

Assignment #1

real analysis

April 4, 2022

2017-11362 통계학과 박건도

Problem 1. Let (X, \mathfrak{M}, μ) be a measure space, let Y be a non-empty measurable subset of X , and let \mathfrak{M}_Y be the collection of subsets of Y belonging to the σ -algebra \mathfrak{M} . Prove that \mathfrak{M}_Y is a σ -algebra and the measure μ restricted to \mathfrak{M}_Y is also a measure on \mathfrak{M}_Y (and thus (Y, \mathfrak{M}_Y, μ) consists a measure space).

Solution. First of all, let's show that \mathfrak{M}_Y is a σ -algebra.

① $Y \subseteq \mathfrak{M}_Y$ is trivial.

② $A \in \mathfrak{M}_Y \Rightarrow A^c \in \mathfrak{M}_Y$

Let $A \in \mathfrak{M}_Y$. Then $A \in \mathfrak{M}$ and so does A^c .

Since $Y \in \mathfrak{M}$, $Y \setminus A = Y \cap A^c \in \mathfrak{M}$.

$\therefore Y \setminus A \in \mathfrak{M}_Y$. ($\because Y \cap A^c \subseteq Y$)

③ $A_1, A_2, \dots \in \mathfrak{M}_Y \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{M}_Y$

$A_1, A_2, \dots \in \mathfrak{M}_Y$ means $A_k \subseteq Y$ for all k , i.e., $\bigcup_{k=1}^{\infty} A_k \subseteq Y$.

Since \mathfrak{M} is a σ -algebra, $\bigcup_{k=1}^{\infty} A_k \in \mathfrak{M}$ for $A_1, A_2, \dots \in \mathfrak{M}$.

Therefore,

$$\begin{aligned} A_1, A_2, \dots \in \mathfrak{M}_Y &\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{M} \text{ and } \bigcup_{k=1}^{\infty} A_k \subseteq Y \\ &\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{M}_Y \end{aligned}$$

Therefore, \mathfrak{M}_Y is a σ -algebra.

Then we have to show that the measure μ restricted to \mathfrak{M}_Y is a measure on \mathfrak{M}_Y . Since $\mu(\emptyset) = 0 < \infty$, countable additivity remains only to prove.

Let $A_1, A_2, \dots \in \mathfrak{M}_Y$ be disjoint sets. Since \mathfrak{M}_Y belonging to \mathfrak{M} , $A_1, A_2, \dots \in \mathfrak{M}$ and $\mu(\bigcup A_k) = \sum \mu(A_k)$.

Therefore, (Y, \mathfrak{M}_Y, μ) is a measure space. ■

Problem 2. Let (X, \mathfrak{M}, μ) be a measure space such that $\mu(X) < \infty$. Define

$$\mathfrak{M}_1 = \{A \in \mathfrak{M} : \mu(A) = 0 \text{ or } \mu(A) = \mu(X)\}. \quad (1)$$

(1) Prove that (X, \mathfrak{M}_1, μ) is a measure space.

(2) Let $f : X \rightarrow \mathbb{R}$ be a measurable function with respect to the σ -algebra \mathfrak{M} . Prove that f is measurable with respect to the σ -algebra \mathfrak{M}_1 if and only if there exists $c \in \mathbb{R}$ such that $f(x) = c$ a.e. $[\mu]$.

Solution. (1) To show that (X, \mathfrak{M}_1, μ) is a measure space, we have to prove following statements:

(i) \mathfrak{M}_1 is a σ -algebra.

(ii) μ restricted to \mathfrak{M}_1 is a measure on (X, \mathfrak{M}_1)

First of all, let's show that \mathfrak{M}_1 is a σ -algebra. It is clear that $X \in \mathfrak{M}_1$. For $A \in \mathfrak{M}_1$, $\mu(A)$ is either 0 or $\mu(X)$. Then, $\mu(A^c) = \mu(X) - \mu(A) = \mu(X)$ or 0, so $A^c \in \mathfrak{M}_1$. And for $A_1, A_2, \dots \in \mathfrak{M}_1$, there are two cases:

$$\begin{cases} \mu(A_1) = \mu(A_2) = \dots = 0 \\ \exists k \text{ s.t. } \mu(A_k) = \mu(X) \end{cases}$$

In first case, $\mu(\bigcup A_i) \leq \sum \mu(A_i) = 0$ means $\bigcup A_i \in \mathfrak{M}_1$. We can also lead the same conclusion from the other case since

$$\mu(X) = \mu(A_k) \leq \mu(\bigcup A_i) \leq \mu(X).$$

Therefore, \mathfrak{M}_1 is a σ -algebra. Next, to show that μ is a measure on (X, \mathfrak{M}_1) , we have to show the countable additivity ($\mu(\emptyset) = 0 < \infty$ is easy to show). Likewise the **Solution** of **Problem 1.**, it is clear that (X, \mathfrak{M}_1, μ) is a measure space.

(2) To show iff condition, we have to show following 2 statements:

(i) $f(x) = c$ a.e. $[\mu] \Rightarrow f$ is measurable on \mathfrak{M}_1 .

$\exists N$ s.t. $\mu(N) = 0$, $f(x) = c$ where $x \in X \setminus N$.

Let's show $\mu(f^{-1}(V)) = 0$ or $\mu(X)$ for open $V \subseteq \mathbb{R}$, so that $f^{-1}(V) \in \mathfrak{M}_1$.

If $c \in V$,

$$\begin{aligned} f^{-1}(V) &= f^{-1}(\{c\} \cup V \setminus \{c\}) \\ &= f^{-1}(\{c\}) \cup f^{-1}(V \setminus \{c\}) \\ &\supseteq X \setminus N \end{aligned}$$

$$\therefore \mu(X \setminus N) = \mu(X) - \mu(N) = \mu(X) \Rightarrow \mu(f^{-1}(V)) = \mu(X)$$

If $c \notin V$, $f(x) = c$ where $x \in X \setminus N$ and it means $f^{-1}(V) \subseteq N$.

$$\therefore \mu(f^{-1}(V)) = 0 (\because \mu(N) = 0).$$

$$\therefore f^{-1}(V) \in \mathfrak{M}_1.$$

(ii) f is measurable on $\mathfrak{M}_1 \Rightarrow f(x) = c$ a.e. $[\mu]$.

Let $c = \sup\{a \in \mathbb{R} : \mu(f^{-1}((a, \infty))) = \mu(X)\}$.

Since $\lim_{n \rightarrow \infty} \mu(f^{-1}((n, \infty))) = \mu(\bigcap_{n=1}^{\infty} f^{-1}((n, \infty))) = \mu(f^{-1}(\emptyset)) = 0$, $\exists N$ s.t.

$$\mu(f^{-1}(N, \infty)) = 0 \text{ and } c < N < \infty.$$

Likewise, $\exists M$ s.t. $\mu(f^{-1}(M, \infty)) = \mu(X)$ and $M < c < N$, i.e., $c \in \mathbb{R}$.

Then for $n \in \mathbb{N}$, $\mu(f^{-1}((c - \frac{1}{n}, \infty))) = \mu(X)$ and $\mu(f^{-1}((c + \frac{1}{n}, \infty))) = 0$ by the definition of supremum.

Then,

$$\begin{aligned} \mu(X) &= \mu\left(f^{-1}\left(\left(c - \frac{1}{n}, \infty\right)\right)\right) - \mu\left(f^{-1}\left(\left(c + \frac{1}{n}, \infty\right)\right)\right) \\ &= \mu\left(f^{-1}\left(\left(c - \frac{1}{n}, c + \frac{1}{n}\right]\right)\right), \\ \mu(f^{-1}(\{c\})) &= \mu\left(\bigcap_{n=1}^{\infty} f^{-1}\left(\left(c - \frac{1}{n}, c + \frac{1}{n}\right]\right)\right) \\ &= \lim_{n \rightarrow \infty} \mu\left(f^{-1}\left(\left(c - \frac{1}{n}, c + \frac{1}{n}\right]\right)\right) = \mu(X) \end{aligned}$$

$$\therefore f(x) = c \text{ a.e. } [\mu]. (\because \mu(f^{-1}(\mathbb{R} \setminus \{c\})) = 0).$$

■

Problem 3. Let (X, \mathfrak{M}) be a measurable space and let $\mu_1, \mu_2, \dots, \mu_k$ be positive measures on this space. Let $c_1, c_2, \dots, c_k \geq 0$ be non-negative reals and define $\mu : \mathfrak{M} \rightarrow \mathbb{R}$ as

$$\mu(E) = \sum_{i=1}^k c_i \mu_i(E) \quad \text{for } E \in \mathfrak{M}$$

Prove that μ is a measure. (This measure is denoted by $c_1 \mu_1 + \dots + c_k \mu_k$).

Solution. $\mu(\emptyset) = \sum c_i \mu_i(\emptyset) = 0 < \infty$.

Let $A_1, A_2, \dots \in \mathfrak{M}$ be disjoint sets. Since each μ_i is a measure, $\mu_i(\bigcup A_j) =$

$$\sum_j \mu_i(A_j).$$

$$\begin{aligned} \mu \left(\bigcup_{j=1}^{\infty} A_j \right) &= \sum_{i=1}^k c_i \mu_i \left(\bigcup_{j=1}^{\infty} A_j \right) = \sum_{i=1}^k c_i \sum_{j=1}^{\infty} \mu_i(A_j) \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^k c_i \mu_i(A_j) \\ &= \sum_{j=1}^{\infty} \mu(A_j) \end{aligned}$$

$\therefore \mu$ is a measure. ■

Problem 4. Suppose that (\mathbb{R}, τ) is the standard topological space, i.e.,

$$\tau = \{A : A \text{ is a countable union of open intervals in } \mathbb{R}\}.$$

Denote by \mathfrak{B} the Borel σ -algebra associated to this topological space. Prove that the following set belongs to \mathfrak{B} .

$$A = \{x \in \mathbb{R} : x = q_1\sqrt{n_1} + \cdots + q_k\sqrt{n_k} \text{ for some } k \in \mathbb{N}, q_1, \dots, q_k \in \mathbb{Q}, \text{ and } n_1, \dots, n_k \in \mathbb{N}\}$$

Solution. For $a \in \mathbb{R}$, $[a, \infty) = \bigcup_{n=1}^{\infty} (a - \frac{1}{n}, \infty)$ and $(-\infty, a] = \bigcup_{n=1}^{\infty} (-\infty, a + \frac{1}{n})$, so $[a, \infty), (-\infty, a] \in \tau$ and therefore $\{a\} \in \mathfrak{B}$. Let

$$A_k = \{x \in \mathbb{R} : x = q_1\sqrt{n_1} + \cdots + q_k\sqrt{n_k} \text{ for some } q_i \in \mathbb{Q}, n_i \in \mathbb{N}, i = 1, \dots, k\}$$

Since there exists an injection s.t. $A_k \rightarrow \mathbb{Q}^k \times \mathbb{N}^k$ and \mathbb{Q}, \mathbb{N} are countable, A_k is at most countable and $A_k = \bigcup_{x \in A_k} \{x\} \in \mathfrak{B}$.

$$\therefore A = \bigcup_{k=1}^{\infty} A_k \in \mathfrak{B}. \quad \blacksquare$$

Problem 5. Let (X, \mathfrak{M}, μ) be a measure space such that $\mu(X) < \infty$ and let $f, g, h \in L^1(\mu)$. For $n \in \mathbb{N}$, define

$$B_n = \{x \in X : |f(x)| + |g(x)| \leq n\} \in \mathfrak{M}.$$

Prove that

$$\lim_{n \rightarrow \infty} \int_{B_n} h \, d\mu = \int_X h \, d\mu.$$

Solution. It is clear that $|h|, (|h|\chi_{B_n})_{n=1}^\infty$ are measurable functions.

Since $B_n \nearrow X$, $|h|\chi_{B_n} \nearrow |h|$.

Also, $|h| \in L^1(\mu)$ and $|h|\chi_{B_n} \leq |h|$, $\forall n \in \mathbb{N}$.

DCT $\Rightarrow \int_X ||h| - |h|\chi_{B_n}| d\mu \rightarrow 0$ as $n \rightarrow \infty$.

$$\begin{aligned} 0 &\leq \left| \int_X h - h\chi_{B_n} d\mu \right| \leq \int_X |h - h\chi_{B_n}| d\mu \\ &= \int_X |h|(1 - \chi_{B_n}) d\mu \\ &= \int_X ||h| - |h|\chi_{B_n}| d\mu \rightarrow 0 \\ \therefore \int_X h d\mu &= \lim_{n \rightarrow \infty} \int_X h\chi_{B_n} d\mu = \lim_{n \rightarrow \infty} \int_{B_n} h d\mu. \end{aligned}$$

■

Problem 6. Let (X, \mathfrak{M}, μ) be a complete measure space and let $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty$ be sequences of measurable functions such that

$$f_m \leq g_n \quad \text{a.e. } [\mu] \quad \text{for all } m, n \in \mathbb{N}.$$

Prove that

$$\sup_{n \in \mathbb{N}} f_n \leq \int_{n \in \mathbb{N}} g_n \quad \text{a.e. } [\mu].$$

Solution. First of all, let's show that $f_m \geq g_n$ for all $m, n \in \mathbb{N}$ a.e. $[\mu]$.

Let $N_{m,n} = \{x \in X : f_m(x) > g_n(x)\}$ so that $\mu(N_{m,n}) = 0$. Then $0 \leq \mu(\bigcup_{m,n} N_{m,n}) \leq \sum_{m,n} \mu(N_{m,n}) = 0$. Thus, for $N = \bigcup_{m,n} N_{m,n}$, $\mu(N) = 0$ and $\exists m, n \in \mathbb{N}$ s.t. $f_m(x) > g_n(x)$ for all $x \in N$, i.e., $f_m(x) \leq g_n(x)$ for $\forall m, n \in \mathbb{N}$, $\forall x \in X \setminus N$.

It suffices to show that $\sup f_n(x) \leq \inf g_n(x)$, $\forall x \in X \setminus N$.

Let $x \in X \setminus N$ is given and assume that $\sup f_n(x) > \inf g_n(x)$.

Then, $\exists M \in \mathbb{N}$ satisfying followings:

$$\begin{cases} f_M(x) > \sup f_n(x) - \frac{1}{2}(\sup f_n(x) - \inf g_n(x)) = \frac{1}{2}(\sup f_n(x) + \inf g_n(x)) \\ g_M(x) < \inf g_n(x) + \frac{1}{2}(\sup f_n(x) - \inf g_n(x)) = \frac{1}{2}(\sup f_n(x) + \inf g_n(x)) \end{cases}$$

It concludes that $g_M(x) < f_M(x)$, which contradicts to $f_m \leq g_n$, $\forall m, n$.

$\therefore \sup f_n(x) \leq \inf g_n(x)$, $\forall x \in X \setminus N$, $\mu(N) = 0$.

$\therefore \sup f_n(x) \leq \inf g_n(x)$ a.e. $[\mu]$.

■

Problem 7. Let (X, \mathfrak{M}, μ) be a measure space, and let $f \in L^1(\mu)$. Prove that, for all $\epsilon > 0$, there exists a simple measurable function s such that

$$\int_X |f - s| d\mu < \epsilon.$$

Solution. First of all, let's show above statement holds on $f \geq 0$ s.t. $f \in L^1(\mu)$. Thm 1.17 $\Rightarrow \exists$ simple $(s_n)_{n=1}^\infty : X \rightarrow [0, \infty)$ s.t. $s_n \nearrow f$. Then $s_n \in L^1(\mu)$. Then, we can see following holds:

$$\begin{cases} (s_n)_{n=1}^\infty, f : X \rightarrow \mathbb{R} \text{ MFs.} \\ \lim_{n \rightarrow \infty} s_n(x) = f(x), \quad \forall x \in X. \\ \exists f \in L^1(\mu) \text{ s.t. } |s_n(x)| \leq f(x), \quad \forall x \in X, \quad \forall n \in \mathbb{N}. \end{cases}$$

DCT $\Rightarrow \int_X |f - s_n| d\mu \rightarrow 0$ as $n \rightarrow \infty$.

$\therefore \exists N \in \mathbb{N}$ s.t. $\int_X |f - s_N| d\mu < \epsilon/2$ and take $s = s_N$.

Since $f = f^+ - f^-$, there exists simple functions s^+, s^- s.t. $\int_X |f^+ - s^+| d\mu < \epsilon/2$ and $\int_X |f^- - s^-| d\mu < \epsilon/2$.

Then for $s = s^+ - s^-$,

$$0 \leq \int_X |f - s| d\mu \leq \int_X |f^+ - s^+| + |f^- - s^-| d\mu < \epsilon$$

■

Problem 8. Let $X = \{1, 2, \dots, 3n\}$ for some positive integer $n \in \mathbb{N}$, and let

$$\mathfrak{M} = \{A \subset X : |\{3k - 2, 3k - 1, 3k\} \cap A| = 0 \text{ or } 3 \text{ for all } 1 \leq k \leq n\}.$$

(1) Prove that (X, \mathfrak{M}) is a measurable space.

(2) Let $(a_k)_{k=1}^{3n}$ be a sequence of non-negative reals. Define $\mu : \mathfrak{M} \rightarrow [0, \infty)$ as

$$\mu(A) = \sum_{i \in A} a_i.$$

Prove that μ is a measure on (X, \mathfrak{M}) and hence (X, \mathfrak{M}, μ) is a measure space.

(3) Suppose that the sequence $(a_k)_{k=1}^{3n}$ given above satisfies $a_1 = a_2 = a_3 = 0$ and $a_k > 0$ for all $k \geq 4$. Prove that the measure space (X, \mathfrak{M}, μ) is not complete and derive the completion of the measure space (X, \mathfrak{M}, μ) .

Solution. (1) To show that (X, \mathfrak{M}) is a measurable space, we have to show three things:

- ① $|\{3k-2, 3k-1, 3k\} \cap X| = 3$ for all $k = 1, 2, \dots, n \Rightarrow X \in \mathfrak{M}$
 ② Assume $A \in \mathfrak{M}$. For $k = 1, 2, \dots, n$,

$$\begin{cases} |\{3k-2, 3k-1, 3k\} \cap A| = 3 & \Rightarrow \{3k-2, 3k-1, 3k\} \subseteq A \\ & \Rightarrow |\{3k-2, 3k-1, 3k\} \cap A^c| = 0 \\ |\{3k-2, 3k-1, 3k\} \cap A| = 0 & \Rightarrow |\{3k-2, 3k-1, 3k\} \cap A^c| = 3 \end{cases}$$

$\therefore A^c \in \mathfrak{M}$.

- ③ $A_1, A_2, \dots \in \mathfrak{M}$ is given.

For $k = 1, 2, \dots, n$,

- i) $|\{3k-2, 3k-1, 3k\} \cap A_i| = 0, \forall i$

$$\begin{aligned} \emptyset &= \bigcup_{i=1}^{\infty} (\{3k-2, 3k-1, 3k\} \cap A_i) \\ &= \{3k-2, 3k-1, 3k\} \cap \left(\bigcup_{i=1}^{\infty} A_i \right) \end{aligned}$$

$$\therefore |\{3k-2, 3k-1, 3k\} \cap \bigcup A_i| = 0$$

- ii) $\exists A_i$ s.t. $|\{3k-2, 3k-1, 3k\} \cap A_i| = 3$

$$\begin{aligned} \{3k-2, 3k-1, 3k\} &\subseteq \bigcup_{i=1}^{\infty} (\{3k-2, 3k-1, 3k\} \cap A_i) \\ &= \{3k-2, 3k-1, 3k\} \cap \bigcup A_i \\ &\subseteq \{3k-2, 3k-1, 3k\} \end{aligned}$$

$$\therefore |\{3k-2, 3k-1, 3k\} \cap \bigcup A_i| = 3$$

$$\therefore \bigcup A_i \in \mathfrak{M}$$

$\therefore (X, \mathfrak{M})$ is a measurable space.

(2) Since $\mu(\emptyset) = 0 < \infty$, it suffices to show countable additivity of μ .

Let $A_1, A_2, \dots \in \mathfrak{M}$ be disjoint sets.

Since $|X| < \infty$ and $\mathfrak{M} \subseteq 2^X$, $|\mathfrak{M}| < \infty$ and $\exists N$ s.t. $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^N A_n$. Then, when we show that $A_1, A_2 \in \mathfrak{M}$ disjoint $\Rightarrow \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$, it can be easily expanded to conclusion by induction.

$$\mu(A_1 \cup A_2) = \sum_{i \in A_1 \cup A_2} a_i = \sum_{i \in A_1} a_i + \sum_{i \in A_2} a_i = \mu(A_1) + \mu(A_2).$$

$\therefore (X, \mathfrak{M}, \mu)$ is a measure space.

(3) First of all, let's show that given measure space (X, \mathfrak{M}, μ) is not complete.

Let $N = \{1, 2, 3\} \in \mathfrak{M}$ and $A = \{1\} \subseteq N$.

Then $\mu(N) = 0$ but $A \notin \mathfrak{M}$ ($\because |\{1, 2, 3\} \cap A| = 1$), so we can conclude that (X, \mathfrak{M}, μ) is not complete.

The completion of above measure space is,

$$\mathfrak{M}^* = \{E \subseteq X : \exists A, B \in \mathfrak{M} \text{ s.t. } A \subseteq E \subseteq B, \mu(B \setminus A) = 0\}$$

Let's find $N \subseteq X$ s.t. $\mu(N) = 0$.

$$\mu(N) = \sum_{i \in N} a_i = 0 \quad \Rightarrow \quad N = \emptyset \text{ or } a_i = 0, \forall i \in N$$

$$\therefore N \in 2^{\{1,2,3\}}$$

$$\therefore \mathfrak{M}^* = \{N \cup E : E \in \mathfrak{M}, N \in 2^{\{1,2,3\}}\}.$$

Extend μ as $\mu(E) = \mu(E \setminus \{1, 2, 3\})$ for $E \in \mathfrak{M}^*$.

Then, (X, \mathfrak{M}^*, μ) is a complete measure space. ■