HW#1

2023-20200 박건도

Theorem 1.6.8 *Suppose $X_n \to X$ a.s. Let g, h be continuous functions with*
(i) $g \geq 0$ and $g(x) \to \infty$ as $|x| \to \infty$,
(ii) $|h(x)|/g(x) \to 0$ as $|x| \to \infty$,
and (iii) $Eg(X_n) \leq K < \infty$ for all n.
Then $Eh(X_n) \to Eh(X)$.

Proof By subtracting a constant from h, we can suppose without loss of generality that $h(0) = 0$. Pick M large so that $P(|X| = M) = 0$ and $g(x) > 0$ when $|x| \geq M$. Given a random variable Y, let $\bar{Y} = Y1_{(|Y| \leq M)}$. Since $P(|X| = M) = 0$, $\bar{X_n} \to \bar{X}$ a.s. Since $h(\bar{X_n})$ is bounded and h is continuous, it follows from the bounded convergence theorem that

(a) $$E(h(\bar{X_n}) \to Eh(\bar{X})$$

To control the effect of the truncation, we use the following:

(b) $$|Eh(\bar{Y}) - Eh(Y)| \leq E|h(\bar{Y}) - h(Y)| \leq E(|h(Y)|; |Y| > M) \leq \epsilon_M Eg(Y)$$

where $\epsilon_M = \sup\{|h(x)|/g(x) : |x| \geq M\}$. To check the second inequality, note that when $|Y| \leq M$, $\bar{Y} = Y$, and we have supposed $h(0) = 0$. The third inequality follows from the definition of ϵ_M.
Taking $Y = X_n$ in (b) and using (iii), it follows that

(c) $$|Eh(\bar{X_n}) - Eh(X_n)| \leq K\epsilon_M$$

To estimate $|Eh(\bar{X}) - Eh(X)|$, we observe that $g \geq 0$ and g is continuous, so Fatou's lemma implies

$$Eg(X) \leq \liminf_{n\to\infty} Eg(X_n) \leq K$$

Taking $Y = X$ in (b) gives

(d) $$|Eh(\bar{X}) - Eh(X)| \leq K\epsilon_M$$

The triangle inequality implies

$$\begin{aligned} |Eh(X_n) - Eh(X)| \leq & \, |Eh(X_n) - Eh(\bar{X}_n)| \\ & + |Eh(\bar{X}_n) - Eh(\bar{X})| + |Eh(\bar{X}) - Eh(X)| \end{aligned}$$

Taking limits and using (a), (c), (d), we have

$$\limsup_{n\to\infty} |Eh(X_n) - Eh(X)| \leq 2K\epsilon_M$$

which proves the desired result since $K < \infty$ and $\epsilon_M \to 0$ as $M \to \infty$. $\square$