Course Information

TEXT: INTRODUCTION TO GRAPH THEORY
(BY DOUGLAS B. WEST)
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GRADING SCHEME:

- Min-term exam: 30%
- Homework: 20%
- Quiz: 20%
- Final exam: 30%

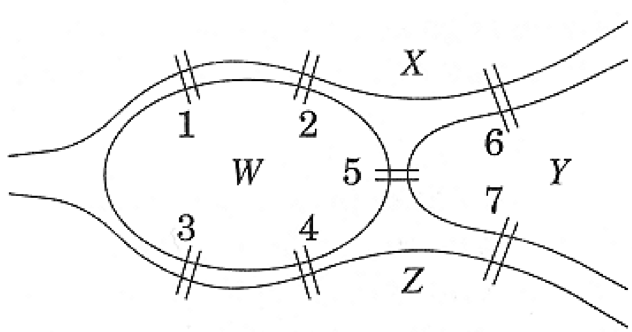
Chap 1 Fundamental Concepts

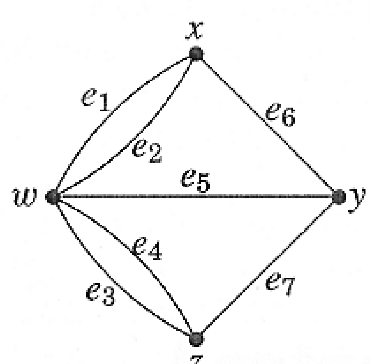
1-1

Def:

A *graph G* is a triple consisting of a vertex set *V*(*G*), an *edge set* *E*(*G*), and a relation that associates with each edge two vertices (not necessarily distinct) called its *endpoints*.

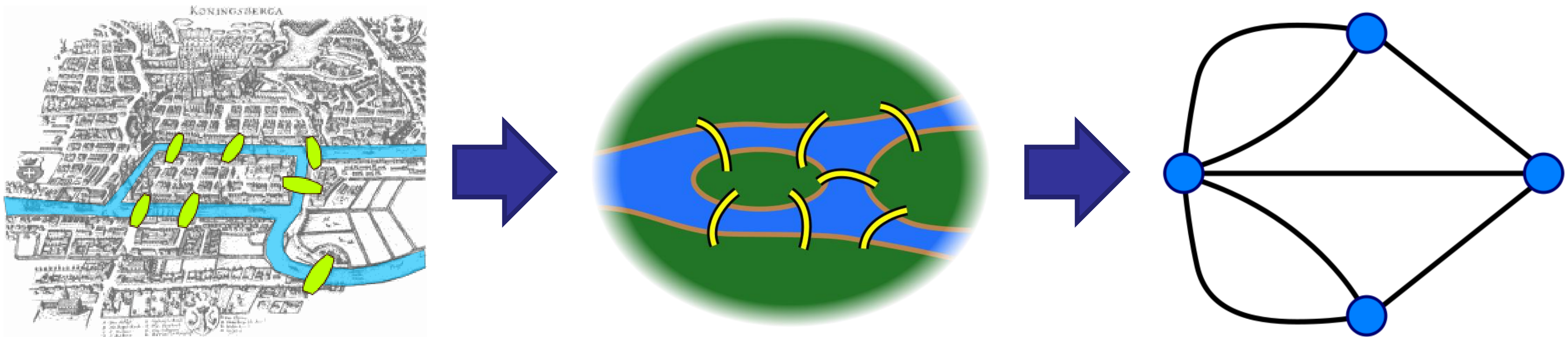
e.g. *G*=({*w*, *x*, *y*, *z*},{e1, e2, e3, e4, e5, e6,e7}, {(w,x), (w,x), (w,z), (w,z), (w,y), (x,y), (y,z)})





柯尼斯堡-七橋問題(The Königsberg Bridge Problem)

- 在東普魯士的**柯尼斯堡**(今俄羅斯的加里寧格勒)市區有一條名為普列戈利亞的河(Pregel river)，河的中心有兩座小島，小島與河的兩岸共有七條



維基百科 – 柯尼斯堡七橋問題

- 1735年萊昂哈德·歐拉證明了在所有橋只能走一遍的前提下，把這七條橋都走遍的方法並不存在，成為圖論史上第一篇重要的文獻。
- **判定法則:**
  - 如果通奇數座橋的地方不止兩個，那麼滿足要求的路線便不存在了。
  - 如果只有兩個地方通奇數座橋，則可從任何一通奇數橋的地方出發，到另一通奇數橋的地方結束，找到所要求的路線。
  - 若沒有一個地方通奇數座橋，則從任何一地出發，所求的路線都能實現。

Def:

☞A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints.

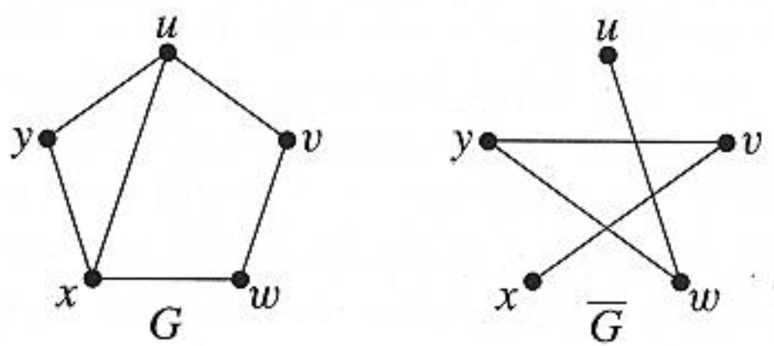
☞A *simple graph* is a graph having no loops or multiple edges.

☞When u and v are the endpoints of an edge, they are *adjacent* and are *neighbors* ($u \leftrightarrow v$).

☞ *finite graph*: its vertex set and edge set are finite;
null graph: its vertex set and edge set are empty.

Def:

☞ *complement* $\overline{G}$ *of a simple graph G*: the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G}) \Leftrightarrow uv \notin E(G)$.

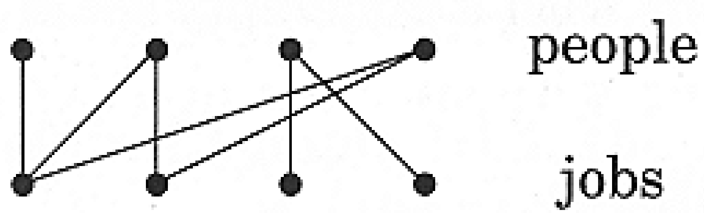


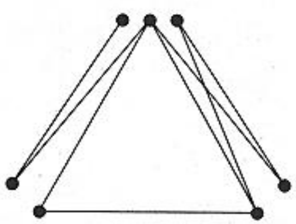
☞ *Clique* in a graph: a set of pairwise adjacent vertices.

☞ *Independent set* (or *stable set*) in a graph: a set of pairwise nonadjacent vertices.

Def:

☞ A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possible empty) independent sets called *partite sets* of G.
p.s. k-partite



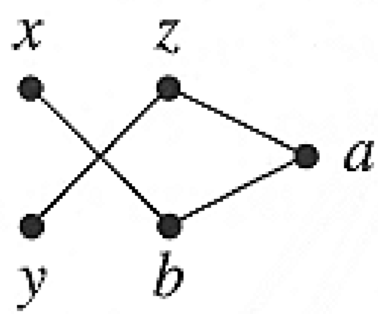


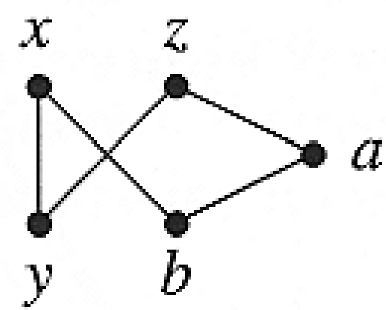
☞ *Chromatic number* of a graph G, $\chi(G)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors.

Def:

☞A *path* $\langle v_0, v_1, \ldots, v_k \rangle$ is a sequence of distinct vertices such that any two consecutive vertices are adjacent; we call v_0 and v_k the end-vertices of the path.

☞A *cycle* $\langle v_0, v_1, \ldots, v_k, v_0 \rangle$ is a sequence of vertices, where $v_0, v_1, \ldots, v_k$ are all distinct such that any two consecutive vertices are adjacent.





Def:

☞A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G.

Notation: $H \subseteq G$, "G contains H".

☞A graph G is *connected* if each pair of vertices in G belongs to a path; otherwise, G is *disconnected*.

Def:

G: loopless graph with $V(G)=\{v_1,v_2,\ldots,v_n\}$ and $E(G)=\{e_1,e_2,\ldots,e_m\}$.

Adjacency matrix $A(G)$ *of* G: n-by-n matrix in which $a_{i,j}$ is the number of edges in G with endpoints $\{v_i,v_j\}$.

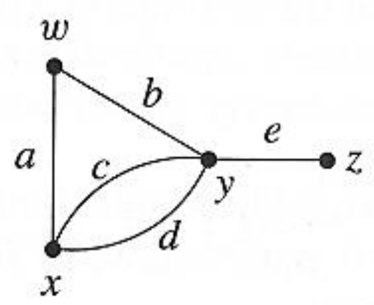
Incidence matrix $M(G)$: n-by-m matrix in which entry $m_{i,j}$ is 1 if v_i is an endpoint of edge e_j and otherwise is 0.

Incident:

Degree:

$$\begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$A(G)$



G

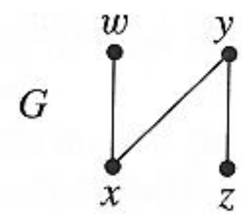
$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

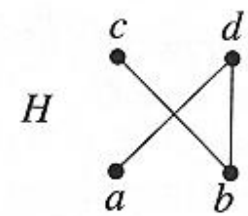
$M(G)$

Def:

- An *isomorphism* from a simple graph G to a simple graph H is a bijection $f: V(G) \to V(H)$ such that $uv \in E(G)$ iff $f(u)f(v) \in E(H)$

- *G is isomorphic to H*, $G \cong H$, if there is an isomorphism from G to H.



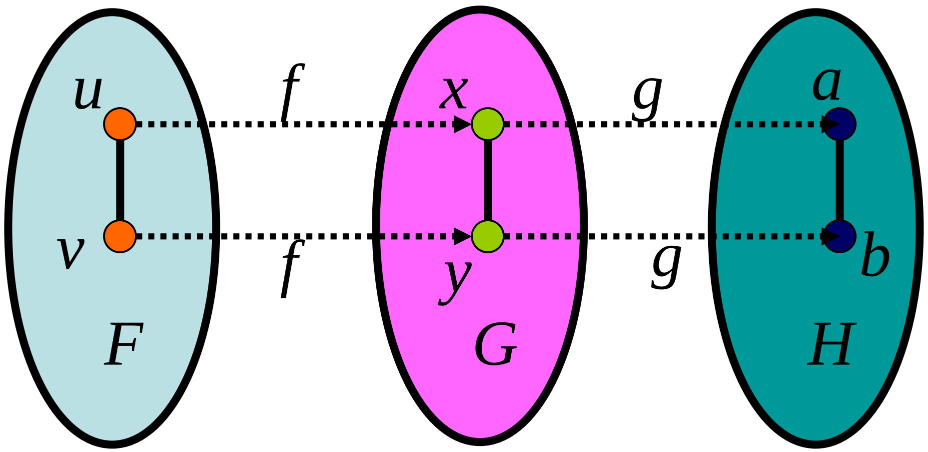


$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ w \\ y \\ z \\ x \end{array}\begin{array}{c} \begin{array}{cccc} w & y & z & x \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Proposition:
The isomorphism relation is an equivalence relation on the set of (simple) graphs.

Pf:
Reflexive. Clearly, $G \cong G$.
Symmetric. Since $uv \in E(G) \Leftrightarrow f(u)f(v) \in E(H)$ yields
$xy \in E(H) \Leftrightarrow f^{-1}(x)f^{-1}(y) \in E(G)$,
$G \cong H \Rightarrow H \cong G$.

Transitive. Let $f:V(F)\rightarrow V(G)$ and $g:V(G)\rightarrow V(H)$ be two isomorphism. Then, $g\circ f$ is an isomorphism from F to H

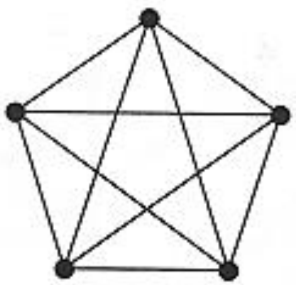


Def:

☞ A *complete* graph is a simple graph whose vertices are pairwise adjacent;
K_n: the (unlabeled) complete graph with n vertices.

☞ A complete bipartite graph $K_{r,s}$ or biclique is a simple bipartite graph such that two vertices are adjacent iff they are in different partite sets.

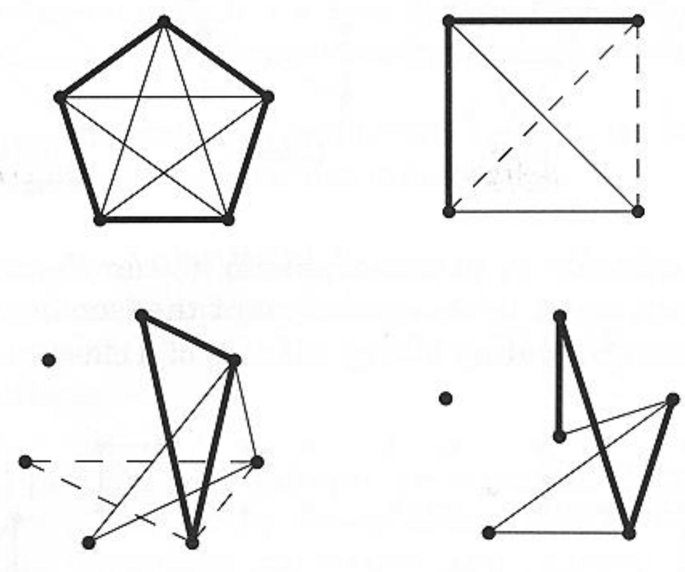
K_5



$K_{2,3}$

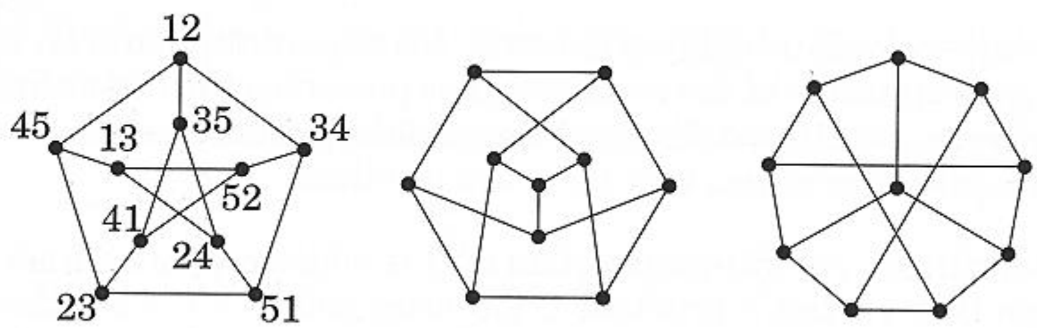
Def:

- ☞ A graph is *self-complementary* if it is isomorphic to its complement.
- ☞ A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.
- ☞ An *n*-vertex graph *H* is self-complementary iff K_n has a decomposition consisting of two copies of *H*.



Def:

- The *Petersen graph*: the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.



- The *girth* of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.

Proposition.

If two vertices are nonadjacent in Petersen graph, then they have exactly one common neighbor.

Proof.

- Nonadjacent vertices are 2-sets sharing one element; their union S has 3.
- A vertex adjacent to both is a 2-set disjoint from both.
- Since the 2-sets are chosen from {1, 2, 3, 4, 5}, there exactly one 2-set disjoint from S.

Corollary.

The Petersen graph has girth 5.

Proof.

- The graph is simple, it has no 1-cycle or 2-cycle.
- A 3-cycle require three pairwise-disjoint 2-sets, which can not occur among 5 elements.
- A 4-cycle in the absence of 3-cycles would require nonadjacent vertices with two common neighbors, which Proposition forbids.
- The vertices 12, 34, 51, 23, 45 yield a 5-cycle, so the girth is 5.

Def:

☞ An *automorphism* of G is an isomorphism from G to G.

☞ A graph G is *vertex-transitive* if for every pair $u,v \in V(G)$ there is an automorphism that maps u to v.

e.g.

G: the path with $V(G)=\{1,2,3,4\}$ and $E(G)=\{12, 23, 34\}$.

G has two automorphisms: the identity permutation and the permutation $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$.

e.g.

$K_{r,s}$ has $r!s!$ automorphisms when $r \neq s$.

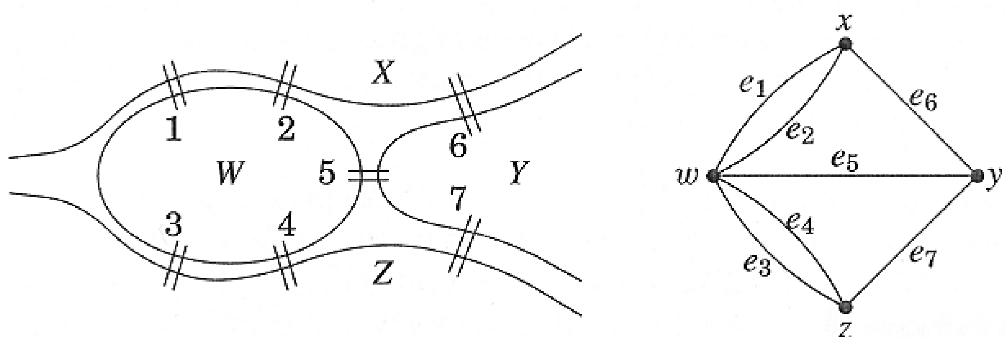
$2(r!)^2$ when $r=s$.

1-2

Def:

☞ A *walk* is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that, for $1 \le i \le k$, the edges e_i has endpoints $v_{i\text{-}1}$ and v_i.

☞ A *trail* is a walk with no repeated edge.



e.g. The list $x, e_2, w, e_5, y, e_6, x, e_1, w, e_2, x$ is a closed walk of length 5 (not a trail).

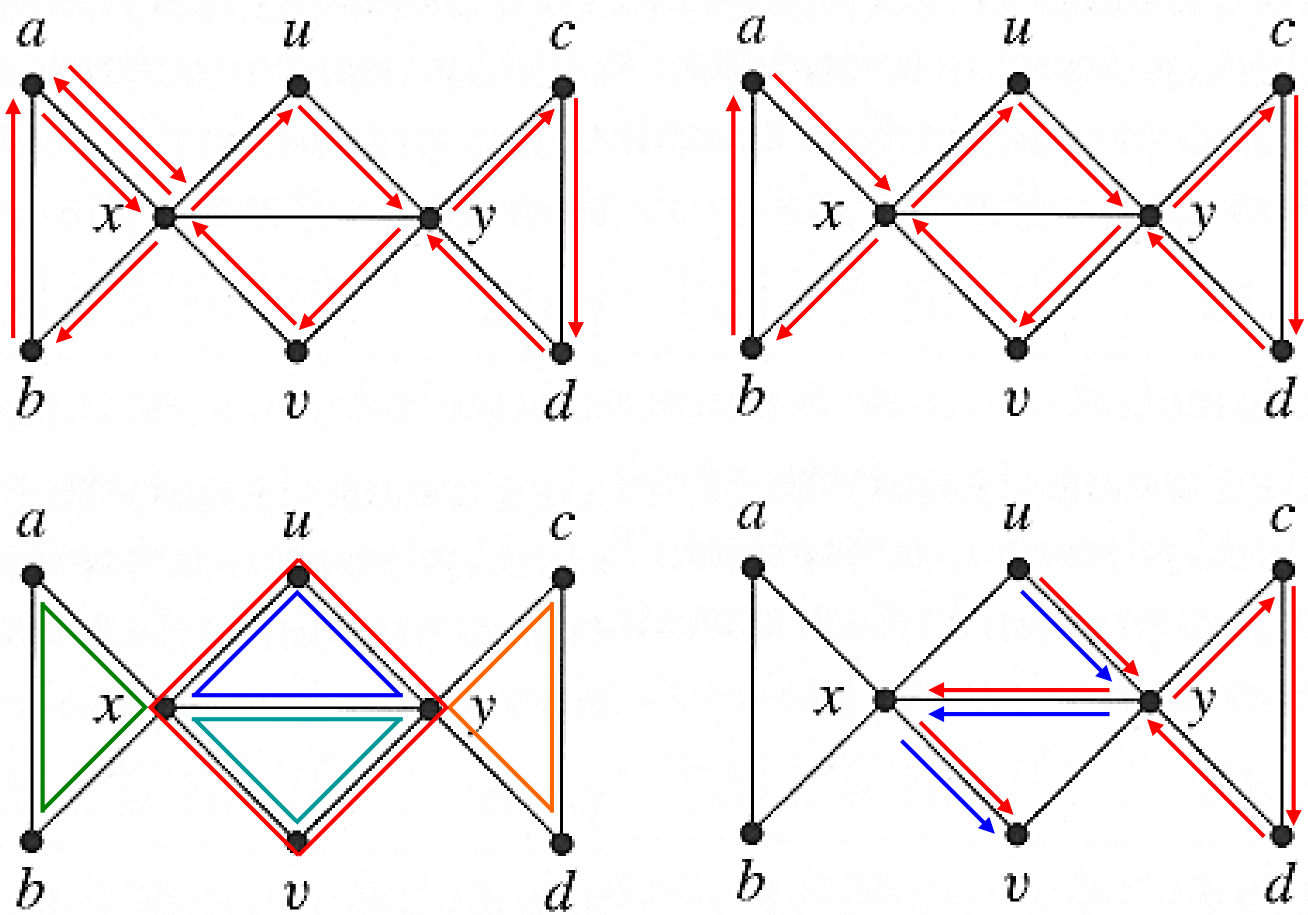
The list $x, e_2, w, e_5, y, e_6, x, e_1, w$ is a trail.

Def:

☞A *u,v-walk* or *u,v-trail* has first vertex *u* and last vertex *v* (endpoints).

☞A *u,v-path* is a path whose vertices of degree 1 are *u* and *v*; the others are *internal vertices*.

The *length* of a walk, trail, path, or cycle is its number of edges. A walk or trail is *closed* if its endpoints are the same.



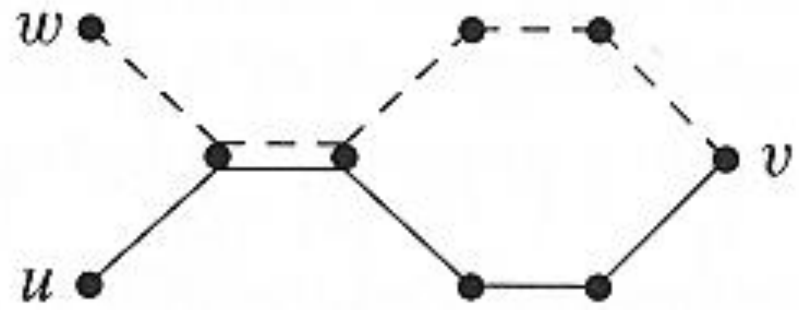
e.g. • closed walk: *a*, *x*, *a*, *x*, *u*, *y*, *c*, *d*, *y*, *v*, *x*, *b*, *a*

• closed trail: *a*, *x*, *u*, *y*, *c*, *d*, *y*, *v*, *x*, *b*, *a*

• cycles: (*a*, *b*, *x*, *a*), (*c*, *y*, *d*, *c*), (*u*, *x*, *y*, *u*), (*x*, *y*, *v*, *x*), (*u*, *x*, *v*, *y*, *u*)

• *u*, *v*-trail *u*, *y*, *c*, *d*, *y*, *x*, *v* contains *u*, *v*-path *u*, *y*, *x*, *v* but not *u*, *y*, *v*

E.g. *u*, *w*-walk contains a *u*, *w*-path.



Lemma. Every *u*, *v*-walk contains a *u*, *v*-path.

Proof: By induction on the length *l* of a *u*, *v*-walk *W*.

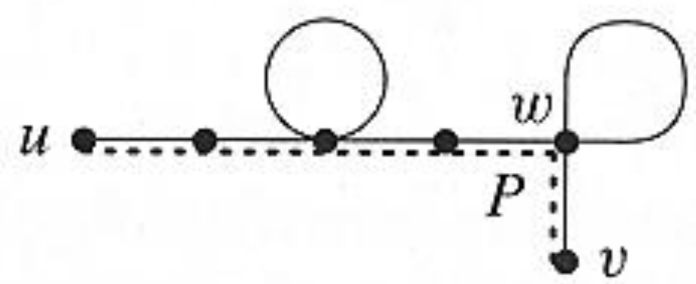
Basis step: *l*=0.

W contains a single vertex (a length-0 path).

Induction step: *l*≥1.

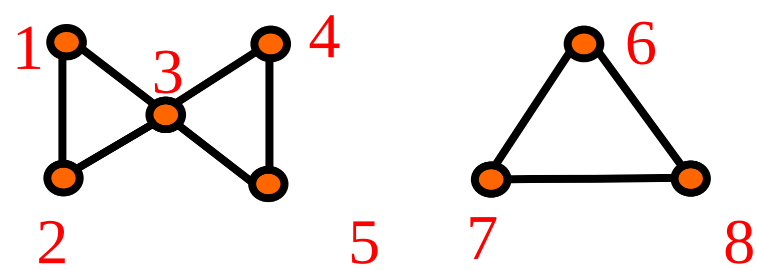
Case 1. No repeated vertex.

Case 2. *W* has a repeated vertex *w*.



Def:

- ☞A graph G is *connected* if it has a u, v-path whenever $u, v \in V(G)$ (otherwise, G is *disconnected*).
- ☞If G has a u,v-path, then u is *connected to v* in G.
- ☞The *connection relation* on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v.



Connection relation: {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7), (8,8),(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),(1,5),(5,1),(2,3),(3,2), (2,4),(4,2),(2,5),(5,2),(3,4),(4,3),(3,5),(5,3),(4,5),(5,4),(6,7), (7,6),(6,8),(8,6),(7,8),(8,7)}

Equivalence classes: {1,2,3,4,5}, {6,7,8}

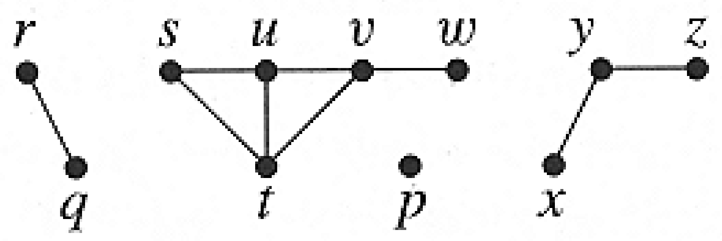
Def:

☞The *components* of a graph *G* are its *maximal connected subgraphs*
(a *maximal* connected subgraph of *G* is a subgraph that is connected and is not contained in any other subgarph).

☞A component is *trivial* if it has no edges.

☞An *isolated vertex* is a vertex of degree 0.

E.g. The equivalence classes of the connection relation on *V*(*G*) are the vertex sets of the components of *G*.



Proposition.
Every graph with n vertices and k edges has at least n-k components.

Proof:
An n-vertex graph with no edges has n components.
Add each edge $\Rightarrow$ reduce the number of components ≤ 1
Add k edges $\Rightarrow$ the number of resulting components $\geq n$-k.

Def:

☞ A *cut-edge* or *cut-vertex* of a graph is an edge or vertex whose deletion increases the number of components.

G-e (resp. G-v): delete an edge e (resp. a vertex v) from G

G-M (resp. G-S): delete a set of edges M (resp. a set of vertices S) from G

☞ The *induced subgraph* $G[T]$ consists of $T \subseteq V(G)$ and all edges whose endpoints are contained in T.

Theorem.

An edge is a cut-edge if and only if it belongs to no cycle.

Proof:

$e=(x, y)$: an edge

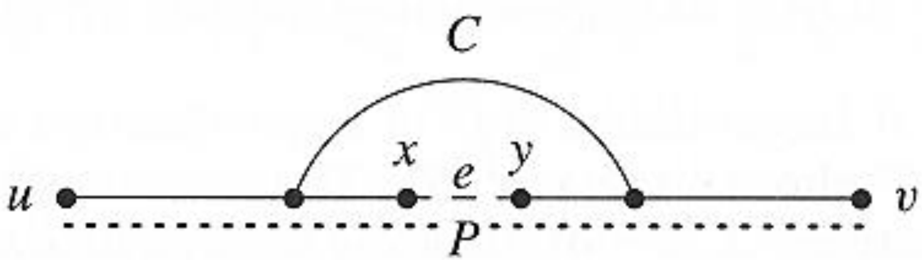
H: the component containing e

***H*-*e* is connected iff *e* belongs to a cycle**

⇒Suppose that H-e is connected. Then, H-e contains an x,y-path $P(x,y)$.

$P(x,y)+e$ is a cycle of G.

⇐



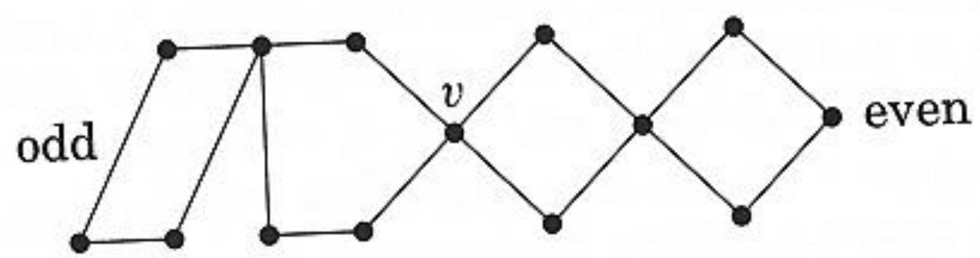
Lemma. *Every closed odd walk contains an odd cycle.*

Proof: By induction on the length l of a closed walk W.
Basis step: l=1 (loop).

Induction step: l>1.
Case 1: W has no repeated vertex (other than first=last).
W itself forms a desired cycle.

Case 2: W contains a repeated vertex v.

E.g.



Def:

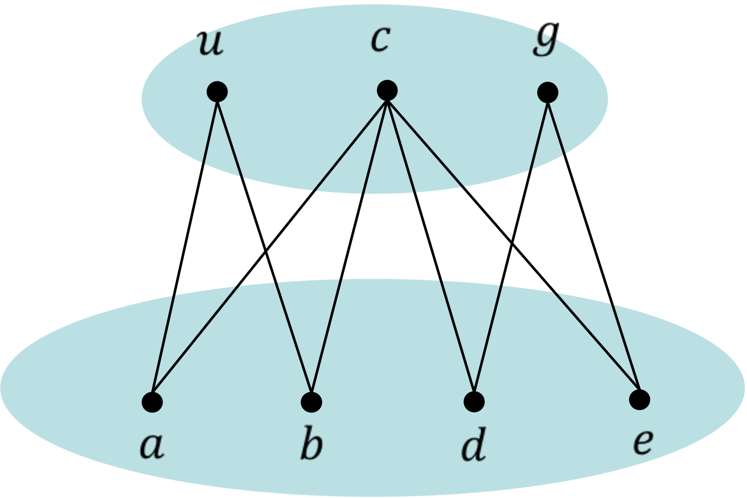
- ☞ A *bipartition* of G is a specification of two disjoint independent sets in G whose union is $V(G)$.
- ☞ An X, Y-bigraph is a bipartite graph with bipartition X, Y.

Theorem. *A graph is bipartite if and only if it has no odd cycle.*

Proof: *Necessity*.

Every walk alternates between two partite sets ⇒ return to the original partite set needs an even number of steps.

e.g.



Sufficiency. Let G be a graph with no odd cycle.

Construct a bipartition of each nontrivial component H:

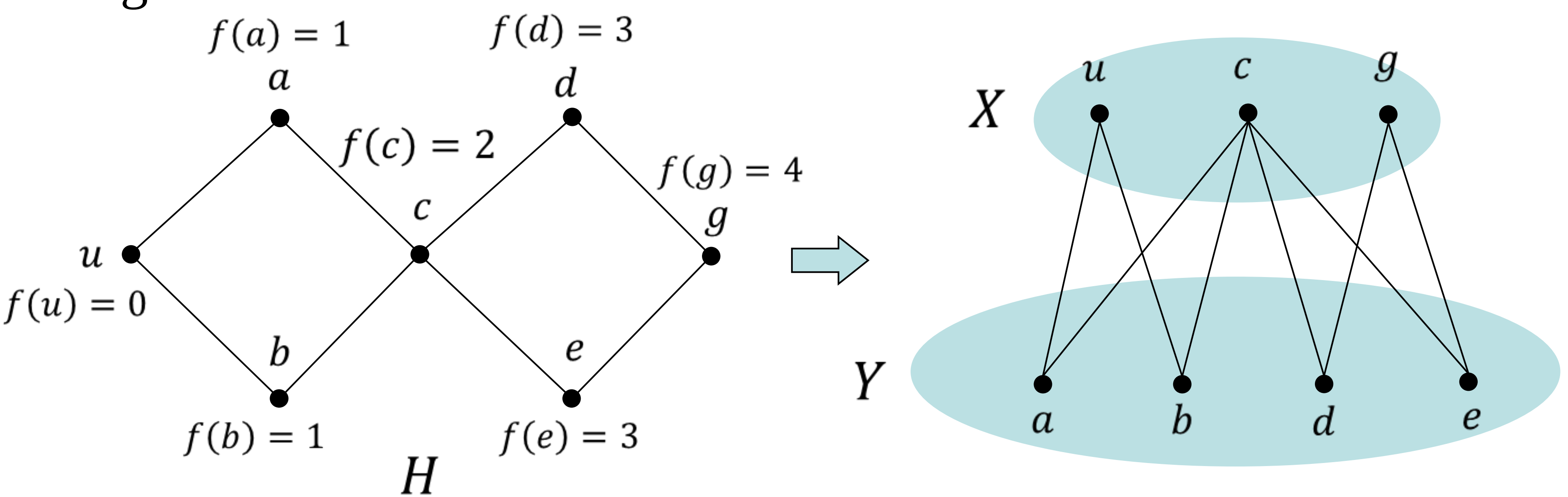
Let u be a vertex in H.

For each $v \in V(H)$, let $f(v)$=the minimum length of a u, v-path.

Let $X=\{v \in V(H): f(v) \text{ is even}\}$ and $Y=\{v \in V(H): f(v) \text{ is odd}\}$.

Then, H is an X,Y-bigraph.

e.g.



Def:

☞ A graph is *Eulerian* if it has a closed trail containing all edges.

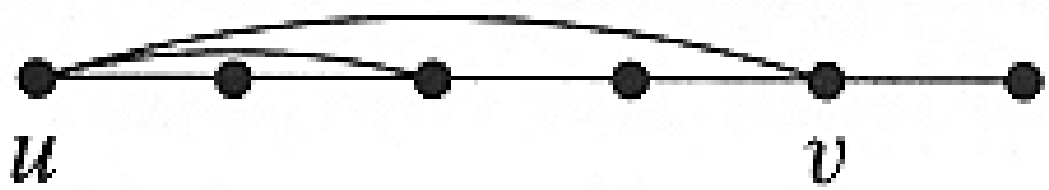
☞ A *circuit* is a closed trail when we do not specify the first vertex but keep the list in cyclic order.

☞ An *Eulerian circuit* or *Eulerian trail* in a graph is a circuit or trail containing all the edges.

Lemma. *If every vertex of a graph G has degree at least 2, then G contains a cycle*.

Proof:

P: a maximal path in *G*



Theorem. *A graph G is Eulerian if and only if it has at most one nontrivial component and its vertices all have even degree.*

Proof:

Necessity. Let C be an Eulerian circuit of G.

• Each passage of C through a vertex uses two incident edges, and the first edge is paired with the last at the first vertex.
$\Rightarrow$Every vertex has even degree.

• Two edges can be in the same trail only when they in the same component$\Rightarrow$ at most one nontrivial component.

Sufficiency. Prove by induction on the number of edges m.

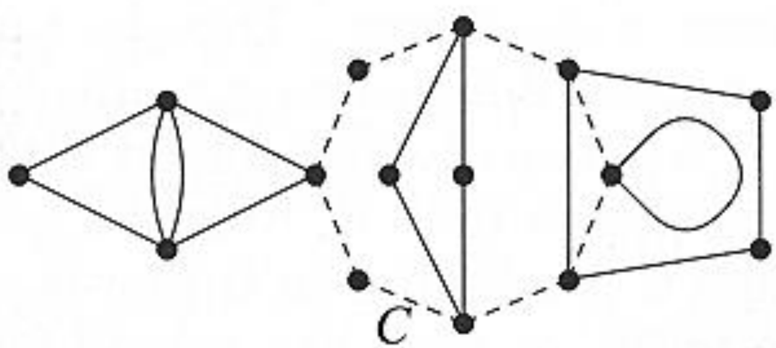
Basis step: $m=0$. G contains only one vertex.

Induction step: $m>0$. The nontrivial component has a cycle C.

Let $G'=G-E(C)$. Note that G' is an even graph.

By induction hypothesis, each component of G' has an Eulerian circuit.

Merge these circuits to obtain an Eulerian circuit of G.



Theorem. For a connected nontrival graph with exactly $2k$ *odd* vertices, the minimum number of trails that decompose it is $\max\{k, 1\}$.

Proof:

1. A trail contributes even degree to every vertex, except a non-closed trail$\Rightarrow$each odd vertex has some non-closed trail ending at it $\Rightarrow$ contains at least k trails
   (Note that G is Eulerian when k=0).

2. Pair up the odd vertices and form *G'* by adding for each pair an edge: *G'* has an Eulerian circuit *C* $\Rightarrow$traverse C and start a new trail when an edge of *G'*-*E*(*G*) $\Rightarrow$obtain *k* trail decomposing *G*.

e.g.

1-3 Vertex Degree and Counting

Def:

☞The degree of vertex *v* in a graph *G*, $d_G(v)$ or *d*(*v*), is the number of edges incident to *v*, except that each loop at *v* counts twice.

☞The maximum degree is $\Delta(G) = \max_{v \in V(G)} \{d_G(v)\}$, the minimum degree is $\delta(G) = \min_{v \in V(G)} \{d_G(v)\}$, and *G* is *regular* if $\Delta(G) = \delta(G)$.

☞It is *k-regular* if the common degree is *k*.

☞The *neighborhood* of *v*, $N_G(v)$ or *N*(*v*), is the set of vertices adjacent to *v*.

Def:

☞The *order* of a graph G, $n(G)$, is the number of vertices of G.

☞The *size* of a graph G, $e(G)$, is the number of edges in G.

Counting and Bijections

Proposition. (Degree-Sum Formula) If G is a graph, then

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Corollary. In a graph G, the average vertex degree is $\frac{2e(G)}{n(G)}$, and hence $\delta(G) \le \frac{2e(G)}{n(G)} \le \Delta(G)$.

Corollary. Every graph has an even number of vertices of odd degree. No graph of odd order is regular with odd degree.

Corollary. A k-regular graph with n vertices has $\frac{nk}{2}$ edges.

Proposition. If $k>0$, then a k-regular bipartite graph has the same number of vertices in each partite set.

Proof:

Let G be an X,Y-bigraph.

Clearly, $e(G) = k\,|X|$ and $e(G) = k\,|Y|$. Thus $|X| = |Y|$.

Proposition. The minimum number of edges in a connected graph with n vertices is n-1.

Proof:

- Every graph with n vertices and k edges has at least n-k components.
- Every n-vertex graph with fewer than n-1 edges has at least two components (disconnected).
- The lower bound is achieved by the path P_n.

Proving that β is the minimum of *f*(*G*) for graphs in a class *G* requires showing two things:

1) $f(G) \geq \beta$ for all $G \in \boldsymbol{G}$.
2) $f(G) = \beta$ for some $G \in \boldsymbol{G}$.

Proposition. If G is a simple n-vertex graph with $\delta(G) \geq \frac{(n-1)}{2}$, then G is connected.

Proof:

Let u, v be a two arbitrary non-adjacent vertices.

Since G is simple, $|N(u)| \geq \delta(G) \geq \frac{(n-1)}{2}$, and similarly for v.

Moreover, $|N(u) \cup N(v)| \leq n-2$.

Therefore,

$$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq$$

$$\frac{n-1}{2} + \frac{n-1}{2} - (n-2) = 1.$$

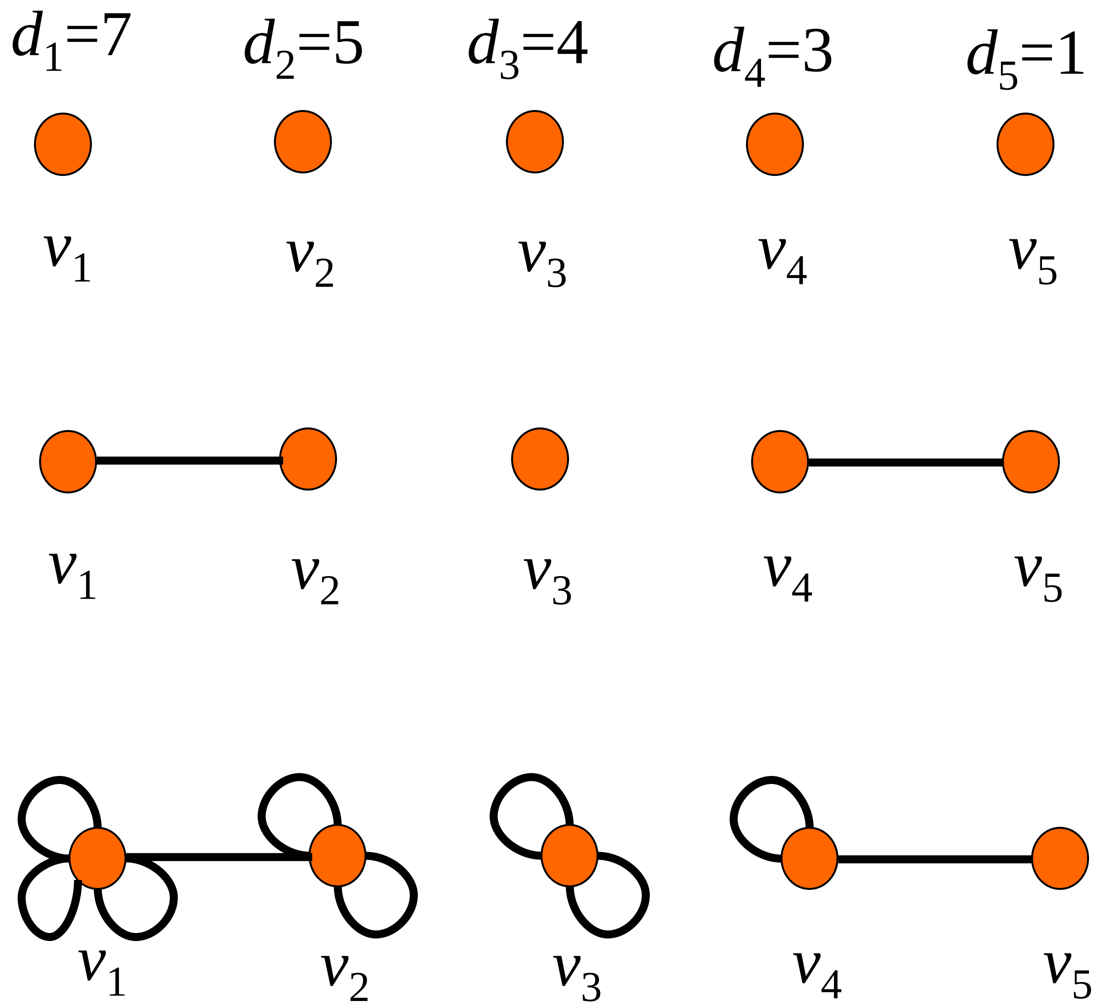
Proposition. The nonnegative integers $d_1, d_2, \ldots, d_n$ are the vertex degrees of some graph if and only if $\sum d_i$ is even.

Pf:

(⇨) $\sum d_i = 2e(G)$

(⇦) Since $\sum d_i$ is even, the number of odd values is even.

1. Form an arbitrary pairing of the vertices in $\{v_i : d_i \text{ is odd}\}$;
2. For each resulting pair u and v, form an edge (u, v);
3. The remaining degree at each vertex v_i is even. Create $\left\lfloor d_i/2 \right\rfloor$ loops at v_i.


d_1=7
d_2=5
d_3=4
d_4=3
d_5=1
v_1
v_2
v_3
v_4
v_5
v_1
v_2
v_3
v_4
v_5
v_1
v_2
v_3
v_4
v_5

1.4. Directed Graphs

Def:

☞ A *directed graph* or *digraph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices.

☞ The first (resp. second) vertex of the ordered pair is the *tail* (resp. *head*); together, they are the *endpoints*.

☞ An edge is from its *tail* to its *head*.

- A *loop* is an edge whose endpoints are equal.

- *Multiple edges* are edges having the same ordered pair of endpoints.

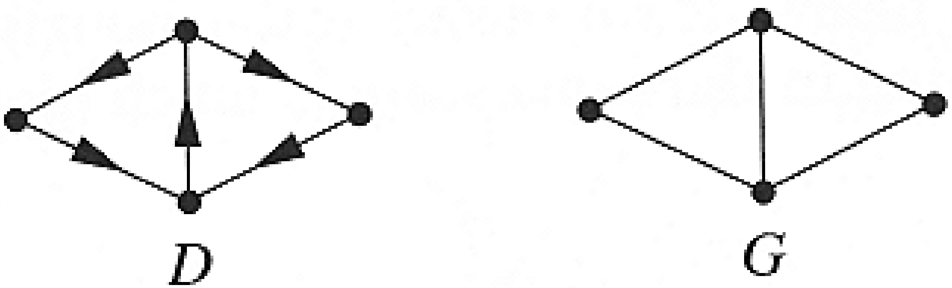
- A digraph is *simple* if each ordered pair is the head and tail of at most one edge; one loop may be present at each vertex.

Def:

- A digraph is a *path* if it's a simple digraph whose vertices can be linearly ordered so that there is an edge with tail *u* and head *v* iff *v* immediately follows *u* in the vertex ordering.

- *directed cycle*

- The *underlying graph* of a digraph *D* is the graph *G* obtained by treating the edges of *D* as unordered pairs.



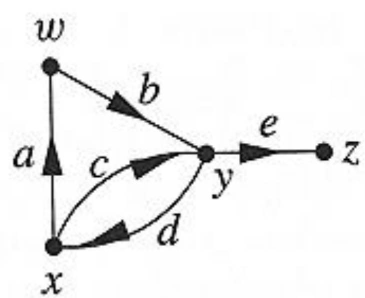
Def.

☞In the *adjacency matrix* $A(G)$ of a digraph G, the entry in positions i, j is the number of edges from v_i to v_j.

☞In the *incidence matrix* $M(G)$ of a loopless digraph G, we set $m_{i,j} = +1$ if v_i is the tail of e_j and $m_{i,j} = -1$ if v_i is the head of e_j .

$$\begin{array}{c|c} & \begin{matrix} w & x & y & z \end{matrix} \\ \begin{matrix} w \\ x \\ y \\ z \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$A(G)$



G

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} w \\ x \\ y \\ z \end{matrix} & \begin{pmatrix} -1 & +1 & 0 & 0 & 0 \\ +1 & 0 & +1 & -1 & 0 \\ 0 & -1 & -1 & +1 & +1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

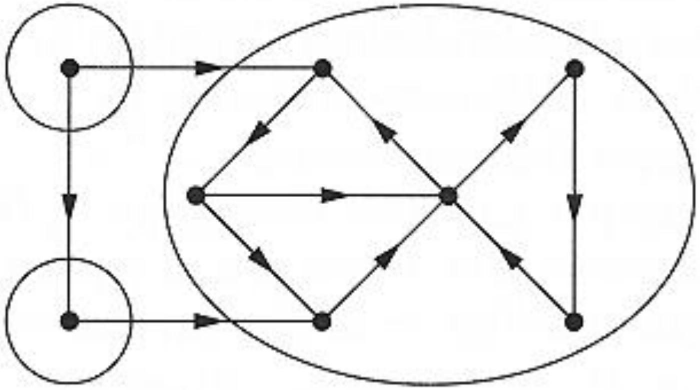
$M(G)$

Def:

☞A digraph is *weakly connected* if its underlying graph is connected.

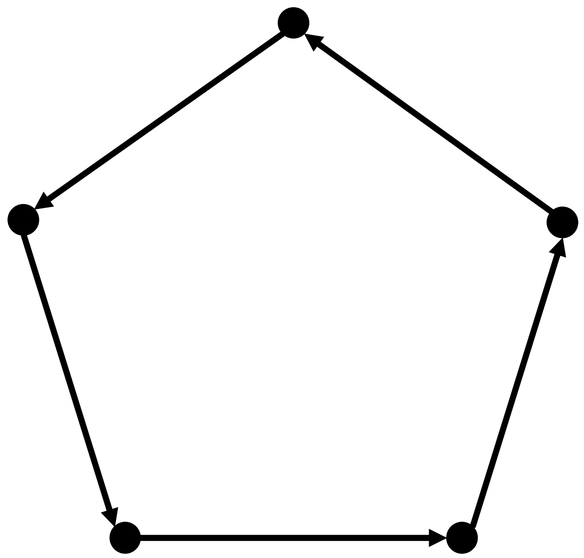
☞A digraph is *strongly connected* or strong if for each ordered pair *u*, *v* of vertices, there is a path from *u* to *v*.

☞The *strong components* of a digraph are its maximal strong subgraphs.



E.g.:
An *n*-vertex path has *n* strong components, but a cycle has only one.

path



cycle

Def: Let *v* be a vertex in digraph.

- ☞The *outdegree* $d^{+}(v)$ is the number of edges with tail *v*.
- ☞The *indegree* $d^{-}(v)$ is the number of edges with head *v*.
- ☞The *out-neighborhood* or *successor set* $N^{+}(v)$ is $\{x \in V(G) : v \to x\}$.
- ☞The *in-neighborhood* or *predecessor set* $N^{-}(v)$ is $\{x \in V(G) : x \to v\}$.
- ☞The minimum and maximum indegree are $\delta^{-}(G)$ and $\Delta^{-}(G)$; for outdegree we use $\delta^{+}(G)$ and $\Delta^{+}(G)$.

Proposition.

In a digraph G,

$$\sum_{v \in V(G)} d^{+}(v) = e(G) = \sum_{v \in V(G)} d^{-}(v).$$

Def:

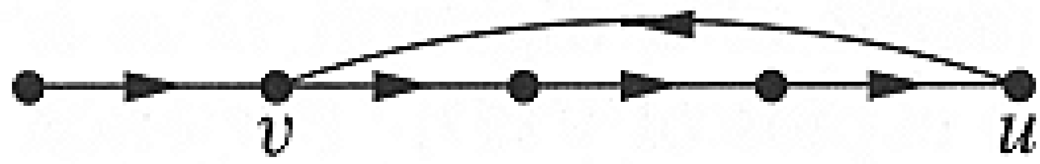
☞ An *Eulerian trail* in a digraph (or graph) is a trail containing all edges.

☞ An *Eulerian circuit* is a closed trail containing all edges.

☞ A digraph is *Eulerian* if it has an Eulerian circuit.

Lemma. If G is a digraph with $\delta^+(G) \geq 1$, then G contains a cycle. The same conclusion holds when $\delta^-(G) \geq 1$.

Proof.

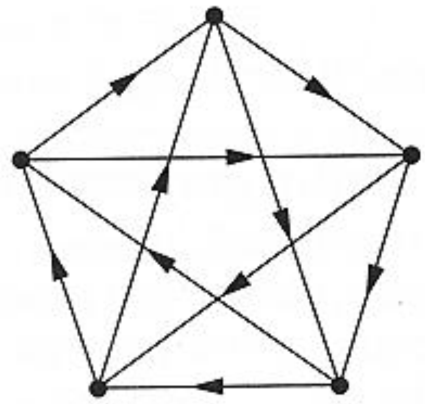


Def:

☞An *orientation* of a graph *G* is a digraph *D* obtained from *G* by choosing an orientation ($x \to y$ or $y \to x$) for each edge $xy \in E(G)$.

☞An *oriented graph* is an orientation of a simple graph.

☞A *tournament* is an orientation of a complete graph.



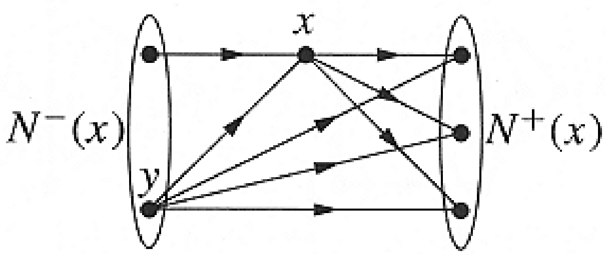
Def: In a digraph, a *king* is a vertex from which every vertex is reachable by a path of length at most two.

Proposition. Every tournament has a king.

Proof:

x: a vertex in a tournament T.

- If x is not a king, then $\exists\, y$ such that $d^+(y) > d^+(x)$.



- If y is not a king, then find z with yet larger outdegree.

Cannot forever obtain vertices of successively higher outdegree $\Rightarrow$ the procedure must terminate when a king is found.