REAL ANALYSIS: ASSIGNMENT 1

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1. Let $(X,\, \mathfrak{M},\, \mu)$ be a measure space, let Y be a non-empty measurable subset of X, and let $\mathfrak{M}_Y$ be the collection of subsets of Y belonging to the σ-algebra $\mathfrak{M}$. Prove that $\mathfrak{M}_Y$ is a σ-algebra and the measure μ restricted to $\mathfrak{M}_Y$ is also a measure on $\mathfrak{M}_Y$ (and thus $(Y,\, \mathfrak{M}_Y,\, \mu)$ consists a measure space).

2. Let $(X,\, \mathfrak{M},\, \mu)$ be a measure space such that $\mu(X) < \infty$. Define

$$\mathfrak{M}_1 = \{A \in \mathfrak{M} : \mu(A) = 0 \text{ or } \mu(A) = \mu(X)\} \, .$$

(1) Prove that $(X,\, \mathfrak{M}_1,\, \mu)$ is a measure space.
(2) Let $f : X \to \mathbb{R}$ be a measurable function with respect to the σ-algebra $\mathfrak{M}$. Prove that f is measurable with respect to the σ-algebra $\mathfrak{M}_1$ if and only if there exists $c \in \mathbb{R}$ such that $f(x) = c$ a.e. $[\mu]$.

3. Let $(X,\, \mathfrak{M})$ be a measurable space and ;et $\mu_1,\, \mu_2,\, \ldots,\, \mu_k$ be positive measures on this space. Let $c_1,\, c_2,\, \cdots,\, c_k \geq 0$ be non-negative reals and define $\mu : \mathfrak{M} \to \mathbb{R}$ as

$$\mu(E) = \sum_{i=1}^{k} c_i \mu_i(E) \quad \text{for } E \in \mathfrak{M}.$$

Prove that μ is a measure. (This measure is denoted by $c_1\mu_1 + \cdots + c_k\mu_k$).

4. Suppose that $(\mathbb{R},\, \tau)$ is the standard topological space, i.e.,

$$\tau = \{A : A \text{ is a countable union of open intervals in } \mathbb{R}\}.$$

Denote by $\mathfrak{B}$ the Borel σ-algebra associated to this topological space. Prove that the following set belongs to $\mathfrak{B}$.

$$A = \Big\{ x \in \mathbb{R} : x = q_1\sqrt{n_1} + \cdots + q_k\sqrt{n_k} \text{ for some } k \in \mathbb{N},\, q_1,\, \ldots,\, q_k \in \mathbb{Q}, \\ \text{and } n_1,\, \ldots,\, n_k \in \mathbb{N} \Big\}$$

5. Let $(X, \mathfrak{M}, \mu)$ be a measure space such that $\mu(X) < \infty$ and let $f, g, h \in L^1(\mu)$. For $n \in \mathbb{N}$, define

$$B_n = \{x \in X : |f(x)| + |g(x)| \leq n\} \in \mathfrak{M} .$$

Prove that

$$\lim_{n\to\infty} \int_{B_n} h \, d\mu = \int_X h \, d\mu .$$

6. Let $(X, \mathfrak{M}, \mu)$ be a complete measure space and let $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$ be sequences of measurable functions such that

$$f_m \leq g_n \quad \text{a.e. } [\mu] \quad \text{for all } m, n \in \mathbb{N} .$$

Prove that

$$\sup_{n\in\mathbb{N}} f_n \leq \inf_{n\in\mathbb{N}} g_n \quad \text{a.e. } [\mu] .$$

7. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $f \in L^1(\mu)$. Prove that, for all $\epsilon > 0$, there exists a simple measurable function s such that

$$\int_X |f - s| d\mu < \epsilon .$$

Here, the simple function s can assume negative value as well; in other words, s can be represented as $s = \sum_{i=1}^{n} a_i \chi_{A_i}$ for some pairwise disjoint measurable sets $A_1, A_2, \ldots, A_n$ and real numbers $a_1, a_2, \ldots, a_n$.

8. Let $X = \{1, 2, \ldots, 3n\}$ for some positive integer $n \in \mathbb{N}$, and let

$$\mathfrak{M} = \{A \subset X : |\{3k-2, 3k-1, 3k\} \cap A| = 0 \text{ or } 3 \text{ for all } 1 \leq k \leq n\} .$$

(1) Prove that $(X, \mathfrak{M})$ is a measurable space.
(2) Let $(a_k)_{k=1}^{3n}$ be a sequence of non-negative reals. Define $\mu : \mathfrak{M} \to [0, \infty)$ as

$$\mu(A) = \sum_{i\in A} a_i .$$

Prove that μ is a measure on $(X, \mathfrak{M})$ and hence $(X, \mathfrak{M}, \mu)$ is a measure space.
(3) Suppose that the sequence $(a_k)_{k=1}^{3n}$ given above satisfies $a_1 = a_2 = a_3 = 0$ and $a_k > 0$ for all $k \geq 4$. Prove that the measure space $(X, \mathfrak{M}, \mu)$ is not complete and derive the completion of the measure space $(X, \mathfrak{M}, \mu)$.

Recommending Problems

1. Let $X = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, \infty\}$ be the extended real number system, and define a topology τ on X as a collection of subsets of X which can be represented as a countable union of intervals of the form

$$(a, b)\ , \quad [-\infty, a)\ , \quad (a, \infty] \text{ for some } a, b \in X.$$

Prove that (X, τ) is a topological space.

2. Prove that an open set in the Euclidean space $\mathbb{R}^d$ can be expressed as a countable union of disjoint open boxes. Here, open box refers to a set of the form

$$(a_1, b_1) \times \cdots \times (a_d, b_d)\ .$$

3. Let $(X, \mathfrak{M}_1)$ and $(X, \mathfrak{M}_2)$ be two measurable spaces such that $\mathfrak{M}_1 \subset \mathfrak{M}_2$ (we say that $\mathfrak{M}_2$ is a refinement of $\mathfrak{M}_1$), and let Y be a topological space. Let $f : X \to Y$ be a measurable function with respect to the space $(X, \mathfrak{M}_1)$. Prove that f is measurable with respect to the space $(X, \mathfrak{M}_2)$ as well.

4. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $f, g \in L^1(\mu)$. Then, define

$$h(x) = \min(f(x), g(x)) \text{ for all } x \in X.$$

(1) Prove that $h \in L^1(\mu)$.
(2) Prove that

$$\int_X h\, d\mu \le \min\Big(\int_X f d\mu, \int_X g d\mu\Big)$$

and explain when the equality holds.

5. Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f \in L^1(\mu)$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative functions in $L^1(\mu)$ such that

$$\lim_{n\to\infty} f_n = f \text{ a.e. } [\mu] \quad \text{and} \quad \lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu\ .$$

For any $A \in \mathfrak{M}$, prove that

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f d\mu\ .$$

(6-8) These problems discuss the local measurability and σ-finite sets. I strongly recommend to spend your time to think about these problems.

6. Let $(X, \mathfrak{M}, \mu)$ be a measure space. A set $E \subset X$ is called **locally measurable** if $E \cap A \in \mathfrak{M}$ for all $A \in \mathfrak{M}$ with $\mu(A) < \infty$. Denote the collection of locally measurable sets by $\mathfrak{M}_\ell$.
(1) Prove that $\mathfrak{M}_\ell$ is a σ-algebra.
(2) A function $f : X \to [-\infty, \infty]$ is called **locally measurable** if, for all $Y \in \mathfrak{M}$ with $\mu(Y) < \infty$, the function f restricted to the set Y is measurable with respect to the measurable space $(Y, \mathfrak{M}_Y)$ introduced in problem **2** (of the mandatory part). Prove that a function f is locally measurable if and only if f is measurable with respect to the σ-algebra $\mathfrak{M}_\ell$.
(3) Prove or disprove: if $(X, \mathfrak{M}, \mu)$ is complete then $(X, \mathfrak{M}_\ell, \mu)$ is complete as well.

7. Let $(X, \mathfrak{M}, \mu)$ be a measure space. A subset $E \subset X$ is called a **σ-finite set** with respect to the measure μ, if there exist a countable decomposition $\{E_n : n \in \mathbb{N}\}$ of E (i.e., $E = \cup_{n=1}^{\infty} E_n$ and $E_i \cap E_j = \phi$ for all $i \neq j$) such that $\mu(E_n) < \infty$ for all n. Prove that, if X is a σ-finite set with respect to the measure μ, then any locally measurable set is measurable.

8. Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f \in L^1(\mu)$. Prove that the set $\{x : f(x) \neq 0\}$ is a σ-finite set with respect to the measure μ.

8. Exercise 3, 5-(1), 6, 7, 9, 12 of Chapter 1