The first three-qubit and six-qubit full quantum multiple error-correcting codes with low quantum costs

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The first three-qubit and six-qubit full quantum multiple error-correcting codes with low quantum costs

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Abstract Quantum Error Correction (QEC) is a rigorous consolidation of facts and figures from both quantum mechanics and classical theory of error correcting codes aimed for a stronger output in the quantum domain. Here, we achieve correcting multiple bit-flip or phase-flip errors using a three-qubit quantum code with an extra qubit. Furthermore, for the first time, we construct a three-qubit full quantum error-correcting code that corrects errors not only on one qubit but multiple qubits. We then extend this approach in order to construct a six-qubit full quantum multiple error-correcting code. The quantum cost is significantly reduced as compared to any existing full quantum error-correcting code.

Keywords Quantum error correction, Shor's nine-qubit code, superposition of quantum states, quantum gates, Pauli operators, Dirac notations, IBM quantum experience

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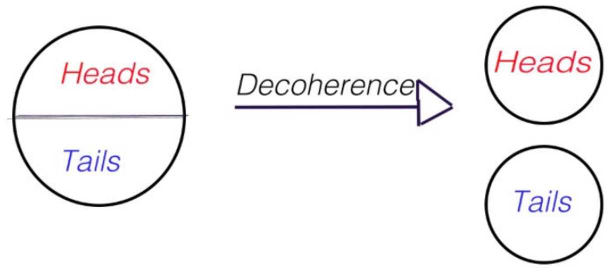

Fig. 1 Decoherence Theory [1].

1 Introduction

1.1 CLASSICAL AND QUANTUM WORLDS

In classical world, events that generally occur happen to be continuous i.e., the events that take place in the classical world progress in smooth, orderly and more importantly, a predictable pattern. One might argue from how it is observed that when a coin is tossed, the probability of outcome of either heads or tails is 1/2. However, if other variables like force applied on the coin, air resistance, impact of gravity etc are considered in this scenario, it is actually possible to predict the outcome of the toss. These unknown quantities are referred to as hidden variables. However, in a quantum world, events in particular are unpredictable. Seemingly random transitions between states are involved in this unpredictability which are also referred as quantum leaps. Moreover, a quantum leap is an all or nothing proposition, like jumping from the roof of one building onto another. One either makes it or breaks it. These quantum leaps are seemingly discontinuous [1]. The one theory that happens to be acting as a bridge between the classical and quantum worlds is "Theory of Decoherence" (Fig. 1). For example, superposition for subatomic particles is like balancing a coin, any small movement, vibration or even sound can affect the coin from being in a neutral state to collapsing on either heads or tails (0 or 1).

The decoherence theory regresses a quantum system to classical through higher degrees of freedom i.e. environmental aspects which eliminate the possessed quantum behaviour of that system. It is due to decoherence that qubits are extremely fragile and hence, the ability of them to stay entangled or in a state of superposition is jeopardized. This means if we don't factor in the attempts for eliminating decoherence, a quantum computer cannot exist.

Quantum computation is a vast field of application of quantum information theory and hence, possesses a lot of potential for development [2]. It has evolved exceptionally over time and there seems a lot more scope of research in this area. Quantum computers hold extreme amount of positive promises and for

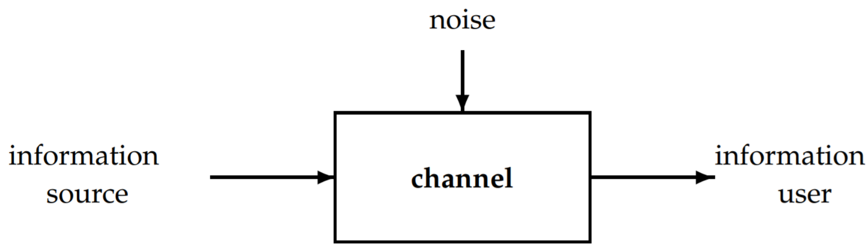

Fig. 2 Error interference [3].

them to work as such, they need some protection from these errors (also called noise) which arise due to decoherence. These errors need to be eliminated and hence, *quantum error-correction* becomes an important aspect in the field of quantum computation and information. Considering the logic that is applied in the field of classical error correction, we try to make stringent changes which go in accordance with the principles of quantum mechanics to construct models of quantum error-correction.

1.2 CLASSICAL ERROR CORRECTION

Error-correcting codes are widely used in the field of communication and data storage frameworks. The basic ideology of communication or data storage revolves around a source which transmits information to the user by means of a channel (Fig. 2). This channel, namely a communication channel, through which this process takes place is usually unfit for 100% efficiency that is to say, it adds some noise or errors to the information during the transmission [3]. Thus, to inoculate information to these errors, sender performs invertible encoding operations that induce redundancy within transmitted information. This is later examined by the receiver for traces of errors and is corrected as possible. Finally, the redundancy added by the sender is removed for a better outcome of low or possibly no errors in the information.

1.2.1 Repetition Code

Repetition code is said to be the simplest classical code which triplicates the information bit. There are two important aspects of any error-correcting algorithm, encoding and decoding. The error-correction happens after the encoding takes place. The repetition code is as follows:

$$\begin{aligned} 0 &\rightarrow 000 \\ 1 &\rightarrow 111 \end{aligned} \tag{1}$$

Usually, cases in which only one bit is flipped are only dealt. Hence, the set of probable errors can be shown as $\{001,010,100,110,101,011\}$. If the error occurs in the first bit and the output is say 001, error-correction code works in such a way that changes 001 to 000, hence in favour of majority. Similarly, 010, 100 get converted to 000 while 110, 101 and 011 to 111. This allows reduction of errors, hence providing better output. The reason we do not consider errors due to two or three flipped bits (if three are encoded) is because when 000 flips to 110, output gets converted to 111. The probability of this happening is a lot lesser than that for one bit-flip. Thus, we deal with cases in which only one bit is flipped.

1.3 QUANTUM ERROR CORRECTION

As written before, considering the logic that is applied in the field of classical error correction, we try to make stringent changes which go in accordance with the principles of quantum mechanics to construct models of quantum error-correction. However, using classical error-correction for correcting qubits cannot be a trivial operation for the fact implied by "No Cloning Theorem [4]" that says quantum states cannot be cloned. As described by Nielson and Chuang [2], quantum error-correction has the following challenges:

1. *Measurement destroys quantum information:* Recovery gets impossible once a quantum state is measured.

2. *No cloning:* No quantum operation can take a state $|\psi\rangle \otimes |0\rangle$ to $|\psi\rangle \otimes |\psi\rangle$. This is true for every arbitrary unknown quantum state $|\psi\rangle$ [4].

3. *Errors are continuous:* Qubits are continuous and hence, different errors on a single qubit form a continuum. Infinite precision is thus required to determine which error occurred in order to correct it.

1.3.1 Single Qubit Errors - Bit and phase-flips

Bit-flip Error: Pauli-X gate is used to perform operations for this error.

$$\begin{aligned} |0\rangle &\rightarrow |1\rangle \\ |1\rangle &\rightarrow |0\rangle \\ \alpha|0\rangle + \beta|1\rangle &\rightarrow \alpha|1\rangle + \beta|0\rangle \end{aligned} \tag{2}$$

Phase-flip Error: Pauli-Z gate is used to perform operations for this error.

$$\begin{aligned} |0\rangle &\rightarrow |0\rangle \\ |1\rangle &\rightarrow -|1\rangle \\ \alpha|0\rangle + \beta|1\rangle &\rightarrow \alpha|0\rangle - \beta|1\rangle \end{aligned} \tag{3}$$

Bit- and Phase-flip Errors: Pauli-Y gate is used to perform operations for this error.

$$\begin{aligned} |0\rangle &\rightarrow |1\rangle \\ |1\rangle &\rightarrow -|0\rangle \\ \alpha|0\rangle + \beta|1\rangle &\rightarrow \alpha|1\rangle - \beta|0\rangle \end{aligned} \tag{4}$$

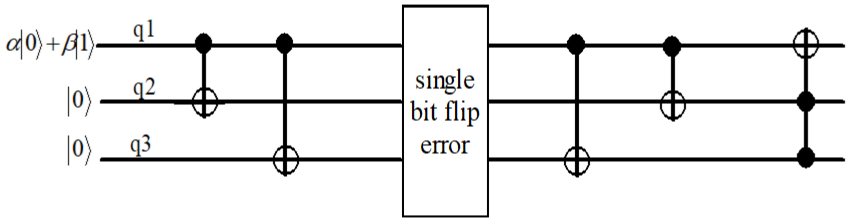

Fig. 3 Shor's three-qubit bit-flip error correcting code [5]

1.4 SHOR'S ERROR CORRECTIONS

In 1995, it was shown that quantum error-correction holds a valid place of consideration in the field of quantum computation and information theory. It was Shor's work [6] that sparked a new hope in the minds of quantum information theorists. He showed that quantum error-correction was indeed a possibility through a three-qubit error-correcting code and later went ahead with a nine-qubit error-correcting code for better outputs which will be discussed later.

1.4.1 Shor's three-qubit code for bit-flip error-correction

Circuit for Shor's three-qubit code for bit-flip error-correction is given in Fig. 3. Encoding in this circuit takes place as follows:

$$\alpha\,|0\rangle + \beta\,|1\rangle \rightarrow \alpha\,|0_1 0_2 0_3\rangle + \beta\,|1_1 1_2 1_3\rangle \tag{5}$$

This state after the error occurs becomes either one of the following:

$$\begin{aligned} &\alpha\,|1_1 0_2 0_3\rangle \,+\, \beta\,|0_1 1_2 1_3\rangle \\ &\alpha\,|0_1 1_2 0_3\rangle \,+\, \beta\,|1_1 0_2 1_3\rangle \\ &\alpha\,|0_1 0_2 1_3\rangle \,+\, \beta\,|1_1 1_2 0_3\rangle \end{aligned} \tag{6}$$

that is to say, one of the three qubits gets flipped. After this error-induced state passes through the last two CNOTs as seen in the Figure, it takes one of following forms respectively.

$$\begin{aligned} &(\alpha\,|1_1\rangle + \beta\,|0_1\rangle) \otimes |1_2 1_3\rangle \\ &(\alpha\,|0_1\rangle + \beta\,|1_1\rangle) \otimes |1_2 0_3\rangle \\ &(\alpha\,|0_1\rangle + \beta\,|1_1\rangle) \otimes |0_2 1_3\rangle \end{aligned} \tag{7}$$

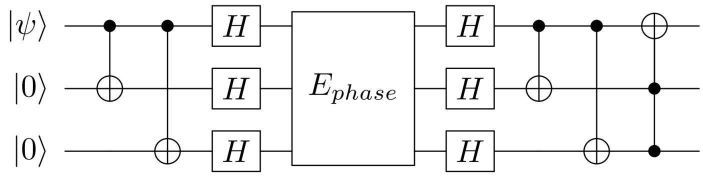

Fig. 4 Shor's three-qubit error correcting code for phase-flip.

CCNOT gate in this case is known as the error-correcting circuit. The output after CCNOT gate for the ordered cases above is as follows:

$$\begin{aligned}&(\alpha\,|0_1\rangle\,+\,\beta\,|1_1\rangle)\otimes|1_2 1_3\rangle\\&(\alpha\,|0_1\rangle\,+\,\beta\,|1_1\rangle)\otimes|1_2 0_3\rangle\\&(\alpha\,|0_1\rangle\,+\,\beta\,|1_1\rangle)\otimes|0_2 1_3\rangle\end{aligned}\tag{8}$$

In all these cases (Eq. (8)) the original state in the first qubit is preserved.

1.4.2 Shor's three-qubit code for phase-flip error-correction

Hadamard gate is used to convert a phase-flip to a bit-flip. As we know from the logic of gates that HZH = X where H, Z and X are Hadamard, Pauli-Z and Pauli-X gates respectively. The circuit of the three-qubit code for phase-flip error correction is given in Fig. 4.

The initial state is given by

$$\alpha\,|0\rangle + \beta\,|1\rangle \tag{9}$$

We know the state after encoding becomes $\alpha\,|0_1 0_2 0_3\rangle + \beta\,|1_1 1_2 1_3\rangle$. Hadamard gates are introduced in order to correct phase-flip errors, if there happen to be any. Thus, after applying Hadamard gates, state changes to $\alpha\,|+_1 +_2 +_3\rangle + \beta\,|-_1 -_2 -_3\rangle$. Phase-flip errors can change the state to one of the following:

$$\begin{aligned}&\alpha\,|-_1 +_2 +_3\rangle\,+\,\beta\,|+_1 -_2 -_3\rangle\\&\alpha\,|+_1 -_2 +_3\rangle\,+\,\beta\,|-_1 +_2 -_3\rangle\\&\alpha\,|+_1 +_2 -_3\rangle\,+\,\beta\,|-_1 -_2 +_3\rangle\end{aligned}\tag{10}$$

It is to be noted that the positive and negative signs have subscripts 1, 2 and 3 for indication of position of the qubits. Conclusively, states can take one of the following forms after the next three Hadamard gates.

$$\begin{aligned}&\alpha\,|1_1 0_2 0_3\rangle\,+\,\beta\,|0_1 1_2 1_3\rangle\\&\alpha\,|0_1 1_2 0_3\rangle\,+\,\beta\,|1_1 0_2 1_3\rangle\\&\alpha\,|1_1 1_2 0_3\rangle\,+\,\beta\,|0_1 0_2 1_3\rangle\end{aligned}\tag{11}$$

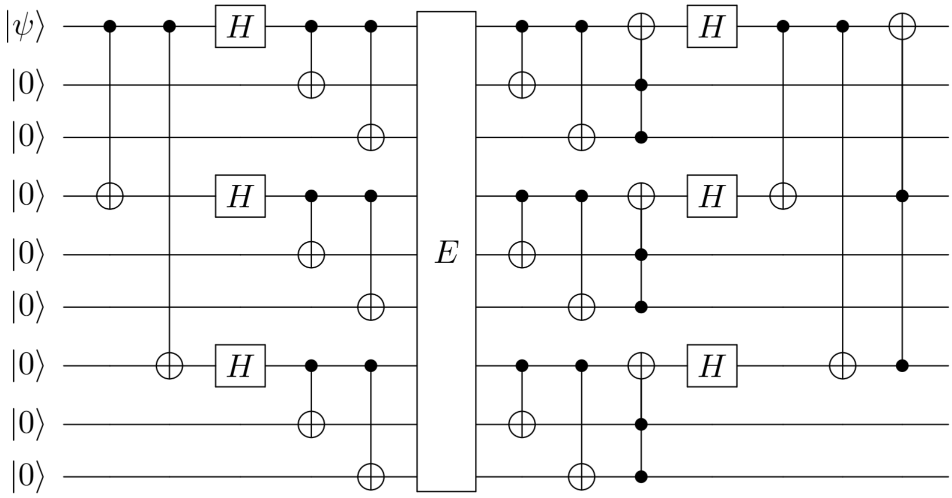

Fig. 5 Shor's nine-qubit error-correction code

respectively. It can be noticed that these are the exact states we corrected for bit-flip errors (Eq. (6)). Hence, bit-flip errors can be corrected for these states in the same process that had been taken into account before. Circuit was constructed accordingly by Shor.

1.4.3 Shor's nine-qubit error-correction code

The first full quantum code was contrived by Shor in the year 1995 [6]. The quantum circuit for the code is given in Fig. 5. As we see at the initial stage of this circuit, encoding takes place as it did in the three-qubit phase-flip code while the succeeding part of encoding section infers three-qubit bit-flip code. This gives the implication that Shor's nine-qubit error-correcting code can be used to correct both bit- and phase-flip errors. After calculations, encoded state of $\alpha|0\rangle + \beta|1\rangle$ becomes,

$$\begin{aligned}|\Psi_L\rangle = \alpha\frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle + |1_1 1_2 1_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5 0_6\rangle + |1_4 1_5 1_6\rangle)\frac{1}{\sqrt{2}}(|0_7 0_8 0_9\rangle + |1_7 1_8 1_9\rangle) \\ + \beta\frac{1}{\sqrt{2}}(|0_1 0_2 0_3\rangle - |1_1 1_2 1_3\rangle)\frac{1}{\sqrt{2}}(|0_4 0_5 0_6\rangle - |1_4 1_5 1_6\rangle)\frac{1}{\sqrt{2}}(|0_7 0_8 0_9\rangle - |1_7 1_8 1_9\rangle)\end{aligned} \tag{12}$$

Phase-flips change the sign of any one of the three blocks of qubits i.e. $|000\rangle + |111\rangle$ to $|000\rangle - |111\rangle$. From Shor's three-qubit phase error correcting code, the idea is clear that Hadamard gates allow correction of phase-flip errors by parity checks, but in a different basis as HZH=X. This code also corrects both bit- and phase-flip errors if they occur in the same block out of the three.

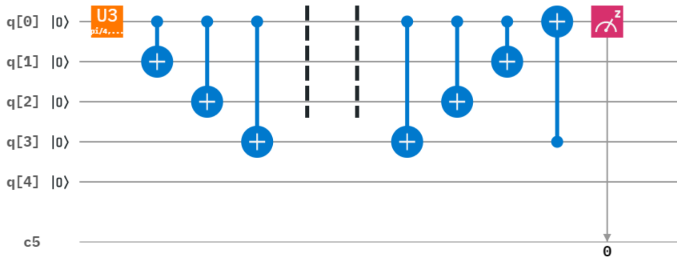

Fig. 6 Circuit for three-qubit multiple bit-flip error-correction. It is to be noted that the space between barriers is just to imply that error occurs in that region.

In brief, Shor's nine-qubit code corrects bit-flips, phase-flips, both bit- and phase-flip errors in a quantum circuit [7].

2 A NEW SCHEME

2.1 THREE-QUBIT MULTIPLE BIT-FLIP ERROR CORRECTION WITH AN EXTRA QUBIT AND LOW QUANTUM COST

As conveyed earlier, Shor's three-qubit bit-flip code corrects errors based on the majority and hence, it deals only with those cases in which only one bit is flipped. However, if one extra (ancilla) qubit is introduced in the circuit, errors can be solved even if there are two or three qubits in which error has occurred. Quantum circuit for the above statement is given in Fig. 6.

An input state, $\alpha |0\rangle + \beta |1\rangle$ is encoded with the help of first two CNOT gates. Hence, the state after encoding becomes

$$\alpha |0_1 0_2 0_3\rangle + \beta |1_1 1_2 1_3\rangle \tag{13}$$

Now, an extra qubit, q[3] is introduced. After the application of third CNOT gate, the total state including the ancilla qubit becomes

$$\alpha |0_1 0_2 0_3 0_A\rangle + \beta |1_1 1_2 1_3 1_A\rangle \tag{14}$$

If suppose error occurs on **one** of the three qubits, the state can become one of the following depending on which qubit the error has occurred.

$$\begin{gathered} \alpha |1_1 0_2 0_3 0_A\rangle + \beta |0_1 1_2 1_3 1_A\rangle \\ \alpha |0_1 1_2 0_3 0_A\rangle + \beta |1_1 0_2 1_3 1_A\rangle \\ \alpha |0_1 0_2 1_3 0_A\rangle + \beta |1_1 1_2 0_3 1_A\rangle \end{gathered} \tag{15}$$

Another CNOT gate is dropped on the ancilla in order to check whether or not error occurred. Hence, the state becomes one of the following respectively

$$\begin{aligned}
&\alpha\,|1_1 0_2 0_3 1_A\rangle + \beta\,|0_1 1_2 1_3 1_A\rangle \\
&\alpha\,|0_1 1_2 0_3 0_A\rangle + \beta\,|1_1 0_2 1_3 0_A\rangle \\
&\alpha\,|0_1 0_2 1_3 0_A\rangle + \beta\,|1_1 1_2 0_3 0_A\rangle
\end{aligned} \tag{16}$$

After the application of two CNOT gates, the state becomes one of the following respectively

$$\begin{aligned}
&(\alpha\,|1_1\rangle + \beta\,|0_1\rangle) \otimes |1_2 1_3\rangle \otimes |1_A\rangle \\
&(\alpha\,|0_1\rangle + \beta\,|1_1\rangle) \otimes |1_2 0_3\rangle \otimes |0_A\rangle \\
&(\alpha\,|0_1\rangle + \beta\,|1_1\rangle) \otimes |0_2 1_3\rangle \otimes |0_A\rangle
\end{aligned} \tag{17}$$

The extra qubit is flipped only in one of the four probable cases and hence, a CNOT gate is applied from ancilla qubit to the first qubit in order to correct that error. Thus, CNOT gate through ancilla works only in the first case of (17) and hence, state on the first qubit becomes $\alpha\,|0\rangle + \beta\,|1\rangle$. The idea of this seems similar to Shor's three-qubit code but following (14), here's what happens when the case of error on **two** qubits is considered. The state can take one of the forms in such a case:

$$\begin{aligned}
&\alpha\,|0_1 1_2 1_3 0_A\rangle + \beta\,|1_1 0_2 0_3 1_A\rangle \\
&\alpha\,|1_1 0_2 1_3 0_A\rangle + \beta\,|0_1 1_2 0_3 1_A\rangle \\
&\alpha\,|1_1 1_2 0_3 0_A\rangle + \beta\,|0_1 0_2 1_3 1_A\rangle
\end{aligned} \tag{18}$$

Following the circuit, state after decoding becomes one of the following cases respectively

$$\begin{aligned}
&(\alpha\,|0_1\rangle + \beta\,|1_1\rangle) \otimes |1_2 1_3\rangle \otimes |0_A\rangle \\
&(\alpha\,|1_1\rangle + \beta\,|0_1\rangle) \otimes |1_2 0_3\rangle \otimes |1_A\rangle \\
&(\alpha\,|1_1\rangle + \beta\,|0_1\rangle) \otimes |0_2 1_3\rangle \otimes |1_A\rangle
\end{aligned} \tag{19}$$

As we can see, when the extra qubit is in the state $|1\rangle$, the state measured on the first qubit happens to be $\alpha\,|1\rangle + \beta\,|0\rangle$ and thus, a CNOT gate is applied from the ancilla on to the first qubit as shown in the circuit. This finally leads to gaining back the input state $\alpha\,|0\rangle + \beta\,|1\rangle$ in spite of considering the case in which error occurred on two qubits. Now, following (14), let's consider a case in which error occurs on all the **three** qubits. Then the state becomes

$$\alpha\,|1_1 1_2 1_3 0_A\rangle + \beta\,|0_1 0_2 0_3 1_A\rangle \tag{20}$$

Following the circuit, the state after decoding becomes $\alpha\,|1_1 0_2 0_3 1_A\rangle + \beta\,|0_1 0_2 0_3 1_A\rangle$ which can also be written as $(\alpha\,|1_1\rangle + \beta\,|0_1\rangle) \otimes |0_2 0_3\rangle \otimes |1_A\rangle$. It can be noticed that the extra qubit is $|1_A\rangle$ and hence, CNOT gate from the extra qubit on to the first qubit works. On measuring state on the first qubit, we get back the input state $\alpha\,|0\rangle + \beta\,|1\rangle$. This concludes the fact that if a bit-flip error occurs

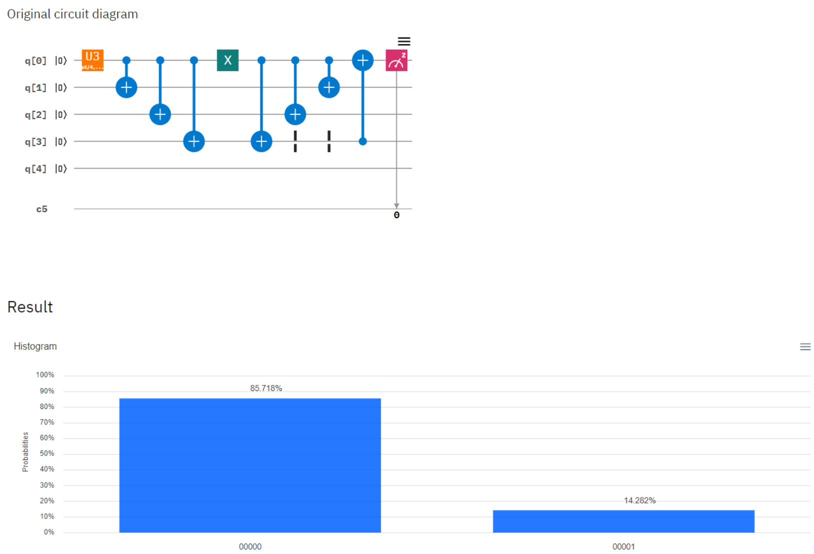

Fig. 7 Bit-flip error on one qubit

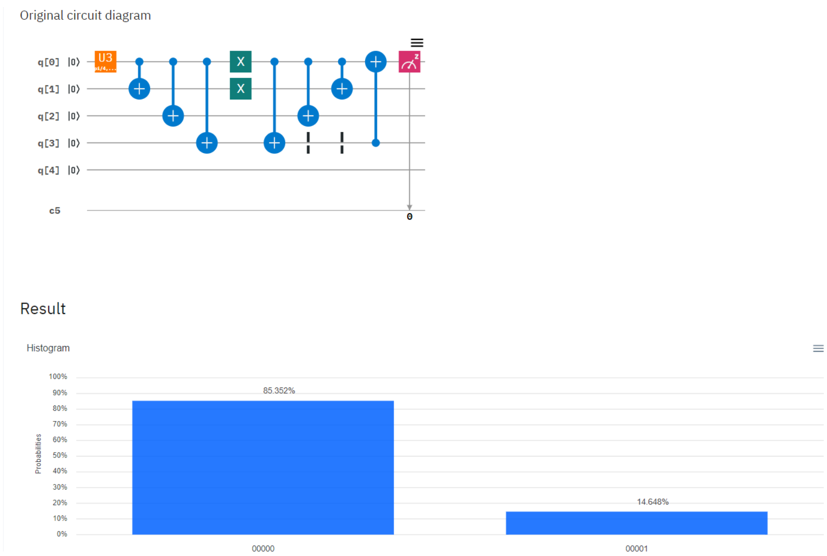

Fig. 8 Bit-flip error on two qubits

on any qubit and on any number of qubits, it can be corrected with the help of this circuit. Implementation on IBM quantum experience's platform was done in order to verify the calculations with the proposed quantum circuits. Outcomes for one of each cases are shown in Figs. 7, 8 and 9 for an arbitrary state $U3(\pi/4, \pi/2, \pi/2)|0\rangle$ i.e. $\sqrt{0.85}\,|0\rangle + i\sqrt{0.15}\,|1\rangle$.

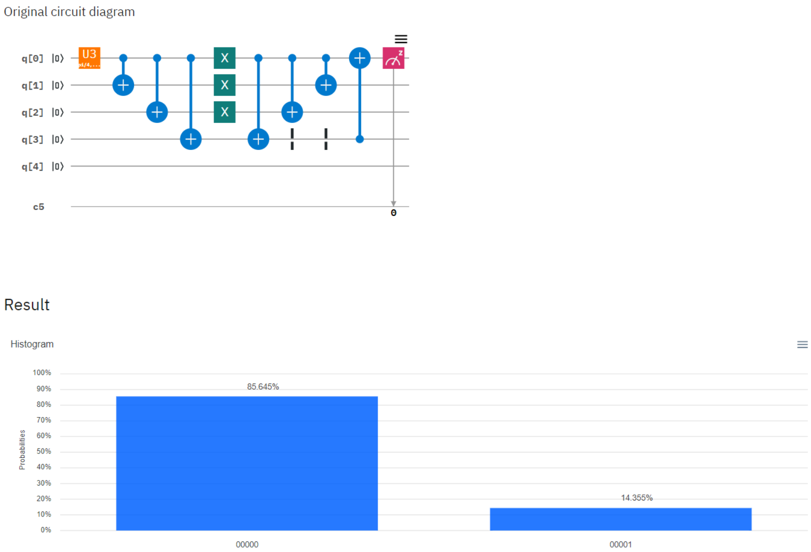

Fig. 9 Bit-flip error on all three qubits

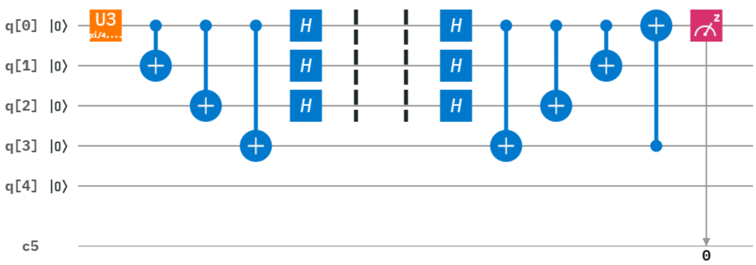

Fig. 10 Circuit for three-qubit multiple phase-flip error-correction. It is to be noted that the space between barriers is just to imply that error occurs in that region.

2.2 THREE-QUBIT MULTIPLE PHASE-FLIP ERROR CORRECTION WITH AN EXTRA QUBIT AND LOW QUANTUM COST

As we already know that HZH=X where H, Z and X are Hadamard, Pauli-Z and Pauli-X respectively, we can also conclude that this circuit works for phase-flip error-corrections. Circuit for three-qubit multiple phase-flip error-correction is given in Fig. 10.

The state after Hadamard gates takes the form

$$\alpha \left|+_1 +_2 +_3 0_A\right\rangle + \beta \left|-_1 -_2 -_3 1_A\right\rangle \tag{21}$$

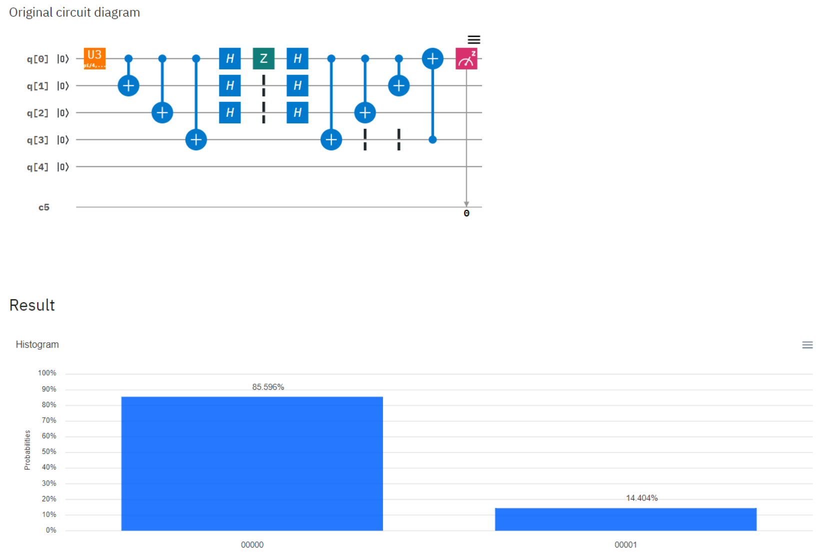

Fig. 11 Phase-flip error on one qubit

If a phase error occurs in **one** of the qubits, the probable outcomes can be one of the following.

$$\begin{aligned} &\alpha\,|-_1 +_2 +_3 0_A\rangle + \beta\,|+_1 -_2 -_3 1_A\rangle \\ &\alpha\,|+_1 -_2 +_3 0_A\rangle + \beta\,|-_1 +_2 -_3 1_A\rangle \\ &\alpha\,|+_1 +_2 -_3 0_A\rangle + \beta\,|-_1 -_2 +_3 1_A\rangle \end{aligned} \tag{22}$$

The state after decoding becomes one of the following respectively

$$\begin{aligned} &(\alpha\,|1_1\rangle + \beta\,|0_1\rangle) \otimes |1_2 1_3\rangle \otimes |1_A\rangle \\ &(\alpha\,|0_1\rangle + \beta\,|1_1\rangle) \otimes |1_2 0_3\rangle \otimes |0_A\rangle \\ &(\alpha\,|0_1\rangle + \beta\,|1_1\rangle) \otimes |0_2 1_3\rangle \otimes |0_A\rangle \end{aligned} \tag{23}$$

As observed, if the ancilla qubit is in $|1\rangle$, the state needs to be flipped. Thus, CNOT gate works with the intention of serving in such scenarios. After this final CNOT gate, if measurement is performed on the first qubit, we get back the input state $\alpha\,|0\rangle + \beta\,|1\rangle$ in spite of having introduced a phase error in it. Similarly, we can prove that this circuit corrects even if **two** or **three** phase-flip errors occur. Implementations of these were done on IBM Q experience's platform for an arbitrary state $U3(\pi/4, \pi/2, \pi/2)|0\rangle$ i.e. $\sqrt{0.85}\,|0\rangle + i\sqrt{0.15}\,|1\rangle$ and results for this are shown for one of each cases in Figs. 11, 12 and 13.

Depending on the number of qubits, a similar pattern of the circuit can be followed if one wants to correct multiple bit-flip or phase-flip errors in that qubit system, for that matter, even two.

It can be noted that, following in case of Shor's three-qubit bit-flip and phase-flip error correcting code, a CCNOT gate is used, which has a quantum

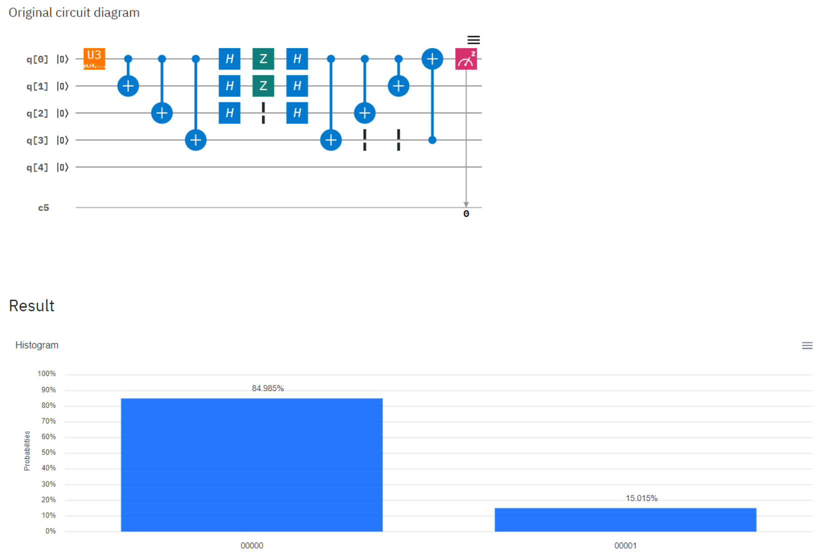

Fig. 12 Phase-flip error on two qubits

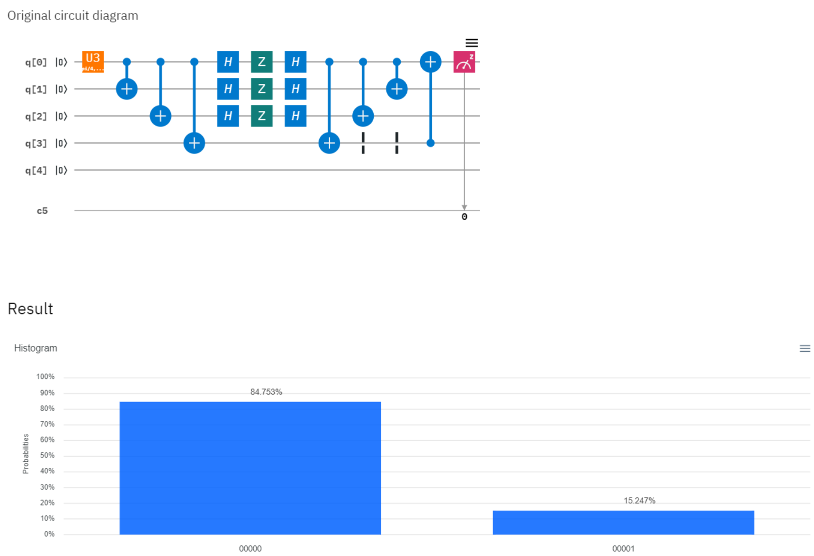

Fig. 13 Phase-flip error on all three qubits

cost of 15 i.e. to decompose the CCNOT gate, 15 quantum gates are required. However, in our above proposed quantum circuits, we only need 3 CNOT gates to correct bit-flip and phase-flip errors, hence reducing the quantum cost by 5 times.

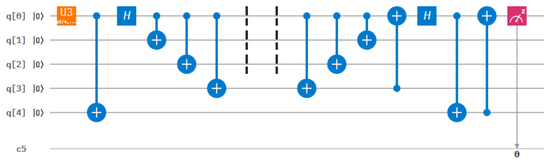

Fig. 14 Three-qubit full quantum multiple error-correcting code. It is to be noted that the space between barriers imply that error occurs in that region.

2.3 THREE-QUBIT FULL QUANTUM MULTIPLE ERROR CORRECTING CODE

An error-correcting code which corrects errors when they occur on more than one qubit is known as a multiple error-correcting code. These errors which occur on multiple qubits need not be the same. Merging both of the previously described concepts, if two extra qubits, A1 and A2 are introduced and a circuit as shown in Fig. 14 is constructed, it can correct both bit- and phase-flip errors for any arbitrary state. This circuit also corrects all possible errors and all possible combinations of errors (X,Y,Z) which might occur in more than one qubit.

Below, a case in which both bit- and phase-flips occur on the first qubit for an arbitrary state is considered. Calculations for this circuit are performed using $\alpha|0\rangle + \beta|1\rangle$ and after encoding, we get

$$\begin{aligned}&\alpha\frac{1}{\sqrt{2}}(|0_1 0_2 0_3 0_{A1} 0_{A2}\rangle + |1_1 1_2 1_3 1_{A1} 0_{A2}\rangle)\\ &+\beta\frac{1}{\sqrt{2}}(|0_1 0_2 0_3 0_{A1} 1_{A2}\rangle - |1_1 1_2 1_3 1_{A1} 1_{A2}\rangle)\end{aligned} \tag{24}$$

Both bit- and phase-flip error occurs on the first qubit and hence, this quantum state changes to

$$\begin{aligned}&\alpha\frac{1}{\sqrt{2}}(|1_1 0_2 0_3 0_{A1} 0_{A2}\rangle - |0_1 1_2 1_3 1_{A1} 0_{A2}\rangle)\\ &+\beta\frac{1}{\sqrt{2}}(|1_1 0_2 0_3 0_{A1} 1_{A2}\rangle + |0_1 1_2 1_3 1_{A1} 1_{A2}\rangle)\end{aligned} \tag{25}$$

Three CNOT gates are applied. Here is where the invertible operations start. State after these gates becomes

$$\begin{aligned}&\alpha\frac{1}{\sqrt{2}}(|1_1 1_2 1_3 1_{A1} 0_{A2}\rangle - |0_1 1_2 1_3 1_{A1} 0_{A2}\rangle)\\ &+\beta\frac{1}{\sqrt{2}}(|1_1 1_2 1_3 1_{A1} 1_{A2}\rangle + |0_1 1_2 1_3 1_{A1} 1_{A2}\rangle)\end{aligned} \tag{26}$$

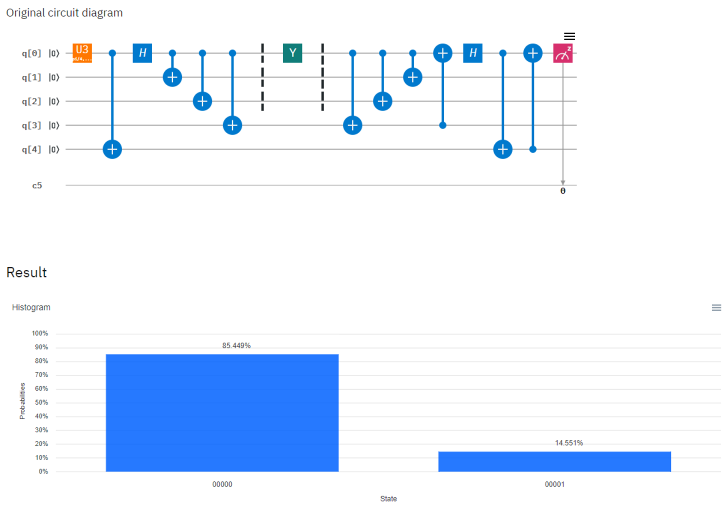

Fig. 15 Result for both bit- and phase-flip error on first qubit. This result supports the statement that the circuit corrects any error which can occur in any one of the qubits.

CNOT operation from A1 is used for bit-flip error correction. The state after this operation is applied becomes

$$\alpha\frac{1}{\sqrt{2}}(|0_1 1_2 1_3 1_{A1} 0_{A2}\rangle - |1_1 1_2 1_3 1_{A1} 0_{A2}\rangle)$$
$$+\beta\frac{1}{\sqrt{2}}(|0_1 1_2 1_3 1_{A1} 1_{A2}\rangle + |1_1 1_2 1_3 1_{A1} 1_{A2}\rangle) \tag{27}$$

In order to correct phase-flip errors, the role of Hadamard gates is vital. After the Hadamard gates are applied, this state changes to

$$\alpha\,|1_1 1_2 1_3 1_{A1} 0_{A2}\rangle + \beta\,|0_1 1_2 1_3 1_{A1} 1_{A2}\rangle \tag{28}$$

CNOT gate is applied from first on to A2. This de-entangles the state to

$$(\alpha\,|1_1\rangle + \beta\,|0_1\rangle) \otimes |1_2 1_3\rangle \otimes |1_{A1}\rangle \otimes |1_{A2}\rangle \tag{29}$$

Final CNOT operation from A2 to first qubit is applied which gives back the input state $\alpha\,|0\rangle + \beta\,|1\rangle$ on measuring the first qubit. Implementation was done for this and a few other cases that are shown in Fig. 15, Fig. 16 and Fig. 17. The state $U3(\pi/4, \pi/2, \pi/2)\,|0\rangle$ i.e. $\sqrt{0.85}\,|0\rangle + \sqrt{0.15}\,|1\rangle$ was used for proceeding with these implementations.

2.4 SIX-QUBIT FULL QUANTUM MULTIPLE ERROR CORRECTING CODE

Circuit for this can be termed as an extension of the approach with which we acquired the three-qubit full quantum multiple error-correcting code. It is

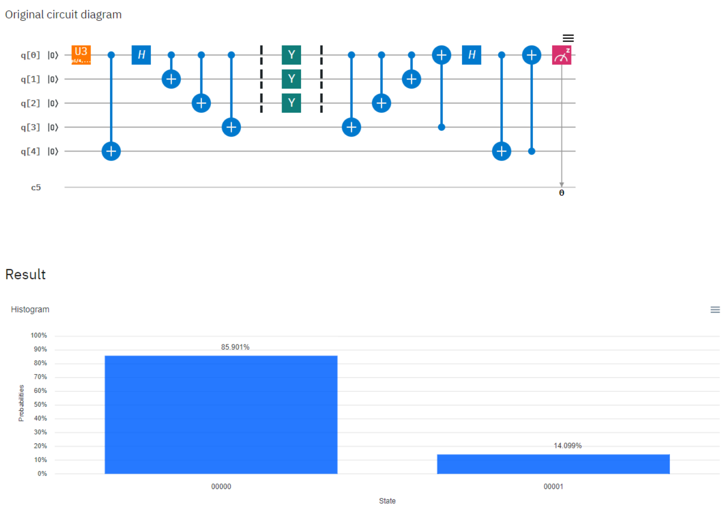

Fig. 16 Result for both bit- and phase-flip errors on all three qubits. This result supports the statement that the circuit corrects any error which can occur simultaneously in any multiple qubits.

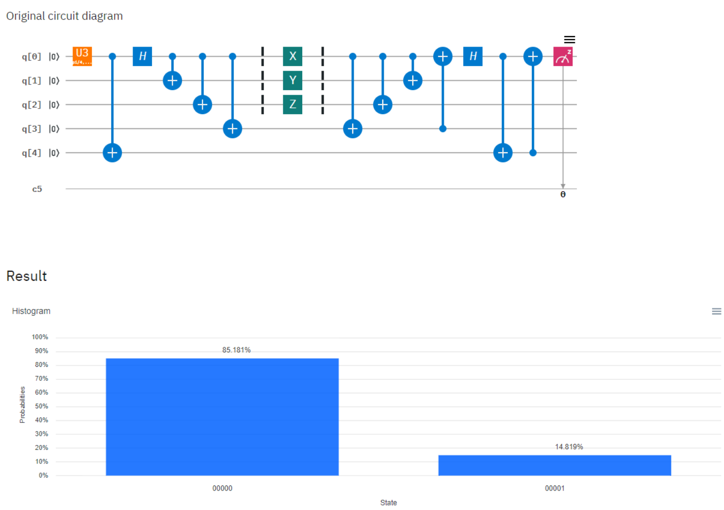

Fig. 17 Result for different errors on different qubits. This result supports the statement that the circuit corrects any combination of errors which can occur simultaneously in any multiple qubits.

given in Fig. 18. For a six-qubit full quantum multiple error-correcting code, three extra qubits are required and here, we take them to be A1, A2 and A3. Encoding for correction of a bit-flip error in a full quantum code preferably happens after Hadamard gates before the error occurs, while decoding and correction of a bit-flip error needs to happen before the Hadamard gates which

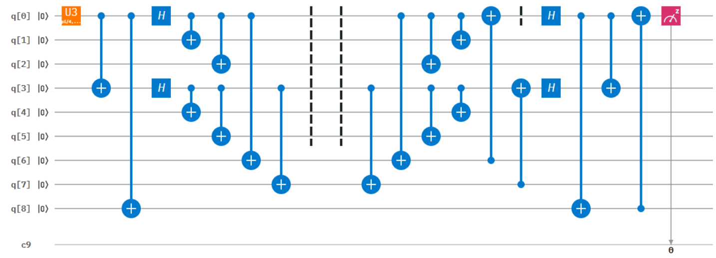

Fig. 18 Circuit for a six-qubit full quantum multiple error-correcting code with three extra qubits. It is to be noted that the space between barriers is to imply that the error occurs in that region.

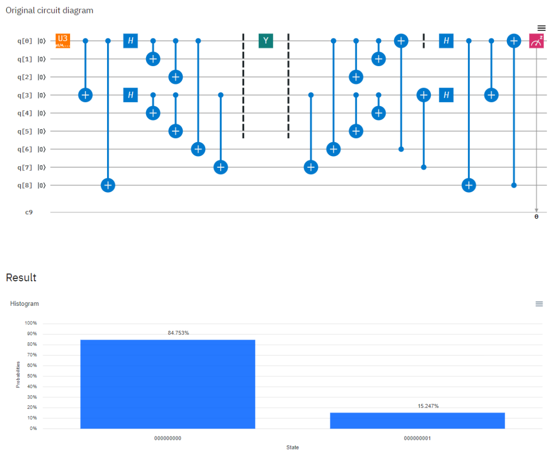

Fig. 19 Result for both bit- and phase-flip error for six-qubit full quantum multiple error-correcting code.

decode phase errors. Encoding for correction of a phase-flip error takes place before the encoding of bit-flip error code, while the decoding of phase-flip error takes place after the bit-flip error is corrected. In Fig. 18, the extra qubits A1, A2 and A3 are q[6], q[7] and q[8] respectively. An extra qubit was introduced in order to correct bit-flip errors in the three-qubit full quantum multiple error-correcting code. Similarly in this case, since there is a pair of three qubits, two extra qubits are required in order to correct bit-flip errors and one extra qubit for phase-flip errors. Implementations were done on IBM Q Experience's platform and the results for two cases are shown in Fig. 19 and Fig. 20

3 CONCLUSION

In this paper, we achieved correcting multiple bit errors using a three-qubit quantum code taken with an extra qubit. We then replaced X with HZH in order to make the three-qubit code solve multiple phase errors. Going further,

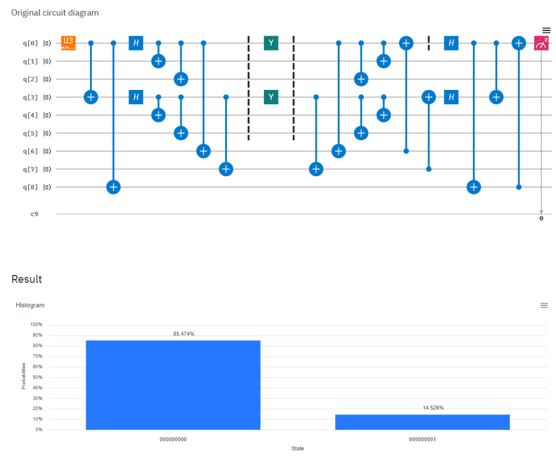

Fig. 20 Result for both bit- and phase-flip error on two qubits for six-qubit full quantum multiple error correcting code

we stringently merged both of these concepts in order to construct the first three-qubit full quantum multiple error-correcting code with two extra qubits. Considerably, an extension of circuit allowed us to also come up with a six-qubit full quantum multiple error-correcting code with three extra qubits. We believe that this approach to constructing full quantum multiple error-correcting codes can be further extended to 9, 12, 15 qubits and so on upto any 3n, with number of extra qubits to be taken accordingly i.e. n+1. Moreover, the quantum cost associated with any of these circuits is significantly lower than any existing full quantum error-correcting code. This is precisely due to the reason that in our circuits, CCNOT gates have not been used. It is so because the quantum cost of one CCNOT gate alone is 15. However, if involvement of extra qubit(s) is to be restrained, then the CNOT gates linked with one of the extra qubit which are responsible for bit-flip error correction can be removed and replaced by a CCNOT gate just before the final Hadamard gate.

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