INTEGRATION OF ODE's (Ordinary Diff. Eq.)

Before discussing algorithms, we must first discuss the formulation of the problem. Since the methods all focus on 1st order ODE's (ie no second derivatives or higher), we must reformulate the given ODE as a set of (possibly coupled) first ~~ordr~~ order ODE's. Consider the following example:

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

We can eliminate 2nd order derivatives by introducing

$$z(x) \equiv \frac{dy}{dx} \; ; \quad \text{then} \quad \frac{d^2y}{dx^2} = \frac{dz}{dx} \; , \quad \text{giving us}$$

two 1st order equations:

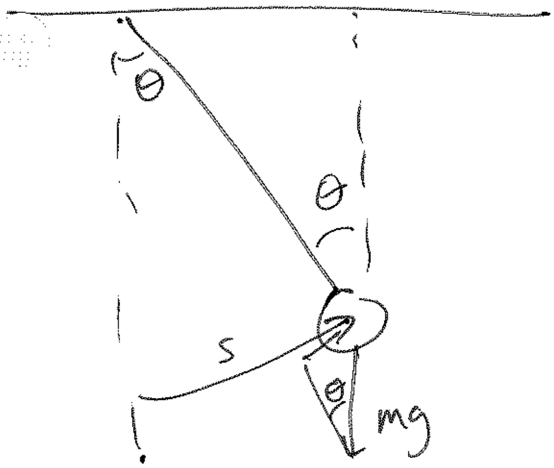
$$\frac{dy}{dx} = z(x) \quad \& \quad \frac{dz}{dx} = r(x) - q(x)z(x)$$

Recasting your problem will always be the first step.

The current assignment will focus on the motion of N-objects interacting via gravity. Therefore ~~of~~ our gravitational DE of motion ~~is~~ for the i^{th} object is

$$m_i \frac{d^2\vec{r}_i(t)}{dt^2} = F_i(t)$$

Numerical Int of Simple Pendulum:



$$F_s = m\frac{d^2s}{dt^2}$$

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad s = l\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \qquad \omega_0^2 \equiv \frac{g}{l}$$

For $\theta \ll 1$, we know $\sin\theta \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \approx \theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \omega_0^2\theta = 0 \quad \Rightarrow \quad \theta(t) = A\cos(\omega_0 t + \phi)$$

We can also potentially include drag $F_d = -bv$

$$= -b\frac{ds}{dt}$$

$$\Rightarrow \quad ml\frac{d^2\theta}{dt^2} = -mg\sin\theta - bl\frac{d\theta}{dt}$$

or $$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2 \cancel{\sin}\theta = 0 \qquad (\sin\theta \approx \theta \text{ if } \theta \text{ small})$$

This is our general problem to solve.

For $\theta \ll 1$, & $Q = \frac{\omega_0}{\gamma} \geq \frac{1}{2}$ we have the "under-damped" solution:

$$\theta(t) = A e^{-\gamma t/2}\cos(\omega t + \phi)$$

where $$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$$ ("damping slow oscillation")

For the assignment today, we will consider the numerical evolution of the pendulum for 3 cases:

1. Using small angle approximation

   $$\sin\theta \approx \theta$$

   with no damping. This is a good starting point since we can check our result against the known solution.

2. Without small angle approximation, so that

   $$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

   This is nice to compare with the approximate analytic result for cases when $\theta \not\ll 1$.

3. No approximation + damping. In this case,

   $$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - \gamma\frac{d\theta}{dt} = -\left(\omega_0^2\sin\theta + \gamma\frac{d\theta}{dt}\right)$$

   This exact solution can again be compared to the approximate analytic solution when θ small

We must recast this as two first order ODE'S:

Let $\boxed{\dot{\theta} = \frac{d\theta}{dt}}$ as an independent fntn

+ GIVEN $\theta(0)$, $\dot{\theta}(0)$.

$$\Rightarrow \boxed{\frac{d\dot{\theta}}{dt} + \gamma\dot{\theta} + \omega_0^2 \sin\theta = 0}$$

These are our two coupled, First order ODE's.

or $\frac{d\dot{\theta}}{dt} = -(\gamma\dot{\theta} + \omega^2\theta)$

Already we derived the "Euler method" from simple Taylor expansion:

$$\frac{d\theta}{dt} = \frac{\theta(t+\delta t) - \theta(t)}{\delta t} \Rightarrow \theta(t+\delta t) = \theta(t) + \frac{d\theta}{dt}\delta t + \mathcal{O}(\delta t^2)$$

$$= \theta(t) + \dot{\theta}(t)\,\delta t$$

evolution of theta.

similarly, $\dot{\theta}(t+\delta t) = \dot{\theta}(t) + \frac{d\dot{\theta}}{dt}\delta t$

most general case

$$= \dot{\theta}(t) - (\gamma\dot{\theta}(t) + \omega_0^2 \sin\theta(t))\,\delta t$$

If we break time interval T into N steps size δt,

$N = \frac{T}{\delta t}$. So N interation have error

$$N\delta t^2 = \frac{T}{\delta t}\delta t^2 = \mathcal{O}(\delta t).$$ Bad!

Runge-Kutta Method : (2nd order)

For a complete derivation of ~~RSG~~ RK, look at Numerical Recipes. Here, we provide it w/o derivation:

$$\theta(t+\delta t) = \theta(t) + \delta t \, \dot{\theta}\left(t+\frac{\delta t}{2}\right)$$

$$\dot{\theta}(t+\delta t) = \dot{\theta}(t) + \delta t \, \underbrace{\frac{d\dot{\theta}}{dt}\left(t+\frac{\delta t}{2}\right)}_{= -\left[\omega_0^2 \sin\theta\left(t+\frac{\delta t}{2}\right) + \gamma\,\dot{\theta}\left(t+\frac{\delta t}{2}\right)\right]}$$

So to get θ & $\dot{\theta}$ at time $t+\delta t$, we must know their values at $t+\frac{\delta t}{2}$ ("the half step"). To get these, we use an Euler move, ie

$$\theta\left(t+\frac{\delta t}{2}\right) = \theta(t) + \frac{\delta t}{2}\,\dot{\theta}(t)$$

$$\dot{\theta}\left(t+\frac{\delta t}{2}\right) = \dot{\theta}(t) + \frac{\delta t}{2}\,\underbrace{\frac{d\dot{\theta}}{dt}(t)}_{= -\left[\omega_0^2 \sin\theta(t) + \gamma\,\dot{\theta}(t)\right]}$$

Note that using the Euler to create half steps does not lead to the accumulation of error that repeated use of Euler alone does.