Assignment #5

real analysis

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Problem 1. *Let X be a topological space and $K \subset X$ be a compact set, let $C(K)$ be the collection of real-valued continuous functions on K (which is a vector space associated with usual addition and scalar multiplication of functions). The sup-norm on $C(K)$ is defined by,*

$$\|f\|_{\sup} = \sup_{x \in K} |f(x)| \quad \text{for } f \in C(K).$$

Prove that the vector space $C(K)$ equipped with this norm is a Banach space.

Solution. First of all, let's show that $C(K)$ is NLS.

(i) $\|f+g\| \leq \|f\| + \|g\|$
$|f(x)+g(x)| \leq |f(x)| + |g(x)|, \ ^{\forall}x \in K.$

$$\begin{aligned}\|f+g\| &= \sup_{x \in K} |f(x)+g(x)| \\ &\leq \sup_{x \in K} |f(x)| + \sup_{x \in K} |g(x)| \\ &= \|f\| + \|g\|\end{aligned}$$

(ii) $\|\alpha f\| = |\alpha| \, \|f\|, \ ^{\forall}\alpha \in \mathbb{R}$
$|\alpha f(x)| = |\alpha| \, |f(x)|, \ ^{\forall}x \in K.$

$$\begin{aligned}\|\alpha f\| &= \sup_{x \in K} |\alpha f(x)| \\ &= |\alpha| \sup_{x \in K} |f(x)| \\ &= |\alpha| \, \|f\|\end{aligned}$$

(iii) $\|f\| \geq 0$, $\|f\| = 0 \iff f = 0.$

$\|f\| \geq 0$ is trivial.

$$\begin{aligned}\|f\| = 0 &\iff \sup_{x\in K} |f(x)| = 0 \\ &\iff 0 \leq |f(x)| < \varepsilon,\ {}^{\forall}x \in K,\ {}^{\forall}\varepsilon > 0 \\ &\iff f = 0\end{aligned}$$

Then, we only have to show that $C(K)$ is complete.
Let $(f_n)_{n=1}^{\infty} \in C(K)$ is Cauchy sequence.
For $\varepsilon > 0$, ${}^{\exists}N$ s.t. $m, n > N \ \Rightarrow\ \|f_m - f_n\| < \varepsilon$.
$m, n > N \ \Rightarrow\ |f_m(x) - f_n(x)| \leq \|f_m - f_n\| < \varepsilon,\ {}^{\forall}x \in K$.
$\therefore x \in K \ \Rightarrow\ {}^{\exists}f(x)$ s.t. $\lim_{n\to\infty} f_n(x) = f(x)$.
We know that $m, n > N \ \Rightarrow\ |f_m(x) - f_n(x)| < \varepsilon,\ {}^{\forall}x \in K$.
As $m \to \infty$, it becomes $n > N \ \Rightarrow\ |f(x) - f_n(x)| \leq \varepsilon < 2\varepsilon,\ {}^{\forall}x \in K$.
Since f_n is continuous,
${}^{\exists}$ open $U \subset K$ s.t. $x, y \in U \ \Rightarrow\ |f_n(x) - f_n(y)| < \varepsilon$.
Then for $x, y \in U$,

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< 2\varepsilon + \varepsilon + 2\varepsilon = 5\varepsilon.\end{aligned}$$

$\therefore f$ is continuous. ■

Problem 2. *Let $p \in [1, \infty)$ and let Λ be a linear functional on $L^p(I)$ defined by*

$$\Lambda(f) = \int_0^1 e^{2x} f(x)dx \quad \text{for } f \in L^p(I).$$

(1) Prove that Λ is a bounded and compute $\|\Lambda\|$.

(2) Find all $f \in L^p(I)$ with $\|\Lambda(f)\| = \|\Lambda\|\|f\|$.

Solution.

(1) Λ is bounded.
By Hölder's Inequality,

$$\int_I |f|\, dm = \int_I |f| \cdot 1\ dm \leq \left[\int_I |f|^p\, dm\right]^{1/p} \left[\int_I 1\ dm\right]^{1/q} = \|f\|_p < \infty.$$

$$
\begin{aligned}
|\Lambda f| = \left|\int_I e^{2x} f(x) dx\right| &\leq e^2 \left|\int_I f(x) dx\right| \\
&\leq e^2 \int_I |f| \, dm \\
&\leq e^2 \|f\|_p
\end{aligned} \tag{1}
$$

$$
\therefore \|\Lambda\| = \sup\left\{\frac{|\Lambda f|}{\|f\|_p} : f \neq 0\right\} \leq e^2 < \infty.
$$

Let's compute $\|\Lambda\|$.

(i) $p = 1$.
Consider $(f_n)_{n=1}^{\infty}$ s.t. $f_n = \chi_{[1-\frac{1}{n},1]}$, $n = 1,\ 2,\ \cdots$.
Since $\|f_n\|_1 = \frac{1}{n} < \infty$, $(f_n)_{n=1}^{\infty} \in L^1(I)$.

$$
\begin{aligned}
|\Lambda f_n| &= \int_{1-\frac{1}{n}}^{1} e^{2x} dx = \frac{e^2}{2}\left(1 - e^{-\frac{2}{n}}\right), \\
\|\Lambda\| &\geq \frac{|\Lambda f_n|}{\|f_n\|_1} = \frac{e^2}{2} n \left(1 - e^{-\frac{2}{n}}\right).
\end{aligned}
$$

Take $\lim_{n\to\infty}$ both sides. Then,

$$
\begin{aligned}
\|\Lambda\| \geq \frac{e^2}{2} \lim_{n\to\infty} n\left(1 - e^{-\frac{2}{n}}\right) &= \frac{e^2}{2} \lim_{h\to 0+} \frac{1 - e^{-2h}}{h} \\
&= \frac{e^2}{2} \lim_{h\to 0+} \frac{2e^{-2h}}{1} \\
&= e^2.
\end{aligned}
$$

$\therefore \|\Lambda\| = e^2$.

(ii) $p > 1$.
Assume $f \geq 0$.
By Hölder's Inequality, for q s.t. $\frac{1}{p} + \frac{1}{q} = 1$,

$$
\begin{aligned}
|\Lambda f| = \int_I e^{2x} f(x) dx &\leq \left[\int_I |f(x)|^p \, dx\right]^{1/p} \left[\int_I e^{2qx} dx\right]^{1/q} \\
&= \|f\|_p \left(\frac{1}{2q}\left(e^{2q} - 1\right)\right)^{1/q}.
\end{aligned}
$$

$$\therefore \|\Lambda\| = \sup_{f\neq 0} \frac{|\Lambda f|}{\|f\|_p} = \left(\frac{1}{2q}\left(e^{2q}-1\right)\right)^{1/q}.$$

($\because$ The equality holds where $f(x) = ce^{2\frac{q}{p}x}$ a.e. for some c and that $f \in L^p(I)$.)

(2) For $f = 0$ a.e., the equality holds.
If $p > 1$, $f(x) = ce^{2\frac{q}{p}x}$ a.e. for $c \in \mathbb{R}$. ($\because$ Hölder's Inequality)
If $p = 1$, $f = 0$ a.e. ($\because$ (1) holds only if $f = 0$ a.e.)

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Problem 3. *Let Λ be a linear functional on $C(I)$ (cf. Problem 1) defined by*

$$\Lambda(f) = \int_0^1 xf(x)dx \quad \text{for } f \in C(I).$$

(1) Prove that Λ is a bounded and compute $\|\Lambda\|$.

(2) Find all $f \in C(I)$ with $\|\Lambda(f)\| = \|\Lambda\|\|f\|$.

(3) Define a subspace X of $C(I)$ as

$$X = \{f \in C(I) \ : \ f(1) = 0\}$$

Denote by $\Lambda_X = \Lambda\,|_X$ the restriction of Λ to X. Prove that $\|\Lambda_X\| = \|\Lambda\|$ and find all $f \in X$ such that $\|\Lambda_X(f)\| = \|\Lambda_X\|\|f\|$.

Solution.

(1) $\|\Lambda\| = \sup\{|\Lambda f| : f \in C(I), \|f\|_{\sup} = 1\}$.
For $f \in C(I)$ s.t. $\|f\|_{\sup} = 1$,

$$\begin{aligned} |\Lambda f| = \left|\int_I xf(x)dx\right| &\leq \int_I |xf(x)|\,dx \\ &= \int_I |x|\,|f(x)|\,dx \\ &\leq \|f\|_{\sup}\int_I |x|\,dx = \frac{1}{2} \end{aligned}$$

For $f = 1$, $\|f\|_{\sup} = 1$ and $|\Lambda f| = \int_I xdx = \frac{1}{2}$.
$\therefore \|\Lambda\| = \frac{1}{2}$.

(2) Claim: $f = c \iff \left|\int_I xf(x)dx\right| = \frac{1}{2}\|f\|_{\sup}$ for $c \in \mathbb{R}$.
($\Rightarrow$) is trivial.
($\Leftarrow$) Assume that $|f(x_0)| \neq \|f\|_{\sup}$ for some $x_0 \in [0,1]$.
WLOG, $f \geq 0$. (Consider f as $|f|$.)
For $\varepsilon = \|f\|_{\sup} - f(x_0) > 0$, $^{\exists}\delta > 0$ s.t.

$$|x - x_0| < \delta \ \Rightarrow \ |f(x) - f(x_0)| < \frac{\varepsilon}{2} = \frac{1}{2}\left(\|f\|_{\sup} - f(x_0)\right).$$

Then, for $x \in (x_0 - \delta, x_0 + \delta)$,

$$f(x) < f(x_0) + \frac{\varepsilon}{2} = \frac{1}{2}\left(\|f\|_{\sup} + f(x_0)\right) = \|f\|_{\sup} - \frac{\varepsilon}{2}.$$

Let $B = (x_0 - \delta, x_0 + \delta) \cap [0,1]$, and $m(B) > 0$. Then,

$$\begin{aligned}\left|\int_I xf(x)dx\right| = \int_I xf(x)dx &= \int_B xf(x)dx + \int_{B^c} xf(x)dx \\ &\leq \int_B x(\|f\|_{\sup} - \frac{\varepsilon}{2})dx + \int_{B^c} x\|f\|_{\sup}dx \\ &= \|f\|_{\sup}\int_I xdx - \frac{\varepsilon}{2}\int_B xdx \\ &< \frac{1}{2}\|f\|_{\sup}.\end{aligned}$$

(3) Consider $(f_n)_{n=1}^{\infty}$ defined by

$$f_n(x) = \begin{cases} 1 & , 0 \leq x < 1 - \frac{1}{n} \\ -n(x-1) & , 1 - \frac{1}{n} \leq x \leq 1 \end{cases}.$$

Then $f_n \in X$ and $\|f_n\|_{\sup} = 1$, $^{\forall}n \in \mathbb{N}$.

$$\begin{aligned}|\Lambda_X f_n| = \int_I xf_n(x)dx &= \int_0^{1-1/n} xdx + \int_{1-1/n}^1 -nx(x-1)dx \\ &= \frac{1}{2}\left(1 - \frac{1}{n}\right)^2 + n\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{1-1/n}^1 \\ &= \frac{1}{2}\left(1 - \frac{1}{n}\right)^2 + \frac{1}{2n} - \frac{1}{3n^2}.\end{aligned}$$

$\therefore \|\Lambda_X\| \geq \lim_{n\to\infty} |\Lambda_X f_n| = \frac{1}{2}$.
From the **Solution** of 3-(1), $\|\Lambda_X\| \leq \frac{1}{2}$. ($\because X \leq C(I)$.)
$\therefore \|\Lambda_X\| = \frac{1}{2} = \|\Lambda\|$.
From the **Solution** of 3-(2), only $f = 0$ satisfies $\|\Lambda_X(f)\| = \|\Lambda_X\|\|f\|$.

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Problem 4. *For $n \in \mathbb{N}$ and $1 \le p < \infty$, let $X_n \subset L^p(I)$ be a collection of polynomials of degree $\le n$ and let $X = \cup_{n=1}^{\infty} X_n$.*

(1) For $n \in \mathbb{N}$, is there a bounded linear functional Λ on $L^p(I)$ such that $\Lambda(f) = f'(0)$ for all $f \in X_n$?

(2) Is there a bounded linear functional Λ on $L^p(I)$ such that $\Lambda(f) = f'(0)$ for all $f \in X$?

Solution.

(1) Consider $\Lambda_n : X_n \to \mathbb{R}$ s.t. $\Lambda_n(f) = f'(0)$ for all $f \in X_n$.
Λ_n is a BLF $\iff$ Λ_n is continuous at $0 \in X_n$.
Let $(f_i)_{i=1}^{\infty} \in X_n$ s.t. $f_i \to 0$ in L^p. Then $\lim_{i\to\infty} \int_I |f_i|^p \, dm = 0$.
$\int_I |f_i| \, dm \le \int_I |f_i|^p \, dm \to 0 \;\Rightarrow\; \int_I |f_i| \, dm \to 0$.
Since $f_i \in X_n$, all coefficients of f_i go to 0 as $i \to \infty$.
$\therefore |\Lambda_n f_i| \to 0$, i.e., .$\Lambda_n$ is continuous at 0.
$\therefore \Lambda_n$ is a BLF.
By Hahn-Banach Theorem, $^\exists$ extension $\Lambda : L^p(I) \to \mathbb{R}$ of Λ_n s.t. Λ is a BLF. (It is obvious that $X_n \le L^p(I)$.)

(2) Consider $(f_n)_{n=1}^{\infty}$ s.t. $f_n(x) = (1-x)^n \in X_n \subseteq X$.
$\int_I |f_n(x)|^p \, dx = \frac{1}{np+1} < \infty$.
For any linear functional Λ s.t. $\Lambda(f) = f'(0)$,

$$|\Lambda f_n| = \left| -n(1-x)^{n-1}\big|_{x=0} \right| = n \to \infty,$$
$$\frac{|\Lambda f_n|}{\|f\|_p} = n(np+1) \to \infty.$$

$\therefore \Lambda$ is not bounded.

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Problem 5. *For $k \in \mathbb{N}$, denote by $C^k(I)$ the space of real-valued functions on I possessing continuous derivatives up to order k on I, including one-sided derivative at the end points 0 and 1. Define, for $p \in [1, \infty)$,*

$$\|f\| = \sum_{i=0}^{k} \|f^{(i)}\|_{L^p([0,1])} \quad \text{for } f \in C^k(I)$$

where $f^{(i)}$ denotes ith derivative of f.

(1) Prove that $\|\cdot\|$ is a norm on $C^k(I)$.

(2) Prove that the space $C^k(I)$ equipped with the norm $\|\cdot\|$ is not a Banach space. (The completion of this space is called a Sobolev space which is a central Banach space in the study of PDEs).

Solution.

(1) Let's show that $\|\cdot\|$ is a norm.

(i) $\|f+g\| \le \|f\| + \|g\|$

$$\begin{aligned} \|f+g\| &= \sum_{i=0}^{k} \|(f+g)^{(i)}\|_p = \sum_{i=0}^{k} \|f^{(i)} + g^{(i)}\|_p \\ &\le \sum_{i=0}^{k} \|f^{(i)}\|_p + \|g^{(i)}\|_p = \|f\| + \|g\|. \end{aligned}$$

(ii) $\|\alpha f\| = |\alpha|\,\|f\|$.

$$\|\alpha f\| = \sum_{i=0}^{k} \|(\alpha f)^{(i)}\|_p = \sum_{i=0}^{k} |\alpha|\,\|f^{(i)}\|_p = |\alpha|\,\|f\|.$$

(iii) $\|f\| \ge 0$, $\|f\| = 0 \iff f = 0$.

$$\|f\| = \sum_{i=0}^{k} \|f^{(i)}\|_p \ge 0$$

$$\begin{aligned} \|f\| = 0 \ &\Rightarrow\ \sum_{i=0}^{k} \|f^{(i)}\|_p = 0 \\ &\Rightarrow\ \|f\|_p = 0 \\ &\Rightarrow\ f = 0. \end{aligned}$$

Opposite side of iff condition is trivial.

(2) Consider $(f_n)_{n=1}^{\infty} \in C^k(I)$ s.t. $f_n(x) = \left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{n}\right)^{k/2}$.

Then, $f_n \to f$ where $f(x) = \left|x - \frac{1}{2}\right|^k$, which is not contained at $C^k(I)$.
For $i = 0, 1, \cdots, k$,
$f_n^{(i)} \to f^{(i)}$ and $f_n^{(i)} \ge f_{n+1}^{(i)}$, which lead to $f_n^{(i)}$ converges to $f^{(i)}$ uniformly.

($\because f_n^{(i)}$ is continuous, Thm 7.13 of PMA.)
$\therefore f_n^{(i)}$ is uniformly Cauchy.
$\therefore f_n$ is Cauchy with the norm $\|\cdot\|$.
Thus, f_n is Cauchy sequence in $C^k(I)$ but $f = \lim f_n$ is not the element in $C^k(I)$, which means that $C^k(I)$ is not a Banach Space.

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Problem 6. *Let X be a Banach space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that, for some $x \in X$,*

$$\lim_{n\to\infty} \ell(x_n) = \ell(x) \text{ for all bounded linear functional } \ell : X \to \mathbb{R}.$$

Prove that

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

Solution. It is trivial for $x = 0$.
For $x \in X \setminus \{0\}$, by **Thm 5.20**,
$\exists$ a bounded linear functional $\ell_x : X \to \mathbb{R}$ s.t. $\ell_x(x) = \|x\|$ and $\|\ell_x\| = 1$.
Then, with the fact that $|\ell_x(x_n)| \le \|\ell_x\|\|x_n\|$,

$$\begin{aligned}
\|x\| = |\ell_x(x)| &= \lim_{n\to\infty} |\ell_x(x_n)| \\
&= \liminf_{n\to\infty} |\ell_x(x_n)| \\
&\le \liminf_{n\to\infty} \|\ell_x\|\|x_n\| \\
&= \liminf_{n\to\infty} \|x_n\|.
\end{aligned}$$

$\blacksquare$