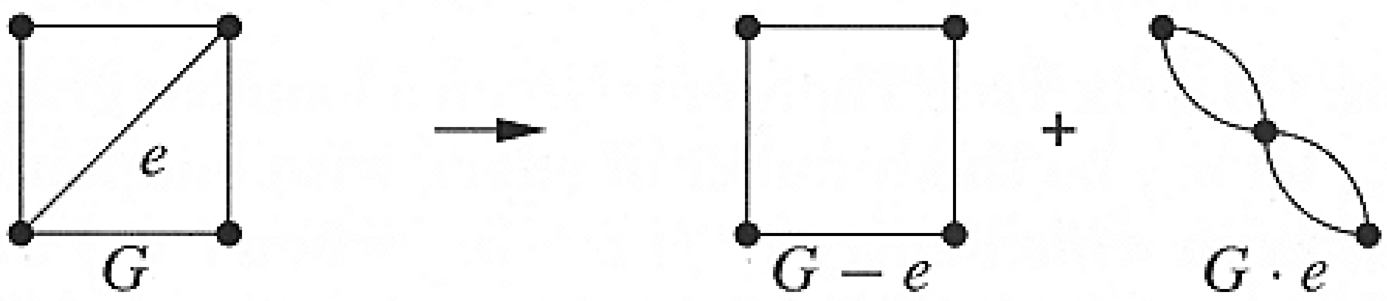
- Eg.



8=4+4

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- We cannot apply Proposition 2.2.8 when *e* is a loop!
- Eg. A graph consisting of one vertex and one loop.

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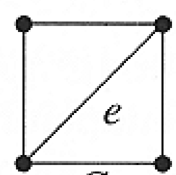
2.2.12 Theorem. (Matrix Tree Theorem) Given a loopless graph G with vertex set $v_1,\ldots,v_n$, let $a_{i,j}$ be the number of edges with endpoints v_i and v_j. Let Q be the matrix in which entry (i, j) is $-a_{i,j}$ when $i \neq j$ and is $d(v_i)$ when $i = j$. If Q^* is a matrix obtained by deleting row s and column t of Q, then $\tau(G) = (-1)^{s+t} \det Q^*$.



G

$$\begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \rightarrow 8$$

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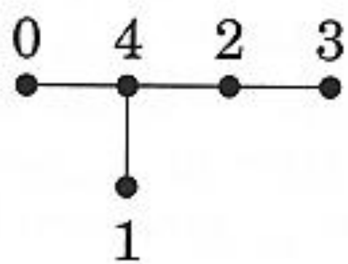
Def:

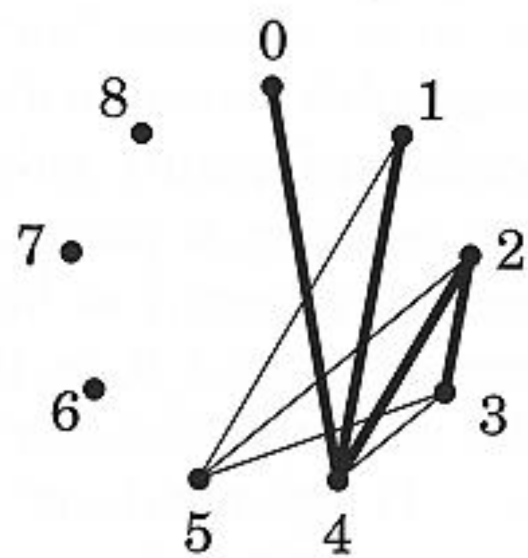
- A **graceful labeling** of a graph G with m edges is a function f : $V(G)\rightarrow\{0,\ldots, m\}$ such that distinct vertices receive distinct numbers and $\{|f(u)-f(v)| : uv \in E(G)\}=\{1,\ldots,m\}$

- A graph is **graceful** if it has a graceful labeling.

Theorem. If a tree T with m edges has a graceful labeling, then K_{2m+1} has a decomposition into $2m+1$ copies of T.

Idea:





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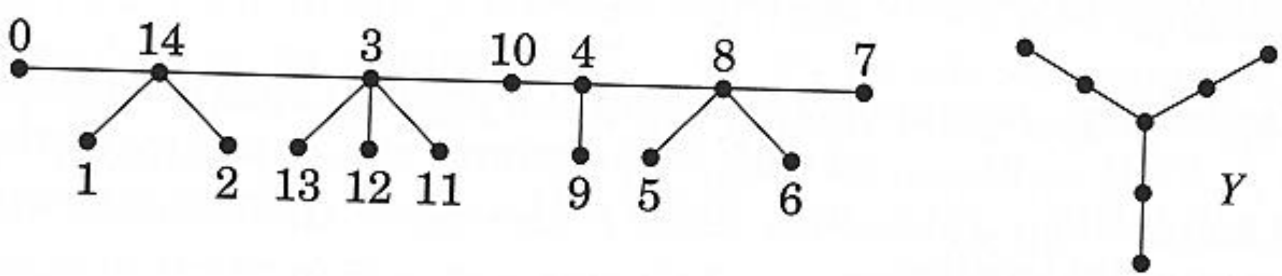
Def:

A *caterpillar* is a tree in which a simple path (the *spine*) is incident to (or contain) every edge.



Theorem. A tree is a caterpillar if and only if it does not contain the tree *Y* above.

Proof:

G' : obtained from G by deleting the leaves of G

G' has a vertex of degree at least 3 iff Y appears in G

$\therefore \Delta(G') \leq 2 \Leftrightarrow G'$ is a path.

Branching and Eulerian Digraphs

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- A **branching** or **out-tree** is an orientation of a tree having a root of indegree 0 and all other vertices of indegree 1.
- An **in-tree** is an out-tree with edges reversed.

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Lemma. In a strong digraph, every vertex is the root of an out-tree (and an in-tree).

Proof: Consider a vertex v.

- Iteratively add edges to grow a branching from v.
- Let S_i=set of vertices reached i edges have been added. Initially, $S_0=\{v\}$.
- Since the digraph is strong $\Rightarrow \exists$ an edge leaving S_i add one such edge to the branching and add its head to S_i to obtain $S_{i+1} \Rightarrow$ until reaching all vertices
- Reverse all edges and apply the same procedure to obtain an in-tree

2.2.24 Algorithm (Eulerian circuit in directed graph)

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- Input: An Eulerian digraph G without isolated vertices and a spanning in-tree T consisting of paths to v
  - **S1**: For each $u \in V(G)$, specify an ordering of the edges that leave u, s.t. for $u \neq v$ the edges leaving u in T comes last.
  - **S2**: Beginning at v, construct an Eulerian circuit by always exiting the current vertex u along the next unused edge in the ordering specified at u

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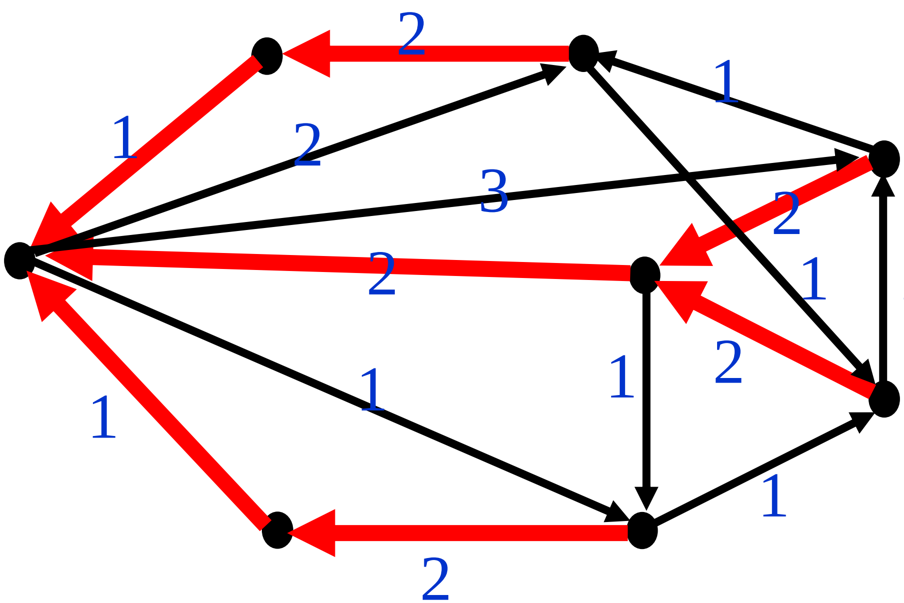

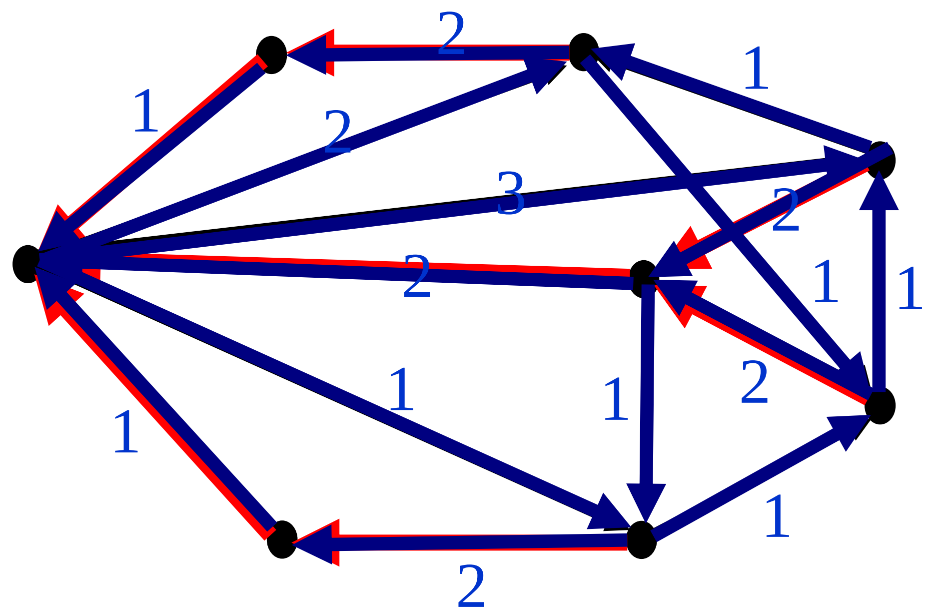

Theorem. Algorithm 2.2.24 always produces an Eulerian circuit.

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Proof:

- Show the trail can end only at v after traversing all edges
- When we enter $u \neq v$, the edge leaving u in T has not yet been used ($d^+(u)=d^-(u)$) $\Rightarrow$ the trail can only end at v
- When we cannot continue $\Rightarrow$ stop at v and have used all exiting edges $\Rightarrow$ also used all edges entering v
- We cannot use an edge of T until it is only remaining edge leaving its tail $\Rightarrow$ cannot use all edges entering v until we have finished all the other vertices

  (T contains a path from each vertex to v)

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- Definition: Two Eulerian circuits are the same if the successive pairs of edges are the same.
- Theorem (van Aardenne-Ehrenfest and de Bruijn [1951]).

  In an Eulerian digraph with $d_i = d^+(v_i) = d^-(v_i)$ the number of Eulerian circuits is $c\Pi\,(d_i - 1)!$, where c counts the in-trees to or out-trees from any vertex.

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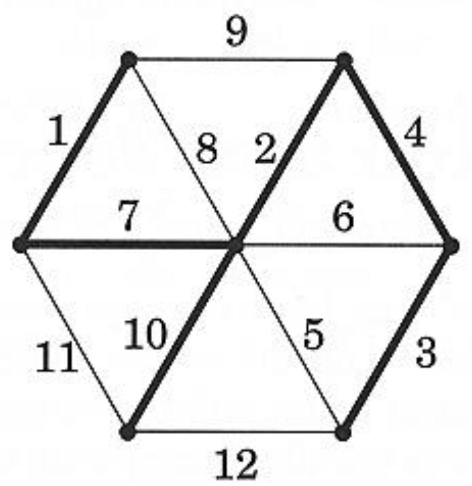
2.3.1.algorithm.(Kruskal's Algorithm – for minimum spanning trees.)

Input: A weighted connected graph.

- Idea: Maintain an acyclic spanning subgraph H, enlarging it by edges with low weight to form a spanning tree. Consider edges in nondecreasing order of weight, breaking ties arbitrarily.
- Initialization: Set $E(H)=\varnothing$.
- Iteration: If the next cheapest edge joins two components of H, then include it; otherwise, discard it. Terminate when H is connected.



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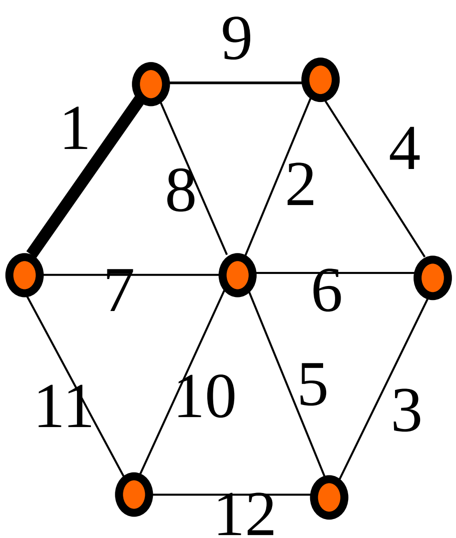

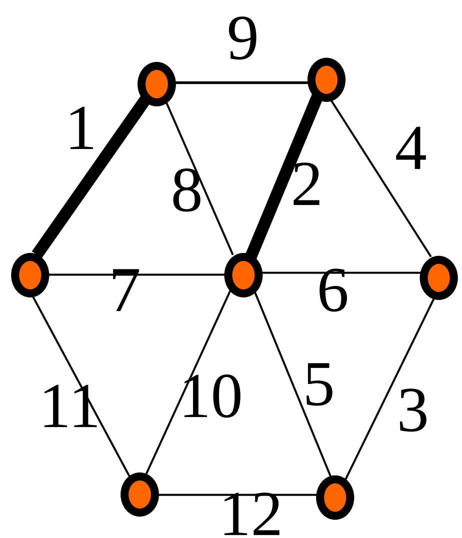

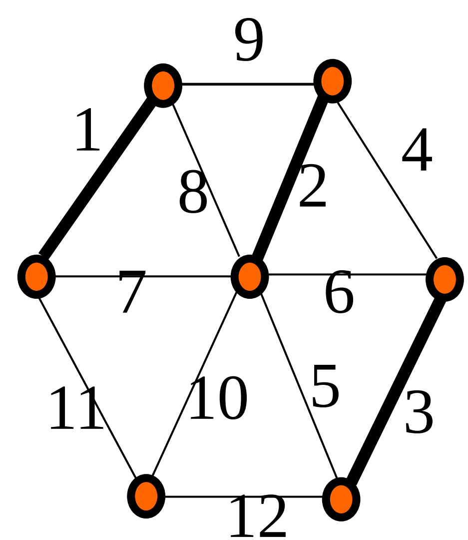

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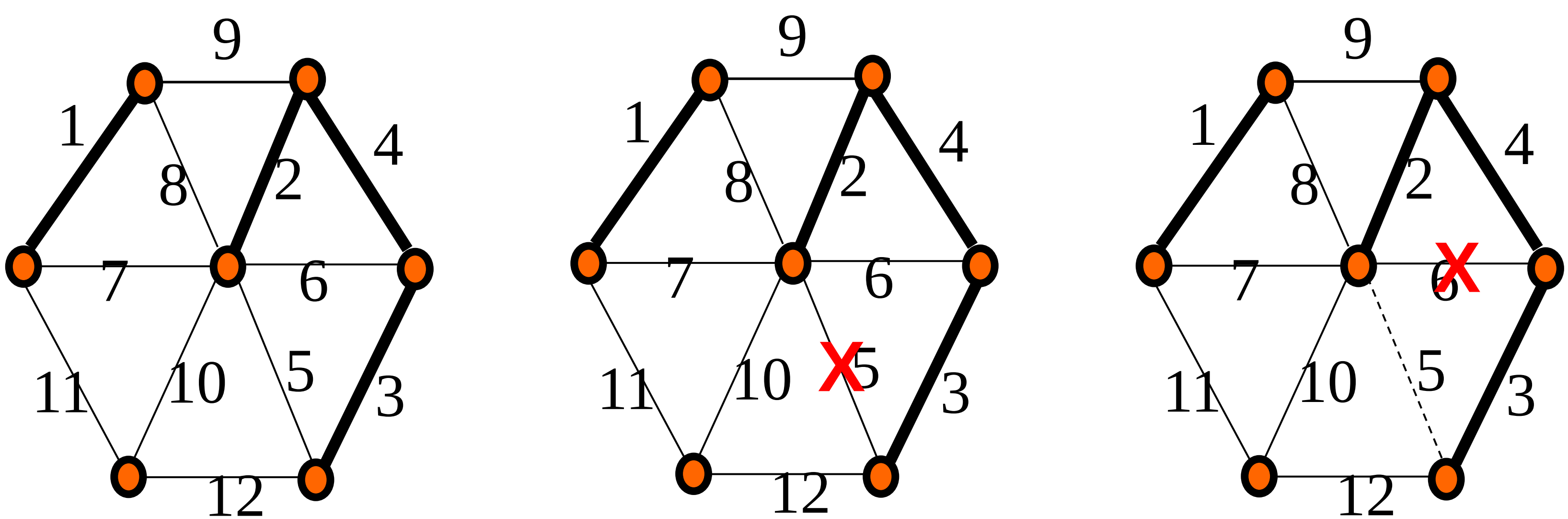

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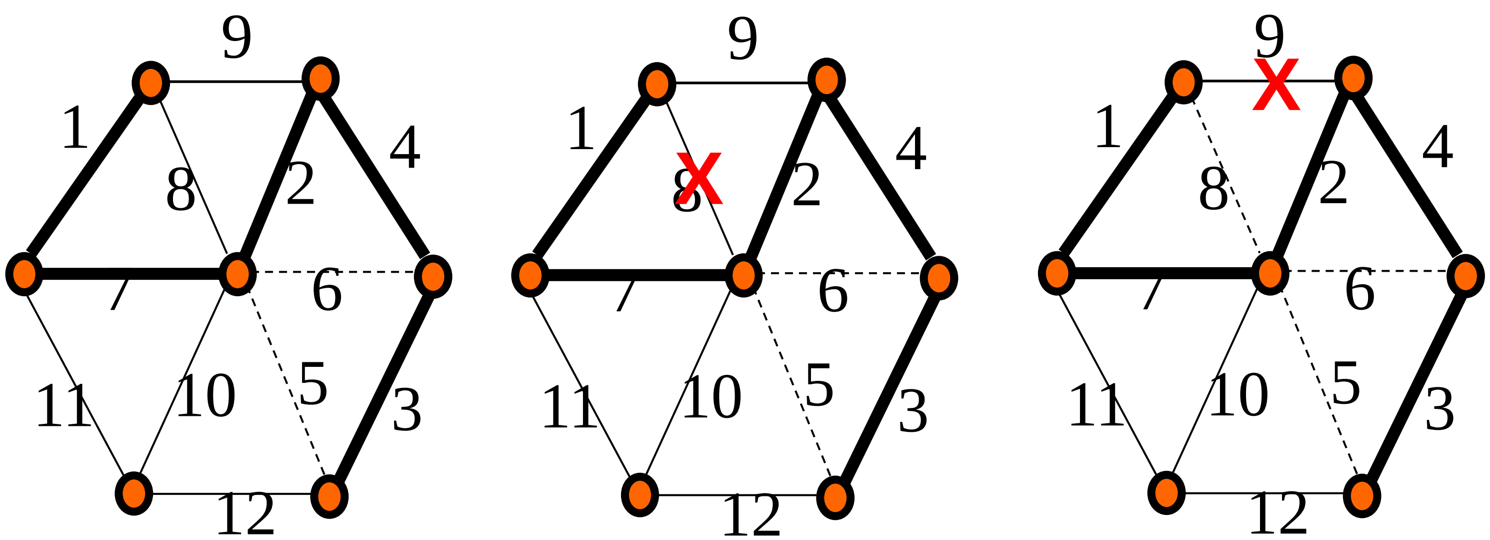

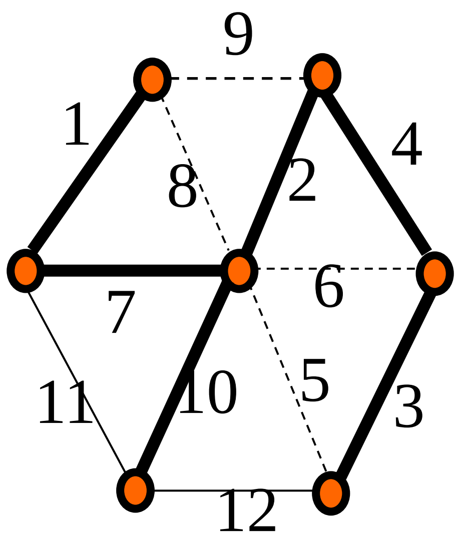
9
1
4
8
2
7
6
11
10
5
3
12

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Theorem. In a connected weighted graph G, Kruskal's Algorithm constructs a minimum-weight spanning tree.

Pf:

- The algorithm produces a connected and acyclic graph.
- T: the resulting tree produced by the algorithm
- T^*: a spanning tree of minimum weight
  - If $T=T^*$, we are done.
  - If $T\neq T^*$, let e be the first edge chosen for T that is not in T^*
    - T^*+e creates a cycle C and C has an edge $e' \notin E(T)$
    - $w(e)\leq w(e')$
    - $T^* +e- e'$ is a spanning tree with weight at most T^*
- Repeat the above yields a minimum-weight spanning tree that agrees completely with T.

Dijkstra's Algorithm-distances from one vertex

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- Input: A graph (or digraph) with nonnegative edge weights and a starting vertex u. The weight of edge xy is $w(xy)$; let $w(xy) = \infty$ if xy is not an edge.
- Initialization: Set $S = \{u\}$; $t(u) = 0$; $t(z) = w(uz)$ for $z \neq u$.
- Iteration:
  - Select a vertex v outside S such that $t(v) = \min_{z \notin S} t(z)$.
  - Add v to S.
  - Explore edges from v to update tentative distance: for each edge vz with $z \notin S$, update $t(z)$ to $\min\{t(z), t(v) + w(vz)\}$.
- The iteration continues until $S=V(G)$ or until $t(z)=\infty$ for every $z \notin S$. At the end, set $d(u, v) = t(v)$ for all v.

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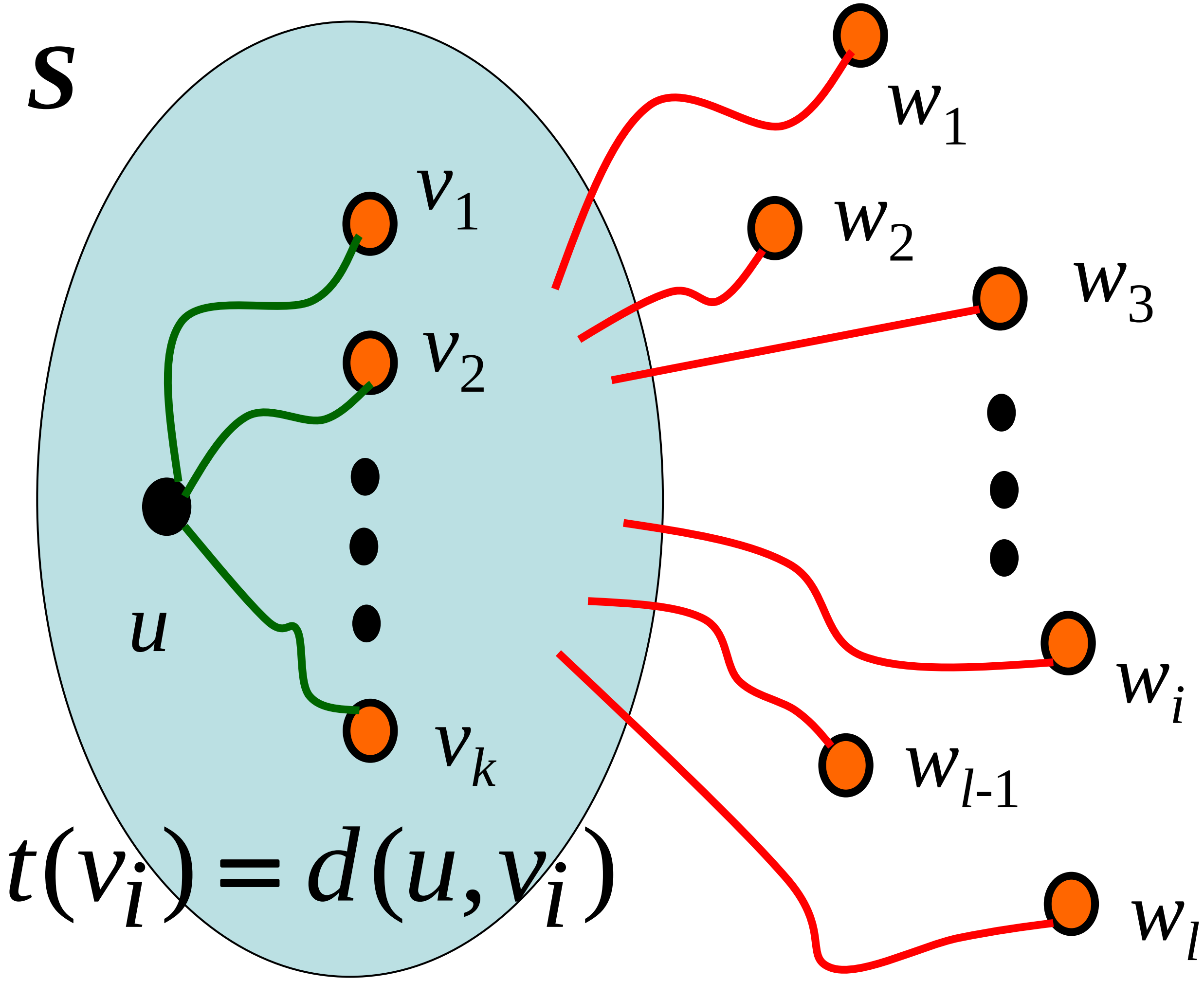
- $t(w_i)$ is the least length of a u,w_i-path reaching w_i from u



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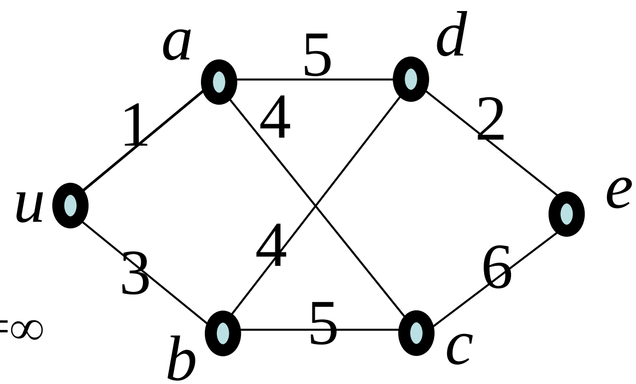
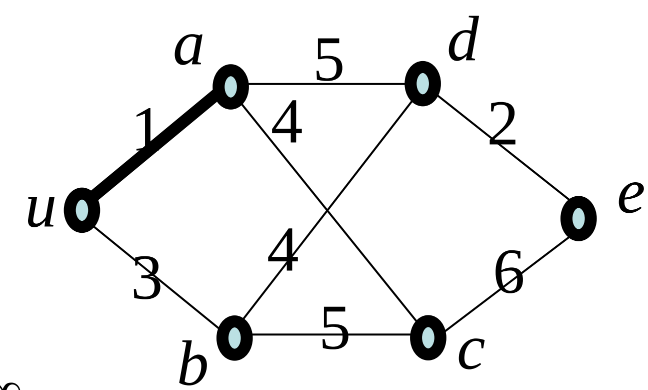
$S=\{u\}$
$t(u)=0$
$t(a)=1$
$t(b)=3$
$t(c)=t(d)=t(e)=\infty$



$S=\{u,a\}$
$t(u)=0$
$t(a)=1$
$t(b)=3$
$t(c)=5$
$t(d)=6 \quad t(e)=\infty$



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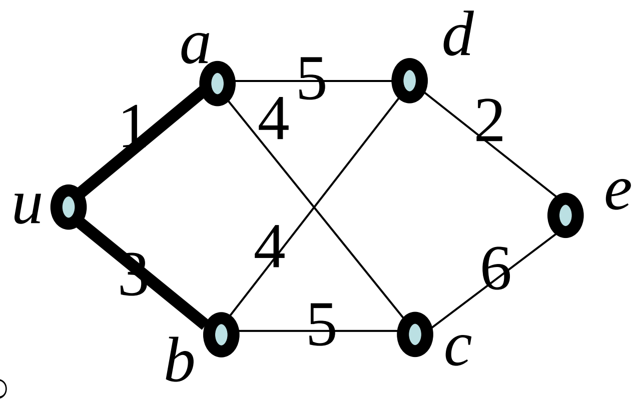
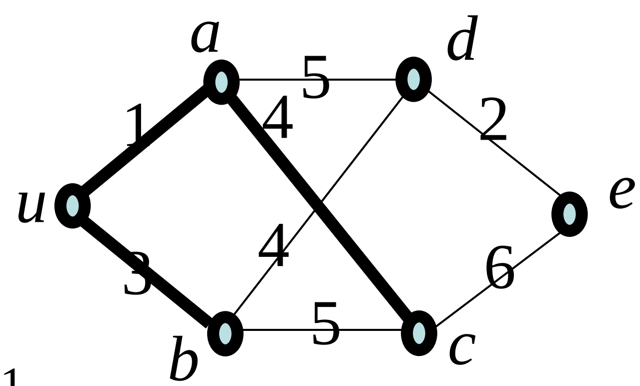
$S=\{u,a,b\}$
$t(u)=0$
$t(a)=1$
$t(b)=3$
$t(c)=5$
$t(d)=6$ $t(e)=\infty$



$S=\{u,a,b,c\}$
$t(u)=0$
$t(a)=1$
$t(b)=3$
$t(c)=5$
$t(d)=6$ $t(e)=11$



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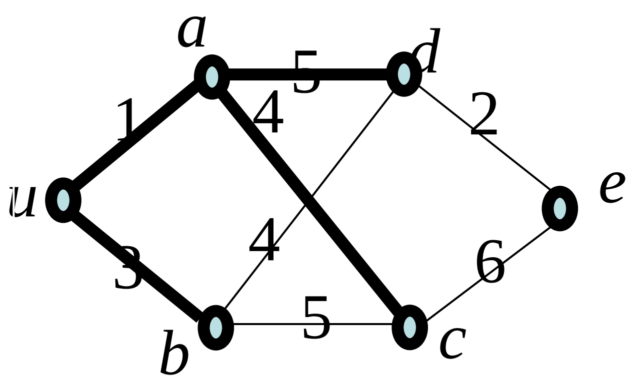
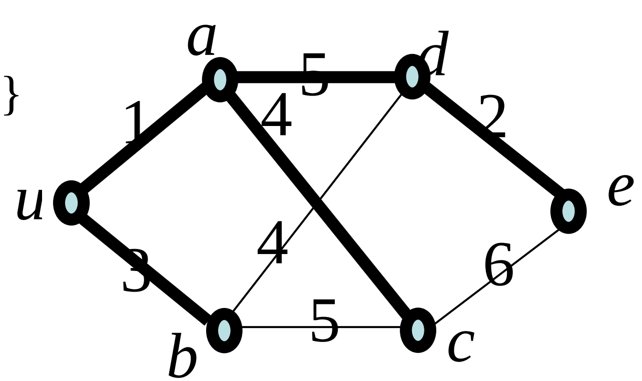
$S=\{u,a,b,c,d\}$
$t(u)=0$
$t(a)=1$
$t(b)=3$
$t(c)=5$
$t(d)=6$ $t(e)=8$



$S=\{u,a,b,c,d,e\}$
$t(u)=0$
$t(a)=1$
$t(b)=3$
$t(c)=5$
$t(d)=6$ $t(e)=8$



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Theorem. Given a (di)graph G and a vertex Dijkstra's Algorithm computes $d(u,z)$ for every $z \in V(G)$

Pf:

By induction on $k=|S|$ to show

(1) For $z \in S$, $t(z) = d(u, z)$,

(2) For $z \notin S$, $t(z)$ is the least length of a u,z-path reaching z from S

- Basis on k=1.
- I.S.: Suppose that $|S|=k$, (1) and (2) are true.

  v: a vertex among $z \notin S$ such that $t(z)$ is smallest.

Let $S' = S \cup \{v\}$

- A shortest u,v-path must exit S before reaching v.
- By I.H., the length of the shortest path from S to v is $t(v)$.
- By I.H. and the choice of v, a path visiting any vertex outside S and later to v has length at least $t(v) \Rightarrow d(u, v) = t(v)$

To show (2):

z: a vertex outside *S* other then *v*

- By I.H., the shortest *u*,*z*-path reaching *z* directly from *S* has length $t(z)$;
- When *v* is added, the desired value for *z* is $\min\{t(z),\mathbf{t(v)+w(vz)}\}$

2.3.8 Algorithm (Breadth-First Search—BFS)

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- Input: An unweighted graph (or digraph) and a start vertex u.
- Idea: Maintain a set R of vertices that have been reached but not searched and a set S of vertices that have been searched. The set R is maintained as a First-In First-Out list (queue), so the first vertices found are the first vertices explored.
- Initialization: $R = \{u\}$, $S = \varnothing$, $d(u, u)=0$.
- Iteration: As long as $R \neq \varnothing$, we search from the first vertex v of R. The neighbors of v not in $S \cup R$ are added to the back of R and assigned distance $d(u, v)+1$, and then v is removed from the front R and placed in S.

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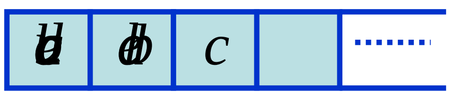
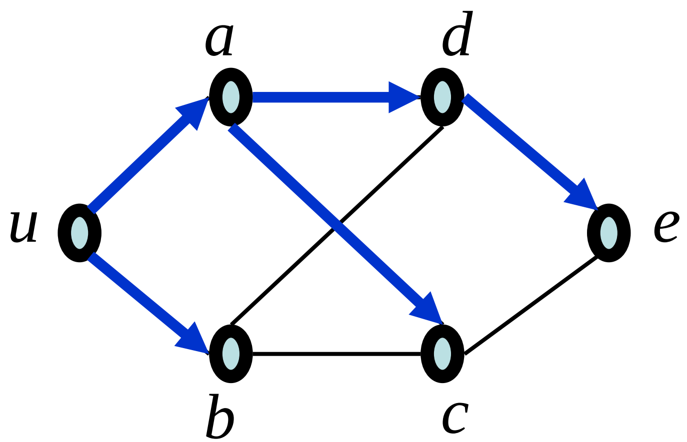
R:





$S = \{u, a, b, d, c, e\}$

$d(u, e) = 3$

$d(u, b) = 1$

Chinese Postman Problem:

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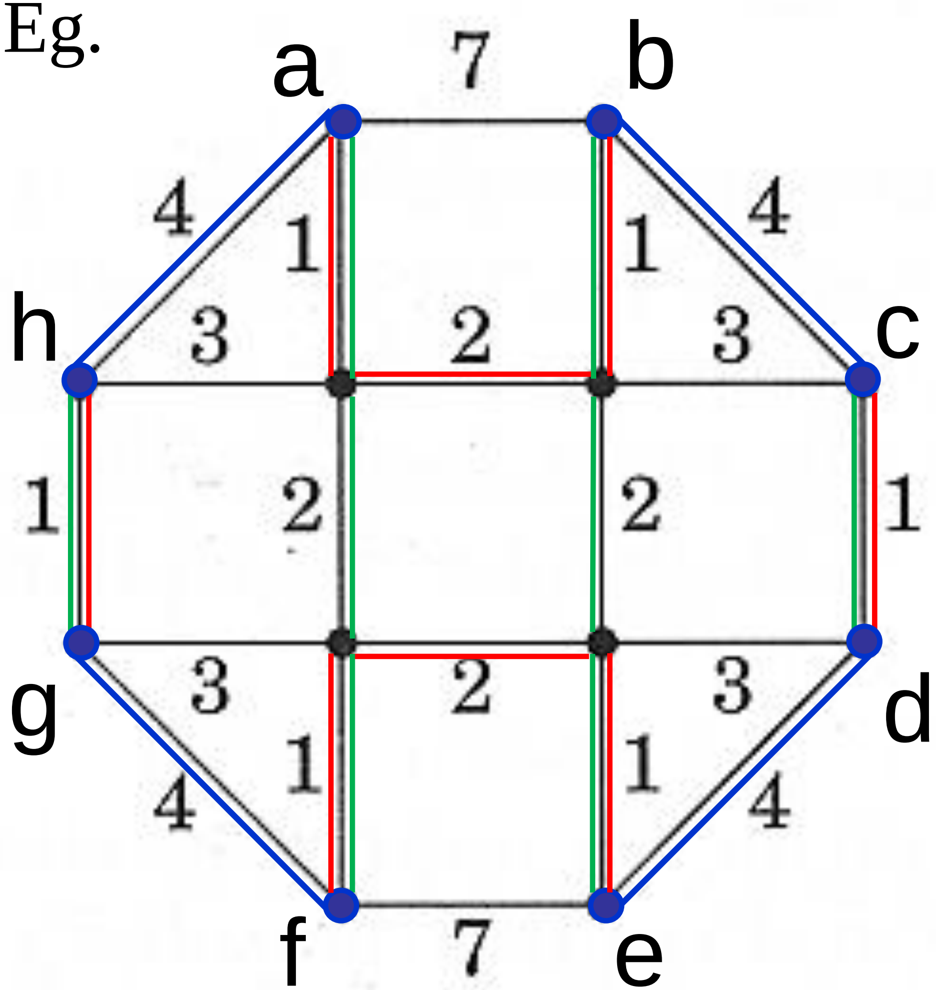
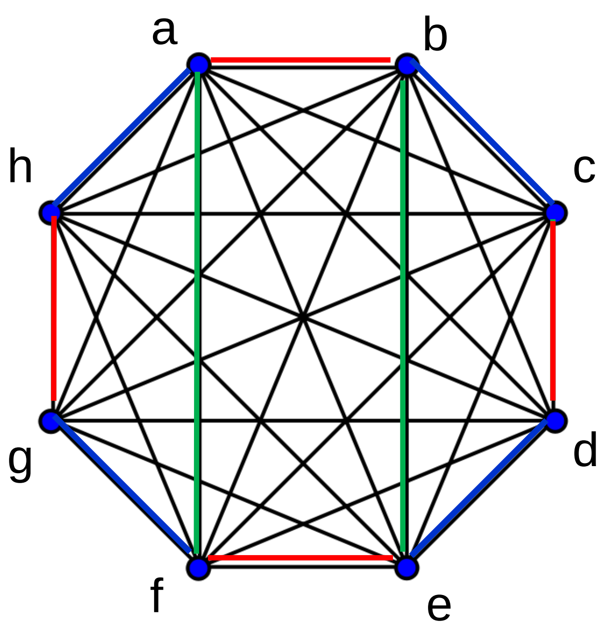
- A mail carrier must traverse all edges in a road network, starting and ending at the Post Office.
- The edges have nonnegative weights representing distance or time.
- We seek a closed walk of minimum total length that uses all the edges.
- Key:
  - If every vertex is even, then the graph is Eulerian and the answer is the sum of the edge weights.
  - Otherwise, we must repeat edges.

- Eg.





4+4+4+4=16

1+1+2+1+1+2+1+1=10

1+2+1+1+2+1+1+1=10

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- If there are only two odd vertices, then use Dijkstra's algorithm to find the shortest path between them.

- If there are $2k$ odd vertices, then
  - Use Dijkstra's algorithm to find the shortest paths connecting each pair of odd vertices;
  - Use these lengths as weights on the edges of K_{2k};
  - Solve the weighted version of the maximum matching problem on K_{2k}.