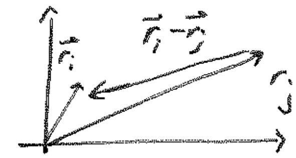
<u>N-body Gravitation</u>

For two masses, we have a gravitational interaction described by the conservative potential,

$$V(r_{ij}) = -\frac{G m_i m_j}{r_{ij}}$$



where $r_{ij} = |\vec{r}_i - \vec{r}_j|$

So for many such objects, we just sum over all possible pairs, ie

$$V_{tot} = \sum_{pairs} V(r_{ij}) = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} V(r_{ij}) \quad \text{Sums over } \underline{\text{distinct}} \text{ pairs}$$

$$= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1\,(j \neq i)}^{N} V(r_{ij})$$

If we want to describe the motion of such a system, we need to solve Newton's eqn of motion, ie

$$m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i \qquad \text{so we need } F_i \text{; for a single pair...}$$

$$\vec{F}_{ij} = -\nabla V(r_{ij}) = -\frac{G m_i m_j}{r_{ij}^2} \hat{r}_{ij} \qquad \hat{r}_{ij} = \frac{\vec{r}_i - \vec{r}_j}{r_{ij}}$$

$$= -\frac{G m_i m_j}{r_{ij}^3} (\vec{r}_i - \vec{r}_j). \qquad \text{(more convenient for simulation).}$$

<u>NB</u>: $F_{ij} = -F_{ji}$, so $F_{ji} = \ldots$

To get total force on i, we sum

$$\vec{F}_i = -G m_i \sum_{j \neq i} \frac{m_j}{r_{ij}^3} (\vec{r}_i - \vec{r}_j)$$

$V \propto \frac{1}{r}$ $\quad V \to \infty$

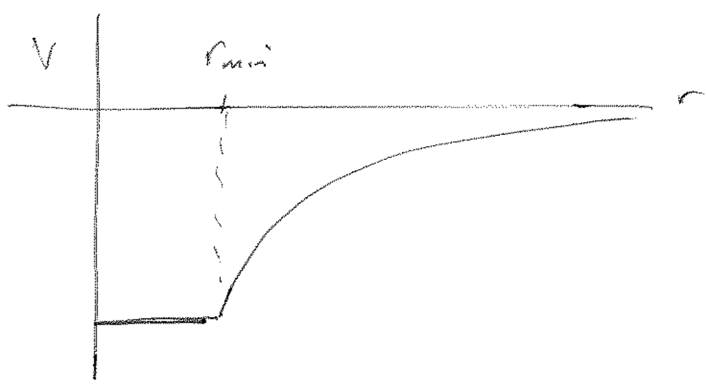
Since $|F| \propto \frac{1}{r^2}$, $F \to \infty$ as $r \to 0$!

To avoid any practical mishaps, we will modify our potential so that

$$V(r) = \begin{cases} -\frac{G m_i m_j}{r_{ij}} & r \geq r_{min} \\ -\frac{G m_i m_j}{r_{min}} & r < r_{min} \quad \text{(constant!)} \end{cases}$$

$$F(r) = \begin{cases} \cdots & r \geq r_{min} \\ 0 & r < r_{min} \end{cases}$$

$\Rightarrow$



To make amenable to numerical solution, we break into two first order ODE's,

$$\vec{v}_i(t) = \frac{d\vec{r}_i(t)}{dt} \qquad \frac{dv_i(t)}{dt} = \vec{F}_i(t) = -Gm_i \sum_{j\neq i} \frac{m_j}{r_{ij}^3(t)} \left(\vec{r}_i(t) - \vec{r}_j(t)\right)$$

This is our set of DE to solve.

Reminding you from last time, we can do this via Runge-Kutta method:

$$\vec{r}_i(t+\delta t) = \vec{r}_i(t) + \delta t\, \vec{v}_i\left(t + \frac{\delta t}{2}\right)$$

$$\vec{v}_i(t+\delta t) = \vec{v}_i(t) + \delta t \frac{1}{m} \vec{F}_i\left(t + \frac{\delta t}{2}\right)$$

where
$$\left.\begin{aligned} \vec{r}_i\left(t+\frac{\delta t}{2}\right) &= \vec{r}_i(t) + \frac{\delta t}{2}\, \vec{v}_i(t) \quad \text{(Euler half-step)} \\ \vec{v}_i\left(t+\frac{\delta t}{2}\right) &= \vec{v}_i(t) + \frac{\delta t}{2} \frac{1}{m} \vec{F}_i(t) \end{aligned}\right\} \text{need temporary storage}$$

We need $r_i(t+\frac{\delta t}{2})$ to evaluate $F(t+\frac{\delta t}{2})$.

We break this into components x & y.

Velocity - Verlet Algorithm :

The velocity - Verlet method is specifically for Hamiltonian systems (such as systems obeying Newton's laws). The algorithm is (in terms of $\vec{r}$ & $\vec{v}$):

$$\vec{r}_i(t+\delta t) = \vec{r}_i(t) + \vec{v}_i(t)\,\delta t + \frac{\vec{F}_i(t)}{2m}\,\delta t^2 + O(\delta t^4)$$

$$\vec{v}_i(t+\delta t) = \vec{v}_i(t) + \frac{\delta t}{2m}\left[\vec{F}_i(t) + \vec{F}_i(t+\delta t)\right] + O(\delta t^2)$$

This algorithm, like Runge-Kutta, requires infomation at 2 different times to generate a single step. In this case, it is only the velocities that require 2 times, so the positions may be propogated, and the information of $\vec{r}_i(t+\delta t)$ is used to calculate $\vec{F}_i(t+\delta t)$, since force is defined in terms of positions.

Verlet & variants are symplectic --- meaning they preserve phase space. As a result, these algorithms have very good long-time energy conservation.

① Calc $F(t)$
② Update $r(t) \to r(t+\delta t) \to$ calc $F(t+\delta t)$
③ Update $v(t) \to v(t+\delta t)$
helpful to recycle info from one step to next.

Calc $f(t)$
loop t
update $r(t)$
update $v(t)$
calc $f(t)$
update $v(t)$
done

Note on units: so that we deal w/ reasonable numbers (ie $O(1-10)$), we choose the following units

length → earth ~~earth~~ radii

mass → earth mass

time → hours

In those units, $G = 19.94$ (much nicer than $6.67 \times 10^{-11} \ldots$)

Position & Velocity Arrays:

These are 2-d arrays!

~~r[NDIM~~ r [NMAX][NDIM] NDIM = 2

so to access x coord of particle 0,

r[0][0] = x of 0
r[0][1] = y of 0

r[1][0] = x of 1
r[1][1] = y of 1 etc...

same for v[ ][ ]