Natural implicative expansions of variants of Kleene's strong 3-valued logic with Gödel-type and dual Gödel-type negation

Gemma Robles[a] and José M. Méndez[b]

[a]Dpto. de Psicología, Sociología y Filosofía, Universidad de León. Campus de Vegazana, s/n, 24071, León, Spain. ORCID: 0000-0001-6495-0388; [b]Edificio FES, Campus Unamuno, Universidad de Salamanca, 37007, Salamanca, Spain; ORCID: 0000-0002-9560-3327.

ABSTRACT

Let $MK3_I$ and $MK3_{II}$ be Kleene's strong 3-valued matrix with only one and two designated values, respectively. Next, let $MK3_G$ (resp., $MK3_{dG}$) be defined exactly as $MK3_I$ (resp., $MK3_{II}$), except that the characteristic Łukasiewicz-type negation of Kleene's strong 3-valued matrix is replaced by a 'Gödel-type' negation (resp., 'dual Gödel-type' negation). The aim of this paper is to axiomatize the logics determined by all natural implicative expansions of $MK3_G$ and $MK3_{dG}$. The axiomatic formulations are defined by using a 'two-valued' Belnap-Dunn semantics.

1. Introduction

Let $MK3_I$ and $MK3_{II}$ be Kleene's strong 3-valued matrix with only one and two designated values, respectively (cf. (Kleene, 1952, §64); Definition 2.2 below). And let $MK3_{I+}$ and $MK3_{II+}$ be the positive submatrices of $MK3_I$ and $MK3_{II}$, that is, the result of deleting the function for negation in $MK3_I$ and $MK3_{II}$. There are exactly 6 (resp., 24) 'natural' implicative expansions of $MK3_{I+}$ (resp., $MK3_{II+}$), where by a 'natural implicative expansion' is meant one defined by a conditional function extending the classical one, satisfying the rule *modus ponens* and assigning a designated value to a conditional whenever its antecedent is assigned a value lesser than or equal to the value assigned to its consequent[1]. Next, let $MK3_G$ (resp., $MK3_{dG}$) be defined exactly as $MK3_I$ (resp., $MK3_{II}$), except that the characteristic Łukasiewicz-type negation of Kleene's strong 3-valued matrix is replaced by a 'Gödel-type' negation (resp., a 'dual Gödel-type' negation) (cf. Definition 2.5[2]). The aim of this paper is to define the logics

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[1]We remark that there are a number of papers devoted to the notion of "natural implication": see (Karpenko & Tomova, 2017; Petrukhin, 2018; Petruhkin & Shanging, 2018, 2020; Tomova, 2012, 2015a,b); see also (Robles & Méndez, 2019, 2020; Robles, Salto & Méndez, 2019, and references in these papers). (We thank a referee of the JANCL for providing some of the items just noted.)

[2]A referee of the JANCL notes that "dual Gödel-type negation" is in fact Bochvar's negation (cf. Bochvar & Bergmann, 1981). Concerning Łukasiewicz-type negation and Gödel-type negation, cf. (Łukasiewicz, 1920) and (Gödel, 1932).

determined by all natural implicative expansions of MK3$_G$ and MK3$_{dG}$ comprehending 6 expansions of MK3$_G$ and 24 expansions of MK3$_{dG}$, as remarked above.

In (Robles & Méndez, 2019; Robles et al., 2019), the logics determined by all natural implicative expansions of MK3$_I$ and MK3$_{II}$ are formulated with Hilbert-style axiomatic systems by using Belnap-Dunn 'two-valued' semantics (BD-semantics). In particular, by simply translating expansions of MK3$_I$ into an underdetermined BD-semantics (cf. Robles et al., 2019) and those of MK3$_{II}$ into overdetermined BD-semantics (cf. Robles & Méndez, 2019). The strategy of the present paper is to relay again on this method in order to provide Hilbert-style axiomatizations for the logics characterized by all natural implicative expansions of MK3$_G$ and MK3$_{dG}$. In this sense, it is remarkable that MK3$_G$ and MK3$_{dG}$ are interpretable in terms of BD-semantics, despite the fact that the negation function characteristic of each one of both matrices do not have a value being the negation of itself, feature upon which the definition of BD-semantics originally depended.

Gödel-type negation is a kind of superintuitionistic negation; and dual Gödel-type negation is dual to Gödel-type negation in a similar sense to which the negation defined by Sylvan for the extension CCω of da Costa's paraconsistent logic Cω is dual to the negation characteristic of intuitionistic logic (cf. Robles & Méndez, 2020, and references therein). In this sense, Gödel-type and dual Gödel-type negation as displayed in the present paper can be viewed as the 3-valued expansion of the general superintuitionistic and dual intuitionistic H-negation and DH-negation defined in (Robles, 2020) and (Robles & Méndez, 2020), respectively, in 3-valued logic.

Leaving aside Gödel famous 3-valued logic and a couple of less famous but still attractive systems (cf. §5), the logics we are introducing in this paper have not previously been particularized in the literature, as far as we know (but cf. §6). However, they have interesting properties comparable to those that the more known logics determined by MK3$_I$ and MK3$_{II}$ enjoy, some of which are examined in section 5 of the paper: paraconsistency, paracompleteness, self-extensionality or the deductive theorem, for instance. In this regard, it is to be expected that the logics to be studied in what follows be of interest, in a similar way to which those characterized by MK3$_I$ and MK3$_{II}$ and displayed in (Robles & Méndez, 2019; Robles et al., 2019), articles of which the present one can be considered a sequel, are.

The paper is organized as follows. In §2, the matrices MK3$_G$ and MK3$_{dG}$ are defined and the 6 natural implicative expansions of MK3$_G$ and the 24 expanding MK3$_{dG}$ are displayed. In §3, the matrix semantics corresponding to each one of these 30 natural implicative expansions is translated into an overdetermined BD-semantics (expansions of MK3$_G$) and an underdetermined BD-semantics (expansions of MK3$_{dG}$). Also, the soundness and completeness proofs are sketched. We follow the strategy set up in (Brady, 1982) and just as it is particularly applied in (Robles & Méndez, 2019; Robles et al., 2019), as pointed out above. Thus, it will not be necessary to go down to each detail and many of the proofs will be referred to our papers (Robles & Méndez, 2019; Robles et al., 2019). In §4, the logics determined by the 30 natural implicative expansions are defined and the soundness and completeness theorems are proved. Soundness is proved w.r.t. the matrix semantics, and completeness w.r.t. the BD-semantics corresponding to each logic. Given the equivalence in each case between the matrix semantics and the BD-semantics, we have soundness and completeness w.r.t. both types of semantics. In §5, we prove some interesting properties predicable of the logics defined in §4, and finally, the paper is ended in §6 with some remarks on the results obtained.

2. The matrices MK3_G, MK3_dG and their natural implicative expansions

In this section, the matrices MK3_G, MK3_dG and their natural implicative expansions are defined.

Definition 2.1 (Some preliminary notions). The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and some or all of the following connectives $\rightarrow$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\sim$ (general negation), $\neg$ (Gödel-type negation —G-negation), $\overset{\bullet}{\neg}$ (dual Gödel-type negation — dG-negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables. Then, logics are formulated as Hilbert-type axiomatic systems, the notions of "theorem" and "proof from a set of premises" being the usual ones, while the following notions are understood in a fairly standard sense (cf., e.g., (Robles & Méndez, 2019) or (Robles et al., 2019)): *extension* and *expansion* of a given logic; *logical matrix* M and *M-interpretation*, *M-consequence*, *M-validity* and, finally, *M-determined* logic. Finally, given a matrix semantics, a conditional is natural if (1) it coincides with the classical conditional when restricted to the classical values T and F, (2) it satisfies the rule modus ponens, and (3) it is assigned a designated value whenever the value assigned to the antecedent is less than or equal to the value assigned to the consequent (cf. (Tomova, 2012), (Robles & Méndez, 2019; Robles et al., 2019) and Definition 4.1).

After defining the matrices MK3_I and MK3_II, we define the matrices MK3_G and MK3_dG below.

Definition 2.2 (The matrices MK3_I and MK3_II). The matrices MK3_I and MK3_II are Kleene's strong 3-valued matrix with only one and two designated values, respectively (cf. Kleene, 1952, §64). The propositional language consists of the connectives $\wedge, \vee$ and $\sim$. The matrix MK3_I is the structure $(\mathcal{V}, D, \text{F})$, where (1) $\mathcal{V} = \{0, 1, 2\}$ with $0 < 1 < 2$; (2) $D = \{2\}$; (3) $\text{F} = \{f_\wedge, f_\vee, f_\sim\}$, where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively, and $f_\sim$ is an involution with $f_\sim(2) = 0, f_\sim(0) = 2$ and $f_\sim(1) = 1$. We display the tables for $\wedge, \vee$ and $\sim$:

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\sim$	0
0	2
1	1
2	0

Next, the matrix MK3_II is the structure $(\mathcal{V}, D, \text{F})$ where $\mathcal{V}, D$ and F are defined exactly as in MK3_I except that now $D = \{1, 2\}$. $\text{MK3}_\text{I+}$ and $\text{MK3}_\text{II+}$ are the positive submatrices of MK3_I and MK3_II, that is, the result of deleting the negation function in MK3_I and MK3_II, respectively.

Remark 2.3 (On designated values in MK3_I and MK3_II). Kleene uses $\mathfrak{t}, \mathfrak{f}$ and $\mathfrak{u}$ instead of $2, 0$ and 1, respectively. Now, he does not seem to have considered designated values in (Kleene, 1952, §64), although he remarks: "The third "truth-value" $\mathfrak{u}$ is thus not on a par with the other two $\mathfrak{t}$ and $\mathfrak{f}$ in our theory. Consideration of its status will show that we are limited to a special kind of truth-value" (Kleene, 1952, p. 333). Priest logic LP (cf. Priest, 1979) is essentially the result of taking $\mathfrak{t}$ and $\mathfrak{u}$ as designated values in Kleene's 3-valued logic. According to Karpenko (cf. Karpenko, 1999, p. 83), the idea of defining such a logic first appeared in (Asenjo, 1966). (The digits $2, 1, 0$ have been chosen in order to use the tester in (González, 2011), in case the reader

needs one. Also, to put in connection the results in the present paper with previous work by us.)

Remark 2.4 (On the ordering of the truth-values in MK3$_\text{I}$ and MK3$_\text{II}$). Regarding the ordering of the truth-values in Kleene's strong 3-valued matrix, Fitting notes: "The informal reading suggests two natural orderings, concerning "amount of knowledge" and "degree of truth"" (Fitting, 1991, p. 797), (cf. also Fitting, 1994). Of course, the elements of $\mathcal{V}$ in Definition 2.1 are ordered according to degree of truth.

Definition 2.5 (The matrices MK3$_\text{G}$ and MK3$_\text{dG}$). Given a 3-valued matrix where $\mathcal{V}$ is defined as in Definition 2.2, 'Gödel-type' negation ($\neg$) and 'dual Gödel-type' negation ($\overset{\bullet}{\neg}$) are understood according to the functions $f_\neg$ and $f_{\overset{\bullet}{\neg}}$ given by the following truth-tables:

$\neg$			$\overset{\bullet}{\neg}$
0	2	0	2
1	0	1	2
2	0	2	0

Then, the matrices MK3$_\text{G}$ and MK3$_\text{dG}$ are defined exactly as MK3$_\text{I}$ and MK3$_\text{II}$, respectively, except that $f_\sim$ is replaced by $f_\neg$ in MK3$_\text{I}$, and by $f_{\overset{\bullet}{\neg}}$ in MK3$_\text{II}$. Or, in other words, MK3$_\text{G}$ (resp., MK3$_\text{dG}$) is the result of expanding MK3$_\text{I+}$ (resp., MK3$_\text{II+}$) with $f_\neg$ (resp., $f_{\overset{\bullet}{\neg}}$). (In §6, we briefly comment on the possibility of changing $f_\sim$ by $f_\neg$ (resp., $f_{\overset{\bullet}{\neg}}$) in MK3$_\text{II}$ (resp., MK3$_\text{I}$).)

In what follows, the natural implicative expansions of MK3$_\text{G}$ and MK3$_\text{dG}$ are defined (cf. Robles & Méndez, 2019; Robles et al., 2019).

Definition 2.6 (Natural implicative expansions of MK3$_\text{dG}$). There are exactly 24 natural implicative expansions of MK3$_\text{dG}$, Mt1, Mt2,..., Mt24, which are defined as follows. Each Mti ($1 \leq i \leq 24$) is the structure $(\mathcal{V}, D, \text{F})$ where $\mathcal{V}, D, f_\wedge, f_\vee$ and $f_{\overset{\bullet}{\neg}}$ are defined exactly as in MK3$_\text{dG}$, whereas $f_\to$ is defined according to the table ti. Tables t1, t2, ..., t24 are displayed below (cf. Robles & Méndez, 2019, Proposition 4.4).

t1

$\to$	0	1	2
0	2	1	2
1	0	1	1
2	0	0	2

t2

$\to$	0	1	2
0	2	1	2
1	0	1	1
2	0	1	2

t3

$\to$	0	1	2
0	2	1	2
1	0	1	1
2	0	2	2

t4

$\to$	0	1	2
0	2	1	2
1	0	1	2
2	0	0	2

t5

$\to$	0	1	2
0	2	1	2
1	0	1	2
2	0	1	2

t6

$\to$	0	1	2
0	2	1	2
1	0	1	2
2	0	2	2

t7

$\to$	0	1	2
0	2	1	2
1	0	2	1
2	0	0	2

t8

$\to$	0	1	2
0	2	1	2
1	0	2	1
2	0	1	2

t9

$\to$	0	1	2
0	2	1	2
1	0	2	1
2	0	2	2

t10

$\to$	0	1	2
0	2	1	2
1	0	2	2
2	0	0	2

t11

$\to$	0	1	2
0	2	1	2
1	0	2	2
2	0	1	2

t12

$\to$	0	1	2
0	2	1	2
1	0	2	2
2	0	2	2

t13 →	0	1	2
0	2	2	2
1	0	1	1
2	0	0	2

t14 →	0	1	2
0	2	2	2
1	0	1	1
2	0	1	2

t15 →	0	1	2
0	2	2	2
1	0	1	1
2	0	2	2

t16 →	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

t17 →	0	1	2
0	2	2	2
1	0	1	2
2	0	1	2

t18 →	0	1	2
0	2	2	2
1	0	1	2
2	0	2	2

t19 →	0	1	2
0	2	2	2
1	0	2	1
2	0	0	2

t20 →	0	1	2
0	2	2	2
1	0	2	1
2	0	1	2

t21 →	0	1	2
0	2	2	2
1	0	2	1
2	0	2	2

t22 →	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

t23 →	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

t24 →	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

Definition 2.7 (Natural implicative expansions of $MK3_G$). There are exactly 6 natural implicative expansions of $MK3_G$, Mt25, Mt26,..., Mt30, which are defined from $MK3_G$ in a similar way to which the 24 natural implicative expansions were defined from $MK3_{dG}$ in Definition 2.6. The implicative tables t25, t26, ..., t30 are displayed below (cf. Robles et al., 2019, Proposition 4.4).

t25 →	0	1	2
0	2	2	2
1	2	2	2
2	0	1	2

t26 →	0	1	2
0	2	2	2
1	2	2	2
2	0	0	2

t27 →	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

t28 →	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

t29 →	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

t30 →	0	1	2
0	2	2	2
1	1	2	2
2	0	0	2

(Notice that t27 (resp., t29) is the same table as t22 (resp., t23) but for the fact that 1 and 2 are designated values in t22 and t23.)

By using Belnap-Dunn semantics (BD-semantics), we will define the logic Lti determined by the matrix Mti ($1 \leq i \leq 30$).

3. Belnap-Dunn semantics for Lti-logics

Let Lti the logic determined by the matrix Mti ($1 \leq i \leq 30$). In this section, we define Belnap-Dunn semantics w.r.t. which Lti is sound and complete. By the term Lti-logics, we will generally refer to the logics characterized by the 30 matrices defined in Definitions 2.6 and 2.7.

Belnap-Dunn semantics (BD-semantics) was originally defined for Anderson and Belnap's *First degree entailment logic* (cf. Anderson & Belnap, 1975). Let T represent truth and F represent falsity. BD-semantics is characterized by the possibility of

assigning T, F, both T and F or neither T nor F to the formulas of a given logical language (cf. Belnap, 1977a,b; Dunn, 1976, 2000). Concerning 3-valued logics, two particular types of BD-semantics can be considered: overdetermined BD-semantics (o-semantics) and underdetermined BD-semantics (u-semantics). Formulas can be assigned T, F or both T and F in the former; T, F or neither T nor F in the latter (cf. Robles & Méndez, 2019; Robles et al., 2019). U-semantics is especially adequate to 3-valued logics determined by matrices with only one designated value; o-semantics, for those determined by matrices where only one value is not designated.

Given an implicative expansion of $MK3_G$, M, with 2 as the only designated value, the idea for defining a u-semantics, M_u, equivalent to the matrix semantics based upon M is simple: a wff A is assigned neither T nor F in M_u iff it is assigned 1 in M. Then, A is assigned T (resp., F) in M_u iff it is assigned 2 (resp., 0) in M. On the other hand, if M is an implicative expansion of $MK3_{dG}$ with 1 and 2 as designated values, then an o-semantics, M_o, equivalent to the matrix semantics based upon M is defined as follows. A is assigned both T and F in M_o iff it is assigned 1 in M. Next, A is assigned T (resp., F) in M_o iff it is not assigned 0 (resp., 2) in M. (Notice that, in the u-semantics, formulas can be assigned neither F nor T but not both F and T, while interpretation of formulas cannot be empty in o-semantics —i.e., formulas can be assigned both T and F.)

The o-semantics (resp., the u-semantics) equivalent to the matrix semantics based upon each one of the 24 (resp., 6) matrices in Definition 2.6 (resp., Definition 2.7) have been built by translating the matrix semantics in question into an o-semantics (or a u-semantics, as the case may be) according to the simple intuitive ideas just exposed.

In the sequel, the notion of an Lti-model and the accompanying notions of Lti-consequence and Lti-validity are defined. For each i ($1 \leq i \leq 24$) (resp., $25 \leq i \leq 30$), the Lti-models and said annexed notions is an o-semantics (resp., u-semantics) (which will be referred to by Lti-semantics) equivalent to the matrix semantics defined upon the matrix Mt_i, in the sense explained above.

Definition 3.1 (Lti-models I. ($1 \leq i \leq 24$)). For all i ($1 \leq i \leq 24$), an Lti-model is a structure (K, I) where (i) $K = \{\{T\}, \{F\}, \{T, F\}\}$, and (ii) I is an Lti-interpretation from the set of all wffs to K, this notion being defined according to the following conditions for each propositional variable p and wffs A, B: (1) $I(p) \in K$; (2a) $T \in I(\overset{\bullet}{\neg}A)$ iff $F \in I(A)$; (2b) $F \in I(\overset{\bullet}{\neg}A)$ iff $F \notin I(A)$; (3a) $T \in I(A \wedge B)$ iff $T \in I(A)$ and $T \in I(B)$; (3b) $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$; (4a) $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$; (4b) $F \in I(A \vee B)$ iff $F \in I(A)$ and $F \in I(B)$. There are two possibilities for assigning $\{T\}$ to conditionals:

- (5a1) $T \in I(A \to B)$ iff $[T \notin I(A)$ or $T \in I(B)]$.
- (5a2) $T \in I(A \to B)$ iff $T \notin I(A)$ or $F \notin I(B)$ or $[T \in I(A)$ & $F \in I(A)$ & $T \in I(B)$ & $F \in I(B)]$.

And there are seventeen different clauses for assigning $\{F\}$ to conditionals:

- (5b1) $F \in I(A \to B)$ iff $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(B)$ & $F \in I(B)]$ or $[F \notin I(A)$ & $T \notin I(B)]$.
- (5b2) $F \in I(A \to B)$ iff $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(B)$ & $F \in I(B)]$ or $[T \in I(A)$ & $F \in I(B)]$.
- (5b3) $F \in I(A \to B)$ iff $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(A)$ & $T \notin I(B)]$ or $[F \in I(A)$ & $T \in I(B)$ & $F \in I(B)]$.
- (5b4) $F \in I(A \to B)$ iff $[T \in I(B)$ & $F \in I(B)]$ or $[T \in I(A)$ & $F \in I(B)]$.

- (5b5) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[F \in I(A)$ & $T \in I(B)$ & $F \in I(B)]$.
- (5b6) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[F \notin I(A)$ & $F \in I(B)]$ or $[T \in I(A)$ & $T \in I(B)$ & $F \in I(B)]$ or $[T \in I(A)$ & $F \in I(A)$ & $F \notin I(B)]$.
- (5b7) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[T \in I(A)$ & $F \in I(A)$ & $F \notin I(B)]$ or $[T \notin I(A)$ & $T \in I(B)$ & $F \in I(B)]$.
- (5b8) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[F \notin I(A)$ & $F \in I(B)]$or $[T \notin I(A)$ & $T \in I(B)$ & $F \in I(B)]$.
- (5b9) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[T \notin I(A)$ & $T \in I(B)$ & $F \in I(B)]$.
- (5b10) $F \in I(A \to B)$ iff $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(A)$ & $F \in I(B)]$.
- (5b11) $F \in I(A \to B)$ iff $[T \in I(A)$ & $F \in I(A)]$ or $[T \in I(A)$ & $T \notin I(B)]$.
- (5b12) $F \in I(A \to B)$ iff $T \in I(A)$ & $F \in I(B)$.
- (5b13) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[T \in I(A)$ & $F \in I(A)$ & $F \in I(B)]$.
- (5b14) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[F \notin I(A)$ & $F \in I(B)]$or $[T \in I(A)$ & $F \in I(A)$ & $F \notin I(B)]$.
- (5b15) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[T \in I(A)$ & $F \in I(A)$ & $F \notin I(B)]$.
- (5b16) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[F \notin I(A)$ & $F \in I(B)]$.
- (5b17) $F \in I(A \to B)$ iff $T \in I(A)$ & $T \notin I(B)$.

Definition 3.2 (Lti-models II. ($25 \le i \le 30$)). For all i ($25 \le i \le 30$), an Lti-model is a structure (K, I) where (i) $K = \{\{T\}, \{F\}, \{\emptyset\}\}$, and (ii) I is an Lti-interpretation from the set of all wffs to K, according to the ensuing conditions (clauses) for each propositional variable p and wffs A, B: (1), (3a), (3b), (4a) and (4b) are as in Definition 2.1, and clauses for $\neg$ and $\to$ are as follows. (2a) $T \in I(\neg A)$ iff $F \in I(A)$; (2b) $F \in I(\neg A)$ iff $T \in I(A)$ or $F \notin I(A)$. There are two different clauses for assigning $\{T\}$ and four different ones for assigning $\{F\}$ to conditionals:

- (5a1) $T \in I(A \to B)$ iff $T \notin I(A)$ or $T \in I(B)$.
- (5a3) $T \in I(A \to B)$ iff $F \in I(A)$ or $T \in I(B)$ or $[T \notin I(A)$ & $F \notin I(B)]$.
- (5b12) $F \in I(A \to B)$ iff $T \in I(A)$ & $F \in I(B)$.
- (5b16) $F \in I(A \to B)$ iff $[T \in I(A)$ & $T \notin I(B)]$ or $[F \notin I(A)$ & $F \in I(B)]$.
- (5b17) $F \in I(A \to B)$ iff $T \in I(A)$ & $T \notin I(B)$.
- (5b18) $F \in I(A \to B)$ iff $F \notin I(A)$ & $F \in I(B)$.

Definition 3.3 (Li-consequence, Li-validity). Let M be an Lti-model ($1 \le i \le 30$). For any set of wffs Γ and wff A, $\Gamma \vDash_M A$ (A is a consequence of Γ in the Lti-model M) iff $T \in I(A)$ whenever $T \in I(\Gamma)$ ($T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$). Then, $\Gamma \vDash_{Lti} A$ (A is a consequence of Γ in Lti-semantics) iff $\Gamma \vDash_M A$ for each Lti-model M. In particular, $\vDash_{Lti} A$ (A is valid in Lti-semantics) iff $\vDash_M A$ for each Lti-model M (i.e., iff $T \in I(A)$ for each Lti-model M). (By $\vDash_{Lti}$ we shall refer to the relation just defined.)

Particular Lti-models are defined by providing the corresponding clauses for assigning $\{T\}$ and $\{F\}$ to conditionals. We have:

Definition 3.4 (Particular Lti-models ($1 \le i \le 30$)). For each i ($1 \le i \le 30$), an Lti-model is an Lti-model with the following clauses for the conditional:

- Lt1-models: (5a2) and (5b1).

- Lt2-models: (5a1) and (5b2).
- Lt3-models: (5a1) and (5b3).
- Lt4-models: (5a2) and (5b4).
- Lt5-models: (5a1) and (5b4).
- Lt6-models: (5a1) and (5b5).
- Lt7-models: (5a2) and (5b6).
- Lt8-models: (5a1) and (5b6).
- Lt9-models: (5a1) and (5b7).
- Lt10-models: (5a2) and (5b8).
- Lt11-models: (5a1) and (5b8).
- Lt12-models: (5a1) and (5b9).
- Lt13-models: (5a2) and (5b10).
- Lt14-models: (5a1) and (5b10).
- Lt15-models: (5a1) and (5b11).
- Lt16-models: (5a2) and (5b12).
- Lt17-models: (5a1) and (5b12).
- Lt18-models: (5a1) and (5b13).
- Lt19-models: (5a2) and (5b14).
- Lt20-models: (5a1) and (5b14).
- Lt21-models: (5a1) and (5b15).
- Lt22-models: (5a2) and (5b16).
- Lt23-models: (5a1) and (5b16).
- Lt24-models: (5a1) and (5b17).
- Lt25-models: (5a1) and (5b12).
- Lt26-models: (5a1) and (5b17).
- Lt27-models: (5a3) and (5b16).
- Lt28-models: (5a3) and (5b12).
- Lt29-models: (5a3) and (5b18).
- Lt30-models: (5a3) and (5b17).

Now, given Definitions 2.6, 2.7, 3.1, 3.2 and 3.4, we easily have:

Proposition 3.5 (Coextensiveness of $\vDash_{\mathrm{Mt}i}$ and $\vDash_{\mathrm{Lt}i}$)**.** *For any i ($1 \leq i \leq 30$), set of wffs Γ and wff A, $\Gamma \vDash_{Mti} A$ iff $\Gamma \vDash_{Lti} A$. In particular, $\vDash_{Mti} A$ iff $\vDash_{Lti} A$.*

Proof. Cf., e.g., the proof of Proposition 7.4 in (Robles et al., 2019). $\square$

The proof of Proposition 3.5 is a mere formalization of the intuitive translation (commented upon above) of the semantics based upon the matrix Mti into its corresponding o-semantics or u-semantics. Nevertheless, this proposition greatly simplifies the soundness and completeness proofs, since we can focus on the relation $\vDash_{\mathrm{Mt}i}$ in the former case, while restricting our attention to the relation $\vDash_{\mathrm{Lt}i}$ in the latter one. Thus, regarding soundness, it suffices to show that, given a matrix Mti, the rules of Lti preserve Mti–validity, whereas Lti-axioms are assigned a designated value. As for completeness, it is proved by means of a canonical model construction. In what follows, we give a general ideal of how the proof is developed.

Suppose that the Lti-logics have been defined (cf. Definition 4.4) and consider, for example, Lt11-models. Let $\mathcal{T}$ be a prime, non-trivial and complete (i.e., with either A or $\overset{\bullet}{\neg}A$ for each wff A) Lt11-theory containing all Lt11 theorems (cf. Definition 4.6). A canonical L11-model is a structure $(K, I_{\mathcal{T}})$ where K is defined as in Definition 3.1 and $I_{\mathcal{T}}$ is a function from the set of all wff to K defined as follows: for each wff A,

$T \in I_{\mathcal{T}}(A)$ iff $A \in \mathcal{T}$, and $F \in I_{\mathcal{T}}(A)$ iff $\neg A \in \mathcal{T}$. It is shown that $(K, I_{\mathcal{T}})$ is a canonical Lt11-model by proving by induction of the structure of A that $I_{\mathcal{T}}$ fulfils clauses (1), (2a), (2b), (3a), (3b), (4a), (4b), (5a1) and (5b8). That is, we have to prove: (a) $B \wedge C \in \mathcal{T}$ iff $B \in \mathcal{T}$ and $C \in \mathcal{T}$; (b) $\overset{\bullet}{\neg}(B \wedge C) \in \mathcal{T}$ iff $\overset{\bullet}{\neg}B \in \mathcal{T}$ or $\overset{\bullet}{\neg}C \in \mathcal{T}$; (c) $B \vee C \in \mathcal{T}$ iff $B \in \mathcal{T}$ or $C \in \mathcal{T}$; (d) $\overset{\bullet}{\neg}(B \vee C) \in \mathcal{T}$ iff $\overset{\bullet}{\neg}B \in \mathcal{T}$ and $\overset{\bullet}{\neg}C \in \mathcal{T}$; (e) $\overset{\bullet\,\bullet}{\neg\neg}B \in t$ iff $\overset{\bullet}{\neg}B \notin t$; (f) $B \rightarrow C \in \mathcal{T}$ iff or $B \notin \mathcal{T}$ or $C \in \mathcal{T}$; (g) $\overset{\bullet}{\neg}(B \rightarrow C)$ iff $[B \in \mathcal{T}$ & $C \notin \mathcal{T}]$ or $[\overset{\bullet}{\neg}B \notin \mathcal{T}$ & $\overset{\bullet}{\neg}C \in \mathcal{T}]$ or $[B \notin \mathcal{T}$ & $C \in \mathcal{T}$ & $\overset{\bullet}{\neg}C \in \mathcal{T}]$. Once canonical Lt11-models are shown Lt11-models, completeness is proved as follows. Suppose that $\Gamma \nvdash_{\mathrm{Lt11}} A$, i.e., A is not included in the set of consequences derivable in Lt11 from Γ ((in symbols, $A \notin \mathrm{Cn}\Gamma[\mathrm{Lt11}]$). Then, $\mathrm{Cn}\Gamma[\mathrm{Lt11}]$ is extended to a prime Lt11-theory $\mathcal{T}$ such that $A \notin \mathcal{T}$. Clearly, $\mathcal{T}$ contains all Lt11-theorems, it is non-trivial and for each wff B, $B \in \mathcal{T}$ or $\overset{\bullet}{\neg}B \in \mathcal{T}$ ($\mathcal{T}$ is prime and $C \vee \overset{\bullet}{\neg}C$ is an Lt11-theorem). Next, the canonical Lt11-model $\mathrm{M} = (K, I_{\mathcal{T}})$ based upon $\mathcal{T}$ is defined, and we have $\Gamma \nvDash_{\mathrm{M}} A$ since $T \in I_{\mathcal{T}}(\Gamma)$ (as $T \in I_{\mathcal{T}}(\mathrm{Cn}\Gamma[\mathrm{Lt11}])$) but $T \notin I_{\mathcal{T}}(A)$, whence $\Gamma \nvDash_{\mathrm{Lt11}} A$ (by Definitions 3.1, 3.3 and 3.4), as it was to be proved.

Completeness of the rest of the Lti-logics is proved in a similar way. However, it has to be remarked that, in the case of Lt25 through Lt30, the function $I_{\mathcal{T}}$ is defined upon a prime, non-trivial and consistent theory containing all theorems, instead of upon a complete one (a theory is consistent if it does not contain contradictions). Of course, this has to do with the fact that Lt1-models through Lt24-models together with the corresponding notions of consequence and validity are examples of o-semantics, while Lt25 through Lt30-models along with the corresponding notions of consequence and validity are instances of u-semantics.

In the following section, we prove the two facts that are required in the completeness proofs, as shown above:

(1) An Lti-theory without a given wff can be extended to a prime Lti-theory without this same wff.
(2) Let $\mathcal{T}$ be a prime, non-trivial and complete Lti-theory ($1 \leq i \leq 24$) (or prime, non-trivial and consistent Lti-theory ($25 \leq i \leq 30$)). Then, the canonical translations of clauses (1), (2a), (2b), (3a), (3b), (4a), (4b) and that of the corresponding clauses for the conditional are provable in $\mathcal{T}$.

As pointed out in the introduction, we follow the strategy set up in (Brady, 1982), as applied in (Robles & Méndez, 2019) and (Robles et al., 2019). Thus, it is possible to be reasonably general about the details and many of the proofs will be referred to the two last papers quoted above.

4. The Lti-logics

In this section the Lti-logics are defined, and the two fundamental facts remarked in the preceding section, upon which the completeness proofs rely, are proved. As pointed out in the introduction, we have tried to formulate the Lti-logics in the most possible general way. Thus, Lti-logics ($1 \leq i \leq 24$) are axiomatized from two basic logics $\mathrm{b}^3_{1\mathrm{D}}$ and $\mathrm{b}^3_{2\mathrm{D}}$ (in their turn, extending a more basic one, $\mathrm{b}^3_{\mathrm{D}}$), whereas Lt$i$-logics ($25 \leq i \leq 30$) are axiomatized, in a parallel way, from logics $\mathrm{b}^3_{1\mathrm{G}}$ and $\mathrm{b}^3_{2\mathrm{G}}$ (in their turn extending the basic logic $\mathrm{b}^3_{\mathrm{G}}$). (In the following section, more economical and conspicuous axiomatizations of some of the Lti-logics are provided.) In the definitions to follow, the general symbol for negation, $\sim$, is used since some axioms and/or rules

are shared by both groups of logics and their respective extensions, but it has to be understood that logics extending b_G^3 are endowed with Gödel-type negation, $\neg$, and those extending b_D^3, with dual Gödel-type negation $\overset{\bullet}{\neg}$.

Definition 4.1 (The basic logics b_D^3 and b_{1D}^3)**.** The logic b_D^3 is axiomatized with the following axioms and rules:

    Axioms:

> A1. $A \rightarrow A$
> A2. $(A \wedge B) \rightarrow A$; $(A \wedge B) \rightarrow B$
> A3. $[(A \rightarrow B) \wedge (A \rightarrow C)] \rightarrow [A \rightarrow (B \wedge C)]$
> A4. $A \rightarrow (A \vee B)$; $B \rightarrow (A \vee B)$
> A5. $[(A \rightarrow C) \wedge (B \rightarrow C)] \rightarrow [(A \vee B) \rightarrow C]$
> A6. $[A \wedge (B \vee C)] \rightarrow [(A \wedge B) \vee (A \wedge C)]$
> A7. $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$
> A8. $[(A \rightarrow B) \wedge A] \rightarrow B$
> A9. $A \vee (A \rightarrow B)$
> A10. $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
> A11. $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
> A12. $\sim\sim A \rightarrow A$
> A13. $\sim A \vee \sim\sim A$
> A14. $(\sim A \wedge \sim\sim\sim A) \rightarrow B$

    Rules of inference:

> Adjunction (Adj): $A, B \Rightarrow A \wedge B$
> Modus Ponens (MP): $A \rightarrow B, A \Rightarrow B$

Then, the logic b_{1D}^3 (resp., b_{2D}^3) is axiomatized by adding to b_D^3 A15 (resp., A16, A17 and A18)

> A15. $B \rightarrow (A \rightarrow B)$
> A16. $[(A \rightarrow B) \wedge \sim B] \rightarrow \sim A$
> A17. $\sim B \vee (A \rightarrow B)$
> A18. $[(A \wedge \sim A) \wedge B] \rightarrow (A \rightarrow B)$

Definition 4.2 (The basic logics b_G^3, b_{1G}^3 and b_{2G}^3)**.** The logic b_G^3 is axiomatized with the following axioms and rules: Adj, MP, A1, A2, A3, A4, A5, A6, A10, A11, A13 in Definition 4.1 and, in addition,

> A19. $A \rightarrow \sim\sim A$
> A20. $(A \wedge \sim A) \rightarrow B$
> Disjunctive Modus Ponens (dMP): $C \vee (A \rightarrow B), C \vee A \Rightarrow C \vee B$
> Disjunctive Transitivity (dTrans): $D \vee (A \rightarrow B), D \vee (B \rightarrow C) \Rightarrow D \vee (A \rightarrow C)$

Then, the logic b_{1G}^3 (resp., b_{2G}^3) is axiomatized by adding to b_G^3 the axioms A8, A9, A15 (resp., A21, A22 and the rules dr1 and dr2)

$$A21. \sim A \to [A \vee (A \to B)]$$
$$A22. (A \vee \sim B) \vee (A \to B)$$
$$dr1. (C \vee B) \Rightarrow C \vee [\sim B \vee (A \to B)]$$
$$dr2. C \vee (A \to B), C \vee \sim B \Rightarrow C \vee \sim A$$

Remark 4.3 (On disjunctive rules). On the need of strengthen some rules to their disjunctive forms, given the strategy here adopted for proving completeness (cf. the end of section 3), see (Robles & Méndez, 2020, §5) and (Robles, 2020, §6).

The Lti-logics are axiomatized by adding to the basic logics some subset of the following set of axioms and rules

$$A23. (A \wedge \sim A) \to \sim(A \to B)$$
$$A24. (B \wedge \sim B) \to \sim(A \to B)$$
$$A25. (A \wedge \sim B) \to \sim(A \to B)$$
$$A26. \sim(A \to B) \to (\sim A \vee \sim B)$$
$$A27. \sim(A \to B) \to (A \vee B)$$
$$A28. \sim(A \to B) \to (A \vee \sim B)$$
$$A29. (B \wedge \sim B) \to [A \vee \sim(A \to B)]$$
$$A30. A \to [B \vee \sim(A \to B)]$$
$$A31. [\sim(A \to B) \wedge B] \to \sim A$$
$$A32. [\sim(A \to B) \wedge B] \to \sim B$$
$$A33. \sim(A \to B) \to \sim B$$
$$A34. [(B \wedge \sim B) \wedge \sim A] \to \sim(A \to B)$$
$$A35. \sim B \to [\sim A \vee \sim(A \to B)]$$
$$A36. (A \wedge \sim A) \to [\sim B \vee \sim(A \to B)]$$
$$A37. [\sim(A \to B) \wedge [(A \wedge \sim A) \wedge (B \wedge \sim B)]] \to C$$
$$A38. [\sim(A \to B) \wedge [(B \wedge \sim B) \wedge A]] \to C$$
$$A39. [\sim(A \to B) \wedge (A \wedge B)] \to C$$
$$A40. [\sim(A \to B) \wedge \sim A] \to A$$
$$A41. \sim(A \to B) \to A$$
$$A42. \sim(A \to B) \to (A \wedge \sim B)$$
$$A43. [(A \wedge \sim A) \wedge \sim B] \to \sim(A \to B)$$
$$A44. [[\sim(A \to B) \wedge (B \wedge \sim B)] \wedge \sim A] \to C$$
$$A45. [\sim(A \to B) \wedge (B \wedge \sim B)] \to C$$
$$A46. [\sim(A \to B) \wedge (\sim A \wedge B)] \to C$$
$$A47. [\sim(A \to B) \wedge B] \to C$$
$$dr3. C \vee \sim(A \to B), C \vee \sim A \Rightarrow C \vee (A \vee B)$$
$$dr4. C \vee \sim(A \to B), C \vee \sim A \Rightarrow C \vee (A \vee \sim B)$$
$$dr5. C \vee \sim(A \to B), C \vee B \Rightarrow C \vee (\sim A \vee \sim B)$$

$$\text{dr6. } C \vee \sim(A \to B), C \vee \sim A \Rightarrow C \vee A$$
$$\text{dr7. } C \vee \sim(A \to B), C \vee B \Rightarrow C \vee \sim B$$
$$\text{dr8. } C \vee (A \wedge \sim B) \Rightarrow C \vee \sim(A \to B)$$

We have:

Definition 4.4 (The Lti-logics $(1 \leq i \leq 30)$)**.** (a) Extensions of b^3_{1D}: The ensuing Lti-logics are axiomatized by adding to b^3_{1D} the following axioms:

- Lt2: A23, A24, A25, A26, A27, A28.
- Lt3: A27, A28, A29, A30, A31.
- Lt5: A24, A25, A27, A33.
- Lt6: A27, A30, A31, A33, A34.
- Lt8: A26, A27, A28, A29, A30, A35, A36, A37.
- Lt9: A27, A28, A29, A30, A31, A36, A38.
- Lt11: A27, A29, A30, A32, A35, A38.
- Lt12: A27, A29, A30, A33, A39.
- Lt14: A23, A25, A26, A40.
- Lt15: A23, A30, A31, A41.
- Lt17: A25, A42.
- Lt18: A30, A31, A42, A43.
- Lt20: A26, A30, A35, A36, A41, A44.
- Lt21: A30, A31, A36, A41, A45.
- Lt23: A30, A35, A42, A46.
- Lt24: A30, A42, A47.

(b) Extensions of b^3_{2D}: The ensuing Lti-logics are axiomatized by adding to b^3_{2D} the following axioms:

- Lt1: A23, A24, A25, A26, dr3, dr4, dr5.
- Lt4: A24, A25, A28, A32, dr3.
- Lt7: A26, A29, A30, A35, A36, A37, dr3, dr4.
- Lt10: A29, A30, A32, A35, A38, dr3.
- Lt13: A23, A25, A26, dr6.
- Lt16: A25, A32, dr6.
- Lt19: A26, A30, A35, A36, A38, dr6.
- Lt22: A30, A32, A35, A46, dr6.

(c) Extensions of b^3_{1G}: The ensuing Lti-logics are axiomatized by adding to b^3_{1G} the following axioms:

- Lt25: A25, A42.
- Lt26: A30, A32, A41.

(d) Extensions of b^3_{2G}: The ensuing Lti-logics are axiomatized by adding to b^3_{2G} the following axioms:

- Lt27: A30, A35, A40, A46, dr7.
- Lt28: A42, dr8.
- Lt29: A33, A35, A40.
- Lt30: A30, A41, dr7.

The following proposition can be useful for proving some of the facts upon which

the completeness proofs are based.

Proposition 4.5 (The rules Trans and dr1′-dr8′)**.** *The rules $A \to B, B \to C \Rightarrow A \to C$ (Trans); $B \Rightarrow \sim B \vee (A \to B)$ (dr1′); $A \to B, \sim B \Rightarrow \sim A$ (dr2′); $\sim(A \to B), \sim A \Rightarrow A \vee B$ (dr3′); $\sim(A \to B), \sim A \Rightarrow A \vee \sim B$ (dr4′); $\sim(A \to B), B \Rightarrow \sim A \vee \sim B$ (dr5′); $\sim(A \to B), \sim A \Rightarrow A$ (dr6′); $\sim(A \to B), B \Rightarrow \sim B$ (dr7′); $A \wedge \sim B \Rightarrow \sim(A \to B)$ (dr8′) are provable from FDE_+ and the respective disjunctive version. (The logic FDE_+ is the negationless fragment of Anderson and Belnap's First degree entailment logic FDE. It can be axiomatized with (cf. Definition 4.1) A1, A2, A4, A6, Trans, MP and A3 and A5 in rule form; (cf. Anderson & Belnap, 1975; Slaney, 1995).)*

Proof. It is immediate (notice that dr2′ is the rule *Modus Tollens*). □

In the sequel, we proceed to prove the facts (1) and (2), instrumental in the completeness proofs, as discussed at the end of the preceding section. We begin with some definitions.

Definition 4.6 (Some preliminary notions)**.** Let us refer by Eb^3 to the family of extensions of the basic logics (cf. Definitions 4.1 and 4.2) and the Lti-logics ($1 \leq i \leq 30$) (cf. Definition 4.4). (In general, by EL, we mean an extension of the logic L.) And let L be an Eb^3-logic. An L-theory is a set of wffs closed under Adjunction (Adj), provable L-entailment (L-ent) and all the rules of L (A theory t is closed under Adj if whenever $A, B \in t$, then $A \wedge B \in t$; it is closed under L-ent iff whenever $A \to B$ is an L-theorem and $A \in t$, then $B \in t$). For example, an Lt27-theory is a set of wffs closed under Adj, L27-ent, dMP, dTrans, dr1, dr2 and dr7. Next, t is *regular* iff it contains all L-theorems; t is *prime* if it enjoys the *disjunction property* (i.e., if $A \vee B \in t$, then $A \in t$ or $B \in t$); t is *complete* iff for each wff A, $A \in t$ or $\sim A \in t$; t is inconsistent iff $A \wedge \sim A \in t$ for some wff A; t is consistent iff it is not inconsistent; finally, t is trivial iff it contains all wffs.

Next, the extensions and primeness lemmas are proved. The proofs, similar to those in (Robles & Méndez, 2019) and (Robles et al., 2019), are referred to these papers.

Definition 4.7 (Disjunctive Eb^3-derivability)**.** Let L be an Eb^3-logic. For any sets of wffs Γ, Θ, Θ is disjunctively derivable from Γ in L (in symbols, $\Gamma \vdash_L^d \Theta$) iff $A_1 \wedge ... \wedge A_n \vdash_L B_1 \vee ... \vee B_m$ for some wffs $A_1, ..., A_n \in \Gamma$ and $B_1, ..., B_m \in \Theta$.

Lemma 4.8 (Preliminary lemma to the extension lemma)**.** *Let L be an Eb^3-logic whose primitive rules of inference are in the set $\rho = \{Adj, MP, dMP, dTrans, dr1, dr2, dr3, dr4, dr5, dr6, dr7, dr8\}$ for any set of wffs $A, B_1, ..., B_n$ if $\{B_1, ..., B_n\} \vdash_L A$, then, for any wff C, $C \vee (B_1 \wedge ... \wedge B_n) \vdash_L C \vee A$.*

Proof. Similar to that in Lemma 6.2 in (Robles et al., 2019). □

Definition 4.9 (Maximal sets)**.** Let L be an Eb^3-logic. Γ is an L-maximal set of wffs iff $\Gamma \vdash_L^d \bar{\Gamma}$ ($\bar{\Gamma}$ is the complement of Γ).

Lemma 4.10 (Extension to maximal sets)**.** *Let L be an Eb^3-logic whose primitive rules of inference are in the set ρ (cf. Lemma 4.8) and let Γ, Θ be sets of wffs such that $\Gamma \nvdash_L^d \Theta$. Then, there are sets of wffs Γ', Θ' such that $\Gamma \subseteq \Gamma', \Theta \subseteq \Theta', \Theta' = \bar{\Gamma}'$ and $\bar{\Gamma} \nvdash_L^d \Theta'$ (that is, Γ' is an L-maximal set such that $\Gamma' \nvdash_L^d \Theta'$).*

Proof. By leaning on Lemma 4.8, as in Lemma 6.4 in (Robles et al., 2019). □

Lemma 4.11 (Primeness). *Let L be an Eb^3-logic whose primitive rules of inference are in the set ρ (cf. Lemma 4.8). If Γ is a L-maximal set, then it is a prime L-theory closed under the rules of L.*

Proof. Similar to that of Lemma 6.5 in (Robles et al., 2019). $\square$

Thus, fundamental fact (1) is proved. Next, we proceed to the proof of fundamental fact (2).

Proposition 4.12 (Conjunction and disjunction in prime Eb^3-theories). *Let L be an Eb^3-logic and t be a prime L-theory. Then, (1) $A \wedge B \in t$ iff $A \in t$ and $B \in t$; (2) $\sim(A \wedge B) \in t$ iff $\sim A \in t$ or $\sim B \in t$; (3) $A \vee B \in t$ iff $A \in t$ or $B \in t$; (4) $\sim(A \vee B) \in t$ iff $\sim A \in t$ and $\sim B \in t$.*

Proof. By De Morgan laws (A10 and A11) together with A2-A5. $\square$

Proposition 4.11 guaranties that the canonical interpretation of conjunction and disjunction works in any Eb^3-logic (clauses (3a), (3b), (4a), (4b) in Definitions 3.1 and 3.2). Regarding negation, more specific logics are needed.

Proposition 4.13 (Gödel neg. in prime cons. Eb_G^3-theories). *Let L be an Eb_G^3-logic and t be a prime consistent L-theory. Then, $\neg\neg A \in t$ iff $A \in t$ or $\neg A \notin t$.*

Proof. ($\Rightarrow$) Suppose $\neg\neg A \in t$. By consistency of t, $\neg A \notin t$. ($\Leftarrow$) Suppose $A \in t$. Then, $\neg\neg A \in t$ by $A \to \neg\neg A$ (A19). On the other hand, suppose $\neg A \notin t$. By $\neg A \vee \neg\neg A$ (A13) and primeness of t, $\neg\neg A \in t$. $\square$

Proposition 4.14 (Dual Gödel neg. in prime non-trivial Eb_D^3-theories). *Let L be an Eb_D^3-logic and t be a prime non-trivial L-theory. Then, $\overset{\bullet\bullet}{\neg\neg} A \in t$ iff $A \in t$ or $\overset{\bullet}{\neg} A \notin t$.*

Proof. ($\Rightarrow$) Suppose $\overset{\bullet\bullet}{\neg\neg} A \in t$ and, for *reductio*, $\overset{\bullet}{\neg} A \in t$. By $(\overset{\bullet}{\neg} A \wedge \overset{\bullet}{\neg}\overset{\bullet}{\neg} A) \to B$ (A14), we have $B \in t$ for arbitrary wff B, contradicting the non-triviality of t. ($\Leftarrow$) Suppose $A \in t$. Then, $\overset{\bullet\bullet}{\neg\neg} A \in t$ by $A \vee \overset{\bullet\bullet}{\neg\neg} A$ (A19). Suppose then $\overset{\bullet}{\neg} A \notin t$. Then, $\overset{\bullet\bullet}{\neg\neg} A \in t$ by $\overset{\bullet}{\neg} A \vee \overset{\bullet\bullet}{\neg\neg} A$ (A13) and primeness of t. $\square$

Proposition 4.13 takes care of the canonical interpretation of clause (2b) in Definition 3.1 (clause (2a) is trivial), and Proposition 4.14, of that of clause (2b) in Definition 3.2 (clause (2a) is again trivial). So, let us proceed to the conditional case.

Proposition 4.15 (The conditional in prime Eb_G^3-theories). *(a) Let L be an Eb_{1G}^3-logic and t be a prime regular L-theory. Then, $A \to B \in t$ iff $A \notin t$ or $B \in t$. (b) (a) Let L be an Eb_{2G}^3-logic and t be a prime regular and consistent L-theory. Then, $A \to B \in t$ iff $\neg A \in t$ or $B \in t$ or $(A \notin t$ and $\neg B \notin t)$.*

Proof. (a) ($\Rightarrow$) Immediate by $[(A \to B) \wedge A] \to B$ (A8). (a) ($\Leftarrow$): Suppose $A \notin t$. $A \to B \in t$ follows by $A \vee (A \to B)$ (A9) together with primeness and regularity of t. Suppose now $B \in t$. Then, $A \to B \in t$ is immediate by $B \to (A \to B)$ (A15).

(b) ($\Rightarrow$): Suppose (1) $A \to B \in t$ and, for *reductio*, (2) $\neg A \notin t$ and $A \in t$ and $B \notin t$ or (3) $\neg A \notin t$, $B \notin t$ and $\neg B \in t$. But 2 and 3 are impossible by closure of t under MP and Modus Tollens (i.e, dr2$'$; cf. Proposition 4.5). (b) ($\Leftarrow$): Suppose (1) $\neg A \in t$ or (2) $B \in t$ or (3) $A \notin t$ and $\neg B \notin t$. We have to prove $A \to B \in t$. Case (1) follows by $\neg A \to [A \vee (A \to B)]$ (A21) and primeness and consistency of t. Case (2) is provable in a similar way by using closure of t under dr1$'$ ($B \Rightarrow \neg B \vee (A \to B)$; cf.

Proposition 4.5). Finally, case (3) follows by $(A \vee \neg B) \vee (A \rightarrow B)$ (A22) and primeness and regularity of t. $\qquad\square$

Proposition 4.14 shows that the canonical interpretation of clauses (5a1) and (5a3) in Definition 3.2 hold.

A corresponding proposition for clauses (5a1) and (5a2) in Definition 3.1 is the following.

Proposition 4.16 (The conditional in prime Eb_D^3-theories). *(a) Let L be an Eb_{1D}^3-logic and t be a prime regular L-theory. Then, $A \rightarrow B \in t$ iff $A \notin t$ or $B \in t$. (b) (a) Let L be an Eb_{2D}^3-logic and t be a prime regular L-theory. Then, $A \rightarrow B \in t$ iff $A \notin t$ or $\overset{\bullet}{\neg} B \notin t$ or $[A \in t$ & $\overset{\bullet}{\neg} A \in t$ & $B \in t$ & $\overset{\bullet}{\neg} B \in t]$.*

Proof. (a) By A8, A9, A15, similarly as in Proposition 4.15(a). (b) By using A8, A9, A16, A17 and A18, similarly as in Proposition 4.15 or as in case (a). $\qquad\square$

It is interesting to note that, as shown in Propositions 4.15 and 4.16, the characteristic axioms of the basic logics extending b_G^3 and b_D^3 suffice to prove said propositions and, consequently, the canonical validity of clauses (5a1), (5a2) and (5a3). Concerning the clauses for assigning $\{F\}$ to conditionals (cf. Definition 3.4), these are proved to hold canonically by using the characteristic axioms and/or rules added to the basic logics in order to define the particular Lti-logics (cf. Definitions 4.4). Let us propose an example.

Proposition 4.17 (Negated conditionals in ELt27-logics). *Let L be an ELt27-logic and t a prime, regular and consistent L-theory. Then, $\neg(A \rightarrow B) \in t$ iff $[A \in t$ & $B \notin t]$ or $[\neg A \notin t$ & $\neg B \in t]$.*

Proof. (a) ($\Rightarrow$) Suppose (1) $\neg(A \rightarrow B) \in t$ and, for *reductio*, that one of the following (2)-(4) holds

$$2.\ A \notin t\ \&\ \neg A \in t$$
$$3.\ A \notin t\ \&\ \neg B \notin t$$
$$4.\ B \in t\ \&\ \neg A \in t$$
$$5.\ B \in t\ \&\ \neg B \notin t$$

But 2, 3, 4, and 5 are impossible by primeness, regularity and non-triviality of t and $[\neg(A \rightarrow B) \wedge \neg A] \rightarrow A$ (A40), $(A \vee \neg B) \vee (A \rightarrow B)$ (A22), $[\neg(A \rightarrow B) \wedge (\neg A \wedge B)] \rightarrow C$ (A46) and $\neg(A \rightarrow B), B \Rightarrow B$ (dr7′), respectively.

(b) ($\Leftarrow$) By primeness and completeness of t and $A \rightarrow [B \vee \neg(A \rightarrow B)]$ (A30) and $\neg B \rightarrow [\neg A \vee \neg(A \rightarrow B)]$ (A35). $\qquad\square$

On the basis of the discussion developed so far in this section, we consider proved the fundamental facts (1) and (2); then, on the basis of the argumentation developed at the end of section 3 together with facts (1) and (2), we think that we are entitled to state the following theorem.

Theorem 4.18 (Soundness and completeness of the Lti-logics). *For any i ($1 \leq i \leq 30$), set of wffs Γ and wff A, (1) $\Gamma \vDash_{Lti} A$ iff $\Gamma \vdash_{Lti} A$; (2) $\Gamma \vDash_{Mti} A$ iff $\Gamma \vdash_{Lti} A$.*

5. Some facts about the Lt*i*-logics

For each i ($1 \leq i \leq 30$), let Mt$i_\to$ and Mti_+ be defined similarly as Mti except that F is restricted to $f_\to$ in the former case and to $f_\wedge$, $f_\vee$ and $f_\to$ in the latter one. In addition, let Mti' be the result of adding to Mti_+ Łukasiewicz-type negation (given by the table $\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline \sim & 2 & 1 & 0 \end{array}$) instead of Gödel type negation or dual Gödel-type negation. Without trying to be exhaustive, we remark some facts concerning the matrices just defined.

- The logics Lt1′, Lt2′, ..., Lt30′ determined by Mt1′, Mt2′, ..., Mt30′, respectively, have been given Hilbert-type formulations in Robles & Méndez (2019) and (Robles et al., 2019). We remark that there are very well known logics among them such as Łukasiewicz's 3-valued logic Ł3 (Lt28′), paraconsistent logic Pac (Lt16′). (Cf. (Robles & Méndez, 2019; Robles et al., 2019) and (Karpenko, 1999, and references therein).)
- Some of the Mt$i_\to$ matrices define important conditionals such as he following: Sobocinski's (Mt16$_\to$), Jaskowski's (Mt17$_\to$), Sette's (Mt24$_\to$), Łukasiewicz's (Mt28$_\to$), Carnielli, Marcos and Amo's (Mt27$_\to$) or, finally, that of Gödel 3-valued logic G3 (Mt29$_\to$) (cf. (Sobociński, 1952), (Jaśkowski, 1948), (Sette, 1973), (Łukasiewicz, 1920), (Carnielli, Marcos, & Amos, 2000), (Gödel, 1932), respectivley; cf. also (Heyting, 1930) and (Karpenko, 1999) and references therein[3]).

Regarding the logics defined in the present paper, it is possible that more than we are aware of have been particularized in the literature (cf. §6). Anyway, we remark those we know that have been studied before. We have, of course, Gödel 3-valued logic G3 (Lt29) (cf. Gödel, 1932), but also two more interesting logics: Lt24 in (Sette, 1973) and Lt27 in (Yang, 2012).

In the sequel, we briefly comment on some of the properties of the logics defined in this paper. Let us refer by Lta (resp., Ltb) to the Lti-logics Lt1, Lt2, ..., Lt24 (resp., Lt25, ..., Lt30).

- *Paraconsistency of the Lta-logics*: All Lta-logics are *paraconsistent* in the sense that the rule ECQ, $A \wedge \overset{\bullet}{\neg} A \Rightarrow B$, fails in each one of them: given a particular matrix Mti, take an Mti-interpretation I such that $I(p) = 1$ and $I(q) = 0$ for distinct propositional variables p, q being substituted instead of A and B.
- *Paracompleteness of Ltb-logics*: All Ltb-logics are *paracomplete* in the sense that not every prime, regular and consistent Ltb-theory t is such that $A \in t$ or $\neg A \in t$ for any wff A: Let L be an Ltb-logic and ThL be the set of all L-theorems. Clearly, $p \vee \neg p \notin$ ThL. Then, ThL is extended to a prime L-theory $\mathcal{T}$ such that $p \vee \neg p \notin \mathcal{T}$ (cf. Lemma 4.11). $\mathcal{T}$ is prime, regular and consistent, but not complete.
- *Ltc-logics: Lti-logics containing C_+*: The logics Lt2, Lt3, Lt5, Lt6, Lt8, Lt9, Lt11, Lt12, Lt14, Lt15, Lt17, Lt18, Lt20, Lt21, Lt23, Lt24, Lt25 and Lt26 contain positive classical propositional logic C_+: in addition to MP, Adj and A2-A5 (cf.

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[3] A referee of the JANCL points out that there are other important conditionals among Mt$i_\to$ matrices, in addition to the ones we have mentioned: another one by Sobociński (Mt11$_\to$), Rescher's implication (Mt27$_\to$), Słupecki, Bryll and Pracnal's (Mt25$_\to$), Bochvar's (Mt26$_\to$) (cf. (Sobociński, 1936), (Rescher, 1969), (Słupecki, Bryll, & Prucnal, 1967) and (Bochvar & Bergmann, 1981), respectively). On the other hand, notice that Rescher's and Carnielli, Marcos and Amo's implication refer to the same table (Mt27$_\to$). Observe also that Jaśkowski himself notes that (Mt17$_\to$) was first studied by Słupecki (Słupecki, 1939).

Definition 4.1), each one of them has the theorems $A \to (B \to A)$, $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$ and $A \vee (A \to B)$. Notice that Lt25 and Lt26 are expansions of C_+ with Gödel-type negation, whereas the rest of the Ltc-logics expand C_+ with dual Gödel-type negation.

- *Inadmissibility of Con in Ltc-logics*: The rule *Contraposition* (Con), $A \to B \Rightarrow {\sim}B \to {\sim}A$, is inadmissible in all Ltc-logics. Consider, for example, the wff $\overset{\bullet}{\neg}(A \to B) \to A$. It is a theorem of Lt20, Lt21, t23 and Lt24, but $\overset{\bullet}{\neg}A \to \overset{\bullet\bullet}{\neg}(A \to B)$ is not ($I[\overset{\bullet}{\neg}p \to \overset{\bullet\bullet}{\neg}(p \to q)] = 0$ for any M$t i$-interpretation I such that $I(p) = 1$ and $I(q) = 0$ —$i \in \{20, 21, 22, 23, 24\}$). The Lt25-theorem $(A \wedge \neg B) \to \neg(A \to B)$ and the Lt26-theorem $[\neg(A \to B) \wedge B] \to \neg B$ can be used in a similar way to show the inadmissibility of Con in Lt25 and Lt26. Finally, inadmissibility of Con in the rest of the Ltc-logics can be proved by using the wff $\overset{\bullet}{\neg}(A \to B) \to (A \vee B)$, a common theorem to all of them (the tester in (González, 2011) can be used in case it is needed).

- *On the alternative axiomatizations of the Ltc-logics*: As they contain C_+, all Ltc-logics can be axiomatized with the following common basis: (a1) $A \to (B \to A)$; (a2) $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$; (a3) $(A \wedge B) \to A$, $(A \wedge B) \to B$; (a4) $A \to [B \to (A \wedge B)]$; (a5) $A \to (A \vee B)$, $B \to (A \vee B)$; (a6) $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$; (a7) $A \vee (A \to B)$; (a8) ${\sim}(A \vee B) \leftrightarrow ({\sim}A \vee {\sim}B)$; (a9) ${\sim}(A \wedge B) \leftrightarrow ({\sim}A \vee {\sim}B)$; (a10) $\overset{\bullet\bullet}{\neg}A \to A$; (a11) $A \vee \overset{\bullet}{\neg}A$; (a12) $({\sim}A \wedge {\sim}{\sim}A) \to B$ and MP as the sole rule of inference. (In the case of Lt25 and Lt26, a10, a11 and a12 are replaced by (a10$'$) $A \to \neg\neg A$, (a11$'$) $\neg A \vee \neg\neg A$ and (a12$'$) $(A \wedge \neg A) \to B$.) Then, each particular logic is axiomatized by adding the corresponding axioms in Definition 4.4 (in most cases, this set of axioms can be simplified by using the common basis).

- *The deduction theorem in Ltc-logics*: All Ltc-logics have the deduction theorem as MP is the sole rule of inference and a1 and a2 are theorems. However, notice that they are not self-extensional logics (they lack the replacement of equivalents theorem), since the rule Con is inadmissible in each one of them.

- *Ltd-logics: Lti-logics with the contraposition axiom or the contraposition rule*: The contraposition axiom, $(A \to B) \to ({\sim}B \to {\sim}A)$, holds in all Lta-logics which are not Ltc-logics, i.e., Lt1, Lt4, Lt7, Lt10, Lt13, Lt16, Lt19 and Lt22. It is also provable in Lt27 and Lt29 whereas in Lt28 and Lt30 it is restricted to the rule form Con, $A \to B \Rightarrow \neg B \to \neg A$ (notice that Lt28 is the expansion with Gödel-type negation of positive Łukasiewicz's 3-valued logic Ł3). Let us refer by Ltd to all the Lti-logics just mentioned and by Ltd1 to the Lta-logics referred to above.

- *On alternative axiomatizations of the Ltd-logics*: Given the presence of the contraposition axiom or the contraposition rule, Ltd-logics can be given axiomatizations simpler than those provided in §4.

  (1) *Ltd1-logics.* Common basis: (a1) $A \to A$; (a2) $(A \wedge B) \to A$ / $(A \wedge B) \to B$; (a3) $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$; (a4) $A \to (A \vee B)$ / $B \to (A \vee B)$; (a5) $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$; (a6) $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$; (a7) $[(A \to B) \wedge (B \to C)] \to (A \to C)$; (a8) $[(A \to B) \wedge A] \to B$; (a9) $A \vee (A \to B)$; (a10) $(\overset{\bullet}{\neg}A \wedge \overset{\bullet}{\neg}B) \to \overset{\bullet}{\neg}(A \vee B)$; (a11) $\overset{\bullet\bullet}{\neg}A \to A$; (a12) $(\overset{\bullet}{\neg}A \wedge \overset{\bullet}{\neg}\,\overset{\bullet}{\neg}A) \to B$; (a13) $[(A \to B) \wedge \overset{\bullet}{\neg}B] \to \overset{\bullet}{\neg}A$; (a14) $\overset{\bullet}{\neg}B \vee \overset{\bullet}{\neg}(A \to B)$; (a15) $[(A \wedge \overset{\bullet}{\neg}A) \wedge B] \to (A \to B)$; (a16) $(A \to B) \to (\overset{\bullet}{\neg}B \to \overset{\bullet}{\neg}A)$ and (a17) $\overset{\bullet}{\neg}A \vee \overset{\bullet\bullet}{\neg}A$ with Adj and MP as rules of inference.

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Then, each particular logic is axiomatized by adding the corresponding axioms and/or rules in Definition 4.4.

(2) *Ltb-logics.* Common basis: a1-a6 as in Ltd1-logics above. Then, (a7) $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$; (a8) $A \rightarrow \neg\neg A$; (a9) $(A \wedge \neg A) \rightarrow B$; (a10) $\neg A \rightarrow [A \vee (A \rightarrow B)]$; (a11) $(A \vee \neg B) \vee (A \rightarrow B)$; (a12) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ and MP, Adj, dMP, dTrans, dr1 and dr2 as rules of inference (a12 is changed for the rule Con in Lt28 and Lt30). The particular logics are axiomatized by adding the corresponding axioms and/or rules in Definition 4.4.

As in the case of Ltc-logics, the axiomatizations of Ltd1-logics and Ltb-logics just defined could be simplified (notice that, for instance, Lt27, Lt28 and Lt29 contain strong positive logics, as pointed out above).

- *Self-extensionality in Ltd-logics*: All Ltd-logics enjoy the replacement of equivalents theorem. We note the following facts:

  (1) Routley and Meyer's basic positive logic B_+ (cf. Routley, Meyer, Plumwood, & Brady, 1982) is not included in any of the Ltd1-logics: either the rule *Suffixing* (Suf), $A \rightarrow B \Rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)$, or the rule *Prefixing* (Pref), $B \rightarrow C \Rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$, or both fail. However, all contain Anderson and Belnap's positive First degree entailment logic FDE_+ (cf. Anderson & Belnap, 1975)).

  (2) All Ltb-logics contain B_+ (actually, Lt27, Lt28 and Lt29 contain the positive fragment of Lewis' S5, Łukasiewicz's Ł3 and Gödel's G3, respectively.

  (3) Although Suf and Pref are not provable in any of the Ltd1-logics, the following rules are: Suppose $A \leftrightarrow B$. Then, $(A \rightarrow C) \leftrightarrow (B \rightarrow C)$, $(C \rightarrow A) \leftrightarrow (C \rightarrow B)$, $(A \wedge C) \leftrightarrow (B \wedge C)$, $(C \wedge A) \leftrightarrow (C \wedge B)$, $(A \vee C) \leftrightarrow (B \vee C)$, $(C \vee A) \leftrightarrow (C \vee B)$.

  It follows from facts 2 and 3 together with the presence of either the contraposition axiom or the contraposition rule, that the replacement of equivalents theorem is provable in all Ltd-logics[4].

- *Quasi-Boolean negations (QB-negations)*: In (Robles, 2020), a QB-negation of 'superintuitionistic character' (H-negation); in (Robles & Méndez, 2020), a QB-negation dual to H (DH-negation) are introduced. The distinctive feature of H-negation is the presence of the ECQ axiom, $(A \wedge \neg A) \rightarrow B$, and its contrapositive form, $B \rightarrow \neg(A \wedge \neg A)$ (cECQ); that of DH-negation, the conditioned 'Principle of Excluded Middle' (CPEM), $B \rightarrow (A \vee \overset{\bullet}{\neg} A)$, and its contrapositive form $\overset{\bullet}{\neg}(A \vee \overset{\bullet}{\neg} A) \rightarrow B$ (cCPEM). Now, CPEM and cCPEM hold in all Lta-logics; and ECQ and cECQ, in all Ltb-logics. Thus, from a general point of view, we can consider Lta-logics as DH-logics, and Ltb-logics as H-logics.

6. Concluding remarks

The paper is ended with some concluding remarks on the results obtained.

(1) In (Robles & Méndez, 2019; Robles et al., 2019), all natural implicative expansions of Kleene's strong matrix MK3 (cf. Definition 2.2) are given Hilbert-style axiomatizations; in the present paper, the same type of formulations is provided for all natural implicative expansions of $MK3_G$ and $MK3_{dG}$ (cf. Definition

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[4]A referee of the JANCL remarks that (Avron, 2017) and (Avron & Béziau, 2017) contain useful information with regard to self-extensionality in 3-valued logic, in general.

2.5). These axiomatic formulations are defined once the respective matrix semantics has been translated into the appropriate BD-semantics. Consequently, BD-semantics constitutes the pivot-point from which logics in (Robles & Méndez, 2019; Robles et al., 2019) and the present paper are considered. In this sense, it is interesting to note that the definition of a BD-semantics for the logics determined by $MK3_G$ and $MK3_{dG}$ is possible although, contrary to what is the case with MK3, the negation function in both matrices lacks the intermediate value upon which the building of BD-semantics hinges. In connection with this question, we note the ensuing remark.

(2) Concerning the expansion of $MK3_{I+}$ (resp., $MK3_{II+}$) with $f_{\overset{\bullet}{\neg}}$ (resp., $f_{\neg}$), the following comments can be noted from the methodological perspective adopted in the present paper.

    (a) Expansions of $MK3_{I+}$ with $f_{\overset{\bullet}{\neg}}$: Mt25 and Mt26 determine propositional classical logics, whereas Mt27, Mt28, Mt29 and Mt30 falsify theses and/or rules needed in the proof of the canonical validity of the conditional clauses (for instance, $\overset{\bullet}{\neg}p \Rightarrow p \vee (p \to q)$ fails for any Mti-interpretation I such that $I(p) = 1$ and $I(q) = 0$ —i.e., $i \in \{27, 28, 29, 30\}$).

    (b) Expansions of $MK3_{II+}$ with $f_{\neg}$: Mt2, Mt3, Mt5, Mt6, Mt8, Mt9, Mt11, Mt12, Mt14, Mt15, Mt17, Mt18, Mt20, Mt21, Mt23 and Mt24 determine propositional classical logic, while Mt1, Mt4, Mt7, Mt10, Mt13, Mt16, Mt19 and Mt22 falsify theses and/or rules needed in the proof of the canonical validity of the conditional clauses, as in case (a) (for instance, $\neg q \vee (p \to q)$ fails in each one of the second group of matrices when assigning 2 to p and 1 to q).

(3) By using the method employed in this paper, it is possible to axiomatize non-natural implicative expansions of $MK3_G$ and $MK3_{dG}$. Below, an example is provided. Let Mt31 and Mt32 be the expansions of $MK3_G$ and $MK3_{dG}$, respectively, with the $f_{\to}$-function given by the following truth-table:

$\to$	0	1	2
0	2	0	2
1	0	2	0
2	0	0	2

(Notice that 1 and 2 are designated values in Mt32, but 1 is the only designated value in Mt31.) Then, the logics Lt31 and Lt32 determined by Mt31 and Mt32 can be axiomatized as follows:

Common basis: (a1) $A \to A$; (a2) $(A \wedge B) \leftrightarrow (B \wedge A)$; (a3) $(A \vee B) \leftrightarrow (B \vee A)$; (a4) $[A \wedge (B \wedge C)] \leftrightarrow [(A \wedge B) \wedge C]$; (a5) $[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C]$; (a6) $[(A \vee B) \wedge (A \vee C)] \leftrightarrow [(A \wedge B) \vee C]$; (a7) $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$; (a8) $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$; (a9) $[(A \to B) \wedge (B \to C)] \to (A \to C)$; (a10) $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$; (a11) $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$; (a12) $(A \to B) \to (\sim B \to \sim A)$. The rules are: Adj, MP, dMP, dE$\wedge$, $C \vee (A \wedge B) \Rightarrow A, B$, and dI$\vee$, $C \vee A \Rightarrow C \vee (A \vee B), C \vee (B \vee A)$.

Lt31: Negation: dDN, $B \vee A \Rightarrow B \vee \neg\neg A$, and dECQ, $C \vee (A \wedge \neg A) \to C \vee B$. Conditional: (a13) $(\neg A \wedge \neg B) \to (A \to B)$; (a14) $(A \vee \neg A) \vee [(B \vee \neg B) \vee (A \to B)]$; (a15) $(A \to B) \vee \neg(A \to B)$. Rules: $C \vee (A \wedge B) \Rightarrow C \vee (A \to B)$; $C \vee (\neg A \wedge B) \Rightarrow C \vee (A \to B)$; $C \vee [(A \to B) \wedge B] \Rightarrow C \vee (A \vee \neg A)$; $C \vee [(A \to B) \wedge \neg B] \Rightarrow C \vee (A \vee \neg A)$; $C \vee [(A \to B) \wedge A] \Rightarrow C \vee (\neg A \vee B)$; $C \vee [(A \to B) \wedge \neg B] \Rightarrow C \vee (\neg A \vee B)$; $C \vee [(A \to B) \wedge \neg A] \Rightarrow C \vee (B \vee \neg B)$.

Lt32: Negation: (a13) $A \vee \overset{\bullet}{\neg} A$ and drECQ, $C \vee (\overset{\bullet}{\neg} A \wedge \overset{\bullet\bullet}{\neg\neg} A) \Rightarrow C \vee B$ —drECQ abbreviates 'disjunctive restricted ECQ'). Conditional: (a14) $(A \vee B) \vee (A \to B)$; (a15) $(A \vee \overset{\bullet}{\neg} B) \vee (A \to B)$; (a16) $(\overset{\bullet}{\neg} A \vee \overset{\bullet}{\neg} B) \vee (A \to B)$; (a17) $(A \to B) \to \overset{\bullet\bullet}{\neg\neg}(A \to B)$; (a18) $[(A \to B) \wedge \overset{\bullet}{\neg} B] \to \overset{\bullet}{\neg} A$. Rules: $C \vee [(A \wedge \overset{\bullet}{\neg} A) \wedge (B \wedge \overset{\bullet}{\neg} B)] \Rightarrow C \vee (A \to B)$; $C \vee [(A \wedge \overset{\bullet}{\neg} A) \wedge (A \to B)] \Rightarrow C \vee \overset{\bullet}{\neg} B$; $C \vee [(A \to B) \wedge (B \wedge \overset{\bullet}{\neg} B)] \Rightarrow C \vee A$.

(4) Some of the logics defined in this paper can be interpreted in reduced Routley-Meyer semantics similarly as some of the logics determined by MK3$_\text{I}$ were interpreted with this semantics in (Robles, 2019).

(5) Maybe, the logics defined in this paper can be given natural deduction formulations following the strategy of (Kooi & Taminga, 2012) and (Taminga, 2014) (cf. also Petrukhin, 2018; Petruhkin & Shanging, 2018, 2020) or cut-free sequent calculus, following methods in (Avron, Ben-Naim, & Konikowska, 2007; Avron, Konikowska, & Zamansky, 2013)[5]. Concerning a brief comparison between these methods, cf. (Petruhkin & Shanging, 2020, §8) and (Robles, Forthcoming, §6).

(6) Most of the logics here presented have not been defined in the literature (to the best of our knowledge; cf. §5). Nevertheless, it is possible that as has just been pointed out, they could be defined by using the methods of (Petruhkin & Shanging, 2018) and (Kooi & Taminga, 2012) or those of (Avron et al., 2007) and (Avron et al., 2013). In the preceding section it has been shown that, in general, they have interesting properties. Hopefully, then, some of them may be proved useful as it has been the case with certain 3-valued and 4-valued logics, since the introduction of Ł3 in the twenties of the last century.

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[5]A referee of the JANCL notes that in (Anshakov & Rychkov, 1995) a method for generating Hilbert-style systems for many-valued logic is presented. However, the referee remarks that said method requires the use of Rosser and Turquette's operators.

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