Assignment #1

real analysis

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Problem 1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let Y be a non-empty measurable subset of X, and let $\mathfrak{M}_Y$ be the collection of subsets of Y belonging to the σ-algebra $\mathfrak{M}$. Prove that $\mathfrak{M}_Y$ is a σ-algebra and the measure μ restricted to $\mathfrak{M}_Y$ is also a measure on $\mathfrak{M}_Y$ (and thus $(Y, \mathfrak{M}_Y, \mu)$ consists a measure space).*

Solution. First of all, let's show that $\mathfrak{M}_Y$ is a σ-algebra.
① $Y \subseteq \mathfrak{M}_Y$ is trivial.
② $A \in \mathfrak{M}_Y \Rightarrow A^c \in \mathfrak{M}_Y$
Let $A \in \mathfrak{M}_Y$. Then $A \in \mathfrak{M}$ and so does A^c.
Since $Y \in \mathfrak{M}$, $Y \setminus A = Y \cap A^c \in \mathfrak{M}$.
$\therefore Y \setminus A \in \mathfrak{M}_Y$. ($\because Y \cap A^c \subseteq Y$)
③ $A_1, A_2, \ldots \in \mathfrak{M}_Y \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{M}_Y$
$A_1, A_2, \ldots \in \mathfrak{M}_Y$ means $A_k \subseteq Y$ for all k, i.e., $\bigcup_{k=1}^{\infty} A_k \subseteq Y$.
Since $\mathfrak{M}$ is a σ-algebra, $\bigcup_{k=1}^{\infty} A_k \in \mathfrak{M}$ for $A_1, A_2, \ldots \in \mathfrak{M}$.
Therefore,

$$\begin{aligned} A_1, A_2, \ldots \in \mathfrak{M}_Y &\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{M} \text{ and } \bigcup_{\mathrm{k=1}}^{\infty} \mathrm{A_k} \subseteq \mathrm{Y} \\ &\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathfrak{M}_Y \end{aligned}$$

Therefore, $\mathfrak{M}_Y$ is a σ-algebra.
Then we have to show that the measure μ restricted to $\mathfrak{M}_Y$ is a measure on $\mathfrak{M}_Y$. Since $\mu(\varnothing) = 0 < \infty$, countable additivity remains only to prove.
Let $A_1, A_2, \ldots \in \mathfrak{M}_Y$ be disjoint sets. Since $\mathfrak{M}_Y$ belonging to $\mathfrak{M}$, $A_1, A_2, \ldots \in \mathfrak{M}$ and $\mu(\bigcup A_k) = \sum \mu(A_k)$.
Therefore, $(Y, \mathfrak{M}_Y, \mu)$ is a measure space. ■

Problem 2. *Let $(X, \mathfrak{M}, \mu)$ be a measure space such that $\mu(X) < \infty$. Define*

$$\mathfrak{M}_1 = \{A \in \mathfrak{M} : \mu(A) = 0 \text{ or } \mu(\mathrm{A}) = \mu(\mathrm{X})\}. \tag{1}$$

(**1**) *Prove that $(X, \mathfrak{M}_1, \mu)$ is a measure space.*
(**2**) *Let $f : X \to \mathbb{R}$ be a measurable function with respect to the σ-algebra $\mathfrak{M}$. Prove that f is measurable with respect to the σ-algebra $\mathfrak{M}_1$ if and only if there exists $c \in \mathbb{R}$ such that $f(x) = c$ a.e. $[\mu]$.*

Solution. (**1**) To show that $(X, \mathfrak{M}_1, \mu)$ is a measure space, we have to prove following statements:

(i) $\mathfrak{M}_1$ is a σ-algebra.

(ii) μ restricted to $\mathfrak{M}_1$ is a measure on $(X, \mathfrak{M}_1)$

First of all, let's show that $\mathfrak{M}_1$ is a σ-algebra. It is clear that $X \in \mathfrak{M}_1$. For $A \in \mathfrak{M}_1$, $\mu(A)$ is either 0 or $\mu(X)$. Then, $\mu(A^c) = \mu(X) - \mu(A) = \mu(X)$ or 0, so $A^c \in \mathfrak{M}_1$. And for $A_1, A_2, \ldots \in \mathfrak{M}_1$, there are two cases:

$$\begin{cases} \mu(A_1) = \mu(A_2) = \cdots = 0 \\ {}^\exists k \text{ s.t. } \mu(\mathrm{A_k}) = \mu(\mathrm{X}) \end{cases}$$

In first case, $\mu(\bigcup A_i) \le \sum \mu(A_i) = 0$ means $\bigcup A_i \in \mathfrak{M}_1$. We can also lead the same conclusion from the other case since

$$\mu(X) = \mu(A_k) \le \mu(\bigcup A_i) \le \mu(X).$$

Therefore, $\mathfrak{M}_1$ is a σ-algebra. Next, to show that μ is a measure on $(X, \mathfrak{M}_1)$, we have to show the countable additivity ($\mu(\varnothing) = 0 < \infty$ is easy to show). Likewise the **Solution** of **Problem 1.**, it is clear that $(X, \mathfrak{M}_1, \mu)$ is a measure space.

(**2**) To show iff condition, we have to show following 2 statements:

(i) $f(x) = c$ a.e. $[\mu] \quad \Rightarrow \quad f$ is measurable on $\mathfrak{M}_1$.

For open $V \subseteq \mathbb{R}$, ${}^\exists N$ s.t. $\mu(N) = 0$, $f(x) = c$ where $x \in X \setminus N$.
Let's show $\mu(f^{-1}(V)) = 0$ or $\mu(X)$, so that $f^{-1}(V) \in \mathfrak{M}_1$.

If $c \in V$,

$$\begin{aligned} f^{-1}(V) &= f^{-1}(\{c\} \cup V \setminus \{c\}) \\ &= f^{-1}(\{c\}) \cup f^{-1}(V \setminus \{c\}) \\ &\supseteq X \setminus N \end{aligned}$$

$\therefore \mu(X \setminus N) = \mu(X) - \mu(N) = \mu(X) \quad \Rightarrow \quad \mu(f^{-1}(V)) = \mu(X)$
If $c \notin V$, $f(x) = c$ where $x \in X \setminus N$ and it means $f^{-1}(V) \subseteq N$.
$\therefore \mu(f^{-1}(V)) = 0$ ($\because \mu(N) = 0$).
$\therefore f^{-1}(V) \in \mathfrak{M}_1$.

(ii) f is measurable on $\mathfrak{M}_1 \quad \Rightarrow \quad f(x) = c$ a.e. $[\mu]$.
Let $c = \sup\{a \in \mathbb{R} : \mu(f^{-1}((a, \infty)) = \mu(X)\}$. Then for $\epsilon > 0$, $\mu(f^{-1}((c - \epsilon, \infty)) = \mu(X)$ and $\mu(f^{-1}((c + \epsilon, \infty)) = 0$ by the definition of supremum. Then,

$$\begin{aligned}
\mu(X) &= \mu(f^{-1}((c-\epsilon, \infty)) - \mu(f^{-1}((c+\epsilon, \infty)) \\
&= \mu(f^{-1}(c-\epsilon, c+\epsilon])) \\
&\to \mu(f^{-1}(\{c\}) \quad (\text{as } \epsilon \to 0).
\end{aligned}$$

$\therefore f(x) = c$ a.e. $[\mu]$. ($\because \mu(f^{-1}(\mathbb{R} \setminus \{c\})) = 0$). ■

Problem 3. *Let $(X, \mathfrak{M})$ be a measureable space and let $\mu_1, \mu_2, \ldots, \mu_k$ be positive measures on this space. Let $c_1, c_2, \ldots, c_k \geq 0$ be non-negative reals and define $\mu : \mathfrak{M} \to \mathbb{R}$ as*

$$\mu(E) = \sum_{i=1}^{k} c_i \mu_i(E) \quad \text{for E} \in \mathfrak{M}$$

Prove that μ is a measure. (This measure is denoted by $c_1\mu_1 + \cdots + c_k\mu_k$).

Solution. $\mu(\varnothing) = \sum c_i \mu_i(\varnothing) = 0 < \infty$.
Let $A_1, A_2, \ldots \in \mathfrak{M}$ be disjoint sets. Since each μ_i is a measure, $\mu_i(\bigcup A_j) = \sum_j \mu_i(A_j)$.

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^{\infty} A_j\right) &= \sum_{i=1}^{k} c_i \mu_i \left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{i=1}^{k} c_i \sum_{j=1}^{\infty} \mu_i(A_j) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{k} c_i \mu_i(A_j) \\
&= \sum_{j=1}^{\infty} \mu(A_j)
\end{aligned}$$

$\therefore \mu$ is a measure. ■

Problem 4. *Suppose that $(\mathbb{R}, \tau)$ is the standard topological space, i.e.,*

$$\tau = \{A : A \text{ is a countable union of open intervals in } \mathbb{R}\}.$$

Denote by $\mathfrak{B}$ the Borel σ-algebra associated to this topological space. Prove that the following set belongs to $\mathfrak{B}$.

$$A = \{\, x \in \mathbb{R} : x = q_1\sqrt{n_1} + \cdots + q_k\sqrt{n_k} \text{ for some } k \in \mathbb{N},\ q_1, \ldots, q_k \in \mathbb{Q}, \\ \text{and } n_1, \ldots, n_k \in \mathbb{N} \,\}$$

Solution. For $a \in \mathbb{R}$, $[a, \infty) = \bigcup\limits_{n=1}^{\infty} \left(a - \frac{1}{n}, \infty\right)$ and $(-\infty, a] = \bigcup\limits_{n=1}^{\infty} \left(-\infty, a + \frac{1}{n}\right)$, so $[a, \infty), (-\infty, a] \in \tau$ and therefore $\{a\} \in \mathfrak{B}$. Let

$$A_k = \{\, x \in \mathbb{R} : x = q_1\sqrt{n_1} + \cdots + q_k\sqrt{n_k} \text{ for some } q_i \in \mathbb{Q},\ n_i \in \mathbb{N}, \\ i = 1, \cdots, k \,\}$$

Since there exists an injection s.t. $A_k \to \mathbb{Q}^k \times \mathbb{N}^k$ and $\mathbb{Q}$, $\mathbb{N}$ are countable, A_k is at most countable and $A_k = \bigcup_{x \in A_k} \{x\} \in \mathfrak{B}$.

$\therefore A = \bigcup\limits_{k=1}^{\infty} A_k \in \mathfrak{B}$. ∎

Problem 5. *Let $(X, \mathfrak{M}, \mu)$ be a measure space such that $\mu(X) < \infty$ and let $f,\ g,\ h \in L^1(\mu)$. For $n \in \mathbb{N}$, define*

$$B_n = \{x \in X :\ |f(x)| + |g(x)| \leq n\} \in \mathfrak{M}.$$

Prove that

$$\lim_{n \to \infty} \int_{B_n} h \ d\mu = \int_X h \ d\mu.$$

Solution. Since $f,\ g \in L^1(\mu)$, $\int_X |f| \ d\mu$, $\int_X |g| \ d\mu < \infty$ and so $\int_X |f| + |g| \ d\mu < \infty$. Then,

$$\begin{aligned} \infty > \int_X |f| + |g| \ d\mu &= \int_{X \cap B_n} |f| + |g| \ d\mu + \int_{X \cap {B_n}^c} |f| + |g| \ d\mu \\ &> \int_{X \cap B_n} |f| + |g| \ d\mu + n \cdot \mu(X \cap {B_n}^c) \end{aligned}$$

$\therefore \mu(X \setminus B_n) \to 0$ as $n \to \infty$.
Then,

$$\begin{aligned}
\int_X |h - h\chi_{B_n}| \, d\mu &= \int_X |h|(1 - \chi_{B_n}) \, d\mu \\
&= \int_X |h| \, d\mu - \int_X |h|\chi_{B_n} \, d\mu \\
&= \int_{B_n} |h| \, d\mu + \int_{X \setminus B_n} |h| \, d\mu - \int_X |h|\chi_{B_n} \, d\mu \\
&= \int_{X \setminus B_n} |h| \, d\mu \to 0 \quad \text{as} \quad n \to \infty \quad (\because h \in L^1(\mu))
\end{aligned}$$

Since $0 \leq \left|\int_X h - h\chi_{B_n} \, d\mu\right| \leq \int_X |h - h\chi_{B_n}| \, d\mu \to 0$, $\lim\limits_{n\to\infty} \int_X h - h\chi_{B_n} \, d\mu = 0$.

$$\therefore \int_X h \, d\mu = \lim_{n\to\infty} \int_X h\chi_{B_n} \, d\mu = \lim_{n\to\infty} \int_{B_n} h \, d\mu.$$

■

Problem 6. *Let $(X, \mathfrak{M}, \mu)$ be a complete measure space and let $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$ be sequences of measurable functions such that*

$$f_m \leq g_n \quad \text{a.e. } [\mu] \quad \text{for all } m, n \in \mathbb{N}.$$

Prove that

$$\sup_{n\in\mathbb{N}} f_n \leq \int_{n\in\mathbb{N}} g_n \quad \text{a.e. } [\mu].$$

Solution. There exists $N \in \mathfrak{M}$ s.t. $\mu(N) = 0$, and ${}^{\exists}m, n \in \mathbb{N}$ s.t. $f_m(x) > g_n(x)$ for all $x \in N$. Then, $f_m(x) \leq g_n(x)$ for ${}^{\forall}m, n \in \mathbb{N}$, ${}^{\forall}x \in X \setminus N$.
It suffices to show that $\sup f_n(x) \leq \inf g_n(x)$, ${}^{\forall}x \in X \setminus N$.
Let $x \in X \setminus N$ is given and assume that $\sup f_n(x) > \inf g_n(x)$.
Then, ${}^{\exists}M \in \mathbb{N}$ satisfying followings:

$$\begin{cases} f_M(x) > \sup f_n(x) - \frac{1}{2}\left(\sup f_n(x) - \inf g_n(x)\right) = \frac{1}{2}\left(\sup f_n(x) + \inf g_n(x)\right) \\ g_M(x) < \inf g_n(x) + \frac{1}{2}\left(\sup f_n(x) - \inf g_n(x)\right) = \frac{1}{2}\left(\sup f_n(x) + \inf g_n(x)\right) \end{cases}$$

It concludes that $g_M(x) < f_M(x)$, which contradicts to $f_m \leq g_n$, ${}^{\forall}m, n$.
$\therefore \sup f_n(x) \leq \inf g_n(x)$, ${}^{\forall}x \in X \setminus N$, $\mu(N) = 0$.
$\therefore \sup f_n(x) \leq \inf g_n(x)$ a.e. $[\mu]$. ■

Problem 7. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $f \in L^1(\mu)$. Prove that, for all $\epsilon > 0$, there exists a simple measurable function s such that*

$$\int_X |f - s| \, d\mu < \epsilon.$$

Solution. If above statment holds on all non-negative lebesgue measurable functions, then since $f = f^+ - f^-$ and $s = s^+ - s^-$,

$$0 \leq \int_X |f - s| \, d\mu \leq \int_X |f^+ - s^+| + |f^- - s^-| \, d\mu < 2\epsilon$$

WLOG, it suffices to show above statement where $f \geq 0$.
Thm 1.17 $\Rightarrow$ $^\exists$ simple $(s_n)_{n=1}^\infty : X \to [0, \infty)$ s.t. $s_n \nearrow f$. Then $s_n \in L^1(\mu)$.
Then, we can see following holds:

$$\begin{cases} (s_n)_{n=1}^\infty, f : X \to \mathbb{R} \text{ MFs.} \\ \lim_{n\to\infty} s_n(x) = f(x), \ ^\forall x \in X. \\ ^\exists f \in L^1(\mu) \text{ s.t. } |s_n(x)| \leq f(x), \ ^\forall x \in X, \ ^\forall n \in \mathbb{N}. \end{cases}$$

DCT $\Rightarrow \int_X |f - s_n| \, d\mu \to 0$ as $n \to \infty$.
$\therefore$ $^\exists N \in \mathbb{N}$ s.t. $\int_X |f - s_N| \, d\mu < \epsilon$ and take $s = s_N$. ■

Problem 8. *Let $X = \{1, 2, \ldots, 3n\}$ for some positive integer $n \in \mathbb{N}$, and let*

$$\mathfrak{M} = \{A \subset X : \ |\{3k - 2, 3k - 1, 3k\} \cap A| = 0 \text{ or } 3 \text{ for all } 1 \leq k \leq n\}.$$

(**1**) *Prove that $(X, \mathfrak{M})$ is a measurable space.*
(**2**) *Let $(a_k)_{k=1}^{3n}$ be a sequence of non-negative reals. Define $\mu : \mathfrak{M} \to [0, \infty)$ as*

$$\mu(A) = \sum_{i \in A} a_i.$$

Prove that μ is a measure on $(X, \mathfrak{M})$ and hence $(X, \mathfrak{M}, \mu)$ is a measure space.
(**3**) *Suppose that the sequence $(a_k)_{k=1}^{3n}$ given above satisfies $a_1 = a_2 = a_3 = 0$ and $a_k > 0$ for all $k \geq 4$. Prove that the measure space $(X, \mathfrak{M}, \mu)$ is not complete and derive the completion of the measure space $(X, \mathfrak{M}, \mu)$.*

Solution. (**1**) To show that $(X, \mathfrak{M})$ is a measurable space, we have to show three things:

① $|\{3k-2, 3k-1, 3k\} \cap X| = 3$ for all $k = 1, 2, \ldots, n \quad \Rightarrow \quad X \in \mathfrak{M}$
② Assume $A \in \mathfrak{M}$. For $k = 1, 2, \ldots, n$,

$$\begin{cases} |\{3k-2, 3k-1, 3k\} \cap A| = 3 & \Rightarrow \quad \{3k-2, 3k-1, 3k\} \subseteq A \\ & \Rightarrow \quad |\{3k-2, 3k-1, 3k\} \cap A^c| = 0 \\ |\{3k-2, 3k-1, 3k\} \cap A| = 0 & \Rightarrow \quad |\{3k-2, 3k-1, 3k\} \cap A^c| = 3 \end{cases}$$

$\therefore A^c \in \mathfrak{M}$.
③ $A_1, A_2, \ldots \in \mathfrak{M}$ is given.
For $k = 1, 2, \ldots, n$,
i) $|\{3k-2, 3k-1, 3k\} \cap A_i| = 0$, ${}^\forall i$

$$\begin{aligned} \varnothing &= \bigcup_{i=1}^{\infty} (\{3k-2, 3k-1, 3k\} \cap A_i) \\ &= \{3k-2, 3k-1, 3k\} \cap \left(\bigcup_{i=1}^{\infty} A_i \right) \end{aligned}$$

$\therefore |\{3k-2, 3k-1, 3k\} \cap \bigcup A_i| = 0$
ii) ${}^\exists A_i$ s.t. $|\{3k-2, 3k-1, 3k\} \cap A| = 3$

$$\begin{aligned} \{3k-2, 3k-1, 3k\} &\subseteq \bigcup_{i=1}^{\infty} (\{3k-2, 3k-1, 3k\} \cap A) \\ &= \{3k-2, 3k-1, 3k\} \cap \bigcup A_i \\ &\subseteq \{3k-2, 3k-1, 3k\} \end{aligned}$$

$\therefore |\{3k-2, 3k-1, 3k\} \cap \bigcup A_i| = 3$
$\therefore \bigcup A_i \in \mathfrak{M}$
$\therefore (X, \mathfrak{M})$ is a measurable space.

(**2**) Since $\mu(\varnothing) = 0 < \infty$, it suffices to show countable additivity of μ.
Let $A_1, A_2, \ldots \in \mathfrak{M}$ be disjoint sets.
Since $|X| < \infty$ and $\mathfrak{M} \subseteq 2^X$, $|\mathfrak{M}| < \infty$ and ${}^\exists N$ s.t. $A_{N+1} = A_{N+2} = \cdots = \varnothing$.
Then, when we show that $A_1, A_2 \in \mathfrak{M}$ disjoint $\Rightarrow \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$, it can be easily expanded to conclusion by induction.

$$\mu(A_1 \cup A_2) = \sum_{i \in A_1 \cup A_2} a_i = \sum_{i \in A_1} a_i + \sum_{i \in A_2} a_i = \mu(A_1) + \mu(A_2).$$

$\therefore$ $(X, \mathfrak{M}, \mu)$ is a measure space.

(**3**) First of all, let's show that given measure space $(X, \mathfrak{M}, \mu)$ is not complete.
Let $N = \{1,2,3\} \in \mathfrak{M}$ and $A = \{1\} \subseteq N$.
Then $\mu(N) = 0$ but $A \notin \mathfrak{M}$ ($\because$ $|\{1,2,3\} \cap A| = 1$), so we can conclude that $(X, \mathfrak{M}, \mu)$ is not compelete.
The completion of above measure space is,

$$\mathfrak{M}^* = \{E \subseteq X : \ {}^{\exists}A, B \in \mathfrak{M} \text{ s.t. } A \subseteq E \subseteq B,\ \mu(B \setminus A) = 0\}$$

Let's find largest $N \subseteq X$ s.t. $\mu(N) = 0$.

$$\mu(N) = \sum_{i \in N} a_i = 0 \quad \Rightarrow \quad N = \varnothing \text{ or } a_i = 0,\ {}^{\forall}i \in N$$

$\therefore$ $N \in 2^{\{1,2,3\}}$
$\therefore$ $\mathfrak{M}^* = \{N \cup E : \ E \in \mathfrak{M},\ N \in 2^{\{1,2,3\}}$.
Extend μ so that satisfies $\mu(N \cup E) = \mu(E)$.
Then, $(X, \mathfrak{M}^*, \mu)$ is a complete measure space. ■